

Supercooling exit from charge supersaturation

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Systems that feature a scalar field ϕ with a quasi scale invariant potential, metastable at $\phi = 0$, can remain trapped, during cosmic evolution, in the ‘wrong’ vacuum because the process of bubble nucleation to the true vacuum is inefficient. If ϕ carries a conserved charge Q , the presence in the system of a nonzero chemical potential for Q offers the possibility of escaping eternal supercooling; when a species decouples from the plasma the balance among the stabilizing effect of temperature and the destabilizing effect of chemical potential can change in favor of the latter, so that ϕ condenses and triggers the transition to the stable phase.

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I. INTRODUCTION

Scalar fields with a quasi scale invariant potential that is metastable at the origin appear in various problems. To mention a few; old models of inflation [1], the Coleman-Weinberg model [2], ultraviolet completions of the electroweak sector with strong dynamics [3], and last but not least the Standard Model at regimes in which v_{EW} can be neglected [4–6].

Potentials of this kind can be problematic in a cosmological context. At very early times, the temperature is so high that the potential is generally stabilized at $\phi = 0$ due to a thermal contribution of the form $\Delta V = aT^2\phi^2$, with $a > 0$. As the universe cools down a local minimum forms at some $v(T) \neq 0$, and the new minimum becomes degenerate with the symmetric one for some critical temperature T_c . Further cooling makes the symmetry breaking minimum energetically favored.

However, due to the flatness of the zero temperature contribution to the potential, thermal effects leave a metastable minimum at $\phi = 0$ that persists at arbitrarily low temperatures, until the cosmic plasma reaches the de Sitter temperature T_{dS} where no further cooling is possible.

If no drastic departure from this picture takes place, the only way in which a phase transition to the stable phase can take place is via bubble nucleation. This process is exponentially suppressed and is not necessarily efficient enough. For a quasi scale invariant potential $V = -\lambda(\phi)\phi^4$, the tunneling rate γ is proportional to $e^{-2\pi^2/3\lambda}$ and therefore

approximately constant for $T_{dS} < T < T_c$. If λ is small then $\gamma \ll H^4$, where H is the Hubble parameter, and the transition to the true vacuum cannot take place. Because of the cooling induced by the Universe expansion, the vacuum energy soon starts dominating the energy density of the system and the universe starts inflating; H becomes constant and, since γ is also approximately constant, nucleation can never become efficient.

For most of the previously mentioned scenarios, it is phenomenologically necessary to eventually transition to the stable vacuum. The exception is the Standard Model, where the false vacuum must survive cosmic evolution.

In the seminal paper [7], in the context of a Coleman-Weinberg implementation of the Higgs sector, it was noted that supercooling proceeds all the way to T_{QCD} , at which temperature the condensation of quarks induces a linear term to the Higgs potential, of the form $\Delta V = y\phi\langle\bar{q}q\rangle$. This term—which comes from the Yukawa coupling of the Higgs to the quarks—destabilizes the symmetric point $\phi = 0$ and drives the transition to the true vacuum, which happens via classical roll-down instead of quantum or thermally assisted tunneling.

The present work proposes an alternative mechanism for exiting eternal supercooling, based on the well-known fact that a chemical potential for ϕ gives a destabilizing contribution to the effective potential. If Q is the generator of a $U(1)$ symmetry under which ϕ transforms as $[Q, \phi(x)] = -q_\phi\phi(x)$, and μ is the chemical potential associated to Q ,¹ the effective potential of ϕ gets a contribution $\Delta V = -q_\phi^2\mu^2|\phi|^2$. When $q_\phi\mu > \sqrt{a}T$ the point $\phi = 0$ turns from a local minimum to a maximum and the field rolls down.

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¹We follow the standard definition, according to which μ enters the grand partition function via $\mathcal{Z} = \text{tr}[e^{-\beta(H-\mu Q)}]$.

The example of [7] shows that, if no physical scale other than temperature breaks the (approximate) scale invariance of the problem, supercooling does not end. On the contrary, the existence of the QCD scale Λ_{QCD} and a coupling between the QCD sector and the Higgs is at the heart of the mechanism for supercooling exit (see also [8,9] for more recent investigations in the context of Coleman-Weinberg-like models and theories with a strongly coupled electro-weak sector, respectively).

In the examples that are going to be presented here the same is observed. Suppose that, at some temperature $T < T_c$, some Q -charge excess is present that makes $\mu \neq 0$. For $\phi = 0$ to be initially an equilibrium point, it must be $q_\phi \mu < \sqrt{a}T$. If the particles that make up the thermal plasma are all massless or have negligible masses with respect to T , the equilibrium evolution of the system implies that the ratio μ/T remains constant. Therefore the coefficient in front of $|\phi|^2$ remains positive and the presence of μ does not change the above qualitative picture.

The situation changes drastically whenever the system cools down to $T \lesssim \Lambda$, with Λ some physical scale that characterizes the plasma (typically the mass of a particle, dynamically generated or not). In this case, the ratio μ/T in general changes and the condition $q_\phi \mu < \sqrt{a}T$ can be violated, making the system unstable.

The remainder of the paper is organized as follows. In Sec. II the stage is set, and the condition for inducing Bose-Einstein condensation for ϕ (which eventually triggers the end of supercooling) is derived. Section III presents formulas for the thermal mass of ϕ at leading order in the couplings that govern its interaction with the other particles of the thermal plasma. Section IV considers several examples of the mechanism and Sec. V draws general conclusions. Appendix A presents a numerical implementation of the mechanism in an explicit model.

II. MECHANISM

We consider a plasma of particles that are in thermal equilibrium at temperature T . Some of the species, including ϕ , are charged under a global conserved charge Q .

In this section we are going to consider regimes in which particles have either $m \gg T$, and so their density is Boltzmann suppressed, or $m \ll T$, so they can be treated as relativistic. Appendixes A 2 and A 3 present in-depth numerical analyses in the regime $T \sim m$.

The presence of a charge unbalance implies that μ , the chemical potential associated to Q , is different than zero. For bosonic species, a large enough chemical potential induces Bose-Einstein condensation. Quantitatively, the condition is that

$$q_\phi \mu > m_\phi, \quad (1)$$

where m_ϕ includes thermal corrections that typically dominate at high temperatures.

The simple condition for condensation in (1) gives a concrete meaning to μ , whose value is driven by the cosmic evolution. The relevant questions are then: What is the path that μ follows in the cosmic evolution? Is it possible to reach the condition (1) if it is initially not satisfied?

To address these questions some assumptions have to be made. It is physically natural and mathematically elegant to assume that (i) the system experiences no drastic departure from equilibrium, so its entropy S does not increase, and (ii) that Q is exactly conserved.

S and Q are uniquely determined by T , μ , and V , with a trivial dependence on the volume ($S, Q \propto V$). Their ratio is both conserved and independent of volume, so one gets an expression like

$$\frac{S}{Q} = f(T, \mu), \quad (2)$$

which can be inverted to determine $\mu(T)|_{S/Q}$. This gives the chemical potential for ϕ as a function of temperature at fixed entropy per charge.

To derive an expression for $\mu(T)|_{S/Q}$ one needs to determine the function f in (2). Let us start from estimating the entropy. This is largely dominated by the relativistic species and is essentially insensitive to μ if $\mu \lesssim T$.² The free theory expression at vanishing μ , i.e.,

$$\frac{S}{V} = \frac{2\pi^2 T^3}{45} \left(N_b + \frac{7}{8} N_f \right) = gT^3, \quad (3)$$

where N_b and N_f are, respectively, the number of relativistic bosons and fermions, is enough to capture it.

Assuming for simplicity that the only particle which is charged under Q is ϕ itself, and that it has charge q_ϕ , one has

$$\frac{Q}{V} = q_\phi (n_\phi - n_{\bar{\phi}}), \quad (4)$$

where the number density of particle ϕ is given by

$$n_\phi = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\sqrt{k^2 + m_\phi^2} - q_\phi \mu)} - 1}, \quad (5)$$

while $n_{\bar{\phi}}$ is obtained by changing $q_\phi \rightarrow q_{\bar{\phi}} = -q_\phi$.

Unlike with the entropy, the role of μ is crucial for determining the total charge Q . In particular, $Q = 0$ when $\mu = 0$ because the contributions of particle and antiparticle exactly cancel (for the entropy, instead, particle and antiparticle contribute with the same sign). With a potential

²In the examples that we are going to consider in Sec. IV, this condition is always satisfied at weak coupling.

for ϕ which is approximately scale invariant at zero temperature, the mass of ϕ comes mostly from thermal effects, and is given by $m_\phi = \sqrt{a}T$, with a some expression of the marginal couplings.³ By simple dimensional analysis one can express $n_\phi = T^3 I(\frac{\mu}{T})$, with

$$I(x) = \frac{1}{2\pi^2} \int_0^\infty dt t^2 \frac{1}{e^{\sqrt{t^2 + a - q_\phi x}} - 1}. \quad (6)$$

The expression in (2) can then be refined by stating that entropy per charge is a function of only *the ratio* of chemical potential and temperature. Precisely,

$$\frac{Q}{S} = g^{-1} q_\phi \left(I\left(\frac{\mu}{T}\right) - I\left(-\frac{\mu}{T}\right) \right). \quad (7)$$

Inverting the expression one finds,

$$\mu = \alpha \left[\frac{S}{gQ}, a \right] T, \quad (8)$$

where S/Q is constant in time if there is no charge nor entropy injection, and similarly a and g unless T crosses some physical mass scale (the form of α is not important for the moment).

The condition that defines the onset of Bose-Einstein condensation, i.e., $q_\phi \mu > \sqrt{a}T$, translates into

$$q_\phi \alpha > \sqrt{a}, \quad (9)$$

which depends on temperature only through g and a .

The question now becomes the following. Is it possible for g and a to change in such a way that condensation is induced when going from higher to lower temperatures?

The physics that governs changes in g is pretty transparent, and makes us lean towards a negative answer to the question. This is because g decreases under cooling when particles become nonrelativistic and their density gets Boltzmann suppressed. A decrease of g , in turn, is equivalent to an increase of entropy per unit charge at g fixed, as can be seen from (8). With more and more entropy per unit charge, the system gets further and further away from criticality [see also (12) and the discussion below] and condensation cannot take place upon cooling unless a compensating effect comes from the change in a .

Before going into a discussion of the physics of m_ϕ and a quantitative determination of a in Sec. III, it is useful to derive an approximate formula for $I(\frac{\mu}{T})$ that makes the dependence on the various parameters explicit.

³The way m_ϕ^2 enters in (5) implies a resummation of the so-called ‘daisy’ diagrams. See for example [10]; see also [11] for a recent analysis on the emergence of a thermal mass in purely S-matrical terms.

A. Simple parametric expressions

A useful approximation to $n_\phi - n_{\bar{\phi}}$ is derived by Taylor expanding (6) and discarding $O(\mu^3/T^3)$ terms. The leading order cancels in the difference $I(\frac{\mu}{T}) - I(-\frac{\mu}{T})$, while the coefficient of the linear term is a function of a with a finite $\lim_{a \rightarrow 0}$. In many scenarios a is a small number and it is a good approximation to just take the limit. Doing the integral one finds simply

$$\frac{Q}{V} = \frac{q_\phi^2}{3} \mu T^2. \quad (10)$$

After taking the ratio with (3) one finds $Sq_\phi^2 \mu = 3gQT$, which can be used to derive the following expression for the chemical potential,

$$\mu = \frac{3gQ}{Sq_\phi^2} T. \quad (11)$$

Condensation takes place when $q_\phi \mu > \sqrt{a}T$, that is when

$$\frac{3gQ}{q_\phi \sqrt{a}S} \equiv \kappa > 1. \quad (12)$$

This expression can be read in many ways. Perhaps most intuitively as a condition on the maximal charge that can be stored in the system before inducing condensation (similarly to supersaturation effects in chemistry). Thinking at Q/S fixed, as appropriate for cosmic evolution, one concludes that the condition for κ to grow under cooling is that the relative decrease in $\sqrt{a}$ is stronger than the relative decrease of g . This is illustrated in Fig. 1. As we are going to see in Sec. III, this means that particles that Boltzmann decouple from the plasma are the ones that give a sizeable contribution to the thermal mass of ϕ .

In the limit of small a , it is immediate to derive an approximate expression for Q when particles other than ϕ are charged,

$$\frac{Q}{V} = \frac{1}{3} \mu T^2 \left(\sum_b q_b^2 + \frac{1}{2} \sum_f q_f^2 \right) \equiv \frac{1}{3} \mu T^2 \mathcal{Q}^2, \quad (13)$$

valid up to $O((\beta\mu)^2)$ corrections, with b a bosonic and f a fermionic species. A decrease in the number of *charged* relativistic species tends to increase μ/T , as one can see by promoting $q_\phi^2 \rightarrow \mathcal{Q}^2$ in (11). Condensation happens when $\kappa = 3gq_\phi Q / (\mathcal{Q}^2 \sqrt{a}S) > 1$. When several scalars i are present, one has a different κ_i factor for each of them, with

$$\kappa_i = \left(\frac{3gQ}{\mathcal{Q}^2 S} \right) \frac{q_i}{\sqrt{a_i}}. \quad (14)$$

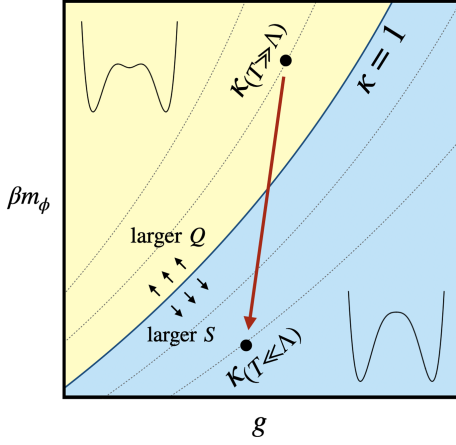


FIG. 1. Contours of κ , defined in (12), at fixed entropy S and charge Q , as a function of the number of active species g and the ratio of the scalar's thermal mass to temperature $\beta m_\phi = \sqrt{a}$ (schematic). Both g and $\sqrt{a}$ are approximately piecewise constant functions of T that decrease under cooling when T crosses a physical scale Λ . The change in these two parameters can be such that the system goes from the yellow region ($\kappa < 1$) to the blue region ($\kappa > 1$) where a condensate is formed. The contours of κ can also be thought of as different critical lines ($\kappa = 1$) at varying Q or S .

Particles with the largest ratio $q_i/\sqrt{a_i}$ are more prone to condensation.

III. THERMAL MASS

Given that ϕ has vanishing or negligible mass at zero temperature, its mass at finite temperature is entirely given by thermal effects and is set by T . In these conditions, at leading order in couplings the thermal mass squared of ϕ is captured by the one-loop self-energy at vanishing momentum, i.e., $m_\phi^2 = -\lim_{\mathbf{p} \rightarrow 0} \Sigma_T^{(1)}(0, \mathbf{p})$, where the suffix T restricts to the thermal part of the self-energy [12].

$\Sigma_T^{(1)}$ can be interpreted as coming from the forward scattering (at tree level) of ϕ with the particles that constitute the thermal bath [13]. The contribution of a given mode of species i that has momentum $\mathbf{k}$ is proportional to its density $n_i(\mathbf{k})$. In formulas,

$$m_\phi^2 = \lim_{\mathbf{p} \rightarrow 0} \sum_i \int \frac{d^3 k}{(2\pi)^3 2E_i(\mathbf{k})} M_{i\phi \rightarrow i\phi}(\mathbf{k}, \mathbf{p}) n_i(\mathbf{k}), \quad (15)$$

where $M_{i\phi \rightarrow i\phi}$ is computed in the forward configuration, with $p_\mu = (0, \mathbf{p})$ and $k_\mu = (E_i(\mathbf{k}), \mathbf{k})$, with $E_i(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m_i^2}$. It is not always safe to simply set $\mathbf{p} = 0$ in the integrand [14].

The most important feature of (15) is that the contribution of a given particle i is exponentially suppressed as $e^{-m_i/T}$ when $T \lesssim m_i$, due to the presence of $n_i(\mathbf{k}) = (e^{\beta E_i(\mathbf{k})} \pm 1)^{-1}$. Given that particles and antiparticles

contribute with the same sign, the effect of a chemical potential is not important in (15).

The effect of the interaction of ϕ with scalars, spin 1/2 fermions and gauge bosons via the usual marginal couplings is going to be presented next.

A. Scalar quartic

The contribution to the thermal mass of ϕ coming from a quartic coupling $\mathcal{L} = -\frac{1}{2} \lambda' |\phi|^2 \varphi^2$ is the easiest to compute, as the forward amplitude $\phi\phi \rightarrow \phi\phi$ is simply given by λ' . If the scalar φ has mass m_φ , one finds [15]

$$m_\phi^2 = \begin{cases} \frac{\lambda' T^2}{24} & T \gg m_\varphi \\ \frac{\lambda'}{2} (2\pi)^{-\frac{3}{2}} \sqrt{m_\varphi} T^3 e^{-m_\varphi/T} & T \ll m_\varphi \end{cases}. \quad (16)$$

Given the exponential damping, the contribution of φ to the thermal mass of ϕ effectively disappears for $T \ll m_\varphi$ and any other contribution quickly dominates over it, be it the zero temperature contribution (assumed here to be negligible for a wide range of temperatures) or the thermal mass coming from those particles of the thermal bath that are still relativistic.

Keeping the same normalization for the $2 \rightarrow 2$ amplitude, one gets an additional factor of 2 if φ is complex and a factor of d_φ if it constitutes a multiplet with d_φ components. Notice that a complex φ can be either charged or neutral under Q , the conserved operator under which we assume ϕ to be charged.

B. Yukawa

Since ϕ is charged under Q , a Yukawa term has to involve two fermionic species, ψ and χ , with interaction Lagrangian

$$\mathcal{L} = \phi \bar{\psi} (y + y' \gamma_5) \chi + \text{c.c.} \quad (17)$$

and masses m_ψ and m_χ (we are assuming for simplicity that ψ and χ are Dirac fermions that admit a Dirac mass term). The charges respect $q_\phi + q_\chi = q_\psi$, so at least one fermionic species is charged under Q .

The thermal contribution to the mass of ϕ induced by (17) can be computed with (15). With this method one needs to be very careful in taking into account the relative signs of contributions coming from four species in the plasma, that is $\psi, \bar{\psi}, \chi$ and $\bar{\chi}$. Another possibility is to derive all four contributions from one single object: the thermal loop of Figure 2.

It can be computed with the usual (Euclidean) Feynman rules, albeit with a modified definition of the loop integration: instead of an integral over k_0 , the time component of the loop momentum, one needs to perform a sum over fermionic Matsubara modes $k_0 = (2n+1)\pi T$, with n integer.

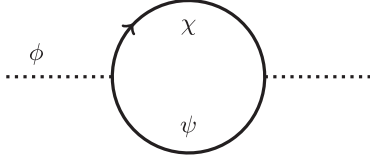


FIG. 2. The one-loop self-energy of ϕ in the model specified by (17) comes from the above diagram. Including thermal effects amounts to compute it with thermal instead of vacuum propagators for ψ and χ .

Setting the external momentum to zero, one has

$$\Sigma^{(1)} = \int_k \frac{\text{tr}[(k + m_\psi)(y + y'\gamma_5)(k + m_\chi)(y - y'\gamma_5)]}{(k^2 + m_\psi^2)(k^2 + m_\chi^2)} \quad (18)$$

where $\int_k \equiv T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3}$, and $\Sigma^{(1)}$ contains also the (not yet separated) zero temperature contribution. Evaluating the trace one gets

$$\text{tr}[\dots] = 4(k^2(y^2 + y'^2) + m_\psi m_\chi(y^2 - y'^2)). \quad (19)$$

With a standard reduction procedure one can rewrite the squared momentum in the numerator as $k^2 = \frac{1}{2}(k^2 + m_\psi^2) + \frac{1}{2}(k^2 + m_\chi^2) - \frac{1}{2}(m_\psi^2 + m_\chi^2)$ and reduce the problem to the evaluation of the following thermal integrals

$$I_1 = \int_k \frac{1}{k^2 + m^2}, \quad (20)$$

$$I_2 = \int_k \frac{1}{(k^2 + m^2)(k^2 + \hat{m}^2)}. \quad (21)$$

Let us review the evaluation of the tadpole integral in detail, to see how the separation of zero temperature effects arises algebraically and the on shell nature of the remaining, purely thermal, contribution.

The key identity to invoke is a version of the Poisson summation formula, which allows us to rewrite the sum over Matsubara modes as

$$\begin{aligned} T \sum_{k_0} \frac{1}{k_0^2 + E_{\mathbf{k}}^2} &= \sum_{\ell \in \mathbb{Z}} (-1)^\ell \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\ell\omega\beta}}{\omega^2 + E_{\mathbf{k}}^2} \\ &= \frac{1}{E_{\mathbf{k}}} \left(\frac{1}{2} + \sum_{\ell > 0} (-1)^\ell e^{-\ell\beta E_{\mathbf{k}}} \right), \end{aligned} \quad (22)$$

with $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. To go to the second line one needs to interpret the integral in $d\omega$ as a contour integral and close the contour in the upper or lower half-plane for respectively negative and positive ℓ . The integral then localizes on the poles $\omega = \pm iE_{\mathbf{k}}$ by virtue of the Cauchy theorem. The sum over $\ell > 0$ can be recognized as *minus* the Fermi-Dirac density $n_f(\mathbf{k})$, while the $\ell = 0$ term is the contribution of

vacuum fluctuations that one computes with the usual zero temperature Feynman rules.

All in all one finds

$$I_1 = \int \frac{d^3k}{(2\pi)^3 E_{\mathbf{k}}} \left(\frac{1}{2} - n_f(\mathbf{k}) \right). \quad (23)$$

With analogous manipulations one finds, for the T -dependent part of I_2 ,

$$I_2|_T = \frac{1}{m^2 - \hat{m}^2} \int \frac{d^3k}{(2\pi)^3} \left(\frac{n_f(\mathbf{k})}{E_{\mathbf{k}}} - \frac{\hat{n}_f(\mathbf{k})}{\hat{E}_{\mathbf{k}}} \right). \quad (24)$$

In the evaluation of I_2 one encounters poles in $\omega = \pm iE_{\mathbf{k}}$ and $\omega = \pm i\hat{E}_{\mathbf{k}}$. For a discussion on the subtleties of the computation of $\Sigma^{(1)}$ for coincident masses, see [14]. Putting everything together, one finds that the one-loop thermal mass of ϕ induced by (17) is given by

$$m_\phi^2 = 4m_\psi \left(\frac{y^2}{m_\psi + m_\chi} + \frac{y'^2}{m_\psi - m_\chi} \right) \int_{\mathbf{k}} \frac{n_\psi}{E_\psi} + (\psi \leftrightarrow \chi), \quad (25)$$

where $\int_{\mathbf{k}} = \int \frac{d^3k}{(2\pi)^3}$. It can be seen that the result fits the general formula (15). At very high temperatures it reduces to

$$m_\phi^2 = \frac{y^2 + y'^2}{6} T^2 (T \gg m_\psi, m_\chi). \quad (26)$$

In the opposite regime one has the usual Boltzmann suppression, captured by the asymptotic formula,

$$\int_{\mathbf{k}} \frac{n(\mathbf{k})}{E_{\mathbf{k}}} \sim \sqrt{\frac{mT^3}{(2\pi)^3}} e^{-m/T}, \quad (27)$$

which is valid for bosons as well as fermions.

It is enough for one among ψ and χ to decouple from the plasma to have a suppression of the thermal mass (not exponential, though). In the regime $m_\chi \ll T \ll m_\psi$ one has

$$m_\phi^2 = \frac{m_\chi}{m_\psi} \frac{y^2 - y'^2}{6} T^2, \quad (28)$$

suppressed by the small ratio m_χ/m_ψ : though χ is abundantly produced thermally and is available for scattering with ϕ , the heavy ψ has to be produced virtually in order for the scattering to take place.

Notice that the term in (25) proportional to y'^2 is not positive definite, as is also clear from (28). The thermal squared-mass induced by the fermion loop of Fig. 2 can therefore become negative in certain regimes, and it can be responsible for inducing ϕ condensation sooner, or for

lower values of Q/S . We are not going to study this interesting possibility further in the present work.

C. Gauge interactions

The scalar ϕ can also be charged under some gauge group G , Abelian or not. The interaction of ϕ with the gauge degrees of freedom, and with itself via the gauge coupling, gives another contribution to its thermal mass. The interaction—mediated by the exchange of a gauge boson—with other species charged under G , instead, gives no contribution at one loop due to an adaptation of Furry's theorem [16].

Using (15) one finds (the covariant derivative is defined as $\partial_\mu - ieA_\mu$)

$$m_\phi^2 = e^2 \int_{\mathbf{k}} \frac{1}{2|\mathbf{k}|} (2n_{\gamma_+} + 2n_{\gamma_-} + n_\phi + n_{\bar{\phi}}) = \frac{e^2 T^2}{4}, \quad (29)$$

where the last expression assumes negligible ϕ mass, and $\gamma_\pm$ denote photons with respectively positive and negative helicity. If G is non-Abelian and ϕ carries representation t^a , such that the covariant derivative is $\partial_\mu - ie t^a A_\mu^a$, one needs to multiply (29) by C_ϕ , with $\sum_a t^a t^a = C_\phi \mathbf{1}$. For example, for ϕ in the fundamental of $SU(N)$, $C_\phi = \frac{1}{2N}(N^2 - 1)$.

IV. EXAMPLES

A. Scalar decoupling

The simplest example we consider is a theory in which ϕ couples to a neutral scalar φ . The relevant physical parameters are captured by the potential,

$$V = -\lambda|\phi|^4 + \frac{1}{2}\lambda'\varphi^2|\phi|^2 + \frac{1}{2}m_0^2\varphi^2 + \lambda''\varphi^4. \quad (30)$$

At finite temperature, on top of the zero temperature loop corrections (fine-tuning issues, which are expected to arise here, are ignored in this discussion) one has to take into account thermal effects. In particular, a thermal mass for ϕ is generated, and the mass of φ gets a correction that dominates for $T \gg m_0$. In this regime, assuming for simplicity that $\lambda' \gg \lambda, \lambda''$, one has

$$m_\phi^2 = \frac{1}{24}\lambda'T^2, \quad (31)$$

$$m_\varphi^2 = m_0^2 + \frac{1}{12}\lambda'T^2. \quad (32)$$

Along with ϕ and φ , let us assume that there is a number of other species which are kept in thermal equilibrium and are either massless or with negligible mass with respect to m_0 . They have couplings to ϕ that are weaker than λ' and overall give a thermal mass to ϕ that has size aT^2 , with $a \ll \lambda'$.

When temperature falls below m_0 , one bosonic degree of freedom decouples from the plasma. More dramatically, the mass to temperature ratio of ϕ drops from $\sqrt{\lambda'/24}$ to $\sqrt{a}$, due to (16).

As a measure of the effect that the decoupling of φ has on the balance among temperature and chemical potential, we consider

$$\rho \equiv \frac{\kappa(T \ll m_0)}{\kappa(T \gg m_0)} = \sqrt{\frac{\lambda'}{24a}} \left(1 - \frac{2\pi^2}{45g}\right), \quad (33)$$

with κ given in (12). The new quantity ρ measures if the drop in the thermal mass of ϕ is more or less sizeable than the decrease in the number of relativistic degrees of freedom. Unlike κ , ρ depends only on the specifics of the model and not on the initial conditions.

For the decoupling of φ to induce condensation, it is necessary (but not sufficient) that $\rho > 1$. If ρ is only slightly larger than 1, condensation is unlikely because one needs the system to be already at the edge of condensation for T above m_0 [that is, $\kappa(T \gg m_0)$ must be very close to 1]. For larger and larger ρ , the range of favorable initial conditions, say on the charge to entropy ratio, gets bigger and bigger. Since the effect is multiplicative, one needs always *some* charge asymmetry for it to operate. Under the natural assumption that $g \gg 1$, if we take $\lambda' \sim 1$ then $\rho \sim 2$ for $a = 10^{-2}$ and $\rho \sim 6.5$ for $a = 10^{-3}$.

While it is natural for the thermal mass to be dominated by the effect of one coupling, it comes as a coincidence that the particle that contributes mostly to m_ϕ decouples first from the plasma.

It is important to stress that the condition of thermal equilibrium, *both kinetic and chemical*, is necessary for the formulas of Secs. II and III to be valid. In an expanding universe, thermal equilibrium holds when the processes that redistribute energy and conserved charges among particles are fast, i.e., when their rate Γ is much larger than H .

In the example of this section it is particularly crucial to check if chemical equilibrium of φ can be maintained for $T \lesssim m_0$. This is because the rates of annihilation processes like $2\varphi \rightarrow \phi\bar{\phi}$, which keep a vanishing chemical potential for φ , become Boltzmann suppressed and switch off for temperatures not too far below m_0 . The number density of φ becomes frozen and the contribution of φ particles to the thermal mass of ϕ is not the one given in the second line of (16).

The conditions for maintaining thermal equilibrium are analyzed in Appendix A 1 for the present model and for that of Sec. IV B, and the effects of deviating from equilibrium are studied numerically in Appendix A 2. The conclusion is that, under very reasonable assumptions, the full picture does not sensibly depart from the one depicted in the main text assuming equilibrium.

As a final remark, we note that the formulas presented in this section are not sufficient to determine the temperature at which condensation occurs; they only provide a criterion for determining whether condensation takes place or not.⁴ How to determine the condensation temperature is the subject of Appendix A 3.

B. Charged fermion multiplet

As a second illustration of the effect we consider the model of (17) with the symmetric assignment $q_\psi = -q_\chi = \frac{1}{2}q_\phi$ and masses $m_\psi = m_\chi$ (the scenario in which $m_\chi \ll m_\psi$ is considered in A 3). To enrich the model we also make ψ and χ multiplets of some non-Abelian group G . For $T \gg m_\psi$, taking $y' = 0$, one has

$$m_\phi^2 = \frac{1}{6}y^2 d_\psi T^2, \quad (34)$$

with d_ψ the length of the fermion multiplets.

When temperature drops below the mass of fermions, three changes take place. First, $8d_\psi$ fermionic degrees of freedom decouple from the thermal plasma. Second, the thermal mass of ϕ drops to aT^2 , with $a \ll y^2$. As a third effect, all of the charge excess of the system gets transferred to ϕ , while at $T \gg m_\psi$ this was democratically distributed among the various degrees of freedom (we are assuming that no other particle in the plasma besides ϕ, ψ and χ carries Q charge). As explained in Sec. A 1, for the excess charge to be depleted from the ψ and χ sectors, freeze-out of ψ and χ must happen sufficiently late. This is typically the case for supercooling at $T_c \sim 10$ TeV, where the freeze-out temperature is roughly given by $m_\psi/20$.

Due to the third phenomenon, which was absent in the previous example, the formula for ρ includes another potentially large ratio, coming from correcting (12) with (13). In this model, it is given by

$$\rho = y \sqrt{\frac{d_\psi}{6a}} \left(1 - \frac{14\pi^2 d_\psi}{45g}\right) \left(1 + \frac{d_\psi}{2}\right), \quad (35)$$

where the last factor is $q_\phi^{-2} \mathcal{Q}^2 (T \gg m_\psi)$, with $\mathcal{Q}^2$ as defined in (13). Even with a sizeable decrease in the number of relativistic degrees of freedom, say by a factor of 2, a large multiplet size induces a much more drastic effect than in the example of Sec. IV A. For $y \sim 1$, with $a = 10^{-2}$ one has $\rho \sim 16$ and $\rho \sim 27$ for, respectively, $d_\psi = 5, 10$. The effect is sizeable also for a less dramatic decrease in the thermal mass. With $a = 10^{-1}$, $\rho \sim 5$ and

$\rho \sim 8.5$ for, respectively, $d_\psi = 5, 10$. As a matter of fact, charge redistribution is a distinct effect that takes place whether or not the thermal mass of ϕ changes significantly. In general, both effects are present and the first gives a boost to the second, or vice versa.

In the present analysis we have not considered the possibility that there are other conserved charges that have an interplay with Q at $T \sim m_\psi$. To give an example, suppose there are no light fermions in the model. If the plasma carries a nonzero fermion number F [to allow the Yukawa term of (17), one has to take $F_\psi = F_\chi$], some amount of ψ and/or χ particles must survive in the plasma. If the depletion is not symmetric for ψ and χ (for non-degenerate masses, this is expected), some amount of Q excess will remain trapped in the fermion sector. For a symmetric depletion $n_\psi = n_\chi$ and the total Q -charge in the fermion sector is zero because $q_\psi = -q_\chi$. Even in this case, however, some amount of energy remains confined in the nonrelativistic sector, and the estimate of the entropy has to be modified. All this is avoided by assuming either that $F = 0$, or that the fermion number excess can be injected into some light species. In general, one needs to know the details of the model, especially what the conserved charges are, and their interplay across mass thresholds, to get an accurate estimate of ρ . See for example the analysis of Appendix A 3.

C. Dynamically generated exit scale

In the present section we explore the possibility that particle decoupling from the plasma is due to a dynamically generated mass scale.

Consider a gauge theory, charged under local $SU(3) \times SU(2)$, whose particle content is that of one generation of Standard Model fermions and one Higgs doublet ϕ . Using the familiar Standard Model nomenclature, let us call the gauge group multiplets $q(3, 2)$, $u^c(\bar{3}, 1)$, $d^c(\bar{3}, 1)$, $\ell(1, 2)$, $e^c(1, 1)$, and $\phi(1, 2)$.

Let us assume that the only Yukawa term that has non-negligible coupling to ϕ is

$$\mathcal{L}_Y = y_e \phi^* \ell e^c. \quad (36)$$

In this scenario there are several conserved global charges, that can be chosen to be the ‘baryon numbers’ B_q, B_u, B_d , which count the number of respectively q, u^c, d^c (minus their antiparticles), the lepton number L , which counts the number of ℓ minus the number of e^c , and a charge Q under which $q_\phi = 1$ and $q_\ell = q_{e^c} = \frac{1}{2}$. We assume that $Q \neq 0$, while $B_q, B_u, B_d, L = 0$.

The interesting scale for inducing condensation is the analogue of Λ_{QCD} in this toy example, which is assumed to have roughly the same features of the real-world QCD (with two flavors of quark, the one-loop beta function of α_s is $\mu \frac{d}{d\mu} \alpha_s = -\frac{\alpha_s^2}{2\pi} (11 - \frac{4}{3})$, making α_s grow in the infrared).

⁴Strictly speaking, the criterion is sharp only if $\kappa(T)$ is a monotonic function, but this is not always the case. For a sharper determination of the region of parameter space where condensation is triggered, one needs to go beyond the determination of κ and ρ and study the dynamics at threshold.

For $T \sim \Lambda_{\text{QCD}}$ confinement is assumed to take place with generation of a chiral condensate [17].

The formation of a quark condensate was crucial in the context of [7] to give a linear term to the potential of ϕ via its Yukawa coupling to the quarks. Assuming that such a direct coupling to the condensate is absent or very small, a more efficient exit mechanism is provided, in the right conditions, by the supersaturation of Q . Let us see how this arises.

The chiral condensate has the same quantum number of the Higgs and it breaks $SU(2)_L$ completely, giving a perturbative mass to the three associated gauge bosons W_μ^i via the Higgs mechanism. The scalar degrees of freedom that are eaten by the $SU(2)_L$ gauge bosons are the three pions π^i , which can be embedded in $U = e^{i\pi^i \sigma^i / f_\pi}$. The canonically normalized gauge covariant kinetic term of U reads $\frac{1}{4} f_\pi^2 \text{tr} D_\mu U^\dagger D_\mu U$, with $D_\mu U = \partial_\mu U - i g_2 W_\mu^i \frac{\sigma^i}{2} U$. Taking $\langle U \rangle = \mathbf{1}$, the kinetic term produces a mass term for the W bosons,

$$\mathcal{L}_{m_W} = \frac{1}{2} \left(\frac{g_2 f_\pi}{2} \right)^2 \delta_{ab} W_\mu^a W_\mu^b. \quad (37)$$

What is important for us is that QCD confinement induces, indirectly via the chiral condensate, a mass to the W bosons. Therefore, for $T \ll m_W = \frac{1}{2} g_2 f_\pi$, the only active species in the plasma are ℓ , e^c , and ϕ .

The drop in the number of active degrees of freedom is considerable, and is quantified by

$$\frac{g'}{g} = \frac{37}{165} \simeq 0.22. \quad (38)$$

If $g_2 \gg y_e$, the thermal mass to temperature ratio also drops significantly, giving a contribution to ρ of order

$$\frac{\sqrt{a}}{\sqrt{a'}} \sim \frac{g_2}{y_e} \gg 1, \quad (39)$$

so that the effect of (38) can be overcome. Let us see how this comes about. For $T \gg m_W$, the main contribution to m_ϕ is given by its interaction with W particles in the plasma and self-interactions mediated by the exchange of W bosons. Using (29) and that $C_\phi = \frac{3}{4}$ for a fundamental of $SU(2)$, one finds $m_\phi^2(T \gg m_W) = \frac{3}{16} g_2^2 T^2$. For $T \ll m_W$ the contribution to m_ϕ coming from interactions with gauge bosons is Boltzmann-suppressed, and the one coming from self-interactions mediated by W is expected to be suppressed by the ratio $T^2/m_W^2 \ll 1$ because W has to be produced with high virtuality. Therefore the main contribution to the thermal mass of ϕ comes from interactions with ℓ and e^c mediated by the Yukawa coupling y_e . Using (26) with $y, y' \rightarrow \frac{y_e}{2}$ one finds $m_\phi^2(T \ll m_W) = \frac{1}{12} y_e^2 T^2$.

In this example no Q -charged particle Boltzmann-decouples from the plasma, so one gets simply,⁵

$$\rho = \frac{37 g_2}{110 y_e}. \quad (40)$$

V. CONCLUSIONS

A long period of supercooling has most of the qualitative features of inflation; it is a period of inflationary expansion that lasts a number $N_e \sim \ln T_c/T_{\text{exit}}$ of e -folds, where $T_{\text{exit}} \sim m_0, m_{\psi, \chi}$ and Λ_{QCD} in the examples of Secs. IV A, IV B, and IV C respectively. When the condition for condensation is reached, the system starts relaxing to the true minimum and all latent heat gets released, increasing the entropy by a factor of roughly $(T_c/T_{\text{exit}})^3$. At the end of the reheating period the ratio Q/S gets therefore diluted by a factor $(T_{\text{exit}}/T_c)^3$. A similar dilution takes place for any other abundance or defect produced before the supercooling epoch.

The transition to the stable vacuum is induced by tachyonic instability [18]. The out-of-equilibrium dynamics that follows the sudden destabilization is potentially rich in implications and deserves further investigation. In the hypotheses of this work, ϕ is $U(1)$ -charged and the transition to the true vacuum is expected to give rise to a string network, similarly to the Peccei-Quinn phase transition [19]. Adding a small $U(1)$ violating term that allows the network to eventually annihilate is not expected to spoil the mechanism described in this work.

The idea that a strong chemical potential can induce the destabilization of a metastable vacuum has been pursued in [20]; the authors considered large and dense systems such as stars as catalysts for the formation of true-vacuum bubbles. Here, instead, the presence of a sizeable charge concentration characterizes the whole Universe at some time in its evolution.

In the context of inflation, the authors of [21] have shown how the large kinetic energy of the inflaton can mimic a chemical potential $\mu \gg T_{\text{dS}}$ which is able to overcome Boltzmann suppression of particles with mass $m > T_{\text{dS}}$; the condition $\mu < m$ is invoked there to avoid tachyonic instabilities. It could be interesting to study an implementation of the mechanism presented here where the destabilizing chemical potential comes from the coupling of ϕ with another scalar φ that has $\dot{\varphi} \neq 0$.

A period of supercooled inflationary expansion has phenomenological consequences even if it happens in a sector that is only gravitationally coupled to the Standard Model; see for instance [22] for a discussion on the

⁵The quoted ρ can of course be seen as the combination of two effects; the decoupling of quarks and gluons from the plasma at $T \sim \Lambda_{\text{QCD}}$ and the decoupling of W bosons at $T \sim m_W \ll \Lambda_{\text{QCD}}$. The total ρ is the product of $\rho_\Lambda = \frac{49}{165}$ and $\rho_W = \frac{111 g_2}{98 y_e}$.

cosmology of dark sectors. As a model-building tool, supersaturation allows a metastable dark sector to persist until it reaches a critical threshold and decays.

The dynamical generation of a confinement scale—what happens in QCD—is a generic mechanism in gauge theories that allows us to accommodate large hierarchies. In the toy example of Sec. IV C, a larger and larger hierarchy between the critical temperature T_c and the exit temperature $T_{\text{exit}} \sim \Lambda_{\text{QCD}}$ implies a longer and longer period of supercooling with inflationary expansion. It would be interesting to study what ingredients are needed, other than $N_e = \ln T_c/T_{\text{exit}} \gtrsim 50$, to embed the supersaturation mechanism in a realistic model for inflation.

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DATA AVAILABILITY

Numerical code was developed with the assistance of Claude (Anthropic), and is available at [23].

APPENDIX: AN EXPLICIT MODEL

The purpose of this Appendix is manifold. After presenting an explicit model, the conditions for maintaining thermal equilibrium in various scenarios are considered in Sec. A 1. Section A 2 studies numerically the freeze-out of φ and the impact on the evaluation of the thermal mass of ϕ . Section A 3 provides analytic formulas for numerically determining the condensation temperature.

For concreteness, let us consider the theory of a neutral scalar φ , a complex scalar ϕ , two Dirac fermions ψ and χ and the photon γ . Let us take ψ and χ to be electrically charged, with charge 1. They are coupled to ϕ via a Yukawa term (all fields are canonically normalized),

$$\mathcal{L}_Y = y\phi\bar{\psi}\chi + \text{c.c.}, \quad (\text{A1})$$

which preserves a global $U(1)$, called Q , with charges $q_\phi = 1$, $q_\psi = \frac{1}{2}$ and $q_\chi = -\frac{1}{2}$. The local $U(1)$, which coincides with the fermion number in this model, is called F .

The neutral scalar couples with ϕ via a quartic interaction

$$\mathcal{L}_4 = -\frac{1}{2}\lambda'\varphi^2|\phi|^2. \quad (\text{A2})$$

By appropriately choosing the mass parameters, it is possible to match this model to both the example of Sec. IV A and that of Sec. IV B.

A straightforward generalization of the model includes N_f generations of fermions ψ_i and χ_i .

1. Conditions for equilibrium

Let us first consider the conditions for equilibrium in three scenarios:

- (i) All particles have masses $\ll T$;
- (ii) Temperature goes below the neutral scalar mass m_0 , the other particles being much lighter than T ;
- (iii) The same as above, now with ψ becoming non-relativistic.

In the large- T regime, charges and energies are distributed efficiently among species, because there are fast processes that execute the task. For example, each species can pairwise annihilate into two other particles at a rate $\Gamma \sim g^4 T$, where g^4 can stand for λ'^2 , $e^2 y^2$, etc., depending on the process. The same is true for elastic $2 \rightarrow 2$ processes that maintain kinetic equilibrium among the species. Adding an extra photon in the final state of an otherwise elastic process only costs an additional e^2 factor to the rate (we are not paying attention to factors of 4π here; including them does not change the picture substantially).

During inflationary expansion, Hubble is constant and roughly given by $H \sim T_c^2/M_P$, where T_c is the critical temperature at which the system becomes metastable (and gives a measure of the energy content of the false vacuum) and M_P is the Planck mass.

The condition for equilibrium, $\Gamma \gg H$, translates into

$$T \gg \frac{1}{g^\#} \left(\frac{T_c}{10 \text{ TeV}} \right)^2 0.01 \text{ eV}, \quad (\text{A3})$$

where $\#$ is 4 or 6 with bremsstrahlung. If the couplings are not artificially small, there is an ample interval of temperatures, say from TeV to keV, in which all species are in equilibrium.

The reason why all rates are proportional to T can be understood by dimensional analysis; with no particle masses and only marginal couplings, the only physical scale is temperature.

The picture changes when a species becomes nonrelativistic. We consider three cases of interest for this work.

a. Kinetic equilibrium of heavy scalar

Nonrelativistic particles that follow their equilibrium distribution have exponentially suppressed density. If they elastically scatter with a relativistic species, their kinetic

equilibrium is still efficiently maintained, because its rate is proportional to the density of the light species it interacts with. For example, the neutral scalar φ has $\mathcal{M}(\varphi\phi \rightarrow \varphi\phi) \sim \lambda'$. Then one has

$$\Gamma_{\text{kin}} \sim \frac{\lambda'^2}{m_0^2} \times T^3, \quad (\text{A4})$$

where the first factor is the cross section, and the second the density of targets (ϕ particles). The condition for equilibrium is now roughly

$$T \gg \frac{1}{\lambda'^{2/3}} \left(\frac{m_0}{100 \text{ GeV}} \right)^{2/3} \left(\frac{T_c}{10 \text{ TeV}} \right)^{2/3} 5 \text{ MeV}. \quad (\text{A5})$$

Kinetic equilibrium is kept for temperatures many orders of magnitude smaller than m_0 .

b. Chemical equilibrium of heavy scalar

With an efficient photon emission and many charged particles in the plasma, $\mu_\gamma = 0$. Particles a that can annihilate into photons then have $\mu_a + \mu_{\bar{a}} = 0$, and particles that can annihilate into $a\bar{a}$ satisfy the same relation, and so on and so forth. The chemical potential of φ is kept to zero in chemical equilibrium thanks, say, to the chain of fast reactions

$$2\varphi \xleftrightarrow{\lambda'} \phi\bar{\phi} \xleftrightarrow{y^2} \psi\bar{\psi} \xleftrightarrow{e^2} 2\gamma, \quad (\text{A6})$$

where the couplings governing the amplitude have been indicated above the arrow.

When T goes below m_0 , however, the density of target particles for φ , which are φ scalars themselves, exponentially drops with $e^{-m_0/T}$. The reaction rate quickly drops and it soon becomes too slow to keep up with the universe expansion, so φ freezes out. This scenario, which is essentially the one studied in Sec. IV A, is analyzed numerically in Sec. A 2.

c. Chemical equilibrium of heavy fermion

Suppose now that the heaviest particle is ψ , all other particles being much lighter. When T drops below m_ψ , some reactions get Boltzmann suppressed, like $\psi\bar{\psi}$ annihilation. However, this is not necessary for maintaining chemical equilibrium, which is kept by the fast reaction,

$$\psi \xleftrightarrow{y} \chi\phi, \quad (\text{A7})$$

whose rate is $\Gamma \sim y^2 T$. Transfer of F and Q from heavy to light particles is therefore very efficient in this case, and ψ does not freeze-out.

2. Departure from equilibrium

The deviation from equilibrium of φ can be studied with standard freeze-out methods. Calling $Y = n_\varphi/T^3$ and $x = m_0/T$, one has⁶

$$\frac{dY}{dx} = -\frac{\lambda'^2 m_0}{32\pi H x^4} (Y^2 - Y_{\text{eq}}^2), \quad (\text{A8})$$

with $Y_{\text{eq}} = x^{\frac{3}{2}} e^{-x}/(2\pi)^{\frac{3}{2}}$. The evolution of $Y(x)$ can be studied numerically. The key dimensionless parameter that governs the deviation from thermal equilibrium is

$$\xi = \frac{\lambda'^2 m_0}{32\pi H} \approx \lambda'^2 \left(\frac{m_0}{100 \text{ GeV}} \right) \left(\frac{0.01 \text{ eV}}{H} \right) \times 10^{11}. \quad (\text{A9})$$

The freeze-out temperature can be estimated by setting $\Gamma = H$, which gives

$$x_f = \ln \left(\frac{\lambda'^2 m_0}{32\pi (2\pi x_f)^{3/2} H} \right) \approx \ln \xi, \quad (\text{A10})$$

where the final expression is valid in the limit $\xi \gg 1$. In scenarios where supercooling is at the 10 TeV scale, $\ln \xi \sim 20 \div 25$.

We show, in the upper panel of Fig. 3, an example of the behaviour of $Y(x)$ in the present model.

a. Thermal mass of ϕ : Deviation from equilibrium formula

The contribution to the thermal mass of ϕ that comes from its interactions with φ particles in the plasma is given by Eq. (15), which reads in this case,

$$m_\phi^2 = \frac{\lambda'}{4\pi^2} \int_0^\infty \frac{dk k^2}{\sqrt{k^2 + m_0^2} (e^{\beta(\sqrt{k^2 + m_0^2} - \mu_\varphi)} - 1)}. \quad (\text{A11})$$

At thermal equilibrium $\mu_\varphi = 0$. If the particle freezes out, however, the distribution ceases to be the equilibrium one, and φ acquires an ‘artificial’ chemical potential, different than zero. The temperature instead remains the one of the plasma, thanks to the fast elastic reactions $\varphi\phi \rightarrow \varphi\phi$ that maintain kinetic equilibrium.

Out of chemical equilibrium, μ_φ is determined by imposing consistency with the freeze-out equation. Deep in the nonrelativistic regime, one has $Y = e^{\beta\mu_\varphi} Y_{\text{eq}}$, so

$$\mu_\varphi(T) = T \ln \left[\frac{Y(x)}{Y_{\text{eq}}(x)} \right], \quad (\text{A12})$$

⁶Scaling n_φ by T^3 instead of the usual entropy density amounts to neglecting the effect that φ decoupling has on $g(T)$, whose relative change in crossing m_0 is given by $\frac{g_{\text{below}}}{g_{\text{above}}} = 1 - \frac{1}{5+7N_f}$.

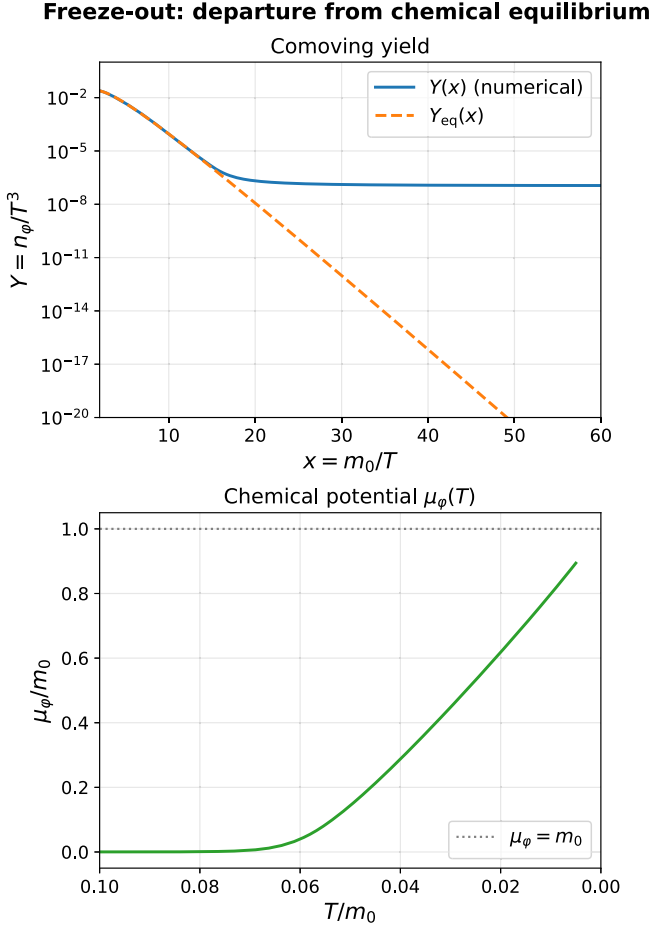


FIG. 3. Comoving yield of field ϕ , computed numerically using (A8) with the central values of (A9) and $\lambda' = 1$ (Above); chemical potential $\mu_\phi(T)$, evaluated with (A12) (Below).

which correctly reproduces the equilibrium result $\mu_\phi = 0$ for $T > T_f$. This is shown in the lower panel of Fig. 3. Using the so obtained $\mu_\phi(T)$, it is possible to numerically compute the deviation of m_ϕ^2 from the thermal equilibrium formula. This is shown in Fig. 4; for $T < T_f$, m_ϕ^2 decays like T^3 instead of exponentially. However, in realistic examples, the contribution to m_ϕ^2 that comes from the other particles in the plasma takes over long before T reaches T_f (see the example of Fig. 4).

3. Determination of condensation temperature

The analysis of Sec. IV is limited to the extreme limits $T \gg \Lambda$ and $T \ll \Lambda$, where Λ is some physical scale at which some particle species decouples from the plasma. Because of this, it could only provide a criterion for establishing if condensation happens or not (the condition is that $\kappa < 1$ above Λ and $\kappa > 1$ below Λ).

Working with the model of A, we provide now formulas that allow us to numerically determine the condensation

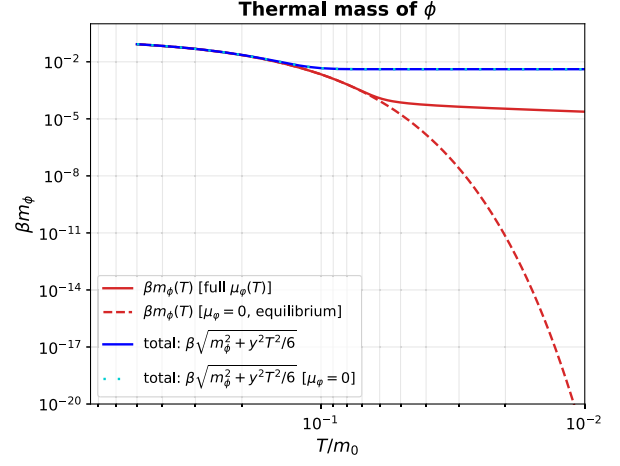


FIG. 4. Comparison between the thermal mass of ϕ in the model of A when (i) $y = 0$ and ϕ is in chemical equilibrium (dashed red), (ii) $y = 0$ and ϕ freezes out (solid red), and (iii) $y = 0.01$, so ϕ gets an additional contribution given by (26) (solid blue). The parameters used to evaluate m_ϕ , whose expression is given in (A11), are the same as in Fig. 3. Even with modest values of y , the shape of the total curve is almost insensitive to the out of equilibrium dynamics of ϕ (dotted cyan).

temperature T_* , which is obtained by solving $q_\phi \mu(T_*) = m_\phi(T_*)$.

Let us consider the scenario in which the N_f species $\psi_{i=1,\dots,N_f}$ have mass m_ψ , and all other particles are comparatively light.

When $T \sim m_\psi$, the entire ψ_i multiplet gets Boltzmann suppressed, efficiently distributing its charges F and Q to the other species. In this case depletion is extremely efficient and it is totally safe to consider equilibrium distributions for all species.

a. Thermal mass of ϕ

It is straightforward to give the formula for the thermal mass of ϕ , taking y' and m_χ to be zero in (25),

$$\beta m_\phi = \sqrt{\frac{\lambda'}{24} + 4N_f y^2 I_m(\beta m_\psi)}, \quad (\text{A13})$$

$$I_m(x) = \frac{1}{2\pi^2} \int_0^\infty \frac{dk k^2}{\sqrt{k^2 + x^2} (e^{\sqrt{k^2 + x^2}} + 1)}. \quad (\text{A14})$$

The integral $I_m(x)$ interpolates between the high temperature regime, with $I_m(0) = 1/24$, and the low-temperature regime, with $I_m(\infty) = 0$.

b. Chemical potential of ϕ

Following the logic of Sec. II, in order to extract $\mu_\phi(T)$ we start by evaluating the entropy of the system. This is poorly sensitive to chemical potentials, especially in the

limit $\mu \ll T$. We get

$$\frac{S}{V} = \frac{\pi^2 T^3}{45} (10 + 7N_f) + 4N_f T^3 I_S(\beta m_\psi), \quad (\text{A15})$$

$$I_S(x) = \frac{1}{6\pi^2} \int_0^\infty \frac{dk k^2 (4k^2 + 3x^2)}{\sqrt{k^2 + x^2} (e^{\sqrt{k^2 + x^2}} + 1)}, \quad (\text{A16})$$

where the integral I_S is a monotonically decreasing function with $I_S(0) = \frac{7\pi^2}{180}$ and $I_S(\infty) = 0$.

On top of entropy, the system has two conserved charges, Q and F . These quantities are sensitive to chemical potentials, and we work at first order in μ_Q and μ_F , which are the Lagrange multipliers associated to the conservation of Q and F , respectively.

It is useful to introduce the individual chemical potentials at equilibrium,

$$\mu_\phi = \mu_Q, \quad (\text{A17})$$

$$\mu_\psi = \frac{1}{2}\mu_Q + \mu_F, \quad (\text{A18})$$

$$\mu_\chi = -\frac{1}{2}\mu_Q + \mu_F. \quad (\text{A19})$$

In terms of these quantities, one has

$$\frac{F}{V} = 2N_f T^2 \left(\mu_\psi I_Q(\beta m_\psi) + \frac{1}{6}\mu_\chi \right), \quad (\text{A20})$$

where the charge integral I_Q decreases monotonically and satisfies $I_Q(0) = \frac{1}{6}$ and $I_Q(\infty) = 0$. It is given by

$$I_Q(x) = \frac{1}{\pi^2} \int_0^\infty \frac{dk k^2}{e^{\sqrt{k^2 + x^2}} + 2 + e^{-\sqrt{k^2 + x^2}}}. \quad (\text{A21})$$

The fermion number cannot be treated as a free parameter in this model, as we assumed it is associated to a local symmetry. Because charges repel each other, $F = 0$ in the bulk. Concretely, this translates into a relation among the chemical potentials of Q and F , given by

$$\mu_F = \frac{1}{2} \times \frac{1 - 6I_Q(\beta m_\psi)}{1 + 6I_Q(\beta m_\psi)} \mu_Q \equiv f(\beta m_\psi) \mu_Q. \quad (\text{A22})$$

The function f interpolates between zero at high temperatures and $\frac{1}{2}$ at low temperatures, and is monotonic.

The condition $F = 0$ has an interesting consequence when T drops below m_ψ ; since n_ψ and $n_{\bar{\psi}}$ become exponentially suppressed, the same must be true for the F charge asymmetry contained in the ψ sector. Because the only other F -charged particle is χ , the F -charge asymmetry in the χ sector, which is proportional to $n_\chi - n_{\bar{\chi}}$, must also

become exponentially suppressed, even though individually $n_\chi, n_{\bar{\chi}} \sim T^3$. This is the reason why μ_F must adjust to $\frac{1}{2}\mu_Q$ —and make $\mu_\chi = 0$, see (A19)—for $T \lesssim m_\psi$. An immediate consequence of this is that also the Q charge asymmetry is almost completely transferred in the ϕ sector, because n_χ and $n_{\bar{\chi}}$ get exponentially close to each other.

With the constraint $F = 0$, the Q charge asymmetry is given by

$$\begin{aligned} \frac{Q}{V} &= T^2 \left(\frac{1}{3}\mu_\phi + 2N_f \mu_\psi I_Q(\beta m_\psi) \right) \\ &= \mu_Q T^2 \left(\frac{1}{3} + 2N_f \left(\frac{1}{2} + f(\beta m_\psi) \right) I_Q(\beta m_\psi) \right), \end{aligned} \quad (\text{A23})$$

where in the second line we have used (A22).

Working at leading order in the chemical potentials we have achieved a nice separation of μ_Q from T . One gets

$$\beta \mu_Q(T) = \frac{Q}{S} \times \frac{\frac{\pi^2}{45} (10 + 7N_f) + 4N_f I_S}{\frac{1}{3} + 2N_f \left(\frac{1}{2} + f \right) I_Q}. \quad (\text{A24})$$

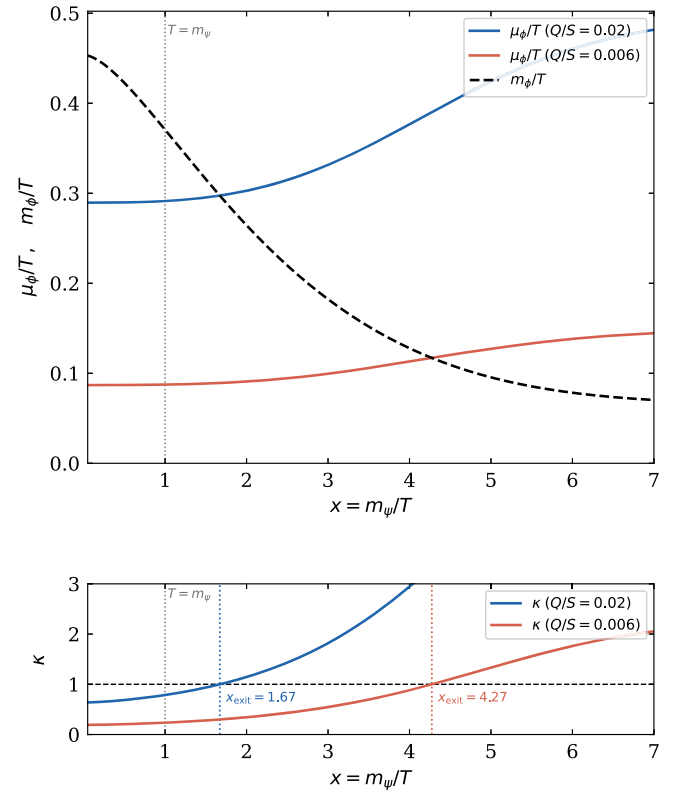


FIG. 5. (Above) Curves describing $\mu_\phi(\beta m_\psi)$ and $m_\phi(\beta m_\psi)$ in the model of Sec. A, with $\lambda' = 0.1$, $y^2 = 0.3$ and $N_f = 4$. The values of charge to entropy ratio are chosen in such a way that $\mu_\phi < m_\phi$ for $T \gg m_\psi$, and $\mu_\phi > m_\phi$ at $T \ll m_\psi$, so condensation is induced at some intermediate temperature T_* . (Below) With the same choice of parameters, $\kappa = \mu_\phi/m_\phi$. The exit temperature T_* is determined by the condition $\kappa(T) = 1$.

Using (A13) and (A24), one can extract T_* by equating them. An example is shown in Fig. 5. For lower and lower charge to entropy ratio, the exit point moves to lower and lower temperatures, until a critical Q is reached below which condensation is not triggered.

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