

Topological sum rules and spectral flows of chiral and gravitational axionlike interactions

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We examine the structure of off-shell effective actions arising from chiral and gravitational anomalies, focusing on the $\langle JJJ_A \rangle$ (vector/vector/axial-vector) and $\langle TTJ_A \rangle$ (stress-energy tensors/axial-vector) correlators, relevant in the analysis of anomaly-driven interactions in perturbation theory. Our approach employs both perturbative methods and techniques in conformal field theory within momentum space, extended to encompass chiral anomalies. The analysis centers on the presence of both particle poles and anomaly poles within these interactions, characterizing their behavior both in the conformal and nonconformal limits. Through explicit computations, we show that universal sum rules in the longitudinal sector regulate these interactions for all kinematic conditions, extending previous analysis. We discuss the resulting spectral flow, an “area law” of the absorptive part of the anomaly form factors as one moves away from or returns to the conformal point. These features are absent in the local effective action of anomaly interactions, commonly used in the description of axionlike particles. Our results further show that anomaly poles correspond to true particle poles only in the conformal limit, when the interaction describes a massless S-matrix process supported on a null surface. Under these conditions, they can be effectively described by two kinetically mixed pseudoscalar fields propagating on the light cone. These studies find application in polarized deeply inelastic scattering, axionlike dark matter, and analog systems such as topological materials.

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I. INTRODUCTION

A direct approach to studying chiral and gravitational anomalies is through perturbative analysis of the one-particle-irreducible (1PI) effective action, particularly the axial-vector/vector/vector (AVV or $\langle JJJ_A \rangle$) and axial-vector/stress-energy/stress-energy (ATT or $\langle TTJ_A \rangle$) correlators. Here, J_A denotes the axial-vector current, J the vector currents, and T the stress-energy tensor, which couples to gravity. The ATT vertex extends anomaly considerations into the gravitational sector.

Axial-vector currents may arise from chiral fermions in external fields or through interactions with gravitational waves [1]. If the early Universe experienced a conformal phase, such anomaly-induced interactions could enhance fermionic chiral asymmetries. In the ATT case, gravitational waves may further amplify chiral chemical potentials [2],

impacting primordial magnetohydrodynamics and the evolution of cosmic magnetic fields [3–5].

In a conformal phase, the three-point functions involving conserved currents and the stress-energy tensor are uniquely determined—up to an overall constant—by conformal symmetry. When chiral anomalies are present, these correlators are likewise uniquely fixed, again modulo an overall constant, by simultaneously imposing conformal constraints and diffeomorphism invariance. Remarkably, the full structure of these solutions can be reproduced entirely within ordinary free field theory realizations.

The study of the effective action associated with these interactions—manifesting in the ultraviolet (UV) regime as the exchange of intermediate states in the form of anomaly poles in the conformal limit—is characterized by the emergence of sum rules rather than of conventional asymptotic states, saturated by spectral densities which are simply proportional to δ functions. These intermediate configurations can be interpreted as ordinary particles only under specific kinematic conditions, namely when the dynamics are constrained to the light cone. In other words, these intermediate states are not ordinary particle states.

A related phenomenon occurs in quantum chromodynamics (QCD) at small values of the Bjorken variable x ,

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where the so-called perturbative pomeron emerges. Unlike ordinary particles, the perturbative pomeron cannot be associated with a well-defined asymptotic state in the conventional sense.

The purpose of this work is to investigate these features for the two correlators discussed above, exploring their full kinematical structure beyond the conformal limit. This is achieved by including perturbative effects due to fermion masses and demonstrating the existence of mass-independent sum rules in all cases. In a related work, we are going to show that a similar result holds for conformal correlators such as the TJJ , involving the stress-energy tensor and non-Abelian vector currents.

A key result of our analysis is that the local effective action for an axionlike particle—originating in the UV through chiral anomaly interactions in a conformal setting—necessarily requires, for its extension at low energy, the inclusion of a nonperturbative intermediate scale in order to be consistently defined as a local action for an asymptotic axion. The only requirement in the UV is described by a conformal interaction. This finding applies not only to standard chiral anomaly interactions but also to gravitational anomalies. Our detailed investigation around the conformal point demonstrates that the presence of a sum rule is essential to reach such a conclusion.

Our analysis focuses on both correlators. In the conformal limit, the framework is based on momentum-space conformal field theory, where anomaly poles in the longitudinal sector of J_A play a central role in solving the conformal Ward identities for both correlators [6–9]. These poles are fundamental to understanding both parity-odd chiral anomalies and parity-even trace anomalies in the conformal limit, and they are intricately linked to the full vertex structure, including the transverse components [10–12].

As we move away from the conformal point, these interactions are characterized by specific sum rules satisfied by the corresponding anomaly form factors, which we examine in detail. Their spectral densities exhibit intriguing cancellation patterns between localized spectral poles and continuum contributions—structures that, to our knowledge, have not been previously analyzed with such precision. The spectral analysis is performed in full generality. As we will demonstrate, these cancellations are closely tied to the structure of the effective action that is expected to reproduce the anomalies under broad kinematic conditions.

A. Local and nonlocal effective actions

A central question addressed in this work is whether anomaly poles appearing in the off-shell effective action—resembling axionlike interactions—can be associated with actual asymptotic axionlike states. This issue is particularly relevant given the strong interest in axionlike particles as potential dark matter candidates. The inclusion of such states on purely phenomenological grounds has often been

proposed to explain various astrophysical observations (see the discussion in [13]).

In standard quantum field theory, these particles are typically characterized by chiral anomaly interactions. In the conformal limit, such interactions can be analyzed using momentum-space conformal field theory (CFT_p), through three-point functions governed by a nonlocal effective action. This action captures the essential features we investigate, including the emergence of anomaly poles and the associated sum rules.

A key part of our analysis is to demonstrate that the effective action is intrinsically nonlocal and does not, in general, support the existence of an asymptotic axionlike state. In particular, obtaining a local action from this structure would require enforcing conformal Ward identities that project out the transverse components of the anomaly vertex. However, doing so invalidates the sum rule central to our findings. The only viable path to a local effective action involves strong coupling effects that evade the UV constraints emerging from the conformal regime—mirroring the situation with standard Peccei-Quinn axions. In those cases, the modification of the functional measure (equivalent to including anomalous Ward identities) is crucial to deriving a consistent local action.

To clarify this further, we investigate the spectral densities of the anomaly form factors that manifest both anomaly poles and continuum contributions. Anomaly poles reflect the topological nature of anomaly vertices: they appear or vanish depending on the kinematic regime, governed by whether the relevant sum rule is saturated by a pole (discrete state) or by a continuum (dispersive cut). Our analysis shows that, in the perturbative regime, these interpolating states can be identified with asymptotic axionlike particles only when the interaction is constrained to the light cone. This unusual behavior stems from the intricate mixing between longitudinal and transverse sectors of the anomaly vertex, governed by Schouten identities.

These identities enable cancellations or enhancements in the residue structure of the form factors. Crucially, anomaly vertices are uniquely characterized by a sum rule always satisfied by their longitudinal form factors—a feature absent in purely local axionlike couplings of the form $a(x)F\tilde{F}$. While such local interactions relate to the full anomaly vertex via a Ward identity, they lack the spectral structure revealed by a dispersive analysis of both the longitudinal and transverse sectors. This distinction underlines the necessity of our nonlocal approach to fully capture the dynamics of anomaly-induced axionlike behavior.

B. Anomaly sum rules

Sum rules in the AVV interaction have been previously studied in various kinematical configurations [14–16]. A more general perspective on the chiral and conformal anomaly in the effective action is offered in [17], with further investigations into the spectral density of

anomaly form factors, including sum rules in the $\mathcal{N} = 1$ superconformal anomaly multiplet and the Konishi anomaly [18].

Sum rules govern the absorptive amplitudes of anomaly vertices and manifest as area laws tied to the longitudinal form factors. In the AVV vertex, we analyze the spectral densities of these form factors, identifying the structure of poles and cuts in the complex momentum plane ($q^2 \equiv t$), and their interplay before and after dispersion integration. In this work, we build on these foundations by exploring different representations using Schouten identities, which can introduce or eliminate anomaly poles in vertex parameterizations.

We focus on their physical relevance, especially near the conformal limit, where anomaly poles are essential for satisfying conformal Ward identities [10–12]. We adopt a minimal form factor basis and use CFT_p methods for a sector decomposition into longitudinal and transverse components.

Notice that our approach leverages minimal decompositions [7], optimized for the massless limit in parity-even trace-anomalous correlators, and recently extended to parity-odd sectors (AVV, ATT) via Schouten-based parameterizations [10–12]. The transverse sectors are characterized by anomaly form factors, while the longitudinal ones encode the anomaly pole and satisfy the anomalous Ward identities. As we deviate from the conformal point, the dispersive analysis remains centered on these longitudinal sectors, where sum rules apply. Finally, we present theoretical evidence supporting the interpretation of the interpolating states—described in [17] as involving pseudoscalar kinetic mixing—as effectively defined only on the light cone.

II. ORDINARY VS ANOMALOUS COUPLINGS OF GOLDSTONE MODES

In order to uncover the difference between ordinary (nonanomalous) Goldstone modes and their coupling to the divergence of a spontaneously broken symmetry, and the axion case, we proceed with a comparison between the two cases.

In theories with spontaneously broken global symmetries and anomalies, Goldstone modes such as axions can couple to gauge fields through interactions like $a(x)F\tilde{F}$. These couplings arise when a global chiral symmetry with a mixed anomaly—such as the axial-vector–vector (AVV) anomaly—is spontaneously broken. The resulting Nambu-Goldstone (NG) boson couples to the anomaly through a mechanism rooted in Ward identities. This connection is often described using a local interaction, though the full anomaly structure in the currents correlator is fundamentally nonlocal.

To illustrate this, consider a model with a fermion Q and a complex scalar field Φ , invariant under a global chiral $U(1)_A$ symmetry that is anomalous. When Φ acquires a

vacuum expectation value, the symmetry is spontaneously broken, and the resulting phase degree of freedom $\pi(x)$ behaves as a Goldstone boson. The fermion gains mass via a Yukawa interaction with Φ , while the chiral current J_A^μ acquires both fermionic and bosonic components.

The Goldstone mode $\pi(x)$ couples to the divergence of the axial current via

$$\mathcal{L} \sim \pi(x) \partial_\mu J_A^\mu. \quad (2.1)$$

If J_A^μ is anomalous, its divergence includes a term like $F\tilde{F}$, and this gives rise to the axion-gauge field coupling

$$\mathcal{L} \sim \frac{a(x)}{v} F\tilde{F}. \quad (2.2)$$

However, this derivation simplifies the full structure of the anomaly. In the perturbative picture, the anomaly is encoded in the longitudinal part of a three-point function (such as the AVV vertex), which contains a $1/q^2$ pole indicating the exchange of a massless pseudoscalar. This is not a true particle pole but rather a signal of the nonlocal nature of the anomalous interaction. Its presence arises from solving the anomalous Ward identity

$$q_\lambda \Delta^{\lambda\mu\nu}(q, p_1, p_2) \sim a_1 \epsilon^{\mu\nu p_1 p_2}, \quad (2.3)$$

which admits a solution of the form

$$\Delta^{\lambda\mu\nu} \sim \frac{q^\lambda}{q^2} \epsilon^{\mu\nu p_1 p_2}. \quad (2.4)$$

When mass terms are included, the divergence of the axial current gains a contribution from the pseudoscalar density

$$\partial_\mu J_A^\mu = a_1 F\tilde{F} + 2m_Q \bar{Q} \gamma_5 Q, \quad (2.5)$$

that modifies the equation of motion for the Goldstone boson, by adding a mass correction via coupling to the pseudoscalar current. Consequently, the effective interaction becomes

$$\mathcal{L} \sim \frac{a(x)}{v} (F\tilde{F} + 2m_Q \bar{Q} \gamma_5 Q). \quad (2.6)$$

In momentum space, this is reflected in a vertex structure involving both the massless anomaly term and a massive correction encoded in the scalar loop integral C_0 of a three-point function [see (3.13)]. Indeed, the full axion-photon-photon vertex becomes

$$V^{\mu\nu} \sim \epsilon^{\mu\nu p_1 p_2} (1 + 2m^2 C_0(p_1^2, p_2^2, q^2, m^2)). \quad (2.7)$$

While the first term corresponds to the anomaly, the second arises from explicit chiral symmetry breaking via mass terms. Importantly, the anomaly pole (i.e., the $1/q^2$

behavior) dominates in the conformal or massless limit. When fermion masses or off-shell momenta are included, the residue of the pole typically vanishes. This implies that the anomaly pole is not a true propagating particle, but it still defines the dominant asymptotic behavior of the vertex at large momentum transfer.

Furthermore, the local interaction $a(x)F\tilde{F}$, while useful phenomenologically (e.g., in axion models like Peccei-Quinn), does not fully capture the rich tensor structure of the anomaly, especially the interplay between longitudinal and transverse components governed by conformal symmetry. The complete off-shell correlator contains both, and they are linked by the anomaly coefficient in a way that the local description obscures.

In contrast to NG modes from nonanomalous symmetries—which couple derivatively and decouple in the infrared—anomalous NG modes retain a distinct signature due to the anomaly pole. However, as we have mentioned above, this pole does not define a physical asymptotic state unless specific kinematic conditions are met that require a light-cone dynamics. Away from these limits, the full vertex must be retained to describe the interaction accurately.

Finally, while the use of Ward identities to derive local interactions is consistent at tree level or in effective field theory, the full quantum structure of anomaly vertices—especially in gravitational or mixed gauge-gravitational contexts—requires a nonlocal formulation. This ensures correct matching to perturbative computations, respects the topological origin of the anomaly, and captures sum rules and dispersion relations not visible in local couplings.

III. RECONSTRUCTING THE AVV CORRELATOR FROM THE ANOMALY POLE AND CONFORMAL SYMMETRY: REVIEW

We begin our analysis in the conformal limit, where the conformal constraints in momentum space, combined with the inclusion of a pole in the solution of these identities for the AVV correlator, play a crucial role. In this regime, the conformal constraints enable the complete identification of parity-odd anomaly interactions using only a single input—an anomaly pole in the longitudinal sector of each correlator. This result is inherently nonperturbative, as it does not rely on any specific Lagrangian formulation. However, once we move away from the conformal point and introduce massive fermions in virtual corrections, this picture undergoes significant modifications. Before delving into these corrections, which will allow for a deeper exploration of the sum rule associated with this correlator, we first provide a brief review of how the presence of a pole, together with the conformal Ward identities, enables the reconstruction of anomaly interactions in the conformal limit. This approach further facilitates the characterization of all form factors in the decomposition of these correlators into longitudinal and transverse sectors.

We start by imposing the external Ward identities in flat space-time for a vector and an axial-vector current respectively as quantum averages

$$\partial_\mu \langle J^\mu \rangle = 0, \quad \partial_\mu \langle J_A^\mu \rangle = a_1 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (3.1)$$

The Ward identities for the AVV are extracted from (3.1) by functional differentiation

$$\begin{aligned} p_{i\mu_i} \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J_A^{\mu_3}(p_3) \rangle &= 0, \quad i = 1, 2, \\ p_{3\mu_3} \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J_A^{\mu_3}(p_3) \rangle &= -8ia_1 \epsilon^{p_1 p_2 \mu_1 \mu_2}, \end{aligned} \quad (3.2)$$

where a_1 is the anomaly coefficient.

Whenever we will be using a symmetric (1, 2, 3) labeling of the momenta, useful for the conformal parametrization of the vertices, all the momenta p_i will be chosen as incoming.

In order to consider also a combined gauge and gravitational background, as in the case for the gravitational anomaly, the quantum average of the axial vector current will be extended with the inclusion of a covariant derivative and curvature related terms

$$\nabla_\mu \langle J_A^\mu \rangle = a_1 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + a_2 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{\alpha\beta} R_{\alpha\beta\rho\sigma}, \quad (3.3)$$

that will be investigated when we will deal with the ATT correlator. As already mentioned, in both the case of the ATT and AVV, these conditions, together with the ordinary Ward identities imposed either from gauge invariance or diffeomorphism invariance, set the boundary conditions for the reconstruction of the corresponding correlators by the inclusion of a single anomaly pole.

In the AVV case, from these relations we construct the general form of the correlator, splitting the operators into a transverse and a longitudinal part as

$$\begin{aligned} J^\mu(p) &= j^\mu(p) + j_{\text{loc}}^\mu(p), \\ j^\mu &= \pi_\alpha^\mu(p) J^\alpha(p), \quad \pi_\alpha^\mu(p) \equiv \delta_\alpha^\mu - \frac{p_\alpha p^\mu}{p^2}, \\ j_{\text{loc}}^\mu(p) &= \frac{p^\mu}{p^2} p \cdot J(p), \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} J_A^\mu(p) &= j_A^\mu(p) + j_{\text{Aloc}}^\mu(p), \\ j_A^\mu &= \pi_\alpha^\mu(p) J_A^\alpha(p), \\ j_{\text{Aloc}}^\mu(p) &= \frac{p^\mu}{p^2} p \cdot J_A(p). \end{aligned} \quad (3.5)$$

Due to (3.2), the correlator is purely transverse in the vector currents. We then have the following decomposition:

$$\begin{aligned} \langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_A^{\mu_3}(p_3) \rangle &= \langle j^{\mu_1}(p_1)j^{\mu_2}(p_2)j_A^{\mu_3}(p_3) \rangle \\ &+ \langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)j_{A\text{loc}}^{\mu_3}(p_3) \rangle \end{aligned} \quad \text{with} \quad (3.6)$$

where the first term is completely transverse with respect to the momenta $p_{i\mu_i}$, $i = 1, 2, 3$ and the second term is the longitudinal part that is proper of the anomaly contribution. Using the anomaly constraint (3.2) on $j_{A\text{loc}}$ we obtain the expression of the longitudinal component of (3.6)

$$\begin{aligned} \langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)j_{A\text{loc}}^{\mu_3}(p_3) \rangle \\ = \frac{p_3^{\mu_3}}{p_3^2} p_{3\alpha_3} \langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_A^{\alpha_3}(p_3) \rangle \\ = \Phi_0 \varepsilon^{p_1 p_2 \mu_1 \mu_2} p_3^{\mu_3} \end{aligned} \quad (3.7)$$

$$\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_A^{\mu_3}(p_3) \rangle = -\frac{8ia_1}{p_3^2} \varepsilon^{p_1 p_2 \mu_1 \mu_2} p_3^{\mu_3} 8ia_1 \pi_{\alpha_1}^{\mu_1}(p_1) \pi_{\alpha_2}^{\mu_2}(p_2) \pi_{\alpha_3}^{\mu_3}(p_3) [p_2^2 I_{3\{1,0,1\}} \varepsilon^{p_1 \alpha_1 \alpha_2 \alpha_3} - p_1^2 I_{3\{0,1,1\}} \varepsilon^{p_2 \alpha_1 \alpha_2 \alpha_3}], \quad (3.9)$$

where, remarkably, the a_1 the anomaly coefficient has turned into a structural factor both for the longitudinal and transverse parts. Unlike the perturbative approach, where the anomaly coefficient is primarily determined by the prefactor in front of a Feynman amplitude, here it emerges as the residue of a pole in the longitudinal sector, as a solution of the anomaly constraint. The conformal Ward identities then propagate this pole to the other sectors of the correlator, specifically the transverse one.

Notably, this parametrization of the AVV correlator is the most compact one, as it not only highlights the central role of the anomaly pole but also reveals additional features that we will discuss shortly. The solution involves the integral

defining the anomaly form factor in the conformal limit. The complete solution can be obtained by solving the special conformal Ward identities and using the Schouten relations in order to parametrize also the transverse sector of (3.6). The procedure involves an intermediate step in which the action of the special conformal transformation on the correlator is projected onto the transverse sector and separated into second order and first order equations using the Schouten relations. Then, the general structure of the transverse part can be further simplified using the Schouten identities, resulting in the expression [11]

$$I_{3\{1,0,1\}} = p_1 p_3 \frac{\partial^2}{\partial p_1 \partial p_3} I_{1\{0,0,0\}}, \quad (3.10)$$

where $I_{1\{0,0,0\}}$ is the master integral (3K integral) related to the three-point function in the theory of a free massless scalar field in $d = 4$ [7]

$$I_{1\{0,0,0\}} = \frac{1}{4} C_0(p_1^2, p_2^2, p_3^2). \quad (3.11)$$

One can then express the 3K integral explicitly as follows:

$$\begin{aligned} I_{3\{1,0,1\}}(p_1^2, p_2^2, p_3^2) &= \frac{1}{\lambda^2} \left\{ -2p_1^2 p_3^2 [p_1^2(p_2^2 - 2p_3^2) + p_1^4 + p_2^2 p_3^2 - 2p_2^4 + p_3^4] C_0(p_1^2, p_2^2, p_3^2) \right. \\ &+ p_1^2 ((p_1^2 - p_2^2)^2 + 4p_2^2 p_3^2 - p_3^4) \log\left(\frac{p_1^2}{p_2^2}\right) + 4p_1^2 p_3^2 (p_1^2 - p_3^2) \log\left(\frac{p_1^2}{p_3^2}\right) \\ &\left. - p_3^2 ((p_2^2 - p_3^2)^2 + 4p_1^2 p_2^2 - p_1^4) \log\left(\frac{p_2^2}{p_3^2}\right) - \lambda(p_1^2 - p_2^2 + p_3^2) \right\} \end{aligned} \quad (3.12)$$

where

$$C_0(q^2, p_1^2, p_2^2) = \frac{1}{i\pi^2} \int d^n k \frac{1}{k^2(k-q)^2(k-p_1)^2}, \quad (3.13)$$

is the ordinary massless scalar master integral. It is given by

$$C_0(s, s_1, s_2) = \frac{1}{s} \Phi(x, y). \quad (3.14)$$

$\Phi(x, y)$ is the special function [19]

$$\Phi(x, y) = \frac{1}{\lambda} \left\{ 2[\text{Li}_2(-\rho x) + \text{Li}_2(-\rho y)] + \ln \frac{y}{x} \ln \frac{1 + \rho y}{1 + \rho x} + \ln(\rho x) \ln(\rho y) + \frac{\pi^2}{3} \right\}, \quad (3.15)$$

where

$$\lambda(1, x, y) = \sqrt{\Delta}, \quad \Delta = (1 - x - y)^2 - 4xy, \quad (3.16)$$

$$\rho(x, y) = 2(1 - x - y + \lambda)^{-1}, \quad x = \frac{s_1}{s}, \quad y = \frac{s_2}{s} \quad s \equiv p_3^2. \quad (3.17)$$

The analysis highlights the pivotal role of the anomaly pole in determining the entire anomaly interaction, particularly in the conformal limit. Remarkably, the complete off-shell correlator is fully specified by a single additional form factor, which is connected to the integral $I_{3\{1,0,1\}}$. Notably, the conformal solution is inherently scale independent and entirely dictated by the anomaly coefficient. If this coefficient is set to zero, the correlator vanishes entirely.

At this juncture, a crucial question arises: does this $1/p_3^2 \equiv 1/q^2$ interaction correspond to an asymptotic state?

For such an interpretation, the anomaly pole must be identified as a massless particle pole. To establish this, one must verify under general kinematic conditions whether the residue of the vertex

$$g_p \equiv \lim_{q^2 \rightarrow 0} q^2 \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J_A^{\mu_3}(q) \rangle \neq 0 \quad (3.18)$$

is nonvanishing. The subtle nature of the anomaly interaction lies in the fact that the longitudinal form factor can exhibit an anomaly pole, even though (3.18) may vanish. As already pointed out in the previous sections, anomaly poles do not necessarily correspond to particle poles. As a result, the on-shell limits used to identify the anomaly coupling in the 1PI vertex are deeply connected to the properties of this vertex in its complete tensorial form rather than just with its longitudinal part.

To move away from the conformal limit, we address the massive fermion case and compute the vertex using ordinary perturbation theory. We can perform a decomposition similar to (3.9) in the form

$$\langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J_A^{\mu_3}(p_3) \rangle = \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) j_{\text{Aloc}}^{\mu_3}(p_3) \rangle + \pi_{\alpha_1}^{\mu_1}(p_1) \pi_{\alpha_2}^{\mu_2}(p_2) \pi_{\alpha_3}^{\mu_3}(p_3) \Delta_T^{\alpha_1 \alpha_2 \alpha_3}(p_1^2, p_2^2, p_3^2, m^2) \quad (3.19)$$

with the longitudinal sector given by

$$\langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) j_{\text{Aloc}}^{\mu_3}(p_3) \rangle \equiv \Phi_0(p_1^2, p_2^2, p_3^2, m^2) \varepsilon^{p_1 p_2 \mu_1 \mu_2} p_3^{\mu_3}. \quad (3.20)$$

The expression of the transverse part is denoted by Δ_T .

To go beyond the conformal limit, we return to perturbation theory by introducing a fermion of mass m . The AVV anomaly tensor $\Delta_{\alpha\mu\nu}(p_1, p_2)$ is defined through the triangle fermion loop

$$\Delta_{\alpha\mu\nu}(p_1, p_2) = - \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left(\frac{1}{\not{k} - \not{p}_1 - m} \gamma_\mu \frac{1}{\not{k} - m} \gamma_\nu \frac{1}{\not{k} + \not{p}_2 - m} \gamma_\alpha \gamma_5 \right) + [(p_1, \mu) \leftrightarrow (p_2, \nu)]. \quad (3.21)$$

The vector current remains conserved

$$p_1^\mu \Delta_{\alpha\mu\nu} = 0, \quad p_2^\nu \Delta_{\alpha\mu\nu} = 0, \quad (3.22)$$

while the axial-vector Ward identity is anomalous,

$$q^\alpha \Delta_{\alpha\mu\nu} = 2m \Delta_{\mu\nu}(p_1, p_2) + \frac{1}{2\pi^2} \epsilon_{\mu\nu\rho\sigma} p_1^\rho p_2^\sigma. \quad (3.23)$$

The tensor structure of $\Delta_{\alpha\mu\nu}$ is parametrized as

$$\Delta_{\alpha\mu\nu} = F_1(s, m^2) q_\alpha \epsilon_{\mu\nu\rho\sigma} p_1^\rho p_2^\sigma + F_2(s, m^2) (p_{2\nu} \epsilon_{\alpha\mu\rho\sigma} - p_{1\mu} \epsilon_{\alpha\nu\rho\sigma}) p_1^\rho p_2^\sigma. \quad (3.24)$$

The anomaly resides entirely in the F_1 form factor, which obeys

$$s F_1(s, m^2) = - \frac{1}{\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{m^2}{m^2 - xys - i\epsilon} + \frac{1}{2\pi^2}. \quad (3.25)$$

The second term on the right-hand side of the equation above defines the anomaly, while the first accounts for explicit chiral symmetry breaking. In the massless limit, only the anomaly term survives and saturates the anomaly sum rule.

The off-shell generalization of F_1 , denoted as Φ_0 , is

$$\Phi_0(p_1^2, p_2^2, p_3^2, m^2) = \frac{m^2}{\pi^2} \frac{1}{q^2} C_0(q^2, p_1^2, p_2^2, m^2) + \frac{1}{2\pi^2} \frac{1}{q^2}, \quad (3.26)$$

with C_0 the standard scalar triangle integral (3.13) parametrized as

$$C_0(q^2, p_1^2, p_2^2, m^2) = \int_0^1 dx \int_0^{1-x} dy \frac{1}{\mathcal{D}(x, y)}, \quad (3.27)$$

where

$$\mathcal{D}(x, y) = m^2 + q^2 y(x + y - 1) - p_1^2 xy + p_2^2 x(x + y - 1). \quad (3.28)$$

While (3.26) exhibits a $1/q^2$ term resembling an anomaly pole, this feature is misleading, since the form factor is characterized by a branch cut rather than a true pole in the massive theory. The full cancellation of this apparent pole will be explored later, and is closely related to the anomaly sum rule. For comparison, the local form factor derived from the axial-vector Ward identity is

$$q_\lambda \langle J^\mu J^\nu J_A^\lambda \rangle \equiv \Phi_W \epsilon^{\mu\nu p_1 p_2}, \quad (3.29)$$

$$\Phi_W = q^2 \Phi_0 = \frac{m^2}{\pi^2} C_0(p_1^2, p_2^2, q^2, m^2) + \frac{1}{2\pi^2}. \quad (3.30)$$

Unlike Φ_0 , the local form Φ_W contains no $1/q^2$ structure and does not obey an anomaly sum rule. It is commonly used in effective axion–gauge field interactions, but lacks the nonlocal and topological properties intrinsic to Φ_0 . Thus, while Φ_W reflects a local vertex structure, only Φ_0 captures the full anomaly content, including its asymptotic $1/q^2$ behavior and relation to the anomaly pole. The distinction between these two forms is central to understanding the anomaly both in conformal and massive regimes.

A. Pole cancellation in Φ_0 for $m \neq 0$ for on-shell vectors

We now prove that there is no pole in Φ_0 when the two vector lines are on-shell and $m \neq 0$. One can show that there is a cancellation between the second and the first contribution in Φ_0 in (3.26). For this purpose we consider the on-shell case, $p_1^2 = p_2^2 = 0$ and use the explicit expression of C_0 in the form

$$C_0(0, 0, q^2, m^2) \equiv C_0(q^2, m^2) = -\frac{1}{2m^2} {}_3F_2\left(1, 1, 1; 2, \frac{3}{2}; x\right), \quad (3.31)$$

where the ${}_3F_2$ hypergeometric function can be written as a series expansion for $q^2/4m^2 < 1$ as

$${}_3F_2\left(1, 1, 1; 2, \frac{3}{2}; x\right) = \sum_{k=0}^{\infty} \frac{(1)_k (1)_k (1)_k}{(2)_k \left(\frac{3}{2}\right)_k} \frac{x^k}{k!}, \quad (3.32)$$

with $x = \frac{p_2^2}{4m^2}$, and where $(a)_k$ is the Pochhammer symbol, defined as

$$(a)_k = a(a+1)(a+2) \cdots (a+k-1), (a)_0 = 1. \quad (3.33)$$

Substituting the Pochhammer symbols for the given parameters

$$\begin{aligned} (1)_k &= 1 \cdot 2 \cdot 3 \cdots (k) = k!, \\ (2)_k &= 2 \cdot 3 \cdot 4 \cdots (k+1) = (k+1)k!, \\ \left(\frac{3}{2}\right)_k &= \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdots \left(\frac{3}{2} + k - 1\right), \end{aligned} \quad (3.34)$$

one derives the expansion

$$C_0(q^2, m^2) = -\frac{1}{2m^2} \left(1 + \frac{q^2}{12m^2} + \frac{q^4}{180m^4} + \mathcal{O}\left(\frac{q^6}{m^6}\right)\right), \quad (3.35)$$

giving an anomaly form factor of the form

$$\Phi_0 = -\frac{1}{4\pi^2} \frac{1}{12m^2} - \frac{1}{4\pi^2} \frac{q^2}{180m^4} + \dots \quad (3.36)$$

Notice that there has been a cancellation between the $1/q^2$ term in (3.26) and a similar term coming from the scalar loop C_0 . In other words, the 1PI effective interaction is entirely mass dependent. Given the condition $q^2 < 4m^2$, the expansion is valid below the cut. Notice that the form factor vanishes as the fermion mass is sent to infinity, indicating the decoupling of the heavy fermion in the perturbative loop.

The area subtended by the spectral density of Φ_0 (see Fig. 1) over the cut plays a crucial role in the analysis of the sum rule. Therefore, we now turn to a more general examination of this form factor.

IV. CHIRAL SUM RULES

In this section, we begin our investigation of the sum rules governing the anomaly form factors, starting with the case of the AVV correlator. The original derivation of the sum rule for the AVV had been discussed for some specific cases. Horejsi [20] investigated the case $p_1^2 = p_2^2 = 0, m \neq 0$. Previous analysis had derived the sum rule when only a single vector line is off-shell ($p_1^2 = 0, p_2^2 \neq 0, m \neq 0$) [14] and for ($p_1^2 = p_2^2 < 0, m = 0$) [15].

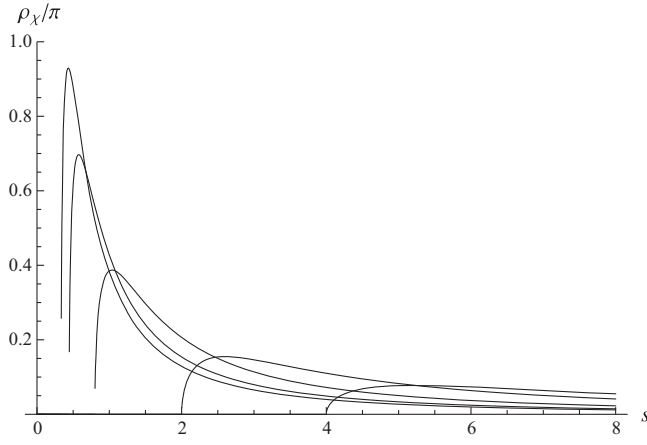


FIG. 1. Spectral density flow in the $p_1^2 = p_2^2 = 0$ case as a function of the fermion mass m . The sum rule is an area law for this spectral flow as one moves toward the conformal point $m \rightarrow 0$.

The area law implicit in this specific feature of the anomaly interactions is, at least in part, a result of the property of analyticity of the vertex but also manifests a correlation between the longitudinal and transverse sectors of the interaction. This point has been observed in [18]. The analyticity conditions of Φ_0 in the complex q^2 plane is summarized by the following two formulas:

$$\oint_C \Phi_0(s, s_1, s_2, m^2) ds = 0, \quad (4.1)$$

and

$$\Phi_0(q^2, s_1, s_2, m^2) = \frac{1}{2\pi i} \oint_C \frac{\Phi_0(s, s_1, s_2, m^2)}{s - q^2} ds, \quad (4.2)$$

where the contour of integration, by Jordan's lemma, is reduced to the $s \geq 0$ axis, where the function, as we are going to show, exhibits poles and a cut for $s > 4m^2$. As we are going to see, the spectral density will contain poles in its rational factors, with a branch cut starting at $4m^2$.

We start by defining the general discontinuity of Φ_0 for $q^2 > 0$, as

$$\text{disc}\Phi_0 = \Phi_0(q^2 + i\epsilon, p_1^2, p_2^2, m^2) - \Phi_0(q^2 - i\epsilon, p_1^2, p_2^2, m^2) \quad (4.3)$$

and its imaginary part, the spectral density Δ , in the s variable as

$$\Delta\Phi_0(s, s_1, s_2, m^2) \equiv \text{Im}\Phi_0(s, s_1, s_2, m^2) = \frac{1}{2i} \text{disc}\Phi_0. \quad (4.4)$$

The vanishing of the boundary integral on the large radius allows one to rewrite the dispersion relation (4.2) as

$$\Phi_0(q^2, p_1^2, p_2^2, m^2) = \frac{1}{\pi} \int_0^\infty \frac{\Delta\Phi_0(s, p_1^2, p_2^2, m^2)}{s - q^2} ds. \quad (4.5)$$

As previously mentioned, the relation holds in the complex q^2 plane at fixed positive values of s_1 and s_2 . Notice that the dispersion relations we derive are unsubtracted, as we enforce vector Ward identities to reexpress the divergent form factors in terms of convergent ones. The computation of the spectral density of the anomaly form factors is carried out using the principal value prescription for the $1/q^2$ pole appearing in their explicit expressions

$$\text{disc} \frac{1}{q^2} \equiv \frac{1}{q^2 + i\epsilon} - \frac{1}{q^2 - i\epsilon} = -2\pi i \delta(q^2). \quad (4.6)$$

Using this relation in (3.26)

$$\begin{aligned} \text{disc}\Phi_0(q^2, p_1^2, p_2^2, m^2) &= \frac{m^2}{\pi^2} \text{disc} \left(\frac{C_0(q^2, p_1^2, p_2^2, m^2)}{q^2} \right) \\ &+ \frac{1}{2\pi^2} \text{disc} \frac{1}{q^2} \end{aligned} \quad (4.7)$$

with the help of (4.6), we can separate the pole and the continuum contributions as

$$\begin{aligned} \text{disc}\Phi_0(q^2, p_1^2, p_2^2, m^2) &= -2i \frac{m^2}{\pi} C_0(q^2, p_1^2, p_2^2, m^2) \delta(q^2) \\ &- \frac{i}{\pi} \delta(q^2) + \frac{m^2}{\pi^2 q^2} \\ &\times \text{disc}C_0(q^2, p_1^2, p_2^2, m^2), \end{aligned} \quad (4.8)$$

showing that the spectral density exhibits two distinct localized terms at $s = 0$ (δ spikes) and a continuum contribution for $q^2 > 4m^2$, with an interplay among these three components, as we are going to show, which is related to the property of analyticity of Φ_0 . We will refer to the localized components as

$$\begin{aligned} \text{spk}_1 &\equiv -\frac{i}{\pi} \delta(q^2), \\ \text{spk}_2 &\equiv -2i \frac{m^2}{\pi} C_0(q^2, p_1^2, p_2^2, m^2) \delta(q^2). \end{aligned} \quad (4.9)$$

The first spike is of constant strength, while the second plays a role in equaling the integral of the density over the cut, causing a cancellation. One can think of these two components as distinct but superimposed resonances at $s = 0$. These features, as we shall demonstrate, either before or after integration, exhibit cancellation patterns tied to the external kinematics of the vertex. Such cancellations arise either within the two spikes themselves or through their interaction of spk_2 with the continuum. The continuum corresponds to the discontinuity in the scalar loop, $\text{disc}C_0$, which is determined by placing two intermediate lines of the

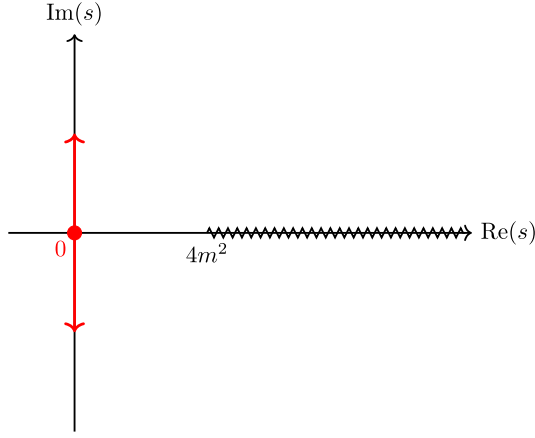


FIG. 2. Analyticity region of the on-shell anomaly form factor in the complex $s \equiv q^2$ plane, with two bold spikes at $s = 0$ of different colors, for visual clarity. The sum rule of the anomaly form factor of the AVV involves a cancellation between the two red spikes at $s = 0$, leaving only the integral over the continuum for $s > 4m^2$.

triangle diagram on-shell. When the external invariants of the vector lines are both vanishing ($p_1^2 = p_2^2 = 0$), the spectral analysis shows that there is a cancellation between the first and the second $\delta(q^2)$ terms on the rhs of (4.8), leaving only the discontinuity along the cut to saturate the sum rule, as shown in Fig. 2. The cancellation takes place before integration for $s \geq 0$. In the general off-shell case, the cancellation between one of the pole contributions and the cut—both marked in black—takes place only after integration, leaving the contribution marked in red, of constant strength, to saturate the sum rule. This is illustrated in Fig. 3.

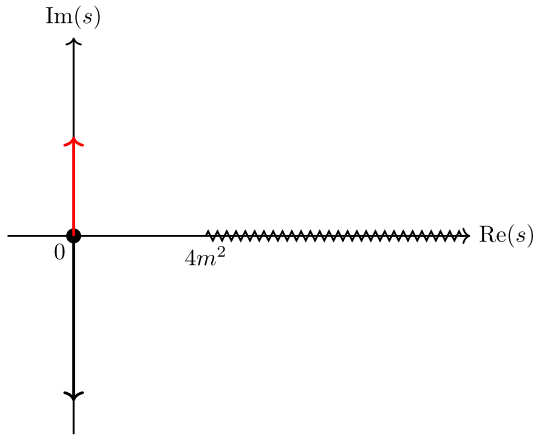


FIG. 3. Analyticity region of the off-shell anomaly form factor in the complex $s \equiv q^2$ plane, with two bold spikes at $s = 0$. In this case there are cancellations between the cut and the contribution of the black spike, whose strength depends on all the external invariants, after integration over the $q^2 > 0$ region. The sum rule then allows one to identify the red spike at $s = 0$ as an anomaly pole.

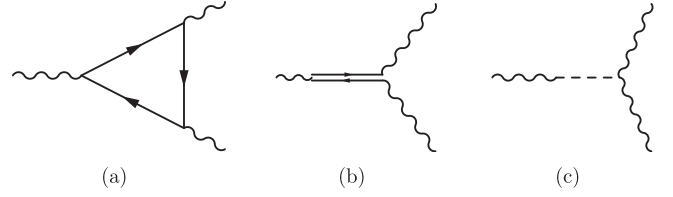


FIG. 4. The fermion loop (a), the collinear region (b), the effective pseudoscalar exchange (c). The $q^2 = 0$ point defining the residue of the interaction can be viewed as a massless state decaying into a massless virtual pseudoscalar state that couples to the anomaly at the last stage.

A. The use of the Laplace/Borel transforms

The on-shell behavior presented above can be discussed for arbitrary kinematics. The pattern of cancellations illustrated in Fig. 2 takes a different form in the general case, when the vector lines are off-shell and we are away from the conformal point. In this case the cancellation is between the integral of spk_2 and the one over the continuum, and is illustrated in Fig. 4. To prove this point we introduced a method that allows one to establish this result starting directly from the parametric integral. This method is essentially based on the use of Borel transforms.

The evaluation of the integral of the spectral density of Φ_0 along the $[4m^2, \infty)$ cut can be computed by a trick that we are going to describe, using the inverse Laplace transform of the same density (indicated by t), followed by the $t \rightarrow 0$ limit on the same variable. For this, in full generality, we define the inverse Laplace transform ($f(t)$) of a generic function $F(s)$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) e^{st} ds, \quad (4.10)$$

and recall that for a rational function $F(s)$ with a pole at $s = s_0$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-s_0}\right\}(t) = e^{s_0 t} \quad (4.11)$$

which provides a nonzero contribution to the dispersion integral as $t \rightarrow 0$. The method allows to compute the integral over the continuum of the spectral density quite efficiently, but it requires some care since extra singular contributions, that can be viewed as subtractions in the reconstruction of a given form factor, are not identified by this approach. For example, double poles of the form $1/(s-s_0)^2$ are not taken into account by the method since one obtains

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-s_0)^2}\right\}(t) = t e^{s_0 t}, \quad (4.12)$$

that vanishes in the same $t \rightarrow 0$ limit. An example can be found in the analysis of the ATT case, where we discuss how this point can be handled (Sec. VIII B).

In our case, for the anomaly form factor Φ_0 , given a spectral representation of the form

$$\Phi_0(q^2) = \frac{1}{\pi} \int_0^\infty \frac{\Delta\Phi_0(s)}{s - q^2} ds \quad (4.13)$$

the action of the inverse transform will generate the expression

$$\mathcal{L}^{-1} \left\{ \frac{1}{\pi} \int_0^\infty \frac{\Delta\Phi_0}{s - q^2} ds \right\} (t) = \frac{1}{\pi} \int_0^\infty \Delta\Phi_0(s) e^{st} ds \quad (4.14)$$

and the integral of the spectral density will be obtained by a limit on t

$$\mathcal{L}^{-1} \{ \Phi_0 \} (t) = - \int_0^1 dx \int_0^{1-x} dy \frac{g^3 \left(m^2 \exp \left(\frac{t(-m^2 + p_1^2 xy - p_2^2 x^2 - p_2^2 xy + p_2^2 x)}{xy + y^2 - y} \right) - 2m^2 + p_1^2 xy - p_2^2 x^2 - p_2^2 xy + p_2^2 x \right)}{\pi^2 (m^2 - p_1^2 xy + p_2^2 x^2 + p_2^2 xy - p_2^2 x)} \quad (4.18)$$

on which we perform the $t \rightarrow 0$ limit as described above, in order to derive directly the value of the integral of the discontinuity over the entire cut

$$\mathcal{L}^{-1} \{ \Phi_0 \} (t \rightarrow 0) = \frac{1}{\pi} \int_{4m^2}^\infty \Delta\Phi_0(s, p_1^2, p_2^2, m^2) ds = \frac{1}{2\pi^2}. \quad (4.19)$$

Equation (4.19) holds in the most general case, for off-shell vector lines and a massive fermion. This result is in agreement with the analysis of [17].

Notice that a sum rule holds only for Φ_0 and not for Φ_W , the form factor extracted from the local formulation of the anomaly interaction (3.29). A direct computation indeed shows that the corresponding parametric integral, in this case,

$$\int_{4m^2}^\infty \Delta\Phi_W(s, p_1^2, p_2^2, m^2) ds = \int_0^1 \left(\int_0^{1-x} \frac{g^3 m^2}{2\pi^2 (xy + y^2 - y)} dy \right) dx \quad (4.20)$$

is indeed divergent.

V. THE RESIDUE OF THE PARTICLE POLE IN THE LIGHT-CONE VARIABLES

The computation of the residue of a possible particle pole in the off-shell massive AVV correlator requires the choice of special noncovariant frames where to proceed with the analysis. This allows one to test if the coupling g_p in (3.18) is nonzero. The computation of the residue of the particle pole of the AVV has been discussed in the past in [21] and it was shown that there is no coupling of this state to the vertex for off-shell photons and a virtual massive fermion in the loop. The analysis was performed both for the massive

$$\lim_{t \rightarrow 0} \mathcal{L}^{-1} \{ \Phi_0 \} (t) = \frac{1}{\pi} \int_0^\infty \Delta\Phi_0 ds. \quad (4.15)$$

This approach allows to derive the expression of the integral of the discontinuity along the cut and verify the sum rules of such form factors.

For this purpose we use the parametric representation

$$C_0(p_1^2, p_2^2, q^2, m^2, m^2, m^2) = \int_0^1 dx \int_0^{1-x} dy \frac{1}{\Delta(x, y)}, \quad (4.16)$$

$$\Delta(x, y) = m^2 + q^2 y(x + y - 1) - p_1^2 xy + p_2^2 x(x + y - 1) \quad (4.17)$$

that we insert in the expression of Φ_0 given in (3.26). A direct computation gives for the inverse Laplace transform the expression

and the massless cases using specific parametrizations of the correlator. We are going to elaborate on this point regarding the way the tensorial limits are performed, not discussed in [21], with the goal of extending these analysis, in the next sections, to the case of the ATT.

We recall that in order to discuss the presence of a massless particle pole interaction in the anomaly vertex we need to perform the limit (3.18) and show that it is nonvanishing independently of the virtualities of the other invariants. We use as notations $(s_1, s_2) \equiv (p_1^2, p_2^2)$ and $s = q^2 = p_3^2$, assuming various configurations for the

external invariants and the value of the fermion mass. We are going to consider two possible ways to approach the $q^2 \rightarrow 0$ limit, by sending the four-momentum q^λ on the light-cone or by performing the soft limit $q^\lambda \rightarrow 0$ in all of its components. In order to perform the light-cone limit on q we parametrize the four-momenta as follows:

$$\begin{aligned} p_3 &\equiv q = q^+ n^+ + q^- n^-, & p_1 &= p_1^+ n^+ + p_1^- n^- + p_\perp, \\ p_2 &= p_2^+ n^+ + p_2^- n^- - p_\perp \end{aligned} \quad (5.1)$$

with light-cone versors defined by $(n^\pm)^2 = 0, n^+ \cdot n^- = 1$. In the light cone limit q^+ is large as well as p_1^+ and p_2^+ , while the components

$$q^- = \frac{q^2}{2q^+}, \quad p_1^- = \frac{p_1^2 + p_\perp^2}{2p_1^+}, \quad p_2^- = \frac{p_2^2 + p_\perp^2}{2p_2^+} \quad (5.2)$$

are small. We introduce a scale $\lambda^2 \sim p_1^2 \sim p_2^2$, with $\lambda \ll \bar{Q} = q^+$. In this limit q_0 and q_3 are both large but fixed, with $q^- = q_0 - q_3$ arbitrarily small. $\bar{Q}$ is assumed of any large but *fixed* size in the light-cone direction of n^+ , and the limit $q^2 \rightarrow 0$ is performed, for a fixed value of $\bar{Q}$, by taking q^- , the component in the n^- direction arbitrarily small. For this reason any power of q^+ can be dropped if multiplied by q^- .

The process for extracting the value of the residue of a tensor structure with open indices involves projecting that specific structure onto the light-cone basis. If, during this procedure, a particular component does not vanish, we can conclude that a massless $1/q^2$ pole develops a residue on the light cone.

It is straightforward to demonstrate that all tensorial structures vanish except for those that include a $1/p_3^2$ pole. This outcome is evident from any parametrization of the correlator. The only case where the residue is nonvanishing is for massless vector lines and a massless virtual fermion. We give in Appendix B 1 more details concerning the technical aspects of this analysis.

VI. THE GRAVITATIONAL CHIRAL ANOMALY VERTEX: REVIEW

In this section, we begin our analysis of the ATT vertex. For simplicity, we are going to set the coupling constant to one. As for the AVV, our study starts in the conformal limit, revisiting earlier work that showed how the vertex can be reconstructed from the corresponding anomaly pole in CFT_p [22]. For the ATT vertex, the conformal realization that we examine in momentum space also encompasses the case of Chern-Simons currents [23], which generate spin-1 anomalies at first discussed long ago and revisited in more recent studies [1,24]. The form factor decomposition remains valid in both conformal and nonconformal cases, with the mass dependence of the correlator determined via standard perturbation theory. In [22], it was shown that the conformal solution derived using CFT_p methods is in full agreement with the perturbative result. Much like in the AVV case, a complete determination of the vertex in the conformal limit is also possible for the ATT. Here too, the entire vertex is fixed by the residue at the pole of the longitudinal component of the axial-vector current.

We start by defining

$$T^{\mu_i \nu_i}(p_i) = t^{\mu_i \nu_i}(p_i) + t_{\text{loc}}^{\mu_i \nu_i}(p_i), \quad (6.1)$$

$$t^{\mu_i \nu_i}(p_i) = \Pi_{\alpha\beta}^{\mu\nu}(p_i) T^{\alpha\beta}(p_i) \quad (6.2)$$

having introduced the transverse-traceless (Π), transverse (π) and longitudinal (Σ) projectors, given respectively by

$$\Pi_{\alpha\beta}^{\mu\nu} = \frac{1}{2}(\pi_\alpha^\mu \pi_\beta^\nu + \pi_\beta^\mu \pi_\alpha^\nu) - \frac{1}{d-1} \pi^{\mu\nu} \pi_{\alpha\beta}, \quad (6.3)$$

$$\begin{aligned} \Sigma_{\alpha_i \beta_i}^{\mu_i \nu_i} &= \frac{p_{i\beta_i}}{p_i^2} \left[2\delta_{\alpha_i}^{(\nu_i} p_i^{\mu_i)} - \frac{p_{i\alpha_i}}{(d-1)} \left(\delta^{\mu_i \nu_i} + (d-2) \frac{p_i^{\mu_i} p_i^{\nu_i}}{p_i^2} \right) \right] \\ &+ \frac{\pi^{\mu_i \nu_i}(p_i)}{(d-1)} \delta_{\alpha_i \beta_i}. \end{aligned} \quad (6.4)$$

Such decomposition allows one to split our correlation function into the following terms:

$$\begin{aligned} \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} J_A^{\mu_3} \rangle &= \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} j_A^{\mu_3} \rangle + \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} j_{5\text{loc}}^{\mu_3} \rangle + \langle T^{\mu_1 \nu_1} t_{\text{loc}}^{\mu_2 \nu_2} J_A^{\mu_3} \rangle + \langle t_{\text{loc}}^{\mu_1 \nu_1} T^{\mu_2 \nu_2} J_A^{\mu_3} \rangle \\ &- \langle T^{\mu_1 \nu_1} t_{\text{loc}}^{\mu_2 \nu_2} j_{A\text{loc}}^{\mu_3} \rangle - \langle t_{\text{loc}}^{\mu_1 \nu_1} t_{\text{loc}}^{\mu_2 \nu_2} J_A^{\mu_3} \rangle - \langle t_{\text{loc}}^{\mu_1 \nu_1} T^{\mu_2 \nu_2} j_{A\text{loc}}^{\mu_3} \rangle + \langle t_{\text{loc}}^{\mu_1 \nu_1} t_{\text{loc}}^{\mu_2 \nu_2} j_{5\text{loc}}^{\mu_3} \rangle \end{aligned} \quad (6.5)$$

which are constrained by ordinary trace Ward identities of the form

$$\begin{aligned} \delta_{\mu_i \nu_i} \langle T^{\mu_1 \nu_1}(p_1) T^{\mu_2 \nu_2}(p_2) J_A^{\mu_3}(p_3) \rangle &= 0, \quad i = \{1, 2\}, \\ p_{i\mu_i} \langle T^{\mu_1 \nu_1}(p_1) T^{\mu_2 \nu_2}(p_2) J_A^{\mu_3}(p_3) \rangle &= 0, \quad i = \{1, 2\}. \end{aligned} \quad (6.6)$$

Notice that the trace identity (the first of the two) will be modified in the presence of mass terms, due to the breaking of conformal symmetry. The second remain invariant, being derived from the conservation of the stress energy tensor. As

already mentioned the TTA does not involve any renormalization and there is no trace anomaly. However, due to the presence of several diagrams of different topologies that are separately divergent, one needs a regularization scheme. Using the Ward identities above, we notice that most of the terms in Eq. (6.5) vanish, yielding

$$\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J_A^{\mu_3} \rangle = \langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} j_A^{\mu_3} \rangle + \langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} j_{A\text{loc}}^{\mu_3} \rangle. \quad (6.7)$$

The longitudinal anomalous Ward identity is given by

$$p_{3\mu_3} \langle T^{\mu_1\nu_1}(p_1) T^{\mu_2\nu_2}(p_2) J_A^{\mu_3}(p_3) \rangle = 4ia_2(p_1 \cdot p_2) \left\{ \left[\varepsilon^{\nu_1\nu_2 p_1 p_2} \left(g^{\mu_1\mu_2} - \frac{p_1^{\mu_2} p_2^{\mu_1}}{p_1 \cdot p_2} \right) + (\mu_1 \leftrightarrow \nu_1) \right] + (\mu_2 \leftrightarrow \nu_2) \right\}, \quad (6.8)$$

with the right-hand side of the equation above given by the Fourier transform of the second functional derivative of the density $R\bar{R}$. As in the case of the AVV, Eq. (6.8) in momentum space is solved by the insertion of an anomaly pole $1/p_3^2$ in the longitudinal sector of J_A

$$\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} j_{A\text{loc}}^{\mu_3} \rangle = 4ia_2 \frac{p_3^{\mu_3}}{p_3^2} (p_1 \cdot p_2) \left\{ \left[\varepsilon^{\nu_1\nu_2 p_1 p_2} \left(g^{\mu_1\mu_2} - \frac{p_1^{\mu_2} p_2^{\mu_1}}{p_1 \cdot p_2} \right) + (\mu_1 \leftrightarrow \nu_1) \right] + (\mu_2 \leftrightarrow \nu_2) \right\}. \quad (6.9)$$

On the other hand, after accounting for the Bose symmetry and all the Schouten identities, the transverse-traceless component of the correlator can be expressed as [22]

$$\begin{aligned} \langle t^{\mu_1\nu_1}(p_1) t^{\mu_2\nu_2}(p_2) j_A^{\mu_3}(p_3) \rangle &= \Pi_{\alpha_1\beta_1}^{\mu_1\nu_1}(p_1) \Pi_{\alpha_2\beta_2}^{\mu_2\nu_2}(p_2) \pi_{\alpha_3}^{\mu_3}(p_3) \left[A_1 \varepsilon^{p_1\alpha_1\alpha_2\alpha_3} p_2^{\beta_1} p_3^{\beta_2} - A_1(p_1 \leftrightarrow p_2) \varepsilon^{p_2\alpha_1\alpha_2\alpha_3} p_2^{\beta_1} p_3^{\beta_2} \right. \\ &\quad \left. + A_2 \varepsilon^{p_1\alpha_1\alpha_2\alpha_3} \delta^{\beta_1\beta_2} - A_2(p_1 \leftrightarrow p_2) \varepsilon^{p_2\alpha_1\alpha_2\alpha_3} \delta^{\beta_1\beta_2} + A_3 \varepsilon^{p_1 p_2 \alpha_1 \alpha_2} p_2^{\beta_1} p_3^{\beta_2} p_1^{\alpha_3} + A_4 \varepsilon^{p_1 p_2 \alpha_1 \alpha_2} \delta^{\beta_1\beta_2} p_1^{\alpha_3} \right], \end{aligned} \quad (6.10)$$

where A_3 and A_4 are antisymmetric under the exchange $(p_1 \leftrightarrow p_2)$. The solution is [22]

$$\begin{aligned} A_1 &= \frac{p_2^2}{24\pi^2 \lambda^4} \left\{ A_{11} + A_{12} \log\left(\frac{p_1^2}{p_2^2}\right) + A_{13} \log\left(\frac{p_1^2}{p_3^2}\right) + A_{14} C_0(p_1^2, p_2^2, p_3^2) \right\}, \\ A_2 &= \frac{p_2^2}{48\pi^2 \lambda^3} \left\{ A_{21} + A_{22} \log\left(\frac{p_1^2}{p_2^2}\right) + A_{23} \log\left(\frac{p_1^2}{p_3^2}\right) + A_{24} C_0(p_1^2, p_2^2, p_3^2) \right\}, \\ A_3 &= 0, \\ A_4 &= 0, \end{aligned} \quad (6.11)$$

with the Källen λ -function given by

$$\lambda \equiv \lambda(p_1, p_2, p_3) = (p_1 - p_2 - p_3)(p_1 + p_2 - p_3)(p_1 - p_2 + p_3) \cdot (p_1 + p_2 + p_3), \quad (6.12)$$

expressed in terms of the magnitudes of the three momenta. In this case, the explicit expression aligns well with perturbation theory, which offers a free field theory realization of the anomalous conformal field theory. The coefficients A_{ij} in (6.11) are polynomial functions of the external invariants q^2, p_1^2, p_2^2 and can be found in [22].

The reconstruction of the interaction highlights the central role of the pole contribution in determining the entire vertex. Importantly, the Ward identities are sufficient to eliminate the apparent divergences in the diagram. Notably, topological contributions do not require any counterterms, provided the remaining conservation Ward

identities are imposed on the vertex. A key point here is that the anomaly pole does not manifest as a residue of the full vertex, as for the AVV. We will explore this phenomenon in detail.

VII. PERTURBATIVE COMPUTATION OF THE MASSIVE $\langle TTJ_A \rangle$ CORRELATOR

The investigation of the sum rule for the anomaly form factors of this vertex requires that we move away from the conformal point, by giving a mass to the fermion. For this reason we turn to the perturbative realization, expanding the analysis presented in [22]. Our calculations are carried

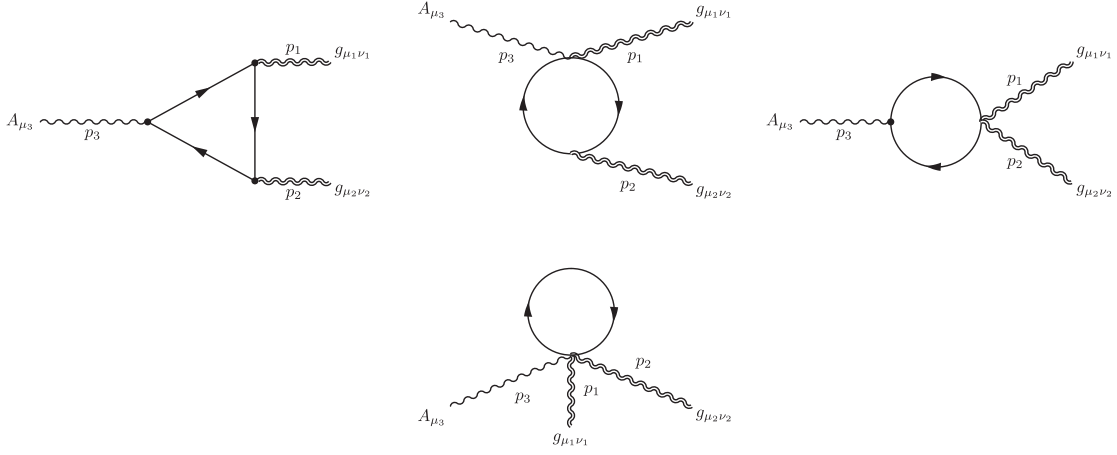


FIG. 5. Feynman diagrams representing triangle, bubble, and tadpole topologies in the perturbative computation of the massive correlator.

out in the Breitenlohner-Maison scheme. To streamline the analysis, we transition to Minkowski space, where the generating functional is defined as

$$e^{iS[g]} = \int [\mathcal{D}\Phi] e^{iS_0[\Phi, g]}, \quad (7.1)$$

with the classical action S_0 involving a fermionic field in a gravitational and axial gauge field background

$$S_0 = \int d^d x e \left[\frac{i}{2} e_a^\mu (\bar{\psi} \gamma^a (D_\mu \psi) - (D_\mu \bar{\psi}) \gamma^a \psi) - m \bar{\psi} \psi \right], \quad (7.2)$$

where e_a^μ is the vielbein, e its determinant, and D_μ the covariant derivative defined as

$$\begin{aligned} D_\mu \psi &= \left(\partial_\mu + i g \gamma_5 B_\mu + \frac{1}{2} \omega_{\mu ab} \Sigma^{ab} \right) \psi, \\ D_\mu \bar{\psi} &= \left(\partial_\mu - i g \gamma_5 B_\mu - \frac{1}{2} \omega_{\mu ab} \Sigma^{ab} \right) \bar{\psi}. \end{aligned} \quad (7.3)$$

Here, Σ^{ab} are the generators of the Lorentz group for spin- $\frac{1}{2}$ fields, and $\omega_{\mu ab}$ is the spin connection

$$\omega_{\mu ab} = e_a^\nu (\partial_\mu e_{\nu b} - \Gamma_{\mu\nu}^\lambda e_{\lambda b}). \quad (7.4)$$

The Greek indices refer to the curved background, while Latin indices correspond to the local flat basis. Substituting the explicit form of Σ^{ab} , the action becomes

$$\begin{aligned} S_0 = \int d^d x e \left[\frac{i}{2} \bar{\psi} e_a^\mu \gamma^a (\partial_\mu \psi) - \frac{i}{2} (\partial_\mu \bar{\psi}) e_a^\mu \gamma^a \psi \right. \\ \left. - g B_\mu \bar{\psi} e_a^\mu \gamma^a \gamma_5 \psi + \frac{i}{4} \omega_{\mu ab} e_c^\mu \bar{\psi} \gamma^{abc} \psi - m \bar{\psi} \psi \right], \end{aligned} \quad (7.5)$$

where $\gamma^{abc} = \{\Sigma^{ab}, \gamma^c\}$. The corresponding vertices can be computed by functional differentiation of the action with respect to the metric and the gauge field, followed by a Fourier transformation to momentum space.

The perturbative computation of the $\langle T T J_A \rangle$ involves three topologically distinct Feynman diagrams, illustrated in Fig. 5.

A. The form factors for nonzero mass

We present the perturbative results of the TTA correlator in the massive case. The TTA can be decomposed into different sectors. In the massive case, the conformal symmetry is broken and there are additional terms that need to be considered in the decomposition. Indeed, the energy-momentum tensor is clearly still conserved, but the correlator exhibits trace terms due to the inclusion of a fermion mass

$$\begin{aligned} p_i \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} J_A^{\mu_3} \rangle &= 0, \quad g_{\mu_1 \nu_1} \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} J_A^{\mu_3} \rangle \neq 0, \\ i &= \{1, 2\}. \end{aligned} \quad (7.6)$$

Therefore, one can write

$$\begin{aligned} \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} J_A^{\mu_3} \rangle &= \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} j_A^{\mu_3} \rangle + \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} j_{A \text{ loc}}^{\mu_3} \rangle \\ &+ \langle t_{\text{loc}}^{\mu_1 \nu_1} t^{\mu_2 \nu_2} j_A^{\mu_3} \rangle + \langle t_{\text{loc}}^{\mu_1 \nu_1} t_{\text{loc}}^{\mu_2 \nu_2} j_A^{\mu_3} \rangle. \end{aligned} \quad (7.7)$$

We comment on this decomposition for clarity, noting the difference between this expansion, which includes only four terms, and the general one given in (6.5). The suffix “loc” in the current indicates the longitudinal sector,

defined along the direction of momentum p_3 . In the case of the stress-energy tensor, $t_{\text{loc}}^{\mu\nu}$ denotes the sector orthogonal to the transverse-traceless one, according to the decompositions in (6.1).

The conservation of the stress-energy tensor $T^{\mu\nu}$ in this decomposition leaves, in the massive fermion case, only a trace component in $t_{\text{loc}}^{\mu\nu}$. This trace cannot be accompanied, in the expansion of the correlator, by an operator such as j_{loc}^μ . The absence of contributions of the form $\langle t_{\text{loc}} t j_{\text{loc}} \rangle$ can also be understood from symmetry considerations: one cannot construct a tensor of odd parity in the longitudinal sector of the axial-vector current, given the symmetry of t and the presence of a trace in t_{loc} , with $j_{\text{loc}}^\mu \sim \frac{p_3^\mu}{p_3^2}$.

Similar arguments show that terms of the form $\langle t_{\text{loc}} t_{\text{loc}} j_{\text{loc}} \rangle$ cannot contribute as parity-odd components of the correlator. Indeed, a perturbative expansion confirms this result

$$\begin{aligned} g_{\mu_1\nu_1} g_{\mu_2\nu_2} \langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J_A^{\mu_3} \rangle &= 0, \\ p_{3\mu_3} g_{\mu_i\nu_i} \langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J_A^{\mu_3} \rangle &= 0, \quad i = \{1, 2\}, \end{aligned} \quad (7.8)$$

proving the consistency of the computation. Of all the terms contained in (7.7), the anomaly form factor of the ATT is present in the longitudinal component

$$\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} j_{A\text{loc}}^{\mu_3} \rangle = \frac{p_3^{\mu_3}}{p_3^2} \Pi_{\alpha_1\beta_1}^{\mu_1\nu_1}(p_1) \Pi_{\alpha_2\beta_2}^{\mu_2\nu_2}(p_2) \epsilon^{\alpha_1\alpha_2 p_1 p_2} (\bar{F}_1 g^{\beta_1\beta_2}(p_1 \cdot p_2) + \bar{F}_2 p_1^{\beta_2} p_2^{\beta_1}). \quad (7.9)$$

This term is determined by the gravitational anomaly, along with the mass corrections. The explicit expressions of the form factors (7.9) are given by

$$\begin{aligned} \bar{F}_1 &= \frac{1}{24\pi^2} + \frac{m^2}{2\pi^2\lambda(s-s_1-s_2)} \{2[\lambda m^2 + s s_1 s_2] C_0(s, s_1, s_2, m^2) + Q(s)[s(s_1+s_2) - (s_1-s_2)^2] \\ &\quad - s_2 Q(s_2)[s+s_1-s_2] - s_1 Q(s_1)[s-s_1+s_2] + \lambda\}, \\ \bar{F}_2 &= -\frac{1}{24\pi^2} + \frac{m^2}{2\pi^2\lambda^2} \{2[\lambda(m^2(-s+s_1+s_2) - 2s_1 s_2) + 3s_1 s_2((s_1-s_2)^2 - s(s_1+s_2))] C_0(s, s_1, s_2, m^2) \\ &\quad + s_1 Q(s_1)[\lambda + 6s_2(s+s_1-s_2)] + s_2 Q(s_2)[\lambda + 6s_1(s-s_1+s_2)] - Q(s)[12s s_1 s_2 + \lambda(s_1+s_2)] \\ &\quad + \lambda(-s+s_1+s_2)\}. \end{aligned} \quad (7.10)$$

The trace term of the correlator, on the other hand, can be written as

$$\begin{aligned} \langle t_{\text{loc}}^{\mu_1\nu_1} t^{\mu_2\nu_2} j_A^{\mu_3} \rangle &\equiv \Sigma_{\alpha_1\beta_1}^{\mu_1\nu_1}(p_1) \Pi_{\alpha_2\beta_2}^{\mu_2\nu_2}(p_2) \pi_{\alpha_3}^{\mu_3}(p_3) \langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J_A^{\mu_3} \rangle \\ &= \frac{\pi^{\mu_1\nu_1}(p_1)}{3} \Pi_{\alpha_2\beta_2}^{\mu_2\nu_2}(p_2) \pi_{\alpha_3}^{\mu_3}(p_3) \langle T_\mu^{\mu} T^{\mu_2\nu_2} J_A^{\mu_3} \rangle \end{aligned} \quad (7.11)$$

where, in the final step, we used the explicit expression for $\Sigma_{\alpha_1\beta_1}^{\mu_1\nu_1}$ given in (6.4) along with the conservation of the energy-momentum tensor. This component is purely a trace term and it is transverse with respect to all the momenta. It can be expressed as

$$\langle t_{\text{loc}}^{\mu_1\nu_1} t^{\mu_2\nu_2} j_A^{\mu_3} \rangle = \frac{\pi^{\mu_1\nu_1}(p_1)}{3} \Pi_{\alpha_2\beta_2}^{\mu_2\nu_2}(p_2) \pi_{\alpha_3}^{\mu_3}(p_3) [\bar{F}_3 \epsilon^{\alpha_2\alpha_3 p_1 p_2} p_1^{\beta_2}] \quad (7.12)$$

where

$$\begin{aligned} \bar{F}_3 &= -\frac{m^2 s_2}{2\pi^2\lambda^2 s} \{2\lambda s + 2Q(s)[2\lambda s + 3s(s_1^2 + 6s_1 s_2 + s_2^2) + 3(s_1+s_2)(\lambda - (s_1-s_2)^2)] \\ &\quad - 2Q(s_1)[s(\lambda + 3s_1(s_1+3s_2)) + 3s_1(\lambda - (s_1-s_2)^2)] - 2Q(s_2)[s(\lambda + 3s_2(3s_1+s_2)) + 3s_2(\lambda - (s_1-s_2)^2)] \\ &\quad + [\lambda(4m^2 + 3(s_1+s_2)) + 24s s_1 s_2(s_1+s_2) - ((s_1-s_2)^2 - \lambda)(\lambda + 12s_1 s_2)] C_0(s, s_1, s_2, m^2)\}. \end{aligned} \quad (7.13)$$

Finally, the transverse-traceless part can be expressed as

$$\begin{aligned} \langle t^{\mu_1 \nu_1}(p_1) t^{\mu_2 \nu_2}(p_2) j_A^{\mu_3}(p_3) \rangle &= \Pi_{\alpha_1 \beta_1}^{\mu_1 \nu_1}(p_1) \Pi_{\alpha_2 \beta_2}^{\mu_2 \nu_2}(p_2) \pi_{\alpha_3}^{\mu_3}(p_3) \left[A_1 \varepsilon^{p_1 \alpha_1 \alpha_2 \alpha_3} p_2^{\beta_1} p_3^{\beta_2} - A_1 (p_1 \leftrightarrow p_2) \varepsilon^{p_2 \alpha_1 \alpha_2 \alpha_3} p_2^{\beta_1} p_3^{\beta_2} \right. \\ &\quad \left. + A_2 \varepsilon^{p_1 \alpha_1 \alpha_2 \alpha_3} \delta^{\beta_1 \beta_2} - A_2 (p_1 \leftrightarrow p_2) \varepsilon^{p_2 \alpha_1 \alpha_2 \alpha_3} \delta^{\beta_1 \beta_2} + A_3 \varepsilon^{p_1 p_2 \alpha_1 \alpha_2} p_2^{\beta_1} p_3^{\beta_2} p_1^{\alpha_3} + A_4 \varepsilon^{p_1 p_2 \alpha_1 \alpha_2} \delta^{\beta_1 \beta_2} p_1^{\alpha_3} \right] \end{aligned} \quad (7.14)$$

where

$$\begin{aligned} A_1 &= -\frac{s_2}{24\pi^2 \lambda^4} [\bar{A}_{11} Q(s_1) + \bar{A}_{12} Q(s_2) + \bar{A}_{13} Q(s) + \bar{A}_{14} C_0(s, s_1, s_2, m^2) + \bar{A}_{15}], \\ A_2 &= -\frac{s_2}{48\pi^2 \lambda^3} [\bar{A}_{21} Q(s_1) + \bar{A}_{22} Q(s_2) + \bar{A}_{23} Q(s) + \bar{A}_{24} C_0(s, s_1, s_2, m^2) + \bar{A}_{25}], \\ A_3 &= 0, \\ A_4 &= 0. \end{aligned} \quad (7.15)$$

The complete expressions of these contributions can be found in Appendix C.

VIII. THE SUM RULE OF THE ATT FOR ON-SHELL AND OFF-SHELL GRAVITONS

Before discussing the sum rule of the gravitational anomaly form factor in full generality, we consider the case in which the two gravitons are on-shell. The on-shell expression of the ATT takes a form close to the AVV. A direct perturbative computation shows that also in this case the only tensor structure present is the longitudinal one, proportional to $R\tilde{R}$

$$\begin{aligned} \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} J_A^{\mu_3} \rangle &= \frac{1}{96\pi^2} \left[1 + \frac{12m^2}{p_3^2} (1 + 2m^2 C_0(p_3^2, 0, 0, m^2)) \right] \\ &\quad \times \frac{p_3^{\mu_3}}{p_3^2} \left\{ \left[\varepsilon^{\nu_1 \nu_2 p_1 p_2} ((p_1 \cdot p_2) g^{\mu_1 \mu_2} - p_1^{\mu_2} p_2^{\mu_1}) + (\mu_1 \leftrightarrow \nu_1) \right] + (\mu_2 \leftrightarrow \nu_2) \right\}. \end{aligned} \quad (8.1)$$

To investigate the presence of a pole in the corresponding form factor, the $C_0(s, 0, 0, m^2)$ integral needs to be expanded around $s = 0$. We can use the expression (3.31) or, equivalently, we can rewrite it in the form

$$C_0(s, 0, 0, m^2) = \frac{\log^2 \left(\frac{\sqrt{s(s-4m^2)+2m^2-s}}{2m^2} \right)}{2s}. \quad (8.2)$$

To better understand the behavior of $C_0(s, 0, 0, m^2)$ near $s = 0$, we perform a series expansion. Expanding $C_0(s, 0, 0, m^2)$ around $s = 0$, as in (3.35), allows for a direct computation that confirms the absence of a pole at $s = 0$. Substituting this expansion into (8.1) and simplifying the expression reveals that the $1/s$ and $1/s^2$ terms cancel out. Specifically, the remaining contributions are finite and free of singularities, explicitly demonstrating the cancellation mechanism. Consequently, in the on-shell spectral function, only the cut contribution remains, in complete analogy with the AVV correlator.

We are going to expand on these analysis, showing the presence of a sum rule in the most general case, extending the approach presented for the AVV in the former sections. The conformal description of the off-shell $(s_1, s_2 \neq 0, m = 0)$ $\langle TTJ_A \rangle$ or $\langle TTJ_{CS} \rangle$ correlators is identical, as the only distinction between the two scenarios lies in the numerical value of the anomaly. In the conformal limit, the pole is inherently present, as demonstrated in (6.9). In the massive case, the same form factor can be computed perturbatively using the diagrams shown in Fig. 5. Similarly, in this case, the perturbative computation does not necessitate any counterterms; however, an explicit regularization scheme is required to handle the γ_5 matrix within the loops. For this purpose, we have employed the Breitenlohner-Maison scheme for γ_5 . We have presented the results of the computation in Sec. VII A. Specifically, the structure of the correlator in the longitudinal part, which defines the anomaly form factor in this context, can be written as

$$\begin{aligned} \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} j_{A\text{loc}}^{\mu_3} \rangle &= (\bar{F}_1 - \bar{F}_2) \frac{p_3^{\mu_3}}{4p_3^2} \{ [\varepsilon^{\nu_1 \nu_2 p_1 p_2} (p_1 \cdot p_2 g^{\mu_1 \mu_2} - p_1^{\mu_2} p_2^{\mu_1}) + (\mu_1 \leftrightarrow \nu_1)] + (\mu_2 \leftrightarrow \nu_2) \} \\ &\quad + (\bar{F}_1 + \bar{F}_2) \frac{p_3^{\mu_3}}{4p_3^2} \{ [\varepsilon^{\nu_1 \nu_2 p_1 p_2} (p_1 \cdot p_2 g^{\mu_1 \mu_2} + p_1^{\mu_2} p_2^{\mu_1}) + (\mu_1 \leftrightarrow \nu_1)] + (\mu_2 \leftrightarrow \nu_2) \} \\ &\quad + (\bar{F}_1 + \bar{F}_2) \frac{(p_1 \cdot p_2)}{p_1^2 p_2^2} \frac{p_3^{\mu_3}}{4p_3^2} \{ [\varepsilon^{\nu_1 \nu_2 p_1 p_2} ((p_1 \cdot p_2) p_1^{\mu_1} p_2^{\mu_2} - p_2^2 p_1^{\mu_1} p_1^{\mu_2} - p_1^2 p_2^{\mu_1} p_2^{\mu_2}) + (\mu_1 \leftrightarrow \nu_1)] + (\mu_2 \leftrightarrow \nu_2) \} \end{aligned} \quad (8.3)$$

where the first contribution, proportional to $\bar{F}_1 - \bar{F}_2$ is related to the second functional derivative of the anomaly $R\bar{R}$ performed with respect to the metric and defines the anomaly form factor

$$\begin{aligned} \phi_{\text{ATT}} = \frac{\bar{F}_1 - \bar{F}_2}{s} = & \frac{m^2 s_1 (s_2 (s_1^2 - s^2) + s_2^2 (5s + s_1) + (s_1 - s)^3 - 3s_2^3) B_0(s_1, m^2)}{\pi^2 s (s - s_1 - s_2) \lambda^2} \\ & + \frac{m^2 s_2 (s_2 (3s^2 + s_1^2) + s_2^2 (s_1 - 3s) - (s - s_1)^2 (s + 3s_1) + s_2^3) B_0(s_2, m^2)}{\pi^2 s (s - s_1 - s_2) \lambda^2} \\ & + \frac{m^2 K_1(s, s_1, s_2) B_0(s, m^2)}{\pi^2 s (s - s_1 - s_2) \lambda^2} + \frac{m^2 K_2(s, s_1, s_2) C_0(s, s_1, s_2, m^2)}{\pi^2 s (s - s_1 - s_2) \lambda^2} \\ & \times \frac{m^2 (s^2 - 2s(s_1 + s_2) + s_1^2 + s_2^2)}{\pi^2 s (s - s_1 - s_2) \lambda} + \frac{1}{12\pi^2 s}. \end{aligned} \quad (8.4)$$

Notice that this form factor at large s behaves as

$$\phi_{\text{ATT}} \sim \frac{1}{12\pi^2 s} + O\left(\frac{m^2}{s^2} \log \frac{s}{m^2}\right), \quad (8.5)$$

similarly to the behavior encountered in the case of the AVV, showing the dominance of the anomaly at large q^2 . The second and the third terms in (8.3), which are related to $\bar{F}_1 + \bar{F}_2$

$$\begin{aligned} \bar{F}_1 + \bar{F}_2 = & \frac{2m^2 s_1 s_2 (s^2 + s(s_1 - 2s_2) - 2s_1^2 + s_1 s_2 + s_2^2) B_0(s_1, m^2)}{\pi^2 (s - s_1 - s_2) \lambda^2} + \frac{2m^2 s_1 s_2 (s_2 (s + s_1) + (s - s_1)^2 - 2s_2^2) B_0(s_2, m^2)}{\pi^2 s (s - s_1 - s_2) \lambda^2} \\ & + \frac{2m^2 s_1 s_2 (-2s^2 + s(s_1 + s_2) + (s_1 - s_2)^2) B_0(s, m^2)}{\pi^2 (s - s_1 - s_2) \lambda^2} + \frac{m^2 s_1 s_2 K_3(s, s_1, s_2) C_0(s, s_1, s_2, m^2)}{\pi^2 (s - s_1 - s_2) \lambda^2} \\ & - \frac{2m^2 s_1 s_2}{\pi^2 (s - s_1 - s_2) \lambda} \end{aligned} \quad (8.6)$$

are generated by the breaking of conformal symmetry, vanishing in the massless case and/or for on-shell gravitons ($s_1 = s_2 = 0$) and after contraction with the graviton polarizers. In Eq. (8.6) the expressions of K_1 , K_2 , and K_3 are given by

$$K_1(s, s_1, s_2) = s_2 (s - s_1) (s^2 + 3s s_1 - 2s_1^2) + s_2^3 (3s + 2s_1) - s_2^2 (s + s_1) (3s + 2s_1) + s_1 (s - s_1)^3 - s_2^4, \quad (8.7)$$

$$K_2(s, s_1, s_2) = 2m^2 \lambda (s^2 - 2s(s_1 + s_2) + s_1^2 + s_2^2) + s_1 s_2 (3s^3 - 5s^2 (s_1 + s_2) + s(s_1 + s_2)^2 + (s_1 - s_2)^2 (s_1 + s_2)), \quad (8.8)$$

$$K_3(s, s_1, s_2) = s_2 (8m^2 (s + s_1) + s^2 - 6s s_1 + s_1^2) + s_2^2 (-4m^2 + s + s_1) - (s - s_1)^2 (4m^2 + s + s_1) - s_2^3. \quad (8.9)$$

We are going to show that the combination $\bar{F}_1 - \bar{F}_2$ defines the correct expression of the anomaly form factor and we will verify the presence of a sum rule also in this case. Notice that the contribution proportional to $\bar{F}_1 + \bar{F}_2$, as we have mentioned, is proportional to the virtualities of the two gravitons and vanishes as we require these to be on-shell [see Eq. (8.1)].

A. Derivation of the general sum rule for the ATT

The anomaly form factor is associated with the first structure proportional to $\bar{F}_1 - \bar{F}_2$ in (8.3). The corresponding tensor structure is the off-shell expression of the second

functional derivative of $R\bar{R}$, Fourier transformed to momentum space. The second contribution, proportional to $\bar{F}_1 + \bar{F}_2$, as we have just discussed, vanishes for on-shell-gravitons and is generated by the explicit breaking of the chiral symmetry for a nonzero mass parameter.

Coming to the spectral function of the form factor in (8.4), we separate the discontinuity into the pole contributions plus the continuum, in the form

$$\text{disc}(\phi_{\text{ATT}}) = \text{disc}(\phi_{\text{ATT}})_{\text{cont}} + \text{disc}(\phi_{\text{ATT}})_{\text{pole}}, \quad (8.10)$$

with the discontinuity over the continuum given by

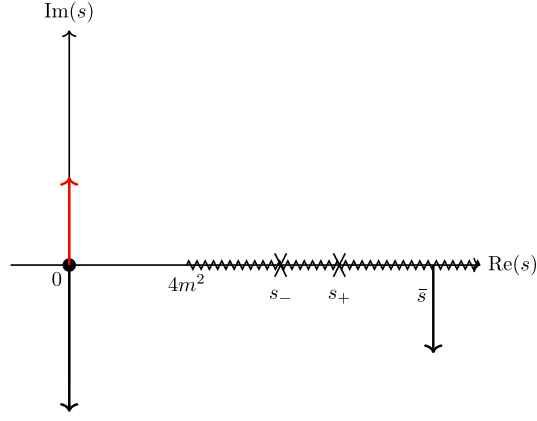


FIG. 6. The spectral density of the off-shell gravitational anomaly form factor in the complex $s \equiv q^2$ plane. Shown are the two spikes at $s = 0$, the anomalous thresholds at $s_{\pm} = (\sqrt{s_1} \pm \sqrt{s_2})^2$, and the pole at $s = \bar{s} = s_1 + s_2$. In this case there is a cancellation in the sum rule between the integral of the density over the cut, the black spike at $s = 0$, of variable strength, and the black spike at $s = \bar{s}$.

$$\text{disc}(\phi_{\text{ATT}})_{\text{cont}} = \frac{m^2 K_1(s, s_1, s_2) \text{disc} B_0(s, m^2)}{\pi^2 s (s - s_1 - s_2) \lambda^2} + \frac{gm^2 K_2(s, s_1, s_2) \text{disc} C_0(s, s_1, s_2, m^2)}{\pi^2 s (s - s_1 - s_2) \lambda^2}. \quad (8.11)$$

The discontinuity of C_0 is given by (A15), while

$$\text{disc} B_0(s, m^2) = 2\pi i \sqrt{1 - \frac{4m^2}{s}} \theta(s - 4m^2). \quad (8.12)$$

The pole contributions split at separate locations

$$\text{disc}(\phi_{\text{ATT}})_{\text{pole}} = \text{disc}(\phi_{\text{ATT}})_0 + \text{disc}(\phi_{\text{ATT}})_{s_1+s_2} + \text{disc}(\phi_{\text{ATT}})_{s_-} + \text{disc}(\phi_{\text{ATT}})_{s_+}, \quad (8.13)$$

where we have indicated the points corresponding to the zeroes of the λ function as

$$s_{\pm} = (\sqrt{s_1} \pm \sqrt{s_2})^2. \quad (8.14)$$

The pole at $s = 0$ is split into two spikes,

$$\text{disc}(\phi_{\text{ATT}})_0 = 2\pi i \frac{\delta(s)}{12\pi^2} - \frac{2\pi i \delta(s)}{12\pi^2 (s_1 - s_2)^3 (s_1 + s_2)} \Psi(s, s_1, s_2, m^2) \quad (8.15)$$

where

$$\begin{aligned} \Psi(s, s_1, s_2, m^2) \equiv & (12m^2 - ((s_1 - s_2)(s_1^2 + s_2^2)B_0(s, m^2)) + s_1(s_1^2 + 2s_1s_2 + 3s_2^2)B_0(s_1, m^2) \\ & - s_2(3s_1^2 + 2s_1s_2 + s_2^2)B_0(s_2, m^2) + (s_1 - s_2)(2m^2(s_1^2 + s_2^2) + s_1s_2(s_1 + s_2))C_0(s, s_1, s_2, m^2)) \\ & + 12m^2(s_1^2 + s_2^2)(s_1 - s_2)). \end{aligned} \quad (8.16)$$

We have outlined the various contributions in Fig. 6. The “black spike” at $s = 0$, which has a constant strength, corresponds to the first term on the right-hand side of (8.15). The second contribution, which depends on the external invariants, is represented by the “red spike” in the same figure.

For $s > 4m^2$, across the cut, we encounter two apparent poles that, as we will demonstrate, do not contribute to the discontinuity. Additionally, there is a pole at $s = s_1 + s_2$, which can lie either above or below the cut (i.e., for $s > 4m^2$ or $s < 4m^2$), depending on the values of s_1 and s_2 . Finally, we observe that in the limit $s \rightarrow 0$,

$$\begin{aligned} \lim_{s \rightarrow 0} \Psi(s, s_1, s_2) = \Psi(0, s_1, s_2, m^2) \equiv & (12m^2 - ((s_1 - s_2)(s_1^2 + s_2^2)B_0^R(0, m^2)) \\ & + s_1(s_1^2 + 2s_1s_2 + 3s_2^2)B_0^R(s_1, m^2) - s_2(3s_1^2 + 2s_1s_2 + s_2^2)B_0^R(s_2, m^2) \\ & + (s_1 - s_2)(2m^2(s_1^2 + s_2^2) + s_1s_2(s_1 + s_2))C_0(0, s_1, s_2, m^2)) + 12m^2(s_1^2 + s_2^2)(s_1 - s_2)) < \infty \end{aligned} \quad (8.17)$$

and Ψ remains finite across the entire light cone.

B. Cancellation of the anomalous thresholds and the sum rule

The expression for the spectral density

$$\Phi_{\text{ATT}} = \frac{F_1 - F_2}{s} \quad (8.18)$$

can be separated into distinct components, representing contributions from regular kinematical thresholds as well as potential anomalous thresholds

$$\Phi_{\text{ATT}} = \Phi_{\text{ATT}}^{\text{reg}} + \Phi_{\text{ATT}}^{(\lambda)}, \quad (8.19)$$

where $\Phi_{\text{ATT}}^{\text{reg}}$ corresponds to regular kinematical thresholds and $\Phi_{\text{ATT}}^{(\lambda)}$ accounts for possible anomalous thresholds. The

regular part, $\Phi_{\text{ATT}}^{\text{reg}}$, includes contributions from standard kinematical thresholds located at

$$q^2 = 0, \quad q^2 = s_1 + s_2, \quad \text{and} \quad q^2 = 4m^2, \quad (8.20)$$

which manifest in the form of simple poles and cuts. These thresholds arise from well-understood physical processes. In contrast, the anomalous part, $\Phi_{\text{ATT}}^{\text{anom}}$, emerges under special conditions when the kinematical factor $\lambda = 0$ which could be present in this computation. Given that

$$\lambda = (s - s_-)(s - s_+), \quad (8.21)$$

the dispersive representation of Φ_{ATT} may allow terms of the form

$$\begin{aligned} \text{disc}\Phi_{\text{ATT}} = & c_0\delta(s) + c_1(s, s_1, s_2)\theta(s - 4m^2)H(s) + c_2(s, s_1, s_2)\delta(s - s_1 - s_2) \\ & + c_3(s, s_1, s_2)\delta'(s - s_-) + c_4\delta'(s - s_+) + c_5\delta(s - s_-) + c_6\delta(s - s_+), \end{aligned} \quad (8.22)$$

where the derivative of the δ function corresponds to higher order poles at $\lambda = 0$. Indeed, the general structure of Φ_{ATT} is

$$\begin{aligned} \Phi_{\text{ATT}} = & \frac{c_0(s, s_1, s_2, m^2)}{s} + \frac{c_1(s, s_1, s_2, m^2)}{s} + (\text{continuum terms}) + \frac{c_2(s, s_1, s_2, m^2)}{s - \bar{s}} + \frac{c_3(s, s_1, s_2, m^2)}{(s - s_-)^2} \\ & + \frac{c_4(s, s_1, s_2, m^2)}{(s - s_+)^2} + \frac{c_5(s, s_1, s_2, m^2)}{s - s_-} + \frac{c_6(s, s_1, s_2, m^2)}{s - s_+}, \end{aligned} \quad (8.23)$$

where $\bar{s} = s_1 + s_2$ and, c_3 and c_4 define possible residues at the double poles of λ^2 . These could, in principle, introduce subtractions that can be analyzed as follows. For example, at $s = s_-$, we have

$$\Phi_{\text{ATT}}^{(\lambda)}(q^2 = s_-) = \frac{1}{2\pi i} \int_0^\infty \frac{c_3(s, s_1, s_2, m^2)\delta'(s - s_-)}{(s - q^2)} ds + \frac{1}{2\pi i} \int_0^\infty \frac{c_5(s, s_1, s_2, m^2)\delta(s - s_-)}{(s - q^2)} ds, \quad (8.24)$$

that simplifies to

$$2\pi i \Phi_{\text{ATT}}^{(\lambda)}(q^2 = s_-) = -\frac{d}{ds} \left(\frac{c_3(s, s_1, s_2, m^2)}{(s - q^2)} \right) \Big|_{s=s_-} - \frac{c_5(s_-, s_1, s_2, m^2)}{s_- - q^2}. \quad (8.25)$$

A similar analysis gives

$$2\pi i \Phi_{\text{ATT}}^{(\lambda)}(q^2 = s_+) = -\frac{d}{ds} \left(\frac{c_4(s, s_1, s_2, m^2)}{(s - q^2)} \right) \Big|_{s=s_+} - \frac{c_6(s_+, s_1, s_2, m^2)}{s_+ - q^2}. \quad (8.26)$$

For the evaluation of $\phi_{\text{ATT}}^{(\lambda)}(q^2 = s_\pm)$, we need to compute the explicit expression of the scalar integral C_0 at these kinematical points. We use the expressions of the scalar integral C_0 at these points. They are given by

$$\begin{aligned} C_0(s_-, s_1, s_2, m^2) = & \frac{1}{s_1 s_2 s_-^2} \left(\sqrt{s_1} \left(\sqrt{s_2(s_2 - 4m^2)} (\sqrt{s_1} - \sqrt{s_2}) \log \left(\frac{\sqrt{s_2(s_2 - 4m^2)} + 2m^2 - s_2}{2m^2} \right) \right. \right. \\ & + \sqrt{s_2} \sqrt{(-2\sqrt{s_1}\sqrt{s_2} + s_1 + s_2)(-4m^2 - 2\sqrt{s_1}\sqrt{s_2} + s_1 + s_2)} \log(S_1) \Big) \\ & + \left. \left. \sqrt{s_2} \sqrt{s_1(s_1 - 4m^2)} (\sqrt{s_2} - \sqrt{s_1}) \log \left(\frac{\sqrt{s_1(s_1 - 4m^2)} + 2m^2 - s_1}{2m^2} \right) \right) \right) \end{aligned} \quad (8.27)$$

and

$$C_0(s_+, s_1, s_2, m^2) = \frac{1}{s_1 s_2 s_+^2} \left(\sqrt{s_1} \left(\sqrt{s_2(s_2 - 4m^2)} (\sqrt{s_1} + \sqrt{s_2}) \log \left(\frac{\sqrt{s_2(s_2 - 4m^2)} + 2m^2 - s_2}{2m^2} \right) \right. \right. \\ \left. \left. - \sqrt{s_2} \sqrt{(\sqrt{s_1} + \sqrt{s_2})^4 - 4m^2(\sqrt{s_1} + \sqrt{s_2})^2} \log(S_2) \right) \right. \\ \left. + \sqrt{s_2} \sqrt{s_1(s_1 - 4m^2)} (\sqrt{s_1} + \sqrt{s_2}) \log \left(\frac{\sqrt{s_1(s_1 - 4m^2)} + 2m^2 - s_1}{2m^2} \right) \right), \quad (8.28)$$

where

$$S_1 = \frac{\sqrt{(-2\sqrt{s_1}\sqrt{s_2} + s_1 + s_2)(-4m^2 - 2\sqrt{s_1}\sqrt{s_2} + s_1 + s_2) + 2m^2 - (\sqrt{s_1} - \sqrt{s_2})^2}}{2m^2} \quad (8.29)$$

and

$$S_2 = \frac{\sqrt{(\sqrt{s_1} + \sqrt{s_2})^4 - 4m^2(\sqrt{s_1} + \sqrt{s_2})^2 + 2m^2 - (\sqrt{s_1} + \sqrt{s_2})^2}}{2m^2}. \quad (8.30)$$

Using these relations one can show by a long computation that

$$\Phi_{\text{ATT}}^{(\lambda)}(q^2 = s_{\pm}) \equiv 0, \quad (8.31)$$

and hence

$$\Phi_{\text{ATT}}^{(\lambda)} = 0. \quad (8.32)$$

Therefore, the only pole to consider at finite s is the one located at $s = s_1 + s_2$. This contributes to the sum rule with a localized density that is given by

$$\text{disc}(\phi_{\text{ATT}})_{s_1+s_2} = \frac{2\pi i m^2 \delta(s - s_1 - s_2)}{4\pi^2(s_1 + s_2)} (-2B_0(s_1 + s_2, m^2) + B_0(s_1, m^2) \\ + B_0(s_2, m^2) + (4m^2 - s_1 - s_2)C_0(s_1, s_2, s_1 + s_2, m^2) + 2). \quad (8.33)$$

Notice that the UV divergences in the B_0 's in the expression above cancel automatically, with rule $-2 + 1 + 1$ in the counting of the $1/\epsilon_{\text{UV}}$ poles in each of these self-energy contributions. Therefore we can replace in (8.33) each B_0 by B_0^R with no change. No IR singularities are encountered unless we consider special values where any of the external invariants vanishes.

In order to compute the sum rule for the longitudinal part of the ATT correlator we can use the same technique used before for the AVV. Therefore, as in the case of the anomaly form factor of the AVV, we apply the inverse Laplace transform on both sides of the dispersive representation of the form factor (Cauchy's relation) to obtain

$$\mathcal{L}^{-1} \left\{ \int_0^\infty \frac{\Delta\phi_{\text{ATT}}}{s - \bar{s}} ds \right\} (t) = \frac{1}{\pi} \int_0^\infty \Delta\phi_{\text{ATT}} e^{st} ds, \quad (8.34)$$

where

$$\Delta\phi_{\text{ATT}} \equiv \frac{1}{2i} \text{disc}\phi_{\text{ATT}} \quad (8.35)$$

and the integral of the spectral density will be obtained by performing the $t \rightarrow 0$ limit

$$\lim_{t \rightarrow 0} \mathcal{L}^{-1} \{ \phi_{\text{ATT}} \} (t) = \frac{1}{\pi} \int_0^\infty \Delta\phi_{\text{ATT}} ds. \quad (8.36)$$

The techniques used to compute the inverse Laplace transform is the same described in the previous case. We have used the parametric form of the two-point and the three-point scalar functions, exchanging the integration with the inverse Laplace transform, and once the limit on t has been performed, we obtain the result of the integral

in Eq. (8.36)

$$\frac{1}{\pi} \int_0^\infty \Delta\Phi_{\text{ATT}} ds = a_g \quad (8.37)$$

with a_g denoting the gravitational anomaly. The pattern of cancellation, in this case, is between the continuum, the pole at $s = \bar{s} = s_1 + s_2$, and the pole at $s = 0$ giving the identity

$$\int_0^\infty \frac{1}{2i} (\text{disc}\Phi_{\text{ATT0}}(s, s_1, s_2, m^2) + \text{disc}\Phi_{\text{ATT}_{s_1+s_2}}(s, s_1, s_2, m^2)) = \int_{4m^2}^\infty ds \frac{1}{2i} \text{disc}\Phi_{\text{ATT}}(s, s_1, s_2, m^2) \quad (8.38)$$

where

$$\begin{aligned} \int_0^\infty \frac{1}{2i} (\text{disc}\Phi_{\text{ATT0}}(s, s_1, s_2, m^2) + \text{disc}\Phi_{\text{ATT}_{s_1+s_2}}(s, s_1, s_2, m^2)) &= \frac{1}{12\pi} - \frac{1}{12\pi(s_1 - s_2)^3(s_1 + s_2)} \Psi(0, s_1, s_2, m^2) \\ &+ \frac{m^2}{4\pi(s_1 + s_2)} (-2B_0^R(s_1 + s_2, m^2) + B_0^R(s_1, m^2) \\ &+ B_0^R(s_2, m^2) + (4m^2 - s_1 - s_2)C_0(s_1, s_2, s_1 + s_2, m^2) + 2), \end{aligned} \quad (8.39)$$

while the right-hand side of (8.38) is given by integrating (8.11). Equation (8.37) is the result of this cancellation. The proof of this identity can be obtained by using the parametric representation of the integrals and the Laplace/Borel method, followed by complete numerical checks.

IX. CONCLUSIONS

In this work, we have examined both local and nonlocal formulations of axion interactions, highlighting their distinctions and limitations—particularly in relation to anomaly dynamics. The local interaction, derived under specific symmetry assumptions, is well suited to contexts such as QCD, where an axion mass emerges at the confinement scale. In such scenarios, the conceptual constraints identified in our analysis can be effectively bypassed.

A central challenge lies in identifying the conditions under which local and nonlocal formulations become equivalent. From the perspective of the one-particle-irreducible effective action, the anomaly form factors associated with nonlocal interactions are constrained by perturbative sum rules. These sum rules govern the dynamics of virtual states and serve as key signatures of anomaly-induced processes. Importantly, we have demonstrated that these rules remain valid even in the presence of gravitational anomalies and for arbitrary fermion masses—despite being violated by the local action.

The sum rules differentiate between contributions from particle poles and branch cuts, offering a comprehensive framework for understanding the off-shell behavior of the interaction. While the local formulation captures on-shell physics, it fails to reflect this richer structure.

We have illustrated the nontrivial interplay between poles and branch cuts, offering a different perspective on the emergence of such interactions, relevant in the context

of QCD factorization and the proton spin, which has been recently debated in several recent works [25–27]. In the conformal limit, anomaly interactions impose nontrivial constraints on both longitudinal and transverse sectors via conformal Ward identities. Spectral density flows further reveal a redistribution of contributions between the continuum and massless poles. This study significantly extends previous investigations by providing explicit off-shell spectral density representations of the chiral anomaly form factors for both correlators. For massless fermions, the anomaly and particle poles coincide only on-shell; off-shell, anomaly-induced interactions behave differently. Notably, no particle pole persists for massive fermions or vector bosons. Nonetheless, at high energies, anomaly interactions remain dominated by pole structures that can be effectively described by light-cone-localized actions involving mixed pseudoscalar fields.

The implications of these results extend beyond conventional axionlike particles (ALPs) in particle physics and cosmology. They also characterize emergent ALP behavior in experimental systems such as artificial gauge fields and topological materials. In these analog systems—driven by thermal gradients and chiral chemical potentials—chiral and gravitational interactions mimic those found in anomaly physics.

We emphasize that local ALP interactions, due to their general couplings, can only be consistently justified in the presence of nonperturbative effects. In such cases, an anomaly pole in the ultraviolet transitions into an asymptotic state below a certain scale.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ABSENCE OF A PARTICLE POLE FOR $q^2 \rightarrow 0$ (s_1, s_2, m^2) $\neq 0$

Using the Schouten relations,

$$\begin{aligned} p_2^{\mu_1} \epsilon^{\mu_2 \mu_3 p_1 p_2} &= p_2^{\mu_2} \epsilon^{\mu_1 \mu_3 p_1 p_2} - p_2^{\mu_3} \epsilon^{\mu_1 \mu_2 p_1 p_2} - p_2^2 \epsilon^{\mu_1 \mu_2 \mu_3 p_1} + (p_1 \cdot p_2) \epsilon^{\mu_1 \mu_2 \mu_3 p_2}, \\ p_1^{\mu_2} \epsilon^{\mu_1 \mu_3 p_1 p_2} &= p_1^{\mu_1} \epsilon^{\mu_2 \mu_3 p_1 p_2} + p_1^{\mu_3} \epsilon^{\mu_1 \mu_2 p_1 p_2} - p_1^2 \epsilon^{\mu_1 \mu_2 \mu_3 p_2} + (p_1 \cdot p_2) \epsilon^{\mu_1 \mu_2 \mu_3 p_1}, \end{aligned} \quad (\text{A1})$$

and by a direct perturbative computation of the vertex, we can express the AVV as follows:

$$\begin{aligned} \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J_A^{\mu_3}(q) \rangle &= F_1(\epsilon^{\mu_1 \mu_2 p_1 p_2} p_1^{\mu_3} + \epsilon^{\mu_1 \mu_2 p_1 p_2} p_2^{\mu_3}), \\ &+ F_2(\epsilon^{\mu_2 \mu_3 p_1 p_2} p_1^{\mu_1} - \epsilon^{\mu_1 \mu_2 \mu_3 p_2} s_1) + F_3(\epsilon^{\mu_1 \mu_3 p_1 p_2} p_2^{\mu_2} - \epsilon^{\mu_1 \mu_2 \mu_3 p_1} s_2), \end{aligned} \quad (\text{A2})$$

with

$$\begin{aligned} F_1 &= \frac{1}{64\pi^2 \lambda(s, s_1, s_2)^2} (2(s^2 - 2(s_1 + s_2)s + (s_1 - s_2)^2)(s - s_1 - s_2) \\ &+ 2((s_1 + s_2)s^2 - 2(s_1^2 - 4s_2s_1 + s_2^2)s + (s_1 - s_2)^2(s_1 + s_2))B_0(s, m^2) \\ &- 2s_1((s - s_1)^2 - 5s_2^2 + 4(s + s_1)s_2)B_0(s_1, m^2) - 2s_2(s^2 + 4s_1s - 2s_2s - 5s_1^2 + s_2^2 + 4s_1s_2)B_0(s_2, m^2) \\ &+ 4(m^2(s - s_1 - s_2)(s^2 - 2(s_1 + s_2)s + (s_1 - s_2)^2) - s_1s_2(-2s^2 + (s_1 + s_2)s + (s_1 - s_2)^2))C_0(s, s_1, s_2, m^2)), \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} F_2 &= \frac{1}{2\pi^2 \lambda(s, s_1, s_2)^2} ((s^2 - 2(s_1 + s_2)s + (s_1 - s_2)^2)(s - s_1 + s_2) + s((s - s_1)^2 - 5s_2^2 + 4(s + s_1)s_2)B_0(s, m^2) \\ &- (s^3 - (2s_1 + s_2)s^2 + (s_1^2 + 8s_2s_1 - s_2^2)s + (s_1 - s_2)^2s_2)B_0(s_1, m^2) \\ &+ (-5s^2 + 4(s_1 + s_2)s + (s_1 - s_2)^2)s_2B_0(s_2, m^2) + 2((s - s_1 + s_2)((s - s_1)^2 + s_2^2 - 2(s + s_1)s_2)m^2 \\ &+ ss_2(s^2 + s_1s - 2s_2s - 2s_1^2 + s_2^2 + s_1s_2))C_0(s, s_1, s_2, m^2)), \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} F_3 &= -\frac{1}{2\pi^2 \lambda(s, s_1, s_2)^2} ((s^2 - 2(s_1 + s_2)s + (s_1 - s_2)^2)(s + s_1 - s_2) \\ &+ s(s^2 + 4s_1s - 2s_2s - 5s_1^2 + s_2^2 + 4s_1s_2)B_0(s, m^2) + s_1(-5s^2 + 4(s_1 + s_2)s + (s_1 - s_2)^2)B_0(s_1, m^2) \\ &- ((s + s_1)(s - s_1)^2 + (s + s_1)s_2^2 - 2(s^2 - 4s_1s + s_1^2)s_2)B_0(s_2, m^2) \\ &+ 2((s + s_1 - s_2)((s - s_1)^2 + s_2^2 - 2(s + s_1)s_2)m^2 + ss_1((s - s_1)^2 - 2s_2^2 + (s + s_1)s_2))C_0(s, s_1, s_2, m^2)). \end{aligned} \quad (\text{A5})$$

Before performing some special kinematical limits on this amplitude, we comment on its UV finiteness, which is not apparent from the results for the three form factors F_1 , F_2 , and F_3 shown above. Notice that in these expressions the bare self-energies B_0 can be interpreted, indifferently, as either renormalized or bare. The renormalization removes the $1/\epsilon_{\text{UV}}$ poles of these integrals by defining

$$B_0(q^2, m^2) \rightarrow B_0^R(q^2, m^2) \equiv B_0(q^2, m^2) - \frac{1}{\epsilon_{\text{UV}}}. \quad (\text{A6})$$

One can explicitly check that by combining all such self-energy contributions and collecting all the polynomial coefficients in front, such ultraviolet divergences cancel, as expected. For this reason, in these expressions one can automatically replace in (A3)–(A5) B_0 with B_0^R . At this point, in order to compute the residue at $q^2 = 0$, given the absence of poles in the representation (A2), we need just to observe that in the expressions above for F_1 , F_2 , and F_3 no pole in $1/q^2$ are present as far as s_1 and s_2 and m are nonzero. Therefore one can perform the light-cone limit, obtaining

$$\lim_{q^2 \rightarrow 0} q^2 \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(q) \rangle = 0. \quad (\text{A7})$$

This result shows that there is no massless particle pole in the vertex for general kinematics. This result remains valid in the infrared limit of q (i.e. $q^\lambda \rightarrow 0$) as well, although the analysis simplifies significantly in this case. As already pointed out, the vanishing of the vertex in the infrared is related to the derivative coupling of the pseudoscalar mode, as for an ordinary NG mode.

1. The continuum $\text{disc}C_0$

The contribution from the continuum is obtained from the s -channel cut by the ordinary cutting rules

$$\text{disc}C_0 = \frac{1}{i\pi^2} \int d^4k \frac{(-2\pi i)^2 \delta_+(k^2 - m^2) \delta_+((k - q)^2 - m^2)}{(k - p_1)^2 - m^2} \quad (\text{A8})$$

that identify the light-cone character of the intermediate state. This integral is evaluated in the rest frame of the invariant $q = (q_0, 0, 0, 0)$. It corresponds to a cut

contribution in which the axial-vector line decays into two back-to-back on-shell fermions. This pair can be effectively treated as a pseudoscalar intermediate state, which subsequently decays into two off-shell spin-1 particles. To evaluate the integral, we can use polar coordinates of $\vec{k}$ around the momentum $|p_1|$ together with the relations

$$\begin{aligned} \delta_+(k^2 - m^2) &= \frac{\delta(k_0 - \sqrt{|k|^2 + m^2})}{2\sqrt{|k|^2 + m^2}}, \\ \delta_+((k - q)^2 - 4m^2) &= \frac{1}{4q_0 \sqrt{q^2 - 4m^2}} \\ &\times \delta\left(|k| - \frac{1}{2}\sqrt{q^2 - 4m^2}\right) \end{aligned} \quad (\text{A9})$$

with

$$k_0 = \frac{\sqrt{q^2}}{2}, \quad |k| = \frac{1}{2}\sqrt{q^2 - 4m^2}. \quad (\text{A10})$$

The integral gets reduced to a simple angular form

$$\int_{-1}^1 \frac{d \cos \theta}{A + B \cos \theta} = \frac{1}{B} \log \left(\frac{A + B}{A - B} \right) \quad \text{where } A = s_1 + s_2 - q^2, \quad B = 2\lambda^{1/2}(q^2, s_1, s_2) \sqrt{q^2 - 4m^2}. \quad (\text{A11})$$

Collecting all the terms together, the spectral density takes the simple form

$$\begin{aligned} \Delta C_0(s, s_1, s_2, m^2) &\equiv \text{Im } C_0(s, s_2, s_2, m^2) = -\frac{\pi}{4s} \log \frac{\chi_+(s, s_1, s_2, m^2)}{\chi_-(s)} \theta(q^2 - 4m^2), \\ \chi_\pm(s, s_1, s_2, m^2) &= s_1 + s_2 - q^2 \pm 2\lambda^{1/2}(q^2, s_1, s_2) \sqrt{q^2 - 4m^2}. \end{aligned} \quad (\text{A12})$$

It is also convenient to introduce the variables

$$z = \frac{1}{2s}(\sqrt{\lambda} + s_1 - s_2 + s), \quad \bar{z} = \frac{1}{2s}(-\sqrt{\lambda} + s_1 - s_2 + s), \quad (\text{A13})$$

with

$$w = \frac{1}{2} \left(1 + \sqrt{1 - \frac{4m^2}{s}} \right), \quad \bar{w} = \frac{1}{2} \left(1 - \sqrt{1 - \frac{4m^2}{s}} \right) \quad (\text{A14})$$

in order to reexpress this density in the form [recall that $\Delta C_0 = \text{disc}C_0/(2i)$]

$$\Delta C_0(s, s_1, s_2, m^2) = -\frac{\pi \log \left(\frac{m^2 - s(w - \bar{z})(\bar{w} - z)}{m^2 - s(w - z)(\bar{w} - \bar{z})} \right)}{s(z - \bar{z})} \theta(s - 4m^2). \quad (\text{A15})$$

For the derivation of the explicit form of the spectral density we also need C_0 at $s = 0$,

$$\begin{aligned} C_0(0, p_1^2, p_2^2, m^2) &= \lim_{q^2 \rightarrow 0} C_0(q^2, p_1^2, p_2^2, m^2) \\ &= \frac{\log^2 \left(\frac{-p_1^2 + \sqrt{p_1^2(p_1^2 - 4m^2) + 2m^2}}{2m^2} \right) - \log^2 \left(\frac{-p_2^2 + \sqrt{p_2^2(p_2^2 - 4m^2) + 2m^2}}{2m^2} \right)}{2(p_1^2 - p_2^2)} \end{aligned} \quad (\text{A16})$$

that can be inserted in the expression of the spectral density

$$\text{disc}\Phi_0(q^2, p_1^2, p_2^2, m^2) = -\frac{1}{2\pi^2} \left(4i\pi m^2 C_0(0, p_1^2, p_2^2, m^2) \delta(q^2) + 2\pi i \delta(q^2) - \frac{2m^2}{q^2} \text{disc}C_0(q^2, p_1^2, p_2^2, m^2) \right), \quad (\text{A17})$$

with the first and third contributions are given respectively by Eqs. (A16) and (A15) (where $\text{disc}C_0 = 2i\Delta C_0$). A direct evaluation of its integral in the $q^2 > 0$ region shows the presence of an interesting pattern of cancellations. An explicit computation, using the inverse Laplace transform, shows that the following identity holds:

$$\frac{2im^2}{\pi} C_0(0, p_1^2, p_2^2, m^2) = \int_{4m^2}^{\infty} ds \rho(s, s_1, s_2, m^2) \quad (\text{A18})$$

where

$$\begin{aligned} \rho(s, s_1, s_2, m^2) &\equiv \frac{m^2}{\pi^2 q^2} \text{disc}C_0(q^2, p_1^2, p_2^2, m^2) \\ &= -\frac{2im^2 \log\left(\frac{m^2-s(w-\bar{z})(\bar{w}-z)}{m^2-s(w-z)(\bar{w}-\bar{z})}\right)}{\pi s^2(z-\bar{z})} \theta(q^2 - 4m^2), \end{aligned} \quad (\text{A19})$$

indicating that the sum rule can be thought of as being saturated by the second constant term on the right-hand side of (A17). Therefore, if we integrate (A17) to compute the sum rule and use (A19), the first and the third term combined contribute zero to the integral, leaving only the constant pole to saturate the sum rule,

$$\int_0^{\infty} ds \text{disc}\Phi_0(q^2, p_1^2, p_2^2, m^2) = \frac{i}{\pi} \int_0^{\infty} ds \delta(s) = \frac{i}{\pi}. \quad (\text{A20})$$

Further insight into this behavior is obtained by analyzing (A18) under a global rescaling of all the external invariants and the mass parameter m^2 . Notice that also in the off-shell correlator the spectral density is self-similar,

$$\rho(\lambda s, \lambda s_1, \lambda s_2, \lambda m^2) = \frac{1}{\lambda} \rho(s, s_1, s_2, m^2), \quad (\text{A21})$$

and that (A18) is scale invariant if we perform a common rescaling of all the invariants and the mass on both of its sides. Notice that in the on-shell limit $(s_1, s_2) \rightarrow (0, 0)$ of (3.9); using in (A16)

$$\lim_{(s_1, s_2) \rightarrow (0, 0)} C_0(0, s_1, s_2, m^2) = -\frac{1}{2m^2} \quad (\text{A22})$$

we obtain

$$\lim_{(s_1, s_2) \rightarrow (0, 0)} \frac{2im^2}{\pi} C_0(0, p_1^2, p_2^2, m^2) = -\frac{i}{\pi} = \int_{4m^2}^{\infty} ds \rho(s, m^2) \quad (\text{A23})$$

and we recover the sum rule that we have discussed above in the on-shell case. From this relation, if we perform the $m^2 \rightarrow 0$ limit, we derive the spectral flow shown in Fig. 1, with $\rho(s, m^2)$ converging to a delta function at $s = 0$,

$$\lim_{m \rightarrow 0} \rho(s, m^2) = -\frac{i}{\pi} \delta(s). \quad (\text{A24})$$

We have seen that in the on-shell case the cancellation between the contributions of the two spikes at $s = 0$ leaves the sum rule to be saturated by the cut, controlled by the mass parameter m . We mention again that the same result, however, can be interpreted in a different way if we perform an integration of both sides of (A17). After integration, the first and the third contributions identically cancel, as shown by (A18), and the entire sum rule appears to be saturated just by the $1/q^2$ isolated pole present in (4.7). In the off-shell case, a similar cancellation takes place after integration, and again, the sum rule appears to be saturated just by the pole, with a constant residue fixed only by the anomaly coefficient.

It is clear that we are confronted, for this interaction, with a rather special behavior, which can be attributed to the topological properties of the vertex, not shared by other interactions.

APPENDIX B: EVALUATION AT THE ANOMALOUS THRESHOLDS

Here we collect the explicit expressions used in the cancellation analysis at the anomalous thresholds $s = s_{\pm}$. These involve the evaluation of the scalar triangle integral C_0 at those kinematic points.

$$\begin{aligned}
C_0(s_-, s_1, s_2, m^2) = & \frac{1}{s_1 s_2 s_-^2} \left(\sqrt{s_1} \left(\sqrt{s_2(s_2 - 4m^2)} (\sqrt{s_1} - \sqrt{s_2}) \log \left(\frac{\sqrt{s_2(s_2 - 4m^2)} + 2m^2 - s_2}{2m^2} \right) \right. \right. \\
& + \sqrt{s_2} \sqrt{(-2\sqrt{s_1}\sqrt{s_2} + s_1 + s_2)(-4m^2 - 2\sqrt{s_1}\sqrt{s_2} + s_1 + s_2)} \log(S_1) \\
& \left. \left. + \sqrt{s_2} \sqrt{s_1(s_1 - 4m^2)} (\sqrt{s_2} - \sqrt{s_1}) \log \left(\frac{\sqrt{s_1(s_1 - 4m^2)} + 2m^2 - s_1}{2m^2} \right) \right) \right) \quad (B1)
\end{aligned}$$

$$\begin{aligned}
C_0(s_+, s_1, s_2, m^2) = & \frac{1}{s_1 s_2 s_+^2} \left(\sqrt{s_1} \left(\sqrt{s_2(s_2 - 4m^2)} (\sqrt{s_1} + \sqrt{s_2}) \log \left(\frac{\sqrt{s_2(s_2 - 4m^2)} + 2m^2 - s_2}{2m^2} \right) \right. \right. \\
& - \sqrt{s_2} \sqrt{(\sqrt{s_1} + \sqrt{s_2})^4 - 4m^2(\sqrt{s_1} + \sqrt{s_2})^2} \log(S_2) \\
& \left. \left. + \sqrt{s_2} \sqrt{s_1(s_1 - 4m^2)} (\sqrt{s_1} + \sqrt{s_2}) \log \left(\frac{\sqrt{s_1(s_1 - 4m^2)} + 2m^2 - s_1}{2m^2} \right) \right) \right), \quad (B2)
\end{aligned}$$

where

$$S_1 = \frac{\sqrt{(-2\sqrt{s_1}\sqrt{s_2} + s_1 + s_2)(-4m^2 - 2\sqrt{s_1}\sqrt{s_2} + s_1 + s_2) + 2m^2 - (\sqrt{s_1} - \sqrt{s_2})^2}}{2m^2}, \quad (B3)$$

$$S_2 = \frac{\sqrt{(\sqrt{s_1} + \sqrt{s_2})^4 - 4m^2(\sqrt{s_1} + \sqrt{s_2})^2 + 2m^2 - (\sqrt{s_1} + \sqrt{s_2})^2}}{2m^2}. \quad (B4)$$

Using these expressions, one can show

$$\Phi_{\text{ATT}}^{(\lambda)}(q^2 = s_{\pm}) = 0.$$

1. The massless and the on-shell limits do not commute

An interesting limit of the amplitude arises when either the fermion mass or the external invariants s_1 and s_2 are sent to zero. As evident from our previous discussions, these two limits do not commute for an obvious reason. If the on-shell limit on s_1 and s_2 is taken first, the interaction is reduced to its longitudinal sector. As shown above, the sum rule then causes the branch cut to collapse into a pole when approaching the conformal limit by setting $m \rightarrow 0$. In this case, the anomaly form factor becomes the only

surviving form factor in the vertex, with a nonzero residue as $q^2 \rightarrow 0$, equaling the numerical coefficient of the anomaly. Conversely, if we first take the $m \rightarrow 0$ limit while keeping the vector lines off-shell, and only afterward send $q^2 \rightarrow 0$, the residue vanishes. This happens because, in this order of limits, both the longitudinal and transverse structures contribute. In the following, we will illustrate this point in detail.

We start from the $m = 0$ case. To show the patterns of cancellations we assume that the two external vector lines are off-shell.

a. $m = 0$

In the $m = 0$ case, a direct computation of the residue, combining all the terms in (A2) gives

$$q^2 \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(q) \rangle = - \frac{(B_0^R(M^2, 0) + M^2 C_0(0, M^2, M^2, 0) - 2) q^{\mu_3} \epsilon^{\mu_1 \mu_2 p_1 p_2}}{8\pi^2} \quad (B5)$$

where we compute the scalar three-point function at $q^2 = 0$ (with $s_1 = s_2 = M^2 < 0$)

$$C_0(0, M^2, M^2, 0) = \lim_{q^2 \rightarrow 0} C_0(q^2, M^2, M^2, 0) = -\frac{1}{\varepsilon_{\text{IR}} M^2} - \frac{\log\left(-\frac{\pi\mu^2}{M^2}\right)}{M^2} + \frac{\gamma}{M^2} + \frac{2\log(\pi)}{M^2} \quad (\text{B6})$$

as well as the scalar two-point function B_0^R

$$B_0^R(0, M^2) \equiv \lim_{q^2 \rightarrow 0} B_0^R(q^2, M^2) = \log\left(-\frac{\mu^2}{\pi M^2}\right) - \gamma + 2, \quad (\text{B7})$$

with $\varepsilon = d - 4$ and γ being the Euler-Mascheroni constant and μ an infrared renormalization scale. Equation (B5) contains a light-cone component $q^{\mu_3} \sim q^+ n^{+\mu_3}$, that remains large but fixed in the light-cone limit that we are considering. At this stage, the vanishing of the residue

$$\lim_{q^2 \rightarrow 0} (B_0^R(q^2, M^2) + M^2 C_0(q^2, M^2, M^2) - 2) = 0 \quad (\text{B8})$$

eliminates such component. Notice that there are no infrared divergences in $1/\varepsilon_{\text{IR}}$ in this limit. Therefore we derive the relation

$$\lim_{q^2 \rightarrow 0} q^2 \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(q) \rangle = 0, \quad (\text{B9})$$

proving the absence of any particle pole in the vertex under these external kinematical constraints.

b. $m \neq 0$

A similar analysis can be performed for a nonzero mass. The result is finite in the $q^2 \rightarrow 0$ limit. We obtain

$$q^2 \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(q) \rangle = \frac{1}{8\pi^2} q^{\mu_3} \varepsilon^{\mu_1 \mu_2 p_1 p_2} (-B_0^R(M^2, m^2) + B_0^R(0, m^2) + (4m^2 - M^2) C_0(0, M^2, M^2, m^2) + 2) \quad (\text{B10})$$

which is purely longitudinal due to the presence of the $q^{\mu_3} \varepsilon^{\mu_1 \mu_2 p_1 p_2}$ tensor, where

$$B_0^R(M^2, m^2) = \log\left(\frac{\mu^2}{\pi m^2}\right) + \frac{\sqrt{M^2(M^2 - 4m^2)} \log\left(\frac{\sqrt{M^2(M^2 - 4m^2)} + 2m^2 - M^2}{2m^2}\right)}{M^2} - \gamma + 2, \quad (\text{B11})$$

$$B_0^R(0, m^2) = \log\left(\frac{\mu^2}{\pi m^2}\right) - \gamma, \quad (\text{B12})$$

and

$$C_0(0, M^2, M^2, m^2) = \frac{\sqrt{-(M^2(4m^2 - M^2))} \log\left(\frac{\sqrt{M^2(M^2 - 4m^2)} + 2m^2 - M^2}{2m^2}\right)}{M^2(4m^2 - M^2)}. \quad (\text{B13})$$

Also in this case there is no particle pole since

$$\lim_{q^2 \rightarrow 0} (-B_0^R(M^2, m^2) + B_0^R(q^2, m^2) + (4m^2 - M^2) C_0(q^2, M^2, M^2, m^2) + 2) = 0 \quad (\text{B14})$$

giving

$$\lim_{q^2 \rightarrow 0} q^2 \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(q) \rangle = 0. \quad (\text{B15})$$

The analysis shows that the transverse sector turns into a longitudinal one when we take the light-cone limit, a fact previously noted in [21].

Starting from the general expression (B5), setting the vector lines on-shell and keeping the mass parameter fixed, results in the absence of a particle pole in the vertex. This is consistent with the observation that, in this case, there is no pole but rather a cut in the longitudinal form factor. Therefore, even in the on-shell case ($s_1 = s_2 = 0$), where the vertex is purely longitudinal, for $m \neq 0$, we obtain

$$q^2 \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(q) \rangle = \frac{(2m^2 C_0(0, 0, 0, m^2) + 1) q^{\mu_3} \epsilon^{\mu_1 \mu_2 p_1 p_2}}{2\pi^2} \quad (\text{B16})$$

which has no pole since, according to (3.35)

$$C_0(0, 0, 0, m^2) = -\frac{1}{2m^2} \quad (\text{B17})$$

and (B16) vanishes. The sum rule, however, indicates that as $m \rightarrow 0$, the spectral density and the area law of the sum rule generate a pole, as in the conformal analysis. At the conformal point the form factor with its longitudinal tensor structure equals the entire vertex

$$\lim_{q^2 \rightarrow 0} q^2 \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(q) \rangle = \frac{1}{2\pi^2} q^{\mu_3} \epsilon^{\mu_1 \mu_2 p_1 p_2} \quad (\text{B18})$$

showing a nonzero residue. This demonstrates that the vertex possesses a particle pole with a nonvanishing

coupling g_{pp} , as defined in (3.29) for the general case, at $s = 0$. This occurs only for on-shell vector lines and a massless fermion. Notice that the limit we are considering is not a soft limit, and clearly characterizes a process that is entirely supported on the light cone.

On the other end, in the infrared case, as $q^\lambda \rightarrow 0$, multiplication by the invariant q^2 leads to a contribution of the form $\epsilon^{\mu\nu p_1 p_2}$. These contributions vanish due to symmetry, given that $p_1 + p_2 \sim 0$. Consequently, one deduces that no particle pole exists in a general sense, except under specific kinematical conditions. While the anomaly form factor is characterized by an anomaly pole, as shown through the analysis of the conformal Ward identities, this does not translate into a general particle pole.

This indicates that topological vertices cannot be directly assimilated into ordinary, nontopological ones. The observed cancellations between the longitudinal and transverse structures of the interaction lead to this conclusion. Moreover, the noncommutativity between the massless and on-shell limits is a property of the entire vertex, stemming from the interrelation between these two sectors. These analyses show that the anomaly pole behaves as a particle pole only in the on-shell and massless case where

$$\langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(q) \rangle = \frac{1}{2\pi^2} \frac{q^{\mu_3}}{q^2} \epsilon^{\mu_1 \mu_2 p_1 p_2} \quad (\text{B19})$$

is present over the entire light cone, getting arbitrarily soft around the apex of the cone where $q^\lambda \rightarrow 0$.

APPENDIX C: THE TRANSVERSE FORM FACTORS OF THE ATT

$$\begin{aligned} \bar{A}_{11} &= -2s_1 \{ 2\lambda m^2 [21s^3 - 3s^2(8s_1 + 7s_2) + s(-15s_1^2 + 100s_1s_2 - 21s_2^2) + 3(s_1 - s_2)^2(6s_1 + 7s_2)] \\ &\quad + s_1[3s^5 + s^4(81s_2 - 12s_1) + 2s^3(9s_1^2 - 40s_1s_2 - 42s_2^2) - 4s^2(3s_1^3 + 20s_1^2s_2 - 76s_1s_2^2 + 21s_2^3) + 3s_2(s_1 - s_2)^4 \\ &\quad + s(s_1 - s_2)^2(3s_1^2 + 82s_1s_2 + 81s_2^2)] \}, \\ \bar{A}_{12} &= -4\lambda m^2 [-3s_2^2(s^2 + 14ss_1 - 17s_1^2) - 3s_2(s - s_1)(s^2 - 21ss_1 - 6s_1^2) + s_2^3(7s - 24s_1) + 2(s - s_1)^3(s + 3s_1) - 3s_2^4] \\ &\quad + 2s_2[s^6 - s^5(14s_1 + 5s_2) + s^4(-11s_1^2 + 28s_1s_2 + 10s_2^2) + 2s^3(52s_1^3 - 102s_1^2s_2 - 5s_2^3) + s^2(-121s_1^4 + 152s_1^3s_2 \\ &\quad + 120s_1^2s_2^2 - 28s_1s_2^3 + 5s_2^4) + s(s_1 - s_2)^2(38s_1^3 + 117s_1^2s_2 + 12s_1s_2^2 - s_2^3) + 3s_1^2(s_1 - s_2)^4], \\ \bar{A}_{13} &= 4\lambda m^2 [3s^4 + s^3(24s_1 - 7s_2) + 3s^2(-17s_1^2 + 14s_1s_2 + s_2^2) + 3s(s_1 - s_2)(6s_1^2 + 21s_1s_2 - s_2^2) + 2(s_1 - s_2)^3(3s_1 + s_2)] \\ &\quad + 2s[-2s_2^3(5s^3 + 102ss_1^2 - 52s_1^3) + s_2^4(10s^2 + 28ss_1 - 11s_1^2) - s_2(s - s_1)^2(s^3 - 12s^2s_1 - 117ss_1^2 - 38s_1^3) \\ &\quad + s_2^2(5s^4 - 28s^3s_1 + 120s^2s_1^2 + 152ss_1^3 - 121s_1^4) + 3s_1^2(s - s_1)^4 - s_2^5(5s + 14s_1) + s_2^6], \\ \bar{A}_{14} &= 24\{\lambda^2 m^4 [s^2 + s(s_1 - 2s_2) - 2s_1^2 + s_1s_2 + s_2^2] + m^2 s_1 [2s_2^3(16s^3 - 15ss_1^2 - 5s_1^3) - 18ss_2^2(s^3 + ss_1^2 - 2s_1^3) - 9s_1s_2^5 \\ &\quad + 2s_2^6 + 3s_2^4(-6s^2 + 3ss_1 + 5s_1^2) + 3s_1s_2(s_1 - 3s)(s_1 - s)^3 + (s - s_1)^5(2s + s_1)] + ss_1^2s_2[3s_2^2(-4s^2 + 4ss_1 + s_1^2) \\ &\quad + 3s_2(s^3 + 4s^2s_1 - 6ss_1^2 + s_1^3) + s_2^3(3s - 7s_1) + (s - s_1)^3(3s + 2s_1) + 3s_2^4]\}, \\ \bar{A}_{15} &= \lambda[4\lambda m^2(5s^2 + 13ss_1 - 10ss_2 - 18s_1^2 + 13s_1s_2 + 5s_2^2) - 2s_2^3(s^2 - 8ss_1 + 8s_1^2) - (s - s_1)^3(s^2 - 5ss_1 - 2s_1^2) \\ &\quad - 2s_2^2(s^3 + 24s^2s_1 - 38ss_1^2 - 5s_1^3) + s_2(s - s_1)(3s^3 + 19s^2s_1 + 95ss_1^2 - s_1^3) + s_2^4(3s + 8s_1) - s_2^5], \end{aligned} \quad (\text{C1})$$

and

$$\begin{aligned}
\bar{A}_{21} &= 4s_1[s_1s_2(s_2^2(9s - 19s_1) - 2s_2(9s + 4s_1)(s - s_1) - (s - s_1)^3 + 10s_2^3) - \lambda m^2(17s^2 - 8s(s_1 + s_2) - 9(s_1 - s_2)^2)], \\
\bar{A}_{22} &= 2ss_2[s^4 - 2s^3(5s_1 + 2s_2) + 6s^2s_2(2s_1 + s_2) + s(26s_1^3 - 60s_1^2s_2 + 6s_1s_2^2 - 4s_2^3) - (s_1 - s_2)^2(17s_1^2 + 6s_1s_2 - s_2^2)] \\
&\quad - 4\lambda m^2[-s_2(s^2 - 30ss_1 + 9s_1^2) + 2(s - s_1)^2(s + 3s_1) - 4ss_2^2 + 3s_2^3], \\
\bar{A}_{23} &= 2s[2s^3(s_1^2 + 5s_1s_2 + 2s_2^2) - 6s^2(s_1^3 - 6s_1^2s_2 + 2s_1s_2^2 + s_2^3) + 2s(s_1 - s_2)(3s_1^3 - 20s_1^2s_2 + s_1s_2^2 - 2s_2^3) \\
&\quad - (s_1 - s_2)^3(2s_1^2 + 5s_1s_2 - s_2^2) - s_2s^4] + 4\lambda m^2[3s^3 + 2s^2(9s_1 - 2s_2) - 2(s_1 - s_2)^3 - s(s_1 - s_2)(19s_1 - s_2)], \\
\bar{A}_{24} &= 12(\lambda m^2 + ss_1s_2)[2\lambda m^2(s + s_1 - s_2) + s_1(3s^3 + s^2(3s_2 - 5s_1) + s(s_1 - s_2)(s_1 + 5s_2) + (s_1 - s_2)^3)], \\
\bar{A}_{25} &= \lambda[20\lambda m^2(s + s_1 - s_2) - s^4 + 2s^3(3s_1 + s_2) + 4s^2s_1(3s_2 - 2s_1) + 2s(s_1 - s_2)(s_1^2 + 8s_1s_2 + s_2^2) + (s_1 - s_2)^4]. \quad (C2)
\end{aligned}$$

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