

Quantum state of interacting primordial inhomogeneities: Desqueezing and decoherence

Amaury Micheli^{1,*}, Yuto Oshima², and Tomo Takahashi³

¹*RIKEN Center for Interdisciplinary Theoretical and Mathematical Sciences (iTHEMS),
Wako, Saitama 351-0198, Japan*

²*Graduate School of Science and Engineering, Saga University, Saga 840-8502, Japan*

³*Department of Physics, Saga University, Saga 840-8502, Japan*



(Received 8 January 2026; accepted 30 April 2026; published 29 June 2026)

We investigate how interactions affect the quantum state of scalar perturbations during inflation and the quantum correlations they may exhibit. Focusing on the case of scalar perturbations in single-field inflation, we model interactions using a Lindblad equation with a nonunitary contribution quadratic in the scalar perturbations and of parametrizable amplitude and time dependence. We compute the quantum state of these interacting perturbations, which is fully described by its purity and squeezing parameters. First, we show that, on top of the well-studied decrease of purity, interactions asymptotically decrease the squeezing parameter by up to 75%. Second, we show that this desqueezing induced by the interactions, in addition to the purity loss, causes a further suppression of quantum correlations. We thus emphasize that the quantum or classical character of the correlations exhibited by the perturbations cannot be correctly determined by computing the effect of interactions on the purity alone. Since the phenomenological framework adopted in this paper encompasses a wide class of possible interactions, our results provide general insights into the nature of decoherence processes in primordial fluctuations.

DOI: [10.1103/q7m4-13jf](https://doi.org/10.1103/q7m4-13jf)

I. INTRODUCTION

The gravitational structures of the Universe, including galaxy distributions and cosmic microwave background (CMB) anisotropies, are believed to originate from vacuum fluctuations amplified during inflation [1,2]. Although these initial fluctuations are quantum, the inhomogeneities they generate are treated as mere classical perturbations in all postinflationary cosmological analyses, e.g., in CMB anisotropy predictions. While this *ad hoc* procedure allows one to successfully predict the CMB anisotropies' Gaussian statistics, the workings and extent of the quantum-to-classical transition of these amplified vacuum fluctuations remain elusive. A first problem is to describe how, if, and when the quantum state initially describing a superposition of possible inhomogeneities “collapsed” to the single pattern of inhomogeneities observed in our Universe [3]. A second distinct problem is to characterize whether the inhomogeneities' quantum state exhibits some distinctly quantum features which, if observed in the realized pattern

of inhomogeneities, would demonstrate the quantum origin of these inhomogeneities. In this work, we will only discuss this second question.

Various properties of the perturbations have been considered as possible probes of their quantum origin, or conversely of their classicalization [4–9]. We refer the readers to the reviews [10–12] for exhaustive presentations. In this work, we follow approaches that quantified the degree of quantum correlations in the distribution of inhomogeneities during inflation by borrowing tools from quantum information, such as Bell inequalities [13–16], nonseparability [17,18], logarithmic negativity [19,20], entanglement entropy [17,19–24], quantum discord [25–28], and others [17]. We will here consider four of these notions: nonseparability, logarithmic negativity, quantum discord, and a Bell inequality violation. Evaluating these quantities requires computing the quantum state of the inhomogeneities and then selecting the degrees of freedom whose correlations we wish to study.

We will consider the quantum state of the perturbations generated by the simplest inflationary dynamics of single-field slow-roll inflation [29]. The inhomogeneities are treated as perturbations about the homogeneous trajectory of the inflaton field and the Friedmann-Lemaître-Robertson-Walker (FLRW) metric describing the homogeneous and isotropic expanding Universe [30]. Decomposing these perturbations into scalar, vector, and

*Contact author: amaury.micheli@riken.jp

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

tensor, one can show that vector perturbations generally decay away while both scalar and tensor perturbations can be decomposed as a collection of independent pairs of opposite momenta $\pm \mathbf{k}$ Fourier modes in two-mode squeezed vacuum states [31], fully characterized by their squeezing parameters r_k and φ_k . In this work, we will study the nonclassicality of correlations present in this state between Fourier modes, where only the opposite momenta $+\mathbf{k}$ and $-\mathbf{k}$ modes within a pair in a two-mode squeezed state are correlated. Note that, while this is the approach taken by most authors [4–9,13,14,16,17,26–28,32], during inflation, there is in general no preferred separation of the Hilbert space of the modes $\pm \mathbf{k}$ into degrees of freedom labeled as $+\mathbf{k}$ and those labeled as $-\mathbf{k}$, and the different definitions lead to different levels of correlations [33]. One approach to avoid this ambiguity is to instead study correlations between local modes [33]. The consensus is that, even for perturbations in a collection of two-mode squeezed vacua, these correlations are small and not increased by expansion [19,20,34,35]. Our approach, detailed in Sec. IV A, is to consider correlations using a definition of Fourier modes which will coincide with, or be close to, the physical one at early times during inflation and at late times after inflation, where the ambiguity is lifted.

In this simple picture where the $\pm \mathbf{k}$ modes are in a two-mode squeezed vacuum, a pure state, most criteria agree [18] and show that the perturbations exhibit quantum correlations [36]. Yet, this state is arrived at when one neglects the effects of the perturbations' interactions with themselves and other degrees of freedom, which are expected to play a key role in their classicalization process [8,9]. Interactions will correlate the otherwise independent $\pm \mathbf{k}$ pairs with other degrees of freedom and are expected to weaken their bipartite quantum correlations, a process known as decoherence [37–39]. To describe the decoherence of the perturbations without having to solve their joint dynamics with specific extra degrees of freedom, we can treat the perturbations as an *open* quantum system with nonunitary dynamics. Numerous works have explored different additional degrees of freedom and different types of interactions with the cosmological perturbations [8,17,40–47]. Beyond the specifics of the decoherence processes considered, as long as the state remains close to Gaussian, homogeneous, and isotropic [17], the state of a pair $\pm \mathbf{k}$ of perturbations can still be fully described using the squeezing parameters r_k , the squeezing angle φ_k , and the purity p_k [28]. We refer collectively to these three quantities as the effective squeezing parameters. In Ref. [28], the authors quantified the quantum correlations of the scalar perturbations by their quantum discord [48,49] expressing it as a function of these parameters. They then analyzed how the specific combination of parameters entering the discord is affected by decoherence in a phenomenological model of interactions based on the Lindblad equation [50–52].

In this paper, we first compute the values of the effective squeezing parameters r_k , φ_k , p_k , which fully characterize the quantum state of the perturbations in the presence of decoherence modeled by the same Lindblad equation considered in Ref. [53] for a large range of parameter space. In doing so, we improve upon previous studies on decoherence of cosmological perturbations which only analyzed the evolution of the purity p_k [28,41,43,47,53] or specific measures of quantum correlations [16,28]. We first demonstrate that, while interactions reduce the purity p_k as expected [28,41,43,47,53], they also typically reduce the squeezing parameter r_k by up to 75% asymptotically; i.e., they desqueeze the perturbations. Second, we use the obtained quantum state to compute several nonclassicality measures. We clarify that these measures are affected by two separate effects: a decreased purity that classicalizes correlations and the desqueezing we previously identified, that diminishes the overall level of correlation. We illustrate the necessity of accounting for both effects by comparing the values of these nonclassicality measures computed with and without taking into account the desqueezing and show they differ significantly. This demonstrates that a correct estimation of quantum correlations requires computing the squeezing parameter, in addition to the purity that previous studies focused on.

The organization of this paper is as follows. In Sec. II, we review the effective squeezing parameters and covariance matrix formalism used to describe the state of the curvature perturbations as well as the Lindblad equation we use to model their dynamics. In Sec. III, we derive, both numerically and analytically, the evolution of the effective squeezing parameters and other parametrizations of the quantum state for this open-system dynamics. This constitutes the first main result of this paper. In Sec. IV, based on these expressions, we compare the degree of quantum correlations as measured by three different quantifiers (quantum discord, Bell operator, and logarithmic negativity), extending Ref. [28]. This is the second main result of this paper. We end by discussing some possible extensions of our analyses.

Throughout this paper, we adopt natural units and set $c = \hbar = 1$, unless otherwise stated.

II. QUANTUM STATE OF CURVATURE PERTURBATIONS IN INFLATION

In this section, we recap the necessary background for the discussion of quantum correlations of a field in curved spacetime, applied to cosmological perturbations during inflation.

A. Free perturbations

We are interested in the evolution of the quantum state of cosmological perturbations during a period of slow-roll inflation. We focus on scalar perturbations in this work, although the same formalism can be equally well applied to

tensor perturbations; see, e.g., Ref. [11] for a review. We assume a spatially flat Universe and describe the inflationary Universe at the background level with a flat FLRW metric,

$$ds^2 = a(\eta)^2(-d\eta^2 + d\mathbf{x}^2), \quad (1)$$

where η denotes conformal time and $a(\eta)$ denotes the scale factor. Working in perturbation theory, in single-field slow-roll inflation, the scalar part of the metric and inflaton perturbations can be fully packaged into a single gauge-invariant scalar field $\hat{v}$, the Mukhanov-Sasaki field [30]. To describe the dynamics using the Hamiltonian formalism, we choose its canonically conjugate field to be $\hat{p} = \partial_\eta \hat{v}$, such that $[\hat{v}(\mathbf{x}), \hat{p}(\mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{x}')$ [54]. Since the dynamics is homogeneous and isotropic, it is convenient to work in Fourier space. The Fourier transform of $\hat{v}$ is defined by

$$\hat{v}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3k e^{-ik\cdot\mathbf{x}} \hat{v}_k, \quad (2)$$

and similarly for $\hat{p}(\mathbf{x})$. At first order in perturbation theory, the dynamics of the perturbation is linear [30] and is thus described by a quadratic Hamiltonian, which in our conventions reads [28]

$$\begin{aligned} \hat{H} &= \frac{1}{2} \int d^3x \left[\hat{p}^2(\mathbf{x}) + (\nabla \hat{v})^2(\mathbf{x}) - \frac{\partial_\eta^2 z}{z} \hat{v}^2(\mathbf{x}) \right], \\ &= \int_{\mathbb{R}^{3+}} d^3k \hat{\mathcal{H}}_{\pm k} = \int_{\mathbb{R}^{3+}} d^3k [\hat{p}_k \hat{p}_k^\dagger + \omega^2(k, \eta) \hat{v}_k \hat{v}_k^\dagger], \end{aligned} \quad (3)$$

with

$$\omega^2(k, \eta) = k^2 - \frac{\partial_\eta^2 z}{z}, \quad (4)$$

where $\mathbb{R}^{3+} = \{\mathbf{k} \in \mathbb{R}^3 | (k_z > 0) \vee (k_z = 0, k_y > 0) \vee (k_z = 0, k_y = 0, k_x > 0)\}$, $z = a\sqrt{2\epsilon_1/\kappa}$, $\kappa = 8\pi G$, $\epsilon_1 = 1 - \partial_\eta \mathcal{H}/\mathcal{H}^2$ and $\mathcal{H} = (\partial_\eta a)/a$. ϵ_1 is the first slow-roll parameter controlling how close we are to a de Sitter phase. In Eq. (3), we dropped the term associated with the $k = 0$ mode that we ignore in the rest of our analysis so that implicitly $k > 0$. Note that, since $\hat{v}(\mathbf{x})$ and $\hat{p}(\mathbf{x})$ are Hermitian, their Fourier amplitudes for $\pm\mathbf{k}$ are not independent. Thus, the sum is only performed over half of the entire space of modes, $\mathbb{R}^{3+}$, and $\hat{\mathcal{H}}_{\pm k}$, which can be checked to be Hermitian, encodes the dynamics of both modes $\pm\mathbf{k}$. Finally, the quantum state of the field is described by its density matrix $\hat{\rho}(\eta)$, and its evolution is given by the Liouville–von Neumann equation

$$i \frac{d\hat{\rho}}{d\eta} = [\hat{H}, \hat{\rho}]. \quad (5)$$

1. Choice of canonical pairs and form of the dynamics

The form (3) of the Hamiltonian is, up to canonical transformations, completely determined by the requirements of linearity, homogeneity, and isotropy [28] of the dynamics. However, it is useful to rewrite this Hamiltonian in terms of different sets of canonically conjugated operators that provide a different perspective on the dynamics. Since it will be crucial in the discussion of correlations in Sec. IV A, we will only use a collection of pairs of Hermitian operators that obey the canonical bosonic commutation relation, e.g., $[\hat{v}(\mathbf{x}), \hat{p}(\mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{x}')$. Each such set of canonical pairs defines a partition of the Hilbert space into a direct sum of Hilbert subspaces. Let us describe three important sets of pairs.

First, we defined the field in real space using the pairs $(\hat{v}(\mathbf{x}), \hat{p}(\mathbf{x}))$, which effectively represent the field as a collection of $\mathbb{R}^3$ bosonic modes localized at each point in space. The Hilbert space is then partitioned as $\mathcal{H} = \bigoplus_{\mathbb{R}^3} \mathcal{H}_{\mathbf{x}}$, and the dynamics is manifestly local as written in the first line of Eq. (3).

Second, since the dynamics respects homogeneity and isotropy, it is best described in Fourier space. Performing the transformation to Fourier space in the second line of Eq. (3), $\hat{H}$ becomes a sum of Hamiltonians $\hat{\mathcal{H}}_{\pm k}$ describing the dynamics of pairs of opposite-momentum Fourier modes $\pm\mathbf{k}$. The dynamics of each of these pairs is thus independent of the others. However, the reality of the fields $\hat{v}(\mathbf{x})$ and $\hat{p}(\mathbf{x})$ implies the relations $\hat{v}_k^\dagger = \hat{v}_{-k}$, $\hat{p}_k^\dagger = \hat{p}_{-k}$ so that the pairs $(\hat{v}_k, \hat{p}_k)$ and $(\hat{v}_{-k}, \hat{p}_{-k})$ are not independent degrees of freedom [43]. A first way to define canonical pairs from these operators is to consider their Hermitian and anti-Hermitian parts (up to a factor). Following Ref. [28], we define

$$\hat{v}_k^R = \frac{\hat{v}_k + \hat{v}_k^\dagger}{\sqrt{2}}, \quad \hat{v}_k^I = \frac{\hat{v}_k - \hat{v}_k^\dagger}{\sqrt{2}i}, \quad (6)$$

$$\hat{p}_k^R = \frac{\hat{p}_k + \hat{p}_k^\dagger}{\sqrt{2}}, \quad \hat{p}_k^I = \frac{\hat{p}_k - \hat{p}_k^\dagger}{\sqrt{2}i}. \quad (7)$$

One can check that these operators are Hermitian and that the sectors R, I are indeed independent, as $[\hat{v}_k^R, \hat{p}_{k'}^I] = [\hat{v}_k^R, \hat{v}_{k'}^I] = [\hat{p}_k^R, \hat{p}_{k'}^I] = 0$. Both satisfy canonical commutation relations, $[\hat{v}_k^s, \hat{p}_{k'}^{s'}] = i\delta(\mathbf{k} - \mathbf{k}')\delta_{s,s'}$ with $s, s' = R, I$. The Hilbert space can then be decomposed as $\mathcal{H} = \bigoplus_{\mathbb{R}^{3+}} \mathcal{H}_{k,R} \oplus \mathcal{H}_{k,I}$, where $\mathcal{H}_{k,s}$ is the Hilbert space on which the pair $\hat{v}_k^s, \hat{p}_k^s$ acts. We call this partition of the Hilbert space R/I. This decomposition allows us to write the dynamics in its simplest form, in which Eq. (3) reads

$$\hat{H} = \int_{\mathbb{R}^{3+}} d^3k \sum_{s=R,I} \hat{\mathcal{H}}_k^s = \int_{\mathbb{R}^{3+}} d^3k \sum_{s=R,I} \frac{(\hat{p}_k^s)^2 + \omega^2(k, \eta)(\hat{v}_k^s)^2}{2}, \quad (8)$$

casting the field dynamics as that of an infinite set of $2 \times \mathbb{R}^{3+} = \mathbb{R}^3$ decoupled modes. Note that this is the correct number of degrees of freedom for a real scalar field.

Finally, a third set of pairs is given by considering the standard quantum field-theoretic procedure to define the particle content of a field. We introduce the standard creation and annihilation operators $\hat{c}_k^{(\dagger)}$, defined by

$$\hat{c}_k = \frac{1}{\sqrt{2}} \left(\sqrt{k} \hat{v}_k + i \frac{\hat{p}_k}{\sqrt{k}} \right). \quad (9)$$

These operators satisfy the expected commutation relations $[\hat{c}_k, \hat{c}_{k'}] = 0$ and $[\hat{c}_k, \hat{c}_{k'}^\dagger] = \delta(\mathbf{k} - \mathbf{k}')$, so that for different $\mathbf{k}$ they act on separate Hilbert spaces. Equation (9) coincides with the standard definition of Fourier modes in the limit where $\omega^2(k, \eta) \rightarrow k^2$; i.e., when the modes are effectively in Minkowski. It leads to the Hilbert-space partition, $H = \bigoplus_{\mathbb{R}^{3+}} H_{\pm k} = \bigoplus_{\mathbb{R}^{3+}} H_k \oplus H_{-k}$, where H_k is the Hilbert space on which the pair $\hat{c}_k, \hat{c}_k^\dagger$ acts. We call this partition $\pm k$. We can then straightforwardly define canonically conjugate bosonic pairs from $\hat{c}_{\pm k}$ as

$$\hat{q}_{\pm k} = \frac{1}{\sqrt{2k}} \left(\hat{c}_{\pm k} + \hat{c}_{\pm k}^\dagger \right) = \frac{1}{\sqrt{2}} \hat{v}_k^R \mp \frac{1}{\sqrt{2k}} \hat{p}_k^I, \quad (10)$$

$$\hat{\pi}_{\pm k} = -i \sqrt{\frac{k}{2}} \left(\hat{c}_{\pm k} - \hat{c}_{\pm k}^\dagger \right) = \frac{1}{\sqrt{2}} \hat{p}_k^R \pm \frac{k}{\sqrt{2}} \hat{v}_k^I. \quad (11)$$

The bosonic operators $\hat{q}_{\pm k}$ and $\hat{\pi}_{\pm k}$ characterize H_{+k} and H_{-k} . Although not convenient for performing computations, it is instructive to rewrite the Hamiltonian $\hat{\mathcal{H}}_{\pm k}$ describing the coupled dynamics of the modes $\pm k$ in terms of these operators. We have

$$\begin{aligned} \hat{\mathcal{H}}_{\pm k} = & \frac{1}{2} [1 + k^{-2} \omega^2(k, \eta)] [\hat{\pi}_k^2 + k^2 \hat{q}_k^2 + \hat{\pi}_{-k}^2 + k^2 \hat{q}_{-k}^2] \\ & + [1 - k^{-2} \omega^2(k, \eta)] [\hat{\pi}_k \hat{\pi}_{-k} - \hat{q}_k \hat{q}_{-k}]. \end{aligned} \quad (12)$$

The terms on the first line are, up to an overall factor, the Hamiltonians of two free oscillators for the modes $+k$ and $-k$. The terms on the second line gives a time-dependent coupling between the $\pm k$ modes. Thus, while the dynamics leaves the degrees of freedom in $H_{\pm k, R}$ and $H_{\pm k, I}$ decoupled, it does generate entanglement between degrees of freedom in H_{+k} and H_{-k} . In the $\pm k$ partition, the dynamics corresponds to a two-mode squeezing, as first noted by Ref. [31], whereas in the R/I partition, it is the direct product of two identical one-mode squeezing

operations. In the $\pm k$ partition, the dynamics takes a vacuum state to a two-mode squeezed vacuum state.

2. Initial state

The most natural choice of initial state for the field at the beginning of inflation is to assume that it contains no excitations, i.e., no particles. Indeed, were the initial state to contain particles, we would be beholden to justify their presence by a preinflationary physical process. Thus, we would only have pushed to an earlier time the problem of selecting initial conditions for the inhomogeneities that inflation is supposed to solve dynamically. However, while in Minkowski space the Poincaré symmetries select a preferred vacuum from which the notion of a particle is defined [55], the essential feature of quantum field theory in curved spacetime is that this notion of vacuum, and thus that of a particle, loses its global character [56]. The noncoincidence of the notion of vacuum in two different spacetime regions is what leads to cosmological particle production and, in our case, to the generation of cosmological inhomogeneities during inflation. For the case at hand, this manifests itself in the time dependence of the Hamiltonian in Eq. (3), which prevents us from defining a ground state. Fortunately, in the asymptotic past $\eta \rightarrow -\infty$, all modes under consideration are sub-Hubble, and we have $\omega^2(k, \eta) \rightarrow k^2$. The frequency is time independent, and we are effectively in a flat-spacetime situation, with a ground state, i.e., a vacuum, given by the usual flat-spacetime methods. This is precisely how we built the $\pm k$ partition in the previous section. Equation (12) then shows that, in the limit $\omega^2(k, \eta) \rightarrow k^2$, the $\pm k$ modes behave as independent free oscillators of frequency k . The vacuum of each mode is uniquely defined as the state $|0\rangle_k$ annihilated by $\hat{c}_k$ as defined in Eq. (9). Applying this constraint to all modes $\mathbf{k}$ defines the Bunch-Davies vacuum $|0\rangle_{\text{BD}} = \prod_{\mathbf{k} \in \mathbb{R}^3} |0\rangle_k$. The Bunch-Davies vacuum thus gives a “minimal” degree of fluctuations in the asymptotic past, and we will assume in the rest of this work that this was the initial state of the perturbations.

3. Reduced dimension dynamics

One can define creation and annihilation operators for the R/I partition related to $\hat{v}_k^s, \hat{p}_k^s$ by the same relations as those linking $\hat{q}_{\pm k}, \hat{\pi}_{\pm k}$ to $\hat{c}_{\pm k}^{(\dagger)}$ in (10). Their associated vacuum matches the Bunch-Davies one, so that $|0\rangle_{\text{BD}} = \prod_{\mathbf{k} \in \mathbb{R}^{3+}, s \in \{R, I\}} |0\rangle_{k, s}$. The initial state of the cosmological perturbations is thus a direct product in the R/I partition, and the separation of the Hamiltonian (8) ensures that this factorization persists at any later time. On the other hand, in the $\pm k$ partition, the dynamics entangles

opposite-momentum modes, so that we can only write the state as a direct product of two-mode subsystems. Using the density-matrix formalism, we have at all times η ,

$$\hat{Q}(\eta) = \bigotimes_{\substack{\mathbf{k} \in \mathbb{R}^{3+} \\ s \in \{\text{R,I}\}}} \hat{Q}_{\mathbf{k},s}(\eta) = \bigotimes_{\mathbf{k} \in \mathbb{R}^{3+}} \hat{Q}_{\pm\mathbf{k}}(\eta), \quad (13)$$

where $\hat{Q}_{\pm\mathbf{k}} = \hat{Q}_{\mathbf{k},\text{R}} \otimes \hat{Q}_{\mathbf{k},\text{I}} \neq \hat{Q}_{\mathbf{k}} \otimes \hat{Q}_{-\mathbf{k}}$ in general. Thanks to this factorization, in the rest of this work, we will focus on the evolution of modes in the two-dimensional Hilbert subspace $H_{\pm\mathbf{k}} = H_{+\mathbf{k}} \oplus H_{-\mathbf{k}} = H_{\mathbf{k},\text{R}} \oplus H_{\mathbf{k},\text{I}}$. The Liouville–von Neumann equation then separates on these different Hilbert subspaces,

$$\frac{\mathcal{V}}{(2\pi)^3} \partial_\eta(\hat{Q}_{\mathbf{k},s}) = -i[\hat{\mathcal{H}}_{\mathbf{k}}^s, \hat{Q}_{\mathbf{k},s}], \quad (14)$$

where $\mathcal{V}$ is the volume of the finite box in real space (e.g., $\mathcal{V} = L^3$) that we must use to make computations well defined and take to infinity eventually. Its inclusion is carefully explained in Appendix D of Ref. [43] and comes from taking the time derivative of a tensor product,

$$\partial_\eta \hat{Q} = \frac{\mathcal{V}}{(2\pi)^3} \int_{\mathbb{R}^{3+}} d^3\mathbf{k} \sum_{s=\text{R,I}} \hat{Q}_{\mathbf{k},s}. \quad (15)$$

We note that, although this factor was missed in the rougher treatment of Ref. [28], it does not affect the subsequent analysis based on transport equations for the elements of the covariance matrix. As will be shown below, Eq. (14) allows one to describe the state separately on each $H_{\mathbf{k},s}$.

4. Covariance matrix

Because the initial Bunch-Davies state is Gaussian, and the state is evolved under a quadratic Hamiltonian (8), it remains Gaussian at all times η . Gaussian states are fully characterized by their covariance matrix; the $2N \times 2N$ matrix of the two-point correlation functions of canonical pairs, where N is the number of modes considered; and the $2N$ -vector of expectation values of these pairs. Our state is centered, and these expectation values vanish, so we only need to consider the covariance matrix. The covariance matrix of a single mode associated with $\mathbf{k}$, s and of state $\hat{Q}_{\mathbf{k},s}(\eta)$ reads

$$\gamma_{\mathbf{k}}^s = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}, \quad (16)$$

where the quantities γ_{ij} are the finite coefficients of the adimensionalized two-point functions [57]

$$2\mathbf{k} \langle \hat{v}_{\mathbf{k}}^s \hat{v}_{\mathbf{k}'}^{s'} \rangle = \gamma_{11} \delta_{s,s'} \delta(\mathbf{k} - \mathbf{k}'), \quad (17a)$$

$$\langle \hat{v}_{\mathbf{k}}^s \hat{p}_{\mathbf{k}'}^{s'} + \hat{p}_{\mathbf{k}}^s \hat{v}_{\mathbf{k}'}^{s'} \rangle = \gamma_{12} \delta_{s,s'} \delta(\mathbf{k} - \mathbf{k}'), \quad (17b)$$

$$\frac{2}{k} \langle \hat{p}_{\mathbf{k}}^s \hat{p}_{\mathbf{k}'}^{s'} \rangle = \gamma_{22} \delta_{s,s'} \delta(\mathbf{k} - \mathbf{k}'). \quad (17c)$$

Since the density matrix factorizes in this partition, the covariance matrix of $\hat{Q}_{\pm\mathbf{k}}$ is simply given by the 4×4 matrix $\gamma_{\mathbf{k}}^{\text{R}} \oplus \gamma_{\mathbf{k}}^{\text{I}}$. We can always express the three covariance-matrix elements γ_{ij} in terms of three real quantities: $r_k (\geq 0)$, the squeezing parameter; φ_k , the squeezing angle; and a third parameter p_k . Together, they are referred to as the effective squeezing parameters. We can write the elements of the covariance matrix as [18]

$$\gamma_{11} = p_k^{-1/2} [\cosh(2r_k) - \cos(2\varphi_k) \sinh(2r_k)], \quad (18a)$$

$$\gamma_{22} = p_k^{-1/2} [\cosh(2r_k) + \cos(2\varphi_k) \sinh(2r_k)], \quad (18b)$$

$$\gamma_{12} = -p_k^{-1/2} \sin(2\varphi_k) \sinh(2r_k). \quad (18c)$$

We can check that p_k^{-1} is given by the determinant of the covariance matrix,

$$p_k^{-1} = \det(\gamma_{\mathbf{k}}^s) = \gamma_{11}\gamma_{22} - \gamma_{12}^2. \quad (19)$$

Since, for Gaussian states [58], we have $\text{Tr}[\hat{\rho}_{\pm\mathbf{k}}^2] = \sqrt{\det(\gamma_{\mathbf{k}}^{\text{R}} \oplus \gamma_{\mathbf{k}}^{\text{I}})^{-1}} = \det(\gamma_{\mathbf{k}}^s)^{-1}$, p_k is the purity of the state $\hat{Q}_{\pm\mathbf{k}}$ and thus satisfies $0 \leq p_k \leq 1$. In the R/I partition, the state $\hat{Q}_{\pm\mathbf{k}}$ thus corresponds to the product $\hat{Q}_{\mathbf{k},\text{R}} \otimes \hat{Q}_{\mathbf{k},\text{I}}$ of two one-mode squeezed thermal states with the same one-mode squeezing parameters r_k, φ_k and purity $\sqrt{p_k}$ given above [59]. On the other hand, in the $\pm\mathbf{k}$ partition considered later, the state is a two-mode squeezed thermal state, see Eq. (59) below, with a two-mode squeezing amplitude given by r_k and purity p_k .

Next, we give a geometrical interpretation to the effective squeezing parameters by considering the Wigner function W associated with the state $\hat{Q}_{\mathbf{k},s}(\eta)$, which is a centered Gaussian with covariance matrix $\gamma_{\mathbf{k}}^s$ [18]. We have

$$W(v_{\mathbf{k}}^s, p_{\mathbf{k}}^s) = \frac{1}{\sqrt{2\pi \det \gamma}} \exp\left(-\frac{V^T \gamma^{-1} V}{2}\right), \quad (20)$$

with

$$V = \begin{pmatrix} v_{\mathbf{k}}^s \\ p_{\mathbf{k}}^s \end{pmatrix}. \quad (21)$$

Then, because W is a centered Gaussian, the locus of points where W reaches a fraction $0 \leq \beta \leq 1$ of its maximum, i.e.,

$$W(v_{\mathbf{k}}^s, p_{\mathbf{k}}^s) = \beta \max_{v_{\mathbf{k}}^s, p_{\mathbf{k}}^s} [W(v_{\mathbf{k}}^s, p_{\mathbf{k}}^s)], \quad (22)$$

is given by an ellipse centered at (0,0). For $\beta = e^{-1/2}$, the lengths of the semimajor axis a_k and of the semiminor axis

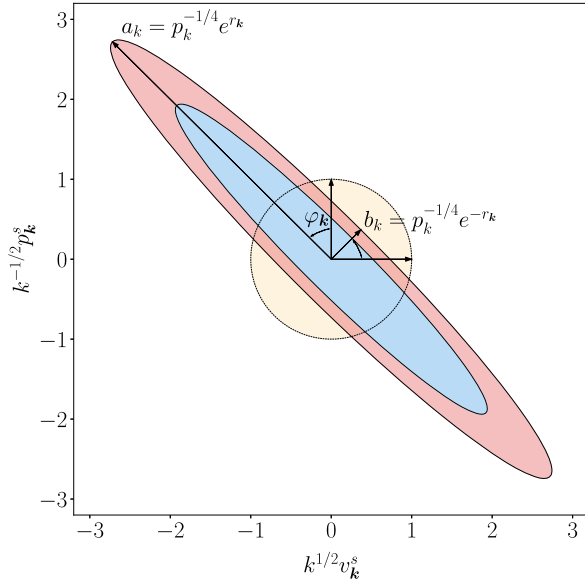


FIG. 1. Squeezing ellipses representing the points where the Wigner function of the mode $\mathbf{k}, s$ given in Eq. (20) reaches $e^{-1/2}$ times its maximum value. The yellow circle corresponds to the vacuum state for which $r_k = \varphi_k = 0$ and $p_k = 1$. The blue ellipse corresponds to a two-mode squeezed vacuum state with $r_k = 1$, $\varphi_k = \pi/4$ and $p_k = 1$. The red ellipse corresponds to a two-mode squeezed thermal state with $r_k = 1$, $\varphi_k = \pi/4$, and $p_k = 1/4$.

b_k are simply given by

$$a_k = p_k^{-1/4} e^{r_k}, \quad b_k = p_k^{-1/4} e^{-r_k}, \quad (23)$$

and the squeezing angle φ_k is the angle between the semiminor axis and the v_k^s axis. Thus, the squeezing parameter r_k controls the aspect ratio of the ellipse, $a_k/b_k = e^{2r_k}$, while the purity p_k gives its area, $S_k = \pi p_k^{-1/2}$. We plot several such squeezing ellipses in Fig. 1 as an illustration.

Finally, note that the Hamiltonian dynamics (14) can be cast directly as a set of three coupled ordinary differential equations for the $\gamma_{ij}s$, or equivalently for r_k, φ_k , and p_k , as given in Ref. [28]. We make a few general remarks on their evolution. First, since the perturbations are initially in their vacuum, which is a pure state, and since unitary dynamics preserve the purity of the state, we have $p_k = 1$ at all times. We thus have one-mode squeezed vacua, rather than generic thermal states. Second, when two quantum modes are in a pure state and correlated, they are necessarily entangled; i.e., all correlations are quantum in this case (see, e.g., Ref. [18]). Thus, as we will discuss in Sec. IV, the $\pm \mathbf{k}$ modes are entangled by the dynamics, and their bipartite correlations violate Bell inequalities and exhibit a large quantum discord. Yet, this equivalence between classical and quantum correlations, as well as that between the different quantumness criteria, breaks down in the presence of interactions which takes the state to a mixed

one [18]. We now study the evolution of perturbations in this context.

B. Interacting perturbations

Describing the evolution of the perturbations by the closed-system dynamics given by the Hamiltonian (8) is an idealization. For instance, the cosmological perturbations would be weakly coupled to other subdominant fields present during inflation [53], and computing the dynamics at next order in perturbation theory would lead to cubic self-interactions of the perturbations [60], which couple any $\pm \mathbf{k}$ pair of modes to all other modes of the curvature perturbation [41]. Retaining the approach of the previous section, we focus on a single $\pm \mathbf{k}$ pair and ask how such interactions would affect their quantum state, as described by the effective squeezing parameters. We rely on an open quantum system approach, in which the degrees of freedom with which the pair interacts are traced out, and the reduced dynamics of the $\pm \mathbf{k}$ pair is described by a master equation. Rather than deriving this equation within a specific interaction model, we adopt the approach of Ref. [53] and assume from the outset that the interactions result in a dynamics described by a specific Lindblad equation; see Eq. (24) below. This allows us to keep simple and explicit expressions for the different parameters, making the effect of interactions transparent.

1. Master equation for cosmological perturbations

We consider the curvature perturbations to couple to their environment through a coupling proportional to $\hat{v}$. Such a linear coupling has two virtues [28,43]: it leads to a master equation that preserves both the separation of the dynamics into independent one-mode systems $H_{\pm \mathbf{k},s}$ and the Gaussianity of the state. The resulting states for pairs of modes $\pm \mathbf{k}$ are two-mode squeezed thermal states, described by the effective squeezing parameters defined in Eq. (B17). We further assume that the evolution of the density matrix is governed by a Lindblad equation [50] of the form obtained in Ref. [53],

$$\frac{\mathcal{V}}{(2\pi)^3} \frac{\partial \hat{\rho}_{\mathbf{k},s}}{\partial \eta} = -i[\hat{\mathcal{H}}_{\mathbf{k},s}^s, \hat{\rho}_{\mathbf{k},s}] - \mathfrak{F}_{\mathbf{k}}(\eta)[k\hat{v}_{\mathbf{k}}^s, [k\hat{v}_{\mathbf{k}}^s, \hat{\rho}_{\mathbf{k},s}]], \quad (24)$$

where

$$\mathfrak{F}_{\mathbf{k}}(\eta) = \frac{1}{2} \left(\frac{k_{\Gamma}}{k} \right)^2 \left(\frac{a}{a_{\star}} \right)^{p-3} \Theta \left(1 - \frac{k\ell_E}{a} \right); \quad (25)$$

Θ is the Heaviside step function; and $k_{\Gamma}(>0)$, p , $a_{\star}$, and $\ell_E(>0)$ are real parameters. Such a Lindblad equation, with a coupling linear in the system degree of freedom, is close to that obtained in the Caldeira-Leggett model [61]. The first term corresponds to the free Hamiltonian dynamics given by the Liouville–von Neumann equation (14). The second term, which is not of Hamiltonian-type and

leads to nonunitary evolution, encodes the action of the environment on the system via the decoherence kernel $\mathfrak{F}_k$ [62], where k_Γ sets the overall amplitude of the kernel. Its time dependence is given by a power law in the scale factor, parametrized by p , with reference scale $a_\star$. The action of the environment becomes efficient when the physical wavelength ak^{-1} becomes larger than $\ell_E(>0)$, the physical autocorrelation length of the environment; then, $k\ell_E/a < 1$, and the Heaviside function is 1. For practical purposes, it is useful to consider the evolution of the expectation value of an operator $\hat{O}$ acting on $H_{\pm k,s}$. Using $\langle \hat{O} \rangle(\eta) = \text{Tr}[\hat{\rho}_{k,s}(\eta)\hat{O}]$, we obtain

$$\frac{\mathcal{V}}{(2\pi)^3} \frac{d\langle \hat{O} \rangle}{d\eta} = -i \left\langle \left[\hat{O}, \hat{\mathcal{H}}_k^s \right] \right\rangle - \mathfrak{F}_k(\eta) \left\langle \left[\left[\hat{O}, k\hat{v}_k^s \right], k\hat{v}_k^s \right] \right\rangle. \quad (26)$$

In Sec. III, we solve the dynamics for Gaussian states by deriving transport equations for 2-point functions using Eq. (26).

Finally, let us briefly comment on the form of the kernel $\mathfrak{F}_k$ given in Eq. (25) and on our use of Eq. (24). First, the functional form of $\mathfrak{F}_k$ in Eq. (25) is a generalization of the form obtained when deriving the master equation form (24) in Ref. [53] from the interaction of a light scalar field (system) with a heavy scalar field (environment), where $p = 5$ in de Sitter. The same form, with $p = 7/2$, is also obtained in Ref. [43], where the authors considered the decoherence of super-Hubble modes by sub-Hubble modes during inflation generated by the second-order dynamics of cosmological perturbations. Since in this paper we only perform an analysis of pairs of opposite momenta modes $\pm k$ separately, we will not need to discuss the k dependence of k_Γ/k . Yet, when a specific microphysical model of interaction is chosen, this dependence is fixed, and, by requiring that the curvature power spectrum, i.e., the two-point function of $\zeta = v/z$, remains compatible with its observed scale-invariance [63], we can constrain model parameters, as done in Ref. [53]. Second, in Appendix A, we sketch a derivation of Eq. (24) starting from an explicit system-environment interaction Hamiltonian. While it is neither complete nor the only possible one (see, for instance, Refs. [43,53] for other derivations), it allows one to emphasize that several assumptions must be satisfied for the system's dynamics to reduce to this form of master equation. Typically, we have to require that the system-environment interaction is perturbative in some sense, that the state of the environment is not affected by its interaction with the system, and that the autocorrelation time of the relevant environment quantities is much shorter than the typical evolution timescale of the dynamics. Imposing these assumptions implies that the range of parameters x_E , k_Γ/k , and p for which Eq. (24) is valid is in practice limited. However, the precise relation between these

effective parameters and the microphysical interaction Hamiltonian can only be specified once the model is fixed. Since in this work we want to be agnostic about which environment and coupling is considered to arrive at Eq. (24), we will solve this master equation as if it were exact. Applying our analyses to a specific model then consists in evaluating the allowed parameter space for this model.

III. SQUEEZING PARAMETERS IN PRESENCE OF DECOHERENCE

The form of the Lindblad equation (24) allows us to still separate the state into single-mode density matrices $\hat{\rho}_{k,s}$ in the R/I partition and to get a reduced dynamics which is quadratic in $\hat{v}_k^s$. The equation therefore admits Gaussian-state solutions. Assuming a Bunch-Davies initial state, the quantum state can still be fully described by covariance matrices of the same form as Eq. (59) and parametrized by its purity and squeezing parameters, as in Eq. (B17). In this section, we discuss how these parameters are affected by the presence of interactions modeled by Eq. (25). In particular, we study numerically the case of a de Sitter background.

A. Transport equations in the Caldeira-Leggett model

First, we derive evolution equations for the covariance-matrix elements using Eq. (26). Note that all volume factors will cancel out. As an illustration, to obtain the evolution equation for γ_{22} , we apply the equation to $\hat{O} = \hat{p}_k^s \hat{p}_{k'}^{s'}$. Then, on the right-hand side, commuting operators give rise to a product of a volume factor and Dirac delta function. For instance, we have

$$\left\langle \left[\hat{p}_k^s \hat{p}_{k'}^{s'}, \hat{\mathcal{H}}_k^s \right] \right\rangle = -i\gamma_{12}\delta_{s,s'}\delta(k-k')\frac{\mathcal{V}}{(2\pi)^3}. \quad (27)$$

On the left-hand side, the expectation value appearing in $\langle \hat{p}_k^s \hat{p}_{k'}^{s'} \rangle$ leads to a Dirac delta function in addition to the already present volume factor. These volume factors and Dirac deltas then factor out. The final equations, already derived in Refs. [28,43], read

$$k^{-1} \frac{\partial \gamma_{11}(k, \eta)}{\partial \eta} = 2\gamma_{12}(k, \eta), \quad (28a)$$

$$k^{-1} \frac{\partial \gamma_{12}(k, \eta)}{\partial \eta} = \gamma_{22}(k, \eta) - k^{-2}\omega^2(k, \eta)\gamma_{11}(k, \eta), \quad (28b)$$

$$k^{-1} \frac{\partial \gamma_{22}(k, \eta)}{\partial \eta} = -2k^{-2}\omega^2(k, \eta)\gamma_{12}(k, \eta) + 4\mathfrak{F}_k(\eta). \quad (28c)$$

The purity p_k is a key quantity for describing the efficiency of the decoherence process. It is thus useful to derive a transport equation for it. Using Eq. (19) together

with the differential equations (28), we get

$$k^{-1} \frac{\partial(p_k^{-1})}{\partial\eta} = 4\mathfrak{F}_k(\eta)\gamma_{11}. \quad (29)$$

B. Covariance matrix and effective squeezing parameters in presence of decoherence

Equations (28) can be solved exactly, and their solutions were obtained in Ref. [28]. They read

$$\gamma_{11}(k, \eta) = |u_k(\eta)|^2 + \mathcal{I}_k(\eta), \quad (30a)$$

$$\gamma_{12}(k, \eta) = k^{-1} \text{Re}[\partial_\eta u_k(\eta) u_k^*(\eta)] + \mathcal{J}_k(\eta), \quad (30b)$$

$$\gamma_{22}(k, \eta) = k^{-2} |\partial_\eta u_k(\eta)|^2 + \mathcal{K}_k(\eta), \quad (30c)$$

with

$$\mathcal{I}_k(\eta) = 4k \int_{\eta_{\text{in}}}^{\eta} \mathfrak{F}_k(\eta') \text{Im}^2[u_k(\eta') u_k^*(\eta)] d\eta', \quad (31a)$$

$$\mathcal{J}_k(\eta) = 4 \int_{\eta_{\text{in}}}^{\eta} \mathfrak{F}_k(\eta') \text{Im}[u_k(\eta') u_k^*(\eta)] \text{Im}[u_k(\eta') \partial_\eta u_k^*(\eta)] d\eta', \quad (31b)$$

$$\mathcal{K}_k(\eta) = \frac{4}{k} \int_{\eta_{\text{in}}}^{\eta} \mathfrak{F}_k(\eta') \text{Im}^2[u_k(\eta') \partial_\eta u_k^*(\eta)] d\eta'. \quad (31c)$$

Here, u_k is a solution of the Mukhanov-Sasaki equation

$$\frac{\partial^2 u_k}{\partial\eta^2} + \omega^2(k, \eta) u_k = 0, \quad (32)$$

with Wronskian normalized to $W = u_k(\partial_\eta u_k^*) - (\partial_\eta u_k) u_k^* = 2ik$. Note that W is constant for any solution, and with this normalization, u_k is dimensionless.

Since the R/I modes are both in one-mode squeezed thermal states with identical parameters, it is convenient to describe the evolution using effective squeezing parameters. These parameters can be computed in terms of the covariance matrix elements γ_{ij} using Eq. (18), and then as a function time using Eq. (30). However, a problem already encountered in Ref. [28] is that, at late times, it is not numerically viable to compute the purity by evaluating the integrals appearing in the covariance-matrix elements in Eqs. (30a)–(30c) and inserting the result into Eq. (19). The final step involves exact cancellations between exponentially large numbers, which fail numerically. We therefore derive an exact expression for the purity p_k that allows a reliable estimate of its late-time behavior. There are two ways to obtain such an expression. One can either insert the exact expression (30a) for γ_{11} into the transport equation (29) and integrate it or perform algebraic manipulations starting from Eq. (19) and insert the exact solutions (30a)–(30c). Both approaches are detailed in Appendix B 2 and lead to

$$p_k^{-1} = 1 + 2\mathcal{L}_k + \mathcal{L}_k^2 - |\mathcal{M}_k|^2, \quad (33)$$

where

$$\mathcal{L}_k = 2k \int_{\eta_{\text{in}}}^{\eta} \mathfrak{F}_k(\eta') |u_k(\eta')|^2 d\eta', \quad (34a)$$

$$\mathcal{M}_k = 2k \int_{\eta_{\text{in}}}^{\eta} \mathfrak{F}_k(\eta') u_k^2(\eta') d\eta'. \quad (34b)$$

Using $\mathcal{L}_k \geq 0$ and the Cauchy-Schwarz inequality $\mathcal{L}_k^2 \geq |\mathcal{M}_k|^2$, one can check that the expression (33) always respects the constraint $0 \leq p_k \leq 1$ on the purity. Finally, we can invert Eq. (18) to obtain expressions for the squeezing parameter r_k and the squeezing angle φ_k in terms of the covariance-matrix elements. The squeezing parameter r_k is given by

$$r_k = \frac{1}{2} \cosh^{-1} \left(\frac{\gamma_{11} + \gamma_{22}}{2p_k^{-1/2}} \right) = \frac{1}{2} \ln \left[\frac{\gamma_{11} + \gamma_{22}}{2p_k^{-1/2}} + \sqrt{\left(\frac{\gamma_{11} + \gamma_{22}}{2p_k^{-1/2}} \right)^2 - 1} \right]. \quad (35)$$

For the squeezing angle φ_k , we can derive expressions for both $\sin(2\varphi_k)$, $\cos(2\varphi_k)$, and $\tan(2\varphi_k)$, which read

$$\sin(2\varphi_k) = -\frac{\gamma_{12}}{\sqrt{\left(\frac{\gamma_{22}-\gamma_{11}}{2}\right)^2 + \gamma_{12}^2}}, \quad (36a)$$

$$\cos(2\varphi_k) = \frac{\frac{\gamma_{22}-\gamma_{11}}{2}}{\sqrt{\left(\frac{\gamma_{22}-\gamma_{11}}{2}\right)^2 + \gamma_{12}^2}}, \quad (36b)$$

$$\tan(2\varphi_k) = -\frac{2\gamma_{12}}{\gamma_{22} - \gamma_{11}}. \quad (36c)$$

The nonlinearity of the relation between the squeezing parameters and the covariance-matrix elements means that, for these parameters, there is no compact exact formula analogous to Eq. (33) for the purity. The squeezing angle encodes the direction of largest fluctuations, and Eq. (36) make the limiting cases transparent. If the fluctuations of $\hat{\delta}_k^s$ are much larger than those of $\hat{p}_k^s$, i.e., $\gamma_{11} \gg \gamma_{22} \geq 0$, then

since $p_k = \gamma_{22}\gamma_{11} - \gamma_{12}^2 \geq 1$, we have $\gamma_{11} \gg |\gamma_{12}|$. Inserting these relations in Eq. (36), we find $\sin(2\varphi_k) \approx 0$ and $\cos(2\varphi_k) \approx -1$, so $\varphi_k \approx \pi/2$ (if we restrict to $[0, 2\pi]$). The squeezing ellipse is thus close to horizontal in Fig. 1, and this gives a graphical representation of the dominance of the fluctuations of $\hat{p}_k^s$. The same reasoning applies when the fluctuations of $\hat{p}_k^s$ are much larger than those of $\hat{v}_k^s$, in which case the ellipse is close to vertical. Note that, since $\gamma_{22}\gamma_{11} \geq \gamma_{12}^2 - 1$, we can never have $|\gamma_{12}| \gg \gamma_{11}, \gamma_{22}$. In the next section, we derive approximations for these quantities in the late-time limit, which is the regime relevant for inflation.

C. Application: Decoherence in a de Sitter phase and late-time approximations

To numerically evaluate the effective squeezing parameters of the cosmological perturbations, we now focus on the de Sitter case, where $H = \text{constant}$ and $a(\eta) = -1/(H\eta)$. Then, ω is given by

$$\frac{\omega^2(k, \eta)}{k^2} = 1 - \frac{2}{x^2}, \quad (37)$$

where we have defined a dimensionless time variable $x = -k\eta (> 0)$, which implicitly depends on our choice of scale k . In addition, given the form of the kernel (25), the perturbations freely evolve until $x_E = (H\ell_E)^{-1}$, such that $a(x_E) = k\ell_E$. The Heaviside function in Eq. (25) therefore reads $\Theta(1 - \frac{k\ell_E}{a}) = \Theta(1 - \frac{x}{x_E})$. We pick η_{in} to be earlier than this time so that the starting time of the integral is always set by x_E . Furthermore, since the perturbations are initially freely evolving, we can work with the standard assumption that they start in the Bunch-Davies vacuum, so that $u_k \underset{x \rightarrow \infty}{\sim} e^{-ix}$. Imposing this initial condition and the Wronskian normalization condition given below Eq. (32) on the solution of the Mukhanov-Sasaki equation, we obtain

$$u_k = \left(1 + \frac{i}{x}\right) e^{ix}. \quad (38)$$

Finally, we also set the reference time in the kernel to $a_\star = a_k = kH^{-1}$, the value of the scale factor when the

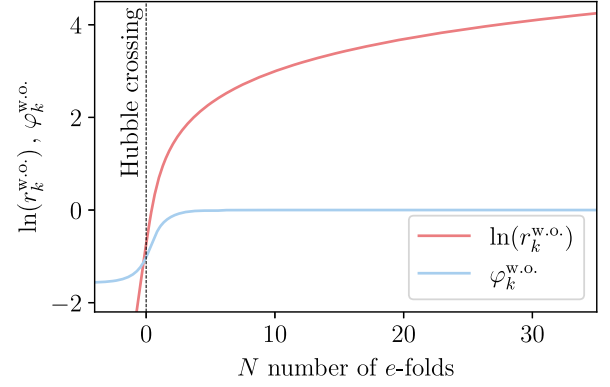


FIG. 2. Evolution of the squeezing parameters as a function of the number of e -folds in de Sitter in the absence of interactions. The red line shows the values of $\ln(r_k^{\text{w.o.}})$ computed using Eq. (40). The blue line shows the values of $\varphi_k^{\text{w.o.}}$ computed using Eq. (41).

modes of interest $\pm k$ cross the Hubble radius. This can always be achieved by a k -dependent redefinition of k_Γ .

1. Pure-state case

For reference, let us state the expressions of the covariance-matrix elements and the squeezing parameters in the absence of decoherence. Our goal is to eventually evaluate quantities at the end of inflation, for which a customary time variable is the number of e -folds $N = \ln[a(\eta)/a_k]$. For de Sitter space, we then have $x = e^{-N}$, and cosmological modes are expected to have undergone at least $N = 60$ e -folds of inflation. Thus, we are required to evaluate quantities for exponentially small x , such that $x = e^{-N} \leq e^{-60} \ll 1$. Therefore, it is useful to give late-time expansions of the pure-state solutions. We have

$$\gamma_{11}^{\text{w.o.}}(\eta) = 1 + \frac{1}{x^2} \underset{x \rightarrow 0}{=} \frac{1}{x^2} + \mathcal{O}(1), \quad (39a)$$

$$\gamma_{12}^{\text{w.o.}}(\eta) = \frac{1}{x^3} \underset{x \rightarrow 0}{=} \frac{1}{x^3}, \quad (39b)$$

$$\gamma_{22}^{\text{w.o.}}(\eta) = 1 - \frac{1}{x^2} + \frac{1}{x^4} \underset{x \rightarrow 0}{=} \frac{1}{x^4} + \mathcal{O}(x^{-2}), \quad (39c)$$

and the associated squeezing parameters are

$$r_k^{\text{w.o.}}(\eta) = -2 \ln x + \frac{1}{2} \ln \left(\frac{1 + \sqrt{1 + 4x^4}}{2} + x^4 \right) \underset{x \rightarrow 0}{=} -2 \ln x + \mathcal{O}(x^4), \quad (40)$$

$$\varphi_k^{\text{w.o.}}(\eta) = -\frac{1}{2} \arctan \left(\frac{2x}{1 - 2x^2} \right) - \frac{\pi}{2} \Theta \left(-k\eta - \frac{1}{\sqrt{2}} \right) \underset{x \rightarrow 0}{=} -x + \mathcal{O}(x^3), \quad (41)$$

where a superscript “w.o.” indicates that the quantity is evaluated in the noninteracting case. Their evolution is shown in Fig. 2.

2. Covariance matrix elements and expansion strategy

We want to evaluate the effective squeezing parameters r_k , φ_k and the purity p_k in the presence of interactions modeled by Eqs. (24) and (25). Since, as explained at the end of Sec. II B 1, in this paper we treat the master equation Eq. (24) as exact, we should explore the whole parameter space of p , x_E , and k_Γ/k and thus be careful that our expansions are valid in general, in particular for both small and large k_Γ/k . All quantities pertaining to the state of the perturbations can be expressed in terms of the covariance matrix elements γ_{ij} . From Eq. (30), these elements are of the form $\gamma_{ij} = \gamma_{ij}^{\text{w.o.}} + (k_\Gamma/k)^2 \alpha_{ij}(x; p, x_E)$, where α_{ij} is given by an integral. Eventually, we want to evaluate quantities, in particular the covariance matrix elements,

at late times, so at exponentially small values of x . This is numerically demanding for at least two reasons. First, computing the γ_{ij} themselves requires integrating oscillatory functions. Second, numerically computing the purity, a key quantity for quantum correlations, involves the near-perfect cancellation of exponentially large terms; see Ref. [28]. We thus have to derive numerically efficient late-time approximations for relevant quantities, starting with the covariance matrix elements γ_{ij} . We proceed by expanding them as power series of $x \ll 1$ and keep both the dominant terms coming from $\gamma_{ij}^{\text{w.o.}}$ and $\alpha_{ij}(x; p, x_E)$. We give detailed computations and reproduce some results of Ref. [28] in Appendix B. First, for γ_{ij} , we have [28]

$$\gamma_{11} = \frac{1}{x^2} \left\{ 1 + \mathcal{O}(x^2) - 2 \left(\frac{k_\Gamma}{k} \right)^2 [B_{11} + \mathcal{O}(x^2) + A_{11}x^{8-p} + \mathcal{O}(x^{10-p})] \right\}, \quad (42a)$$

$$\gamma_{12} = \frac{1}{x^3} \left\{ 1 - 2 \left(\frac{k_\Gamma}{k} \right)^2 [B_{12} + \mathcal{O}(x^2) + A_{12}x^{8-p} + \mathcal{O}(x^{10-p})] \right\}, \quad (42b)$$

$$\gamma_{22} = \frac{1}{x^4} \left\{ 1 + \mathcal{O}(x^2) - 2 \left(\frac{k_\Gamma}{k} \right)^2 [B_{22} + \mathcal{O}(x^2) + A_{22}x^{8-p} + \mathcal{O}(x^{10-p})] \right\}, \quad (42c)$$

with expressions for the p - and x_E -dependent coefficients A_{ij} ($i, j = 1, 2$) and $B_{11} = B_{12} = B_{22} < 0$ when $p < 8$ given in Eqs. (B5)–(B10). One can check that Eq. (42) contain the asymptotically dominant term in the limit $x \rightarrow 0$, for all values of p and for both terms in $(k_\Gamma/k)^0$ and $(k_\Gamma/k)^2$. For instance, in the term in $(k_\Gamma/k)^2$, for $p < 8$, the term in B_{ij} dominates at $x \rightarrow 0$, while for $p > 8$, it is that in $A_{ij}x^{8-p}$.

In practice, we want to use these asymptotic expansions to approximate γ_{ij} at late but finite time $x \ll 1$, typically after 30 or 60 e -folds of inflation. First, we should thus check that the above expansions do capture the largest contribution to γ_{ij} for the values of x , p , x_E , and k_Γ/k we consider in this work. To do so, we should show that none of the contributions with larger powers of x , included as $\mathcal{O}$ or neglected, can dominate over the terms we have explicitly included. Such a situation could only occur if the included terms nearly cancelled out to give a contribution of the same magnitude as the neglected ones. In Appendix B 4, we show that, in principle, it could happen at transition regimes such that $k_\Gamma/k \approx 1$, or $p \approx 8$ so that $x^{8-p} \approx x^0 \approx 1$, and yet that, in practice, for the values of p , x_E , and k_Γ/k considered, it does not happen. Second, we emphasize that for numerical evaluation in a large range of parameters p , x_E , and k_Γ/k , one should use expressions such as Eq. (42) where the possible dominant terms in x are kept for *all orders* in k_Γ/k , rather than only keeping the asymptotically dominant one, such as in Eq. (44) below.

To illustrate this point, let us consider the expression for γ_{11} . For simplicity, we will assume from now on that the factors A_{ij} , B_{ij} , and similar ones below, are of order unity. We thus effectively do not consider the effect of x_E on the accuracy of our approximations and focus on the k_Γ/k dependence. For $p > 8$, the asymptotically dominant term in Eq. (42a) is in x^{8-p} , and the first subdominant contribution is in x^0 . Their ratio is given by

$$\frac{\text{subdominant term in } x^0}{\text{dominant term in } x^{8-p}} = \frac{-\frac{1}{2} \left(\frac{k_\Gamma}{k} \right)^{-2} + B_{11}}{A_{11}} x^{p-8}.$$

While going to 0 when $x \rightarrow 0$, for finite late-time $x \ll 1$, this ratio is large for a very suppressed interaction $k_\Gamma/k \ll 1$ and A_{11} , B_{11} of order unity. Consider, for instance, the following point in parameter space covered in the color maps below such as in Fig. 3: we evaluate the perturbations 60 e -folds after Hubble crossing, i.e., $x = e^{-60}$, for an interaction with $p = 8.1$ and $k_\Gamma/k = 10^{-3}$. Then, $\left(\frac{k_\Gamma}{k} \right)^{-2} x^{p-8} \simeq 10^3$; thus, the ratio is large, and dropping the first term ($= 1$) to only keep the asymptotically dominant term in x in the expression of Eq. (42a) would give a poor numerical approximation of γ_{11} despite having $x \ll 1$. As a result, for numerical evaluations, we will rely on expressions containing all possible dominant terms in x for all orders in k_Γ/k , such as Eq. (45) below, rather than

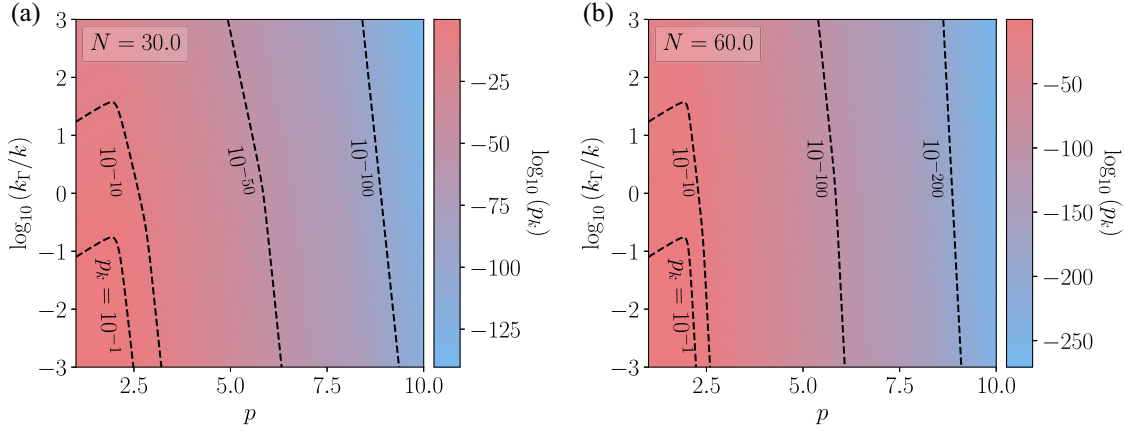


FIG. 3. Values of the purity p_k as a function of p and $\log_{10}(k_\Gamma/k)$ after $N = 30.0$ (a) and $N = 60$ (b) e -folds of inflation. p_k is approximated by the expression (43) where all $\mathcal{O}$ are set to 0.

those keeping only the asymptotically dominant one for a range of parameters, such as Eq. (46) below.

3. Purity

We now derive a late-time approximation similar to Eq. (42) for the purity p_k . One way to derive such an

approximation is to insert the expansions Eq. (42) into Eq. (19). While one can obtain the leading-order term in this way, it does not provide an optimal estimate of the order of the next contribution. To estimate this contribution, we use the exact expression (33) and expand the integrals $\mathcal{L}_k$ and $\mathcal{M}_k$ appearing there [64]. We get

$$p_k^{-1} = 1 + \left(\frac{k_\Gamma}{k}\right)^2 \left[A_\sigma^{(1)} + B_\sigma^{(1)} x^{2-p} + \mathcal{O}(x^{4-p}) \right] + \left(\frac{k_\Gamma}{k}\right)^4 \left[A_\sigma^{(2)} + B_\sigma^{(2)} x^{2-p} + \mathcal{O}(x^{4-p}) + D_\sigma^{(2)}(p, x_E) x^{10-2p} + \mathcal{O}(x^{12-2p}) \right], \quad (43)$$

with the expressions of the coefficients given in Eqs. (B32)–(B37). Equation (43) differs from Eq. (5.17) for the same quantity derived in Ref. [28] [called $\sigma^2(0)$ there] by the inclusion of the contributions in $(k_\Gamma/k)^4$. These contributions can dominate either when $k_\Gamma/k \gg 1$, a situation not considered in Ref. [28], or for $k_\Gamma/k \leq 1$ and $p > 8$ at late enough time $x \ll 1$ so that $(k_\Gamma/k)^4 x^{10-2p} \gg (k_\Gamma/k)^2 x^{2-p}$, a case not excluded by Ref. [28]. Equation (43) thus corrects Eq. (5.17) of

Ref. [28] for $k_\Gamma/k \leq 1$ and $p > 8$ and extends it to arbitrary values of k_Γ/k . Note that, on the other hand, the covariance matrix elements only receive contributions linear in $\mathfrak{F}_k$, so their expressions are strictly of order $(k_\Gamma/k)^2$ in the parametrization (25) and all terms were already included in Ref. [28].

Next, to get a more explicit formula, we distinguish cases on the value of p and select the asymptotically dominant term in each case. We get

$$\begin{aligned} p_k^{-1} &= 1 + \left(\frac{k_\Gamma}{k}\right)^2 A_\sigma^{(1)} + \left(\frac{k_\Gamma}{k}\right)^4 A_\sigma^{(2)} + \mathcal{O}(x^{2-p}) && \text{for } p < 2, \\ &= \left[\left(\frac{k_\Gamma}{k}\right)^2 B_\sigma^{(1)} + \left(\frac{k_\Gamma}{k}\right)^4 B_\sigma^{(2)} \right] x^{2-p} + \mathcal{O}(1) + \mathcal{O}(x^{4-p}) + \mathcal{O}(x^{10-2p}) && \text{for } 2 < p < 8, \\ &= \left(\frac{k_\Gamma}{k}\right)^4 D_\sigma^{(2)}(p, x_E) x^{10-2p} + \mathcal{O}(x^{2-p}) + \mathcal{O}(x^{12-2p}) && \text{for } p > 8. \end{aligned} \quad (44)$$

As detailed in Sec. III C 2, the above expression is not suitable to numerically evaluate p_k at finite time x for generic values of k_Γ/k because the asymptotically subdominant term in $(k_\Gamma/k)^2 x^{2-p}$ for $p < 8$ can still be larger than the dominant one in $(k_\Gamma/k)^4 x^{10-2p}$ for small values of

k_Γ/k . Instead, we should use the less explicit Eq. (43) and simultaneously evaluate the dominant term for each term in k_Γ/k . In Appendix B 4, we checked that Eq. (43) does provide a good approximation of the purity in the studied parameter range. In Fig. 3, we plot the value of the purity p_k

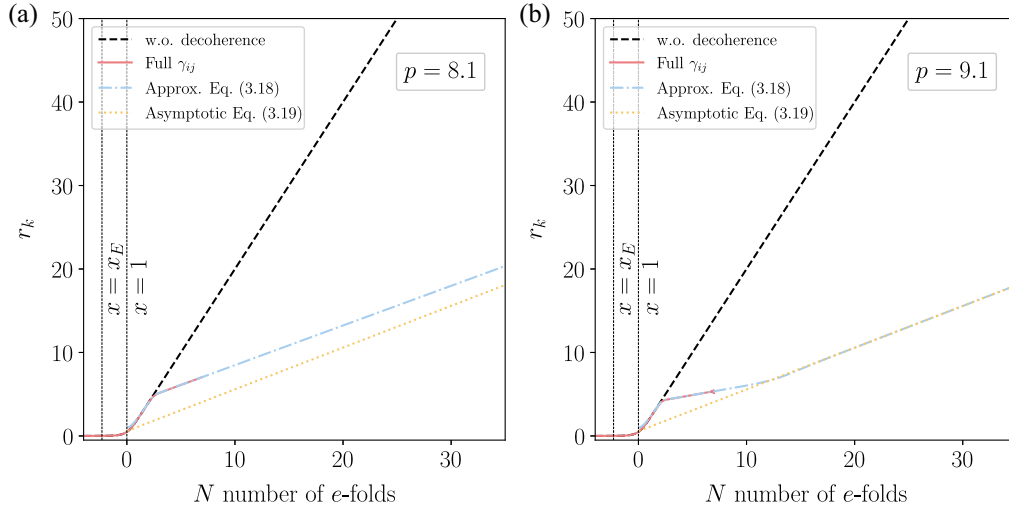


FIG. 4. Evolution of the squeezing parameter r_k as a function of the number of e -folds in the presence of interactions for $k_\Gamma/k = 10^{-3}$, $p = 8.1$ [panel (a)], and $p = 9.1$ [panel (b)]. The full red line is obtained using Eq. (35) with the covariance-matrix elements γ_{ij} computed with Eq. (30) where the integrals of Eq. (31) are numerically evaluated. We stop plotting this line around $N = 7$ e -folds when it starts to show numerical errors for $p = 9.1$. The full light blue line is computed using the late-time approximation (45) of r_k with the covariance-matrix elements evaluated from Eq. (42). The yellow dotted line is computed using Eq. (46). As explained at the end of Sec. III C 2, it performs poorly when p is close to the threshold value $p = 8$ and the interaction suppressed.

as a function of p and k_Γ/k after $N = 30$ and $N = 60$ e -folds from the horizon exit of the pivot scale, i.e., $x = e^{-30}$ and $x = e^{-60}$. We picked an arbitrary value $x_E = 10$ for the coherence length, but changing its value is straightforward as it only fixes the starting point of decoherence. Figure 3 shows that, even for very small values of k_Γ/k , when $p > 2$, the purity will eventually be strongly suppressed.

4. Squeezing parameter

The smallness of the purity at the end of inflation is generally interpreted as the cosmological perturbations having classicalized. Yet, as we discuss in Sec. IV below, the purity alone is not sufficient to correctly assess the level of quantum correlations in the state. One also needs to compute the values of the other squeezing parameters in the presence of decoherence, which we now proceed to do, starting with the squeezing parameter r_k . We first derive an approximation of r_k in the presence of decoherence using only the fact that, for $x \ll 1$, the fluctuations along $\hat{p}_k^s$ are dominant, i.e., $\gamma_{12}/\gamma_{22} \ll 1$ and $\gamma_{11}/\gamma_{22} \ll 1$. We then rearrange terms in Eq. (35) to isolate the asymptotically dominant contribution. The details of the expansion are given in Appendix B 5, and we obtain

$$r_k = \frac{1}{2} \ln(p_k^{1/2} \gamma_{22}) + \frac{1}{2} \left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 \frac{\gamma_{11}}{\gamma_{22}} \right] + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^4 \right]. \quad (45)$$

First note that this equation does not require the covariance matrix elements to be of the specific form derived in Eq. (30) and that a similar formula can be derived if the fluctuations of $\hat{v}_k^s$ rather than $\hat{p}_k^s$ dominate, i.e., $\gamma_{11} \gg \gamma_{22} \gg \gamma_{12}$ instead of $\gamma_{22} \gg \gamma_{11}$. Second, we show in Appendix B 4 that the ratios of covariance matrix elements appearing in Eq. (45) are indeed small for the form of kernel (25) for the values of parameters considered in this paper. More precisely, for $x < e^{-30}$ and $(\log_{10}(k_\Gamma/k), p) \in [-3, 3] \times [2, 10]$, we find $\gamma_{12}/\gamma_{22} \approx x$ and $\gamma_{11}/\gamma_{22} \approx x^2$. Moreover, we also check that in this range parameter we have $p_k^{1/2} \gamma_{22} \geq x^{-1}$. This term is thus the largest contribution to Eq. (45). Since it is also growing with time, Eq. (45) tells us that squeezing always grows despite the presence of decoherence.

To gain more insight into the behavior of the squeezing parameter in the presence of decoherence, we plot in Fig. 4 the expression of r_k as a function of time for a couple of parameter values. We confirm that r_k , although reduced compared to its pure-state value $r_k^{\text{w.o.}}$, still diverges asymptotically: the perturbations remain very squeezed even in the presence of interactions. For a systematic confirmation of this fact and a quantitative estimate of the squeezing reduction caused by interactions, we use the expansion (42c) for γ_{22} and expansion (43) for p_k to expand the first term and obtain a more explicit formula. Which of the terms in the expression dominates for $x \rightarrow 0$ then depends on p . We have

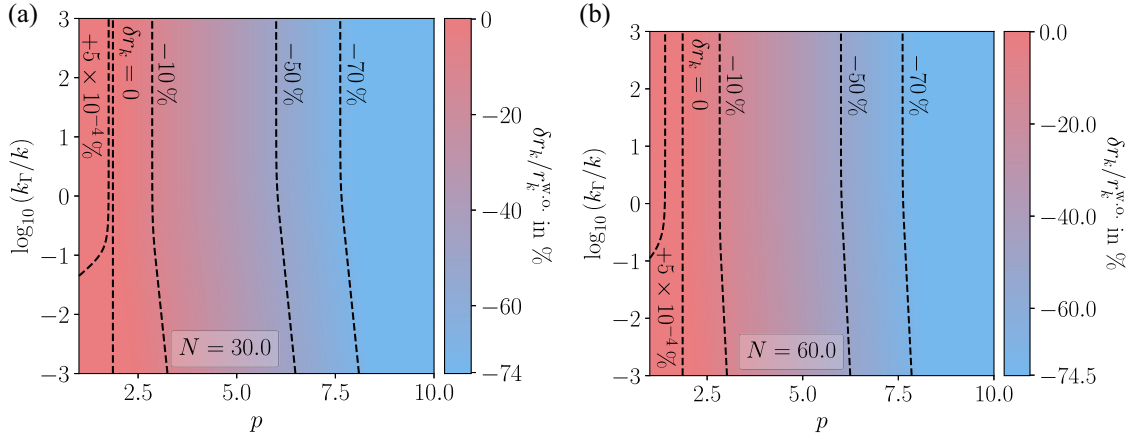


FIG. 5. Values of $\delta r_k / r_k^{w.o.} = r_k / r_k^{w.o.} - 1$ in percentage as a function of p and $\log_{10}(k_\Gamma/k)$ for $N = 30.0$ (a) and $N = 60$ (b) e -folds. $\delta r_k = r_k - r_k^{w.o.}$ is computed using Eq. (45) for r_k and numerically evaluating the covariance matrix elements using their exact expressions Eq. (42).

$$\begin{aligned}
 r_k &= -2 \ln(x) + \frac{1}{2} \ln \left[\frac{1 - 2 \left(\frac{k_\Gamma}{k} \right)^2 B_{22}(p, x_E)}{\sqrt{1 + A_\sigma(p, x_E)}} \right] + \mathcal{O}(x^2) & \text{for } p < 2, \\
 &= -\frac{1}{4} (10 - p) \ln(x) + \frac{1}{2} \ln \left[\frac{1 - 2 \left(\frac{k_\Gamma}{k} \right)^2 B_{22}(p, x_E)}{\sqrt{B_\sigma(p, x_E)}} \right] + \mathcal{O}(x^2) & \text{for } 2 < p < 8, \\
 &= -\frac{1}{2} \ln(x) + \frac{1}{2} \ln \left[\frac{-2 \left(\frac{k_\Gamma}{k} \right)^2 A_{22}(p)}{\sqrt{D_\sigma(p, x_E)}} \right] + \mathcal{O}(x^2) & \text{for } p > 8.
 \end{aligned} \tag{46}$$

We checked numerically that all terms appearing inside the logarithms and square roots are positive in the parameter ranges where they are evaluated. Note that for $p > 8$ the change in the leading-order power of the γ_{ij} compensates that in the leading power of p_k in such a way that the leading power in r_k no longer depends on p . Equation (46) confirms the result of Eq. (43) that the perturbations are always squeezed since asymptotically $r_k \rightarrow +\infty$ and gives an explicit estimate of that squeezing. Moreover, in Fig. 4, we compare the value of the squeezing given by the two equations, Eq. (45) and this rougher approximation (46), to illustrate the failure of the latter. In the panel (a) with $p = 8.1$ and $k_\Gamma/k = 10^{-3}$, Eq. (46) gives a poor approximation of r_k , which confirms the situation described after Eq. (42); when the interaction is sufficiently suppressed by $k_\Gamma/k \ll 1$, the asymptotically dominant term for $p = 8 + \epsilon > 8$ is suppressed compared to subdominant ones even for $x > e^{-30}$. On the other hand, the approximation (45) always matches the results obtained by numerically evaluating

r_k using the exact form (35) and the exact covariance matrix elements (30).

Next, we isolate the effect of the interaction on the squeezing parameter r_k by computing the difference with respect to the pure-state case (i.e., without decoherence), $\delta r_k = r_k - r_k^{w.o.}$, where the pure-state expression $r_k^{w.o.}$ is given in Eq. (40). In Fig. 5, we plot this quantity normalized by the pure-state value $r_k^{w.o.}$ as a function of p and k_Γ/k . The squeezing parameter in the presence of decoherence r_k is evaluated using the approximation (45) where the covariance matrix elements γ_{ij} and their ratios are estimated using Eq. (42). First, Fig. 5 shows that $\delta r_k < 0$ except for the region around $p < 2$; in this model, perturbations are typically desqueezed by interactions. Second, both panels look very similar so that in percent the degree of this desqueezing appears to reach a constant final value. To gain an analytical understanding of these behaviors, we use the expression of r_k in terms of x given in Eq. (46). We find

$$\begin{aligned}
 \delta r_k &= \frac{1}{2} \ln \left[\frac{1 - 2 \left(\frac{k_\Gamma}{k} \right)^2 B_{22}(p, x_E)}{\sqrt{1 + A_\sigma(p, x_E)}} \right] + \mathcal{O}(x^2) & \text{for } p < 2, \\
 &= \frac{1}{4} (p - 2) \ln x + \frac{1}{2} \ln \left[\frac{1 - 2 \left(\frac{k_\Gamma}{k} \right)^2 B_{22}(p, x_E)}{\sqrt{B_\sigma(p, x_E)}} \right] + \mathcal{O}(x^2) & \text{for } 2 < p < 8, \\
 &= \frac{3}{2} \ln x + \frac{1}{2} \ln \left[\frac{-2 \left(\frac{k_\Gamma}{k} \right)^2 A_{22}(p)}{\sqrt{D_\sigma(p, x_E)}} \right] + \mathcal{O}(x^2) & \text{for } p > 8.
 \end{aligned} \tag{47}$$

First, this confirms that when $p > 2$ interactions asymptotically reduce the squeezing $\delta r_k < 0$. Second, when $p < 2$, the squeezing can be increased, but only by a bounded, time-independent amount which vanishes as $k_\Gamma/k \rightarrow 0$ and tends to a constant in the large k_Γ/k limit. Third, note that the expressions Eqs. (46) and (47) are

$$\begin{aligned} \frac{\delta r_k}{r_k^{\text{w.o.}}} &= -\frac{1}{4} \frac{\ln \left[\frac{1-2\left(\frac{k_\Gamma}{k}\right)^2 B_{22}(p, x_E)}{\sqrt{1+A_\sigma(p, x_E)}} \right]}{\ln x} + \mathcal{O}\left(\frac{x^2}{\ln x}\right) & \text{for } p < 2, \\ &= -\frac{1}{8}(p-2) - \frac{1}{4} \frac{\ln \left[\frac{1-2\left(\frac{k_\Gamma}{k}\right)^2 B_{22}(p, x_E)}{\sqrt{B_\sigma(p, x_E)}} \right]}{\ln x} + \mathcal{O}\left(\frac{x^2}{\ln x}\right) & \text{for } 2 < p < 8, \\ &= -\frac{3}{4} - \frac{1}{4} \frac{\ln \left[\frac{-2\left(\frac{k_\Gamma}{k}\right)^2 A_{22}(p)}{\sqrt{D_\sigma(p, x_E)}} \right]}{\ln x} + \mathcal{O}\left(\frac{x^2}{\ln x}\right) & \text{for } p > 8. \end{aligned} \quad (48)$$

These equations show that the level of desqueezing goes asymptotically to a constant between 0% and 75%. This is compatible with the numerical observations of Fig. 5, which shows both that for $p < 2$ the positive values of $\delta r_k/r_k^{\text{w.o.}}$ decrease when going from $N = 30$ to $N = 60$ e -folds and that the maximal amount of desqueezing reached at large p gets closer to 75%. Note that by definition desqueezing has to be less than 100% since a negative squeezing parameter only corresponds to an orthogonal change of the squeezing direction $\varphi_k \rightarrow \varphi_k + \pi/2$.

5. Squeezing angle

To compute the effect of decoherence on the squeezing angle φ_k , we start by deriving expansions for both $\sin(2\varphi_k)$ and $\cos(2\varphi_k)$. The details are given in Appendix B 6. First, following the same approach as used to derive Eq. (45), we expand Eq. (36) in terms of small ratios of covariance matrix elements

$$\cos(2\varphi_k) = 1 - 2\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2 + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2 \frac{\gamma_{11}}{\gamma_{22}}\right], \quad (49a)$$

$$\sin(2\varphi_k) = -2\frac{\gamma_{12}}{\gamma_{22}} \left\{ 1 + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right] + \mathcal{O}\left(\frac{\gamma_{11}}{\gamma_{22}}\right) \right\}. \quad (49b)$$

As for Eq. (45), Eq. (49) are valid beyond the specific form derived in Eq. (30) and only require one quadrature's fluctuations to dominate. On the other hand, we recall that we did check in Appendix B 4 that the ratios are small, $\gamma_{12}/\gamma_{22} \approx x$ and $\gamma_{11}/\gamma_{22} \approx x^2$, for matrix elements of the form Eq. (30) with kernel (25) and in the range of parameters $x < e^{-30}$ and $(\log_{10}(k_\Gamma/k), p) \in [-3, 3] \times [2, 10]$ considered in this paper. To obtain an explicit

continuous in p . Indeed, first, it is straightforward to check that the coefficient of $\ln x$ is continuous, and, in addition, the next-order term is also continuous by taking the limits $p \rightarrow 2$ and $p \rightarrow 8$ on both sides of the equality. Finally, we take the ratio with the pure-state value $r_k^{\text{w.o.}}$ and get

expansion in x , we then insert the expansions of γ_{ij} and distinguish between the values of p . We obtain

$$\cos(2\varphi_k) = 1 - 2x^2 \left[\Theta(8-p) + \frac{A_{12}^2}{A_{22}^2} \Theta(p-8) \right] + \mathcal{O}(x^4), \quad (50a)$$

$$\sin(2\varphi_k) = -2x \left[\Theta(8-p) + \frac{A_{12}}{A_{22}} \Theta(p-8) \right] + \mathcal{O}(x^3). \quad (50b)$$

One can check numerically that these expressions are continuous at $p = 8$. We have $A_{12}/A_{22} > 0$ for $p > 8$, so that, asymptotically, $\sin(2\varphi_k) < 0$ and $\cos(2\varphi_k) > 0$. Thus, $2\varphi_k \in [-\pi/2, 0]$, and so $2\varphi_k = \arctan[\tan(2\varphi_k)]$ without additional π terms. We obtain

$$\varphi_k = -x \left[\Theta(8-p) + \frac{A_{12}}{A_{22}} \Theta(p-8) \right] + \mathcal{O}(x^3). \quad (51)$$

To quantify the effect of decoherence on the squeezing angle, we compute the difference with the pure-state value $\delta\varphi_k = \varphi_k - \varphi_k^{\text{w.o.}}$. From Eq. (51), we immediately see that we always have $\delta\varphi_k \rightarrow 0$: decoherence, when modeled by Eq. (24), does not affect the asymptotic behavior of the squeezing angle φ_k . Therefore, a color map similar to Fig. 5 would not be very informative, and we instead plot the values of $\sin(2\varphi_k)$ and $\cos(2\varphi_k)$ as functions of time, labeled by the number of e -folds. In Fig. 6, we show their evolution for different values of p at fixed $k_\Gamma/k = 10$, and in Fig. 7, we show their evolution for different values of k_Γ/k at fixed p . In particular, Fig. 7(a), corresponding to $k_\Gamma/k = 10^{-6}$ and $p = 9.1$, demonstrates that the asymptotic forms Eq. (50) perform poorly for $p > 8$ and small k_Γ/k , while Eq. (49)

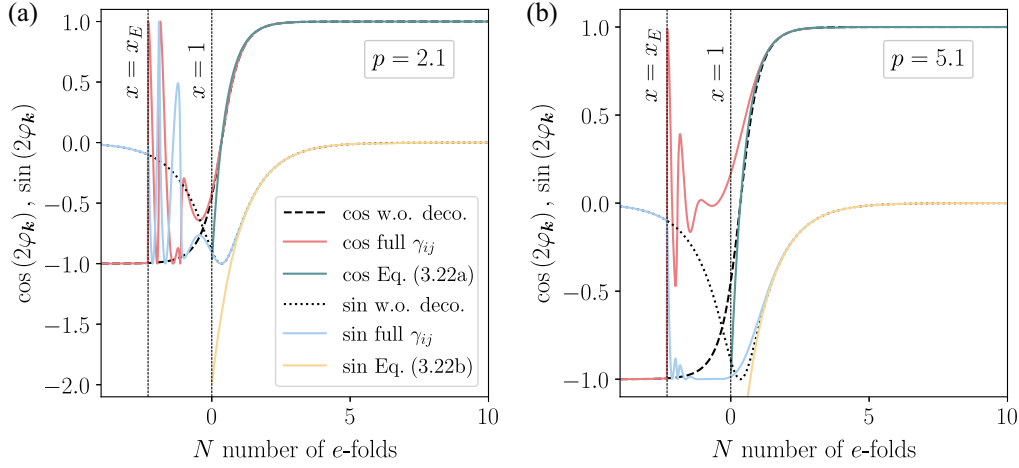


FIG. 6. Evolution of $\cos(2\varphi_k)$ and $\sin(2\varphi_k)$ in the presence of decoherence modelled by Eq. (24) as functions of N , the number of e -folds. The overall amplitude of the effect of the environment is fixed to $k_\Gamma/k = 10$ while its time dependence is varied by changing the exponent p : $p = 2.1$ (a) and $p = 5.1$ (b). The dashed (respectively, dotted) black line shows the values of $\cos$ (respectively, $\sin$) without decoherence. The red (respectively, blue) line represents the values of $\cos$ (respectively, $\sin$) in the presence of decoherence calculated using Eq. (24), where the covariance matrix elements are computed with Eq. (36) by numerically evaluating the integrals (31). In addition, the green (respectively, yellow) lines show the same quantities $\cos$ (respectively, $\sin$) evaluated using the asymptotic expressions of Eq. (50). The good agreement at late times with the exact expressions is a check that our asymptotic expressions are correct. These approximate expressions are only plotted after the Hubble-crossing time shown by a vertical dotted line at $x = 1$.

remain accurate. This illustrates again the situation described below Eq. (42), where, because the interaction is very suppressed, $k_\Gamma/k \ll 1$, the asymptotically dominant term is still smaller than a subdominant one which is not captured by Eq. (50).

Finally, Fig. 6 features some early oscillatory behavior between $x = x_E$ and $x = 1$. These oscillations arise from oscillations in the values of $(\gamma_{22} - \gamma_{11})/2$ and γ_{12} of comparable amplitude at these early times. We interpret these oscillations as fluctuations in the squeezing direction given by φ_k , since all fluctuations γ_{11} , γ_{12} , and γ_{22} are still of comparable amplitude then. At later times, the fluctuations along $\hat{p}_k^s$ quantified by γ_{22} come to dominate, and $\varphi_k \rightarrow 0$.

6. Quantum state in other parametrizations

It can be useful to understand the effect of decoherence on the state in terms of different parameters from the squeezing ones. First, directly related to the squeezing parameters are the sizes of the semimajor a_k and semiminor b_k axes defined in Eq. (23). In the absence of decoherence, we have $p_k = 1$, and thus from Eq. (40),

$$a_k^{\text{w.o.}} = e^{+r_k^{\text{w.o.}}} = x^{-2} + \mathcal{O}(x^2), \quad (52a)$$

$$b_k^{\text{w.o.}} = e^{-r_k^{\text{w.o.}}} = x^2 + \mathcal{O}(x^6). \quad (52b)$$

In the presence of decoherence, we start from Eq. (45) and obtain

$$\ln a_k = \frac{1}{2} \ln(\gamma_{22}) + \frac{1}{2} \left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 \frac{\gamma_{11}}{\gamma_{22}} \right] + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^4 \right], \quad (53a)$$

$$\ln b_k = \frac{1}{2} \ln \left(\frac{p_k^{-1}}{\gamma_{22}} \right) - \frac{1}{2} \left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 \frac{\gamma_{11}}{\gamma_{22}} \right] + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^4 \right]. \quad (53b)$$

One can check that the asymptotic expressions for a_k , Eq. (53a), and b_k , Eq. (53b), satisfy $a_k b_k = \sqrt{p_k}$ and $\ln b_k - \ln a_k = -2r_k$. We can then expand p_k using Eq. (43) and γ_{22} using Eq. (42c) and distinguish cases in p to obtain

$$\begin{aligned} \ln a_k &= -4 \ln x + \mathcal{O}(x^2) & \text{for } p < 8, \\ &= -(p-4) \ln x + \mathcal{O}(x^{6-p}) & \text{for } p > 8, \end{aligned} \quad (54)$$

and

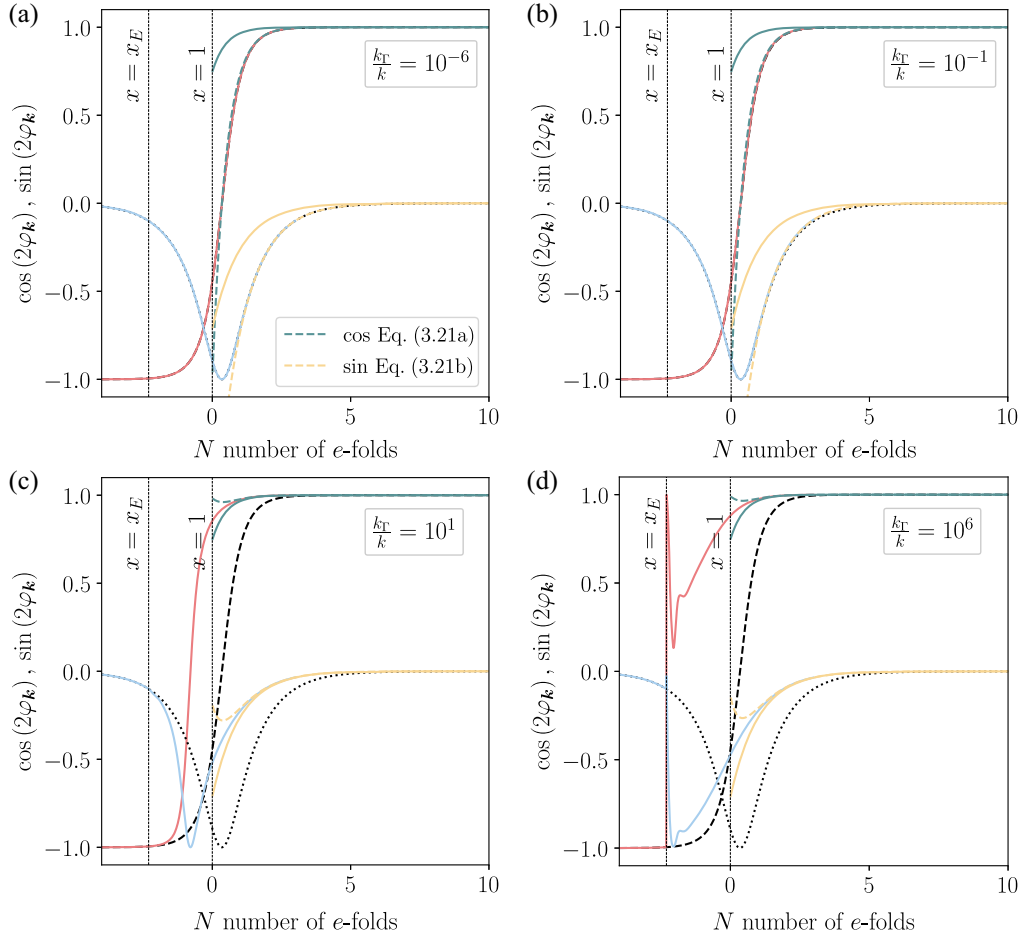


FIG. 7. Evolution of $\cos(2\varphi_k)$ and $\sin(2\varphi_k)$ in the presence of decoherence modeled by Eq. (24) as functions of N , the number of e -folds. The time dependence of the effect of the environment is fixed by taking $p = 9.1$ while its overall amplitude k_Γ/k is varied. Lines of the same color and style have the same meaning as in Fig. 6. In addition, the dashed green (respectively, yellow) lines show a second approximation for the $\cos$ (respectively, $\sin$) in the presence of decoherence. These values are computed using the late-time expansions in Eq. (49), where the covariance-matrix elements γ_{ij} are evaluated using their late-time expansions given in Eq. (42). While this second approximation performs well for all values of $k_\Gamma/k > 1$ and matches the exact expressions evaluated numerically, the approximations of Eq. (50) fail for a small number of e -folds when $k_\Gamma/k < 1$ at this large value of p . The reason for this failure is explained in the text at the end of Sec. III C 2.

$$\begin{aligned}
 \ln b_k &= \frac{1}{2} \ln \left[\frac{x^4(1 + A_\sigma)}{1 - 2(\frac{k_\Gamma}{k})^2 B_{22}} \right] + \mathcal{O}(x^{2-p}) & \text{for } p < 2, \\
 &= \frac{1}{2} \ln \left[\frac{x^{6-p} B_\sigma}{1 - 2(\frac{k_\Gamma}{k})^2 B_{22}} \right] + \mathcal{O}(x^{p-2}) + \mathcal{O}(x^2) & \text{for } 2 < p < 8, \\
 &= \frac{1}{2} \ln \left[\frac{x^{6-p} D_\sigma}{-2(\frac{k_\Gamma}{k})^2 A_{22}} \right] + \mathcal{O}(x^2) & \text{for } 8 < p.
 \end{aligned} \tag{55}$$

One can check that these expressions are continuous across the transitions at $p = 2$ and 8 . Equations (54) and (55) show that decoherence has a stronger effect on b_k than on a_k . On the one hand, a_k still increases exponentially as in the absence of decoherence, at the same rate for $p < 8$ and even at a larger rate for $p > 8$. On the other hand, b_k decreases exponentially as in the absence of decoherence

only for $p < 6$, albeit at a reduced rate for $p > 2$, and increases exponentially for $p > 6$. $p = 6$ is thus a critical value for the behavior of b_k beyond which there is no subfluctuant direction despite the presence of squeezing.

Finally, let us describe the state of the modes $\pm k$ as a collection of correlated particles with opposite momenta. We stress that, as detailed for the degree of squeezing of the

perturbations in Sec. IV A 2 below, this number of particles is ambiguous while the mode is super-Hubble. It would only become physically meaningful when the scale k reenters the Hubble radius and we recover a Minkowskian preferred notion of vacuum. The number and pair correlation of these particles defined by the operators Eq. (9) read

$$n_k = \langle \hat{c}_{\pm k}^\dagger \hat{c}_{\pm k} \rangle, \quad c_k = \langle \hat{c}_{\pm k} \hat{c}_{\mp k} \rangle, \quad (56)$$

and are related to the covariance matrix elements and effective squeezing parameters by [28]

$$2n_k + 1 = \frac{\gamma_{11} + \gamma_{22}}{2} = p_k^{-1/2} \cosh(2r_k), \quad (57a)$$

$$2c_k = \frac{\gamma_{11} - \gamma_{22}}{2} + i\gamma_{12} = -p_k^{-1/2} \sinh(2r_k) e^{2i\varphi_k}. \quad (57b)$$

They can then be related to the sizes of the semimajor and semiminor axes

$$a_k^2 = 2\left(n_k + \frac{1}{2} + |c_k|\right), \quad b_k^2 = 2\left(n_k + \frac{1}{2} - |c_k|\right). \quad (58)$$

Using the late-time expansion of a_k and b_k , it is then straightforward to see that, on the one hand, n_k and $|c_k|$ both behave in first approximation as a_k . Thus, the number of particles in the $\pm k$ modes and their pair correlation grow exponentially, and their ratio goes to 1, even in the presence of decoherence. This might appear surprising since we saw that decoherence desqueezes the perturbations, i.e., reduces r_k , and the number of particles varies exponentially in r_k . However, this reduction of r_k can be compensated by the simultaneous decrease of the purity p_k , which enters as $p_k^{-1/2}$ in the expression of n_k and $|c_k|$. As a result, n_k and $|c_k|$ are either, for $p < 8$, of the same order as their values in the pure-state case, or larger for $p > 8$. On the other hand, their difference $n_k - |c_k|$ is strongly affected by decoherence because it is directly related to b_k . It is this difference, typically much smaller than n_k and $|c_k|$, that captures quantum correlations; see below.

IV. EFFECT OF DECOHERENCE ON QUANTUM CORRELATIONS

While most studies of decoherence focus on computing how interactions degrade the purity of the state, e.g., Refs. [43,46,47], it is in principle necessary to know the full quantum state in order to properly diagnose the presence or absence of quantum features. In particular, for the class of states and partitions considered here, quantifiers of quantum correlations depend on the relative magnitudes of the purity p_k and the squeezing parameter r_k [18]. Since we have shown in the previous section that the

squeezing parameter r_k can also be strongly affected by decoherence, we analyze here how this modification translates into the evaluation of three quantum-correlation criteria: logarithmic negativity, quantum discord, and a Bell-inequality violation. First, we define the correlations we will study and briefly present the criteria use to test them. Then, we use these criteria to evaluate the quantumness of correlations in the presence of interactions using the results of Sec. III.

A. Definition, choice, and relation to observations of studied correlations

1. Defining degrees of freedom

Once an initial state is chosen, the state of the cosmological perturbations is fully determined by solving the Liouville–von Neumann equation (5) for the Hamiltonian (3). However, discussing the presence of quantum correlations in this state requires us to further specify *which* degrees of freedom we intend to study the correlations of. At the quantum level, a bosonic degree of freedom is described by a pair of canonically conjugate operators, and, as discussed in Sec. II A 1, our real scalar field has $\mathbb{R}^3$ degrees of freedom. Thus, we first need to choose a collection of $\mathbb{R}^3$ canonically conjugate pairs representing the field and then select subsets of degrees of freedom within this collection whose correlations we wish to analyze. There are infinitely many such collections to represent the field, and these are related by canonical transformations. If we represent each collection by its (infinite) vector of operator pairs $\hat{r} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \dots)$, then the vectors $\hat{r}$ and $\hat{r}'$ of two collections connected by a (linear) canonical transformation are related by an (infinite) symplectic matrix S as $\hat{r}' = S \cdot \hat{r}$; see Ref. [58].

While all collections define mathematically equally valid degrees of freedom whose correlations we can study, we should pick a collection that defines degrees of freedom with an unambiguous physical meaning and which, preferably, are related to operators we are able to measure [28,33]. Since in cosmology we are dealing with a homogeneous background, the analysis of cosmological perturbations is usually done in Fourier space and connected to observations made on the celestial sphere decomposed on spherical multipoles (see, e.g., Ref. [65]). In this case, the system is represented by the collection $\bigoplus_{k \in \mathbb{R}^{3+}} (\hat{q}_k, \hat{p}_k, \hat{q}_{-k}, \hat{p}_{-k})$, where $\bigoplus$ refers to con-

catenation, and the dynamics and quantum state of the system separate into independent sectors associated to $\pm k$ modes; see Eqs. (8) and (13). Since all $\pm k$ pairs are independent, it is natural to consider the correlations generated by the dynamics between k and $-k$ in one such pair [66]. This is the approach taken by most authors [4–9,13,14,16,17,26–28,32]. However, because the expansion of the Universe makes the Hamiltonian $\hat{\mathcal{H}}_{\pm k}$ time

dependent [see, for instance, Eq. (12)], there is *a priori* no preferred definition of which degrees of freedom are to be labeled as the mode $\mathbf{k}$ and which are to be labeled as the mode $-\mathbf{k}$. As a result, one might argue that we have no motivation to analyze the correlations between the $\mathbf{k}$ and $-\mathbf{k}$ modes defined by (10), rather than another set of modes, for instance, the R/I modes defined by Eq. (6), which would not lead to any generation of correlations [33].

2. Making sense of Fourier modes correlations

We now detail how the ambiguity in the definition of the $\mathbf{k}$ and $-\mathbf{k}$ modes manifests mathematically and explain our motivation to consider the correlations of these modes rather than others. First, recall that the $\pm\mathbf{k}$ modes are defined by the quadrature operators $\hat{q}_{\pm\mathbf{k}}, \hat{\pi}_{\pm\mathbf{k}}$ of Eq. (10), or equivalently by the creation and annihilation operators $\hat{c}_{\pm\mathbf{k}}^{(\dagger)}$ from which they are constructed. These creation and annihilation operators are defined by their commutation relations and by requiring that they diagonalize $\hat{\mathcal{H}}_{\pm\mathbf{k}}$, i.e., put it in the form of a linear combination of number operators $\hat{n}_{\pm\mathbf{k}}$. This procedure completely determines the operators $\hat{c}_{+\mathbf{k}}^{(\dagger)}$ (respectively, $\hat{c}_{-\mathbf{k}}^{(\dagger)}$) up to a phase, which give the physical definition of a particle with momentum $+\mathbf{k}$ (respectively, $-\mathbf{k}$). Yet, the time dependence of $\hat{\mathcal{H}}_{\pm\mathbf{k}}$ implies that there is in general no such operators that can diagonalize the Hamiltonian beyond an instant η . There is therefore no preferred choice of operators $\hat{c}_{\pm\mathbf{k}}^{(\dagger)}$, and infinitely many pairs satisfying the commutation relations can be found. These pairs, including pairs diagonalizing $\hat{\mathcal{H}}_{\pm\mathbf{k}}$ at two different instants η_1 and η_2 , are related by a symplectic transformation, which in general mixes the modes $\mathbf{k}$ and $-\mathbf{k}$. Consider two such sets of operators constructed from either $\hat{c}_{\mathbf{k}}^{(1)}$ or $\hat{c}_{\mathbf{k}}^{(2)}$. Then, $\hat{c}_{\mathbf{k}}^{(1)}$ will be a combination of both $\hat{c}_{\mathbf{k}}^{(2)}$ and $(\hat{c}_{-\mathbf{k}}^{(2)})^\dagger$. Thus, the quadratures $\hat{q}_{\mathbf{k}}^{(2)}, \hat{\pi}_{\mathbf{k}}^{(2)}$ defined from $\hat{c}_{\mathbf{k}}^{(2)}$ will contain both $\hat{q}_{\mathbf{k}}^{(1)}, \hat{\pi}_{\mathbf{k}}^{(1)}$, which define the $\mathbf{k}$ mode in the first set of operators, and $\hat{q}_{-\mathbf{k}}^{(1)}, \hat{\pi}_{-\mathbf{k}}^{(1)}$, which define the $-\mathbf{k}$ mode in the first set of operators. In summary, because there is no preferred definition of the $+\mathbf{k}$ and $-\mathbf{k}$ creation operators, the physical notion of particles having momentum $+\mathbf{k}$ or $-\mathbf{k}$ is ambiguous. As a result, there is no preferred splitting of the Hilbert space $\mathcal{H}_{\pm\mathbf{k}}$ into a mode labeled as $+\mathbf{k}$ and one labeled $-\mathbf{k}$, and at this stage, any splitting defined by a symplectic transformation from Eqs. (10) and (11) is an equally physically valid choice for studying the correlations in the state.

However, this ambiguity is only *transitory*, and eventually Eqs. (10) and (11) (or a simple deformation of them) do define the physical notion of modes $+\mathbf{k}$ and $-\mathbf{k}$, and it thus makes sense to study the correlations in the state in terms of these modes. First, for every wavelength k , there is a time $\eta_{\text{in}}(k)$ such that for $\eta \ll \eta_{\text{in}}$ we have $k^2 \gg 2/\eta^2$ and so $\omega(k, \eta) \approx k$. Thus, for $\eta \ll \eta_{\text{in}}$, the Hamiltonian is

effectively time independent, and there is no difficulty in this regime: we have a preferred notion of vacuum, creation, and annihilation operators, and the modes $\mathbf{k}$ and $-\mathbf{k}$ have a preferred time-independent definition. This is our “in” regime. Second, while this cannot be inferred from the form of the frequency (4) which is only valid during inflation, all wavelengths k^{-1} accessible today will eventually reenter the Hubble radius after inflation, either during radiation or matter domination; i.e., for every k , there exists a time $\eta_{\text{out}}(k)$ such that for $\eta > \eta_{\text{out}}(k)$ we have $a(\eta)k^{-1} < H^{-1}(\eta)$. In the simplest treatment of scalar perturbations in the radiation era [30,32], the Hamiltonian takes the same form as (3) up to a nonunity speed of sound $k \rightarrow c_s k$ and a different time dependence of the ϵ_1 parameter; see Eqs. (1), (2), (42), and (43) in Ref. [32]. For modes deep enough in this sub-Hubble regime during radiation domination, $\eta \gg \eta_{\text{out}}(k)$, we will then again have a time-independent Hamiltonian with $\omega(k, \eta) \approx c_s k$. Thus, for $\eta \gg \eta_{\text{out}}(k)$, the modes $\mathbf{k}$ and $-\mathbf{k}$ have a preferred definition given by performing $k \rightarrow c_s k$ in Eqs. (10) and (11). This gives an “out” regime. In summary, in the cosmological context, there are in and out regions, corresponding to deep sub-Hubble early- and late-time regimes, where the notion of particles with momentum $\mathbf{k}$ and $-\mathbf{k}$, and thus their correlations, become unambiguous.

Finally, while these regions are guaranteed to exist, the analysis of this paper is limited to the end of inflation, which precedes the reentry of the modes into the Hubble radius, i.e., $\eta_{\text{end}} < \eta_{\text{out}}(k)$. We thus leave pending the question of the evolution of the quantum state of the $\pm\mathbf{k}$ pairs from the end of inflation η_{end} to its late-time sub-Hubble regime $\eta \gg \eta_{\text{out}}(k)$, where the notion of $\mathbf{k}$ and $-\mathbf{k}$ modes becomes physically unambiguous and the correlations between these modes computed during inflation become physically “actualized.” Computing the $\pm\mathbf{k}$ correlations after reentry requires extra analyses to take into account the extra evolution of the quantum state during reheating (whether continuously or by applying junction conditions [68] under an instantaneous reheating assumption), radiation domination, and possibly matter domination, which is not described by our equations. Such analyses were conducted for some aspects of the perturbations in the absence of decoherence in other works; see, for instance, Refs. [4,11,32]. Note that the treatment of tensor perturbations in the absence of decoherence is relatively straightforward under an instantaneous reheating assumption, since in the radiation era we have $\omega(k, \eta) = 0$, i.e., all modes instantaneously reenter the Hubble radius, their preferred definition is given by the equivalent of Eq. (10), and their quantum correlations match those at the end of inflation; see Eq. (103) and Fig. 4 in Ref. [11].

3. Observability of Fourier modes correlations?

Before turning to their quantitative analysis, let us briefly discuss whether these $\pm\mathbf{k}$ correlations can be probed with cosmological observations. First, at a very general level,

one can ask how much information can be extracted by measuring local observables about the state of nonlocal degrees of freedom. In particular, can we probe the correlations between Fourier modes, which are degrees of freedom of infinite extension, i.e., nonlocal, by measuring cosmological observables which are always supported in a finite spacetime region? On the one hand, it is clear that the full information about a Fourier mode can never be recovered, as we always observe a finite volume of spacetime, at maximum given by the size of the observable Universe H_0^{-1} . On the other hand, if we restrict attention to Fourier modes of wavelengths $\lambda = 2\pi k^{-1}$ much smaller than the size of the system $V^{-1/3} < H_0^{-1}$, since the volume would contain many wavelengths $\lambda \ll V^{-1/3}$, it does not seem unreasonable to assume that we can reconstruct well some of the properties of the $\pm k$ modes. For instance, if we are able to accurately measure the power spectra constituting the covariance matrix (16), then, under a Gaussianity assumption, we have complete knowledge of the quantum state of the $\pm k$ modes. Here as well, a separate analysis would be required to first assess, given some local modes, how accurately the spectra can be measured. Second, given this incomplete information on the spectra, we further need to check what can be inferred about the entanglement of the nonlocal modes $\pm k$.

Next, we can ask at a more practical level whether cosmological observations allow measuring the relevant quantities to estimate the level of $\pm k$ correlations. This question was, for instance, discussed in Refs. [4,14] and briefly in the conclusion of Ref. [36]. First, neglecting the decaying mode of cosmological perturbations, measurements of CMB temperature anisotropies $\delta T/T$ are proportional to the value of the Mukhanov-Sasaki variable $\delta T/T \propto v/z$. We can thus constrain the power spectrum of $\hat{v}_k$ given by γ_{11} . Yet, it is not clear which observations would allow measuring the value of its time-derivative $p = v'$, necessary to compute the other spectra γ_{12} and γ_{22} [14]. It thus seems challenging to experimentally reconstruct the state of the $\pm k$ correlations even under a Gaussianity assumption. Moreover, it seems even harder to measure some observables that would directly evidence the quantum nature of the perturbations. For instance, the pseudospin operators introduced below in Eq. (70) to construct a Bell inequality for the $\pm k$ modes are not simple combinations of the canonical conjugate operators $(\hat{q}_k, \hat{p}_k)$ [14]. Unfortunately, since the state of perturbations obtained here is Gaussian, it is necessary to consider such operators, which are nonpolynomial in $\hat{v}$ and $\hat{p}$, in order to violate a Bell inequality [10,11,69]. Since CMB temperature anisotropies are proportional to $\hat{v}$, measuring the value of such nonpolynomial operators seems to require access to a putative, completely different probe of the early Universe. Finally, note that the practical problem of accessing observables other than $\hat{v}$ would apply equally to probing the correlations between degrees of freedom

other than Fourier modes, for instance, when considering correlations in real space [19,20,34,35].

Given this lack of experimental protocol to probe the quantum correlations of the cosmological perturbations, in this paper, we focus on assessing whether, in the first place, there are quantum correlations between Fourier modes to be measured and how large they are. We now introduce quantities allowing us to do so.

B. Quantifiers of quantum correlations

1. Local symplectic invariants for $\pm k$ correlations

Recall that we want to study the correlations in the $\pm k$ partition $H = \oplus_{\mathbb{R}^{3+}} H_k \oplus H_{-k}$, initially defined from the modes $+k$ and $-k$ which are represented using the pairs $(\hat{q}_{+k}, \hat{p}_{+k})$ and $(\hat{q}_{-k}, \hat{p}_{-k})$ given in Eq. (10). The quantum state of the modes is given by $\hat{\rho}_{\pm k} = \hat{\rho}_{k,R} \otimes \hat{\rho}_{k,I}$. The covariance matrix $\gamma_{\pm k}$ in this partition is obtained from that in the R/I partition $\gamma_k^R \oplus \gamma_k^I$, with γ_k^s given in Eq. (16), by a symplectic transformation. The details of this transformation are derived in detail in Ref. [28], and the covariance matrix in the $\pm k$ partition reads

$$\gamma_{\pm k} = \begin{pmatrix} \tilde{\gamma}_{11} & \tilde{\gamma}_{12} \\ \tilde{\gamma}_{12}^T & \tilde{\gamma}_{22} \end{pmatrix} \quad (59)$$

with

$$\begin{aligned} \tilde{\gamma}_{11} &= \tilde{\gamma}_{22} = \frac{\gamma_{11} + \gamma_{22}}{2} I_2, \\ \tilde{\gamma}_{12} &= \frac{1}{2} \begin{pmatrix} \gamma_{11} - \gamma_{22} & 2\gamma_{12} \\ 2\gamma_{12} & -(\gamma_{11} - \gamma_{22}) \end{pmatrix}. \end{aligned} \quad (60)$$

Unlike in the R/I partition, the matrix is not block diagonal in this case, and the off-diagonal block $\tilde{\gamma}_{12}$ encodes the correlations between the modes $+k$ and $-k$.

While all correlation measures between the $+k$ and $-k$ modes can be computed from the data of Eq. (59), this form of the covariance matrix is obtained for the specific choice of canonical pairs $(\hat{q}_{+k}, \hat{p}_{+k})$ and $(\hat{q}_{-k}, \hat{p}_{-k})$ representing the degrees of freedom $+k$ and $-k$. Yet, just as in classical Hamiltonian mechanics, different sets of operators can be used to describe the same physical degrees of freedom, e.g., the action-angle variables for a harmonic oscillator. As for the different collections of operators describing the whole fields $\hat{v}(\mathbf{x})$ and $\hat{p}(\mathbf{x})$, different variables used to represent the mode $+k$ are related by a symplectic transformation, and similarly for the mode $-k$. These different choices of variable to represent the modes lead to *a priori* different covariance matrices [58]. Since we want our characterization of the correlations between the degrees of freedom $+k$ and $-k$ not to depend on the operators used to represent them, we have to identify quantities which are invariant under *local* symplectic transformations, i.e., symplectic transformations that change the operators used to represent

each mode while respecting the Hilbert space splitting $H_k \oplus H_{-k}$. These transformations are the direct product of two one-mode symplectic transformations $S_{+k}^{(1)} \otimes S_{-k}^{(1)}$, with $S_{\pm k}^{(1)} \in \text{Sp}(2, \mathbb{R})$. The rules of transformation of the two-mode covariance matrix (59) under this symplectic subgroup, and the quantities left invariant, are known [58]. For a two-mode system, there are in general three such local symplectic invariants, given by $\det \tilde{\gamma}_{11}$, $\det \tilde{\gamma}_{22}$, and $\det \tilde{\gamma}_{12}$, and one global symplectic invariant, $\det \gamma_{\pm k}$. The global symplectic invariant is simply the purity $\det \gamma_{\pm k} = \det^2(\gamma_k^s) = p_k^{-2}$. For our reduced class of states, the local symplectic invariants are combinations of the purity and the two-mode squeezing parameter $\det \tilde{\gamma}_{11} = \det \tilde{\gamma}_{22} = p_k^{-1} \cosh^2(2r_k)$ and $\det \tilde{\gamma}_{12} = p_k^{-1} \sinh^2(2r_k)$. Thus, there are two local symplectic invariant quantities, p_k and r_k , which characterize the correlations between the $\pm k$ modes independently of the operators used to represent them. On the other hand, the squeezing angle φ_k is not invariant under local symplectic transformations. In the rest of this section, we compute the evolution during inflation of quantities testing for the quantum nature of the $\pm k$ correlations. Except for the Bell operator $\hat{B}_{\pm k}$, see below Eq. (76), these are built from p_k and r_k , and thus give a characterization of the evolution of the nonclassicality of correlations between the modes $+k$ and $-k$ that is invariant under the choice of operators used to represent them.

2. Nonseparability and logarithmic negativity

First, we consider whether the state is separable or not for the partition $\pm k$. The $\pm k$ modes are said to be in a nonseparable state if their density matrix cannot be written as a sum of products of density matrices, i.e., we cannot find coefficients $\{\lambda_i\}$ and density matrices $\{\hat{\rho}_{+k}^i, \hat{\rho}_{-k}^i\}$, acting, respectively, on H_{+k} and H_{-k} , such that $\hat{\rho}_{\pm k} = \sum_i \lambda_i \hat{\rho}_{+k}^i \otimes \hat{\rho}_{-k}^i$. For two-mode Gaussian states, it has been shown [70] that the Peres-Horodecki criterion [71,72] provides a necessary and sufficient condition for separability, which can be simply expressed in terms of covariance-matrix elements. In the $\pm k$ partition, the condition takes a particularly elegant form. One can show [17,18] that the state is nonseparable if and only if the semiminor axis of the squeezing ellipse is shorter than in the vacuum, i.e.,

$$b_k = p_k^{-1/4} e^{-r_k} < 1, \quad (61)$$

where we used the expression (23) for b_k . To obtain a quantifier of the degree of this entanglement, we can use the logarithmic negativity [73,74], which for this class of states is simply given by [75]

$$\text{LN}_{\pm k} = \max[0, -\log_2(b_k^2)]. \quad (62)$$

3. Quantum discord

Second, we discuss the quantum discord introduced in Refs. [48,49], which allows one to distinguish classical from quantum correlations between two systems, here the modes $+k$ and $-k$. Briefly, the quantum discord is built from two expressions, $\mathcal{I}_{\pm k}$ and $\mathcal{J}_{\pm k}$, for the mutual information, an entropy-based quantifier of correlations, between the systems $+k$ and $-k$. The expression for $\mathcal{I}_{\pm k}$ only involves the entropies of $+k$, $-k$ and of the joint system $\pm k$, whereas that of $\mathcal{J}_{\pm k}$ involves conditional entropies. These two expressions coincide in their classical definitions thanks to Bayes's theorem but can differ in the quantum setting where, intuitively, conditional entropies require specifying a set of measurement operators that disturb the state. The quantum discord $\mathcal{D}_{\pm k}$ is defined as their difference, $\mathcal{D}_{\pm k} = \mathcal{I}_{\pm k} - \mathcal{J}_{\pm k}$, which can be shown to be positive. Since its classical counterpart vanishes, it quantifies the strictly quantum correlations between $+k$ and $-k$. Properties and detailed expressions of the discord can be found in Refs. [48,49,76]. The quantum discord is in general difficult to evaluate; however, exact expressions are known for Gaussian states in terms of the covariance-matrix elements [58]. Its expression for scalar cosmological perturbations in the presence of the decoherence channel (24) is derived in Ref. [28]. For the partition $\pm k$, we have

$$\mathcal{D}_{\pm k} = f[\sigma(p_k, r_k)] - 2f(p_k^{-1/2}) + f\left[\frac{\sigma(p_k, r_k) + p_k^{-1}}{\sigma(p_k, r_k) + 1}\right], \quad (63)$$

where

$$\sigma(p_k, r_k) = p_k^{-1/2} \cosh(2r_k). \quad (64)$$

The function $f(x)$ is defined for $x \geq 1$ by

$$f(x) = \left(\frac{x+1}{2}\right) \log\left(\frac{x+1}{2}\right) - \left(\frac{x-1}{2}\right) \log\left(\frac{x-1}{2}\right). \quad (65)$$

Note that, since $p_k \leq 1$, we have $\sigma(p_k, r_k) \geq 1$, and all arguments of f in Eq. (63) are therefore larger than 1.

4. Asymptotic form of the discord

Before moving to the specific case of cosmological perturbations in de Sitter space, we derive an asymptotic expression for the discord of a generic two-mode thermal squeezed state in the regime of large squeezing, $e^{-r_k} \ll 1$, and small purity, $p_k \ll 1$. This approximation therefore applies to the late-time state of cosmological perturbations evolving via Eq. (24) for $p > 2$ in our setup (see Sec. III). The first step is to identify the relevant contributions to the discord by singling out the terms that are negligible because they depend only on

e^{-4r_k} and p_k . For this purpose, we follow Ref. [18] and rewrite the discord in terms of the lengths of the semiminor axis b_k and semimajor axis a_k . Note that, while large squeezing, $e^{-r_k} \ll 1$, and small purity, $p_k \ll 1$, imply a large semimajor axis,

$a_k \gg 1$, they do not fix the length of the semiminor axis b_k . This suggests that this quantity may control the magnitude of the discord in this regime. We then rewrite the discord in terms of b_k , p_k , and e^{-r_k} (see Appendix C for the derivation) as

$$\mathcal{D}_{\pm k} = \log_2 \left[1 + \frac{1}{2b_k^2} - p_k + \frac{p_k^{1/2} e^{-2r_k}}{2} - 2p_k^{1/2} e^{-2r_k} \frac{1 - p_k}{1 + e^{-4r_k} + 2p_k^{1/2} e^{-2r_k}} \right] + g \left(\frac{1 + e^{-4r_k}}{2p_k^{1/2} e^{-2r_k}} \right) - 2g(p_k^{-1/2}) + g \left(1 + 2b_k^2 \frac{1 - p_k}{1 + e^{-4r_k} + 2p_k^{1/2} e^{-2r_k}} \right), \quad (66)$$

where

$$g(x) = f(x) - \log_2 \left(\frac{x}{2} \right) - \frac{1}{\ln 2} \quad (67)$$

is a negative, bounded, and strictly increasing function. This equation is still exact, but, similarly to Eq. (45), it is useful for numerical evaluations since the dominant contributions are clearly identified in the first term of each evaluation of f and g and are only corrected by adding small contributions. We then expand the different functions in the limits $e^{-r_k} \ll 1$ and $p_k \ll 1$, see Eqs. (C2) and (C3), and obtain

$$\mathcal{D}_{\pm k} = \log_2 \left[1 + \frac{1}{2b_k^2} - p_k + \frac{3}{2} p_k^{1/2} e^{-2r_k} + \mathcal{O}(p_k^{3/2} e^{-2r_k}) + \mathcal{O}(p_k^{1/2} e^{-6r_k}) + \mathcal{O}(p_k e^{-4r_k}) \right] + \frac{p_k}{3 \ln 2} + \mathcal{O}(p_k^2) + \mathcal{O}(p_k e^{-4r_k}) + g \left\{ 1 + 2b_k^2 [1 - p_k - e^{-4r_k} - 2a_k^{-2} + \mathcal{O}(p_k^{3/2} e^{-2r_k}) + \mathcal{O}(p_k^{1/2} e^{-6r_k}) + \mathcal{O}(p_k e^{-4r_k})] \right\}. \quad (68)$$

Equation (68) is a refinement of Eq. (31) of Ref. [18], where we explicitly give the subdominant contributions for small but finite p_k and e^{-r_k} . It shows that, for a generic two-mode squeezed thermal state with small purity and large squeezing, the value of the discord is controlled by the size of the semiminor axis b_k . Since b_k at late times for cosmological perturbations is either very large or very small (see Sec. IV C), we further expand Eq. (68) in these two limits. We obtain

$$\mathcal{D}_{\pm k} = -2 \log_2 b_k - \frac{1}{\ln 2} + \mathcal{O}(b_k^2 \ln b_k) + \mathcal{O}(p_k) \quad \text{for } b_k \ll 1, \\ = \frac{b_k^{-2}}{2 \ln 2} + \mathcal{O}(b_k^{-4}) + \mathcal{O}(p_k) \quad \text{for } b_k \gg 1. \quad (69)$$

It is natural that the value of the discord is controlled by the size of the semiminor axis b_k . First, since it is a ratio between the squeezing and the purity, $b_k^2 = e^{-2r_k} / \sqrt{p_k}$, it characterizes the competition between squeezing and decoherence. Moreover, as shown in Eq. (62), it controls the separability of the state and the value of the logarithmic negativity. The coincidence of these different correlation measures is illustrated in Fig. 8. Finally, note that the relevance of the size of the semiminor axis for several nonclassicality criteria has already been pointed out in Ref. [17].

5. GKMR Bell inequality

Finally, we discuss the violation of a Bell inequality as a quantifier of quantum correlations. In its most common Clauser-Horne-Shimony-Holt form [77], the inequality involves a combination of expectation values for spin degrees of freedom. While we are dealing here with bosonic modes, we can define pseudospin operators, obeying a spin algebra, for which we can write a Bell inequality in the usual form. Several such pseudospin operators can be defined; here, we adopt those proposed in Ref. [78]. For a bosonic pair $\hat{q}, \hat{\pi}$, they are defined using the eigenstates $|q\rangle$ of the position operator $\hat{q}$. We have

$$\hat{\sigma}_x = \int_{-\infty}^{\infty} \text{sign}(q) |q\rangle \langle q| dq, \quad (70a)$$

$$\hat{\sigma}_y = -i \int_{-\infty}^{\infty} \text{sign}(q) |q\rangle \langle -q| dq, \quad (70b)$$

$$\hat{\sigma}_z = - \int_{-\infty}^{\infty} |q\rangle \langle -q| dq, \quad (70c)$$

and one can check that these operators satisfy the spin commutation relations

$$[\hat{\sigma}_\mu, \hat{\sigma}_\nu] = 2i\epsilon^{\mu\nu\lambda} \hat{\sigma}_\lambda. \quad (71)$$

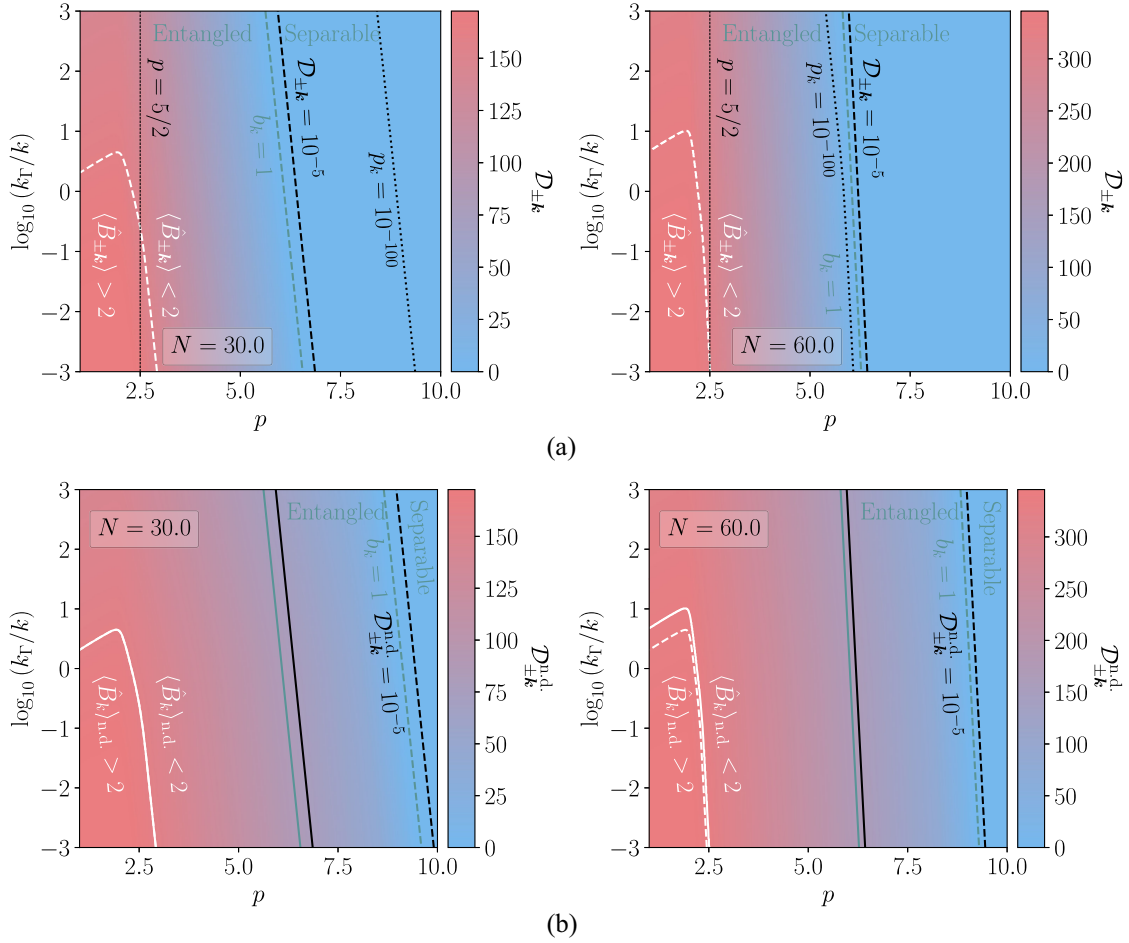


FIG. 8. Values of the quantum discord $\mathcal{D}_{\pm k}$ as a function of p and k_T/k after $N = 30$ (left panel) and $N = 60$ (right panel) e -folds of inflation. The top panels show values computed using the effective squeezing parameters in the presence of interaction. More precisely, we use Eqs. (43), (53a), and (53b), inserted into the discord formula Eq. (66). The dashed green lines show the entanglement threshold, corresponding to $b_k = 1$, computed using Eq. (53b). The dashed white lines show $\langle \hat{B}_{\pm k} \rangle = 2$, i.e., the violation threshold for the Bell inequality (73). The expectation value of the Bell operator is computed using Eq. (82). The lower panels show the values of the quantum discord computed using the purity in the presence of interaction, as given by Eq. (43), and the squeezing parameters in the absence of interaction, Eqs. (40) and (41). These three parameters are then used to compute a_k and b_k via Eq. (23), and the discord is obtained using Eq. (66). The entanglement and Bell-inequality violation thresholds are computed in the same way as in the top panels. In addition, for comparison, we show as solid lines the contours from the top panels for the same values of $N = 30$ (left panel) and $N = 60$ (right panel). (a) Effect of interactions taken into account for all parameters r_k , φ_k , and p_k . (b) Effect of interactions only taken into account for p_k .

We denote $\hat{\sigma}_i^k$, the operators constructed from the position operator $\hat{q}_k$ of the pair $\hat{q}_k, \hat{\pi}_k$ associated to the mode k . Then, using the operators for the pair of modes $\pm k$, we can define a Bell operator $\hat{B}_{\pm k}$ such that its expectation value is given by [78]

$$\langle \hat{B}_{\pm k} \rangle = 2\sqrt{\langle \hat{\sigma}_z^k \hat{\sigma}_z^{-k} \rangle^2 + \langle \hat{\sigma}_x^k \hat{\sigma}_x^{-k} \rangle^2}. \quad (72)$$

One can show that, if the measured values of the spins are described by a joint probability distribution, then $\langle \hat{B}_{\pm k} \rangle \leq 2$. Thus, a quantum state for which

$$\langle \hat{B}_{\pm k} \rangle > 2, \quad (73)$$

that is, for which the Bell inequality is violated, exhibits features that cannot be explained by a classical probability distribution. One can show that, for a two-mode squeezed thermal state, such as that obtained for the cosmological perturbations considered here, $\langle \hat{\sigma}_z^k \hat{\sigma}_z^{-k} \rangle$ and $\langle \hat{\sigma}_x^k \hat{\sigma}_x^{-k} \rangle$ can be expressed in terms of the squeezing parameter r_k , the squeezing angle φ_k , and the purity p_k as [18]

$$\langle \hat{\sigma}_z^1 \hat{\sigma}_z^2 \rangle = p_k, \quad (74)$$

$$\langle \hat{\sigma}_x^1 \hat{\sigma}_x^2 \rangle = -\frac{2}{\pi} \arcsin[|\cos(2\varphi_k)| \tanh(2r_k)]. \quad (75)$$

The expectation value of the Bell operator then reads

$$\langle \hat{B}_{\pm k} \rangle = 2\sqrt{p_k^2 + \frac{4}{\pi^2} \arcsin^2 [|\cos(2\varphi_k)| \tanh(2r_k)]}. \quad (76)$$

Note that, while not a local symplectic invariant (see Sec. IV A), the squeezing angle φ_k does enter the expression for the Bell operator expectation value in Eq. (76). This is because the pseudospin operators are constructed in Eq. (70) starting from a choice of two canonical pairs $(\hat{q}_{\pm k}, \hat{p}_{\pm k})$, and their definition need not be invariant under canonical transformations. When designing a Bell inequality, we design a concrete experimental setup in which a certain set of operators, namely, the pseudospin operators, are measured on each subsystem and the resulting expectation values are used to gauge the correlations between these subsystems. As a result, this does not provide a symplectic-invariant

assessment of the degree of correlations between the subsystems.

C. Decoherence in de Sitter

1. Late-time expansions

We now discuss how the open dynamics of Eq. (24) affect the measures of quantum correlations as functions of the phenomenological parameters k_Γ/k and p . We are particularly interested in the expressions of these measures at late times, $x \rightarrow 0^+$.

First, we consider the logarithmic negativity, which, when nonvanishing, is equal to the logarithm of the size of the semiminor axis $\ln b_k$; see Eq. (62). The late-time expansion of $\ln b_k$ was already derived in Eq. (53), and from it, we can directly read off the behavior of the logarithmic negativity

$$\text{LN}_{\pm k} = \max \left\{ 0, -\log_2 \left(\frac{p_k^{-1}}{\gamma_{22}} \right) - \frac{1}{\ln 2} \left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 \frac{\gamma_{11}}{\gamma_{22}} \right] + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^4 \right] \right\}. \quad (77)$$

Working at first order in x , we find

$$\begin{aligned} \text{LN}_{\pm k} &= \log_2 \left[\frac{1-2\left(\frac{k_\Gamma}{k}\right)^2 B_{22}}{x^4(1+A_\sigma)} \right] + \mathcal{O}(x^{2-p}) & \text{for } p < 2, \\ &= \log_2 \left[\frac{1-2\left(\frac{k_\Gamma}{k}\right)^2 B_{22}}{x^{6-p} B_\sigma} \right] + \mathcal{O}(x^{p-2}) + \mathcal{O}(x^2) & \text{for } 2 < p < 6, \\ &= 0 & \text{for } 6 < p. \end{aligned} \quad (78)$$

Therefore, $p = 6$ is a critical value: at late times, for $p < 6$, the state is entangled, while it is separable for $p > 6$. Moreover, for $p < 2$, the first-order term matches the pure-state case (52), so interactions do not affect the logarithmic negativity at first order in this case. For $p < 6$, we can obtain an approximation for the separability threshold as a function of x , k_Γ , and p by solving $b_k = 1$ in the above expression.

Similarly, we derive a late-time expansion for the quantum discord. In the absence of decoherence, we have $p_k = 1$, and thus

$$\mathcal{D}_{\pm k}^{\text{w.o.}} = f \left[1 + \frac{1}{2x^4} \right] = -4\log_2 x + \frac{1}{\ln 2} - 2 + \mathcal{O}(x^4). \quad (79)$$

In the presence of decoherence, we start from Eq. (66), which can be numerically evaluated using the asymptotic expressions for b_k , p_k , and e^{-4r_k} . To obtain an explicit form, we again have to distinguish cases in p to identify the dominant term. First, at late times, for $p > 2$, the purity is small $p_k \ll 1$, and the squeezing is large $e^{-r_k} \ll 1$. We can thus directly use Eq. (69) together with the late-time approximation (53b). Second, for $p < 2$, we need to expand other terms more carefully; this is done in Appendix C, and the resulting expression is given in Eq. (C16). Combining the two, we obtain

$$\begin{aligned} \mathcal{D}_{\pm k} &= -4\log_2 x + \log_2 \left[1 - 2\left(\frac{k_\Gamma}{k}\right)^2 B_{22} \right] + \frac{1}{\ln 2} - 2 - 2f[\sqrt{1+A_\sigma} + \mathcal{O}(x^{2-p})] + \mathcal{O}(x^{2-p}) & \text{for } p < 2 \\ &= -(6-p)\log_2 x + \log_2 \left[\frac{1-2\left(\frac{k_\Gamma}{k}\right)^2 B_{22}}{B_\sigma} \right] - \frac{1}{\ln 2} + \mathcal{O}(x^{p-2}) + \mathcal{O}(x^2) + \mathcal{O}(x^{6-p} \ln x) & \text{for } 2 < p < 6, \\ &= x^{p-6} \frac{1-2\left(\frac{k_\Gamma}{k}\right)^2 B_{22}}{2B_\sigma \ln 2} + \mathcal{O}(x^{p-4}) & \text{for } 6 < p < 8, \\ &= x^{p-6} \frac{-\left(\frac{k_\Gamma}{k}\right)^2 A_{22}}{D_\sigma \ln 2} + \mathcal{O}(x^{p-4}) & \text{for } 8 < p. \end{aligned} \quad (80)$$

One can straightforwardly check that this expression is continuous in p except at $p = 2$, which requires more careful treatment. Note that, for $p < 2$, as for the logarithmic negativity, the first-order term matches the pure-state case (79).

Finally, we turn to the expectation value of the Bell-operator given in Eq. (76). In the absence of decoherence, we have

$$\frac{\langle \hat{B}_{\pm k} \rangle_{\text{w.o.}}^2}{4} = 2 - \frac{8x}{\pi} + \mathcal{O}(x^2). \quad (81)$$

In the presence of decoherence, we approximate this expectation value using the late-time expansions of the squeezing parameter r_k and of $\cos(2\varphi_k)$, as detailed in Appendix C. We then obtain

$$\frac{\langle \hat{B}_{\pm k} \rangle^2}{4} = 1 + p_k^2 - \frac{8}{\pi} \sqrt{\frac{\gamma_{11}}{\gamma_{22}}} + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right]. \quad (82)$$

By distinguishing cases, we write a more explicit expression using the expansion of the purity (43) and of the covariance-matrix elements (42)

$$\begin{aligned} \frac{\langle \hat{B}_{\pm k} \rangle^2}{4} &= 1 + \frac{1}{[1 + A_\sigma(p, x_E)]^2} - \frac{8x}{\pi} + \mathcal{O}(x^{2-p}) && \text{for } p < 2, \\ &= 1 + \frac{x^{2p-4}}{B_\sigma(p, x_E)^2} - \frac{8x}{\pi} + \mathcal{O}(x^{2p-2}) + \mathcal{O}(x^2) && \text{for } 2 < p < 8, \\ &= 1 - \frac{8x}{\pi} \sqrt{\frac{A_{11}}{A_{22}}} + \mathcal{O}(x^2) && \text{for } 8 < p. \end{aligned} \quad (83)$$

At first order in x , for $p < 2$, the Bell inequality is violated asymptotically since $\langle \hat{B}_{\pm k} \rangle \rightarrow 2 + 2/(1 + A_\sigma)^2 > 2$. For $2 < p < 5/2$, the first term in the second line of Eq. (83), which is positive, dominates, and thus the expectation value approaches 2 from above, $\langle \hat{B}_{\pm k} \rangle \rightarrow 2^+$. Although the inequality is not violated asymptotically in this case, it is always slightly violated for any finite x . On the other hand, for $p > 5/2$, the second term, which is negative, dominates, and we have $\langle \hat{B}_{\pm k} \rangle \rightarrow 2^-$. The Bell inequality therefore stops being violated at a finite time. Finally, for $p > 8$, the term coming from the purity is of order $\mathcal{O}(x^{4p-20})$ and is thus negligible even compared with the first-order term $\mathcal{O}(x^3)$. As a result, the approximation always gives $\langle \hat{B}_{\pm k} \rangle < 2$, and we cannot identify the critical point where $\langle \hat{B}_{\pm k} \rangle = 2$. Such a point exists but would correspond either to an extremely small number of e -folds after coherence-length crossing, or to an extremely small value of k_Γ/k , in order to suppress the effect of decoherence. In summary, asymptotically $p = 5/2$ is a critical value: for $p < 5/2$, the Bell inequality is violated as $x \rightarrow 0^+$, and it is not violated for $p > 5/2$.

2. Discussion

We now summarize our results on these quantum-correlation quantifiers by plotting in Fig. 8 their values in the p - $\log(k_\Gamma/k)$ plane after $N = 30$ (left panel) and $N = 60$ (right panel) e -folds of inflation. Let us comment on their content.

First, Fig. 8(a) reproduces the known hierarchy between the different quantum-correlation quantifiers. While the Bell inequality is only violated for entangled states and entangled states have a nonvanishing quantum discord, a

nonvanishing quantum discord does not imply that the state is entangled and entangled state does not necessarily violate a Bell inequality. Second, Fig. 8(a) shows that quantum correlations become weaker, at fixed time x , when the interaction is stronger, either because it is more time dependent, i.e., p increases, or, to a lesser extent, because its overall magnitude k_Γ/k is larger. Third, by comparing the left and right panels, we see that as time increases the dependence on k_Γ/k becomes less relevant. This is in agreement with the findings of Sec. IV C 1, where we identified that the asymptotic behavior of the quantum correlations mostly depends on the value of p . Turning to the logarithmic negativity and the quantum discord, the figures confirm that $p = 6$ is a critical value: for $p < 6$, the state has a large negativity, is entangled, and has a large quantum discord at late times, while it is separable and has a small quantum discord for $p > 6$. Going from the left panel to the right panel of Fig. 8(a), we observe the emergence of this asymptotic behavior as the contours for the value of the discord and for $b_k = 1$ become more vertical. Moreover, although not plotted here, the contour level for $\mathcal{D}_{\pm k} = 1$ lies on top of that for $b_k = 1$. This confirms that the behavior of the discord is asymptotically controlled by the value of b_k and is in agreement with the results of Ref. [18]. To make quantitative sense of this overlap, we use Eq. (66) to evaluate the discord by taking $b_k = 1$ and setting to zero all terms proportional to powers of p_k and e^{-r_k} . We find that the quantum discord is indeed of order unity, $\mathcal{D}_{\pm k} \approx f(3) - 1/\ln 2 \approx 0.55$. Similarly, for the Bell inequality, we observe that the contour level $\langle \hat{B}_{\pm k} \rangle = 2$ becomes more vertical at later times, and closer to the critical value $p = 5/2$ identified in Sec. IV C 1. However, the convergence to this asymptotic behavior is

slower than for the quantum discord and the logarithmic negativity, and even for $N = 60$ (right panel), the contour still strongly depends on the value of k_Γ/k . A fourth important observation is that there is no direct relation between the value of the purity p_k and the level of quantum correlations. In particular, we see that for $N = 60$ even states with a purity as small as 10^{-100} can still be entangled. Thus, it is incorrect to infer that a state has only classical properties solely based on the smallness of its purity.

Since most studies on the decoherence of cosmological perturbations only evaluate their purity, e.g., Refs. [43,46,47], a natural approach is to estimate the level of quantum correlations by assuming that the squeezing parameters r_k and φ_k are not significantly affected by the interactions leading to decoherence. One then represents the state by the parameters $(p_k, r_k^{w.o.}, \varphi_k^{w.o.})$ and uses them to evaluate the correlation quantifiers of Sec. IV B. This was the approach adopted in the conclusion of Ref. [36]. However, we showed in Sec. III that interactions modeled by Eq. (24) typically induce a reduction of r_k , i.e., a desqueezing of the perturbation. As a result, an estimate based on $(p_k, r_k^{w.o.}, \varphi_k^{w.o.})$ leads to an overestimation of the quantum correlations of the state. To assess the magnitude of this error, we plot in Fig. 8(b) the same quantum-correlation quantifiers as in Fig. 8(a), using a quantum state with effective squeezing parameters $(p_k, r_k^{w.o.}, \varphi_k^{w.o.})$; i.e., we only take into account the effect of decoherence on the purity p_k and keep the pure-state values $r_k^{w.o.}$ and $\varphi_k^{w.o.}$ for the squeezing parameters. In Fig. 8(b), we also reproduce (as solid lines) some of the contour levels from Fig. 8(a) to facilitate the comparison.

We first notice that the contour of the expectation value of the Bell operator is not strongly affected by these incorrect estimates of r_k and φ_k . This can be understood by noting that in Eq. (83), in the region $p < 5/2$ where the contour lies, the dominant contribution always comes from the purity p_k , while the subdominant term coming from the squeezing parameters r_k and φ_k matches the contribution obtained from the pure-state values $r_k^{w.o.}$ and $\varphi_k^{w.o.}$. As a result, the contours estimated from states with parameters $(p_k, r_k^{w.o.}, \varphi_k^{w.o.})$ and (p_k, r_k, φ_k) agree at leading order, as observed in Fig. 8(b). On the other hand, we find that the values of the quantum discord and the separability threshold are strongly affected by an incorrect estimate of the squeezing parameter r_k . For instance, for any values of p and k_Γ/k lying between the solid and dashed green lines in Fig. 8(b), the state is found to be entangled when using the squeezing $r_k^{w.o.}$, while it is in fact separable once the desqueezing is properly taken into account.

V. CONCLUSION

In this paper, we have analyzed the effects of interactions of primordial inhomogeneities on their quantum state,

parametrized by effective squeezing parameters, and on their degree of quantum correlations as measured by several quantifiers. We have adopted a simple interaction model in which the dynamics of the perturbations is described by a Lindblad equation with a nonunitary term quadratic in the Mukhanov-Sasaki variable, parametrized by two parameters k_Γ/k and p , borrowed from Ref. [53].

At the technical level, one of our main results is the derivation of late-time approximations for two sets of quantities. First are those contained in Sec. III C, for the effective squeezing parameters r_k , φ_k , p_k , and related quantities such as the sizes of the semiaxes a_k and b_k . Second are those contained in Secs. IV B and IV C for the measures of quantum correlations, namely, the logarithmic negativity $\text{LN}_{\pm k}$, the quantum discord $\mathcal{D}_{\pm k}$, and the expectation value of a Bell operator $\langle \hat{B}_{\pm k} \rangle$. These expansions are of two types. First, some expansions only require making assumptions on the relative magnitude of fluctuations of the different quadratures, such as Eq. (45) for $\gamma_{22} \gg |\gamma_{12}|$ and $|\gamma_{12}| \gg \gamma_{11}$, and/or a small purity, such as Eq. (68). Second, these can be specified to dynamics of the form Eq. (24) with kernel (25) to be made more explicit, such as Eqs. (46) and (80). These expressions improve upon previous studies in the following way. While Ref. [28] has already considered the evolution of the purity p_k and the quantum discord $\mathcal{D}_{\pm k}$ under Eq. (24), by computing the squeezing parameter r_k and squeezing angle φ_k in addition to p_k , our work gives for the first time a complete characterization of the quantum state of the system, which allows one to compute the expectation values of any quantity. As an application, we have evaluated the logarithmic negativity and the degree of violation of a Bell inequality by the $\pm k$ correlations of cosmological perturbations in this model, which also constitutes a new result obtained in this work. The Bell inequality has already been considered in the pure-state case by Ref. [14] and in the presence of a different decoherence channel in Ref. [16]. Moreover, our work improves upon known formulas for the quantities p_k and $\mathcal{D}_{\pm k}$ which were already computed in this model. For the purity p_k , our Eq. (33) gives an exact form in the presence of decoherence, and our late-time expansion Eq. (43) corrects Eq. (5.17) of Ref. [28] for $p > 8$. For $\mathcal{D}_{\pm k}$, our formula (68) improves in precision over Eq. (31) of Ref. [18], itself an improvement in generality over Eq. (4.27) of Ref. [28]. Moreover, in this work, we give a clear analytic approximation of the discord when the evolution is modeled by (24), while Ref. [28] only discussed leading behaviors in Sec. 5.2.3.

Using these new formulas, we have improved our understanding of the workings of decoherence as modeled by (24). First, while it is expected and well known that interactions suppress the purity p_k of the state, we have also demonstrated that our interaction model also reduces r_k and can do so by up to 75% with respect to the pure-state case; see Fig. 5. On the other hand, although some deviations

from the case without decoherence can be observed at early times, the squeezing angle φ_k is essentially independent of the values of p and k_Γ . Second, when it comes to quantum correlations between opposite-momentum pairs $\pm \mathbf{k}$, we showed that, at leading order, the asymptotic late-time behavior of these quantifiers depends only on the scale-factor dependence of the interaction and not on its overall strength k_Γ/k . This behavior is illustrated in Fig. 8(a).

Furthermore, we demonstrated that estimating the degree of quantum correlations in the presence of interactions based solely on a computation of the purity p_k , supplemented by the pure-state values $r_k^{\text{w.o.}}$ and $\varphi_k^{\text{w.o.}}$, leads to a gross overestimation of these correlations when using the quantum discord or the logarithmic negativity. This is evident from a comparison between Figs. 8(a) and 8(b), where the same quantifiers are plotted for a state in which only the effect of interactions on p_k is taken into account, while the squeezing parameters are fixed to their pure-state values $r_k^{\text{w.o.}}$ and $\varphi_k^{\text{w.o.}}$. This result highlights that determining whether a state has been classicalized by its interactions, i.e., whether it no longer exhibits quantum correlations, requires knowledge of its full quantum state, and not only of its purity.

Our treatment of decoherence was intentionally broad, encompassing a wide class of possible models. Once a specific framework or set of assumptions is chosen, however, one can explicitly formulate the interaction between the system and the environment, as is done, for example, in Refs. [43,44,47]. Applying our analysis to these cases, which consider a minimal level of interaction induced by the second-order dynamics of cosmological perturbations, would allow one to derive an upper bound on the degree of quantum correlations at the end of inflation. While the master equation of Ref. [43] has a form similar to our Eq. (24), the more complete equation of Ref. [47] contains additional nonunitary terms that would need to be included. However, the main difficulty in applying our results to these interactions is that the “Lamb shift” terms appearing in Refs. [43,47] contain divergences; see Eq. (3.36) of Ref. [47]. The authors argue that these divergences can be renormalized by introducing counterterms generated by $1/M_{\text{Pl}}^2$ corrections to the Einstein-Hilbert action. Such a procedure, beyond the scope of the present paper, must be carried out before the present analysis can be applied to this interaction.

Finally, we may ask about the generality of our investigation. First, it is not obvious whether the desqueezing we have observed is universal or specific to an interaction proportional to the Mukhanov-Sasaki field $\hat{v}$. Second, the validity of our formulation may warrant careful scrutiny, particularly in regions of parameter space corresponding to large values of p and k_Γ/k . Although we have briefly addressed this issue in Appendix A 2, further clarification is required. A comprehensive treatment of these questions is deferred to future work.

ACKNOWLEDGMENTS

This work was supported by JSPS KAKENHI Grant No. 25K01004 (T.T.) and MEXT KAKENHI Grants No. 23H04515 (T.T.) and 25H01543 (T.T.). A. M. thanks Greg Kaplanek for discussions about divergences appearing in the open quantum treatment of inflationary perturbations and Jérôme Martin, Fumiya Sanno, and Vincent Vennin for discussions on topics related to the present article.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: APPLICABILITY DOMAIN OF OUR LINDBLAD EQUATION

In this appendix, we sketch how one can arrive at Eq. (24) from a specific field interaction model. This will allow us to discuss its domain of applicability and explain the origin of the different terms appearing in it. A fuller and more rigorous derivation is given in Ref. [53] in general and in Ref. [43] for the case of decoherence induced by the second-order dynamics of the cosmological perturbations.

1. Derivation

First, consider the interaction Hamiltonian with a linear coupling

$$\hat{H}_{\text{int}}(\eta) = G(\eta) \int \hat{v}(\mathbf{x}) \otimes \hat{E}(\eta, \mathbf{x}) d^3\mathbf{x} = G(\eta) \int_{\mathbb{R}^{3+}} \sum_{s=\text{R,I}} \hat{v}_k^s \otimes \hat{E}_k^s(\eta) d^3\mathbf{k}, \quad (\text{A1})$$

where $G(\eta)$ is a time-dependent coupling constant and $\hat{E}(\eta, \mathbf{x})$ is an operator acting only on the environment Hilbert space; their concrete expressions define the chosen interaction model. In the second equality, we transformed the Hamiltonian into Fourier space and defined the Hermitian and anti-Hermitian parts of the operators. Our goal is to characterize how the $\pm k$ correlations generated by the free evolution are modified when the interaction (A1) is taken into account. This is, in general, a difficult problem, as it requires first solving the coupled Liouville equation (14) for the system and the environment and then tracing out the environmental degrees of freedom to obtain the reduced density matrix of the system $\hat{\rho}(\eta)$. Since we are

only concerned with the dynamics of the system, we can instead trace out the environmental degrees of freedom from the start and effectively treat the evolution of the system as open, rather than solving the coupled closed-system dynamics of the system and the environment. Under a set of assumptions on the coupling and the state of the environment, we can perform this trace without knowing the state of the environment exactly and obtain an approximate dynamics for $\hat{\rho}(\eta)$ in the form of a nonunitary equation called a master equation. The first step is to go to the interaction picture and iteratively integrate the full dynamics to cast the equation in an integrodifferential form,

$$\begin{aligned}\hat{\rho}_{S/E}(\eta + \delta\eta) - \hat{\rho}_{S/E}(\eta) &= -i \int_{\eta}^{\eta+\delta\eta} [\tilde{H}_{\text{int}}(\eta'), \tilde{\rho}_{S/E}(\eta')] d\eta', \\ &= -i \int_{\eta}^{\eta+\delta\eta} [\tilde{H}_{\text{int}}(\eta'), \tilde{\rho}_{S/E}(\eta)] d\eta' - \int_{\eta}^{\eta+\delta\eta} \int_{\eta}^{\eta'} [\tilde{H}_{\text{int}}(\eta'), [\tilde{H}_{\text{int}}(\eta''), \tilde{\rho}_{S/E}(\eta'')]] d\eta' d\eta'',\end{aligned}\quad (\text{A2})$$

where $\tilde{\rho}_{S/E}$ is the density matrix of the full system and $\tilde{O}(\eta)$ refers to operators in the interaction picture. The derivation then proceeds by tracing out the environment on both sides:

$$\begin{aligned}\hat{\rho}(\eta + \delta\eta) - \hat{\rho}(\eta) &= -i \int_{\eta}^{\eta+\delta\eta} \text{Tr}_E \{ [\tilde{H}_{\text{int}}(\eta'), \tilde{\rho}_{S/E}(\eta)] \} d\eta' \\ &\quad - \int_{\eta}^{\eta+\delta\eta} \int_{\eta}^{\eta'} \text{Tr}_E \{ [\tilde{H}_{\text{int}}(\eta'), [\tilde{H}_{\text{int}}(\eta''), \tilde{\rho}_{S/E}(\eta'')]] \} d\eta' d\eta''.\end{aligned}\quad (\text{A3})$$

By definition, we have $\text{Tr}_E[\tilde{\rho}_{S/E}(\eta)] = \tilde{\rho}(\eta)$, and we define the reduced density matrix of the environment as $\tilde{\rho}_E(\eta) = \text{Tr}_S[\tilde{\rho}_{S/E}(\eta)]$. The interaction correlates the system and the environment, so that $\tilde{\rho}_{S/E}(\eta) \neq \tilde{\rho}(\eta) \otimes \tilde{\rho}_E(\eta)$. Yet, since in the absence of interactions they evolve independently, $\tilde{\rho}_{S/E}(\eta) = \tilde{\rho}(\eta) \otimes \tilde{\rho}_E(\eta)$, the nonfactorizable part of the density matrix must arise from terms in $\tilde{H}_{\text{int}}$. We assume that the interaction is perturbative, i.e., $\tilde{H}_{\text{int}} \ll 1$ in some sense, so that this contribution is small, and, for simplicity, that the first term is of order $\mathcal{O}(\tilde{H}_{\text{int}}^2)$, so that

$$\tilde{\rho}_{S/E}(\eta) = \tilde{\rho}(\eta) \otimes \tilde{\rho}_E(\eta) + \mathcal{O}(\tilde{H}_{\text{int}}^2). \quad (\text{A4})$$

The first term in Eq. (A3) is *a priori* of order $\mathcal{O}(\tilde{H}_{\text{int}})$, but it can be removed by an appropriate redefinition of the operator $\hat{E}$. To see how, first consider the trace appearing in this term, which reads

$$\text{Tr}_E \{ [\tilde{H}_{\text{int}}(\eta'), \tilde{\rho}_{S/E}(\eta)] \} = G(\eta) \int \text{Tr}_E [\tilde{E}(\eta, \mathbf{x}) \tilde{\rho}_E(\eta)] [\hat{v}(\mathbf{x}), \tilde{\rho}(\eta)] d^3\mathbf{x} + \mathcal{O}(\tilde{H}_{\text{int}}^3). \quad (\text{A5})$$

Assuming that the environment is left unaffected by the interaction with the system and remains stationary, its reduced state is time independent and identical to that of the free theory: we write $\tilde{\rho}_E(\eta) \approx \hat{\rho}_E^{\text{free}}$. We then have

$$\text{Tr}_E [\tilde{E}(\eta, \mathbf{x}) \tilde{\rho}_E(\eta)] = \text{Tr}_E [\hat{E}(\eta, \mathbf{x}) \hat{\rho}_E^{\text{free}}] = \langle \hat{E}(\eta, \mathbf{x}) \rangle, \quad (\text{A6})$$

where in the last equality we identified the expectation value of the operator in the free theory. By redefining the operator appearing in the interaction Hamiltonian as $\hat{E} \rightarrow \hat{E} - \langle \hat{E} \rangle \hat{1}$, we can remove this term. As a result, the remainder of the first term in Eq. (A3) is of order $\mathcal{O}(\tilde{H}_{\text{int}}^3)$ and is thus subdominant compared to the second term in the equation, which is of order $\mathcal{O}(\tilde{H}_{\text{int}}^2)$. Let us now analyze this term. We have

$$\hat{\rho}(\eta + \delta\eta) - \hat{\rho}(\eta) \approx - \int_{\eta}^{\eta+\delta\eta} \int_{\eta}^{\eta'} \text{Tr}_E \{ [\tilde{H}_{\text{int}}(\eta'), [\tilde{H}_{\text{int}}(\eta''), \tilde{\rho}(\eta'') \otimes \hat{\rho}_E^{\text{free}}]] \} d\eta' d\eta''. \quad (\text{A7})$$

Using the form of the interaction Hamiltonian (A1), the contribution of the environment in the second term is encoded in the autocorrelation function

$$C_B(\eta, \eta'; \mathbf{x}, \mathbf{x}') = \text{Tr}_E[\hat{\rho}_E^{\text{free}} \hat{E}(\eta, \mathbf{x}) \hat{E}(\eta', \mathbf{x}')]. \quad (\text{A8})$$

Note that, by cyclicity of the trace and because $\hat{E}$ is Hermitian, we have $C_B(\eta, \eta'; \mathbf{x}, \mathbf{x}')^* = C_B(\eta', \eta; \mathbf{x}', \mathbf{x})$. The equation then reads

$$\begin{aligned} \hat{\rho}(\eta + \delta\eta) - \hat{\rho}(\eta) \approx & - \int_{\eta}^{\eta+\delta\eta} \int_{\eta}^{\eta'} G(\eta') G(\eta'') \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \{ C_B(\eta', \eta''; \mathbf{x}', \mathbf{x}'') [\tilde{v}(\mathbf{x}', \eta'), \tilde{v}(\mathbf{x}'', \eta'') \tilde{\rho}(\eta'')] \\ & + C_B(\eta', \eta''; \mathbf{x}', \mathbf{x}'')^* [\tilde{\rho}(\eta'') \tilde{v}(\mathbf{x}'', \eta''), \tilde{v}(\mathbf{x}', \eta')] \} d\eta' d\eta'' d\mathbf{x}' d\mathbf{x}''. \end{aligned} \quad (\text{A9})$$

The key final step is to assume that $C_B(\eta, \eta'; \mathbf{x}, \mathbf{x}')$ is sharply peaked and supported around $\eta = \eta'$. This allows us to simplify the time integrals and obtain an equation whose right-hand side depends only on $\tilde{\rho}$ evaluated at η , rather than on its entire history. Such an equation is dubbed Markovian. Assuming that the system evolves on a timescale much longer than the typical autocorrelation time t_c over which C_B is supported, we have

$$\int_{\eta}^{\eta'} d\tau C_B(\eta', \eta'') f(\eta', \eta'') \approx f(\eta', \eta') \int_{-\infty}^{\eta'} d\tau C_B(\eta', \eta''); \quad (\text{A10})$$

i.e., the autocorrelation selects only the contribution with $\eta' = \eta''$. In addition, since C_B is supported around $\eta' = \eta''$, we can safely extend the integration domain on the right-hand side to $-\infty$ without changing the value of the integral. This allows us to suppress the explicit dependence on η . Using this approximation in Eq. (A9) and taking $\delta\eta \rightarrow 0$, we obtain

$$\begin{aligned} \partial_{\eta} \hat{\rho} \approx & - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} d\mathbf{x}' d\mathbf{x}'' \{ \text{Re}[\mathfrak{F}(\eta; \mathbf{x}', \mathbf{x}'')] [\tilde{v}(\mathbf{x}', \eta), [\tilde{v}(\mathbf{x}'', \eta), \tilde{\rho}(\eta)]] - i \text{Im}[\mathfrak{F}(\eta; \mathbf{x}', \mathbf{x}'')] [\tilde{v}(\mathbf{x}', \eta) \tilde{v}(\mathbf{x}'', \eta), \tilde{\rho}(\eta)] \\ & - i \text{Im}[\mathfrak{F}(\eta; \mathbf{x}', \mathbf{x}'') - \mathfrak{F}(\eta; \mathbf{x}'', \mathbf{x}')] \tilde{v}(\mathbf{x}', \eta) \tilde{\rho}(\eta) \tilde{v}(\mathbf{x}'', \eta) \}, \end{aligned} \quad (\text{A11})$$

with

$$\mathfrak{F}(\eta; \mathbf{x}', \mathbf{x}'') = \int_{-\infty}^{\eta} G(\eta) G(\eta') C_B(\eta, \eta'; \mathbf{x}', \mathbf{x}'') d\eta'. \quad (\text{A12})$$

Finally, let us assume that the environment is homogeneous and isotropic so that the correlation function reads

$$C_B(\eta, \eta'; \mathbf{x}, \mathbf{x}') = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \mathcal{C}_B(\eta, \eta'; k), \quad (\text{A13})$$

where $\mathcal{C}_B(\eta, \eta'; k)$ depends only on the modulus of k . Thus, in particular, $C_B(\eta, \eta'; \mathbf{x}, \mathbf{x}') = C_B(\eta', \eta; \mathbf{x}', \mathbf{x})$. This symmetry cancels the last term in Eq. (A11). Then, going to Fourier space and separating into R and I sectors, we obtain

$$\begin{aligned} \frac{\mathcal{V}}{(2\pi)^3} \frac{\partial \tilde{\rho}_{k,s}}{\partial \eta} = & -i \text{Im}[\mathfrak{F}_k(\eta)] [(\tilde{v}_k^s)^2, \tilde{\rho}_{k,s}] \\ & - \text{Re}[\mathfrak{F}_k(\eta)] [\tilde{v}_k^s, [\tilde{v}_k^s, \tilde{\rho}_{k,s}]], \end{aligned} \quad (\text{A14})$$

where

$$\mathfrak{F}_k(\eta) = (2\pi)^{3/2} \int_{-\infty}^{\eta} G(\eta) G(\eta') \mathcal{C}_B(\eta, \eta'; k) d\eta'. \quad (\text{A15})$$

Let us make some comments on this final equation. First, note that Eq. (A14) contains two additions with respect to the free dynamics given by Eq. (14). The first term is a Lamb-shift-type term arising from the imaginary part of the kernel, which contributes to the unitary dynamics of the $\pm \mathbf{k}$ modes and can be interpreted as a modification of the bare Hamiltonian. The second term comes from the real part of the kernel and gives a nonunitary contribution that generates dissipation and decoherence for the $\pm \mathbf{k}$ modes. This is the general form of a Lindblad equation.

Yet, Eq. (A14) still does not match Eq. (24). To obtain this equation, we follow the simple, but restrictive, derivation of Ref. [53], where it is assumed that the environment is in a state where C_B decays exponentially in time with a rate $t_c(\eta)$ and is real, so that no Lamb shift term appears. These assumptions transform Eq. (A14) to an equation of the simplified form (24), which we use in this work. In addition, we can write the resulting kernel as

$$\mathfrak{F}_k(\eta) \propto G(\eta)^2 t_c(\eta) \mathcal{C}_B(\eta, 0; k). \quad (\text{A16})$$

This form shows that the magnitude and time dependence of the decoherence kernel, which we parametrized using k_{Γ}/k and $a^{p-3}(\eta)$, depend on the interaction constant $G(\eta)$

but also on the environmental autocorrelation function $\mathcal{C}_B$ and the timescale t_c .

2. Validity

The derivations of Eqs. (A14) and (24) rest crucially on the assumptions that the interaction is perturbative and that the environment leads to an approximately Markovian dynamics. These assumptions must be carefully checked when considering a concrete interaction model. In general, it is tricky to estimate for which values of the phenomenological parameters k_Γ/k and p Eq. (24) is valid. For instance, the perturbative assumption requires $\tilde{H}_{\text{int}}$ to be small so that p and k_Γ cannot be arbitrarily large. However, there is no one-to-one relation between the magnitude of $\tilde{H}_{\text{int}}$ and that of $\mathfrak{F}_k$. For example, even in the simple model of Ref. [53], $\mathfrak{F}_k$ also contains the autocorrelation time t_c and autocorrelation function $\mathcal{C}_B$, which can be time dependent; see Eq. (A16). Therefore, in this paper, we remain completely agnostic about the range of validity of the equation and solve it in full generality.

To close this appendix, we mention two specific situations in which the assumptions required to derive the master equation were shown to be valid. First, in Ref. [53], the authors demonstrated that an environment made up of a heavy scalar field interacting with the cosmological perturbations during slow-roll inflation satisfies these assumptions, provided the coupling is small enough. They derived Eq. (24) with a time dependence $p \approx 5$. Second, in

Ref. [43], the authors considered the self-coupling of inflationary perturbations in slow-roll inflation, obtained at second order in perturbation theory [60]. These interactions thus set the minimal amount of decoherence expected in any single-field slow-roll inflation. Considering the interaction of a pair of modes $\pm k$ corresponding to CMB scales, interacting with shorter, but still super-Hubble, scales, they identified a dominant interaction channel among the different cubic terms. This channel gives a Hamiltonian of the form (A1) and leads to a master equation of the form Eq. (A14). The time dependence of the imaginary part also gives $p \approx 5$. This computation was refined and corrected in Ref. [47], where the authors took into account all leading-order terms and derived a generalized equation in which one must consider additional kernels in $\hat{p}_k^2$ and cross-terms $\hat{v}_k \hat{p}_{-k}$. We further comment on the application of our results to those of Refs. [43,47] in the conclusion; see Sec. V.

APPENDIX B: LATE-TIME EXPANSIONS IN DE SITTER IN PRESENCE OF DECOHERENCE

1. Covariance matrix elements

Exact expressions can be obtained for the covariance-matrix elements in de Sitter, where the mode function is given by Eq. (38), and in the presence of interactions modeled by Eq. (24). Their form is derived in Ref. [28] and depends on the evaluation of the following integral,

$$A_\alpha(x; x_E) = A_\alpha^R + iA_\alpha^I = \int_{x_E}^x e^{2iy} y^\alpha dy = -2^{-1-\alpha} (-i)^{-1-\alpha} [\Gamma(1+\alpha, -2ix) - \Gamma(1+\alpha, -2ix_E)], \quad (\text{B1})$$

where we first split A_α into its real and imaginary parts, and used $\Gamma(a, z) = \int_z^{+\infty} t^{a-1} e^{-t} dt$, the incomplete Gamma function [79]. We are interested in obtaining expressions valid at late times $x \rightarrow 0^+$, where the integral is convergent. We thus define

$$\mathcal{A}_\alpha^R = A_\alpha^R(0; x_E), \quad \mathcal{A}_\alpha^I = A_\alpha^I(0; x_E), \quad (\text{B2})$$

making the x_E dependence implicit. To obtain the leading-order terms in the expressions of the effective squeezing parameters and correlation measures, we need to expand A_α to sufficiently high order as follows:

$$\begin{aligned} A_\alpha(x; x_E) = & \mathcal{A}_\alpha^R + \frac{x^{1+\alpha}}{1+\alpha} - \frac{2x^{3+\alpha}}{3+\alpha} - \frac{2x^{5+\alpha}}{3(5+\alpha)} - \frac{4x^{7+\alpha}}{45(7+\alpha)} + \mathcal{O}(x^{9+\alpha}) \\ & + i \left[\mathcal{A}_\alpha^I + \frac{2x^{2+\alpha}}{2+\alpha} - \frac{4x^{3+\alpha}}{3(4+\alpha)} + \frac{4x^{6+\alpha}}{15(6+\alpha)} - \frac{8x^{8+\alpha}}{315(8+\alpha)} + \mathcal{O}(x^{10+\alpha}) \right]. \end{aligned} \quad (\text{B3})$$

Using this expansion, we can obtain corresponding expansions for the covariance-matrix elements. Details are given in Appendix D of Ref. [28] (the main text only quotes the small- x_E expansions). The above discussion gives Eq. (42), which we reproduce here,

$$\gamma_{11} = \frac{1}{x^2} \left\{ 1 + \mathcal{O}(x^2) - 2 \left(\frac{k_\Gamma}{k} \right)^2 [B_{11} + \mathcal{O}(x^2) + A_{11}x^{8-p} + \mathcal{O}(x^{10-p})] \right\}, \quad (\text{B4a})$$

$$\gamma_{12} = \frac{1}{x^3} \left\{ 1 - 2 \left(\frac{k_\Gamma}{k} \right)^2 [B_{12} + \mathcal{O}(x^2) + A_{12}x^{8-p} + \mathcal{O}(x^{10-p})] \right\}, \quad (\text{B4b})$$

$$\gamma_{22} = \frac{1}{x^4} \left\{ 1 + \mathcal{O}(x^2) - 2 \left(\frac{k_\Gamma}{k} \right)^2 [B_{22} + \mathcal{O}(x^2) + A_{22}x^{8-p} + \mathcal{O}(x^{10-p})] \right\}, \quad (\text{B4c})$$

where

$$A_{11}(p) = -\frac{2}{(p-8)(p-5)(p-2)}, \quad (\text{B5})$$

$$B_{11}(p, x_E) = \frac{1}{2} \left[\frac{(\mathcal{L}_E H)^{p-4}}{p-4} + \frac{(\mathcal{L}_E H)^{p-2}}{p-2} - \mathcal{A}_{1-p}^R - 2\mathcal{A}_{2-p}^I + \mathcal{A}_{3-p}^R \right], \quad (\text{B6})$$

$$A_{12}(p) = -\frac{(p-6)}{(p-8)(p-5)(p-2)}, \quad (\text{B7})$$

$$B_{12}(p, x_E) = B_{11}, \quad (\text{B8})$$

$$A_{22}(p) = -\frac{26 + p(p-11)}{(p-8)(p-5)(p-2)}, \quad (\text{B9})$$

$$B_{22}(p, x_E) = B_{11}. \quad (\text{B10})$$

Two remarks are in order. First, we checked that for $p > 8$ the term neglected in γ_{11} is of order $\mathcal{O}(x^{8-p})$, despite being indicated as $\mathcal{O}(x^{7-p})$ in Ref. [28]. Second, by definition, $\gamma_{11} > 0$, irrespective of the value of k_Γ and p . Thus, for $p < 8$, where the constant term dominates, it must always be positive. Taking $k_\Gamma/k \rightarrow \infty$, this implies that $B_{11}(p, x_E) < 0$ for $p < 8$ and arbitrary x_E .

2. Exact expression of purity in presence of decoherence

A first method to obtain an exact expression for the purity is to integrate Eq. (29) from η_{in} to η . We have

$$\begin{aligned} p_k^{-1} &= 1 + 4k \int_{\eta_{\text{in}}}^{\eta} \mathfrak{F}_k(\eta') \gamma_{11}(\eta') d\eta', \\ &= 1 + 4k \int_{\eta_{\text{in}}}^{\eta} \mathfrak{F}_k(\eta') |u_k|^2(\eta') d\eta' + 16k^2 \int_{\eta_{\text{in}}}^{\eta} \mathfrak{F}_k(\eta') \left\{ \int_{\eta_{\text{in}}}^{\eta'} \mathfrak{F}_k(\eta'') \text{Im}^2[u_k(\eta') u_k^*(\eta'')] d\eta'' \right\} d\eta', \\ &= 1 + 4k \int_{\eta_{\text{in}}}^{\eta} \mathfrak{F}_k(\eta') |u_k|^2(\eta') d\eta' + 8k^2 \int_{\substack{\eta_{\text{in}} \leq \eta' \leq \eta \\ \eta_{\text{in}} \leq \eta'' \leq \eta}} \mathfrak{F}_k(\eta') \mathfrak{F}_k(\eta'') \text{Im}^2[u_k(\eta') u_k^*(\eta'')] d\eta' d\eta', \end{aligned} \quad (\text{B11})$$

where, moving from the second line to the third line, we used the symmetry of the integrand to double the integration domain from $\eta_{\text{in}} \leq \eta' \leq \eta'' \leq \eta$ to $(\eta_{\text{in}} \leq \eta' \leq \eta'' \leq \eta) \cup (\eta_{\text{in}} \leq \eta'' \leq \eta' \leq \eta) = \{(\eta', \eta'') \in [\eta_{\text{in}}, \eta]^2\}$. Expanding the imaginary part, we obtain

$$\begin{aligned} \text{Im}^2[u_k(\eta') u_k^*(\eta'')] &= \frac{1}{4} \{ 2|u_k|^2(\eta') |u_k|^2(\eta'') \\ &\quad - [u_k^2(\eta') (u_k^*)^2(\eta'') + \text{c.c.}] \}. \end{aligned} \quad (\text{B12})$$

Inserting this expression under the integral and switching the dummy indices $\eta'' \leftrightarrow \eta'$, the two terms on the right-hand side give the same contribution, and we get

$$\begin{aligned} &\int_{\substack{\eta_{\text{in}} \leq \eta' \leq \eta \\ \eta_{\text{in}} \leq \eta'' \leq \eta}} \mathfrak{F}_k(\eta') \mathfrak{F}_k(\eta'') \text{Im}^2[u_k(\eta') u_k^*(\eta'')] d\eta' d\eta'' \\ &= \frac{1}{2} (\mathcal{L}_k^2 - |\mathcal{M}_k|^2), \end{aligned} \quad (\text{B13})$$

where $\mathcal{L}_k$ and $\mathcal{M}_k$ are given by Eqs. (34a) and (34b), which yields Eq. (33).

A second method to obtain Eq. (33) is to directly insert the exact expressions for γ_{ij} into Eq. (19). It is instructive to see how the computation proceeds. First, we have

$$\begin{aligned} p_k^{-1} &= \frac{1}{k^2} (|u_k|^2 |u'_k|^2 - \text{Re}^2[u_k(u'_k)^*]) + \mathcal{I}_k \frac{|u'_k|^2}{k^2} \\ &\quad + \mathcal{K}_k |u_k|^2 - 2\mathcal{J}_k \frac{\text{Re}[u_k(u'_k)^*]}{k} + \mathcal{I}_k \mathcal{K}_k - \mathcal{J}_k^2, \end{aligned} \quad (\text{B14})$$

and we can compute each of these terms directly. An important relation is

$$|u_k|^2 |u'_k|^2 - \text{Re}^2[u_k(u'_k)^*] = -\frac{W}{4} = k^2, \quad (\text{B15})$$

where we identified the Wronskian W and used its normalization $W = 2ik$ for the solution considered here, as explained below Eq. (32). By simple but tedious algebra, expanding the real and imaginary parts in the integrals, factoring out the functions depending on η' and η'' , and rearranging the terms in u_k using the Wronskian, we obtain

$$\mathcal{I}_k \frac{|u'_k|^2}{k^2} + \mathcal{K}_k |u_k|^2 - 2\mathcal{J}_k \frac{\text{Re}[u_k(u'_k)^*]}{k} = 2\mathcal{L}_k, \quad (\text{B16})$$

$$\mathcal{I}_k \mathcal{K}_k - \mathcal{J}_k^2 = \mathcal{L}_k^2 - |\mathcal{M}_k|^2. \quad (\text{B17})$$

One can check that inserting these expressions into Eq. (B14) indeed reproduces Eq. (33).

3. Late-time expansion of purity

We now consider the case of de Sitter space, where the mode function u_k is given by Eq. (38), and take the coupling to be of the form given in Eq. (25). We can then explicitly evaluate the integrals entering Eq. (33) as follows:

$$\mathcal{L}_k = \left(\frac{k_\Gamma}{k}\right)^2 \left(\frac{x_E^{2-p}}{2-p} + \frac{x_E^{4-p}}{4-p} - \frac{x^{2-p}}{2-p} - \frac{x^{4-p}}{4-p} \right), \quad (\text{B18})$$

$$\mathcal{M}_k = -\left(\frac{k_\Gamma}{k}\right)^2 (-A_{1-p} + A_{3-p} + 2iA_{2-p}). \quad (\text{B19})$$

In the main text, we computed the purity for exponentially small values of x . To perform this computation numerically, we need an analytic approximation of the above expressions. As explained in the main text above Eq. (43), this is most easily achieved by starting from Eq. (33) and expanding the integrals $\mathcal{L}_k$ and $\mathcal{M}_k$ in powers of x , while keeping all orders in k_Γ and x_E . In practice, this only requires expanding the terms in A_α using Eq. (B3). Doing so leads to

$$\mathcal{L}_k = \left(\frac{k_\Gamma}{k}\right)^2 (A_{\mathcal{L}} + B_{\mathcal{L}} x^{2-p} + C_{\mathcal{L}} x^{4-p}), \quad (\text{B20})$$

with

$$A_{\mathcal{L}} = \frac{x_E^{4-p}}{4-p} + \frac{x_E^{2-p}}{2-p}, \quad B_{\mathcal{L}} = -\frac{1}{2-p}, \quad C_{\mathcal{L}} = -\frac{1}{4-p}, \quad (\text{B21})$$

and

$$\begin{aligned} \mathcal{M}_k = & -i \left(\frac{k_\Gamma}{k}\right)^2 [A_{\mathcal{M},\text{I}} + B_{\mathcal{M},\text{I}} x^{5-p} + C_{\mathcal{M},\text{I}} x^{7-p} + \mathcal{O}(x^{9-p})] \\ & - \left(\frac{k_\Gamma}{k}\right)^2 [A_{\mathcal{M},\text{R}} + B_{\mathcal{M},\text{R}} x^{2-p} + C_{\mathcal{M},\text{R}} x^{4-p} + D_{\mathcal{M},\text{R}} x^{6-p} + E_{\mathcal{M},\text{R}} x^{8-p} + \mathcal{O}(x^{10-p})], \end{aligned} \quad (\text{B22})$$

where

$$A_{\mathcal{M},\text{R}} = -\mathcal{A}_{1-p}^R + \mathcal{A}_{3-p}^R - 2\mathcal{A}_{2-p}^I, \quad (\text{B23})$$

$$B_{\mathcal{M},\text{R}} = -\frac{1}{2-p}, \quad (\text{B24})$$

$$C_{\mathcal{M},\text{R}} = -\frac{1}{4-p}, \quad (\text{B25})$$

$$D_{\mathcal{M},\text{R}} = 0, \quad (\text{B26})$$

$$E_{\mathcal{M},\text{R}} = \frac{2}{9(8-p)}, \quad (\text{B27})$$

$$A_{\mathcal{M},\text{I}} = -\mathcal{A}_{1-p}^I + \mathcal{A}_{3-p}^I + 2\mathcal{A}_{2-p}^R, \quad (\text{B28})$$

$$B_{\mathcal{M},\text{I}} = -\frac{2}{3(5-p)}, \quad (\text{B29})$$

$$C_{\mathcal{M},\text{I}} = -\frac{4}{15(7-p)}. \quad (\text{B30})$$

Note that $D_{\mathcal{M},\text{R}}$ vanishes due to exact cancellations of terms coming from different integrals. Inserting the expressions in Eqs. (B20) and (B22) into Eq. (33), we obtain

$$p_k^{-1} = 1 + \left(\frac{k_\Gamma}{k}\right)^2 [A_\sigma^{(1)} + B_\sigma^{(1)} x^{2-p} + \mathcal{O}(x^{4-p})] + \left(\frac{k_\Gamma}{k}\right)^4 [A_\sigma^{(2)} + B_\sigma^{(2)} x^{2-p} + \mathcal{O}(x^{4-p}) + D_\sigma^{(2)}(p, x_E) x^{10-2p} + \mathcal{O}(x^{12-2p})]. \quad (\text{B31})$$

The coefficients of this expansion are

$$A_\sigma(p, x_E) = \left(\frac{k_\Gamma}{k}\right)^2 A_\sigma^{(1)} + \left(\frac{k_\Gamma}{k}\right)^4 A_\sigma^{(2)}, \quad (\text{B32})$$

$$A_\sigma^{(1)} = 2A_{\mathcal{L}}, \quad A_\sigma^{(2)} = A_{\mathcal{L}}^2 - A_{\mathcal{M},\text{R}}^2 - A_{\mathcal{M},\text{I}}^2, \quad (\text{B33})$$

$$B_\sigma(p, x_E) = \left(\frac{k_\Gamma}{k}\right)^2 B_\sigma^{(1)} + \left(\frac{k_\Gamma}{k}\right)^4 B_\sigma^{(2)}, \quad (\text{B34})$$

$$B_\sigma^{(1)} = 2B_{\mathcal{L}}, \quad B_\sigma^{(2)} = 2(A_{\mathcal{L}}B_{\mathcal{L}} - A_{\mathcal{M},\text{R}}B_{\mathcal{M},\text{R}}), \quad (\text{B35})$$

$$C_\sigma(p, x_E) = C_{\mathcal{L}}^2 - C_{\mathcal{M},\text{R}}^2 + 2B_{\mathcal{M},\text{R}}D_{\mathcal{M},\text{R}}, \quad (\text{B36})$$

$$D_\sigma(p, x_E) = \left(\frac{k_\Gamma}{k}\right)^2 D_\sigma^{(1)} + \left(\frac{k_\Gamma}{k}\right)^4 D_\sigma^{(2)}, \quad (\text{B37})$$

$$D_\sigma^{(1)} = 0, \quad D_\sigma^{(2)} = -(2C_{\mathcal{M},\text{R}}D_{\mathcal{M},\text{R}} + 2B_{\mathcal{M},\text{R}}E_{\mathcal{M},\text{R}} + B_{\mathcal{M},\text{I}}^2). \quad (\text{B38})$$

Although it does not appear in the final expression, we have defined the term C_σ , which would be associated with x^{8-2p} . This term vanishes, but this is due to a nontrivial combination of cancellations between $\mathcal{L}_k$ and $\mathcal{M}_k$, as well as the accidental cancellation of $D_{\mathcal{M},\text{R}}$. In addition, by inserting the expansions for the covariance-matrix elements (42) into the expression for the purity, Eq. (19), and identifying the different terms in the series, one can check that the following relations hold:

$$B_\sigma = 2\left(\frac{k_\Gamma}{k}\right)^2 \left\{ 2A_{12} \left[1 - 2\left(\frac{k_\Gamma}{k}\right)^2 B_{12}(p, x_E) \right] - A_{11}A_{22} \left[1 - 2\left(\frac{k_\Gamma}{k}\right)^2 B_{22}(p, x_E) \right] \right. \\ \left. \times \left[1 - 2\left(\frac{k_\Gamma}{k}\right)^2 B_{11}(p, x_E) \right] \right\}, \quad (\text{B39})$$

$$D_\sigma = 4\left(\frac{k_\Gamma}{k}\right)^4 [A_{11}A_{22} - A_{12}^2]. \quad (\text{B40})$$

4. Accuracy of numerical approximations and estimates

a. Small- x expansions of covariance matrix elements and purity

Let us check that the expansions we derived for γ_{ij} and p_k do give good numerical approximations of these quantities, i.e., that they capture the largest contribution to the quantity when evaluated at small but finite $x \ll 1$, and for all values of k_Γ/k and p considered. We start with the expansions of Eq. (42) for the covariance matrix elements γ_{ij} . We have to check whether and when the terms fully computed in the expression are comparable to or smaller than the contributions appearing as $\mathcal{O}$. To do so, we check that the ratio of the maximum of the $\mathcal{O}$ -terms to the terms used to approximate γ_{ij} is less than unity, i.e., that

$$x^2 \frac{\max \left[1, \left(\frac{k_\Gamma}{k}\right)^2, \left(\frac{k_\Gamma}{k}\right)^2 x^{8-p} \right]}{1 - 2\left(\frac{k_\Gamma}{k}\right)^2 (B_{ij} + A_{ij}x^{8-p})} \leq 1. \quad (\text{B41})$$

First, let us study when this ratio could be large in principle. The terms in the numerator, corresponding to the $\mathcal{O}$ -terms, are suppressed by an additional factor of x^2 with respect to those in the denominator, corresponding to the terms fully computed in the estimate of γ_{ij} . Moreover, except for this x^2 factor, the numerator and denominator have the same structure in $(k_\Gamma/k)^a x^b$. Thus, if one such term is much larger than the others, it dominates in both the numerator and denominator, and the ratio is of order x^2 , hence small. Therefore, to escape these asymptotic cases and have a large ratio, some cancellations must occur between the different terms in the denominator (and not simultaneously occur in the numerator). Let us identify exhaustively the cases for which this can happen. For the first two terms in the denominator to cancel (assuming again B_{ij} of order unity), we must have $k_\Gamma/k \approx 1$. For the second and third terms to cancel, we should have $x^{8-p} \approx 1$ so $p \approx 8$. For the first and third terms, we should have $x^{8-p}(k_\Gamma/k)^2 \approx 1$, or equivalently $(8-p) \ln x = -2 \log_{10}(k_\Gamma/k)$. Finally, if all terms are to be of the same order, we have to consider $p \approx 8$ and $k_\Gamma/k \approx 1$. Outside these regimes of parameters, one term dominates, and the fraction is of order unity, and the whole ratio is of order x^2 . In summary, at a given late time $x \ll 1$, the expansion can only be inaccurate around the lines $(8-p) \ln x = -2 \log_{10}(k_\Gamma/k)$, $\log_{10}(k_\Gamma/k) = 0$, and $p = 8$. Next, we explicitly check whether the required cancellations do happen in practice. To do so, we numerically evaluate the ratio (B41) at $x = e^{-30}$ and $x = e^{-60}$, and for the parameter-space region $(\log_{10}(k_\Gamma/k), p) \in [-3, 3] \times [2, 10]$ considered in the figures of the text, see, for instance, Fig. 3. We find that the maximum value of the ratio in Eq. (B41) in the sample of points we used for our numerical evaluation of γ_{11} , γ_{12} , and γ_{22} is less than 10^{-24} for $x = e^{-30}$ and 10^{-50} for $x = e^{-60}$. Hence, no significant cancellations occur, and the ratio in Eq. (B41) is always very small.

Similarly, for the purity, we consider the ratio

$$\frac{\left(\frac{k_\Gamma}{k}\right)^2 x^{4-p} \max \left[1, \left(\frac{k_\Gamma}{k}\right)^2, \left(\frac{k_\Gamma}{k}\right)^2 x^{8-p} \right]}{1 + A_\sigma + B_\sigma x^{2-p} + D_\sigma x^{10-2p}} \leq 1, \quad (\text{B42})$$

and we checked as well that for $x = e^{-30}$ and $x = e^{-60}$ and for $(\log_{10}(k_{\Gamma}/k), p) \in [-3, 3] \times [2, 10]$ it is less than 10^{-24} for $x = e^{-30}$ and less than 10^{-50} for $x = e^{-60}$. In summary, approximating γ_{ij} and p_k^{-1} by dropping the $\mathcal{O}$ terms in Eqs. (42) and (43) provides very good approximations in the studied parameter space, since these $\mathcal{O}$ -terms are negligible by several orders of magnitude.

b. Estimates of ratios

Next, we consider ratios of covariance matrix elements γ_{12}/γ_{22} and γ_{11}/γ_{12} since they appear when computing the effective squeezing parameters. First, note that when Eq. (42) give good approximations γ_{ij}^{est} of the covariance matrix elements γ_{ij} in the sense that

$$|r\gamma_{ij}| = \left| \frac{\gamma_{ij}^{\text{est}} - \gamma_{ij}}{\gamma_{ij}} \right| \ll 1$$

then their ratios give good approximations of the ratios γ_{ij}/γ_{kl} . Indeed,

$$\frac{\left| \frac{\gamma_{ij}}{\gamma_{kl}} - \frac{\gamma_{ij}^{\text{est}}}{\gamma_{kl}^{\text{est}}} \right|}{\left| \frac{\gamma_{ij}}{\gamma_{kl}} \right|} = \frac{|r\gamma_{kl} - r\gamma_{ij}|}{|1 + r\gamma_{kl}|} \ll 1.$$

This is generally valid for any ratio of well-approximated quantities. Moreover, we want to show that the specific ratios γ_{12}/γ_{22} and γ_{11}/γ_{12} can generically be taken as being of order x when $x \ll 1$. The same reasoning applies to both ratios, and we consider the first one for concreteness. We have

$$\frac{\gamma_{12}}{\gamma_{22}} = x \frac{1 - 2\left(\frac{k_{\Gamma}}{k}\right)^2 [B_{11} + \mathcal{O}(x^2) + A_{12}x^{8-p} + \mathcal{O}(x^{10-p})]}{1 + \mathcal{O}(x^2) - 2\left(\frac{k_{\Gamma}}{k}\right)^2 [B_{11} + \mathcal{O}(x^2) + A_{22}x^{8-p} + \mathcal{O}(x^{10-p})]}. \quad (\text{B43})$$

x appears as a factor of a fraction that we now argue is typically of order 1. First, since numerator and denominator have the same type of terms in x and k_{Γ}/k with coefficients of order unity, if any term is much larger than the others in the numerator, for instance, $(k_{\Gamma}/k)^2 x^{8-p} A_{12}$, then the corresponding term $(k_{\Gamma}/k)^2 x^{8-p} A_{22}$ similarly dominates in the denominator. Factoring out these terms, we are left with a fraction close to 1 multiplied by a factor of order unity, here A_{12}/A_{22} . The same reasoning applies starting from terms in the denominator. Second, the only situations where no single term dominates over all the others correspond to those studied previously in Appendix B 4 a for possible cancellations, i.e., around $(8-p)\ln x = -2\ln(k_{\Gamma}/k)$, $\ln(k_{\Gamma}/k) = 0$, and $p = 8$. We showed that no significant cancellations occur. Thus, on the one hand, if two of the three terms dominate and their combination does not cancel, remaining of the same order, e.g., $(k_{\Gamma}/k)^2 (B_{11} + A_{12}x^{8-p})$ of order $(k_{\Gamma}/k)^2$, then the same combination dominates in the denominator and the previous reasoning applies; the fraction is of order unity. On the other hand, if all three terms are of the same order, then they are of order unity, and, again because no significant cancellation occurs, so is their combination. Since the same simultaneously happens in the denominator, the fraction is of order unity. In summary, we schematically have

$$\frac{1}{x} \frac{\gamma_{ij}}{\gamma_{kl}} \approx \frac{1 - 2\left(\frac{k_{\Gamma}}{k}\right)^2 (B_{11} + A_{ij}x^{8-p})}{1 - 2\left(\frac{k_{\Gamma}}{k}\right)^2 (B_{11} + A_{kl}x^{8-p})} \approx 1. \quad (\text{B44})$$

Moreover, we numerically checked that for $(\log_{10}(k_{\Gamma}/k), p) \in [-3, 3] \times [2, 10]$ the ratios are numerically close to 1. For $x = e^{-30}$ and $x = e^{-60}$, we have $0.24 \leq \gamma_{12}/x\gamma_{22} \leq 1 + 10^{-14}$ and $0.49 \leq \gamma_{11}/x\gamma_{12} \leq 1 + 10^{-14}$. Finally, we consider the ratio $\gamma_{22}/p_k^{-1/2}$. We want to show it is at least of order x for the range of parameters considered in this paper. First, we can run an analytical argument similar to that for the ratios γ_{ij}/γ_{kl} by considering all cases where one term dominates in the numerator and another in the denominator. We skip this lengthy analysis here and directly compute numerically the ratio $x\gamma_{22}/p_k^{-1/2}$ using the approximations Eqs. (42c) and (43). We find that for $(\log_{10}(k_{\Gamma}/k), p) \in [-3, 3] \times [2, 10]$ and $x > e^{-30}$, $x\gamma_{22}/p_k^{-1/2} \geq 2.5$, confirming that in the limit $x \ll 1$ we can in practice take

$$p_k^{1/2}\gamma_{22} \geq x^{-1}. \quad (\text{B45})$$

5. Squeezing parameter r_k

To derive an expansion for r_k , we start from Eq. (35) and express the inverse hyperbolic cosine as a logarithm. We have

$$\begin{aligned} r_k &= \frac{1}{2} \ln \left[\frac{\gamma_{22} + \gamma_{11}}{2p_k^{-1/2}} + \sqrt{\left(\frac{\gamma_{22} + \gamma_{11}}{2p_k^{-1/2}} \right)^2 - 1} \right], \\ &= \frac{1}{2} \ln \left(\frac{\gamma_{22}}{p_k^{-1/2}} \right) + \frac{1}{2} \ln \left[\frac{1}{2} + \frac{\gamma_{11}}{2\gamma_{22}} + \frac{1}{2} \left(1 - \frac{\gamma_{11}}{\gamma_{22}} \right) \sqrt{1 + \left(\frac{2\gamma_{12}}{\gamma_{22}} \frac{1}{1 - \frac{\gamma_{11}}{\gamma_{22}}} \right)^2} \right], \end{aligned} \quad (\text{B46})$$

where we used the expression of the purity in terms of the covariance-matrix elements, Eq. (19). Rewriting r_k in the form of Eq. (B46) isolates the large contribution in the first term, while the second term only depends on small ratios. Using the smallness of ratios characterized by Eq. (B43), we can expand the argument of the second term and obtain

$$\frac{1}{2} + \frac{\gamma_{11}}{2\gamma_{22}} + \frac{1}{2} \left(1 - \frac{\gamma_{11}}{\gamma_{22}} \right) \sqrt{1 + \left(\frac{2\gamma_{12}}{\gamma_{22}} \frac{1}{1 - \frac{\gamma_{11}}{\gamma_{22}}} \right)^2} = 1 + \left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 \frac{\gamma_{11}}{\gamma_{22}} \right] + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^4 \right]. \quad (\text{B47})$$

Inserting this expansion into Eq. (B46), we recover Eq. (45). This expression is the most useful for numerical approximation but is unfortunately not very explicit, since we still have to compare the behaviors of γ_{22}^2 and p_k^{-1} . To obtain a more explicit formula, we need to use the expansions of γ_{22} and p_k^{-1} for small x and expand the first term, which gives

$$r_k = -2 \ln(x) + \frac{1}{2} \ln \left[\frac{1 - 2\left(\frac{k_F}{k}\right)^2 B_{22} + \mathcal{O}(x^2) - 2\left(\frac{k_F}{k}\right)^2 A_{22} x^{8-p} + \mathcal{O}(x^{10-p})}{\sqrt{1 + A_\sigma + B_\sigma x^{2-p} + \mathcal{O}(x^{4-p}) + D_\sigma x^{10-2p} + \mathcal{O}(x^{12-2p})}} \right], \quad (\text{B48})$$

where the dominant term inside the brackets depends on the value of p . Note that the $\mathcal{O}(x^2)$ term arising from the expansion of the second term has been reabsorbed into the logarithm. First, for $p < 2$, the denominator corresponding to p_k^{-1} tends to a nonvanishing constant, and so does the numerator. Second, for $2 < p < 8$, the numerator still tends to a constant, while the denominator diverges and the term in x^{2-p} dominates. Third, for $p > 8$, both the numerator and the denominator diverge: the term in x^{8-p} dominates in the numerator, and that in x^{10-2p} dominates in the denominator. The resulting expressions are given in Eq. (46).

6. Squeezing angle φ_k

Similarly to our approach for r_k , we use the hierarchy between the values of the γ_{ij} to isolate the small contributions to the asymptotic values of the cosine and sine. In this case, we only need to note that for $x \ll 1$, $\gamma_{22} > \gamma_{11}$, so that we can factor out terms under the square root in Eq. (36) to obtain

$$\sin(2\varphi_k) = \frac{\tan(2\varphi_k)}{\sqrt{1 + \tan^2(2\varphi_k)}}, \quad (\text{B49a})$$

$$\cos(2\varphi_k) = \frac{1}{\sqrt{1 + \tan^2(2\varphi_k)}}. \quad (\text{B49b})$$

We then expand the tangent in terms of small ratios of the covariance matrix elements

$$\begin{aligned} \tan(2\varphi_k) &= -2 \frac{\gamma_{12}}{\gamma_{22}} \left[1 - \frac{\gamma_{11}}{\gamma_{22}} \right]^{-1}, \\ &= -2 \frac{\gamma_{12}}{\gamma_{22}} \left[1 + \mathcal{O} \left(\frac{\gamma_{11}}{\gamma_{22}} \right) \right], \end{aligned} \quad (\text{B50})$$

where we have used the order of the ratios Eq. (B43). Inserting this into the expressions for the cosine and sine, we recover Eq. (49). Finally, taking the arctan of the expression for the tangent, we obtain

$$\varphi_k = -x \frac{\gamma_{12}}{\gamma_{22}} \left\{ 1 + \mathcal{O} \left(\frac{\gamma_{11}}{\gamma_{22}} \right) + \mathcal{O} \left[\left(\frac{\gamma_{12}}{\gamma_{22}} \right)^2 \right] \right\}, \quad (\text{B51})$$

We now have to use Eqs. (42b) and (42c) and distinguish cases in p in order to obtain an explicit expression. If $p < 8$, the constant terms in γ_{12} and γ_{22} dominate in the numerator and denominator; otherwise, the term in x^{8-p} dominates. In both cases, the fraction tends to a constant, and we recover Eq. (51). The same procedure leads to Eqs. (50a) and (50b). Note that once the expression for the sine is known that for the cosine can be derived directly using $\cos^2(2\varphi_k) + \sin^2(2\varphi_k) = 1$.

APPENDIX C: USEFUL APPROXIMATIONS OF QUANTUM CORRELATIONS MEASURES IN PRESENCE OF DECOHERENCE

In this appendix, we derive useful approximations of the quantum discord given in Eq. (63) and the Bell operator expectation value given in Eq. (76).

1. Discord for small purity and large squeezing

We start by deriving an expression for the discord in the large-squeezing and small-purity regime in which cosmological perturbations are typically found. Following Ref. [18], we first rewrite the discord in terms of the semimajor and semiminor axis lengths, a_k and b_k defined in Eq. (23). We have

$$\mathcal{D}_{\pm k} = f\left[\frac{a_k^2 + b_k^2}{2}\right] - 2f(a_k b_k) + f\left(1 + 2b_k^2 \frac{1 - \frac{1}{a_k^2 b_k^2}}{1 + \frac{b_k^2}{a_k^2} + \frac{2}{a_k^2}}\right). \quad (\text{C1})$$

Next, we want to expand the discord in the small-squeezing $b_k/a_k = e^{-2r_k} \ll 1$ and small-purity $(a_k b_k)^{-1} = \sqrt{p_k} \ll 1$ limits. Note that, while in this case $a_k \gg 1$, the asymptotic behavior of b_k is not fixed. This is a hint that b_k will be the relevant parameter controlling the value of the discord. Since its value is a ratio involving the squeezing parameter r_k and the purity p_k , whether it is large or small depends on the relative magnitude of squeezing and decoherence. Before expanding, we rearrange the terms in the discord to single out contributions that can be large and write the remainder as combinations of small quantities. This gives Eq. (66), which involves both the functions f , defined in Eq. (65), and g , defined in Eq. (67). Their asymptotic behaviors are given by

$$f(x) \underset{x \rightarrow 1}{=} -\frac{1}{2}(x-1)\log_2(x-1) + \mathcal{O}(x-1), \quad (\text{C2a})$$

$$f(x) \underset{x \rightarrow +\infty}{=} \log_2(x) + \frac{1}{\ln 2} - 1 + \mathcal{O}\left(\frac{1}{x^2}\right), \quad (\text{C2b})$$

and

$$g(x) \underset{x \rightarrow 1}{=} 1 - \frac{1}{\ln 2} + \mathcal{O}[(x-1)\ln(x-1)], \quad (\text{C3a})$$

$$g(x) \underset{x \rightarrow +\infty}{=} -\frac{1}{6\ln 2} \frac{1}{x^2} + \mathcal{O}\left(\frac{1}{x^4}\right), \quad (\text{C3b})$$

Using these approximations, at first order in p_k and e^{-2r_k} , we obtain Eq. (68). Finally, we expand the function g in the

small- b_k and large- b_k cases. First, for $b_k \ll 1$, the largest small contribution is given by the terms in p_k , since $e^{-4r_k} = b_k^2 a_k^{-2} \ll a_k^{-2} = p_k b_k^2 \ll p_k$. We thus have

$$\mathcal{D}_{\pm k} = -2\log_2 b_k - \frac{1}{\ln 2} + \mathcal{O}(b_k^2 \ln b_k) + \mathcal{O}(p_k), \quad \text{for } b_k \ll 1. \quad (\text{C4})$$

The case $b_k \gg 1$ is more subtle, because the logarithmic term then gives a contribution proportional to b_k^{-2} that is small and need not be dominant compared to the other small terms in p_k and e^{-4r_k} . First, reversing the inequalities of the previous case, we find the ordering of the small contributions

$$e^{-4r_k} = b_k^2 a_k^{-2} \gg a_k^{-2} = p_k b_k^{-2} \gg p_k. \quad (\text{C5})$$

Second, we can check that $b_k^{-2} = a_k^2 p_k \gg p_k$. Third, if the purity is small enough compared to the squeezing, $p_k < e^{-12r_k}$, then the terms related to the squeezing parameter dominant over that in b_k , $b_k^{-2} < e^{-4r_k}$. However, in this expansion, the terms proportional to e^{-2r_k} are always further suppressed by powers of p_k , which is smaller than b_k^{-2} . Thus, the term proportional to b_k^{-2} is indeed dominant. Therefore, we have

$$\mathcal{D}_{\pm k} = \frac{b_k^{-2}}{2\ln 2} + \mathcal{O}(b_k^{-4}), \quad \text{for } b_k \gg 1. \quad (\text{C6})$$

Combining both cases gives Eq. (69).

2. Discord at late time

We derive a late-time approximation of the quantum discord. For $p > 2$, the purity goes to zero, and we can directly use Eqs. (53) and (68). However, for $p < 2$, we must reexpand all three evaluations of the function f in Eq. (63). The first term depends on the asymptotic behavior of $\sigma(p_k, r_k)$. The second term depends on that of p_k , which we already computed in Eq. (43). The third depends on $p_k \sigma(p_k, r_k)$. We start with $\sigma(p_k, r_k)$, which can be rewritten as

$$\sigma(p_k, r_k) = \frac{a_k^2}{2}(1 + e^{-4r_k}). \quad (\text{C7})$$

Since the squeezing is always positive asymptotically, the asymptotic behavior of $\sigma(p_k, r_k)$ is exponentially close to that of a_k , given in Eq. (53a). Next, we obtain an estimate for the small parameter e^{-4r_k} . Taking the exponential of Eq. (45), we obtain

$$e^{-4r_k} = \left(\frac{p_k^{-1/2}}{\gamma_{22}}\right)^2 \left\{1 + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right]\right\} = \left(\frac{p_k^{-1/2}}{\gamma_{22}}\right)^2 + \mathcal{O}\left[\left(\frac{p_k^{-1/2}}{\gamma_{22}}\right)^2 \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right]. \quad (\text{C8})$$

Using Eq. (B45), we have $p_k^{-1}/\gamma_{22}^2 \leq x^2$, so that e^{-4r_k} is at most of order x^2 . Inserting the two expansions, Eqs. (53a) and (C8), into the expression of $\sigma(p_k, r_k)$ in terms of a_k and e^{-4r_k} , we obtain

$$\begin{aligned} \ln \sigma(p_k, r_k) &= 2 \ln a_k - \ln 2 + e^{-4r_k} + \mathcal{O}(e^{-8r_k}), \\ &= \ln \gamma_{22} - \ln 2 + \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2 + e^{-4r_k} \\ &\quad + \mathcal{O}\left[\frac{p_k^{-1}}{\gamma_{22}^2} \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right] + \mathcal{O}\left(\frac{p_k^{-2}}{\gamma_{22}^4}\right), \end{aligned} \quad (\text{C9})$$

so

$$\sigma(p_k, r_k) = \frac{\gamma_{22}}{2} \left\{1 + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right] + \mathcal{O}(e^{-4r_k})\right\}. \quad (\text{C10})$$

The first term in the expression of $\mathcal{D}_{\pm k}$ thus reads

$$\begin{aligned} f[\sigma(p_k, r_k)] &= \log_2 \sigma(p_k, r_k) + \frac{1}{\ln 2} - 1 + \mathcal{O}[\sigma(p_k, r_k)^{-2}], \\ &= \log_2 \gamma_{22} + \frac{1}{\ln 2} - 2 + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right]. \end{aligned} \quad (\text{C11})$$

The term proportional to σ^{-2} is of order γ_{22}^{-2} , which is always smaller than $(\gamma_{12}/\gamma_{22})^2$ for $x \ll 1$ and thus negligible. The second term in the expression of $\mathcal{D}_{\pm k}$ is immediately obtained from Eq. (C2) as

$$f(p_k^{-1/2}) = \log_2(p_k^{-1/2}) + \frac{1}{\ln 2} - 1 + \mathcal{O}(p_k). \quad (\text{C12})$$

Finally, we move to the third term in the expression of $\mathcal{D}_{\pm k}$. The argument of f in the third term reads

$$\begin{aligned} \frac{\sigma(p_k, r_k) + p_k^{-1}}{\sigma(p_k, r_k) + 1} &= 1 + \frac{p_k^{-1} - 1}{\sigma(p_k, r_k)} \frac{1}{1 + \sigma(p_k, r_k)^{-1}}, \\ &= 1 + \frac{2(p_k^{-1} - 1)}{\gamma_{22}} \\ &\quad \times \left\{1 + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right] + \mathcal{O}(e^{-4r_k})\right\}, \end{aligned} \quad (\text{C13})$$

where we used the fact that $\sigma(p_k, r_k)^{-1}$ is of order γ_{22}^{-1} to neglect the dominant contribution of $1/[1 + \sigma(p_k, r_k)^{-1}]$ compared to the subdominant ones coming from $(p_k^{-1} - 1)/\sigma(p_k, r_k)$. Since the argument tends to 1 for $p < 6$ and to infinity for $p > 6$, we must now distinguish cases in order to expand f . Using Eq. (C2), we find for $p < 6$

$$\begin{aligned} f\left[\frac{\sigma(p_k, r_k) + p_k^{-1}}{\sigma(p_k, r_k) + 1}\right] &= -\frac{p_k^{-1} - 1}{\gamma_{22}} \log_2\left(\frac{p_k^{-1} - 1}{\gamma_{22}}\right) \\ &\quad + \mathcal{O}\left(\frac{p_k^{-1} - 1}{\gamma_{22}}\right) \end{aligned} \quad (\text{C14})$$

and for $p > 6$

$$\begin{aligned} f\left[\frac{\sigma(p_k, r_k) + p_k^{-1}}{\sigma(p_k, r_k) + 1}\right] &= \log_2\left(\frac{p_k^{-1} - 1}{\gamma_{22}}\right) + \frac{1}{\ln 2} \\ &\quad + \mathcal{O}\left[\left(\frac{p_k^{-1} - 1}{\gamma_{22}}\right)^{-1}\right] + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right] \\ &\quad + \mathcal{O}(e^{-4r_k}). \end{aligned} \quad (\text{C15})$$

Finally, combining all three terms in Eq. (63), we obtain the following expression for the discord:

$$\begin{aligned} \mathcal{D}_{\pm k} &= \log_2(\gamma_{22}) + \frac{1}{\ln 2} - 2 - 2f(p_k^{-1/2}) + \mathcal{O}\left[\frac{p_k^{-1} - 1}{\gamma_{22}} \log_2\left(\frac{p_k^{-1} - 1}{\gamma_{22}}\right)\right] \quad \text{for } p < 2, \\ &= \log_2(p_k \gamma_{22}) - \frac{1}{\ln 2} + \mathcal{O}(p_k) + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right] + \mathcal{O}\left[\frac{p_k^{-1} - 1}{\gamma_{22}} \log_2\left(\frac{p_k^{-1} - 1}{\gamma_{22}}\right)\right] \quad \text{for } 2 < p < 6, \\ &= \mathcal{O}(p_k) + \mathcal{O}\left[\left(\frac{p_k^{-1} - 1}{\gamma_{22}}\right)^{-1}\right] + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right] + \mathcal{O}(e^{-4r_k}) \quad \text{for } p > 6. \end{aligned} \quad (\text{C16})$$

Equation (C16) gives the expression of the discord for $p < 2$, where we cannot use Eq. (69), which requires the purity to be small. For $2 < p < 6$, one can check using Eq. (53) that this expression matches Eq. (69). On the other

hand, for $p > 6$, the leading terms cancel, and no estimate is obtained. Fortunately, for $p > 6$, we can use Eq. (69). We summarize the late-time approximation of the discord for arbitrary p in Eq. (80).

3. Bell operator at late time

We expand the expectation value of the Bell operator given in Eq. (76). The first term is given by p_k , and the second contains $\tanh(2r_k)|\cos(2\varphi_k)|$, which we must expand. First, we expand the hyperbolic tangent using Eq. (C8) and obtain

$$\begin{aligned} \tanh(2r_k) &= 1 - 2e^{-4r_k} + \mathcal{O}(e^{-8r_k}), \\ &= 1 - 2\left(\frac{p_k^{-1/2}}{\gamma_{22}}\right)^2 + \mathcal{O}\left[\left(\frac{p_k^{-1/2}}{\gamma_{22}}\right)^2 \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right]. \end{aligned} \quad (\text{C17})$$

Since asymptotically $\cos(2\varphi_k) > 0$, we can remove the absolute value in $\tanh(2r_k)|\cos(2\varphi_k)|$ and use Eq. (49a). We then obtain

$$\begin{aligned} |\cos(2\varphi_k)| \tanh(2r_k) &= 1 - 2\frac{\gamma_{11}}{\gamma_{22}} + \mathcal{O}\left[\left(\frac{p_k^{-1/2}}{\gamma_{22}}\right)^2 \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right] \\ &\quad + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2 \frac{\gamma_{11}}{\gamma_{22}}\right], \end{aligned} \quad (\text{C18})$$

where we used $p_k^{-1} = \gamma_{11}\gamma_{22} - \gamma_{12}^2$ and all $\mathcal{O}$ terms are of order $\mathcal{O}(x^4)$ at the largest. Expanding the arcsine close to 1, we obtain

$$\frac{4}{\pi^2} \arcsin^2 y = 1 - \frac{4\sqrt{2}}{\pi} \sqrt{1-y} + \mathcal{O}(1-y). \quad (\text{C19})$$

Combining this with the expansion (C18), we obtain

$$\begin{aligned} \frac{4}{\pi^2} \arcsin^2 [|\cos(2\varphi_k)| \tanh(2r_k)] \\ = 1 - \frac{8}{\pi} \sqrt{\frac{\gamma_{11}}{\gamma_{22}}} + \mathcal{O}\left[\left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\right], \end{aligned} \quad (\text{C20})$$

where the term in $\mathcal{O}$ is order $\mathcal{O}(x^2)$. When Eq. (C20) is inserted into the expression (76) for the expectation value, gives Eq. (82). Further expanding the purity finally yields Eq. (83).

-
- [1] A. H. Guth, The inflationary universe: A possible solution to the horizon and flatness problems, *Phys. Rev. D* **23**, 347 (1981).
 - [2] V. F. Mukhanov and G. V. Chibisov, Quantum fluctuations and a nonsingular Universe, *JETP Lett.* **33**, 532 (1981).
 - [3] D. Sudarsky, Shortcomings in the understanding of why cosmological perturbations look classical, *Int. J. Mod. Phys. D* **20**, 509 (2011).
 - [4] A. Albrecht, P. Ferreira, M. Joyce, and T. Prokopec, Inflation and squeezed quantum states, *Phys. Rev. D* **50**, 4807 (1994).
 - [5] D. Polarski and A. A. Starobinsky, Semiclassicality and decoherence of cosmological perturbations, *Classical Quantum Gravity* **13**, 377 (1996).
 - [6] J. Lesgourgues, D. Polarski, and A. A. Starobinsky, Quantum to classical transition of cosmological perturbations for nonvacuum initial states, *Nucl. Phys. B* **497**, 479 (1997).
 - [7] C. Kiefer and D. Polarski, Emergence of classicality for primordial fluctuations: Concepts and analogies, *Ann. Phys. (Berlin)* **510**, 137 (1998).
 - [8] C. Kiefer, D. Polarski, and A. A. Starobinsky, Quantum to classical transition for fluctuations in the early Universe, *Int. J. Mod. Phys. D* **07**, 455 (1998).
 - [9] C. Kiefer, J. Lesgourgues, D. Polarski, and A. A. Starobinsky, The coherence of primordial fluctuations produced during inflation, *Classical Quantum Gravity* **15**, L67 (1998).
 - [10] J. Martin, Cosmic inflation, quantum information and the pioneering role of John S Bell in cosmology, *Universe* **5**, 92 (2019).
 - [11] A. Micheli and P. Peter, Quantum Cosmological gravitational waves?, in *Handbook of Quantum Gravity*, edited by C. Bambi, L. Modesto, and I. Shapiro (Springer Nature, Singapore, 2023), pp. 1–66, [10.1007/978-981-19-3079-9_10-1](https://doi.org/10.1007/978-981-19-3079-9_10-1).
 - [12] A. Belfiglio, O. Luongo, and S. Mancini, Quantum entanglement in cosmology, *Phys. Rep.* **1146**, 1 (2025).
 - [13] D. Campo and R. Parentani, Inflationary spectra and violations of Bell inequalities, *Phys. Rev. D* **74**, 025001 (2006).
 - [14] J. Martin and V. Vennin, Obstructions to Bell CMB experiments, *Phys. Rev. D* **96**, 063501 (2017).
 - [15] J. Maldacena, A model with cosmological Bell inequalities, *Fortschr. Phys.* **64**, 10 (2016).
 - [16] C. M. Sou, J. Wang, and Y. Wang, Cosmological Bell tests with decoherence effects, *J. Cosmol. Astropart. Phys.* **10** (2024) 084.
 - [17] D. Campo and R. Parentani, Inflationary spectra and partially decohered distributions, *Phys. Rev. D* **72**, 045015 (2005).
 - [18] J. Martin, A. Micheli, and V. Vennin, Comparing quantumness criteria, *Europhys. Lett.* **142**, 18001 (2023).
 - [19] I. Agullo, B. Bonga, and P. Ribes-Metidieri, Inflation does not create entanglement in local observables, *Classical Quantum Gravity* **43**, 01LT01 (2026).

- [20] P. Ribes-Metidieri, I. Agullo, and B.onga, Entanglement and correlations between local observables in de Sitter spacetime, *Phys. Rev. D* **113**, 065001 (2026).
- [21] Y. Nambu, Entanglement of quantum fluctuations in the inflationary Universe, *Phys. Rev. D* **78**, 044023 (2008).
- [22] Y. Nambu and Y. Ohsumi, Entanglement of coarse grained quantum field in the expanding Universe, *Phys. Rev. D* **80**, 124031 (2009).
- [23] J. Maldacena and G. L. Pimentel, Entanglement entropy in de Sitter space, *J. High Energy Phys.* **02** (2013) 038.
- [24] S. Brahma, O. Alaryani, and R. Brandenberger, Entanglement entropy of cosmological perturbations, *Phys. Rev. D* **102**, 043529 (2020).
- [25] E. A. Lim, Quantum information of cosmological correlations, *Phys. Rev. D* **91**, 083522 (2015).
- [26] T. J. Hollowood and J. I. McDonald, Decoherence, discord and the quantum master equation for cosmological perturbations, *Phys. Rev. D* **95**, 103521 (2017).
- [27] J. Martin and V. Vennin, Quantum discord of cosmic inflation: Can we show that CMB anisotropies are of quantum-mechanical origin?, *Phys. Rev. D* **93**, 023505 (2016).
- [28] J. Martin, A. Micheli, and V. Vennin, Discord and decoherence, *J. Cosmol. Astropart. Phys.* **04** (2022) 051.
- [29] B. A. Bassett, S. Tsujikawa, and D. Wands, Inflation dynamics and reheating, *Rev. Mod. Phys.* **78**, 537 (2006).
- [30] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions, *Phys. Rep.* **215**, 203 (1992).
- [31] L. P. Grishchuk and Y. V. Sidorov, Squeezed quantum states of relic gravitons and primordial density fluctuations, *Phys. Rev. D* **42**, 3413 (1990).
- [32] R. de Putter and O. Doré, In search of an observational quantum signature of the primordial perturbations in slow-roll and ultraslow-roll inflation, *Phys. Rev. D* **101**, 043511 (2020).
- [33] I. Agullo, B.onga, and P. R. Metidieri, Does inflation squeeze cosmological perturbations?, *J. Cosmol. Astropart. Phys.* **09** (2022) 032.
- [34] J. Martin and V. Vennin, Real-space entanglement in the cosmic microwave background, *J. Cosmol. Astropart. Phys.* **10** (2021) 036.
- [35] L. Espinosa-Portalés and V. Vennin, Real-space Bell inequalities in de Sitter, *J. Cosmol. Astropart. Phys.* **07** (2022) 037.
- [36] A. Micheli, Entanglement and decoherence in cosmology and in analogue gravity experiments, Ph.D. thesis, IJCLab, Orsay, 2023, [10.70675/a99c7bcadz462z4586zaf3zee-f3e64a2618](https://arxiv.org/abs/10.70675/a99c7bcadz462z4586zaf3zee-f3e64a2618).
- [37] W. H. Zurek, Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?, *Phys. Rev. D* **24**, 1516 (1981).
- [38] W. H. Zurek, Environment induced superselection rules, *Phys. Rev. D* **26**, 1862 (1982).
- [39] E. Joos and H. D. Zeh, The emergence of classical properties through interaction with the environment, *Z. Phys. B* **59**, 223 (1985).
- [40] A. L. Matacz, The coherent state representation of quantum fluctuations in the early Universe, *Phys. Rev. D* **49**, 788 (1994).
- [41] C. P. Burgess, R. Holman, and D. Hoover, Decoherence of inflationary primordial fluctuations, *Phys. Rev. D* **77**, 063534 (2008).
- [42] E. Nelson, Quantum decoherence during inflation from gravitational nonlinearities, *J. Cosmol. Astropart. Phys.* **03** (2016) 022.
- [43] C. P. Burgess, R. Holman, G. Kaplanek, J. Martin, and V. Vennin, Minimal decoherence from inflation, *J. Cosmol. Astropart. Phys.* **07** (2023) 022.
- [44] C. M. Sou, D. H. Tran, and Y. Wang, Decoherence of cosmological perturbations from boundary terms and the non-classicality of gravity, *J. High Energy Phys.* **04** (2023) 092.
- [45] F. Sano and J. Tokuda, False and genuine decoherence in the early Universe: A local observer and time-averaged observables, *J. High Energy Phys.* **07** (2025) 266.
- [46] F. Lopez and N. Bartolo, Quantum signatures and decoherence during inflation from deep subhorizon perturbations, [arXiv:2503.23150](https://arxiv.org/abs/2503.23150).
- [47] C. P. Burgess, R. Holman, and G. Kaplanek, Inflationary decoherence from the gravitational floor, *J. Cosmol. Astropart. Phys.* **02** (2026) 042.
- [48] H. Ollivier and W. H. Zurek, Quantum discord: A measure of the quantumness of correlations, *Phys. Rev. Lett.* **88**, 017901 (2001).
- [49] L. Henderson and V. Vedral, Classical, quantum and total correlations, *J. Phys. A* **34**, 6899 (2001).
- [50] G. Lindblad, On the generators of quantum dynamical semigroups, *Commun. Math. Phys.* **48**, 119 (1976).
- [51] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Colloquium: Non-Markovian dynamics in open quantum systems, *Rev. Mod. Phys.* **88**, 021002 (2016).
- [52] D. Manzano, A short introduction to the Lindblad master equation, *AIP Adv.* **10**, 025106 (2020).
- [53] J. Martin and V. Vennin, Observational constraints on quantum decoherence during inflation, *J. Cosmol. Astropart. Phys.* **05** (2018) 063.
- [54] This choice is a matter of convention: it does not change the quantum expectation values of observables. However, it does change how these values are expressed in terms of squeezing parameters, and thus the values of these parameters; see Sec. 2.2 of [36]. Here we follow the conventions of [28,43].
- [55] S. Weinberg, *The Quantum Theory of Fields. Vol. 1: Foundations* (Cambridge University Press, Cambridge, England, 2005), [10.1017/CBO9781139644167](https://arxiv.org/abs/10.1017/CBO9781139644167).
- [56] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1982), [10.1017/CBO9780511622632](https://arxiv.org/abs/10.1017/CBO9780511622632).
- [57] Note that different conventions exist here. For instance, Ref. [43] does not consider symmetrized and adimension-alized anticommutators, so that their expectation values differ by numerical factors from ours. In our convention, their power spectra P_{vv} and P_{pp} are related to our covariance-matrix elements as $\gamma_{11} = 2kP_{vv}$, $\gamma_{12} = 2P_{vv}$, and $\gamma_{22} = 2P_{pp}/k$.
- [58] G. Adesso, S. Ragy, and A. R. Lee, Continuous variable quantum information: Gaussian states and beyond, *Open Syst. Inf. Dyn.* **21**, 1440001 (2014).

- [59] To avoid confusion let us explain our use of the terminology of thermal state. Since any centred Gaussian one-mode state is fully described by its average occupation number $\bar{n} = \langle \hat{c}^\dagger \hat{c} \rangle$, it has the same density matrix as a thermal state, i.e., a state whose occupation number follows Bose-Einstein statistics, with the same average occupation number $\bar{n}$. Thus we refer in a loose sense to any centred Gaussian one-mode state which is non-pure, i.e., $\bar{n} > 0$, as a thermal state, even though the environment leading to the state being mixed rather than pure might not be a thermal bath. Then two-mode squeezed thermal states are states produced by the application of a two-mode squeezing operation to two modes that are in independent one-mode thermal states with the same average occupation [80].
- [60] J. M. Maldacena, Non-Gaussian features of primordial fluctuations in single field inflationary models, *J. High Energy Phys.* **05** (2003) 013.
- [61] A. O. Caldeira and A. J. Leggett, Path integral approach to quantum Brownian motion, *Physica (Amsterdam)* **121A**, 587 (1983).
- [62] Using that $\hat{v}(\mathbf{x})$ has the dimension of an energy per unit length, one can check that k_F must have the dimension of a wavenumber for the equation to be dimensionally consistent.
- [63] Y. Akrami *et al.* (Planck Collaboration), Planck 2018 results. X. Constraints on inflation, *Astron. Astrophys.* **641**, A10 (2020).
- [64] Another approach would be to use Eq. (29) and integrate the right-hand side containing γ_{11} from x to $x_{\text{in}} = -k\eta_{\text{in}}$. However, x_{in} needs not be small, and thus the leading-order approximation Eq. (42a) may not be valid over the whole integration domain. We therefore have to use the full expression Eq. (30a).
- [65] S. Dodelson and F. Schmidt, *Modern Cosmology* (Academic Press, New York, 2020), [10.1016/C2017-0-01943-2](https://doi.org/10.1016/C2017-0-01943-2).
- [66] Note that when going beyond quadratic dynamics the state ceases to be Gaussian and more modes are correlated [67].
- [67] S. S. Haque, G. Jafari, and B. Underwood, Inflation is not magic, [arXiv:2512.10126](https://arxiv.org/abs/2512.10126).
- [68] N. Deruelle and V. F. Mukhanov, On matching conditions for cosmological perturbations, *Phys. Rev. D* **52**, 5549 (1995).
- [69] M. Revzen, P. A. Mello, A. Mann, and L. M. Johansen, Bell's inequality violation with non-negative Wigner functions, *Phys. Rev. A* **71**, 022103 (2005).
- [70] R. Simon, Peres-Horodecki separability criterion for continuous variable systems, *Phys. Rev. Lett.* **84**, 2726 (2000).
- [71] A. Peres, Separability criterion for density matrices, *Phys. Rev. Lett.* **77**, 1413 (1996).
- [72] M. Horodecki, P. Horodecki, and R. Horodecki, On the necessary and sufficient conditions for separability of mixed quantum states, *Phys. Lett. A* **223**, 1 (1996).
- [73] G. Vidal and R. F. Werner, Computable measure of entanglement, *Phys. Rev. A* **65**, 032314 (2002).
- [74] M. B. Plenio, Logarithmic negativity: A full entanglement monotone that is not convex, *Phys. Rev. Lett.* **95**, 090503 (2005).
- [75] A. J. Brady, I. Agullo, and D. Kranas, Symplectic circuits, entanglement, and stimulated Hawking radiation in analogue gravity, *Phys. Rev. D* **106**, 105021 (2022).
- [76] A. Bera, T. Das, D. Sadhukhan, S. S. Roy, A. Sen(De), and U. Sen, Quantum discord and its allies: A review of recent progress, *Rep. Prog. Phys.* **81**, 024001 (2017).
- [77] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden variable theories, *Phys. Rev. Lett.* **23**, 880 (1969).
- [78] G. Gour, F. C. Khanna, A. Mann, and M. Revzen, Optimization of Bell's inequality violation for continuous variable systems, *Phys. Lett. A* **324**, 415 (2004).
- [79] *NIST Handbook of Mathematical Functions*, edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Cambridge University Press: NIST, Cambridge, New York, 2010).
- [80] M. S. Kim, F. A. M. de Oliveira, and P. L. Knight, Properties of squeezed number states and squeezed thermal states, *Phys. Rev. A* **40**, 2494 (1989).