

Numerical extraction of crosscap coefficients in microscopic models for $(2+1)$ D conformal field theory

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Conformal field theory (CFT) can be placed on disparate space-time manifolds to facilitate investigations of their properties. For $(2+1)$ -dimensional $[(2+1)\text{D}]$ theories, one useful choice is the real projective space $\mathbb{RP}^3$ obtained by identifying antipodal points on the boundary sphere of a three-dimensional ball. One-point functions of scalar primary fields on this manifold generally do not vanish and encode the so-called crosscap coefficients. These coefficients also manifest on the sphere as the overlaps between certain crosscap states and CFT primary states. Taking the $(2+1)\text{D}$ Ising CFT as a concrete example, we demonstrate that crosscap coefficients can be extracted from microscopic models. We construct crosscap states in both lattice models defined on polyhedrons and continuum models in Landau levels, where the degrees of freedom at antipodal points are entangled in Bell-type states. By computing their overlaps with the eigenstates of many-body Hamiltonians, we obtain results consistent with those from conformal bootstrap. Importantly, our approach directly reveals the absolute values of crosscap overlaps, whereas bootstrap calculations typically yield only their ratios. Furthermore, we investigate the finite-size scaling of these overlaps and their evolution under perturbations away from criticality.

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Introduction. Phase transitions are intriguing phenomena that arise from the interplay of a large number of constituents in many-body systems. At the critical points in many continuous transitions, conformal symmetry may emerge even if it does not exist at the microscopic level [1]. Building upon this feature, the powerful framework of conformal field theory (CFT) was established [2]. Besides critical phenomena, CFTs have also been applied to study particle physics and quantum gravity. From a mathematical perspective, CFTs can be defined rigorously unlike general quantum field theory. In two-dimensional classical systems or $(1+1)$ -dimensional $[(1+1)\text{D}]$ quantum systems, local conformal transformations are described by an infinite-dimensional algebra, so they place tight constraints on physical properties. This allows for nonperturbative analytical calculations of many interesting quantities.

As we turn to higher spatial dimensions, CFTs become much more difficult because the associated algebra is finite dimensional [3]. One successful approach is conformal bootstrap in which correlation functions are analyzed by imposing suitable constraints derived from conformal symmetry [4,5]. Another approach is the direct numerical simulations of microscopic models, which have also played a crucial role in studying CFTs. This method faces challenges about how to design suitable models, how to determine if there are continuous transitions, and how to extract the CFT data at critical points [6,7]. In the past three years, an important progress has been made in quantum critical model design and extraction of CFT data [8]. Its essence is to put electrons on a sphere with a magnetic monopole at its center [9–11]. The single-particle eigenstates on the sphere are organized into Landau levels (LLs) each having a finite degeneracy. If the cyclotron gap is sufficiently large, the electrons can be restricted to the lowest LL, which makes numerical calculations feasible. More importantly, spherical symmetry is preserved exactly so energy spectra can be analyzed using the state-operator correspondence [12,13]. This idea has been employed and further developed in subsequent works [14–35]. For the three-dimensional (3D) Ising CFT, numerical results obtained by this approach, and conformal bootstrap are remarkably close to each other (when both are available).

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Another related progress is the identification of the Gross-Neveu-Yukawa criticality in interacting Dirac fermions on the sphere without magnetic monopole [36].

Shortly after the inception of CFT, it was recognized that global topological features, such as boundary conditions or nontrivial manifolds, can lead to rich and universal phenomena in finite-size systems [37–41]. One way to explore their effects is through the study of conformal boundary states. In this work, we focus on one special class of conformal boundary states, known as crosscap states [42–61]. The name stems from their analogy with the geometric object crosscap and is most easily visualized in one spatial dimension. A crosscap is obtained from a circle by gluing together both ends of each diameter. For one-dimensional (1D) lattice or continuum models, crosscap states are constructed by putting each pair of antipodal points into entangled states [see Fig. 1(a)]. This definition extends to integrable models that are not described by CFTs. In some cases, they are called entangled antipodal states [53,54]. An intriguing feature of the crosscap states is that their overlaps with some CFT eigenstates are universal. In this paper, we construct crosscap states for $(2+1)$ -dimensional $[(2+1)\text{D}]$ CFT and perform numerical calculations to obtain their overlaps in finite-size systems. We mainly focus on the spherical geometry but shall also consider polyhedrons [62].

Crosscap overlap in $(2+1)$ D CFT. We consider a $(2+1)$ D CFT with the quantum Hamiltonian defined on the surface of a sphere with radius R . The space-time manifold is $\mathbb{R} \times S^2$, with coordinate $(v\tau, \vec{x})$, where v is the velocity and $\vec{x}$ denotes the spatial coordinate constrained to the spherical surface $|\vec{x}| = R$. The spherical surface can be parametrized as $\vec{x} = R\vec{\omega}$, where $\vec{\omega} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is a unit vector specified by the angular coordinates θ and ϕ . The radial quantization scheme is realized via the conformal transformation $\vec{r} = e^{v\tau/R}\vec{\omega}$ [13], which maps the space-time manifold to the flat Euclidean space $\mathbb{R}^3$. Under this transformation, the spatial manifold at the $\tau = 0$ time slice becomes a unit sphere. In this scheme, there is a state-operator correspondence: each CFT primary state $|\mathcal{O}_i\rangle$ is associated with a scalar primary field $\mathcal{O}_i(\vec{r})$ with $\vec{r}$ being the space-time coordinate [12]. We aim to construct crosscap states $|\mathcal{C}\rangle$ on the sphere and compute the crosscap overlaps between them and the CFT eigenstates. The physical picture for 1D crosscap states shall provide valuable guidance for this quest. For the primary states, state-operator correspondence tells us that

$$\langle \mathcal{C} | \mathcal{O}_i \rangle = \lim_{|\vec{r}| \rightarrow 0} \langle \mathcal{C} | \mathcal{O}_i(\vec{r}) | 0 \rangle, \quad (1)$$

where the CFT vacuum is denoted as $|0\rangle$. In the path integral formalism, the crosscap state is associated with a boundary condition that identifies antipodal points on the surface of a unit ball. This identification converts the

space-time manifold to the 3D real projective space $\mathbb{RP}^3$. Many properties about CFTs on $\mathbb{RP}^3$ have been studied before [63–65]. The crosscap overlap in Eq. (1) can be reinterpreted as the one-point function

$$\langle \mathcal{C} | \mathcal{O}_i \rangle = \langle \mathcal{O}_i(\vec{r} = \vec{0}) \rangle_{\mathbb{RP}^3} \quad (2)$$

on the $\mathbb{RP}^3$ manifold, with primary field $\mathcal{O}_i(\vec{r})$ inserted at the origin.

Since translation symmetry is explicitly broken, the one-point functions of primary fields generally do not vanish on $\mathbb{RP}^3$. In fact, their forms are fully constrained by conformal transformations generated by $\vec{P} + \vec{K}$ as they preserve the geometry of $\mathbb{RP}^3$. Here, $\vec{P}$ and $\vec{K}$, respectively, denote the generators of translations and special conformal transformations, whose explicit expressions are given by

$$\begin{aligned} \vec{P} &= -i \frac{\partial}{\partial \vec{r}}, \\ \vec{K} &= -i \left[2\vec{r} \left(\vec{r} \cdot \frac{\partial}{\partial \vec{r}} \right) - |\vec{r}|^2 \frac{\partial}{\partial \vec{r}} \right]. \end{aligned} \quad (3)$$

It is helpful to study the following finite conformal transformation:

$$\begin{aligned} \vec{f}(\vec{r}) &= e^{i\vec{a} \cdot (\vec{P} + \vec{K})}(\vec{r}) \\ &= \frac{1 + (\tan |\vec{a}|)^2}{1 + (|\vec{r}| \tan |\vec{a}|)^2 - 2 \tan |\vec{a}| \frac{(\vec{r} \cdot \vec{a})}{|\vec{a}|}} \vec{r} \\ &\quad + \frac{(1 - |\vec{r}|^2) \tan |\vec{a}| - 2 \tan |\vec{a}| \frac{(\vec{r} \cdot \vec{a})}{|\vec{a}|}}{1 + (|\vec{r}| \tan |\vec{a}|)^2 - 2 \tan |\vec{a}| \frac{(\vec{r} \cdot \vec{a})}{|\vec{a}|}} \vec{a}. \end{aligned} \quad (4)$$

Under this conformal transformation, the origin is mapped to $\vec{r} \equiv \vec{f}(\vec{0}) = \tan |\vec{a}| \frac{\vec{a}}{|\vec{a}|}$, and a scalar primary field transforms as $\mathcal{O}_i(\vec{r}) = |\vec{f}'(\vec{0})|^{-\Delta_i/3} \mathcal{O}_i(\vec{0})$, where Δ_i is the scaling dimension of this field and $|\vec{f}'(\vec{0})| = (1 + \tan^2 |\vec{a}|)^3$ is the Jacobian determinant of the conformal transformation. One thus concludes that one-point functions on $\mathbb{RP}^3$ must take the form

$$\langle \mathcal{O}_i(\vec{r}) \rangle_{\mathbb{RP}^3} = \frac{\mathcal{A}_i}{(1 + |\vec{r}|^2)^{\Delta_i}} \quad (5)$$

with $\mathcal{A}_i$ being a universal coefficient that is identified with the crosscap overlap

$$\mathcal{A}_i = \langle \mathcal{C} | \mathcal{O}_i \rangle. \quad (6)$$

Here, $\mathcal{A}_i$ is the so-called crosscap coefficient.

For $(1+1)$ D rational CFTs, the crosscap coefficients encode conformal data on flat space and can be computed analytically [43,45,66]. Their values have also been

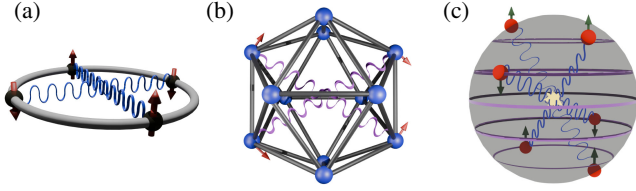


FIG. 1. Schematics of crosscap states in (a) one-dimensional systems, (b) the icosahedron, and (c) LLs on the sphere.

extracted via crosscap overlaps in a variety of microscopic models [55,56,67–76]. In contrast, analytical results about crosscap coefficients in $(2+1)$ D CFTs are generally unavailable. Nevertheless, the *ratios* between crosscap coefficients of some primary fields and that of the identity primary field have been obtained by numerical calculations based on the conformal bootstrap approach [63]. Instead, absolute values of the crosscap coefficients—not just their ratios—can be extracted directly from the crosscap overlaps in Eq. (6). The feasibility of this approach is demonstrated below using concrete microscopic models that realize the $(2+1)$ D Ising CFT. For quantum many-body Hamiltonians that are defined on the sphere or its discrete counterpart, low-lying eigenstates can be obtained by exact diagonalization. It is not *a priori* clear how to construct crosscap states in these models. According to the intuitive picture mentioned above, the degrees of freedom on the two ends of each diameter of the sphere should form a maximally entangled state.

Lattice models on icosahedron. We begin with a two-dimensional quantum Ising model defined on discrete lattice sites. The intuition accumulated during the past about 1D crosscap states can be extended easily to these cases. However, the usual planar lattice models with or without periodic boundary conditions are not suitable for studying crosscap states. Instead, highly symmetric polyhedrons may be considered [62]. As shown in Fig. 1(b), the transverse-field Ising (TFI) model

$$H_{\text{TFI}} = -\sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x - h \sum_i \sigma_i^z \quad (7)$$

is defined on the 12 vertices of icosahedron. An inspection of the excited-state energy levels suggests that $h = 4.375$ is the critical point and several primary states can be pinpointed [62]. We are interested in crosscap overlaps of the ground state, the ϵ primary state, and the ϵ' primary state, which are denoted as $\mathcal{A}_{\mathbb{I}}$, $\mathcal{A}_\epsilon$, and $\mathcal{A}_{\epsilon'}$, respectively. It is possible to define a point as the center of icosahedron that has the same distance to all vertices. This allows one to construct a fictitious sphere such that the vertices can be divided into six groups, each having two vertices on a diameter of the fictitious sphere. For each group with two vertices denoted as i and $\bar{i}$, the Bell state $|\uparrow\rangle_i |\uparrow\rangle_{\bar{i}} + |\downarrow\rangle_i |\downarrow\rangle_{\bar{i}}$

TABLE I. Comparison between the crosscap coefficient results for the 3D Ising CFT in conformal bootstrap calculations and microscopic lattice and continuum models. NA means not available.

Method	$\mathcal{A}_{\mathbb{I}}$	$\mathcal{A}_\epsilon/\mathcal{A}_{\mathbb{I}}$	$\mathcal{A}_{\epsilon'}/\mathcal{A}_{\mathbb{I}}$
Bootstrap	NA	0.667	0.896
Icosahedron	1.133	0.690	1.141
Sphere (extrapolated)	1.129	0.674	0.898

is imposed. It is natural to accept that

$$|C_{\mathbb{I}}\rangle = \prod_{i \in \text{half}} (|\uparrow\rangle_i |\uparrow\rangle_{\bar{i}} + |\downarrow\rangle_i |\downarrow\rangle_{\bar{i}}) \quad (8)$$

is a crosscap state for the whole system. This expression gives rise to $\mathcal{A}_{\mathbb{I}} = 1.133$, $\mathcal{A}_\epsilon = 0.782$, and $\mathcal{A}_{\epsilon'} = 1.293$ at the critical point. Using the bootstrap method, Ref. [63] found that $\mathcal{A}_\epsilon/\mathcal{A}_{\mathbb{I}} = 0.667$ and $\mathcal{A}_{\epsilon'}/\mathcal{A}_{\mathbb{I}} = 0.896$. The former value is not far from ours, but the latter value is quite different. We summarize the results in Table I for comparison.

Continuum models in landau levels. The results on icosahedron are quite impressive given the small number of sites. In this section, we turn to the main topic of this paper. We employ the model constructed in Ref. [8] to study Ising phase transitions on the sphere [see Fig. 1(c)]. The single-particle eigenstates are monopole harmonics [9,10]. If the magnetic fluxes through the sphere is $2Q$, explicit expressions for the lowest LL are

$$Y_{Q,m}(\mathbf{\Omega}) = \left[\frac{2Q+1}{4\pi} \binom{2Q}{Q-m} \right]^{\frac{1}{2}} (-1)^{Q-m} \exp(im\phi) \times \cos^{Q+m}(\theta/2) \sin^{Q-m}(\theta/2), \quad (9)$$

where θ is the azimuthal angle, ϕ is the radial angle, $\mathbf{\Omega} = (\theta, \phi)$ is the solid angle, and m is the z component of angular momentum. There are $2Q+1$ Landau orbitals with $m \in [-Q, -Q+1, \dots, Q-1, Q]$. In units of the magnetic length, the radius of the sphere is $R = \sqrt{Q}$. The electrons are endowed with an internal degree of freedom called spin with two possible values $\uparrow$ and $\downarrow$. For the Landau orbital with spin index σ and angular momentum m , the annihilation (creation) operator is denoted as $C_{\sigma,m}$ ($C_{\sigma,m}^\dagger$). In real space, their counterparts are $\Psi_\sigma(\mathbf{\Omega}) = \sum_m Y_{Q,m}(\mathbf{\Omega}) C_{\sigma,m}^\dagger$ and $\Psi_\sigma^\dagger(\mathbf{\Omega}) = \sum_m [Y_{Q,m}(\mathbf{\Omega})]^* C_{\sigma,m}$. The system has $N_e = 2Q+1$ electrons that interact with each other via the two-body potential $V_{\sigma\tau}(\mathbf{\Omega}_1, \mathbf{\Omega}_2)$. It should satisfy the criterion: when this is the only term in the many-body Hamiltonian, the system has two degenerate ground states in which all electrons have the same spin index. This is the ferromagnetic side of the Ising transition. We choose $V_{\uparrow\uparrow} = V_{\downarrow\downarrow} = 0$ and $V_{\uparrow\downarrow}(\mathbf{\Omega}_1, \mathbf{\Omega}_2) = g_0 \delta(\mathbf{\Omega}_1 - \mathbf{\Omega}_2) + g_1 \nabla^2 \delta(\mathbf{\Omega}_1 - \mathbf{\Omega}_2)$ as

in Ref. [8]. The second-quantized form of this potential is [77]

$$H_a = 4 \sum_{\{m_i\}} V_{m_1 m_2 m_3 m_4} C_{\uparrow, m_1}^\dagger C_{\downarrow, m_2}^\dagger C_{\downarrow, m_4} C_{\uparrow, m_3}, \quad (10)$$

where the two-body matrix elements are expressed in terms of the Haldane pseudopotentials V_0 , V_1 [11] and Clebsch-Gordon coefficients

$$\begin{aligned} V_{m_1 m_2 m_3 m_4} &= \int d\Omega_1 \int d\Omega_2 [Y_{Q, m_1}(\Omega_1)]^* [Y_{Q, m_2}(\Omega_2)]^* V_{\uparrow\downarrow}(\Omega_1, \Omega_2) Y_{Q, m_3}(\Omega_2) Y_{Q, m_4}(\Omega_1) \\ &= \delta_{m_1+m_2, m_3+m_4} V_0 \langle Q, m_1; Q, m_2 | 2Q, m_1+m_2 \rangle \langle 2Q, m_3+m_4 | Q, m_3; Q, m_4 \rangle \\ &\quad + \delta_{m_1+m_2, m_3+m_4} V_1 \langle Q, m_1; Q, m_2 | 2Q-1, m_1+m_2 \rangle \langle 2Q-1, m_3+m_4 | Q, m_3; Q, m_4 \rangle. \end{aligned} \quad (11)$$

To induce a phase transition, we introduce a tunneling term,

$$H_b = -h \sum_m (C_{\uparrow, m}^\dagger C_{\downarrow, m} + C_{\downarrow, m}^\dagger C_{\uparrow, m}), \quad (12)$$

between spin-up and spin-down orbitals at the same angular momentum. The full Hamiltonian $H = H_a + H_b$ respects $SO(3)$ rotational symmetry so its eigenstates can be labeled by total angular momentum $L(L+1)$. In the limit of $h \rightarrow \infty$, each electron is driven into an equal superposition of spin-up and spin-down. This is the paramagnetic side of the Ising transition. The critical point with minimal finite-size effects was located at $V_0 = 4.75$, $V_1 = 1$, and $h_c = 3.16$ [8]. Exact diagonalization has been performed in multiple systems with up to $N_e = 18$ electrons. The energy spectrum of the largest system at the critical point is displayed in Fig. 2(a). We have identified three primary states σ , ϵ , and ϵ' following the procedure described in Ref. [8].

It is not immediately clear how to define crosscap states on the sphere at the microscopic level. We conjecture that

$$\begin{aligned} |\mathcal{C}_{LL}\rangle &= \exp \left\{ \int_{\text{upper}} d\Omega [\Psi_{\uparrow}^\dagger(\theta, \phi) \Psi_{\uparrow}^\dagger(\pi - \theta, \pi + \phi) \right. \\ &\quad \left. + \Psi_{\downarrow}^\dagger(\theta, \phi) \Psi_{\downarrow}^\dagger(\pi - \theta, \pi + \phi)] \right\} |\emptyset\rangle \end{aligned} \quad (13)$$

is a crosscap state, where the integral is restricted to the upper half-sphere and $|\emptyset\rangle$ is the vacuum state. While $\phi + \pi$ could be larger than 2π , periodicity of the monopole harmonics ensures that they are still well defined. To mimic a 1D circle that is split as two semicircles, we divide the sphere into the upper and lower halves. It is possible to consider other bipartitions, but the present choice is the most convenient one. For the point with coordinates (θ, ϕ) in the upper half, a diameter of the sphere connects it with another point $(\pi - \theta, \pi + \phi)$ on the lower half. Inside the square bracket, the combination of operators indicates that electrons are created at these two points with their spins forming a Bell state. We cannot

multiply these entangled states together due to the continuous nature of the sphere. Instead, putting these operators in the exponent leads to a well-defined state. For a crosscap state in continuum field theory, a certain part of the conformal symmetry should be preserved. However, the state proposed above does not have this property. This is not surprising because the spherical microscopic model does not have exact conformal symmetry. It only emerges in the thermodynamic limit at the critical point. If a Taylor expansion is performed on Eq. (13), the M th-order term would contain $2M$ electrons that are decomposed to M pairs of maximally entangled states. It could have nonzero overlaps with many-body eigenstates when $2M = N_e$.

When the real-space operators are projected to the lowest Landau level, the proposed state becomes

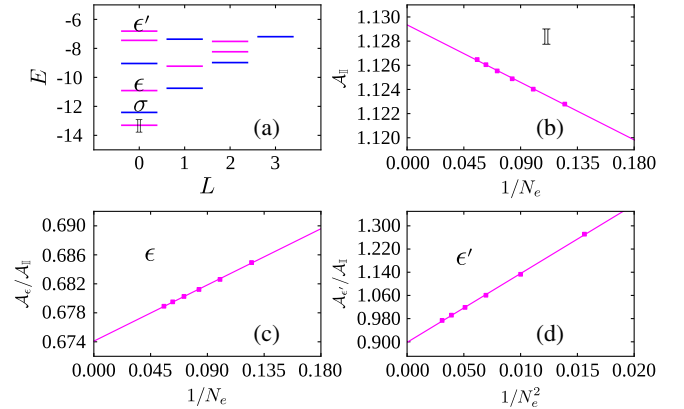


FIG. 2. Numerical results at the critical point. (a) Energy spectrum of the system with $N_e = 18$. The ground state $\mathbb{I}$ and three primary states σ , ϵ , ϵ' are indicated. Eigenstates with even and odd Z_2 parity are represented by magenta and blue colors, respectively. (b) Ground state crosscap overlaps. (c) Ratios between the ϵ state crosscap overlaps and the ground state crosscap overlaps. (d) Ratios between the ϵ' state crosscap overlaps and the ground state crosscap overlaps. The solid lines in panels (b) and (c) are linear fitting versus $1/N_e$, and the one in panel (d) is linear fitting versus $1/N_e^2$.

$$|\mathcal{C}_{LL}\rangle = \exp \left\{ \sum_{\sigma} \sum_{m,n} F_{m,n}^* C_{\sigma,m}^{\dagger} C_{\sigma,n}^{\dagger} \right\} |\emptyset\rangle. \quad (14)$$

The coefficient $F_{m,n}$ can be evaluated as

$$\begin{aligned} & \int_0^{\pi/2} \sin \theta d\theta \int_0^{2\pi} d\phi Y_{Q,m}(\theta, \phi) Y_{Q,n}(\pi - \theta, \pi + \phi) \\ &= \delta_{m,-n} (-1)^{2Q} e^{-im\pi} I_{1/2}(Q - m + 1, Q + m + 1) \end{aligned} \quad (15)$$

with $I_x(a, b)$ being the regularized incomplete beta function. If $2Q + 1$ is even, the crosscap state is

$$|\mathcal{C}_{LL}\rangle = \exp \left[\sum_{\sigma} \sum_{m>0} (-1)^{m+1} C_{\sigma,m}^{\dagger} C_{\sigma,-m}^{\dagger} \right] |\emptyset\rangle. \quad (16)$$

Its fermion parity is even, so there could be nonzero overlaps with the ground state, ϵ state, and ϵ' state. This is indeed the case as confirmed by explicit calculations. We show in Fig. 2(b) the overlaps for the ground states. For the 1D TFI model at its critical point, there is no finite-size correction to the crosscap overlaps [56]. In contrast, the crosscap overlaps on the sphere varies with N_e , but they can be fitted very well using a linear function of $1/N_e$. It yields 1.129 in the thermodynamic limit, which is quite close to the value on icosahedron. For the ϵ and ϵ' states, crosscap overlaps for each N_e divided by the ground-state overlap in the same system are displayed in Figs. 2(c) and 2(d). These numbers can be compared directly with the bootstrap results [63]. Finite-size effects for the ϵ state is quite weak, and the result is already quite close to 0.667 even at the smallest system. When a linear fitting versus $1/N_e$ is performed on the data, we obtain 0.674 in the thermodynamic limit. The ratios for the ϵ' state exhibit strong variations as N_e increases. For the largest system with $N_e = 18$, its value 0.974 is still quite different from the bootstrap result. Linear fitting of finite-size data versus $1/N_e$ does not work well. Instead, the coefficient of determination is 0.9996 for linear fitting versus $1/N_e^2$, and the extrapolated number is 0.898. We stress that linear or quadratic fittings for the crosscap overlaps and ratios are purely empirical without theoretical underpinning at present.

For a $(1+1)$ D CFT perturbed by a relevant operator, crosscap overlap of the ground state was shown to be a universal scaling function of the dimensionless coupling related to this operator [74]. This observation can be extended to $(2+1)$ D systems in a straightforward manner. Let us consider the Hamiltonian

$$H = H_{\text{CFT}} - vg \int_{|\vec{x}|=R} d\vec{x} \varphi(\vec{x}) \quad (17)$$

defined on a sphere with radius R . H_{CFT} is the unperturbed CFT Hamiltonian, φ is an operator with scaling dimension $\Delta_{\varphi} < 3$ for it to be relevant in renormalization group flow, v is the velocity (encoded in H_{CFT}), and g is the coupling

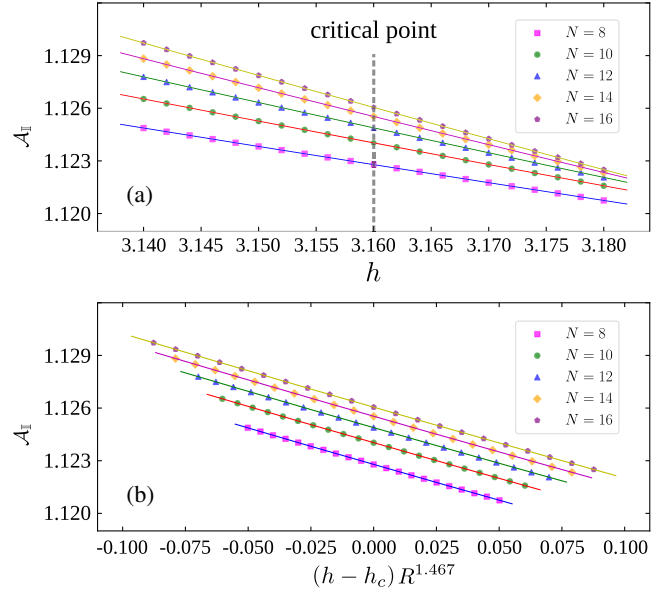


FIG. 3. Numerical results off the critical point. (a) Ground state crosscap overlaps for $h \in [3.14, 3.18]$. (b) The same data in panel (a) plotted versus $(h - h_c) R^{1.467}$. The solid lines in both panels are linear fitting results.

constant. It is apparent that H_{CFT} has dimension $[v/R]$ and φ has dimension $[R]^{-\Delta_{\varphi}}$, so g must have dimension $[R]^{\Delta_{\varphi}-3}$ to match the two terms in Eq. (17). We thus conclude that the whole theory is governed by a single *dimensionless* coupling $s = gR^{3-\Delta_{\varphi}}$ and its ground state can be written as $|G(s)\rangle$. The crosscap overlap $\langle \mathcal{C} | G(s) \rangle$ only depends on the dimensionless coupling and can be captured by a universal scaling function. In the spherical Ising model, the tunneling term becomes a relevant perturbation when h is tuned away from the critical point. For many cases with $h \in [3.14, 3.16]$, the ground-state crosscap overlaps have been evaluated. As one can see from Fig. 3(a), the values in each system are remarkably linear versus $h - h_c$. Based on these results, we conjecture that crosscap coefficients change monotonically when relevant perturbations are turned on to induce a spontaneous symmetry breaking phase transition. For each fitting line in Fig. 3(a), its slope is denoted as $\mu(N_e) \equiv \mu(R)$. These values can be fitted very well using the formula $\mu(R) = 0.041 R^{1.467}$. It is instructive to replot the overlaps versus $(h - h_c) R^{1.467}$ as the fitting lines would become parallel to each other [see Fig. 3(b)]. From the CFT perspective, the system in the infrared limit is described by the Ising CFT perturbed by the ϵ primary field (“thermal perturbation”). According to our general analysis, the overlaps should depend on a dimensionless coupling that is proportional to $(h - h_c) R^{3-\Delta_{\epsilon}}$. This exponent is 1.587 since bootstrap calculations found $\Delta_{\epsilon} \approx 1.413$ [4], which is reasonably close to our numerical result.

Summary and outlook. To sum up, crosscap states have been constructed in microscopic models for the $(2+1)$ D

Using CFT and extracted crosscap coefficients using their overlaps. The results are consistent with the ratios between crosscap coefficients obtained by conformal bootstrap calculations [63]. We hope that the validity of our crosscap state can be further substantiated by additional investigations. Since the numerical values depend on the system size, it would be useful if larger numbers of electrons can be accessed by other methods such as density matrix renormalization group [78]. On the analytical side, direct calculations about the finite-size behaviors of crosscap coefficients are obviously desirable. In a previous work by some of the authors, conformal perturbation theory was developed for $(1+1)$ D CFTs to understand finite-size corrections to crosscap overlaps [74]. It was found that irrelevant and marginal terms generally lead to different types of finite-size corrections. If similar results can be established in $(2+1)$ D CFTs, data fitting would be much more reliable. Besides the crosscap coefficients, one- and two-point correlation functions associated with the proposed crosscap state should also be studied. The results can be compared with previous results obtained by conformal bootstrap [63].

Many other problems about crosscap states and coefficients are largely unclear and deserve further investigations. It would be interesting to explore if bootstrap or other methods can also yield the absolute values of the crosscap overlaps. For several other paradigmatic $(2+1)$ D CFTs, little is known about the crosscap coefficients. In particular, do they encode certain physical quantities? For the $(1+1)$ D compactified boson CFT, which is the low-energy theory of Tomonaga-Luttinger liquids, crosscap coefficients are directly related to the compactification radius and can be used to extract the Luttinger parameter [55].

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Data availability. The data that support the findings of this article are openly available [80].

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- [1] A. M. Polyakov, Conformal symmetry of critical fluctuation, *JEPT Lett.* **12**, 381 (1970), http://www.jetpletters.ru/ps/1737/article_26381.shtml.
 - [2] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nucl. Phys.* **B241**, 333 (1984).
 - [3] S. Rychkov, *EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions*, Lecture Notes in Physics Vol. 959 (Springer International Publishing, New York, 2017).
 - [4] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Solving the 3D Ising model with the conformal bootstrap, *Phys. Rev. D* **86**, 025022 (2012).
 - [5] D. Poland, S. Rychkov, and A. Vichi, The conformal bootstrap: Theory, numerical techniques, and applications, *Rev. Mod. Phys.* **91**, 015002 (2019).
 - [6] Y. Deng and H. W. J. Blöte, Conformal invariance of the Ising model in three dimensions, *Phys. Rev. Lett.* **88**, 190602 (2002).
 - [7] M. Schuler, S. Whitsitt, L.-P. Henry, S. Sachdev, and A. M. Läuchli, Universal signatures of quantum critical points from finite-size torus spectra: A window into the operator content of higher-dimensional conformal field theories, *Phys. Rev. Lett.* **117**, 210401 (2016).
 - [8] W. Zhu, C. Han, E. Huffman, J. S. Hofmann, and Y.-C. He, Uncovering conformal symmetry in the 3D Ising transition: State-operator correspondence from a quantum fuzzy sphere regularization, *Phys. Rev. X* **13**, 021009 (2023).
 - [9] T. T. Wu and C. N. Yang, Dirac monopole without strings: Monopole harmonics, *Nucl. Phys.* **B107**, 365 (1976).
 - [10] T. T. Wu and C. N. Yang, Some properties of monopole harmonics, *Phys. Rev. D* **16**, 1018 (1977).
 - [11] F. D. M. Haldane, Fractional quantization of the Hall effect: A hierarchy of incompressible quantum fluid states, *Phys. Rev. Lett.* **51**, 605 (1983).
 - [12] J. L. Cardy, Conformal invariance and universality in finite-size scaling, *J. Phys. A* **17**, L385 (1984).
 - [13] J. L. Cardy, Universal amplitudes in finite-size scaling: Generalisation to arbitrary dimensionality, *J. Phys. A* **18**, L757 (1985).
 - [14] L. Hu, Y.-C. He, and W. Zhu, Operator product expansion coefficients of the 3D Ising criticality via quantum fuzzy spheres, *Phys. Rev. Lett.* **131**, 031601 (2023).
 - [15] C. Han, L. Hu, W. Zhu, and Y.-C. He, Conformal four-point correlators of the three-dimensional Ising transition via the quantum fuzzy sphere, *Phys. Rev. B* **108**, 235123 (2023).
 - [16] L. Hu, Y.-C. He, and W. Zhu, Solving conformal defects in 3D conformal field theory using fuzzy sphere regularization, *Nat. Commun.* **15**, 3659 (2024).
 - [17] C. Han, L. Hu, and W. Zhu, Conformal operator content of the Wilson-Fisher transition on fuzzy sphere bilayers, *Phys. Rev. B* **110**, 115113 (2024).
 - [18] J. S. Hofmann, F. Goth, W. Zhu, Y.-C. He, and E. Huffman, Quantum Monte Carlo simulation of the 3D Ising transition on the fuzzy sphere, *SciPost Phys. Core* **7**, 028 (2024).
 - [19] Z. Zhou, D. Gaiotto, Y.-C. He, and Y. Zou, The g -function and defect changing operators from wavefunction overlap on a fuzzy sphere, *SciPost Phys.* **17**, 021 (2024).

- [20] L. Hu, W. Zhu, and Y.-C. He, Entropic F -function of 3D Ising conformal field theory via the fuzzy sphere regularization, *Phys. Rev. B* **111**, 155151 (2025).
- [21] G. Fardelli, A. L. Fitzpatrick, and E. Katz, Constructing the infrared conformal generators on the fuzzy sphere, *SciPost Phys.* **18**, 086 (2025).
- [22] R. Fan, Note on explicit construction of conformal generators on the fuzzy sphere, [arXiv:2409.08257](#).
- [23] Z. Zhou and Y. Zou, Studying the 3D Ising surface CFTs on the fuzzy sphere, *SciPost Phys.* **18**, 031 (2025).
- [24] M. Dedushenko, Ising BCFT from fuzzy hemisphere, [arXiv:2407.15948](#).
- [25] Z. Zhou, L. Hu, W. Zhu, and Y.-C. He, $SO(5)$ deconfined phase transition under the fuzzy-sphere microscope: Approximate conformal symmetry, pseudo-criticality, and operator spectrum, *Phys. Rev. X* **14**, 021044 (2024).
- [26] B.-B. Chen, X. Zhang, Y. Wang, K. Sun, and Z. Y. Meng, Phases of $(2+1)D$ $SO(5)$ nonlinear sigma model with a topological term on a sphere: Multicritical point and disorder phase, *Phys. Rev. Lett.* **132**, 246503 (2024).
- [27] C. Voinea, R. Fan, N. Regnault, and Z. Papić, Regularizing 3D conformal field theories via anyons on the fuzzy sphere, *Phys. Rev. X* **15**, 031007 (2025).
- [28] S. Yang, Y.-G. Yue, Y. Tang, C. Han, W. Zhu, and Y. Chen, Microscopic study of 3D Potts phase transition via fuzzy sphere regularization, *Phys. Rev. B* **112**, 024436 (2025).
- [29] A. M. Läuchli, L. Herviou, P. H. Wilhelm, and S. Rychkov, Exact diagonalization, matrix product states and conformal perturbation theory study of a 3D Ising fuzzy sphere model, *SciPost Phys.* **19**, 076 (2025).
- [30] R. Fan, J. Dong, and A. Vishwanath, Simulating the non-unitary Yang-Lee conformal field theory on the fuzzy sphere, [arXiv:2505.06342](#).
- [31] E. A. Cruz, I. R. Klebanov, G. Tarnopolsky, and Y. Xin, Yang-Lee quantum criticality in various dimensions, [arXiv:2505.06369](#).
- [32] J. E. Miro and O. Delouche, Flowing from the Ising model on the fuzzy sphere to the 3D Lee-Yang CFT, *J. High Energy Phys.* **10** (2025) 037.
- [33] Y.-C. He, Free real scalar CFT on fuzzy sphere: Spectrum, algebra and wavefunction ansatz, [arXiv:2506.14904](#).
- [34] J. Taylor, C. Voinea, Z. Papić, and R. Fan, Conformal scalar field theory from Ising tricriticality on the fuzzy sphere, [arXiv:2506.22539](#).
- [35] S. Yang, L.-d. Hu, C. Han, W. Zhu, and Y. Chen, Conformal operator flows of the deconfined quantum criticality from $SO(5)$ to $O(4)$, [arXiv:2507.01322](#).
- [36] Z.-Q. Gao, T. Wang, and D.-H. Lee, Interacting Chern insulator transition on the sphere: Revealing the Gross-Neveu-Yukawa criticality, [arXiv:2504.15338](#).
- [37] J. L. Cardy, Conformal invariance and surface critical behavior, *Nucl. Phys.* **B240**, 514 (1984).
- [38] J. L. Cardy, Effect of boundary conditions on the operator content of two-dimensional conformally invariant theories, *Nucl. Phys.* **B275**, 200 (1986).
- [39] J. L. Cardy, Boundary conditions, fusion rules and the verlinde formula, *Nucl. Phys.* **B324**, 581 (1989).
- [40] N. Ishibashi, The boundary and crosscap states in conformal field theories, *Mod. Phys. Lett. A* **04**, 251 (1989).
- [41] I. Affleck and A. W. W. Ludwig, Universal noninteger “ground-state degeneracy” in critical quantum systems, *Phys. Rev. Lett.* **67**, 161 (1991).
- [42] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory* (Springer, Berlin, Heidelberg, 2009).
- [43] D. Fioravanti, G. Pradisi, and A. Sagnotti, Sewing constraints and non-orientable open strings, *Phys. Lett. B* **321**, 349 (1994).
- [44] G. Pradisi, A. Sagnotti, and Y. S. Stanev, Completeness conditions for boundary operators in 2D conformal field theory, *Phys. Lett. B* **381**, 97 (1996).
- [45] J. Fuchs, L. Huiszoon, A. Schellekens, C. Schweigert, and J. Walcher, Boundaries, crosscaps and simple currents, *Phys. Lett. B* **495**, 427 (2000).
- [46] I. Brunner and K. Hori, Notes on orientifolds of rational conformal field theories, *J. High Energy Phys.* **07** (2004) 023.
- [47] G. Y. Cho, C.-T. Hsieh, T. Morimoto, and S. Ryu, Topological phases protected by reflection symmetry and crosscap states, *Phys. Rev. B* **91**, 195142 (2015).
- [48] J. Caetano and S. Komatsu, Crosscap states in integrable field theories and spin chains, *J. Stat. Phys.* **187**, 30 (2022).
- [49] C. Ekman, Crosscap states in the XXX spin-1/2 spin chain, [arXiv:2207.12354](#).
- [50] T. Gombor, Integrable crosscap states in $gl(n)$ spin chains, *J. High Energy Phys.* **10** (2022) 096.
- [51] T. Gombor, Integrable crosscaps in classical sigma models, *J. High Energy Phys.* **03** (2023) 146.
- [52] M. He and Y. Jiang, Integrable crosscap states: From spin chains to 1D Bose gas, *J. High Energy Phys.* **08** (2023) 079.
- [53] Y. Chiba and Y. Yoneta, Exact thermal eigenstates of nonintegrable spin chains at infinite temperature, *Phys. Rev. Lett.* **133**, 170404 (2024).
- [54] Y. Yoneta, Thermal pure states for systems with antiunitary symmetries and their tensor network representations, *Phys. Rev. Res.* **6**, L042062 (2024).
- [55] B.-Y. Tan, Y. Zhang, H.-C. Zhang, W. Tang, L. Wang, H.-H. Tu, and Y.-H. Wu, Extracting the Luttinger parameter from a single wave function, *Phys. Rev. Lett.* **134**, 076501 (2025).
- [56] Y. Zhang, Y.-H. Wu, L. Wang, and H.-H. Tu, Crosscap states and duality of Ising field theory in two dimensions, [arXiv:2409.11046](#).
- [57] K. Chalas, P. Calabrese, and C. Rylands, Quench dynamics of entanglement from crosscap states, *SciPost Phys.* **19**, 132 (2025).
- [58] Z. Wei, Holographic dual of crosscap conformal field theory, *J. High Energy Phys.* **03** (2025) 086.
- [59] Z. Wei and Y. Yoneta, Crosscap quenches and entanglement evolution, [arXiv:2412.18610](#).
- [60] Y.-Z. Li, Y. Xie, and S. He, Holographic correlators of boundary/crosscap CFTs in two dimensions, *J. High Energy Phys.* **07** (2025) 010.
- [61] S. Mohapatra, S. Moudgalya, and A. C. Balram, Exact volume-law entangled zero-energy eigenstates in a large class of spin models, *Phys. Rev. Lett.* **134**, 210403 (2025).
- [62] B.-X. Lao and S. Rychkov, 3D Ising CFT and exact diagonalization on icosahedron: The power of conformal perturbation theory, *SciPost Phys.* **15**, 243 (2023).
- [63] Y. Nakayama, Bootstrapping critical Ising model on three dimensional real projective space, *Phys. Rev. Lett.* **116**, 141602 (2016).

- [64] N. Andrei, A. Bissi, M. Buican, J. Cardy, P. Dorey, N. Drukker, J. Erdmenger, D. Friedan, D. Fursaev, A. Konechny, C. Kristjansen, I. Makabe, Y. Nakayama, A. O'Bannon, R. Parini, B. Robinson, S. Ryu, C. Schmidt-Colinet, V. Schomerus, C. Schweigert, and G. M. T. Watts, Boundary and defect CFT: Open problems and applications, *J. Phys. A* **53**, 453002 (2020).
- [65] S. Giombi, H. Khanchandani, and X. Zhou, Aspects of CFTs on real projective space, *J. Phys. A* **54**, 024003 (2020).
- [66] G. Pradisi, A. Sagnotti, and Y. Stanev, Planar duality in SU (2) WZW models, *Phys. Lett. B* **354**, 279 (1995).
- [67] H.-H. Tu, Universal entropy of conformal critical theories on a Klein bottle, *Phys. Rev. Lett.* **119**, 261603 (2017).
- [68] W. Tang, L. Chen, W. Li, X. C. Xie, H.-H. Tu, and L. Wang, Universal boundary entropies in conformal field theory: A quantum Monte Carlo study, *Phys. Rev. B* **96**, 115136 (2017).
- [69] L. Chen, H.-X. Wang, L. Wang, and W. Li, Conformal thermal tensor network and universal entropy on topological manifolds, *Phys. Rev. B* **96**, 174429 (2017).
- [70] W. Tang, X. C. Xie, L. Wang, and H.-H. Tu, Klein bottle entropy of compactified boson conformal field theory, *Phys. Rev. B* **99**, 115105 (2019).
- [71] W. Tang, H.-H. Tu, and L. Wang, Continuous matrix product operator approach to finite temperature quantum states, *Phys. Rev. Lett.* **125**, 170604 (2020).
- [72] Z.-Q. Li, L.-P. Yang, Z. Y. Xie, H.-H. Tu, H.-J. Liao, and T. Xiang, Critical properties of the two-dimensional q -state clock model, *Phys. Rev. E* **101**, 060105 (2020).
- [73] R. Vanhove, L. Lootens, H.-H. Tu, and F. Verstraete, Topological aspects of the critical three-state Potts model, *J. Phys. A: Math. Theor.* **55**, 235002 (2022).
- [74] Y. Zhang, A. Hulsch, H.-C. Zhang, W. Tang, L. Wang, and H.-H. Tu, Universal scaling of Klein bottle entropy near conformal critical points, *Phys. Rev. Lett.* **130**, 151602 (2023).
- [75] H. Shimizu and A. Ueda, Tensor network simulations for non-orientable surfaces, [arXiv:2402.15507](https://arxiv.org/abs/2402.15507).
- [76] T. Liu, Y.-H. Wu, H.-H. Tu, and T. Xiang, Bridging conformal field theory and parton approaches to $SU(n)_k$ chiral spin liquids, *Phys. Rev. B* **111**, 205137 (2025).
- [77] The prefactor 4 is chosen to match the convention in Ref. [8].
- [78] S. R. White, Density matrix formulation for quantum renormalization groups, *Phys. Rev. Lett.* **69**, 2863 (1992).
- [79] Diagraph, https://nick-ux.org/diagraph/index.php/Main_Page.
- [80] <https://data.mendeley.com/datasets/b3fvc8kfmq/1>