

Light S -wave pentaquarks on the light front

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(Received 20 October 2025; accepted 6 February 2026; published 9 March 2026)

We construct an explicit basis set for pentaquark states on a regular 4-simplex, that diagonalizes the Hamiltonian for light pentaquarks with confinement on the light front (LF). The ensuing eigenstates are free of the center of mass motion and satisfy exact Dirichlet boundary conditions. Hyperfine interactions in the form of color-spin or flavor-spin are shown to lift the degeneracy of the 16 pentastates, with a spectrum that compares fairly with some of the empirical nucleon excited states. The quark PDF for the light pentastates is discussed.

DOI: [10.1103/q4gt-2g69](https://doi.org/10.1103/q4gt-2g69)

I. INTRODUCTION

Constituent quark models emerge from our current understanding of the quantum chromodynamics (QCD) vacuum at low resolution, with running constituent masses and effective couplings with spontaneously broken chiral symmetry and bound light hadrons. They yield to a modern understanding of current spectroscopy in the center of mass frame.

To address issues related to the partonic constituents of hadrons, a formulation on the light front is needed. In the latter, light and heavy degrees of freedom are treated democratically, both in terms of straight Wilson lines thanks to the infinite momentum frame. In addition what is a cloud of mesons in the rest frame, translates to a parton sea by boosting, in the form of a multi-Fock component of a hadron.

For a baryon with 3-quarks the simplest Fock extension is a state containing a nucleon–meson component. In a recent analysis [1], two of us have shown that the S - and P -shell five quarks states mix with the standard 3-quark states through induced scalar and pseudoscalar interactions (meson clouds), in the center-of-mass (CM) frame. This mixing is central for the explanation of the emergence of a large orbital contribution to the baryon angular momentum, and the strong d-u flavor asymmetry in nucleons.

The aim of this paper is to construct S -wave pentaquark states by diagonalization of the 5-quark light cone Hamiltonian, which is free of the CM motion. In contrast to other approaches [2], where the center of mass motion is present in the light cone eigenbasis but removed only by projection through a Lagrange multiplier proportional to CM motion added to the Hamiltonian, our construction of the longitudinal and transverse eigenbases is inherently free of CM motion. In particular, the longitudinal eigenbasis satisfies the exact Dirichlet boundary conditions on a regular pentachoron or a 4-simplex, appropriate for a five-quark state. This construction, complements our recent results using variational analyses [3].

The paper is organized as follows. In Sec. II we develop the bare LF Hamiltonian for light pentaquarks in Jacobi coordinates, after removing the CM motion. In Sec. III we detail the construction of the eigenbasis on a simplex borrowing the exact solutions to identical and impenetrable fermions on a line. The pentaquark states are then constructed by direct diagonalization of their pertinent LF Hamiltonian. In Sec. IV the hyperfine splitting of the S -states pentaquarks are worked out for total antisymmetrization of the color-spin-flavor states. The color-spin and flavor-spin splittings yield light pentastates which are comparable to the measured nucleon states. The numerical results for the ground state eigenfunctions and hyperfine splitting of s -wave states are presented in Sec. V. Our conclusions are in Sec. VI.

II. LF HAMILTONIAN AND JACOBI COORDINATES

Our description of the light pentaquarks $qqqq\bar{q}$ will follow that of the light baryons qqq on the LF [4]. The light

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front Hamiltonian for the pentaquark state includes the kinetic and confining terms

$$H'_{LF} = \sum_{i=1}^5 \left(\frac{k_{i\perp}^2 + m_Q^2}{x_i} + 2\sigma_T((i\partial/\partial x_i)^2 + M^2 r_{i\perp}^2)^{1/2} \right) \quad (1)$$

where σ_T is the string tension and (in lowest order) $M \approx 5m_Q$, with m_Q the constituent quark mass. Here x_i are momentum fractions for quarks, $i = 1..5$ and $r_{i\perp}$ are transverse coordinates.

The momentum conservation conditions for the pentaquark state are $\sum_{i=1}^5 \vec{k}_{i\perp} = 0$ for transverse momenta in the transverse center-of-mass frame of reference and $\sum_{i=1}^5 x_i = 1$ for longitudinal momentum fraction, $0 \leq x_i \leq 1$. Using the einbein trick, one can get rid of the square roots in the confining term,

$$H'_{LF} = \sum_{i=1}^5 \frac{k_{i\perp}^2 + m_Q^2}{x_i} + \sigma_T \left(5a + \frac{1}{a} \sum_{i=1}^5 ((i\partial/\partial x_i)^2 - M^2 (\partial/\partial \vec{k}_{i\perp})^2) \right) \quad (2)$$

where the transverse coordinates have been replaced by $r_{i\perp}^2 = -(\partial/\partial \vec{k}_{i\perp})^2$. Following [4], the symmetric einbein parameter a will be fixed below by minimization of the resulting masses, with a value that depends on the constituent quark mass.

To eliminate the motion of center mass, we use the following Jacobi coordinates for the longitudinal part,

$$\begin{aligned} x_\alpha &= \frac{x_1 - x_2}{\sqrt{2}}, \\ x_\beta &= \frac{x_1 + x_2 - 2x_3}{\sqrt{6}}, \\ x_\gamma &= \frac{x_1 + x_2 + x_3 - 3x_4}{\sqrt{12}}, \\ x_\delta &= \frac{x_1 + x_2 + x_3 + x_4 - 4x_5}{\sqrt{20}} \end{aligned} \quad (3)$$

One can transfer the longitudinal momentum fractions x_i to the Jacobi coordinates using the above equations with the momentum conservation condition $\sum_{i=1}^5 x_i = 1$,

$$\begin{aligned} x_1 &= \frac{1}{30} (15\sqrt{2}x_\alpha + 5\sqrt{6}x_\beta + 3\sqrt{5}x_\delta + 5\sqrt{3}x_\gamma) + \frac{1}{5}, \\ x_2 &= \frac{1}{30} (-15\sqrt{2}x_\alpha + 5\sqrt{6}x_\beta + 3\sqrt{5}x_\delta + 5\sqrt{3}x_\gamma) + \frac{1}{5}, \\ x_3 &= \frac{1}{30} (-10\sqrt{6}x_\beta + 3\sqrt{5}x_\delta + 5\sqrt{3}x_\gamma) + \frac{1}{5}, \\ x_4 &= \frac{1}{10} (\sqrt{5}x_\delta - 5\sqrt{3}x_\gamma) + \frac{1}{5}, \\ x_5 &= -\frac{2\sqrt{5}}{5}x_\delta + \frac{1}{5} \end{aligned} \quad (4)$$

The transverse part can also be similarly rewritten into the Jacobi coordinates, after enforcing momentum conservation $\sum_{i=1}^5 \vec{k}_{i\perp} = 0$. With the help of the einbein transform, the square-root in the confining potential (2) can be unwound

$$\begin{aligned} &((i\partial/\partial x_i)^2 + M^2 r_{i\perp}^2)^{1/2} \\ &= -\left(\frac{\partial^2}{\partial x_\alpha^2} + \frac{\partial^2}{\partial x_\beta^2} + \frac{\partial^2}{\partial x_\gamma^2} + \frac{\partial^2}{\partial x_\delta^2} + 5 \frac{\partial^2}{\partial X^2} \right) \\ &\quad - M^2 \left(\frac{\partial^2}{\partial \vec{k}_\alpha^2} + \frac{\partial^2}{\partial \vec{k}_\beta^2} + \frac{\partial^2}{\partial \vec{k}_\gamma^2} + \frac{\partial^2}{\partial \vec{k}_\delta^2} + 5 \frac{\partial^2}{\partial \vec{K}^2} \right) \\ &\rightarrow -\left(\frac{\partial^2}{\partial x_\alpha^2} + \frac{\partial^2}{\partial x_\beta^2} + \frac{\partial^2}{\partial x_\gamma^2} + \frac{\partial^2}{\partial x_\delta^2} \right) \\ &\quad - M^2 \left(\frac{\partial^2}{\partial \vec{k}_\alpha^2} + \frac{\partial^2}{\partial \vec{k}_\beta^2} + \frac{\partial^2}{\partial \vec{k}_\gamma^2} + \frac{\partial^2}{\partial \vec{k}_\delta^2} \right) \\ &= -\nabla_L - M^2 \nabla_T \end{aligned} \quad (5)$$

where $\frac{\partial^2}{\partial X^2}$ and $\frac{\partial^2}{\partial \vec{K}^2}$ represent the longitudinal CM motion and transverse CM motion, respectively. These terms are omitted when defining the longitudinal Laplacian operator ∇_L and transverse Laplacian operator ∇_T . The final expression for the light front Hamiltonian, free of CM, is then given by

$$H_{LF} \equiv \sum_{i=1}^5 \frac{k_{i\perp}^2 + m_Q^2}{x_i} + 5\sigma_T a - \frac{\sigma_T}{a} \sum_{\xi=\alpha,\beta,\gamma,\delta} ((\partial/\partial x_\xi)^2 + M^2 (\partial/\partial \vec{k}_{\xi\perp})^2) \quad (6)$$

III. CONSTRUCTION OF AN EIGENBASIS

To diagonalize the Light front Hamiltonian in Eq. (6), one needs to find the proper basis. For the longitudinal part, the momentum constraints

$$X = \sum_{i=1}^N x_i = 1 \quad \text{with} \quad 1 \geq x_i \geq 0 \quad (7)$$

For $N = 3$ (the baryons) the enclosed domain is equilateral triangle, and for $N = 5$ (pentaquarks) it is a regular 4-dimensional simplex known as *pentachoron*, or “standard simplex” $\Delta_N[x_1, \dots, x_N]$ defined by the Jacobi coordinates. It has 5 “corners” and 10 “faces.” Note that the WFs should vanish on all its faces (Dirichlet boundary conditions), which corresponds to some parton momentum vanishing $x_i = 0$, due to the singular terms proportional to $1/x_i$ in the Hamiltonian (6).

A. General case for longitudinal momenta

We now note that the standard simplex $\Delta_N[x_1, \dots, x_N]$ we need to study is geometrically isomorphic to the “ordered simplex” $\Delta_N^*[s_1, \dots, s_N]$ in R^N

$$s_1 \leq s_2 \leq \dots \leq s_N \leq s_1 + a \quad (8)$$

where parameter a is fixed by the length of the side of simplex. The faces of the ordered simplex occur when two successive coordinates are equal $s_i = s_{i+1}$. The LFWFs are then amenable to the WFs of N particles (fermions) moving on a circle of unit circumference

$$\varphi[x] \rightarrow \varphi[s] \quad \text{with} \quad \varphi[s|s_i = s_j] = 0 \quad (9)$$

These N impenetrable particles map onto N free fermions, thanks to the Pauli exclusion principle. The

N -fermion WFs are Slater determinants composed of circular waves [5]

$$\varphi_n[s] = \frac{1}{N^{4/3}} \begin{vmatrix} e^{i\tilde{n}_1 s_1} & e^{i\tilde{n}_1 s_2} & \dots & e^{i\tilde{n}_1 s_N} \\ e^{i\tilde{n}_2 s_1} & e^{i\tilde{n}_2 s_2} & \dots & e^{i\tilde{n}_2 s_N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\tilde{n}_N s_1} & e^{i\tilde{n}_N s_2} & \dots & e^{i\tilde{n}_N s_N} \end{vmatrix} \quad (10)$$

with $\tilde{n}_i = 2\pi n_i$. They have support on disjoint simplexes under permutation, with eigenenergies

$$E[n] = \sum_{i=1}^N \tilde{n}_i^2 \quad (11)$$

A similar approach was used to diagonalize the free fermion problem with Dirichlet boundary conditions, on an irregular tetrahedron [6].

The ordering of the coordinates $s_i \in [0, 1]$, suggests their identification with the coordinates of equidistant particles on a string of length 1 in 1-dimension, undergoing longitudinal vibrations. With this in mind and to eliminate the CM of mass coordinate, we shift $s_j = \frac{S}{N} + \tilde{s}_j$ with $S = \sum_{j=1}^N s_j$ in (10), and using the rearrangement of the rows and columns in the Slater determinant, we obtain

$$\begin{vmatrix} e^{i\tilde{n}_1 s_1} & e^{i\tilde{n}_1 s_2} & \dots & e^{i\tilde{n}_1 s_N} \\ e^{i\tilde{n}_2 s_1} & e^{i\tilde{n}_2 s_2} & \dots & e^{i\tilde{n}_2 s_N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\tilde{n}_N s_1} & e^{i\tilde{n}_N s_2} & \dots & e^{i\tilde{n}_N s_N} \end{vmatrix} = \begin{vmatrix} e^{i\tilde{n}_1 \frac{S}{N}} e^{i\tilde{n}_1 \tilde{s}_1} & e^{i\tilde{n}_1 \frac{S}{N}} e^{i\tilde{n}_1 \tilde{s}_2} & \dots & e^{i\tilde{n}_1 \frac{S}{N}} e^{i\tilde{n}_1 \tilde{s}_N} \\ e^{i\tilde{n}_2 \frac{S}{N}} e^{i\tilde{n}_2 \tilde{s}_1} e^{i\tilde{n}_{21} \tilde{s}_1} & e^{i\tilde{n}_2 \frac{S}{N}} e^{i\tilde{n}_2 \tilde{s}_2} e^{i\tilde{n}_{21} \tilde{s}_2} & \dots & e^{i\tilde{n}_2 \frac{S}{N}} e^{i\tilde{n}_2 \tilde{s}_N} e^{i\tilde{n}_{21} \tilde{s}_N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\tilde{n}_N \frac{S}{N}} e^{i\tilde{n}_N \tilde{s}_1} e^{i\tilde{n}_{N1} \tilde{s}_1} & e^{i\tilde{n}_N \frac{S}{N}} e^{i\tilde{n}_N \tilde{s}_2} e^{i\tilde{n}_{N1} \tilde{s}_2} & \dots & e^{i\tilde{n}_N \frac{S}{N}} e^{i\tilde{n}_N \tilde{s}_N} e^{i\tilde{n}_{N1} \tilde{s}_N} \end{vmatrix} \quad (12)$$

and pulling common factors with the center-of-mass contribution,

$$e^{i\tilde{n}_1 \frac{S}{N}} e^{i\tilde{n}_2 \frac{S}{N}} \dots e^{i\tilde{n}_N \frac{S}{N}} e^{i\tilde{n}_1 \tilde{s}_1} e^{i\tilde{n}_1 \tilde{s}_2} \dots e^{i\tilde{n}_1 \tilde{s}_N} = e^{i\frac{S}{N} \sum_{i=1}^N \tilde{n}_i} \quad (13)$$

yield the result

$$\varphi_n[s] = \frac{1}{N^{4/3}} e^{i\frac{S}{N} \sum_{i=1}^N \tilde{n}_i} \begin{vmatrix} 1 & 1 & \dots & 1 \\ e^{i\tilde{n}_{21} \tilde{s}_1} & e^{i\tilde{n}_{21} \tilde{s}_2} & \dots & e^{i\tilde{n}_{21} \tilde{s}_N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\tilde{n}_{N1} \tilde{s}_1} & e^{i\tilde{n}_{N1} \tilde{s}_2} & \dots & e^{i\tilde{n}_{N1} \tilde{s}_N} \end{vmatrix} \equiv e^{i\frac{S}{N} \sum_{i=1}^N \tilde{n}_i} \varphi_{\tilde{n}}[\tilde{s}] \quad (14)$$

with $\tilde{n}_{i1} = \tilde{n}_i - \tilde{n}_1$ for $i = 2, \dots, N$. The coefficients $\tilde{n}_{i1}$ are ordered as $0 < \tilde{n}_{21} < \tilde{n}_{31} < \tilde{n}_{41} \dots$ to avoid repeated counting of the same state. One can get rid of the center mass and get the eigenvalues for the reduced WFs $\varphi_{\tilde{n}}[\tilde{s}]$, which are only a function of the $N - 1$ coordinates, with energies

$$\begin{aligned}
\tilde{E}[\tilde{n}] &= \frac{1}{\varphi_{\tilde{n}}[\tilde{s}]} \left[\left(\frac{\partial}{\partial s_i} \right)^2 - N \left(\frac{\partial}{\partial S} \right)^2 \right] \varphi_{\tilde{n}}[\tilde{s}] \\
&= \sum_{i=1}^N \tilde{n}_i^2 - \frac{1}{N} \left(\sum_{i=1}^N \tilde{n}_i \right)^2 \\
&= \sum_{i=2}^N \tilde{n}_{i1}^2 - \frac{1}{N} \left(\sum_{i=2}^N \tilde{n}_{i1} \right)^2
\end{aligned} \tag{15}$$

Since the set of Slater determinants is orthonormal and complete, the solutions $\varphi_{\tilde{n}}[\tilde{s}]$ map onto a complete basis set. The normal mode coordinates $\tilde{s}$ can be related to the standard longitudinal coordinates by using $\tilde{s}_i - \tilde{s}_{i+1} = x_i$. These, in turn, are connected to the Jacobi coordinates via their standard defining relation. We will show how to construct the longitudinal basis using $\varphi_{\tilde{n}}[\tilde{s}]$ in the next section for the $N = 5$ case.

B. Longitudinal basis for $N=5$

Specifically, we are interested in the pentaquark states in this study. The orthonormal basis obtained after removing the CM motion, can be recast as

$$\varphi_{\tilde{n}}[\tilde{s}] = \frac{1}{5^{4/3}} \begin{vmatrix} 1 & 1 & \dots & 1 \\ e^{i\tilde{n}_{21}\tilde{s}_1} & e^{i\tilde{n}_{21}\tilde{s}_2} & \dots & e^{i\tilde{n}_{21}\tilde{s}_5} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\tilde{n}_{51}\tilde{s}_1} & e^{i\tilde{n}_{51}\tilde{s}_2} & \dots & e^{i\tilde{n}_{51}\tilde{s}_5} \end{vmatrix} \tag{16}$$

To enforce the Dirichlet boundary conditions on the pentachoron formed by the longitudinal momenta

$$\sum_{i=1}^5 x_i = 1 \quad 0 \leq x_i \leq 1 \tag{17}$$

and keep the determinantal form of the solutions, we construct the linear map

$$\begin{aligned}
\tilde{s}_1 - \tilde{s}_2 &= s_1 - s_2 = x_1 \geq 0 \\
\tilde{s}_2 - \tilde{s}_3 &= s_2 - s_3 = x_2 \geq 0 \\
\tilde{s}_3 - \tilde{s}_4 &= s_3 - s_4 = x_3 \geq 0 \\
\tilde{s}_4 - \tilde{s}_5 &= s_4 - s_5 = x_4 \geq 0
\end{aligned} \tag{18}$$

expression Jacobi coordinates with differences of circle coordinates. The above conditions ensure that the eigenbasis defined in Eq. (16) vanishes at $x_i = 0$. Note that those are subject to the CM constraint

$$\sum_{i=1}^5 \tilde{s}_i = 0, \quad \sum_{i=1}^5 s_i = S$$

which preserves the ordering. Note that the left out coordinate x_5 is fixed by the longitudinal momentum constraint (17), with $x_{1,\dots,4}$ defined in terms of Jacobi variables defined in Eq. (4) we can express $\tilde{s}_i$ and s_i in (18) using Jacobi variables as

$$\begin{aligned}
s_1 &= \tilde{s}_1 + \frac{S}{5} = \frac{1}{10} (2\sqrt{5}x_\delta + 2\sqrt{3}x_\gamma + 4 + \sqrt{2}x_\alpha + \sqrt{6}x_\beta) + \frac{S}{5} \\
s_2 &= \tilde{s}_2 + \frac{S}{5} = \frac{1}{30} (-12\sqrt{2}x_\alpha - 2\sqrt{6}x_\beta + 3\sqrt{5}x_\delta + \sqrt{3}x_\gamma + 6) + \frac{S}{5} \\
s_3 &= \tilde{s}_3 + \frac{S}{5} = \frac{1}{30} (3\sqrt{2}x_\alpha - 7\sqrt{6}x_\beta - 4\sqrt{3}x_\gamma) + \frac{S}{5} \\
s_4 &= \tilde{s}_4 + \frac{S}{5} = \frac{1}{10} (\sqrt{2}x_\alpha + \sqrt{6}x_\beta - \sqrt{5}x_\delta - 3\sqrt{3}x_\gamma - 2) + \frac{S}{5} \\
s_5 &= \tilde{s}_5 + \frac{S}{5} = \frac{1}{10} (-2\sqrt{5}x_\delta + 2\sqrt{3}x_\gamma - 4 + \sqrt{2}x_\alpha + \sqrt{6}x_\beta) + \frac{S}{5}
\end{aligned} \tag{19}$$

The basis in Eq. (16) can be constructed directly from the $\tilde{s}_i$ defined above. Moreover, the Laplacian now takes the form of the following elliptic operator

$$\begin{aligned}
\sum_{i=1}^5 \partial_{s_i}^2 &= 5\partial_S^2 + \frac{1}{6} (18\partial_{x_\alpha}^2 + 14\partial_{x_\beta}^2 + 13\partial_{x_\gamma}^2 + 15\partial_{x_\delta}^2 - 4\sqrt{3}\partial_{x_\alpha}\partial_{x_\beta} + \sqrt{6}\partial_{x_\alpha}\partial_{x_\gamma} + 3\sqrt{10}\partial_{x_\alpha}\partial_{x_\delta} \\
&\quad - 7\sqrt{2}\partial_{x_\beta}\partial_{x_\gamma} + \sqrt{30}\partial_{x_\beta}\partial_{x_\delta} - 2\sqrt{15}\partial_{x_\gamma}\partial_{x_\delta}) \\
&= 5\partial_S^2 + \partial_{X'}^2
\end{aligned} \tag{20}$$

The replacement of $\tilde{s}_i$ in Eq. (16) with the longitudinal Jacobi coordinates $x_{\alpha,\beta,\gamma,\delta}$ using Eq. (19), yields a complete and orthonormal set of Slater determinants that vanish on the 10 equilateral faces of the pentachoron, as defined by

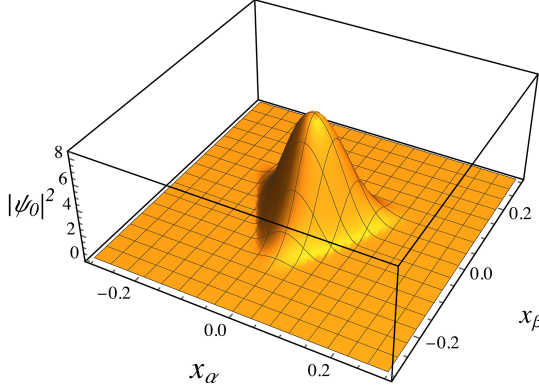


FIG. 1. The ground state of the Laplacian operator obtained by diagonalizing the operator using the basis defined in Eq. (16). The variables x_γ and x_δ are set to $x_\gamma = x_\delta = -0.2$.

the 4-simplex Eq. (17), and solely in terms of the Jacobi coordinates $x_\alpha, x_\beta, x_\gamma, x_\delta$. The eigenvalues of $\partial_{x'}^2$ are consistent with those given in Eq. (15).

The resulting set of Slater determinants can be used as a basis set to diagonalize the LC Hamiltonian where the Laplacian is now a general elliptic operator (20) plus a potential term. Note that the general elliptic operator becomes the standard Laplacian operator when $N = 3$, the eigenvectors obtained using this method are consistent with the ones used in [4]. For the $N = 5$ case it is not, but one can still use the basis defined in Eq. (16) to diagonalize the Laplacian operator with the Dirichlet boundary condition on a regular 4-simplex.

The ground state eigenfunction of the longitudinal Laplacian operator $-\nabla_L$ defined in in Eq. (5) is shown in Fig. 1 in the x_α, x_β plane. Both remaining coordinates, x_γ and x_δ , have been set to $x_\gamma = x_\delta = -0.2$.

C. Transverse basis for $N = 5$

The LF Hamiltonian in (6) includes the nonfactorizable part of the potential at the origin of the pentachoron or 4-simplex

$$V = \sum_{i=1}^5 \vec{k}_i^2 \left(\frac{1}{x_i} - 5 \right) \quad (21)$$

The subtracted mean parton momentum $\frac{1}{5}$ centers the longitudinal momenta around the center of the

4-simplex. With this in mind, the LF Hamiltonian in (6) can be recast as

$$H_{LF} = \sum_{i=1}^5 \frac{m_Q^2}{x_i} - \sum_{\xi=\alpha,\beta,\gamma,\delta} \frac{\sigma_T}{a} (\partial/\partial x_\xi)^2 + 5\sigma_T a + V + \sum_{\xi=\alpha,\beta,\gamma,\delta} -\frac{\sigma_T M^2}{a} \left(\frac{\partial}{\partial \vec{k}_{\xi\perp}} \right)^2 + 5\vec{k}_{\xi\perp}^2 \quad (22)$$

The last contribution amounts to the sum of transverse oscillators in Jacobi coordinates. The corresponding eigenfunctions are generalized Laguerre polynomials,

$$\psi_{n,m}(k_{\xi\perp}, \beta) = \frac{1}{\sqrt{\pi}} \beta^{1/4} \sqrt{\frac{n!}{(n+|m|)!}} \times e^{-\frac{k_{\xi\perp}^2 \beta^{1/2}}{2} + im\phi(k_{\xi\perp} \beta^{1/4})} |L_n^{|m|}(\beta^{1/2} k_{\xi\perp}^2)| \quad (23)$$

with $\beta = 5a/(\sigma_T M^2)$, and eigenvalues

$$E_{n,m} = 2\sqrt{5}(\sigma_T M^2/a)^{1/2}(2n + |m| + 1).$$

Note that the basis in Eq. (23) depends only on the Jacobi coordinates, so it is free of the CM motion.

IV. HYPERFINE INTERACTIONS

In addition to the basic light front Hamiltonian used so far (kinetic energy plus confinement), we consider the hyperfine interactions which split the degeneracy of the light pentaquark states, with the spin-isospin assignments shown in Table I. As the spin-orbit and tensor interactions vanish in the *S*-shell, we will consider the color-spin exchange expected from perturbative one-gluon exchange, or instanton induced interactions between massive quarks. For comparison with earlier analyses [7,8], we will also carry the analysis for flavor-spin hyperfine interactions, although some arguments still persist on their importance [9,10].

A. Interactions

With this in mind, the color-spin and flavor-spin hyperfine interactions are, respectively

$$\begin{aligned} \mathbb{V}_{1g}(1, 2, 3, 4, 5) &= -\sum_{i<j\leq 5} \frac{V_{1g}(r_{ij})}{m_Q^2} \lambda_{ij} \Sigma_{ij} \rightarrow -\frac{V_{1g}}{m_Q^2} \left(\sum_{i<j\leq 4} \lambda_i \cdot \lambda_j \sigma_i \cdot \sigma_j + \sum_{i<5} \lambda_i \cdot \bar{\lambda}_5 \sigma_i \cdot \bar{\sigma}_5 \right) \\ \mathbb{V}_f(1, 2, 3, 4, 5) &= -\sum_{i<j\leq 5} C_\chi(r_{ij}) T_{ij} \Sigma_{ij} \rightarrow -C_\chi \left(\sum_{i<j\leq 4} \tau_i \cdot \tau_j \sigma_i \cdot \sigma_j + \sum_{i<5} \tau_i \cdot \bar{\tau}_5 \sigma_i \cdot \bar{\sigma}_5 \right) \end{aligned} \quad (24)$$

TABLE I. Spin-isospin assignments and their degeneracies for the light s -wave pentastates. S and I represent the spin and isospin, respectively.

I/S	$1/2$	$3/2$	$5/2$
$1/2$	3	3	1
$3/2$	3	3	1
$5/2$	1	1	0

are sandwiched between the pertinent color-spin wave functions in (25). Note that when the exchange involves the antiquark labeled by 5, the replacement of the Gell-Mann matrices $\bar{\lambda}_5 = -\lambda^*$ is assumed, and similarly $\bar{\sigma}_5 = -\sigma^*$ for the spin below (the isospin matrix τ is also a Pauli matrix). The rightmost result refers to the local approximation to be used below for a spectrum estimate. In this work, our focus is on the S -wave pentaquark state, with no effects from spin-orbit interactions. The contributions from spin-orbit interactions have been studied in [1].

B. Representations of permutation group S_4

In this work we consider the case of pentaquarks formed of only the light u, d quark flavors of equal mass. The ground state wave function carries $L = 0$ or space symmetric S -state. The ground state can be constructed using the irreducible representation of the permutation group of four quarks S_4 . The main results will be summarized below, and more details can be found in the Appendix of Ref. [1]. More specifically, the totally antisymmetric color-spin-flavor wave function in the S -state reads [1]

$$\Psi_P^S = \frac{\varphi_{00L}}{\sqrt{3}} ([C[211]_\beta SF[31]_\alpha - C[211]_\alpha SF[31]_\beta + C[211]_\gamma SF[31]_\gamma] C[11] SF[1]) \quad (25)$$

where $C[211]$ are the irreducible representation of S_4 for the color indices. Their Young tableaux are

$$C[211]_\alpha = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \quad (26)$$

$$C[211]_\beta = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \quad (27)$$

$$C[211]_\gamma = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \quad (28)$$

For the ground state wave function with $L = 0$, the 4-quarks spin-flavor mixed representation conjugate to the 4-quarks color representation is also 3-degenerate, with the Jacobi-like nomenclature

$$SF[31]_\alpha = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array} \quad (29)$$

$$SF[31]_\beta = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array} \quad (30)$$

$$SF[31]_\gamma = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \quad (31)$$

These mixed representations follow from the product representations

$$[q^4]_S \otimes [q^4]_F \quad (32)$$

with a mixed $[31]_{S,F}$ net component. The spin flavor configurations of $[q^4]$ follow from the standard $SU(2)$ Young tableaux,

$$[q^4]_S = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \equiv S[4] \oplus S[31] \oplus S[22] \quad (33)$$

and similarly for $[q^4]_F$ with $S \rightarrow F$. It follows from the character representations of S_4 that the product representations (32) using (33) are

$$\begin{aligned} S[4] \otimes F[31] &= SF[31] \\ S[31] \otimes F[4] &= SF[31] \\ S[31] \otimes F[31] &= SF[4] \oplus SF[31] \oplus SF[22] \\ S[31] \otimes F[22] &= SF[31] \\ S[22] \otimes F[22] &= SF[4] \oplus SF[22] \end{aligned} \quad (34)$$

and the spin-flavor exchanged products

$$\begin{aligned}
 F[4] \otimes S[31] &= SF[31] \\
 F[31] \otimes S[31] &= SF[4] \oplus SF[31] \oplus SF[22] \\
 F[31] \otimes S[22] &= SF[31] \\
 F[31] \otimes S[4] &= SF[31]
 \end{aligned} \tag{35}$$

hence

$$SF[31] \subset S[4] \otimes F[31], S[31] \otimes F[4], S[31] \otimes F[31], S[31] \otimes F[22], S[22] \otimes F[31] \tag{36}$$

C. Matrix elements

In Table II, we show the matrix element of color-spin $\lambda_{ij}\Sigma_{ij}$ interaction for the different pentastates. To distinguish the states with same spin(S), isospin(I) originating from different spin-isospin configurations, we will use the following notations

$$\begin{aligned}
 S[I] = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_a : S[31] \otimes F[31], & \quad \frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_b : S[31] \otimes F[22], & \quad \frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_c : S[22] \otimes F[31] \\
 \frac{1}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_a : S[31] \otimes F[4], & \quad \frac{1}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_b : S[22] \otimes F[31], & \quad \frac{1}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_c : S[31] \otimes F[31] \\
 \frac{3}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_a : S[4] \otimes F[31], & \quad \frac{3}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_b : S[31] \otimes F[22], & \quad \frac{3}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_c : S[31] \otimes F[31] \\
 \frac{3}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_a : S[31] \otimes F[4], & \quad \frac{3}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_b : S[4] \otimes F[31], & \quad \frac{3}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_c : S[31] \otimes F[31] \\
 \frac{1}{2} \begin{bmatrix} 5 \\ 2 \end{bmatrix} : S[31] \otimes F[4], & \quad \frac{3}{2} \begin{bmatrix} 5 \\ 2 \end{bmatrix} : S[31] \otimes F[4], & \\
 \frac{5}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} : S[4] \otimes F[31], & \quad \frac{5}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix} : S[4] \otimes F[31] &
 \end{aligned} \tag{37}$$

On the LF, the quantum states carry fixed S_z in the ground state. On the other hand, the spin-color interaction is the same for fixed S but different S_z . In Table II we show the color-spin interaction for fixed S_z . For example, the states with $I = 1/2$ and $S_z = 1/2$ include the states $\frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_a, \frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_b, \frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_c, \frac{3}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_a, \frac{3}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_b, \frac{3}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_c$, and $\frac{5}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The matrix entries of $\lambda_{ij}\Sigma_{ij}$ between these states are

$$\left\langle S_z = \frac{1}{2}, I = \frac{1}{2} \left| \sum_{i < j \leq 5} \lambda_{ij} \Sigma_{ij} \right| S_z = \frac{1}{2}, I = \frac{1}{2} \right\rangle = \begin{pmatrix} -\frac{16}{3} & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & \frac{56}{3} & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\sqrt{10} & 0 \\ 0 & 0 & 0 & 0 & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 4\sqrt{10} & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{40}{3} \end{pmatrix} \tag{38}$$

In Table III we show the expectations of the spin-flavor interaction $\sum_{i < j \leq 5} \lambda_{ij} \Sigma_{ij}$ in the states with $L = 0$, spin S and isospin I . The values inside the outer brackets represent the total spin and isospin of the state, respectively. The subscripts a, b, and c indicate different combinations of spin and isospin representations. The matrix entries of $T_{ij}\Sigma_{ij}$ between these states are

TABLE II. The expectational value of $\sum_{i<j\leq 5} \lambda_{ij} \Sigma_{ij}$ in the states with $L = 0$, spin S and isospin I . The number outside and inside the brackets represent the total spin and isospin of the state, respectively. The subscripts a , b , and c indicate different combinations of spin and isospin representations. An empty entry indicates that the corresponding expectation value is zero.

	$\frac{1}{2} [\frac{1}{2}]_a$	$\frac{1}{2} [\frac{1}{2}]_b$	$\frac{1}{2} [\frac{1}{2}]_c$	$\frac{1}{2} [\frac{3}{2}]_a$	$\frac{1}{2} [\frac{3}{2}]_b$	$\frac{1}{2} [\frac{3}{2}]_c$	$\frac{1}{2} [\frac{5}{2}]$	$\frac{3}{2} [\frac{1}{2}]_a$	$\frac{3}{2} [\frac{1}{2}]_b$	$\frac{3}{2} [\frac{1}{2}]_c$	$\frac{3}{2} [\frac{3}{2}]_a$	$\frac{3}{2} [\frac{3}{2}]_b$	$\frac{3}{2} [\frac{3}{2}]_c$	$\frac{3}{2} [\frac{5}{2}]$	$\frac{5}{2} [\frac{1}{2}]$	$\frac{5}{2} [\frac{3}{2}]$
$\frac{1}{2} [\frac{1}{2}]_a$	$-\frac{16}{3}$		8													
$\frac{1}{2} [\frac{1}{2}]_b$		$\frac{56}{3}$														
$\frac{1}{2} [\frac{1}{2}]_c$	8		0													
$\frac{1}{2} [\frac{3}{2}]_a$				$-\frac{88}{3}$												
$\frac{1}{2} [\frac{3}{2}]_b$					0	8										
$\frac{1}{2} [\frac{3}{2}]_c$					8	$-\frac{16}{3}$										
$\frac{1}{2} [\frac{5}{2}]$							$-\frac{88}{3}$									
$\frac{3}{2} [\frac{1}{2}]_a$								0		$4\sqrt{10}$						
$\frac{3}{2} [\frac{1}{2}]_b$									$-\frac{4}{3}$							
$\frac{3}{2} [\frac{1}{2}]_c$								$4\sqrt{10}$		$-\frac{4}{3}$						
$\frac{3}{2} [\frac{3}{2}]_a$											$-\frac{40}{3}$					
$\frac{3}{2} [\frac{3}{2}]_b$												0	$4\sqrt{10}$			
$\frac{3}{2} [\frac{3}{2}]_c$												$4\sqrt{10}$	$-\frac{4}{3}$			
$\frac{3}{2} [\frac{5}{2}]$														$-\frac{40}{3}$		
$\frac{5}{2} [\frac{1}{2}]$															$-\frac{40}{3}$	
$\frac{5}{2} [\frac{3}{2}]$																$-\frac{40}{3}$

TABLE III. Similar as Table II but for the expectational value of spin-flavor interaction $\sum_{i<j\leq 5} T_{ij} \Sigma_{ij}$.

	$\frac{1}{2} [\frac{1}{2}]_a$	$\frac{1}{2} [\frac{1}{2}]_b$	$\frac{1}{2} [\frac{1}{2}]_c$	$\frac{1}{2} [\frac{3}{2}]_a$	$\frac{1}{2} [\frac{3}{2}]_b$	$\frac{1}{2} [\frac{3}{2}]_c$	$\frac{1}{2} [\frac{5}{2}]$	$\frac{3}{2} [\frac{1}{2}]_a$	$\frac{3}{2} [\frac{1}{2}]_b$	$\frac{3}{2} [\frac{1}{2}]_c$	$\frac{3}{2} [\frac{3}{2}]_a$	$\frac{3}{2} [\frac{3}{2}]_b$	$\frac{3}{2} [\frac{3}{2}]_c$	$\frac{3}{2} [\frac{5}{2}]$	$\frac{5}{2} [\frac{1}{2}]$	$\frac{5}{2} [\frac{3}{2}]$
$\frac{1}{2} [\frac{1}{2}]_a$	14	-4	-4													
$\frac{1}{2} [\frac{1}{2}]_b$	-4	10	8													
$\frac{1}{2} [\frac{1}{2}]_c$	-4	8	10													
$\frac{1}{2} [\frac{3}{2}]_a$				4	$-2\sqrt{10}$	$-2\sqrt{10}$										
$\frac{1}{2} [\frac{3}{2}]_b$				$-2\sqrt{10}$	10	2										
$\frac{1}{2} [\frac{3}{2}]_c$				$-2\sqrt{10}$	2	2										
$\frac{1}{2} [\frac{5}{2}]$							-6									
$\frac{3}{2} [\frac{1}{2}]_a$								4	$-2\sqrt{10}$	$-2\sqrt{10}$						
$\frac{3}{2} [\frac{1}{2}]_b$								$-2\sqrt{10}$	10	2						
$\frac{3}{2} [\frac{1}{2}]_c$								$-2\sqrt{10}$	2	2						
$\frac{3}{2} [\frac{3}{2}]_a$											-5	5	$\sqrt{10}$			
$\frac{3}{2} [\frac{3}{2}]_b$											5	-5	$\sqrt{10}$			
$\frac{3}{2} [\frac{3}{2}]_c$											$\sqrt{10}$	$\sqrt{10}$	8			
$\frac{3}{2} [\frac{5}{2}]$														0		
$\frac{5}{2} [\frac{1}{2}]$															-6	
$\frac{5}{2} [\frac{3}{2}]$																0

$$\left\langle S_z = \frac{1}{2}, I = \frac{1}{2} \left| \sum_{i < j \leq 5} T_{ij} \Sigma_{ij} \right| S_z = \frac{1}{2}, I = \frac{1}{2} \right\rangle = \begin{pmatrix} 14 & -4 & -4 & 0 & 0 & 0 & 0 \\ -4 & 10 & 8 & 0 & 0 & 0 & 0 \\ -4 & 8 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -2\sqrt{10} & -2\sqrt{10} & 0 \\ 0 & 0 & 0 & -2\sqrt{10} & 10 & 2 & 0 \\ 0 & 0 & 0 & -2\sqrt{10} & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{pmatrix} \quad (39)$$

V. NUMERICAL RESULTS

In this section, we will combine the results given above, for the basis states and color-spin interactions, and present the numerical results for the resulting wave functions, the eigenstate of full light cone Hamiltonian. Additionally, we will show the spectrum of light pentaquark state and comparison to the experimental results.

A. Light cone wave function

To diagonalize the light front Hamiltonian (22), we choose the eigenbasis as

$$\Psi_{\tilde{n},n,m}(x_\xi, k_{\xi\perp}) = \varphi_{\tilde{n}}[\tilde{s}] \psi_{n,m}(k_{\xi\perp}, \beta) \quad (40)$$

where $\xi = \alpha, \beta, \gamma, \delta$ represents the label of Jacobi coordinates, $\varphi_{\tilde{n}}[\tilde{s}]$ and $\psi_{n,m}(k_{\xi\perp}, \beta)$ are given in (16) and (23), respectively. The longitudinal and transverse eigenstates are truncated using the following constraints

$$E_L = \sum_{i=2}^5 \tilde{n}_{i1}^2 - \frac{1}{5} \left(\sum_{i=2}^5 \tilde{n}_{i1} \right)^2 \leq \frac{1184\pi^2}{5},$$

$$E_T = \sum_{i=1}^4 (2n_i + |m|_i + 1) \leq 14 \quad (41)$$

In the following, we will set $m_i = 0$ since we are interested in the S -wave pentastates. The above cutoffs include 134 states and 126 states in the longitudinal and transverse directions, respectively. Eventually, the ground state eigenfunction can be expressed as

$$\Phi(x_\xi, k_{\xi\perp}) = \sum_{\tilde{n},n} C_{\tilde{n},n} \Psi_{\tilde{n},n,0}, \quad (42)$$

where the coefficient $C_{\tilde{n},n}$ can be obtained by diagonalizing the light cone Hamiltonian defined in Eq. (6) using the eigenbasis we defined before. The constituent in the vacuum at low resolution runs with the momentum, but at zero momentum it is tied to the vacuum condensate [11]

$$m_Q = \frac{g_S |\langle \bar{\psi} \psi \rangle|}{N_c} \rightarrow 383 \pm 39 \text{ MeV} \quad (43)$$

The string tension is fixed either from quarkonia spectroscopy or from meson Reggion trajectories (by the rho mass)

$$\sigma_T = \frac{m_\rho^2}{\pi} \rightarrow (0.44 \text{ GeV})^2 \quad (44)$$

In Fig. 2, we show the dependence of the ground mass on the einbein variable a . The minimal value of the ground state mass sets in around 7.59, which we will use from hereon.

In Fig. 3 we show the longitudinal wave function of the ground pentastate obtained by diagonalizing the full Hamiltonian. The longitudinal variables x_γ and x_δ are set to $x_\gamma = x_\delta = -0.2$. The corresponding Jacobi transverse momenta are labeled as $k_\perp = (|k_\alpha|, |k_\beta|, |k_\gamma|, |k_\delta|)$.

B. Parton distribution function in light pentastates

The parton distribution function(PDF) carried by the 5-antiquark inside the pentastate follows from the LF wave function $\Psi_{\tilde{n},n,m}$,

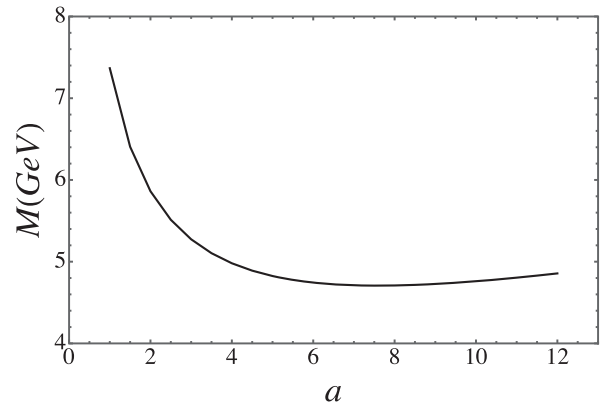


FIG. 2. Dependence of ground pentastate mass on the einbein parameter a .

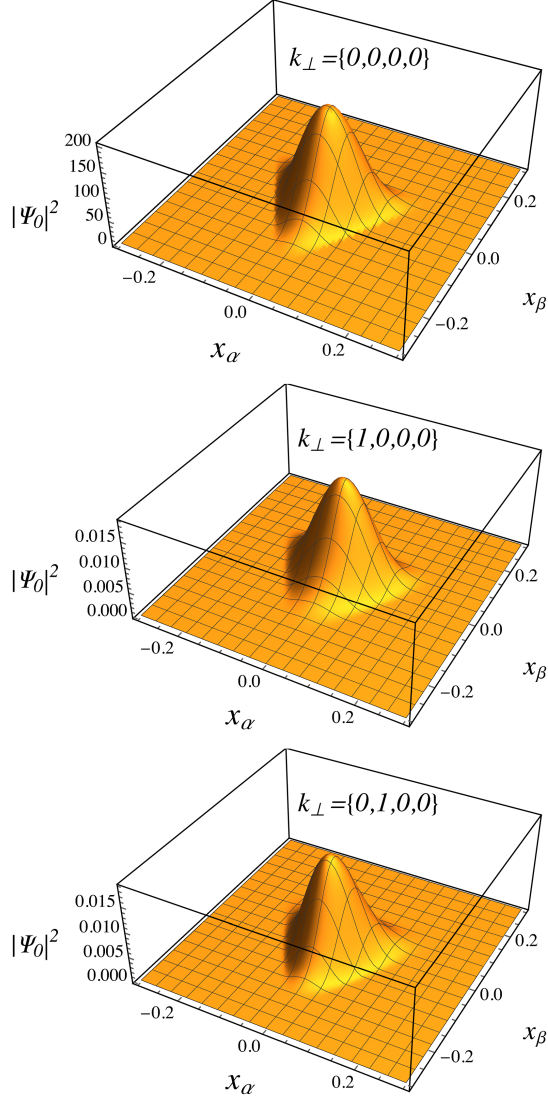


FIG. 3. The ground state wave function squared of full Hamiltonian as a function of two Jacobi coordinates x_α , x_β . (Other longitudinal variables x_γ and x_δ are set to $x_\gamma = x_\delta = -0.2$.) The corresponding values of the transverse momentum are labeled in the legend as $k_\perp = (|k_\alpha|, |k_\beta|, |k_\gamma|, |k_\delta|)$.

$$f(x_5) = \frac{\sqrt{5}}{2} \int dx_\xi d^2 \vec{k}_{\xi\perp} |\Phi(x_\xi, \vec{k}_{\xi\perp})|^2 \delta\left(x_\delta + \frac{5x_5 - 1}{2\sqrt{5}}\right) \quad (45)$$

where $dx_\xi = dx_\alpha dx_\beta dx_\gamma dx_\delta$ and similarly for $d^2 \vec{k}_{\xi\perp}$. The delta-constraint follows from (4). The above PDF satisfies the normalization condition $\int_0^1 dx_5 f(x_5) = 1$, which follows the LF wave function normalization

$$\int dx_\xi d^2 \vec{k}_{\xi\perp} |\Phi(x_\xi, \vec{k}_{\xi\perp})|^2 = 1.$$

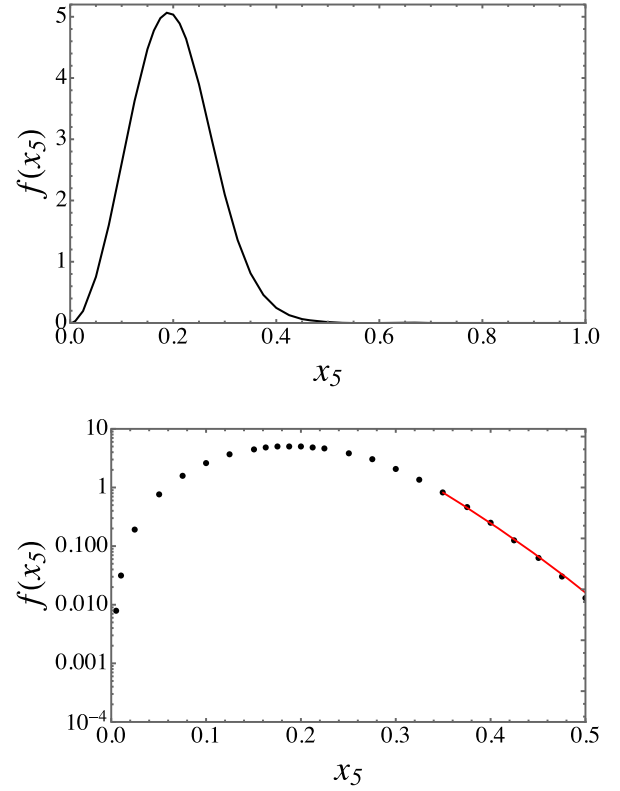


FIG. 4. The upper panel shows the quark PDF for S -wave pentastates. The lower panel shows a semilog plot of the same PDF. The solid red curve shows the asymptotic behavior at large x_5 , which is given by $(1 - x_5)^{15}$.

The quark PDF in the light pentastate $f(x_5)$ is shown in Fig. 4 (upper panel), with a tail of $(1 - x_5)^{15}$ (lower panel). Most of the PDF support is within $0 < x < 0.6$, with a peak at about $x_5 \sim 0.2$, consistent with the expectation of equal momentum fraction between the 5 constituents. The PDF at large x_5 asymptotes $\sim (1 - x_5)^{15}$. Note that for the S -wave analyzed here, there is no difference between the quark and antiquark PDFs in the pentastate.

C. Light pentastates spectrum

To estimate the pentastate spectra, we now fix the strengths of the hyperfine interactions. Assuming the Delta-nucleon mass splitting to be totally due to the color-spin interaction, we have

$$M_\Delta - M_N = \frac{16V_{1g}}{m_Q^2} \rightarrow (1232 - 939) \text{ MeV} \quad (46)$$

For the spin-flavor interaction we set [8]

$$C_\chi = 21 \text{ MeV} \quad (47)$$

The pentastate spectrum extracted from the LF analysis is shown in Table IV. The splittings for both the color-spin

TABLE IV. The S -state pentastates with isospin $I = \frac{1}{2}$ and spin assignments $\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$ and comparison to the experimental result (Exp) from PDG. The Spectrum1 and Spectrum2 represent the hadron mass spectrum obtained by considering spin-color interaction and spin-flavor interactions, respectively. The second and third columns show the energy splitting due to these interactions. The overall parameters c_0 and c_1 are determined by fixing the mass of $\mathbf{N}(1650)$.

	Spectrum1 (color-spin)	Spectrum2 (flavor-spin)	Spectrum1 ($c_0 = 1756$)	Spectrum2 ($c_1 = 1860$)	Exp
$(\frac{1}{2})^-$	$c_0 - 342$	$c_1 - 462$	1414	1398	$\mathbf{N}(1535)$
	$c_0 - 106$	$c_1 - 210$	1650	1650	$\mathbf{N}(1650)$
	$c_0 + 203$	$c_1 - 42$	1959	1818	$\mathbf{N}(1895)$
$(\frac{3}{2})^-$	$c_0 - 220$	$c_1 - 336$	1536	1524	$\mathbf{N}(1520)$
	$c_0 + 24$	$c_1 - 84$	1780	1776	$\mathbf{N}(1700)$
	$c_0 + 244$	$c_1 + 84$	2000	1944	$\mathbf{N}(1875)$
$(\frac{5}{2})^-$	$c_0 + 244$	$c_1 + 126$	2000	1986	$\mathbf{N}(1675)$

interaction (spetrum1) and the spin-flavor interaction (spetrum2), are listed for comparison. The results shown in the second and third column of Table IV, follow from the diagonalization of Eqs. (38) and (39), respectively. The color-spin and spin-flavor splittings are given in MeV.

The centroids parameters c_0 and c_1 are fitted. The reason is that our LF Hamiltonian is defined modulo a constant free shift. Indeed, the minimal pentastate following from our un-shifted LF Hamiltonian in Fig. 2 is about 4.7 GeV. This high mass value is readily shifted down by the free and unfixed constant.

To compare with the experimental results, we fix the overall parameter c_0 and c_1 using the mass of $\mathbf{N}(1650)$. The forth and fifth columns show our results for the different states with isospin $I = \frac{1}{2}$ and spin assignments $\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$. Most of these states are comparable with the empirically listed nucleon states in the PDG [12]. The exception is the spin $\frac{5}{2}^-$, which is 300 MeV heavier than the empirical one.

VI. CONCLUSIONS

The main content of the paper is an explicit construction of eigenbasis for the LF Hamiltonian with confining interactions. It is based on a map from another simplex corresponding to 5 fermions on the circle, and expressed by Slater determinants. In terms of Jacobi coordinates the basis states all satisfy exact Dirichlet boundary conditions on the regular 4-simplex or pentachoron. This method can be extended to any N particle states.

This eigenbasis is free from the CM motion both for longitudinal and transverse motion, contrary to other approaches using basis LF Hamiltonians [2,13,14]. We further constructed the light pentastates on the LF using

Hamiltonian as a matrix in this basis, with subsequent diagonalization.

In addition, we also considered the hyperfine color-spin and flavor-spin interactions. In the absence of hyperfine interactions, there are 16 degenerate S -pentastates. They are constructed using the representation of the permutation group on S_4 , to enforce Pauli statistics among the 4 light quark constituents. The degeneracy is lifted in the presence of color-spin or flavor-spin interactions. A generic set of ultralocal hyperfine interactions, yield pentastates that are in fair agreement with empirically reported nucleon excited states carrying the same quantum numbers.

The ground state pentastate leads to antiquark PDF with a peak at $x \approx 0.2$, consistent with the expectation that the five constituents share the LF momentum equally. This PDF displays a hard edge of the form $(1-x)^{15}$ at large x .

ACKNOWLEDGMENTS

F. H. is supported by the National Science Foundation under Award No. PHY-2412963. W. W. is supported by Tsinghua Scholarship for Overseas Graduate Studies. E. S. and I. Z. are supported by the Office of Science, U.S. Department of Energy under Contract No. DE-FG-88ER40388. This research is also supported in part within the framework of the Quark-Gluon Tomography (QGT) Topical Collaboration, under Contract No. DE-SC0023646.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

- [1] N. Miesch, E. Shuryak, and I. Zahed, *Phys. Rev. D* **112**, 074017 (2025).
- [2] P. Wiecki, Y. Li, X. Zhao, P. Maris, and J. P. Vary, *Phys. Rev. D* **91**, 105009 (2015).
- [3] N. Miesch, E. Shuryak, and I. Zahed, [arXiv:2510.23404](#).
- [4] E. Shuryak and I. Zahed, *Phys. Rev. D* **107**, 034026 (2023).
- [5] M. Girardeau, *J. Math. Phys. (N.Y.)* **1**, 516 (1960).
- [6] H. R. Krishnamurthy, H. Mani, and H. C. Verma, *J. Phys. A* **15**, 2131 (1982).
- [7] L. Y. Glozman and D. O. Riska, *Phys. Rep.* **268**, 263 (1996).
- [8] C. Helminen and D. O. Riska, *Nucl. Phys.* **A699**, 624 (2002).
- [9] N. Isgur, *Phys. Rev. D* **62**, 054026 (2000).
- [10] L. Y. Glozman, [arXiv:nucl-th/9909021](#).
- [11] A. Kock, Y. Liu, and I. Zahed, *Phys. Rev. D* **102**, 014039 (2020).
- [12] S. Navas *et al.* (Particle Data Group), *Phys. Rev. D* **110**, 030001 (2024).
- [13] J. P. Vary, H. Honkanen, J. Li, P. Maris, S. J. Brodsky, A. Harindranath, G. F. de Teramond, P. Sternberg, E. G. Ng, and C. Yang, *Phys. Rev. C* **81**, 035205 (2010).
- [14] S. Jia and J. P. Vary, *Phys. Rev. C* **99**, 035206 (2019).