

Five benefits of grand unified $SU(5)$ brane world scenario

Masato Arai^{1,*}, Filip Blaschke^{2,3,†}, Minoru Eto^{1,4,5,‡} and Masaki Kawaguchi[§]

¹*Faculty of Science, Yamagata University, Kojirakawa-machi 1-4-12, Yamagata, Yamagata 990-8560, Japan*

²*Research Centre for Theoretical Physics and Astrophysics, Institute of Physics, Silesian University in Opava, Bezručovo nám. 1150/13, 746 01 Opava, Czech Republic*

³*Institute of Experimental and Applied Physics, Czech Technical University in Prague, Husova 240/5, 110 00 Prague 1, Czech Republic*

⁴*Department of Physics, and Research and Education Center for Natural Sciences, Keio University, 4-1-1 Hiyoshi, Yokohama, Kanagawa 223-8521, Japan*

⁵*International Institute for Sustainability with Knotted Chiral Meta Matter (WPI-SKCM²), Hiroshima University, 1-3-2 Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8511, Japan*



(Received 2 May 2026; accepted 4 August 2026; published 21 August 2026)

We construct an $SU(5)$ grand unified theory on domain walls in the five-dimensional space-time. In this setup, we introduce an adjoint scalar field and a singlet that together form a set of five domain-wall solutions, realizing a dynamical brane world. The same scalar fields also localize chiral fermion zero modes around the walls via the Jackiw-Rebbi mechanism, break $SU(5)$ down to the Standard Model gauge group via geometric Higgs mechanism and simultaneously trap gauge fields through a field-dependent gauge kinetic term. Furthermore, they enable localization of the Higgs field, providing a novel solution to the doublet-triplet splitting problem. As a result, all essential ingredients of the model are realized by a single adjoint scalar field and a singlet, making the construction very economical. We propose two realizations of the Higgs sector, derive the four-dimensional effective theory, and demonstrate that the Standard Model Yukawa couplings at the weak scale can be reproduced from the five-dimensional Yukawa couplings by the renormalization group analysis with a suitable choice of parameters.

DOI: [10.1103/q7kr-cyqz](https://doi.org/10.1103/q7kr-cyqz)

I. INTRODUCTION

The hypothesis of extra dimensions has been extensively explored in high-energy physics and cosmology. Within this context, the brane world framework provides a compelling paradigm in which our observable universe is realized as a three-dimensional hypersurface (3-brane) embedded in a higher-dimensional space-time. Prominent realizations of this idea include the model of Arkani-Hamed, Dimopoulos, and Dvali [1], which introduces large compact extra dimensions to lower the fundamental Planck scale; its embedding into the superstring theory is studied in [2]. The Randall-Sundrum model [3,4] employs warped extra dimensions to generate a hierarchy with an exponential

factor. Both scenarios illustrate how higher-dimensional geometries can offer novel mechanisms to address long-standing puzzles of the Standard Model (SM), most notably the gauge hierarchy problem.

However, this scenario suffers from fundamental problems: the existence of branes and the localization of SM fields are simply assumed. These problems can be resolved by introducing topological solitons. The dynamical origin of branes in extra dimensions arises from spontaneous symmetry breaking, which generates a topologically stable soliton that hosts our four-dimensional universe. This topological structure not only guarantees the stability of the brane but also naturally explains the localization of chiral matter fields [5,6]. Remarkably, gravitons can likewise be trapped [7–12], ensuring that both matter and gravity are confined to the same four-dimensional surface. Thus, topological solitons provide a compelling and elegant mechanism that bridges the gap between higher-dimensional theories and the observed four-dimensional world.

Despite this success, the localization of massless gauge bosons remains a highly nontrivial problem. When the gauge symmetry H is spontaneously broken in the bulk (the Higgs phase) and restored only in the vicinity of a topological soliton, one might anticipate the emergence of

*Contact author: [arai\(at\)sci.kj.yamagata-u.ac.jp](mailto:arai(at)sci.kj.yamagata-u.ac.jp)

†Contact author: [filip.blaschke\(at\)physics.slu.cz](mailto:filip.blaschke(at)physics.slu.cz)

‡Contact author: [meto\(at\)sci.kj.yamagata-u.ac.jp](mailto:meto(at)sci.kj.yamagata-u.ac.jp)

§Contact author: ddwbb.daigaku@gmail.com

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

massless gauge bosons localized within the soliton's core. However, this expectation is not realized in general: massless modes acquire masses proportional to the inverse width of the soliton [13]. The reason is that the bulk region, apart from the soliton core, behaves as a non-Abelian superconductor. Although the gauge symmetry is restored inside the soliton, any electric flux generated by a charged probe placed within it is absorbed into the surrounding superconducting medium. Consequently, gauge fields can propagate only over distances comparable to the soliton width, implying that they are massive [2,13]. This reasoning is physically well-motivated, and thus, the localization of massless gauge bosons on a soliton appears, in general, to be a highly nontrivial challenge.

A crucial insight to overcome this difficulty was introduced in Ref. [13], where a mechanism based on electromagnetic duality was proposed.¹ If a probe magnetic charge is placed inside the soliton embedded in a superconducting bulk, the magnetic flux is expelled from the bulk by the Meissner effect. Owing to flux conservation, the field lines are then confined to extend along the soliton, effectively to infinity [2,13]. In Ref. [13], this situation was reformulated in a dual picture: the Higgs phase in the bulk is replaced by a confining phase, realized by the condensation of magnetic charges. Under this assumption, a massless gauge field can be localized inside the soliton when the gauge symmetry H remains unbroken within the soliton but is embedded into a larger non-Abelian group G in the confining bulk. This localization mechanism, known as the Dvali-Shifman (DS) mechanism, was rigorously demonstrated in four-dimensional super Yang-Mills theory [13]. Its extension to higher dimensions, however, remains unproven, as the nature of confinement in dimensions beyond four is not yet fully understood. Following Ref. [13], fat-brane scenarios based on this idea have been studied [40–43]. However, most of these works assume that the DS mechanism is valid.

On the other hand, a simpler mechanism for gauge field localization was proposed by Ohta and Sakai [44]. In this mechanism, a field-dependent gauge kinetic term (field-dependent permeability) is introduced:

$$\mathcal{L}_{\text{gauge}} = -\beta(T)^2 F_{MN} F^{MN}, \quad (1.1)$$

where M, N denote indices of the full space-time and T is a scalar field. The field strength is given as $\mathcal{F}_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N]$. This represents a semiclassical realization of the confining phase [1,13,45–49], rather than the Higgs phase that exists outside the solitons. When $\beta(T)^2$ is square-integrable over the entire extra-dimensional space, the massless four-dimensional gauge fields become

¹Following [13], numerous studies on the localization of massless gauge bosons have been carried out [14–39]. Each offers certain advantages but also facing specific limitations, and a comprehensive understanding is still lacking.

localized on the topological soliton. Because square-integrability is independent of the specific details of a model, the resulting localization mechanism is quite robust. We have confirmed this robustness in a number of explicit five- and six-dimensional models [50–55] and have also provided a formal proof valid in arbitrary spacetime dimensions [56,57]. The same localization mechanism is used to construct the five-dimensional brane world models in [58–61].

In [52], we have constructed an $SU(5)$ grand unified theory (GUT) in five dimensions. In this model, two types of scalar fields S and T are introduced, where the T sector consists of an adjoint scalar field $\hat{T}$ and a singlet scalar field T^0 . The scalar field S is responsible for localizing gauge fields and T is for localizing matter fields. More specifically, T sector plays three important roles: (1) it generates the domain-wall configuration, (2) it consequently traps chiral fermion zero modes around the wall, and (3) it breaks $SU(5)$ down to the SM gauge group. It was further shown that proton decay is suppressed due to the reduced overlap of the zero mode functions of the fermions and gauge fields. However, the $SU(5)$ model proposed in [52] remains phenomenologically incomplete, since neither the Higgs sector nor the Yukawa couplings were incorporated. As a result, it was not possible to address electroweak symmetry breaking or to derive the mass spectrum of the matter fields. Moreover, the model could not deal with the doublet-triplet splitting problem in the $SU(5)$ GUT.

In this paper, we construct a five-dimensional $SU(5)$ brane world model based on domain-wall solutions with a Higgs sector and Yukawa couplings. In contrast to the previous model [52], the present framework employs only the T sector, consisting of the adjoint scalar field $\hat{T}$ and the singlet scalar field T^0 , without introducing the additional scalar field S . The T sector is responsible not only for the three tasks (1)–(3) explained above but also for (4) the localization of gauge fields which was achieved by the S field in [52], making the construction more economical. Furthermore, we will show that the T field can also contribute to (5) localization of the SM Higgs field on the walls avoiding the doublet-triplet splitting problem. Theories with extra dimensions provide an attractive framework to solve the doublet-triplet splitting problem [62–68]. Within the higher-dimensional context, we propose two distinct realizations of the Higgs sector that successfully resolve the splitting problem. We further derive the four-dimensional effective theory and show that the SM Yukawa couplings can be reproduced from the five-dimensional Yukawa interactions through renormalization group evolution of the SM.

An additional advantage of the present setup concerns the fermion mass structure. In minimal four-dimensional $SU(5)$ GUTs, the unified Yukawa structure generically predicts the relation $Y_d = Y_e^T$ at the unification scale, where Y_d and Y_e denote the Yukawa matrices for down-type quarks and charged leptons, respectively. This relation

leads to phenomenologically unrealistic fermion mass relations. In our framework this problem is avoided without extending the Higgs sector (for instance by introducing additional Higgs representations such as a **45** as in the Georgi-Jarlskog mechanism [69]) or invoking higher-dimensional operators that modify the Yukawa structure [70]. Although the underlying five-dimensional Yukawa couplings respect $SU(5)$ unification, the effective four-dimensional Yukawa matrices arise from overlap integrals of localized fermion zero modes in the extra dimension. These generation-dependent overlaps naturally relax the usual $SU(5)$ mass relations, allowing a realistic pattern of fermion masses and mixings.

The organization of this paper is as follows. In Sec. **II**, we provide an overview of the model. Section **III** is devoted to the discussion of the domain-wall sector. In Sec. **IV**, we describe the Higgs sector, with particular focus on the two models that provide solutions to the doublet-triplet splitting problem. Specifically, Sec. **V** presents the remaining sectors and the derivation of the four-dimensional effective theory of the model; Sec. **VA** deals with the gauge sector and its four-dimensional effective theory, Sec. **VD** derives the four-dimensional effective theories of the two Higgs-sector models, Sec. **VB** discusses the fermion sector and its effective theory, and Sec. **VC** covers the Yukawa sector and the corresponding four-dimensional effective theory. In Sec. **VI** we summarize our results. In **A**, we provide a derivation of the effective potential in the domain-wall sector. **B** gives proofs of the decompositions used in Eq. (5.20) and Eq. (5.56). **C** summarizes the Standard Model renormalization group equations used in the analysis of the Yukawa couplings in Sec. **VC**.

II. MODEL

We consider a five-dimensional $SU(5)$ GUT with a Higgs $\mathcal{H}$ in a fundamental representation, fermionic matters $\Psi_{5,A}$ in an antifundamental representation and $\Psi_{10,A}$ in an antisymmetric representation, where the index $A = 1, 2, 3$ denotes the generation. In addition, we introduce two scalar fields $\hat{T}$ in an adjoint representation and a singlet T^0 . They transform under the gauge transformation with $U \in SU(5)$ as

$$\mathcal{H} \rightarrow U\mathcal{H}, \quad (2.1)$$

$$\Psi_{\bar{5}} \rightarrow U^*\Psi_{\bar{5}}, \quad (2.2)$$

$$\Psi_{10} \rightarrow U\Psi_{10}U^t, \quad (2.3)$$

$$\hat{T} \rightarrow U\hat{T}U^\dagger. \quad (2.4)$$

We decompose the full Lagrangian into the following parts:

$$\mathcal{L} = \mathcal{L}_T + \mathcal{L}_H + \mathcal{L}_{\text{gauge}} + \mathcal{L}_f + \mathcal{L}_{\text{Yukawa}}. \quad (2.5)$$

We briefly explain each part in the following.

The first part, $\mathcal{L}_T$, serves a role of providing domain wall background on which the SM is realized in the low energy limit. This background allows for several configurations of domain walls and of spontaneous gauge symmetry breaking patterns. We will show that the symmetry breaking from $SU(5)$ to $SU(3) \times SU(2) \times U(1)$ is realized as the most energetically favored for robust choices of parameters in the model. We will provide an explicit form of $\mathcal{L}_T$ and a detailed analysis to obtain the domain walls in Sec. **III**. The effective Lagrangian of this sector will be discussed together with that of the gauge sector in Sec. **VA**.

The second part, $\mathcal{L}_H$, is the Lagrangian for the Higgs fields, which is one of the main focuses of this paper. This part provides a mechanism to localize the doublet part of $\mathcal{H}$, which becomes the SM Higgs field, while the triplet part remains delocalized or only appears in the low-energy effective theory through very heavy modes. This is a solution to the doublet-triplet splitting problem. We will propose two models to solve this problem in Secs. **IVA** and **IVB**. We will also derive the effective Lagrangian of these two models in Sec. **VD**.

The third part, $\mathcal{L}_{\text{gauge}}$, describes the kinetic term of the $SU(5)$ gauge field and the localization of SM gauge fields around a certain region of the fifth-dimension through the Ohta-Sakai mechanism [44]. In this Lagrangian, a nontrivial function of $\hat{T}$ and T^0 appears as a coefficient of the gauge kinetic term. This means that the coefficient which is normally an inverse of a gauge coupling constant is generalized to a positive semidefinite function of $\hat{T}$ and T_0 . This function plays an important role to localize gauge fields. We will describe $\mathcal{L}_{\text{gauge}}$ in Sec. **VA**, where the effective Lagrangian of $\mathcal{L}_{\text{gauge}}$ will be derived as well.

The fourth part, $\mathcal{L}_f$, includes the kinetic terms of the fermion fields and their couplings to the $\hat{T}$ and T_0 fields, by which the fermion can localize around the domain wall. The fifth part, $\mathcal{L}_{\text{Yukawa}}$, contains the 5-dimensional Yukawa terms that give the masses to the fermions after the electroweak symmetry breaking. These two sectors will be discussed, including the derivation of their effective Lagrangians, in Secs. **VB** and **VC**. In Sec. **VC**, we perform the renormalization group analysis of the Yukawa couplings and show that the Yukawa couplings of the SM at Z -boson scale are realized by suitable choice of the Yukawa couplings and parameters in the five-dimensional model.

III. DOMAIN WALL SECTOR

A. Domain wall background

For convenience, let us combine the $SU(5)$ adjoint field $\hat{T}$ and the scalar field T_0 into a single entity T

$$T \equiv \hat{T} + \frac{T_0}{5} \mathbf{1}_5. \quad (3.1)$$

We then organize the $\mathcal{L}_T$ as follows²

$$\mathcal{L}_T = \text{Tr}[\mathcal{D}_M T \mathcal{D}^M T] - V_0 - V_1, \quad (3.2)$$

where the covariant derivative is given by $\mathcal{D}_M T = \partial_M T + i[\mathcal{A}_M, T]$ with $\mathcal{A}_M$ being an $SU(5)$ gauge field. Note that the kinetic term is not canonically normalized for T_0 field, i.e. $\text{Tr}[\mathcal{D}_M T \mathcal{D}^M T] = \text{Tr}[\mathcal{D}_M \hat{T} \mathcal{D}^M \hat{T}] + \frac{1}{5} \partial_M T_0 \partial^M T_0$.

The first part of the potential V_0 is chosen in such a way as to facilitate domain walls (and antiwalls) in the background and it is assumed that it is dominant over the second part. The role of V_1 is to merely modify the shape and dynamics of the (anti)walls slightly and mostly in a generic manner. In particular, let us choose

$$V_0 = \lambda \text{Tr}(v^2 \mathbf{1}_5 - T^2)^2. \quad (3.3)$$

If we set $V_1 = 0$ for the moment, it is easy to see how V_0 gives rise to domain (anti)walls. First, we observe that there are 2^5 discrete vacua described by

$$T_{\text{vac}} \equiv v U \text{diag}(s_1, s_2, \dots, s_5) U^\dagger, \quad (3.4)$$

where U is an arbitrary $SU(5)$ matrix and $s_i \in \{-1, 1\}$ are a quintuplet of signs, while v is the vacuum expectation value.

For further purpose, let us organize the canonical vacua, given by Eq. (3.4) with fixed $U = \mathbf{1}_5$, into three sets $\langle 0 \rangle$, $\langle 1 \rangle$, and $\langle 2 \rangle$ according to how they break $SU(5)$ gauge symmetry:

$$\langle 0 \rangle \equiv T_{\text{vac}}/v \in \{\mathbf{1}_5, -\mathbf{1}_5\} \quad SU(5) \text{ preserving} \quad (3.5)$$

$$\langle 1 \rangle \equiv T_{\text{vac}}/v \in \left\{ \begin{pmatrix} -1 & \\ & \mathbf{1}_4 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -\mathbf{1}_4 \end{pmatrix}, + \text{permutations} \right\} \quad SU(5) \rightarrow U(1) \times SU(4) \quad (3.6)$$

$$\langle 2 \rangle \equiv T_{\text{vac}}/v \in \left\{ \begin{pmatrix} -\mathbf{1}_2 & \\ & \mathbf{1}_3 \end{pmatrix}, \begin{pmatrix} \mathbf{1}_2 & \\ & -\mathbf{1}_3 \end{pmatrix}, + \text{permutations} \right\} \quad SU(5) \rightarrow U(1) \times SU(2) \times SU(3) \quad (3.7)$$

While there are only two elements in the set $\langle 0 \rangle$, there are 10 in $\langle 1 \rangle$ and 20 in $\langle 2 \rangle$, totaling $2^5 = 32$ elements.

Next let us consider the domain walls. Since all the vacua mentioned above are discrete, stable domain walls exist. We choose the $SU(5)$ unbroken vacua $\langle 0 \rangle$ and take the following boundary conditions

$$T \xrightarrow{y \rightarrow -\infty} -v \mathbf{1}_5, \quad T \xrightarrow{y \rightarrow \infty} v \mathbf{1}_5. \quad (3.8)$$

A general static solution with these boundary conditions can be obtained in a closed form as

$$T_{\text{walls}} = v \text{diag}(\tanh(\Omega y - R_i)), \quad i = 1, \dots, 5, \quad (3.9)$$

where $\Omega \equiv \sqrt{\lambda} v$. This solution describes a configuration of five domain walls at (dimensionless) positions R_i along the extra-dimensional coordinate $y \equiv x^4$.

Depending on the number of coincident walls and the ordering of their positions on the y -axis, there are ten patterns of possible configurations. A phenomenologically interesting case is when three coincident walls and two coincident domain walls are placed at different positions, namely, $R_1 = R_2 = R_3 < R_4 = R_5$. We write this

configuration as **3-2**, where the numerals denote a number of coincident walls and the hyphen means the space separation. This configuration is depicted in Fig. 1. We can think about this configuration as consisting of regions of two asymptotic outer vacua and one approximate inner vacuum, in which different symmetry breaking patterns are realized. The $SU(5)$ is unbroken far away from the domain walls, but it is spontaneously broken to $SU(3) \times SU(2) \times U(1)$ in the middle region, where $T \approx v \text{diag}(1, 1, 1, -1, -1)$. There are, of course, many

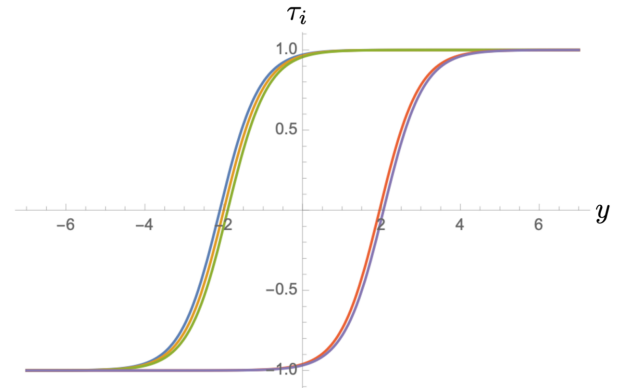


FIG. 1. Illustration of the **3-2** split configuration of domain walls. Here, $v = \Omega = 1$ and the position moduli are given as $R_1 = R_2 = R_3 = -2, R_4 = R_5 = 2$.

²We take the metric to be $\eta_{MN} = \text{diag}(1, -1, -1, -1, -1)$, where $M, N = 0, \dots, 3, 4$ stand for the five-dimensional space-time indices.

other configurations such as **4-1** and **2-1-2**, etc. As we will see in the next subsection, the **3-2** configuration is energetically favored compared to any other as V_1 is switched. Furthermore, we will show that this remains true in a large region of the parameter space and hence represents a generic case.

B. Moduli stabilization

Let us now turn our attention to the question of how the choice of V_1 affects this picture. We consider the following potential.

$$V_1 = \lambda(-\mu^2 v^2 \text{Tr}(\hat{T}^2) + \alpha \text{Tr}(\hat{T}^4)). \quad (3.10)$$

In fact, it turns out that we can be quite general in choosing the extra pieces, provided that they are small compared with V_0 and hence can be treated perturbatively. Before we discuss generic case, however, let us illustrate the situation with specific examples.

This potential (here μ and α are dimensionless couplings) can be characterized as an extra double-well potential for the adjoint part $\hat{T}$. First of all, it changes the structure of the vacua. Namely, the additional energy due to V_1 for the three sets of vacua defined in the previous subsection are given as

$$V_1(\langle 0 \rangle) = 0, \quad (3.11)$$

$$V_1(\langle 1 \rangle) = \frac{16\lambda v^4}{125} (52\alpha - 25\mu^2), \quad (3.12)$$

$$V_1(\langle 2 \rangle) = \frac{24\lambda v^4}{125} (28\alpha - 25\mu^2). \quad (3.13)$$

As we see, which set corresponds to a minimum depends on the values of α and μ . In the following, we consider $\alpha > 25\mu^2/28$, which realizes a phenomenologically interesting case, namely that $SU(5)$ -preserving vacua are the true minima of the full potential with the total energy $V_0(\langle 0 \rangle) + V_1(\langle 0 \rangle) = 0$. As there are only two vacua in the $\langle 0 \rangle$ set, all diagonal entries of the T field must be either walls or antiwalls. Without the loss of generality, let us take $s_1 = \dots = s_5 = 1$.

Treating V_1 as perturbation, we can ascertain which configuration of static domain walls has the lowest energy by plugging (3.9) into V_1 and integrating the result over y . Furthermore, we obtain the effective potential as a function of distances between walls $R_i - R_j$:

$$V_{\text{eff}}(R_i - R_j) \equiv \int_{-\infty}^{\infty} dy V_1(\hat{T}_{\text{walls}}). \quad (3.14)$$

Notice that due to invariance of V_1 with respect to permutation of diagonal entries, we can assume $R_1 < R_2 < \dots < R_5$ without loss of generality. The effective

potential $V_{\text{eff}}(R_i - R_j)$ can be obtained exactly, which we do in Appendix A.

However, for our purposes, let us demonstrate our claim using a simple argument. Naively, we may be tempted to think that a fully coincident configuration of walls, i.e.

$$T_{\text{coin}} = v \tanh(x) \mathbf{1}_5, \quad (3.15)$$

is the minimum of the energy, since $V_1(\hat{T}_{\text{coin}}) = 0$, where $x \equiv \Omega y$. Departing slightly from the coincident point, however, we can show that T_{coin} must be in fact a local maximum. Indeed, given that $\hat{T}$ would be small, we can neglect the quartic term in V_1 and easily integrate the quadratic term (see A) which gives

$$\begin{aligned} & -\lambda v^2 \mu^2 \int_{-\infty}^{\infty} dy \text{Tr}(\hat{T}_{\text{wall}}^2) \\ &= \sqrt{\lambda} v^3 \mu^2 \left(8 - \frac{4}{5} \sum_{i>j} \frac{R_i - R_j}{\tanh(R_i - R_j)} \right). \end{aligned} \quad (3.16)$$

As this is negative—as evident from the fact that the right-hand side is maximal at the fully coincident limit, $R_i = R_j$, for which it equals zero—the effective potential implies a *repulsive* force between any pair of walls for small separations. Hence, we see that the coincident point is unstable.

The other extreme—when all walls are well separated—has the opposite effect, namely an *attractive* force due to the fact that field values in the gaps will be in the $\langle 1 \rangle$ set for the two outer gaps and in the $\langle 2 \rangle$ set for the two inner gaps. These gaps roughly gives a constant positive energy per unit length, which follows from (3.11) and (3.12). Thus, these gaps would tend to collapse for large separations, but as we have seen, the fully coincident configuration is disfavored too. Hence, the true minimum should be a partially coincident configuration. Furthermore, the field value in the gap should be from the $\langle 2 \rangle$ set, since it is energetically favorable over the set $\langle 1 \rangle$.

It is easy to establish that out of the two simplest possibilities—the **3-2** splitting and the **4-1** splitting configurations—

$$\begin{aligned} T_{\mathbf{3-2}} &= v \begin{pmatrix} \tanh(x) \mathbf{1}_3 & \\ & \tanh(x - R) \mathbf{1}_2 \end{pmatrix}, \\ T_{\mathbf{4-1}} &= v \begin{pmatrix} \tanh(x) \mathbf{1}_4 & \\ & \tanh(x - R) \end{pmatrix}, \end{aligned} \quad (3.17)$$

the **3-2** splitting has the least energy. This can be checked by direct calculation of the effective potentials

$$V_{\text{eff}}^{3-2} \equiv \int_{-\infty}^{\infty} dy V_1(\hat{T}_{3-2}) = \frac{112v^3\sqrt{\lambda}\alpha}{125} \left(-11 + \frac{6R}{\tanh(R)} - \frac{15}{\sinh^2(R)} \left(1 - \frac{R}{\tanh(R)} \right) \right) + \frac{24v^3\sqrt{\lambda}\mu^2}{5} \left(1 - \frac{R}{\tanh(R)} \right), \quad (3.18)$$

$$V_{\text{eff}}^{4-1} \equiv \int_{-\infty}^{\infty} dy V_1(\hat{T}_{4-1}) = \frac{416v^3\sqrt{\lambda}\alpha}{375} \left(-11 + \frac{6R}{\tanh(R)} - \frac{15}{\sinh^2(R)} \left(1 - \frac{R}{\tanh(R)} \right) \right) + \frac{16v^3\sqrt{\lambda}\mu^2}{5} \left(1 - \frac{R}{\tanh(R)} \right). \quad (3.19)$$

We see that V_{eff}^{3-2} is always bellow V_{eff}^{4-1} (except the point $R = 0$). We display these effective potentials on Fig. 2, showing that the minimum is reached at some nonzero separations of coincident triplet and coincident doublet of walls.

A further analysis can be made by considering more complicated configurations, such as **2-1-2** double-split background. Omitting the details, the effective potential shows that minimum is achieved by collapsing one of the gaps, thus reaching **3-2** configuration. In the following section, and throughout the paper, we consider the **3-2** configuration as our domain wall background solution.

IV. HIGGS SECTOR

In this section, we propose two models for the Higgs sector, which solve the double-triplet splitting problem. One model does not need fine-tuning (model 1), while the other model (model 2) does.

A. Model 1

We introduce a nonminimal kinetic term for the Higgs field $\mathcal{H}$

$$\mathcal{L}_H = \text{Tr}[\beta_H(\hat{T})^2 \mathcal{D}_M \mathcal{H} \mathcal{D}^M \mathcal{H}^\dagger], \quad (4.1)$$

where $\mathcal{D}_M \mathcal{H} = \partial_M \mathcal{H} - iA_M \mathcal{H}$, and we assume β_H is transposed as $\beta_H \rightarrow U\beta_H U^\dagger$ under $SU(5)$. This is a generalization of Ohta and Sakai idea [44] in Eq. (1.1) to the Higgs sector. As we will shortly show, no other terms are needed to solve the doublet-triplet splitting problem.

To be concrete, let us take a specific example

$$\beta_H(\hat{T}) = \exp(-c\hat{T}^{-1}), \quad (4.2)$$

with c being a constant. We use this example to demonstrate the idea, but we will comment about the general conditions on β_H later.

We consider the **3-2** background wall configuration given by

$$T \equiv \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_2 \end{pmatrix} = v \begin{pmatrix} \tanh \Omega \left(y + \frac{d}{2} \right) \mathbf{1}_3 & 0 \\ 0 & \tanh \Omega \left(y - \frac{d}{2} \right) \mathbf{1}_2 \end{pmatrix}. \quad (4.3)$$

From this, we have

$$\hat{T} = \delta\tau \text{diag} \left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, -\frac{3}{5}, -\frac{3}{5} \right), \quad (4.4)$$

$$\delta\tau = v \tanh \Omega \left(y + \frac{d}{2} \right) - v \tanh \Omega \left(y - \frac{d}{2} \right). \quad (4.5)$$

Then, the explicit form of β_H reads

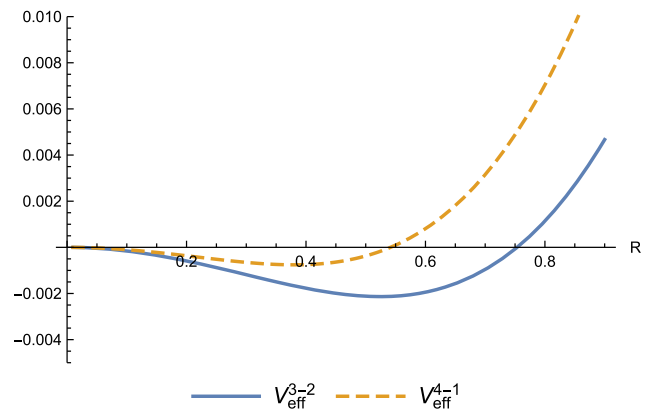
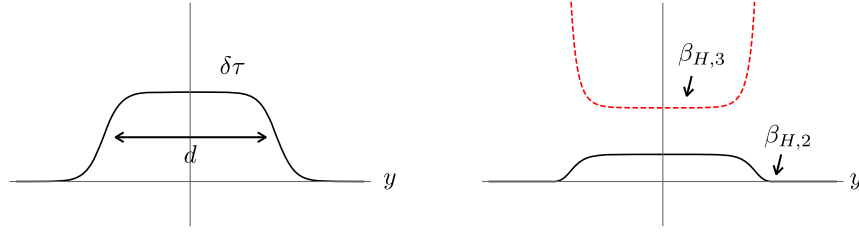


FIG. 2. Comparison between effective potentials V_{eff}^{3-2} and V_{eff}^{4-1} showing that **3-2** splitting background is energetically preferable. Here, $v = \lambda = 10\alpha = 100\mu^2 = 1$.

FIG. 3. The left panel shows $\delta\tau$, and the right panel shows $\beta_{H,3}$ and $\beta_{H,2}$.

$$\beta_H = \begin{pmatrix} \beta_{H,3} \mathbf{1}_3 \\ \beta_{H,2} \mathbf{1}_2 \end{pmatrix} = \begin{pmatrix} \exp\left(-\frac{5c}{2\delta\tau}\right) \mathbf{1}_3 \\ \exp\left(\frac{5c}{3\delta\tau}\right) \mathbf{1}_2 \end{pmatrix}. \quad (4.6)$$

In the following, we assume $c > 0$ and $d > 0$ (but $c < 0$ and $d < 0$ would be equally valid). As a consequence, $\beta_{H,3}$ and $\beta_{H,2}$ acquire shapes depicted in Fig. 3. The sign combination of c and d affects the normalizability (square integrability) of β_H . When $cd > 0$ as shown in Fig. 3, $\beta_{H,2}$ is normalizable but $\beta_{H,3}$ is non normalizable, and vice versa for $cd < 0$.

This naturally leads to decomposing the 5-plet $\mathcal{H}$ into the colored triplet $\mathcal{H}_3$ and the electroweak doublet $\mathcal{H}_2$ as

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_3 & \mathcal{H}_2 \end{pmatrix}^T, \quad (4.7)$$

resulting in the decoupled equations of motion

$$\partial_M(\beta_{H,I}^2 \partial^M \mathcal{H}_I) = 0, \quad (I = 2, 3). \quad (4.8)$$

Since $\beta_{H,I}$ only depends on y , this can be expanded as

$$\beta_{H,I}^2 \partial_\mu \partial^\mu \mathcal{H}_I - \partial_y(\beta_{H,I}^2 \partial_y \mathcal{H}_I) = 0. \quad (4.9)$$

Next, let us define a canonical Higgs fields H_I defined by

$$H_I(x^M) = \beta_{H,I}(y) \mathcal{H}_I(x^M), \quad (4.10)$$

and the equations of motion for H_I read

$$\partial_\mu \partial^\mu H_I + \left(-\frac{d^2}{dy^2} + \frac{\beta_{H,I}''}{\beta_{H,I}} \right) H_I = 0. \quad (4.11)$$

Let us expand H_I in to the complete basis of functions

$$H_I(x^M) = \sum_n f_{I,n}(y) h_{I,n}(x^\mu), \quad (4.12)$$

where $f_{I,n}$ are the eigenfunctions of the operator $-\frac{d^2}{dy^2} + \frac{\beta_{H,I}''}{\beta_{H,I}}$, i.e.,

$$\left(-\frac{d^2}{dy^2} + \frac{\beta_{H,I}''}{\beta_{H,I}} \right) f_{I,n}(y) = m_{I,n}^2 f_{I,n}(y). \quad (4.13)$$

Clearly, $m_{I,n}$ defines the four-dimensional mass for the $h_{I,n}$.

Let us examine the mass spectra. First of all, we can easily show the mass operator is real and positive semi-definite by rewriting it in the following form

$$-\frac{d^2}{dy^2} + \frac{\beta_{H,I}''}{\beta_{H,I}} = Q_{H,I}^\dagger Q_{H,I}, \quad (4.14)$$

where

$$Q_{H,I} \equiv -\frac{d}{dy} + (\log \beta_{H,I})',$$

$$Q_{H,I}^\dagger \equiv \frac{d}{dy} + (\log \beta_{H,I})'. \quad (4.15)$$

Thus, the four-dimensional masses satisfy $m_{H,n}^2 \geq 0$. Second, irrespective of any details of $\beta_{H,I}$, the zero mode always exists

$$Q_{H,I} f_{I,n=0} = 0 \Rightarrow f_{I,n=0} = N_{H,I} \beta_{H,I}. \quad (4.16)$$

where $N_{H,I}$ is a normalization constant which is determined by a condition

$$\int dy f_{I,0}(y) f_{I,0}(y) = 1. \quad (4.17)$$

This gives

$$N_{H,I} = \frac{1}{\sqrt{\int dy \beta_{H,I}^2}}. \quad (4.18)$$

However, only the electroweak doublet $h_{2,0}$ turns out to be physical, since $\beta_{H,2}$ is only normalizable. The colored triplet $h_{3,0}$ is unphysical due to the fact that $\beta_{H,3}$ is non-normalizable. The presence of the massive bound states depends on the details of the model parameters, but they are always superheavy because all the dimensionful parameters are of the order of the GUT scale. Thus, the doublet-triplet splitting puzzle is solved without any fine-tuning in this model.

Let us make comments on our accomplishment in this subsection. The presence of the massless mode is robust in the sense that a small deformation of the model cannot eliminate it. We can understand it as an analogy to a topological edge mode appearing on an interface between topologically distinct regions. For this, we focus on the mass operator, which was found to appear in the very specific form $Q_{H,I}^\dagger Q_{H,I}$. This reminds us of the domain wall fermions via the so-called Jackiw-Rebbi mechanism [6]. Briefly speaking, the Jackiw-Rebbi mechanism states that the massless fermion appears whenever the position-dependent fermion mass $M_\psi(y)$ changes its sign on a kink center. Typically, the mass appears in the first order differential operator $Q_\psi = -\frac{d}{dy} + M_\psi$ and $Q_\psi^\dagger = \frac{d}{dy} + M_\psi$. Comparing the Jackiw-Rebbi fermion and our Higgs boson, we reach the correspondence

$$M_\psi \Leftrightarrow M_{H,I} \equiv (\log \beta_{H,I})' = \frac{\beta'_{H,I}}{\beta_{H,I}}. \quad (4.19)$$

Looking at Fig. 3, one sees that $M_{H,I}$ changes the sign at the center. This, namely the fact that $M_{H,I}(y)$ changes the sign, is a necessary condition for having the massless Higgs doublet. Any details are not important for the presence of the zero mode. In this sense, the massless Higgs doublet is a sort of the topological edge state. In summary, the 3-2 splitting of the 5 domain walls solves the doublet-triplet splitting puzzle of $SU(5)$ GUT.

From the above analysis, if we replace the parameter combination $cd > 0$ by $cd < 0$, we immediately see that the colored Higgs triplet provides the massless physical mode instead of the doublet. Unfortunately, we cannot control whether it is the triplet or the doublet that gives the massless degrees of freedom at this stage. It is our opinion that it is not a high price to pay, compared to some other $SU(5)$ brane world models, which need parity assignment to all the bulk and brane fields. We do not need any parity assignment, but we need the condition $cd > 0$.

The example (4.6) illustrates the general idea behind the localization mechanism for the Higgs doublet: the non-minimal coupling term $\beta_H(\hat{T})$ depends on the domain-wall background in such a way that distinguishes the doublet from triplet via exponent-type dependence that—due to the sign difference—makes the colored triplet part $\beta_{H,3}$ non-normalizable, while localizing the electroweak doublet $\beta_{H,2}$. Let us stress that any type of profile $\beta_H(\hat{T})$ that has the same exponent-type sensitivity would produce the same result, making our scheme, in this sense, robust. However, at this point, it seems difficult to formulate general conditions on β_H that would guarantee the same results.

Furthermore, there is an obvious question to what degree is the presented construction natural. In this regard, we can only offer the above-mentioned similarity to Jackiw-Rebbi mechanism for fermionic modes, making the model 1 a

bosonic analog for a topological edge-state. However, we leave the full motivation as a possible future project.

B. Model 2

In this subsection,³ we propose a model for the Higgs sector that needs fine-tuning. The Lagrangian is given by

$$\mathcal{L}_H = (\mathcal{D}_M \mathcal{H})^\dagger \mathcal{D}^M \mathcal{H} - V^{(4)} - V^{(2)}. \quad (4.20)$$

In order to achieve the triplet-doublet splitting, we need to use the adjoint scalar field, $\hat{T}$, to interact with $\mathcal{H}$ not via the kinetic terms but via the potential terms. We assume the simplest choice of the potential as

$$V^{(4)} = \lambda_H (\mathcal{H}^\dagger \mathcal{H})^2, \quad V^{(2)} = \mathcal{H}^\dagger (\Omega_H^2 \mathbf{1}_5 + b \hat{T}) \mathcal{H}, \quad (4.21)$$

where the quartic coupling $\lambda_H > 0$ assures the stability of the potential at large values of $\mathcal{H}$. The parameter Ω_H is the five-dimensional mass for $\mathcal{H}$, and b is the coupling of the adjoint scalar $\hat{T}$ with $\mathcal{H}$. The interaction $\mathcal{H}^\dagger \hat{T} \mathcal{H}$ with the coupling b is chosen as an operator with the lowest mass dimension.

The adjoint scalar $\hat{T}$ for the 3-2 split domain wall is given as in Eq. (4.4). It again leads to a natural separation of the 5-plet $\mathcal{H}$ as in (4.7). We obtain the decoupled quadratic terms in the potential as

$$V^{(2)} = U_2(y) \mathcal{H}_2^\dagger \mathcal{H}_2 + U_3(y) \mathcal{H}_3^\dagger \mathcal{H}_3, \quad (4.22)$$

$$U_2(y) = \Omega_H^2 - \frac{U_0}{\cosh^2 \Omega y - \tilde{d}^2 \sinh^2 \Omega y}, \quad (4.23)$$

$$U_3(y) = \Omega_H^2 + \frac{2}{3} \frac{U_0}{\cosh^2 \Omega y - \tilde{d}^2 \sinh^2 \Omega y}, \quad (4.24)$$

where the wall separation is parametrized by a dimensionless separation $\tilde{d}$

$$\tilde{d} = \tanh \frac{\Omega d}{2}. \quad (4.25)$$

The strength of the potential U_0 is given as

$$U_0 = \frac{6}{5} b v \tanh \frac{\Omega d}{2} > 0. \quad (4.26)$$

Note that we have chosen the sign of the parameter b to make $U_0 > 0$. Because of the repulsive potential for the triplet $\mathcal{H}_3$, there are no localized modes and a large mass gap of order Ω_H without light modes. Therefore we only need to consider doublet $\mathcal{H}_2$ to find possible light or tachyonic modes, see Fig. 4.

³This section was written with significant contributions from the late Professor Norisuke Sakai.

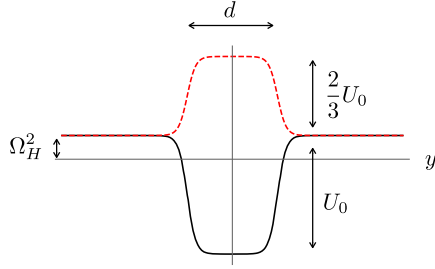


FIG. 4. The quadratic mass terms U_2 and U_3 for the electro-weak Higgs doublet (black and solid) and the colored Higgs triplet (red and dashed), respectively.

The mass spectra of the doublet Higgs scalar $\mathcal{H}_2$ can be obtained by a linearized field equation

$$\mathcal{D}_M \mathcal{D}^M \mathcal{H}_2 + U_2(y) \mathcal{H}_2 = 0. \quad (4.27)$$

The wave equation in the extra-dimension y is assumed to give a complete set of eigenfunctions $u_n(y)$ with the eigenvalue m_n^2 as

$$[-\partial_y^2 + U_2(y)]u_n(y) = m_n^2 u_n(y). \quad (4.28)$$

We can decompose the field in terms of this complete set to obtain a tower of the Kaluza-Klein fields $H_n(x)$, which appear as fields with the mass squared m_n^2 in the effective four-dimensional field theory

$$\mathcal{H}_2(x, y) = \sum_{n=0}^{\infty} h_{2,n}(x) u_n(y), \quad (4.29)$$

$$\partial_\mu \partial^\mu h_{2,n}(x) + m_n^2 h_{2,n}(x) = 0, \quad (4.30)$$

where μ stands for the four-dimensional spacetime indices.

It is not easy to obtain the mode functions $u_n(y)$ exactly for arbitrary separations. However, since we are interested in wall configurations with separations of a fraction of the wall width $1/\Omega$, we can solve the eigenproblem (4.28) in the limit of vanishing separation, i.e., $\tilde{d} \rightarrow 0$ as a leading order approximation.

When $\tilde{d} = 0$, the eigenproblem simplifies into

$$\left[-\partial_y^2 + \Omega_H^2 - \frac{U_0}{\cosh^2 \Omega y} \right] u_n(y) = m_n^2 u_n(y). \quad (4.31)$$

We define an auxiliary variable s

$$s = \frac{1}{2} \left(\sqrt{1 + \frac{4U_0}{\Omega^2}} - 1 \right), \quad (4.32)$$

to count the discrete eigenstates with the mass m_n

$$m_n^2 = \Omega_H^2 - \Omega^2(s - n)^2, \quad n = 0, 1, \dots, [s], \quad (4.33)$$

where $[s]$ is the largest integer less or equal to s . The corresponding eigenfunctions are given by the ${}_2F_1$ hypergeometric function

$$\begin{aligned} u_n(y) &= \frac{N_n}{(\cosh \Omega y)^{s-n}} {}_2F_1 \left[-n, 2s - n + 1; s - n + 1; \frac{1 - \tanh \Omega y}{2} \right] \\ &= \frac{N_n}{(\cosh \Omega y)^{s-n}} \sum_{k=0}^n \frac{n! \Gamma(2s - n + k + 1) \Gamma(s - n + 1)}{k! (n - k)! \Gamma(2s - n + 1) \Gamma(s - n + k + 1)} \left(\frac{\tanh \Omega y}{2} - 1 \right)^k, \end{aligned} \quad (4.34)$$

with the normalization constant N_n defined by $\int dy |u_n(y)|^2 = 1$. In particular, the lowest mode is given as

$$m_0^2 = \Omega_H^2 - (s\Omega)^2, \quad u_0(y) = \frac{N_0}{(\cosh \Omega y)^s}, \quad |N_0|^2 = \frac{\Omega \Gamma(s + \frac{1}{2})}{\sqrt{\pi} \Gamma(s)}. \quad (4.35)$$

It is localized between 3 and 2 walls, and it becomes massless $m_0 = 0$ if the parameters satisfy the condition

$$U_0 = \Omega_H(\Omega_H + \Omega). \quad (4.36)$$

The mass gap between the lightest and the higher excited modes is of order Ω , and above the threshold at Ω_H^2 , we have a continuum spectra.

The electroweak symmetry breaking occurs when $m_0^2 < 0$. This is realized when parameters satisfy the following inequality

$$-\sqrt{\frac{\Omega^2}{4} + U_0} + \frac{\Omega}{2} < \Omega_H < \sqrt{\frac{\Omega^2}{4} + U_0} - \frac{\Omega}{2}. \quad (4.37)$$

We conclude that in this model, the triplet Higgs $\mathcal{H}_3$ is safely decoupled and the electroweak symmetry breaking is realized by condensation of the remaining double Higgs $\mathcal{H}_2$.

V. EFFECTIVE LAGRANGIAN

In this section, we give the Lagrangian of the gauge sector, the fermion sector, and the Yukawa coupling in (2.5), and derive the four-dimensional effective Lagrangian of (2.5).

A. Effective Lagrangian of gauge sector

1. Setup

We consider the following Lagrangian for the gauge sector

$$\mathcal{L}_{\text{gauge}} = -\text{Tr}[\beta^2(T_0, \hat{T}) \mathcal{F}_{MN} \mathcal{F}^{MN}], \quad (5.1)$$

The coefficient $\beta^2(T_0, \hat{T})$ of the gauge kinetic term, which is normally an inverse of a gauge coupling constant but it is a positive semidefinite function of T_0 and $\hat{T}$ in the above Lagrangian. This generalization gives a semiclassical realization of the confinement phase outside the domain wall and the localization of the gauge field is achieved. We require that $\beta^2(T_0, \hat{T})$ covariantly transforms under $SU(5)$ as β_H to maintain the gauge invariance.

For the derivation of the four-dimensional effective Lagrangian, we do not need to consider a concrete $\beta(T_0, \hat{T})$. But we only decompose $\beta(T_0, \hat{T})$ in a block-diagonal form, given that the background solution (3.9) is block-diagonal as well. That is we have

$$\beta(T_0, \hat{T})|_{\text{bg}} = \begin{pmatrix} \beta_3(\tau_3)^2 \mathbf{1}_3 & 0 \\ 0 & \beta_2(\tau_2)^2 \mathbf{1}_2 \end{pmatrix}, \quad \tau_i \equiv v \tanh(\Omega(y - \mathcal{Y}_i)) \quad (i = 2, 3). \quad (5.2)$$

As mentioned before, the β_3 and β_2 realize the localization of the gauge fields by acquiring a profile on the **3-2**-split background that is approaching zero everywhere, except around the domain walls τ_3 and τ_2 , respectively. Namely we assume square integrability of β_3 and β_2 as

$$\frac{1}{g_3^2} \equiv \int dy \beta_3^2 < \infty, \quad \frac{1}{g_2^2} \equiv \int dy \beta_2^2 < \infty. \quad (5.3)$$

Let us now decompose the gauge fields as

$$\mathcal{A}_M = \begin{pmatrix} \mathcal{A}_{3M} & \frac{\mathcal{X}_M}{\sqrt{2}} \\ \frac{\mathcal{X}_M^\dagger}{\sqrt{2}} & \mathcal{A}_{2M} \end{pmatrix} + \mathcal{A}_{1M} \sqrt{\frac{3}{5}} \begin{pmatrix} \frac{1}{3} \mathbf{1}_3 & 0 \\ 0 & -\frac{1}{2} \mathbf{1}_2 \end{pmatrix}, \quad (5.4)$$

where $\mathcal{A}_3^M$, $\mathcal{A}_2^M$, and $\mathcal{A}_1^M$ are would-be $SU(3)$, $SU(2)$, and $U(1)$ gauge fields, and $\mathcal{X}_M$ is a 2×3 rectangular complex matrix.

In addition to the fluctuations of the gauge fields (5.4), we introduce small fluctuations to the scalar field T . Since T is a 5 by 5 Hermitian matrix, there are 25 real fluctuations. We shall decompose them into three parts (ρ, Ψ, Φ) as

$$T = e^{i\Phi}(T_0 + \rho)e^{-i\Phi} = T_0 + \rho + i[\Phi, T_0] \cdots, \quad (5.5)$$

with

$$\rho = \begin{pmatrix} \rho_3 & 0 \\ 0 & \rho_2 \end{pmatrix}, \quad \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \varphi \\ \varphi^\dagger & 0 \end{pmatrix}. \quad (5.6)$$

Here $\rho_3(\rho_2)$ is a 3×3 (2×2) Hermitian matrix, and φ is 3×2 rectangular, complex matrix. ρ_3 , ρ_2 and φ have 9, 4, and 12 degrees of freedom. Summing up, we have 25 degrees of freedom as expected.

The fields ρ_3 and ρ_2 become massive after the moduli are stabilized as is done in Sec. III B and they naturally have GUT scale masses. Thus, they are integrated out in the low energy Lagrangian. The third term in the second equality in (5.5) is nothing but the gauge transformation by the broken generators, so that φ contains the corresponding Nambu-Goldstone fields.

For later convenience, let us prepare

$$\beta_1 \equiv \sqrt{\frac{3\beta_2^2 + 2\beta_3^2}{5}}, \quad \beta_x \equiv \sqrt{\frac{\beta_2^2 + \beta_3^2}{2}}, \quad \beta_\phi \equiv \tau_3 - \tau_2. \quad (5.7)$$

Here β_1 also satisfies the square integrability condition since β_3 and β_2 satisfy it:

$$\frac{1}{g_1^2} \equiv \int dy \beta_1^2 < \infty. \quad (5.8)$$

We shall substitute (5.4) and (5.5) into (5.1) and (3.2), and pick up only terms of the quadratic order of the fluctuations. The resultant Lagrangian is then given by

$$\mathcal{L}_{\text{gauge}} + \text{Tr}[\mathcal{D}_M T \mathcal{D}^M T] = \sum_{\alpha=1}^3 \mathcal{L}_\alpha + \mathcal{L}_X + \mathcal{L}_\phi, \quad (5.9)$$

with

$$\mathcal{L}_\alpha \equiv \text{Tr}[\mathcal{A}_{\alpha\mu} \{ \beta_\alpha^2 (\eta^{\mu\nu} \square - \partial^\mu \partial^\nu) - \eta^{\mu\nu} \partial_y \beta_\alpha^2 \partial_y \} \mathcal{A}_{\alpha\nu} - 2(\partial^\mu \mathcal{A}_{\alpha\mu}) \partial_y (\beta_\alpha^2 \mathcal{A}_{\alpha y}) - \mathcal{A}_{\alpha y} (\beta_\alpha^2 \square) \mathcal{A}_{\alpha y}], \quad (5.10)$$

$$\mathcal{L}_X \equiv \text{Tr}[\mathcal{X}_\mu^\dagger [\beta_X^2 (\eta^{\mu\nu} \square - \partial^\mu \partial^\nu) - \eta^{\mu\nu} \partial_y \beta_X^2 \partial_y + \eta^{\mu\nu} \beta_\phi^2] \mathcal{X}_\nu - 2(\partial^\mu \mathcal{X}_\mu^\dagger) [\partial_y (\beta_X^2 \mathcal{X}_y) + \beta_\phi^2 \varphi] - \mathcal{X}_y^\dagger (\beta_X^2 \square + \beta_\phi^2) \mathcal{X}_y], \quad (5.11)$$

$$\begin{aligned} \mathcal{L}_\phi = & \text{Tr}[-\varphi^\dagger(\beta_\phi^2 \square - \partial_y \beta_\phi^2 \partial_y) \varphi + \varphi^\dagger \partial_y (\beta_\phi^2 \mathcal{X}_y) \\ & - \beta_\phi^2 \mathcal{X}_y^\dagger \partial_y \varphi], \end{aligned} \quad (5.12)$$

where $\mathcal{L}_\alpha$ and $\mathcal{L}_X$ are the Lagrangians for the unbroken and broken gauge fields, respectively. The indices μ, ν denote the four-dimensional space-time, y is a five-dimensional coordinate, and $\alpha = 1, 2, 3$ represents indices of $U(1)$, $SU(2)$ and $SU(3)$ gauge fields. It should be noticed that there are mixing terms between $\mathcal{X}_y$ the Nambu-Goldstone field φ in (5.11) and also in (5.12). Note that the trace in (5.10) for $\mathcal{A}_{1\mu}$ should be understood as replacing Tr with $\frac{1}{2}$.

Together with the above Lagrangians, we need the following gauge fixing terms:

$$\begin{aligned} \mathcal{L}_A^{(\text{gf})} &= \sum_{\alpha=1}^3 \mathcal{L}_{\alpha A}^{(\text{gf})}, \\ \mathcal{L}_{\alpha A}^{(\text{gf})} &= -\frac{\beta_\alpha^2}{\xi} \text{Tr} \left[\left\{ \partial^\mu \mathcal{A}_{\alpha\mu} - \frac{\xi}{\beta_\alpha^2} \partial_y (\beta_\alpha^2 \mathcal{A}_{\alpha y}) \right\}^2 \right], \end{aligned} \quad (5.13)$$

As mentioned above Tr is replaced with $\frac{1}{2}$ for $\alpha = 1$. We also need another gauge fixing term

$$\begin{aligned} \mathcal{L}_{X\phi}^{(\text{gf})} &= -\frac{\beta_X^2}{\xi} \text{Tr} \left[\left\{ \partial^\mu \mathcal{X}_M - \frac{\xi}{\beta_X^2} (\partial_y (\beta_X^2 \mathcal{X}_y) + \beta_\phi^2 \varphi) \right\} \right. \\ &\quad \left. \times (\text{H.c.}) \right], \end{aligned} \quad (5.14)$$

where ξ is a gauge fixing parameter similar to a standard R_ξ gauge fixing procedure.

2. Mode analysis for the unbroken gauge fields

The above quadratic Lagrangian has a complicated form because of the existence of the factor $\beta_{\alpha X}$. In the following, we will bring the Lagrangian to a standard form. First we shall focus on the unbroken part, $\mathcal{L}_\alpha$ and the corresponding gauge fixing term $\mathcal{L}_{\alpha A}^{(\text{gf})}$. In the beginning, let us define

$$A_{\alpha M} \equiv \beta_\alpha \mathcal{A}_M^\alpha, \quad (\alpha = 1, 2, 3), \quad (5.15)$$

where the index α is not summed. We also introduce differential operators

$$D_\alpha \equiv -\beta_\alpha \partial_y \frac{1}{\beta_\alpha} = -\partial_y + (\beta_\alpha^{-1} \partial_y \beta_\alpha), \quad (\alpha = 1, 2, 3), \quad (5.16)$$

where the index α is again not summed. An adjoint of D_α is given by

$$D_\alpha^\dagger = \beta_\alpha^{-1} \partial_y \beta_\alpha = \partial_y + (\beta_\alpha^{-1} \partial_y \beta_\alpha), \quad (\alpha = 1, 2, 3), \quad (5.17)$$

where again we do not sum over α .

Now we are ready to rewrite the effective Lagrangian (5.10) by using the above differential operators:

$$\begin{aligned} \mathcal{L}_\alpha = & \text{Tr} [A_{\alpha\mu} (\eta^{\mu\nu} \square - \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta_\alpha) A_{\alpha\nu} \\ & - 2(\partial^\mu A_\mu)(D_\alpha^\dagger A_{\alpha y}) - A_{\alpha y} \square A_{\alpha y}], \end{aligned} \quad (5.18)$$

with $\Delta_\alpha \equiv D_\alpha^\dagger D_\alpha$ and $\square = \partial_\mu \partial^\mu$. Next we rewrite the gauge fixing Lagrangian as

$$\mathcal{L}_{\alpha A}^{(\text{gf})} = -\frac{1}{\xi} \text{Tr} [(\partial^\mu A_{\alpha\mu} - \xi D_\alpha^\dagger A_{\alpha y})^2]. \quad (5.19)$$

Summing up these, we obtain relatively simple formula for the unbroken gauge fields

$$\begin{aligned} \mathcal{L}_{\text{ub}} &\equiv \sum_{\alpha=1}^3 (\mathcal{L}_\alpha + \mathcal{L}_{\alpha A}^{(\text{gf})}) \\ &= \sum_{\alpha=1}^3 \text{Tr} \left[A_{\alpha\mu} \left(\eta^{\mu\nu} \square - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta_\alpha \right) A_{\alpha\nu} \right. \\ &\quad \left. - A_{\alpha y} (\square + \xi D_\alpha D_\alpha^\dagger) A_{\alpha y} \right], \end{aligned} \quad (5.20)$$

where we have used

$$(D_\alpha^\dagger A_{\alpha y}) D_\alpha^\dagger A_{\alpha y} = A_{\alpha y} D_\alpha D_\alpha^\dagger A_{\alpha y}, \quad (5.21)$$

where we do not sum over α .

It is possible to decompose the extra-dimensional gauge field $A_{\alpha y}$ into two scalar fields B_α and C_α as

$$A_{\alpha y} = D_\alpha B_\alpha + C_\alpha, \quad (5.22)$$

where C_α satisfies the divergence-free condition

$$D_\alpha^\dagger C_\alpha = 0. \quad (5.23)$$

We prove this decomposition in B.

Multiplying $D_\alpha^\dagger$ on (5.22) leads to the following equation

$$\Delta_\alpha B_\alpha = D_\alpha^\dagger A_{\alpha y}. \quad (5.24)$$

We can determine B_α and C_α by solving the Eqs. (5.23) and (5.24). It is easy to solve Eq. (5.23) as

$$C_\alpha \propto \beta_\alpha^{-1}. \quad (5.25)$$

From the square integrability conditions (5.3) and (5.8), we see that C_α is non-normalizable. The scalar B_α can be derived for a given $A_{\alpha y}$ in (5.24). However, there is an ambiguity. We can determine B_α only up to solutions of the following equation

$$\Delta_\alpha B_\alpha^{(0)} = 0. \quad (5.26)$$

A solution to this equation is the kernel of D_α , which is given by

$$B_\alpha^{(0)} \propto \beta_\alpha. \quad (5.27)$$

Note that this is normalizable from the square integrability conditions (5.3) and (5.8).

By using the decomposition (5.22) we can rewrite (5.20) as

$$\begin{aligned} \mathcal{L}_{\text{ub}} = \sum_{\alpha=1}^3 \text{Tr} \left[A_{\alpha\mu} \left(\eta^{\mu\nu} \square - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta_\alpha \right) A_{\alpha\nu} \right. \\ \left. - C_\alpha \square C_\alpha - B_\alpha \Delta_\alpha (\square + \xi \Delta_\alpha) B_\alpha \right]. \end{aligned} \quad (5.28)$$

We redefine the scalar B_α by

$$b_\alpha \equiv \sqrt{\Delta_\alpha} B_\alpha. \quad (5.29)$$

Substituting this into (5.28), we have

$$\begin{aligned} \mathcal{L}_{\text{ub}} = \sum_{\alpha=1}^3 \text{Tr} \left[A_{\alpha\mu} \left(\eta^{\mu\nu} \square - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta_\alpha \right) A_{\alpha\nu} \right. \\ \left. - b_\alpha (\square + \xi \Delta_\alpha) b_\alpha \right], \end{aligned} \quad (5.30)$$

where C_α are dropped since they are non-normalizable.

We should emphasize that b_α does not include the zero mode. It is found that the zero mode for B_α is obtained from $\Delta_\alpha B_\alpha = 0$ [see (5.28)]. This is nothing but (5.26) and its solution is given by (5.27). However, the zero mode is projected out by the definition (5.29) and therefore it does not exist in b_α . Now we decompose b_α by the eigenfunctions of Δ_α

$$b_\alpha(x^M) = \sum_{n>0} b_\alpha^{(n)}(x^\mu) B_\alpha^{(n)}(y), \quad (5.31)$$

where $b_\alpha^{(n)} = f_{an} m_{an}$ with f_{an} being an expansion coefficient (a field in four dimensions) and m_{an} being the four-dimensional masses. The mode functions $B_\alpha^{(n)}(y)$ satisfy

$$\Delta_\alpha B_\alpha^{(n)} = m_{an}^2 B_\alpha^{(n)}, \quad (5.32)$$

Note that $n = 0$ is not included in the expansions because of the explanation above. We set the normalization of the mode functions as usual

$$\int dy B_\alpha^{(m)} B_\alpha^{(n)} = \delta^{mn}. \quad (5.33)$$

In contrast to the extra-dimensional gauge fields, there exists the zero mode of Δ_α for the four-dimensional gauge fields. Therefore, when we decompose $A_{\alpha\mu}$ in terms of the eigenfunctions of Δ_α , we have to include $n = 0$ as

$$A_{\alpha\mu}(x^M) = \sum_{n \geq 0} A_{\alpha\mu}^{(n)}(x^\mu) B_\alpha^{(n)}(y). \quad (5.34)$$

The mode function for the zero mode is given by (5.27). They are localized around the domain walls, that is the simple and robust resolution of the longstanding problem in the brane world models. Note also that the domain walls work as trigger of spontaneous symmetry breaking from GUT to SM symmetries. The normalization is fixed by (5.3) and (5.8) and we have

$$B_\alpha^{(0)} = g_\alpha \beta_\alpha, \quad (5.35)$$

Substituting (5.31) and (5.34) into (5.30) and integrating it over y , we obtain the effective Lagrangian for the Kaluza-Klein (KK) towers as

$$\begin{aligned} \mathcal{L}_{\text{ub}} = \sum_{n \geq 0} \text{Tr} \left[A_{\alpha\mu}^{(n)} \left(\eta^{\mu\nu} \square - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} (m_{an})^2 \right) A_{\alpha\nu}^{(n)} \right. \\ \left. - \sum_{n>0} b_\alpha^{(n)} (\square + \xi (m_{an})^2) b_\alpha^{(n)} \right]. \end{aligned} \quad (5.36)$$

Note that the scalar b_α is coming from the extra-dimensional component of $A_{\alpha y}$ and it appears with the mass squared $\xi(m_{an})^2$. This is nothing but the standard Lagrangian in the so-called R_ξ gauge. This means that the scalar b_α plays a role as a Nambu-Goldstone boson to give a mass to the KK gauge fields through the Higgs mechanism.

Since the KK gauge fields $A_{\alpha\mu}^{(n)}$ for $n > 0$ become supermassive, they are integrated out and the effective Lagrangian for the zero modes is given by

$$\mathcal{L}_{\text{ub}}^{\text{eff}} = \sum_{\alpha=1}^3 \text{Tr} \left[A_{\alpha\mu}^{(0)} \left(\eta^{\mu\nu} \square - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right) A_{\alpha\nu}^{(0)} \right]. \quad (5.37)$$

This is the quadratic Lagrangian of massless gauge fields. In order to recover the self-interaction of the non-Abelian gauge fields, we go back to (5.15).

$$\mathcal{A}_{\alpha\mu} = \beta_\alpha^{-1} A_{\alpha\mu} = g_\alpha A_{\alpha\mu}^{(0)} + \cdots, \quad (5.38)$$

where the dots denote the massive modes. Keeping only the zero mode in (5.38), the field strength corresponding to $\mathcal{A}_{\alpha\mu}$ becomes

$$\mathcal{F}_{\alpha\mu\nu} = g_\alpha F_{\alpha\mu\nu}^{(0)}, \quad (5.39)$$

where

$$F_{\alpha\mu\nu}^{(0)} = \partial_\mu A_{\alpha\nu}^{(0)} - \partial_\nu A_{\alpha\mu}^{(0)} + ig_\alpha [A_{\alpha\mu}^{(0)}, A_{\alpha\nu}^{(0)}]. \quad (5.40)$$

This should be compared with the original field strength $\mathcal{F}_{MN} = \partial_M \mathcal{A}_N - \partial_N \mathcal{A}_M + i[\mathcal{A}_M, \mathcal{A}_N]$ where the gauge coupling is absorbed in the gauge field $\mathcal{A}_M$. On the other hand, the four-dimensional gauge coupling appears in (5.40) as a consequence of integrating over the extra-dimensional coordinate y . As a result, the effective Lagrangian for the zero modes including the self-interaction is given by

$$\begin{aligned} \mathcal{L}_{\text{ub}}^{\text{eff}} &= \int dy \sum_{\alpha=1}^3 \text{Tr} \left[-\frac{1}{2} \beta_\alpha^2 \mathcal{F}_{\alpha\mu\nu} \mathcal{F}_\alpha^{\mu\nu} \right] \\ &= -\frac{1}{2} \text{Tr}(G_{\mu\nu} G^{\mu\nu}) - \frac{1}{2} \text{Tr}(W_{\mu\nu} W^{\mu\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \end{aligned} \quad (5.41)$$

where we have defined $A_{\alpha\mu}^{(0)}(x) = (A_{1\mu}^{(0)}, A_{2\mu}^{(0)}, A_{3\mu}^{(0)}) \equiv (B_\mu, W_\mu, G_\mu)$ being the four-dimensional massless gauge fields for $U(1), SU(2), SU(3)$ gauge groups and corresponding field strength

$$G^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu + ig_3 [G^\mu, G^\nu], \quad (5.42)$$

$$W^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu + ig_2 [W^\mu, W^\nu], \quad (5.43)$$

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu. \quad (5.44)$$

3. Mode analysis for the broken gauge fields

In this subsection, we consider the effective Lagrangian of $\mathcal{L}_X$ and $\mathcal{L}_\phi$. First we prepare canonical fields as

$$X_M \equiv \beta_X \mathcal{X}_M, \quad X_5 \equiv \beta_\phi \varphi. \quad (5.45)$$

It is convenient to define a two-component vector containing the extra-dimensional components of the unbroken gauge field and the Nambu-Goldstone field as

$$\mathbf{X} \equiv \begin{pmatrix} X_4 \\ X_5 \end{pmatrix}, \quad (5.46)$$

where we have denoted $X_4 \equiv X_y$. As in the unbroken gauge part, we introduce differential operators

$$D_X \equiv -\beta_X \partial_y \frac{1}{\beta_X} = -\partial_y + (\beta_X^{-1} \partial_y \beta_X), \quad (5.47)$$

$$D_\phi \equiv -\beta_\phi \partial_y \frac{1}{\beta_\phi} = -\partial_y + (\beta_\phi^{-1} \partial_y \beta_\phi). \quad (5.48)$$

Note that D_X can be expressed via D_ϕ as

$$D_X = \mathcal{M}^{-1} D_\phi \mathcal{M}, \quad \mathcal{M} \equiv \frac{\beta_\phi}{\beta_X}. \quad (5.49)$$

Then, we define a two-component operator as

$$\mathbf{D} \equiv \begin{pmatrix} D_X \\ \mathcal{M} \end{pmatrix}, \quad (5.50)$$

and

$$\mathbf{D}_R = (\mathcal{M}, -D_\phi). \quad (5.51)$$

With the use of (5.46), (5.47), (5.48), and (5.50), we can rewrite the sum of (5.11) and (5.12) in a compact form

$$\begin{aligned} \mathcal{L}_X + \mathcal{L}_\phi &= \text{Tr} [X_\mu^\dagger (\eta^{\mu\nu} \square - \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta) X_\nu \\ &\quad - 2(\partial^\mu X_\mu^\dagger) \mathbf{D}^\dagger \mathbf{X} - \mathbf{X}^\dagger (\square + \tilde{\Delta}) \mathbf{X}], \end{aligned} \quad (5.52)$$

where

$$\Delta = \mathbf{D}^\dagger \mathbf{D}, \quad (5.53)$$

$$\tilde{\Delta} = \mathbf{D}_R^\dagger \mathbf{D}_R = \begin{pmatrix} \mathcal{M}^2 & -\mathcal{M} D_\phi \\ -D_\phi^\dagger \mathcal{M} & D_\phi^\dagger D_\phi \end{pmatrix}. \quad (5.54)$$

Similarly, the gauge fixing Lagrangian (5.14) is expressed as

$$\mathcal{L}_{X\phi}^{(\text{gf})} = -\frac{1}{\xi} \text{Tr} [(\partial^\nu X_\nu - \xi \mathbf{D}^\dagger \mathbf{X})^\dagger (\partial^\mu X_\mu - \xi \mathbf{D}^\dagger \mathbf{X})]. \quad (5.55)$$

Summing up (5.52) and (5.55), we have the following quadratic Lagrangian

$$\begin{aligned} \mathcal{L}_b &\equiv \mathcal{L}_X + \mathcal{L}_\phi + \mathcal{L}_{X\phi}^{(\text{gf})} \\ &= \text{Tr} \left[X_\mu^\dagger \left(\eta^{\mu\nu} \square - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta \right) X_\nu \right. \\ &\quad \left. - \mathbf{X}^\dagger (\square + \tilde{\Delta} + \xi \mathbf{D} \mathbf{D}^\dagger) \mathbf{X} \right], \end{aligned} \quad (5.56)$$

where we have used

$$(\mathbf{D} \mathbf{X})^\dagger \mathbf{D}^\dagger \mathbf{X} = \mathbf{X}^\dagger \mathbf{D} \mathbf{D}^\dagger \mathbf{X}. \quad (5.57)$$

As in the case of the extra-dimensional component of the unbroken gauge fields in Eq. (5.22), we can show that $\mathbf{X}$ is decomposed by a real scalar Y and a two-component complex vector $\mathbf{Z}$ as

$$\mathbf{X} = \mathbf{D} Y + \mathbf{Z}, \quad (5.58)$$

where Y and $\mathbf{Z}$ satisfy

$$\mathbf{D}_R \mathbf{D} \mathbf{Y} = 0, \quad (5.59)$$

$$\mathbf{D}^\dagger \mathbf{Z} = 0. \quad (5.60)$$

The proof of the decomposition (5.58) is given in B. Acting $\mathbf{D}^\dagger$ on (5.58), we find

$$\mathbf{D}^\dagger \mathbf{D} \mathbf{Y} = \mathbf{D}^\dagger \mathbf{X}, \quad (5.61)$$

by which $\mathbf{Y}$ is determined. Note that we shall exclude the kernel of $\mathbf{D}$ from $\mathbf{Y}$ because it cannot participate in the physical field $\mathbf{X}$. Substituting (5.58) into (5.56), we have

$$\begin{aligned} \mathcal{L}_b = \text{Tr} \left[X_\mu^\dagger \left\{ \eta^{\mu\nu} \square - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta \right\} X_\nu \right. \\ \left. - Y \Delta (\square + \xi \Delta) Y - \mathbf{Z}^\dagger (\square + \tilde{\Delta}) \mathbf{Z} \right]. \end{aligned} \quad (5.62)$$

We make this Lagrangian canonical by redefining $\mathbf{Y}$ as

$$\mathcal{Y} = \sqrt{\Delta} \mathbf{Y}. \quad (5.63)$$

We excluded the kernel of $\mathbf{D}$ from $\mathbf{Y}$ and so it is true for $\mathcal{Y}$. Substituting this into (5.62), we have the final form of the quadratic Lagrangian of the broken sector

$$\begin{aligned} \mathcal{L}_b = \text{Tr} \left[X_\mu^\dagger \left\{ \eta^{\mu\nu} \square - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta \right\} X_\nu \right. \\ \left. - \mathcal{Y} (\square + \xi \Delta) \mathcal{Y} - \mathbf{Z}^\dagger (\square + \tilde{\Delta}) \mathbf{Z} \right]. \end{aligned} \quad (5.64)$$

We shall discuss mass spectra of $X_\mu, \mathcal{Y}$ and $\mathbf{Z}$. The mass spectrum of X_μ is controlled by the operator $\Delta = \mathbf{D}^\dagger \mathbf{D} = D_X^\dagger D_X + \mathcal{M}^2$. This means that there is no state satisfying $\Delta X_\mu = 0$ and therefore there are no massless modes in X_μ . This is, of course, expected since X_μ corresponds to the gauge fields associated with the broken generators. Indeed, X_μ becomes massive by absorbing $\mathcal{Y}$. We are lead to this conclusion by the fact that X_μ and $\mathcal{Y}$ share the same mass operator Δ except for the extra constant factor ξ for $\mathcal{Y}$. Again we come across a generalized form of the R_ξ gauge. This means that $\mathcal{Y}$ plays a role of infinite tower of Nambu-Goldstone bosons and they are eaten by infinite tower of the KK modes of X_μ including the bottom of the tower.

Recalling the definition (5.61), the source for $\mathcal{Y}$ is $\mathbf{D}^\dagger \mathbf{X} = D_X^\dagger X_4 + \mathcal{M} \beta_\phi \phi$. Therefore X_μ eats not pure Nambu-Goldstone field ϕ but the mixture of $D_X^\dagger X_4$ and ϕ . It is in general difficult to specify the lowest mass of Δ except for the special case, where $\mathcal{M}$ is a constant. In this case, since $\Delta = D_X^\dagger D_X + \mathcal{M}^2$, Δ differs from $D_X^\dagger D_X$ by just constant $\mathcal{M}^2$. The zero mode of $D_X^\dagger D_X$ is given by β_X and therefore the lowest mass of Δ is the constant $\mathcal{M}$ itself.

So if we set $\mathcal{M}$ to GUT scale, the broken gauge field X_μ is superheavy.

Let us turn to the mass spectra of $\mathbf{Z}$. Note that $\tilde{\Delta}$ is positive semidefinite because $\mathbf{Z}^\dagger \tilde{\Delta} \mathbf{Z} = \mathbf{Z}^\dagger \mathbf{D}_R^\dagger \mathbf{D}_R \mathbf{Z} = (\mathbf{D}_R \mathbf{Z})^\dagger \mathbf{D}_R \mathbf{Z}$. The zero modes of $\tilde{\Delta}$ are kernel of $\mathbf{D}_R$, but one can show that $\mathbf{Z}$ does not include the zero mode. It is proved in B.

In summary, $X_\mu, \mathcal{Y}$ and $\mathbf{Z}$ do not have zero modes and their KK towers start from modes with masses. These modes are integrated out at low energies so that we can conclude that modes of $X_\mu, \mathcal{Y}$, and $\mathbf{Z}$ do not appear in the low-energy effective Lagrangian.

B. Effective Lagrangian of fermion sector 1

In this subsection, we give the Lagrangian of the fermion sector and derive its effective Lagrangian. The Lagrangian of the fermion part $\mathcal{L}_f$ in (2.5) is written as

$$\begin{aligned} \mathcal{L}_f = i \bar{\Psi}_{\bar{5}A} \Gamma^M D_M \Psi_{\bar{5}A} + i \text{Tr} [\bar{\Psi}_{10A} \Gamma^M D_M \Psi_{10A}] \\ + h_{5A} \bar{\Psi}_{\bar{5}A} \hat{T}^A \Psi_{\bar{5}A} + \tilde{h}_{5A} T^0 \bar{\Psi}_{\bar{5}A} \Psi_{\bar{5}A} \\ + h_{10A} \text{Tr} [\bar{\Psi}_{10A} \hat{T}^A \Psi_{10A}] + \tilde{h}_{10A} T^0 \text{Tr} [\bar{\Psi}_{10A} \Psi_{10A}], \end{aligned} \quad (5.65)$$

where $h_{5A}, \tilde{h}_{5A}, h_{10A}$, and $\tilde{h}_{10A}$ are Yukawa-type coupling constants with $\hat{T}$ and T^0 , $A = 1, 2, 3$ are the generation index, and Γ^M is the five-dimensional gamma matrices defined as

$$\Gamma^\mu = \gamma^\mu, \quad \Gamma^5 = i\gamma^5, \quad \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad (5.66)$$

where γ^μ are the four-dimensional gamma matrices. The Dirac conjugate is defined by $\bar{\Psi} = \Psi^\dagger \Gamma^0$. The covariant derivatives are

$$D_M \Psi_{\bar{5}A} = \partial_M \Psi_{\bar{5}A} - i \mathcal{A}_M^* \Psi_{\bar{5}A}, \quad (5.67)$$

$$D_M \Psi_{10A} = \partial_M \Psi_{10A} + i \mathcal{A}_M \Psi_{10A} + i \Psi_{10A} \mathcal{A}_M^t. \quad (5.68)$$

To derive the effective Lagrangian for $\mathcal{L}_f$ we first decompose the $SU(5)$ representations as $\bar{5} \rightarrow (\bar{3}, 1)_{1/3} \oplus (1, 2)_{-1/2}$ and $10 \rightarrow (\bar{3}, 1)_{-2/3} \oplus (3, 2)_{1/6} \oplus (1, 1)_1$ under $SU(3) \times SU(2) \times U(1)$

$$\Psi_{\bar{5}} = \begin{pmatrix} \Psi_{\bar{3}} \\ \Psi_2 \end{pmatrix}, \quad \Psi_{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{\bar{3}'} & \Psi_{3,2} \\ -\Psi_{3,2}' & \Psi_1 \end{pmatrix}. \quad (5.69)$$

Hereafter we will suppress the generation index A for ease of notation. The next step is expanding the fermions into Kaluza-Klein modes by which the Lagrangian $\mathcal{L}_f$ is diagonalized. For $\Psi_{\bar{3}}$, we have

$$\Psi_{\bar{3}} = \sum_n [f_{\bar{3},L}^{(n)}(y)\psi_{\bar{3},L}^{(n)}(x) + f_{\bar{3},R}^{(n)}(y)\psi_{\bar{3},R}^{(n)}(x)], \quad (5.70)$$

where $f_{\bar{3},L}^{(n)}(y)$ and $f_{\bar{3},R}^{(n)}(y)$ are wave functions satisfying the mode equation

$$Q_{\bar{3}}^\dagger Q_{\bar{3}} f_{\bar{3}}^{(n)}(y) = (m_{\bar{3}}^{(n)})^2 f_{\bar{3}}^{(n)}(y), \quad (5.71)$$

where

$$Q_{\bar{3}} = \partial_y + h_{\bar{3}}(y), \quad Q_{\bar{3}}^\dagger = -\partial_y + h_{\bar{3}}(y),$$

$$h_{\bar{3}}(y) = \frac{2}{5} h_5 \beta_\phi(y) + \tilde{h}_5 \tau_+(y). \quad (5.72)$$

Here β_ϕ is defined in (5.7), $\tau_+ \equiv 3\tau_3 + 2\tau_2$. The asymptotic behaviors of y -dependent Yukawa coupling $h_{\bar{3}}(y)$ are $h_{\bar{3}}(y = +\infty) = 5v\tilde{h}_5$ and $h_{\bar{3}}(y = -\infty) = -5v\tilde{h}_5$. The most important fact is that the fermion mass gap is opened at $y = \pm\infty$ whereas it is closed at somewhere between $y = -\infty$ and $+\infty$. $m_{\bar{3}}^{(n)}$ are the masses of the four-dimensional chiral fermions $\psi_{\bar{3},L}^{(n)}, \psi_{\bar{3},R}^{(n)}$ which satisfy

$$\gamma^5 \psi_{\bar{3},L}^{(n)} = -\psi_{\bar{3},L}^{(n)}, \quad \gamma^5 \psi_{\bar{3},R}^{(n)} = \psi_{\bar{3},R}^{(n)}. \quad (5.73)$$

As it turns out, for appropriate choice of parameters h_5 and $\tilde{h}_5$, the normalizable zero modes appear only for the left-handed components as

$$f_{\bar{3},L}^{(0)} = \tilde{N}_{\bar{3}} \exp\left(-\int^y h_{\bar{3}}(y') dy'\right), \quad (5.74)$$

where $\tilde{N}_{\bar{3}}$ is the normalization constant, which is determined by a normalization condition

$$\int dy (f_{\bar{3},L}^{(0)})^2 = 1. \quad (5.75)$$

We identify the zero modes of the chiral fermion $\psi_{\bar{3},L}^{(n)}$ with the down quarks as

$$\Psi_{\bar{3}} \supset \psi_{\bar{3},L}^{(0)} \rightarrow d_a = (d_1^c, d_2^c, d_3^c), \quad (5.76)$$

where $a = 1, 2, 3$ is the $SU(3)$ index and where the superscript c denotes charge conjugation. The same analysis is done for $\Psi_2, \Psi'_3, \Psi_{3,2}$ and Ψ_1 similarly. The zero modes for them are given by

$$f_{2,L}^{(0)} = \tilde{N}_2 \exp\left(-\int^y h_2(y') dy'\right),$$

$$h_2 = -\frac{3}{5} h_5 \beta_\phi + \tilde{h}_5 \tau_+, \quad (5.77)$$

$$f_{\bar{3}',L}^{(0)} = \tilde{N}_{\bar{3}'} \exp\left(-\int^y h_{\bar{3}'}(y') dy'\right),$$

$$h_{\bar{3}'} = \frac{2}{5} h_{10} \beta_\phi + \tilde{h}_{10} \tau_+, \quad (5.78)$$

$$f_{3,2,L}^{(0)} = \tilde{N}_{3,2} \exp\left(-\int^y h_{3,2}(y') dy'\right),$$

$$h_{3,2} = -\frac{1}{10} h_{10} \beta_\phi + \tilde{h}_5 \tau_+, \quad (5.79)$$

$$f_{1,L}^{(0)} = \tilde{N}_1 \exp\left(-\int^y h_1(y') dy'\right),$$

$$h_1 = -\frac{3}{5} h_{10} \beta_\phi + \tilde{h}_5 \tau_+, \quad (5.80)$$

respectively. All these are normalizable due to the fact that the y -dependent Yukawa coupling h 's have opposite signs at $y = -\infty$ and $+\infty$, and therefore they are physical massless fermions. Then, the other quarks and leptons are identified as

$$\Psi_2 \supset \psi_{2,L}^{(0)} \rightarrow l_a = \begin{pmatrix} e^- \\ -\nu_e \end{pmatrix}, \quad (5.81)$$

$$\Psi_{\bar{3}'} \supset \psi_{\bar{3}',L}^{(0)} \rightarrow U_{\alpha\beta}^c = \epsilon_{\alpha\beta\gamma} u_\gamma^c = \begin{pmatrix} 0 & u_3^c & -u_2^c \\ -u_3^c & 0 & u_1^c \\ u_2^c & -u_1^c & 0 \end{pmatrix}, \quad (5.82)$$

$$\Psi_{3,2} \supset \psi_{3,2,L}^{(0)} \rightarrow q_{aa} = \begin{pmatrix} u_1 & d_1 \\ u_2 & d_2 \\ u_3 & d_3 \end{pmatrix},$$

$$\Psi_1 \supset \psi_{1,L}^{(0)} \rightarrow E_{ab}^+ = \begin{pmatrix} 0 & e^+ \\ -e^+ & 0 \end{pmatrix}. \quad (5.83)$$

Note that the generation index A is suppressed as we mentioned above.⁴

Let us now derive the four-dimensional effective action. We treat T as the background and ignore its fluctuations because they are higher order corrections. Substituting the domain wall background

$$\hat{T} = \begin{pmatrix} \frac{2}{5} \beta_\phi \mathbf{1}_3 & 0 \\ 0 & -\frac{3}{5} \beta_\phi \mathbf{1}_2 \end{pmatrix}, \quad T^0 = \tau_+. \quad (5.84)$$

and (5.70) into the Lagrangian, picking up the left-handed zero modes, and integrating over y , we find

⁴The mass dimensions: [5D fermion] = 2, [5D Yukawa] = -1/2, [fermion zero mode function] = 1/2, [4D fermion] = 3/2, and [4D Yukawa] = 0.

$$\begin{aligned}
i \int dy \bar{\Psi}_3 (\Gamma^M D_M \Psi_3 - i h_3 \Psi_3) &\rightarrow i \int dy f_{3,L}^{(0)}(y) \bar{\psi}_{3,L}^{(0)}(x) \gamma^\mu \left(\partial_\mu - i g_3 A_{3\mu}^{(0)*} - i \frac{1}{3} \sqrt{\frac{3}{5}} g_1 A_{1\mu}^{(0)*} \right) f_{3,L}^{(0)}(y) \psi_{3,L}^{(0)}(x) \\
&+ \int dy f_{3,L}^{(0)}(y) \bar{\psi}_{3,L}^{(0)}(x) (i \Gamma^5 \partial_y + h_3) f_{3,L}^{(0)}(y) \psi_{3,L}^{(0)}(x) \\
&= i \bar{\psi}_{3,L}^{(0)}(x) \gamma^\mu \left(\partial_\mu - i g_3 A_{3\mu}^{(0)*} - i \frac{1}{3} \sqrt{\frac{3}{5}} g_1 A_{1\mu}^{(0)*} \right) \psi_{3,L}^{(0)}(x) \\
&+ \bar{\psi}_{3,L}^{(0)}(x) \psi_{3,L}^{(0)}(x) \int dy f_{3,L}^{(0)}(y) Q_{3,L} f_{3,L}^{(0)}(y) \\
&= i \bar{d}^c \gamma^\mu \left(\partial_\mu - i g_3 G_\mu^* - i g_1 \frac{1}{3} \sqrt{\frac{3}{5}} B_\mu^* \right) d^c, \tag{5.85}
\end{aligned}$$

where we have used (5.71) and (5.73). Repeating the similar calculation for Ψ_2 , Ψ'_3 , $\Psi_{3,2}$, and Ψ_1 , we find the resultant effective Lagrangian for $\mathcal{L}_f$ as

$$\mathcal{L}_f^{\text{eff}} = i \bar{d}^c \gamma^\mu D_\mu d^c + i \bar{l} \gamma^\mu D_\mu l + i \text{Tr}[\bar{q} \gamma^\mu D_\mu q] + \frac{i}{2} \text{Tr}[\bar{U}^c \gamma^\mu D_\mu U^c] + \frac{i}{2} \text{Tr}_2[E^+ \gamma^\mu D_\mu E^+], \tag{5.86}$$

where the covariant derivatives are defined by

$$D_\mu d^c = \partial d^c - i g_3 G_\mu^* d^c - i g_1 \frac{1}{3} \sqrt{\frac{3}{5}} B_\mu^* d^c, \tag{5.87}$$

$$D_\mu l = \partial l - i g_2 W_\mu^* l + i g_1 \frac{1}{2} \sqrt{\frac{3}{5}} B_\mu^* l, \tag{5.88}$$

$$D_\mu q = \partial_\mu q + i g_3 G_\mu q + i g_2 q W_\mu^t + i g_1 \frac{1}{6} \sqrt{\frac{3}{5}} B_\mu q, \tag{5.89}$$

$$D_\mu U^c = \partial_\mu U^c + 2 i g_3 U^c G_\mu^t + i g_1 \frac{2}{3} \sqrt{\frac{3}{5}} B_\mu U^c, \tag{5.90}$$

$$D_\mu E^+ = \partial_\mu E^+ + i g_1 \sqrt{\frac{3}{5}} g_1 B_\mu E^+. \tag{5.91}$$

C. Effective Lagrangian of fermion sector 2: SM Yukawa sector

In this subsection, we consider the Yukawa sector with $\mathcal{H}$. The Yukawa Lagrangian $\mathcal{L}_{\text{Yukawa}}$ is described by

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} &= \Psi_{5A}^t C Y_{AB}^{(1)} \Psi_{10B} \mathcal{H}^\dagger + \epsilon_{ijklm} \Psi_{10A}^{ijt} Y_{AB}^{(2)} C \Psi_{10B}^{kl} \mathcal{H}^m \\
&+ \text{H.c.}, \tag{5.92}
\end{aligned}$$

where $C = \gamma_1 \gamma_3$ is the charge conjugation matrix in five dimensions, $\epsilon_{ijklm}(i, j, k, l, m = 1, \dots, 5)$ is the

five-dimensional Levi-Civita tensor, and $Y_{AB}^{(1)}$ and $Y_{AB}^{(2)}$ are Yukawa couplings.⁵

We derive the effective Lagrangian for the Yukawa sector (5.92). In particular, we focus on model 1 of the Higgs sector described in Sec. IV A.⁶ Substituting (4.7) and (5.69) into (5.92), we have

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} &= \frac{1}{\sqrt{2}} \Psi_3^t Y^{(1)} \Psi_{3,2} \mathcal{H}_2^* + \frac{1}{\sqrt{2}} \Psi_2^t Y^{(1)} \Psi_1 \mathcal{H}_2^* \\
&+ \epsilon_{\alpha\beta\gamma} \Psi_{3'}^{\alpha\beta} (Y^{(2)} + Y^{(2)t}) \Psi_{3,2}^{\gamma a} \epsilon_{ab} \mathcal{H}_2^b + \text{H.c.}, \tag{5.93}
\end{aligned}$$

where $\epsilon_{\alpha\beta\gamma}(\epsilon_{ab})$ is the 3(2)-dimensional Levi-Civita tensor. The generation indices are suppressed. We drop the Yukawa couplings including the triplet Higgs since they do not have a normalizable zero mode as shown in Sec. IV A. First we substitute (4.12) together with (4.16), (4.18) and the expansion of the fermions such as (4.29) into (5.93). Next we only pick up the zero modes of Higgs and the fermions and integrate over y . Then, we obtain the effective Lagrangian of (5.92) as

⁵Since $\mathcal{H}$ vanishes in the background configuration, $\mathcal{L}_{\text{Yukawa}}$ does not affect the fermion zero modes in the previous subsection. Also, we will find that the $\mathcal{H}$ -Yukawa couplings $Y_{AB}^{(1,2)}$ are smaller than T -Yukawa couplings $h_{5,5,10,10}$, so their contributions can be safely ignored in the previous subsection.

⁶We expect qualitatively the same results for the model 2, as the only difference is the Higgs wave functions.

$$\mathcal{L}_{\text{Yukawa}}^{\text{eff}} = d^c Y_d^{\text{eff}} q h_{2,0}^* + l Y_e^{\text{eff}} e^+ h_{2,0}^* + u^c Y_u^{\text{eff}} q h_{2,0} + \text{H.c.}, \quad (5.94)$$

where Y_u^{eff} , Y_d^{eff} , and Y_e^{eff} are the effective Yukawa couplings given by

$$Y_u^{\text{eff}} = 2N_{H,2} \int dy f_{\bar{3},L}^{(0)} (Y^{(2)} + Y^{(2)\dagger}) f_{3,2,L}^{(0)}, \quad (5.95)$$

$$Y_d^{\text{eff}} = \frac{N_{H,2}}{2} \int dy f_{\bar{3},L}^{(0)} Y^{(1)} f_{3,2,L}^{(0)}, \quad (5.96)$$

$$Y_e^{\text{eff}} = \frac{N_{H,2}}{2} \int dy f_{2,L}^{(0)} Y^{(1)} f_{1,L}^{(0)}, \quad (5.97)$$

where the mode functions such as $f_{\bar{3},L}$ are given in (5.74) and (5.77)–(5.80). We assume that this effective Lagrangian is defined at the GUT scale $M_X = 2.0 \times 10^{16}$ GeV (As for this scale, see, for example, [71]). Since the normalization constant $N_{H,2}$ can be absorbed into the Yukawa matrices, the choice of $N_{H,2}$ is merely a convention.

In the following, we show that the effective Yukawa couplings in (5.95)–(5.97) can partially realize the four-dimensional Yukawa couplings in the SM, which are experimentally determined at the weak scale, by choosing the parameters in the five-dimensional theory. The strategy is as follows. We first run the four-dimensional Yukawa couplings in the SM to the GUT scale M_X from the weak scale defined at $M_Z = 91.2$ GeV by the renormalization group equation (RGE) of the SM as $Y_{u,d,e}^{\text{SM}}(M_Z) \rightarrow Y_{u,d,e}^{\text{SM}}(M_X)$. Next we find the parameters, such as v , Ω , $\mathcal{Y}_{2,3}$, h , and $\tilde{h}$, in the five-dimensional theory such that the effective Yukawa couplings $Y_{u,d,e}^{\text{eff}}$ (5.95)–(5.97) correctly reproduce to the four-dimensional Yukawa coupling $Y_{u,d,e}^{\text{SM}}(M_X)$ in the SM at the GUT scale.

However, since dealing with all the Yukawa couplings are a bit complicated, in this paper we simply look at the squared Yukawa couplings $H_a^{\text{SM}} = Y_a^{\text{SM}} Y_a^{\text{SM}\dagger}$ ($a = u, d, e$) only. The H -RGEs for the squared Yukawa couplings H_a^{SM} at two-loop level are given in [72]. In our analysis, we use the one-loop RGEs which are summarized in C. We first take the SM fermion diagonal mass matrices M_a at the renormalization scale $\mu = M_Z$ [73]

$$M_u(M_Z) = \text{diag}(1.38 \times 10^{-3}, 0.638, 172.1) \text{ GeV}, \quad (5.98)$$

$$M_d(M_Z) = \text{diag}(2.82 \times 10^{-3}, 57 \times 10^{-3}, 2.86) \text{ GeV}, \quad (5.99)$$

$$M_e(M_Z) = \text{diag}(0.487 \times 10^{-3}, 103 \times 10^{-3}, 1750 \times 10^{-3}) \text{ GeV}. \quad (5.100)$$

The diagonalized SM Yukawa couplings in the mass basis can be extract as $D_a^{\text{SM}}(M_Z) = \sqrt{2} M_a(M_Z) / v_{\text{SM}}$ where $v_{\text{SM}} = 246$ GeV. The SM Yukawa couplings $Y_a^{\text{SM}}(M_Z)$ in the flavor basis are related to D_a^{SM} via a bi-unitary transformations $D_a = U_L^a Y_a U_R^{a\dagger}$ [$U_{L,R}^a \in SU(3)$]. As usual, this leads to the appearance of the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V_{\text{CKM}} = U_L^u U_L^{d\dagger}$ in the quark charged currents in the mass basis, originating from the mismatch of the left-handed rotations in the up- and down-type sectors. However, in order to use the H -RGE, we need to redefine the down-type quark fields so as to absorb the CKM matrix in the quark charged currents, namely, $d_L \rightarrow V_{\text{CKM}} d_L$, such that all data for the flavor mixing resides in the down-type Yukawa matrix (the down-sector mixing basis) as,

$$Y_{u,e} \rightarrow D_{u,e} \rightarrow D_{u,e}, \quad Y_d \rightarrow D_d \rightarrow V_{\text{CKM}} D_d \equiv Y'_d. \quad (5.101)$$

Here, we suppressed the superscript SM and the energy scale (M_Z). As a result, we have the squared Yukawa couplings H_a^{SM} in the down-sector mixing basis at $\mu = M_Z$:

$$H_u^{\text{SM}}(M_Z) = D_u^2, \quad D_u = \sqrt{2} v_{\text{SM}}^{-1} M_u(M_Z), \quad (5.102)$$

$$H_d^{\text{SM}}(M_Z) = Y'_d Y'^{\dagger}_d = V_{\text{CKM}} D_d^2 V_{\text{CKM}}^{\dagger}, \quad D_d = \sqrt{2} v_{\text{SM}}^{-1} M_d(M_Z), \quad (5.103)$$

$$H_e^{\text{SM}}(M_Z) = D_e^2, \quad D_e = \sqrt{2} v_{\text{SM}}^{-1} M_e(M_Z), \quad (5.104)$$

where V_{CKM} is the CKM matrix given by

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \quad (5.105)$$

Here we denote $s_{ij} \equiv \sin \theta_{ij}$, $c_{ij} \equiv \cos \theta_{ij}$ and $s_{12} = 0.22500$, $s_{13} = 0.00369$, $s_{23} = 0.04182$, $\delta = 1.144$.

Now, we run H_a^{SM} at M_Z to the GUT scale at M_X . Note that $H_a(M_X)$ is not diagonalized in general, but we can diagonalize it again by using U_L^a only as $H_a \rightarrow (U_L^a Y_a U_R^a)(U_L^a Y_a U_R^a)^{\dagger} = U_L^a H_a U_L^{a\dagger}$. The resultant diagonal matrices $\sqrt{H_a}(M_X)$ are found as

$$\sqrt{H_u}(M_X)|_{\text{diag}} = \text{diag}(3.24 \times 10^{-6}, 1.50 \times 10^{-3}, 4.61 \times 10^{-1}), \quad (5.106)$$

$$\sqrt{H_d}(M_X)|_{\text{diag}} = \text{diag}(6.84 \times 10^{-6}, 1.38 \times 10^{-4}, 6.08 \times 10^{-3}), \quad (5.107)$$

$$\sqrt{H_e}(M_X)|_{\text{diag}} = \text{diag}(2.75 \times 10^{-6}, 5.80 \times 10^{-4}, 9.86 \times 10^{-3}). \quad (5.108)$$

Let us compare these originated from the SM with $\sqrt{H_a^{\text{eff}}} = \sqrt{Y_a^{\text{eff}} Y_a^{\text{eff}\dagger}}$ with the effective Yukawa couplings Y_a^{eff} in (5.96), (5.97), (5.95) originated by the five dimensional theory. Since our focus is solely on reproducing the fermion mass spectrum, we compare the eigenvalues of $\sqrt{H}$. We find that the following set of parameters in the five-dimensional Lagrangian can reproduce good agreement:

$$v = M_X^{\frac{3}{2}}, \quad \Omega = M_X, \quad \mathcal{Y}_2 = -\mathcal{Y}_3 = 2M_X^{-1}, \quad c = 0.13M_X^{\frac{3}{2}}, \quad (5.109)$$

where $\mathcal{Y}_2 = \mathcal{Y}_3 = 2M_X^{-1}$ means the separation of the walls is $d = 4M_X^{-1}$ as shown in Fig. 1. The parameter c is in the $\beta_H(\hat{T})$ given in (4.2) and it gives $N_{H,2} = 0.2M_X^{1/2}$ by (4.18). We also choose the 5 dimensional Yukawa couplings for $\hat{T}$ and T_0 in the following way

$$h_{5,1} = -0.227, \quad \tilde{h}_{5,1} = 0.800, \quad h_{5,2} = -8.00, \quad \tilde{h}_{5,2} = 6.00, \quad h_{5,3} = 1.18, \quad \tilde{h}_{5,3} = 0.170, \quad (5.110)$$

$$h_{10,1} = -9.06, \quad \tilde{h}_{10,1} = 1.00, \quad h_{10,2} = -6.6, \quad \tilde{h}_{10,2} = 1.91, \quad h_{10,3} = -1.85, \quad \tilde{h}_{10,3} = 1.00, \quad (5.111)$$

together with the 5-dimensional Yukawa couplings for $\mathcal{H}$ in the mass basis as

$$Y^{(1)} = \text{diag}(x_1, x_2, x_3), \quad Y^{(2)} = \text{diag}(y_1, y_2, y_3). \quad (5.112)$$

These are all in the unit of $M_X^{-\frac{1}{2}}$. The fermion zero mode functions are shown in Fig. 5. By using Eqs. (5.95)–(5.97), we find

$$\sqrt{H_u^{\text{eff}}} = \text{diag}(|y_1| \times 1.89 \times 10^{-4}, |y_2| \times 3.55 \times 10^{-3}, |y_3| \times 3.67 \times 10^{-1}), \quad (5.113)$$

$$\sqrt{H_d^{\text{eff}}} = \text{diag}(|x_1| \times 7.31 \times 10^{-2}, |x_2| \times 1.55 \times 10^{-2}, |x_3| \times 7.25 \times 10^{-2}), \quad (5.114)$$

$$\sqrt{H_e^{\text{eff}}} = \text{diag}(|x_1| \times 2.74 \times 10^{-6}, |x_2| \times 5.80 \times 10^{-4}, |x_3| \times 2.36 \times 10^{-4}). \quad (5.115)$$

We find $|x_a|$ and $|y_a|$ that provides closest match between $\sqrt{H_a^{\text{SM}}}(M_X)$ and $\sqrt{H_a^{\text{eff}}}$ up to three decimals. It is not trivial because we have overdetermined system of nine equations for six variables $|x_a|$ and $|y_a|$. Choosing the six parameters in the following way,

$$(|x_1|, |x_2|, |x_3|) = (9.35 \times 10^{-5}, 8.91 \times 10^{-3}, 8.39 \times 10^{-2}), \quad (5.116)$$

$$(|y_1|, |y_2|, |y_3|) = (1.72 \times 10^{-2}, 4.21 \times 10^{-1}, 1.25), \quad (5.117)$$

the resultant numerical matrices are found to be

$$\sqrt{H_u^{\text{eff}}} = \text{diag}(3.24 \times 10^{-6}, 1.50 \times 10^{-3}, 4.61 \times 10^{-1}), \quad (5.118)$$

$$\sqrt{H_d^{\text{eff}}} = \text{diag}(6.84 \times 10^{-6}, 1.38 \times 10^{-4}, 6.08 \times 10^{-3}), \quad (5.119)$$

$$\sqrt{H_e^{\text{eff}}} = \text{diag}(2.75 \times 10^{-6}, 5.80 \times 10^{-4}, 9.86 \times 10^{-3}), \quad (5.120)$$

which are in perfect agreement with the RGE-evolved SM Yukawa matrices in Eqs. (5.106)–(5.108) within the displayed numerical precision.

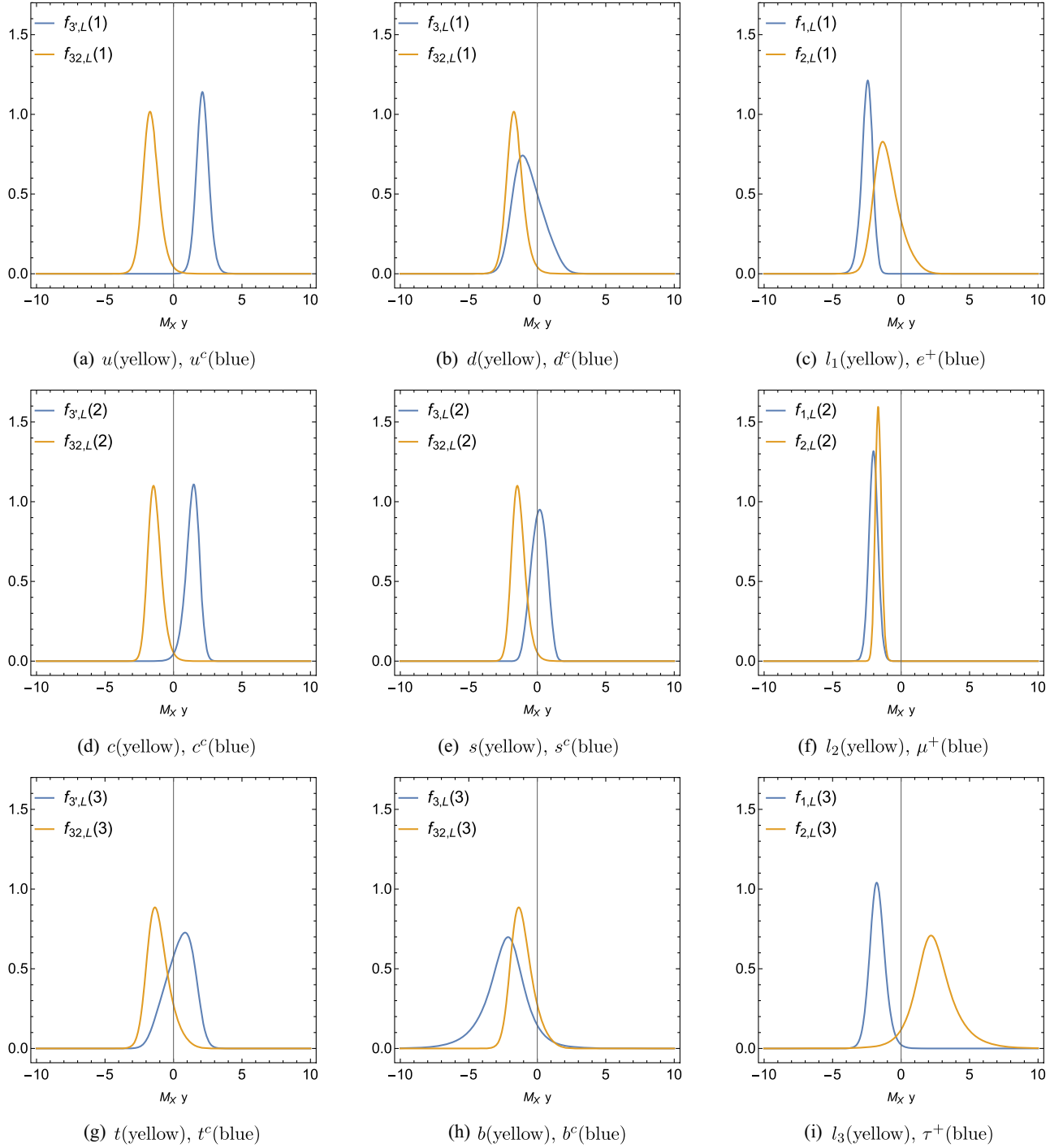


FIG. 5. Nine overlaps of the normalized fermion zero mode functions.

Importantly, in minimal four-dimensional $SU(5)$ GUTs, the unified Yukawa structure generically predicts the relation $Y_d = Y_e^T$ at the unification scale, leading to phenomenologically unrealistic fermion mass relations. Conventional resolutions typically involve extending the Higgs sector, for example by introducing a **45** representation as in the Georgi-Jarlskog mechanism [69], or by invoking higher-dimensional operators that modify the Yukawa structure [70].

In contrast, our framework avoids this problem without introducing such ingredients. The model employs only the adjoint **24** representation for $SU(5)$ breaking, as in minimal $SU(5)$, and does not rely on higher-dimensional operators. Instead, the theory is formulated in five dimensions, where the effective four-dimensional Yukawa matrices arise from overlap integrals of localized fermion zero-mode functions in the extra dimension. Although the

underlying five-dimensional Yukawa couplings respect $SU(5)$ unification, the generation-dependent wave-function overlaps induce nonuniversal corrections to the effective four-dimensional Yukawa matrices. As a result, the usual $SU(5)$ fermion mass relations are relaxed, allowing for a realistic pattern of fermion masses and mixings.

D. Effective Lagrangian for the Higgs sector

Finally, we shall discuss the electroweak symmetry breaking in the Higgs sector in the effective Lagrangian.

1. Model 1

We shall modify the five-dimensional Lagrangian for the model, (4.1). We first note that the final result in this section is valid for any β_H . We propose a starting Lagrangian by adding a scalar potential as a perturbation which will be a seed of the electroweak symmetry breaking at a low energy:

$$\mathcal{L}_H = \text{Tr}[\beta_H(\hat{T}, T_0)^2\{(\mathcal{D}_M \mathcal{H})^\dagger \mathcal{D}^M \mathcal{H} - V_{\text{pert}}\}], \quad (5.121)$$

where

$$V_{\text{pert}} = -\tilde{\mu}^2 \mathcal{H}^\dagger \mathcal{H} + \tilde{\lambda}(\mathcal{H}^\dagger \mathcal{H})^2. \quad (5.122)$$

We assume that this potential becomes relevant at energies which are much lower than the GUT scale. Therefore V_{pert} does not affect the results concerning the double-triplet splitting problem, which are mentioned above. Decomposing the Higgs as (4.7) and dropping the triplet part $\mathcal{H}_3$, we have the effective Lagrangian for the standard model Higgs $\mathcal{H}_2$:

$$\mathcal{L}_H = \beta_{H,2}^2\{(D_\mu \mathcal{H}_2)^\dagger D^\mu \mathcal{H}_2 - V'\}, \quad (5.123)$$

where $D_\mu \mathcal{H}_2 = \partial_\mu \mathcal{H}_2 - iA_{2\mu} \mathcal{H}_2 + i\frac{1}{2}\sqrt{\frac{3}{5}}A_{1\mu} \mathcal{H}_2$ and

$$V' = \beta_{H,2}^2\{(\partial_y \mathcal{H}_2)^\dagger \partial_y \mathcal{H}_2 - \tilde{\mu}^2 \mathcal{H}_2^\dagger \mathcal{H}_2 + \tilde{\lambda}(\mathcal{H}_2^\dagger \mathcal{H}_2)^2\}. \quad (5.124)$$

From (4.10) and (4.12) with (4.16), we can write

$$\mathcal{H}_2 = N_{H,2} h_{2,0} + \text{massive modes}. \quad (5.125)$$

Here $N_{H,2} = 1/\sqrt{\int dy \beta_{H,2}^2}$ as is given in Eq. (4.18). Substituting (5.125) into (5.123) and integrating over y , we obtain

$$\mathcal{L}_H^{\text{eff}} = (D_\mu h_{2,0})^\dagger D^\mu h_{2,0} - V'^{\text{eff}}, \quad (5.126)$$

where $D_\mu h_{2,0} = \partial_\mu h_{2,0} - ig_2 W_\mu h_{2,0} + ig_1 \frac{1}{2}\sqrt{\frac{3}{5}}B_\mu h_{2,0}$ and

$$V'^{\text{eff}} = -\tilde{\mu}^2 h_{2,0}^\dagger h_{2,0} + \tilde{\lambda}'(h_{2,0}^\dagger h_{2,0})^2, \quad (5.127)$$

with $\tilde{\lambda}' = N_{H,2}^2 \tilde{\lambda}$.

2. Model 2

We start with the five-dimensional Lagrangian (4.20). In the effective Lagrangian the colored Higgs $\mathcal{H}_3$ is simply dropped since it is so heavy. For the standard model Higgs $\mathcal{H}_2$, we only consider the lowest mode. Then, the Lagrangian (4.20) becomes

$$\mathcal{L}_H = (D_M \mathcal{H}_2)^\dagger D^M \mathcal{H}_2 - \mathcal{H}_2^\dagger U_2 \mathcal{H}_2 - \lambda_H (\mathcal{H}_2^\dagger \mathcal{H}_2)^2, \quad (5.128)$$

where $D_M \mathcal{H}_2 = \partial_M \mathcal{H}_2 - iA_{2M} \mathcal{H}_2 + i\frac{1}{2}\sqrt{\frac{3}{5}}A_{1M} \mathcal{H}_2$. Note that we do not need to add additional potential terms because the starting Lagrangian (4.20) already has a seed of the electroweak symmetry breaking. We derive the effective Lagrangian in the $\tilde{d} \rightarrow 0$ limit, so that we can utilize the analytic form of the mode functions (4.34). Substituting (4.29) and (5.38), integrating over y , and picking up the zero mode $h_{2,0}$ in Eq. (4.29), we obtain

$$\mathcal{L}_H^{\text{eff}} = (D_\mu h_{2,0})^\dagger D^\mu h_{2,0} - V_h, \quad (5.129)$$

where $D_\mu h_{2,0} = \partial_\mu h_{2,0} - ig_2 W_\mu h_{2,0} + ig_1 \frac{1}{2}\sqrt{\frac{3}{5}}B_\mu h_{2,0}$ and

$$V_h = \int dy \{ (h_{2,0}^\dagger h_{2,0}) (\partial_y u_0(y))^2 + h_{2,0}^\dagger U_0 h_{2,0} (u_0(y))^2 + \lambda_H (h_{2,0}^\dagger h_{2,0})^2 (u_0(y))^4 \} \quad (5.130)$$

$$= \Omega_H^2 h_{2,0}^\dagger h_{2,0} + V_K + V_P + V_4, \quad (5.131)$$

$$V_K = h_{2,0}^\dagger h_{2,0} \int dy (\partial_y u_0(y))^2, \quad (5.132)$$

$$V_P = h_{2,0}^\dagger h_{2,0} \int dy (u_0(y))^2 \times \frac{-U_0}{\cosh \Omega(y - \frac{\mathcal{Y}_2 - \mathcal{Y}_3}{2}) \cosh \Omega(y + \frac{\mathcal{Y}_2 - \mathcal{Y}_3}{2})}, \quad (5.133)$$

$$V_4 = \lambda_H (h_{2,0}^\dagger h_{2,0})^2 \int dy (u_0(y))^4. \quad (5.134)$$

Note that $\mathcal{A}_{a5}$ is simply dropped in the covariant derivative since their zero modes do not exist. We can analytically evaluate (5.132), (5.133), and (5.134) with the use of (4.35) as

$$V_K = \frac{s^2}{2s+1} \Omega^2 h_{2,0}^\dagger h_{2,0}, \quad (5.135)$$

$$V_P = -U_0 \frac{2s}{2s+1} h_{2,0}^\dagger h_{2,0}, \quad (5.136)$$

$$V_4 = \lambda_H \Omega \frac{B(\frac{1}{2}, 2s)}{(B(\frac{1}{2}, s))^2} (h_{2,0}^\dagger h_{2,0})^2, \quad (5.137)$$

where $B(p, q)$ is the beta function. Collecting the above results, we have the effective potential as

$$V_h = \mu^2 h_{2,0}^\dagger h_{2,0} + \frac{\lambda}{2} (h_{2,0}^\dagger h_{2,0})^2, \quad (5.138)$$

with

$$\mu^2 = \frac{s^2}{2s+1} \Omega^2 - \frac{2s}{2s+1} U_0 + \Omega_H^2, \quad (5.139)$$

$$\frac{\lambda}{2} = \lambda_H^2 \Omega \frac{B(\frac{1}{2}, 2s)}{(B(\frac{1}{2}, s))^2}. \quad (5.140)$$

VI. CONCLUSION

In this work, we have constructed a five-dimensional $SU(5)$ grand unified theory formulated on domain walls, in which both gauge and matter fields are dynamically localized. The framework naturally realizes a brane world scenario without assuming the existence of branes by hand, as the four-dimensional universe emerges on a domain wall configuration generated by spontaneous symmetry breaking in the bulk. The overall structure of the present framework is summarized schematically in Fig. 6. The top panel of Fig. 6 illustrates the energetically preferred 3-2 split domain-wall configuration, which spontaneously breaks $SU(5)$ down to the SM gauge group $SU(3) \times SU(2) \times U(1)$. This configuration stabilizes by the inclusion of an additional potential term, leading to a robust domain-wall background suitable for localizing the SM fields as discussed in Sec. III B.

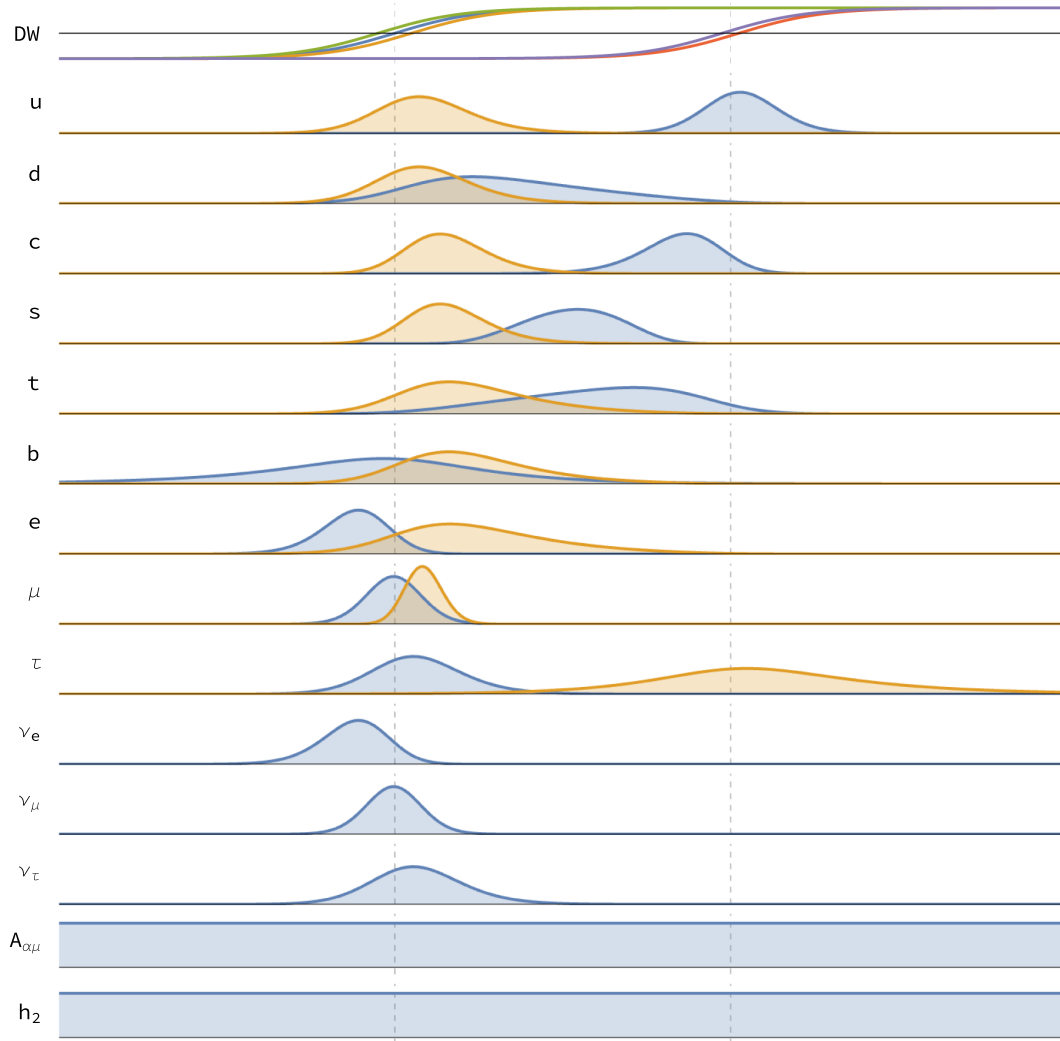


FIG. 6. The summary of our $SU(5)$ GUT in the five dimensions. The top line corresponds to the 3-2 splitting domain walls, followed by the six quarks and six leptons. The last two rows show the flat zero mode functions for the gauge fields and the SM Higgs doublet in the model 1, respectively.

The middle panels of Fig. 6 summarize the localization of the quark and lepton zero modes. Their localized wave-function profiles provide the origin of the hierarchical Yukawa couplings in the four-dimensional effective theory.

In Sec. VA, the localization of gauge fields is achieved by introducing a field-dependent gauge kinetic term, effectively realizing a semiclassical analog of the confinement in the bulk. This mechanism ensures that massless gauge bosons are localized to the vicinity of the domain wall, independent of the detailed form of the field-dependent coupling. We should emphasize that the localization of the gauge field is accomplished in terms of “dressed gauge field” $A_{a\mu}(x^\mu, y) \simeq g_a \beta_a(y) A_{a\mu}^{(0)}(x^\mu) + \dots$ with the zero mode function $\beta_a(y)$ which can be any square integrable function. This leads to the fact that the zero mode function for the original gauge field is flat in the y coordinate as $\mathcal{A}_{a\mu}(x^\mu, y) = \beta_a(y)^{-1} A_{a\mu}(x^\mu, y) \simeq g_a A_{a\mu}^{(0)}(x^\mu) + \dots$. The flatness is physically important because it ensures the universality of the gauge coupling constants g_a for the non-Abelian gauge symmetries. The flat zero mode function is shown in the row at the second from the bottom of Fig. 6.

The most distinctive feature of our construction is the treatment of the Higgs sector as discussed in Sec. IV, which resolves the long-standing doublet-triplet splitting problem of $SU(5)$ GUTs. We have proposed two distinct models. In model 1, the doublet-triplet splitting is achieved dynamically through a nonminimal, topological coupling between the Higgs field and the domain-wall background. The electroweak Higgs doublet possesses a normalizable zero mode $h_{2,0}(x^\mu)$ localized between the separated walls, while the colored triplet mode $h_{3,0}(x^\mu)$ remains non-normalizable and hence decoupled as shown in Fig. 3. Similarly to the localization of the gauge fields, the localized zero mode function for $h_{2,0}(x^\mu)$ appears in the “dressed Higgs field” as $H_I(x^\mu, y) = N_{H,I} \beta_{H,I}(y) h_{I,0}(x^\mu) + \dots$. Therefore, the zero mode function of the original Higgs field is flat as $\mathcal{H}_I(x^\mu, y) = \beta_{H,I}(y)^{-1} H_I(x^\mu, y) = N_{H,I} h_{I,0}(x^\mu) + \dots$. The flatness of the Higgs doublet zero mode function is not necessary but it makes the calculation of the effective Yukawa couplings quite simple as we seen in Sec. VC. The flat zero mode function is shown at the bottom row of Fig. 6. Note that the massless doublet appears as a topologically protected edge state, analogous to the Jackiw-Rebbi fermion, and no fine-tuning of parameters is required. In model 2, a potential coupling between the Higgs and the adjoint scalar fields is introduced. The resulting potential repels the colored Higgs triplet from the domain wall, so that it possesses neither localized nor light modes. Consequently, only the Higgs doublet needs to be examined for possible light or tachyonic modes. The correct pattern of electroweak symmetry breaking can then be realized by an appropriate fine-tuning of the model parameters.

We have also derived the four-dimensional effective theory by integrating out the extra dimension. The effective Lagrangian successfully reproduces the SM structure at low energies, including the correct gauge kinetic terms, fermion sectors, and Yukawa interactions. Furthermore, renormalization group analysis demonstrates that the SM Yukawa couplings at the weak scale can be reproduced from appropriate choices of the five-dimensional parameters, establishing the phenomenological viability of the framework.

Although the present work has focused on the theoretical construction of a realistic dynamical $SU(5)$ brane world scenario, the completion of the Higgs sector and Yukawa interactions considerably broadens its phenomenological scope. In particular, it will be interesting to revisit proton decay by extending the analysis of Ref. [52] to include the Higgs sector and Yukawa interactions developed in this work. The present framework also provides a realistic $SU(5)$ setting for investigating phenomenological aspects of dynamical brane world models, including collider signatures of Kaluza-Klein gauge bosons and fermions, and Higgs phenomenology and flavor observables associated with nonuniversal Kaluza-Klein gauge couplings [58–60]. These issues are left for future work.

ACKNOWLEDGMENTS

The authors (M. A. and M. E.) wish to acknowledge the support of the KA171/Erasmus+ program, which made it possible to stay at the Silesian University in Opava, where this work was brought to completion. This work is supported in part by JSPS Grant-in-Aid for Scientific Research KAKENHI Grant No. JP21K03565 and JP25K07275 (M. A.). This work has been also supported by the Grant No. SGS/24/2024 Astrophysical processes in strong gravitational and electromagnetic fields of compact object (F. B.).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: THE EFFECTIVE POTENTIAL FOR DOMAIN WALLS

In this Appendix, we discuss the calculation of the effective potential of Sec. III.

The problem at hand is to calculate the integral

$$\int_{-\infty}^{\infty} dx V_1(T_{\text{wals}}, \text{Tr}(T_{\text{walls}})) = V_{\text{eff}}(R_1, \dots, R_5), \quad (\text{A1})$$

where we denoted that V_1 can be function of both T_{walls} and its trace. In general, such integral can be broken down to

evaluation of many pieces of the form

functions:

$$\int_{-\infty}^{\infty} dx \tanh^{m_i}(x - R_i) \tanh^{m_j}(x - R_j) \dots \tanh^{m_k}(x - R_k), \quad (A2)$$

$$G_1(i) \equiv \int_{-\infty}^{\infty} dx (\tanh(x - R_i) - \tanh(x)), \quad (A3)$$

where m_i 's are some integers. All of these integrals can be related (by differentiation with respect to R_i 's or by taking limits) to some combination of the following generating

$$G_2(i, j) \equiv \int_{-\infty}^{\infty} dx (\tanh(x - R_i) \tanh(x - R_j) - 1), \quad (A4)$$

$$G_3(i, j, k) \equiv \int_{-\infty}^{\infty} dx (\tanh(x - R_i) \tanh(x - R_j) \tanh(x - R_k) - \tanh(x)), \quad (A5)$$

$$\begin{aligned} G_4(i, j, k, l) &\equiv \int_{-\infty}^{\infty} dx (\tanh(x - R_i) \tanh(x - R_j) \tanh(x - R_k) \tanh(x - R_l) - 1), \\ &\vdots \end{aligned} \quad (A6)$$

Notice that for odd number of R_i 's we subtracted $\tanh(x)$, while for even number we subtracted 1 to have manifest convergence.

Using the identity

$$\tanh(x - R_i) \tanh(x - R_j) - 1 = \frac{\tanh(x - R_i) - \tanh(x - R_j)}{\tanh(R_i - R_j)}, \quad (A7)$$

it is possible to establish recurrence relations:

$$G_n(i, j, k, \dots, l) = \frac{G_{n-1}(i, k, \dots, l) - G_{n-1}(j, k, \dots, l)}{\tanh(R_i - R_j)} + G_{n-2}(k, \dots, l), \quad (A8)$$

that, in principle, allow for the computation of any given G_n starting from $G_1(i) = -2R_i$.

After a bit of algebra we can obtain explicit forms of a first few generating functions as

$$G_2(i, j) = -2 \left(\frac{R_i}{\tanh_{i-j}} + \frac{R_j}{\tanh_{j-i}} \right), \quad (A9)$$

$$G_3(i, j, k) = -2 \left(\frac{R_i}{\tanh_{i-j} \tanh_{i-k}} + \frac{R_j}{\tanh_{j-i} \tanh_{j-k}} + \frac{R_k}{\tanh_{k-i} \tanh_{k-j}} \right), \quad (A10)$$

$$G_4(i, j, k, l) = -2 \left(\frac{R_i}{\tanh_{i-j} \tanh_{i-k} \tanh_{i-l}} + \frac{R_j}{\tanh_{j-i} \tanh_{j-k} \tanh_{j-l}} + \frac{R_k}{\tanh_{k-i} \tanh_{k-j} \tanh_{k-l}} + \frac{R_l}{\tanh_{l-i} \tanh_{l-j} \tanh_{l-k}} \right), \quad (A11)$$

where we used a shorthand $\tanh_{i-j} \equiv \tanh(R_i - R_j)$. From this, one can make a general guess

$$G_n(i_1 i_2 \dots i_n) = -2 \sum_{j=i_1}^{i_n} \frac{R_j}{\prod_{\substack{k=i_1 \\ k \neq j}}^{i_n} \tanh(R_j - R_k)}. \quad (A12)$$

This expression can be then used to check the validity of the recurrence (A8).

Note that all G_n 's are totally symmetric functions of their arguments. This fact is useful when summing over all positions (which occurs due to presence of traces of T_{wall} 's). To be concrete, the following identities (that holds for any symmetric function) are useful when evaluating the effective potential:

$$\sum_{i,j} G_2(i,j) = 2 \sum_{i>j} G_2(i,j) + \sum_i G_2(i,i), \quad (\text{A13})$$

$$\sum_{i,j,k} G_3(i,j,k) = 6 \sum_{i>j>k} G_3(i,j,k) + 3 \sum_{i>j} (G_3(i,j,j) + G_3(i,i,j)) + \sum_i G_3(i,i,i), \quad (\text{A14})$$

$$\begin{aligned} \sum_{i,j,k,l} G_4(i,j,k,l) &= 24 \sum_{i>j>k>l} G_4(i,j,k,l) + 12 \sum_{i>j>k} (G_4(i,j,k,k) + G_4(i,j,j,k) + G_4(i,i,j,k)) \\ &\quad + \sum_{i>j} (6G_4(i,i,j,j) + 4G_4(i,j,j,j) + 4G_4(i,i,i,j)) + \sum_i G_4(i,i,i,i). \end{aligned} \quad (\text{A15})$$

Let us illustrate our methods on evaluating

$$V_{\text{eff}} = v^2 \lambda \int_{-\infty}^{\infty} dy (\alpha \text{Tr}(\hat{T}_{\text{wall}}^4) - v^2 \mu^2 \text{Tr}(\hat{T}_{\text{wall}}^2)). \quad (\text{A16})$$

This integral only converges when the outside vacua belong to the $\langle 0 \rangle$ set. Hence, we can set $s_1 = \dots s_5 = 1$ without loss of generality. In order to simplify our task, let us evaluate each term separately. The second term gives

$$\begin{aligned} -\lambda v^2 \mu^2 \int_{-\infty}^{\infty} dy \text{Tr}(\hat{T}_{\text{wall}}^2) &= \frac{\sqrt{\lambda} v^3 \mu^2}{5} \sum_{i,j} \int_{-\infty}^{\infty} dx (\tanh(x - R_i) \tanh(x - R_j) - \tanh^2(x - R_i)) \\ &= \frac{\sqrt{\lambda} v^3 \mu^2}{5} \sum_{i,j} (G_2(i,j) - G_2(i,i)) \\ &= \sqrt{\lambda} v^3 \mu^2 \left(8 - \frac{4}{5} \sum_{i>j} \frac{R_i - R_j}{\tanh(R_i - R_j)} \right). \end{aligned} \quad (\text{A17})$$

This piece of the effective potential describes a repulsion for any pair of walls.

The second piece reads

$$\begin{aligned} \lambda \alpha \int_{-\infty}^{\infty} dy \text{Tr}(\hat{T}_{\text{wall}}^4) &= \frac{\sqrt{\lambda} v^3 \alpha}{125} \sum_{i,j,k,l} \int_{-\infty}^{\infty} dx (-3 \tanh(x - R_i) \tanh(x - R_j) \tanh(x - R_k) \tanh(x - R_l) \\ &\quad + 2(\tanh^2(x - R_i) \tanh(x - R_j) \tanh(x - R_k) + \tanh(x - R_i) \tanh^2(x - R_j) \tanh(x - R_k) \\ &\quad + \tanh(x - R_i) \tanh(x - R_j) \tanh^2(x - R_k)) - 4 \tanh^3(x - R_i) \tanh(x - R_j) + \tanh^4(x - R_i)) \\ &= \frac{\sqrt{\lambda} v^3 \alpha}{125} \sum_{i,j,k,l} (-3G_4(i,j,k,l) + 2G_3^{\text{sym}}(i,j,k) - 4G_4(i,i,i,j) + G_4(i,i,i,i)) \\ &= \frac{\sqrt{\lambda} v^3 \alpha}{125} \left(-72 \sum_{i>j>k>l} G_4(i,j,k,l) + 24 \sum_{i>j>k} G_3^{\text{sym}}(i,j,k) + 42 \sum_{i>j} G_4(i,i,j,j) - 84 \sum_{i>j} G_4(i,i,i,j) - \frac{2080}{3} \right), \end{aligned} \quad (\text{A18})$$

where

$$\begin{aligned} G_3^{\text{sym}}(i,j,k) &\equiv G_4(i,i,j,k) + G_4(i,j,j,k) + G_4(i,j,k,k) \\ &= -2 \left(1 + \frac{R_i - R_j}{\tanh_{i-j}} + \frac{R_i - R_k}{\tanh_{i-k}} + \frac{R_j - R_k}{\tanh_{j-k}} \right), \end{aligned} \quad (\text{A19})$$

$$G_4(i, i, j, j) = -4 \left(1 + \frac{1}{\sinh_{i-j}^2} \left(1 - \frac{R_i - R_j}{\tanh_{i-j}} \right) \right), \quad (\text{A20})$$

$$G_4(i, i, i, j) = -2 \frac{R_i - R_j}{\tanh_{i-j}} + \frac{2}{\sinh_{i-j}^2} \left(1 - \frac{R_i - R_j}{\tanh_{i-j}} \right), \quad (\text{A21})$$

$$G_4(i, i, i, i) = -\frac{8}{3}. \quad (\text{A22})$$

This can be simplified to

$$\begin{aligned} \lambda \alpha \int_{-\infty}^{\infty} dy \text{Tr}(\hat{T}_{\text{wall}}^4) &= \frac{\sqrt{\lambda} v^3 \alpha}{125} \left(-\frac{8560}{3} + 64 \sum_{i>j} \frac{R_i - R_j}{\tanh(R_i - R_j)} \right. \\ &\quad \left. - \sum_{i>j} \frac{376}{\sinh^2(R_i - R_j)} \left(1 - \frac{R_i - R_j}{\tanh(R_i - R_j)} \right) - 72 \sum_{i>j>k>l} G_4(i, j, k, l) \right). \end{aligned} \quad (\text{A23})$$

Combining the above, we get the final result

$$\begin{aligned} V_{\text{eff}} &= \frac{v^3 \sqrt{\lambda}}{125} \left(1000 \mu^2 - \frac{8650}{3} \alpha + (64\alpha - 100\mu^2) \sum_{i>j} \frac{R_i - R_j}{\tanh(R_i - R_j)} - \sum_{i>j} \frac{376\alpha}{\sinh^2(R_i - R_j)} \right. \\ &\quad \left. \times \left(1 - \frac{R_i - R_j}{\tanh(R_i - R_j)} \right) - 72\alpha \sum_{i>j>k>l} G_4(i, j, k, l) \right). \end{aligned} \quad (\text{A24})$$

APPENDIX B: PROOF OF DECOMPOSITION (5.22) AND (5.58)

Here we give a proof that the extra-dimensional gauge field $A_{\alpha y}$ can be decomposed into $D_\alpha C_\alpha$ and the divergence-free component as in Eq. (5.22). Firstly, we define a projection operator as

$$P_\alpha = D_\alpha (\Delta_\alpha)^{-1} D_\alpha^\dagger. \quad (\text{B1})$$

This is a projection operator since it satisfies $P_\alpha^2 = P_\alpha$. Therefore we can always decompose $A_{\alpha y} = P_\alpha A_{\alpha y} + (1 - P_\alpha) A_{\alpha y}$. The first term is actually $D_\alpha B_\alpha$ in (5.22) if we take $B_\alpha = \Delta_\alpha^{-1} D_\alpha^\dagger A_{\alpha y}$. However, P is well defined only when Δ_α^{-1} is well-defined. A dangerous case occurs if Δ_α^{-1} acts on its zero mode $\Delta_\alpha B^{(0)} = 0 (D_\alpha B^{(0)}) = 0$ as is given in (5.32). This is not the case, however, since Δ_α^{-1} acts right after $D_\alpha^\dagger$.

Let E be an arbitrary scalar and let us take an inner product

$$(B_0, D_\alpha^\dagger E) = (D_\alpha B_0, E) = 0, \quad (\text{B2})$$

where we have defined the inner product by $(A, B) = \int dy A^* B$. Hence we conclude that $D_\alpha^\dagger E$ does not include B_0 , and $\Delta_\alpha^{-1} D_\alpha^\dagger$ is always well defined, so is P_α .

Let us next treat the second term $(1 - P_\alpha) A_{\alpha y}$. This is denoted as C_α in (5.22). Indeed C_α satisfy the divergence-free condition (5.23)

$$\begin{aligned} D_\alpha^\dagger C_\alpha &= D_\alpha^\dagger (1 - P_\alpha) A_{\alpha y} = (D_\alpha^\dagger - D_\alpha^\dagger D_\alpha \Delta_\alpha^{-1} D_\alpha^\dagger) A_{\alpha y} \\ &= 0. \end{aligned} \quad (\text{B3})$$

This is a proof of the decomposition (5.22).

We can repeat the same procedure for the decomposition of X . Let us first introduce the following projection operator

$$P = D \Delta^{-1} D^\dagger. \quad (\text{B4})$$

Note that Δ^{-1} is well defined because $\Delta = D_X^\dagger D_X + \mathcal{M}^2$. Then a vector X can be always decomposed into $X = P X + (1 - P) X$. The first term is written as $D Y$ if we defined $Y = \Delta^{-1} D^\dagger X$. The second term is clearly divergence-free since

$$D^\dagger (1 - P) X = (D^\dagger - D^\dagger D \Delta^{-1} D^\dagger) X = 0. \quad (\text{B5})$$

Thus the second term is identified with Z in (5.58). This is proof of decomposition (5.58).

Lastly we show that $Z = (1 - P) X$ does not include the zero mode. We decompose it as

$$(1 - P)X = \sum_{n \geq 0} \chi_n C_n, \quad (\text{B6})$$

where C_n stands for the eigenvector of $\tilde{\Delta}$ as

$$\tilde{\Delta} C_n = \tilde{M}_n^2 C_n. \quad (\text{B7})$$

We assign $n = 0$ for the lowest eigenvalue of the non-negative definite operator $\tilde{\Delta}$. Note that the lowest eigenstate is the kernel of D_R , and it can be expressed as

$$C_0 \propto D\Lambda, \quad (\text{B8})$$

with Λ being an arbitrary function. Now let us verify if $(1 - P)X$ is orthonormal to C_0 by the following calculation

$$\begin{aligned} ((1 - P)X, C_0) &\propto ((1 - P)X, D\Lambda) \\ &= (D^\dagger(1 - P)X, \Lambda) = 0. \end{aligned} \quad (\text{B9})$$

Hence the expansion (B6) does not include the lowest eigenstate $n = 0$, namely Z does not have the zero mode.

APPENDIX C: ONE LOOP RGE OF SM

The SM gauge couplings $\alpha_i = \frac{g_i^2}{4\pi}$ at one loop levels are given by

$$\alpha_i(\mu)^{-1} = \frac{1}{\alpha_i(M_Z)} - \frac{b_i}{2\pi} \ln \frac{\mu}{M_Z}, \quad (\text{C1})$$

where $i = 1, 2, 3$ correspond to the gauge groups $U(1)$, $SU(2)$, $SU(3)$, respectively. Here $b_1 = \frac{41}{10}$, $b_2 = -\frac{19}{6}$, $b_3 = -7$

and $\alpha_1(M_Z) = \frac{5\alpha_{\text{EM}}}{3\cos^2\theta_W}$, $\alpha_2(M_Z) = \frac{\alpha_{\text{EM}}}{\sin^2\theta_W}$, $\alpha_3(M_Z) = 0.1176$ where $\alpha_{\text{EM}} = \frac{1}{127.909}$ and $\sin^2\theta_W = 0.23119$.

The RGEs for Yukawa couplings are given by

$$\frac{d}{dt} H_a = \frac{1}{16\pi^2} \beta_a^{(1)} H_a + H_a \frac{1}{16\pi} \beta_a^{(1)\dagger}, \quad (\text{C2})$$

where $a = u, d, e$ and $t = \ln(\mu/M_Z)$. $\beta_a^{(1)}$ is give as

$$\beta_a^{(1)} = c_a^{(1)} \mathbf{1} + \sum_b a_a^b H_b, \quad (\text{C3})$$

where $c_a^{(1)} = T_a - G_a$ with

$$G_u = \frac{17}{20} g_1^2 + \frac{9}{4} g_2^2 + 8g_3^2, \quad (\text{C4})$$

$$G_d = \frac{1}{4} g_1^2 + \frac{9}{4} g_2^2 + 8g_3^2, \quad (\text{C5})$$

$$G_e = \frac{9}{4} g_1^2 + \frac{9}{4} g_2^2, \quad (\text{C6})$$

$$T_u = T_d = T_e = 3\text{Tr}(H_u + H_d) + \text{Tr}H_e, \quad (\text{C7})$$

$$a_u^u = a_d^d = \frac{3}{2}, \quad (\text{C8})$$

$$a_u^d = a_d^u = -\frac{3}{2}, \quad (\text{C9})$$

$$a_e^e = \frac{3}{2}. \quad (\text{C10})$$

-
- [1] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, The hierarchy problem and new dimensions at a millimeter, *Phys. Lett. B* **429**, 263 (1998).
 - [2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, New dimensions at a millimeter to a Fermi and superstrings at a TeV, *Phys. Lett. B* **436**, 257 (1998).
 - [3] L. Randall and R. Sundrum, A large mass hierarchy from a small extra dimension, *Phys. Rev. Lett.* **83**, 3370 (1999).
 - [4] L. Randall and R. Sundrum, An alternative to compactification, *Phys. Rev. Lett.* **83**, 4690 (1999).
 - [5] V. A. Rubakov and M. E. Shaposhnikov, Do we live inside a domain wall?, *Phys. Lett.* **125B**, 136 (1983).
 - [6] R. Jackiw and C. Rebbi, Solitons with fermion number 1/2, *Phys. Rev. D* **13**, 3398 (1976).
 - [7] M. Cvetič, S. Griffies, and S. J. Rey, Static domain walls in $N = 1$ supergravity, *Nucl. Phys.* **B381**, 301 (1992).
 - [8] O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, Modeling the fifth-dimension with scalars and gravity, *Phys. Rev. D* **62**, 046008 (2000).
 - [9] C. Csaki, J. Erlich, T. J. Hollowood, and Y. Shirman, Universal aspects of gravity localized on thick branes, *Nucl. Phys.* **B581**, 309 (2000).
 - [10] M. Eto, N. Maru, N. Sakai, and T. Sakata, Exactly solved BPS wall and winding number in $N = 1$ supergravity, *Phys. Lett. B* **553**, 87 (2003).
 - [11] M. Eto, S. Fujita, M. Naganuma, and N. Sakai, BPS multiwalls in five-dimensional supergravity, *Phys. Rev. D* **69**, 025007 (2004).
 - [12] M. Eto and N. Sakai, Solvable models of domain walls in $N = 1$ supergravity, *Phys. Rev. D* **68**, 125001 (2003).
 - [13] G. R. Dvali and M. A. Shifman, Domain walls in strongly coupled theories, *Phys. Lett. B* **396**, 64 (1997); **407**, 452(E) (1997).

- [14] G. R. Dvali, G. Gabadadze, and M. A. Shifman, (Quasi) localized gauge field on a brane: Dissipating cosmic radiation to extra dimensions?, *Phys. Lett. B* **497**, 271 (2001).
- [15] A. Kehagias and K. Tamvakis, Localized gravitons, gauge bosons and chiral fermions in smooth spaces generated by a bounce, *Phys. Lett. B* **504**, 38 (2001).
- [16] S. L. Dubovsky and V. A. Rubakov, On models of gauge field localization on a brane, *Int. J. Mod. Phys. A* **16**, 4331 (2001).
- [17] K. Ghoroku and A. Nakamura, Massive vector trapping as a gauge boson on a brane, *Phys. Rev. D* **65**, 084017 (2002).
- [18] E. K. Akhmedov, Dynamical localization of gauge fields on a brane, *Phys. Lett. B* **521**, 79 (2001).
- [19] I. I. Kogan, S. Mouslopoulos, A. Papazoglou, and G. G. Ross, Multilocalization in multibrane worlds, *Nucl. Phys. B* **615**, 191 (2001).
- [20] H. Abe, T. Kobayashi, N. Maru, and K. Yoshioka, Field localization in warped gauge theories, *Phys. Rev. D* **67**, 045019 (2003).
- [21] M. Laine, H. B. Meyer, K. Rummukainen, and M. Shaposhnikov, Localization and mass generation for non-Abelian gauge fields, *J. High Energy Phys.* **01** (2003) 068.
- [22] N. Maru and N. Sakai, Localized gauge multiplet on a wall, *Prog. Theor. Phys.* **111**, 907 (2004).
- [23] B. Batell and T. Gherghetta, Yang-Mills localization in warped space, *Phys. Rev. D* **75**, 025022 (2007).
- [24] R. Guerrero, A. Melfo, N. Pantoja, and R. O. Rodriguez, Gauge field localization on brane worlds, *Phys. Rev. D* **81**, 086004 (2010).
- [25] W. T. Cruz, M. O. Tahim, and C. A. S. Almeida, Gauge field localization on a dilatonic deformed brane, *Phys. Lett. B* **686**, 259 (2010).
- [26] A. E. R. Chumbes, J. M. Hoff da Silva, and M. B. Hott, A model to localize gauge and tensor fields on thick branes, *Phys. Rev. D* **85**, 085003 (2012).
- [27] C. Germani, Spontaneous localization on a brane via a gravitational mechanism, *Phys. Rev. D* **85**, 055025 (2012).
- [28] T. Delsate and N. Sawado, Localizing modes of massive fermions and a $U(1)$ gauge field in the inflating baby-skyrmion branes, *Phys. Rev. D* **85**, 065025 (2012).
- [29] W. T. Cruz, A. R. P. Lima, and C. A. S. Almeida, Gauge field localization on the Bloch brane, *Phys. Rev. D* **87**, 045018 (2013).
- [30] A. Herrera-Aguilar, A. D. Rojas, and E. Santos-Rodriguez, Localization of gauge fields in a tachyonic de Sitter thick braneworld, *Eur. Phys. J. C* **74**, 2770 (2014).
- [31] Z. H. Zhao, Y. X. Liu, and Y. Zhong, $U(1)$ gauge field localization on a Bloch brane with Chumbes-Hoff da Silva-Hott mechanism, *Phys. Rev. D* **90**, 045031 (2014).
- [32] C. A. Vaquera-Araujo and O. Corradini, Localization of Abelian gauge fields on thick branes, *Eur. Phys. J. C* **75**, 48 (2015).
- [33] G. Alencar, R. R. Landim, M. O. Tahim, and R. N. Costa Filho, Gauge field localization on the brane through geometrical coupling, *Phys. Lett. B* **739**, 125 (2014).
- [34] G. Alencar, R. R. Landim, C. R. Muniz, and R. N. Costa Filho, Nonminimal couplings in Randall-Sundrum scenarios, *Phys. Rev. D* **92**, 066006 (2015).
- [35] G. Alencar, I. C. Jardim, R. R. Landim, C. R. Muniz, and R. N. Costa Filho, Generalized nonminimal couplings in Randall-Sundrum scenarios, *Phys. Rev. D* **93**, 124064 (2016).
- [36] G. Alencar, C. R. Muniz, R. R. Landim, I. C. Jardim, and R. N. Costa Filho, Photon mass as a probe to extra dimensions, *Phys. Lett. B* **759**, 138 (2016).
- [37] G. Alencar, Hidden conformal symmetry in Randall-Sundrum 2 model: Universal fermion localization by torsion, *Phys. Lett. B* **773**, 601 (2017).
- [38] N. Pantoja, Self-gravitating $SU(5)$ Higgs domain walls as a braneworld, *Phys. Rev. D* **100**, 125011 (2019).
- [39] C. E. Fu and H. Guo, Stueckelberg-like Kaluza-Klein modes in 6D brane world, *Symmetry* **18**, 336 (2026).
- [40] R. Davies, D. P. George, and R. R. Volkas, The SM on a domain-wall brane, *Phys. Rev. D* **77**, 124038 (2008).
- [41] A. Davidson, D. P. George, A. Kobakhidze, R. R. Volkas, and K. C. Wali, $SU(5)$ grand unification on a domain-wall brane from an $E(6)$ -invariant action, *Phys. Rev. D* **77**, 085031 (2008).
- [42] J. E. Thompson and R. R. Volkas, $SO(10)$ domain-wall brane models, *Phys. Rev. D* **80**, 125016 (2009).
- [43] B. D. Callen and R. R. Volkas, Fermion masses and mixing in a $4 + 1$ -dimensional $SU(5)$ domain-wall brane model, *Phys. Rev. D* **83**, 056004 (2011).
- [44] K. Ohta and N. Sakai, Non-Abelian gauge field localized on walls with four-dimensional world volume, *Prog. Theor. Phys.* **124**, 71 (2010); **127**, 1133(E) (2012).
- [45] M. A. Luty and N. Okada, Almost no scale supergravity, *J. High Energy Phys.* **04** (2003) 050.
- [46] J. B. Kogut and L. Susskind, Vacuum polarization and the absence of free quarks in four-dimensions, *Phys. Rev. D* **9**, 3501 (1974).
- [47] R. Fukuda, String-like Phase in Yang-Mills theory, *Phys. Lett.* **73B**, 305 (1978); **74B**, 433(E) (1978).
- [48] R. Fukuda, Stability of the vacuum and dielectric model of confinement in QCD, *Mod. Phys. Lett. A* **24**, 251 (2009).
- [49] R. Fukuda, Derivation of dielectric model of confinement in QCD, *arXiv:0805.3864*.
- [50] M. Arai, F. Blaschke, M. Eto, and N. Sakai, Matter fields and non-Abelian gauge fields localized on walls, *Prog. Theor. Exp. Phys.* **2013**, 013B05 (2013).
- [51] M. Arai, F. Blaschke, M. Eto, and N. Sakai, Stabilizing matter and gauge fields localized on walls, *Prog. Theor. Exp. Phys.* **2013**, 093B01 (2013).
- [52] M. Arai, F. Blaschke, M. Eto, and N. Sakai, Grand unified brane world scenario, *Phys. Rev. D* **96**, 115033 (2017).
- [53] M. Arai, F. Blaschke, M. Eto, and N. Sakai, Non-Abelian gauge field localization on walls and geometric Higgs mechanism, *Prog. Theor. Exp. Phys.* **2017**, 053B01 (2017).
- [54] M. Arai, F. Blaschke, M. Eto, and N. Sakai, Localization of the standard model via the Higgs mechanism and a finite electroweak monopole from non-compact five dimensions, *Prog. Theor. Exp. Phys.* **2018**, 083B04 (2018).
- [55] M. Arai, F. Blaschke, M. Eto, M. Kawaguchi, and N. Sakai, Standard model gauge fields localized on non-Abelian vortices in six dimensions, *Prog. Theor. Exp. Phys.* **2021**, 123B07 (2021).

- [56] M. Arai, F. Blaschke, M. Eto, and N. Sakai, Localized non-Abelian gauge fields in non-compact extra-dimensions, *Prog. Theor. Exp. Phys.* **2018**, 063B02 (2018).
- [57] M. Eto and M. Kawaguchi, Localization of gauge bosons and the Higgs mechanism on topological solitons in higher dimensions, *J. High Energy Phys.* **10** (2019) 098.
- [58] N. Okada, D. Raut, and D. Villalba, Domain-Wall Standard Model in non-compact 5D and LHC phenomenology, *Mod. Phys. Lett. A* **34**, 1950080 (2019).
- [59] N. Okada, D. Raut, and D. Villalba, Aspects of the domain-wall standard model, *Prog. Theor. Exp. Phys.* **2024**, 023B01 (2024).
- [60] N. Okada, D. Raut, and D. Villalba, Fermion mass hierarchy and phenomenology in the 5D domain wall standard model, *J. High Energy Phys.* **10** (2019) 259.
- [61] P. Das, N. Okada, D. Raut, and D. Villalba, Nambu–Goldstone boson phenomenology in domain-wall standard model, *Eur. Phys. J. C* **85**, 1424 (2025).
- [62] Y. Kawamura, Triplet doublet splitting, proton stability and extra dimension, *Prog. Theor. Phys.* **105**, 999 (2001).
- [63] M. Kakizaki and M. Yamaguchi, Splitting triplet and doublet in extra dimensions, *Prog. Theor. Phys.* **107**, 433 (2002).
- [64] A. Hebecker and J. March-Russell, A minimal $S^1/(Z_2 \times Z'_2)$ orbifold GUT, *Nucl. Phys.* **B613**, 3 (2001).
- [65] N. Maru, Doublet—triplet splitting and fat branes, *Phys. Lett. B* **522**, 117 (2001).
- [66] N. Haba and N. Maru, Light Higgs triplets in extra dimensions, *Phys. Lett. B* **532**, 93 (2002).
- [67] N. Haba and N. Maru, Fat branes, orbifolds and doublet triplet splitting, *Eur. Phys. J. C* **33**, 457 (2004).
- [68] F. J. de Anda and S. F. King, Orbifold modular GUT of flavor, *Phys. Rev. D* **109**, 095046 (2024).
- [69] H. Georgi and C. Jarlskog, A new lepton–quark mass relation in a unified theory, *Phys. Lett.* **86B**, 297 (1979).
- [70] J. R. Ellis and M. K. Gaillard, Fermion masses and Higgs representations in SU(5), *Phys. Lett.* **88B**, 315 (1979).
- [71] A. Hebecker and J. March-Russell, Proton decay signatures of orbifold GUTs, *Phys. Lett. B* **539**, 119 (2002).
- [72] H. Fusaoka and Y. Koide, Updated estimate of running quark masses, *Phys. Rev. D* **57**, 3986 (1998).
- [73] Z. z. Xing, H. Zhang, and S. Zhou, Impacts of the Higgs mass on vacuum stability, running fermion masses and two-body Higgs decays, *Phys. Rev. D* **86**, 013013 (2012).