

Erratum: Action-angle variables of a binary black hole with arbitrary eccentricity, spins, and masses at 1.5 post-Newtonian order
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This erratum serves to correct mistakes in the original publication in regard to calculating the leading post-Newtonian (PN) order contribution of the fifth action integral.

The sketch of the calculation is laid out in Sec. V of the original article in steps 1–5. The first correction pertains to step 2. Replace “As a first step, series expand the cubic expression of Eq. (A21), and its roots, keeping terms up to $\mathcal{O}(\epsilon^2)$. Expansion of the roots up to $\mathcal{O}(\epsilon^2)$ is necessary because the turning points f_1 and f_2 coincide at lower orders.” with “As a first step, series expand the roots of the cubic expression of Eq. (A21), keeping terms up to $\mathcal{O}(\epsilon^2)$.” The above correction basically tells the reader not to series expand the cubic expression but only its roots up to $\mathcal{O}(\epsilon^2)$. This is the main error in the article, from which other errors originate.

We now proceed to correct the expression of the fifth action’s leading PN order contribution (given by Eqs. (47)–(49) in the original article). With the following new definitions,

$$\mathcal{A} = -J^2 + L^2 + S_1^2 + S_2^2, \quad (1)$$

$$\mathcal{B}_1 = 2S_{\text{eff}} \cdot L + \mathcal{A}\sigma_1, \quad (2)$$

$$\mathcal{B}_2 = 2S_{\text{eff}} \cdot L + \mathcal{A}\sigma_2, \quad (3)$$

$$\mathcal{C}_1 = \sqrt{L^2 S_2^2 - \frac{\mathcal{B}_1^2}{4(\sigma_1 - \sigma_2)^2}}, \quad (4)$$

$$\mathcal{C}_2 = \sqrt{L^2 S_1^2 - \frac{\mathcal{B}_2^2}{4(\sigma_1 - \sigma_2)^2}}, \quad (5)$$

$$\mathcal{D} = \sqrt{-\left(\frac{-\mathcal{B}_1^2 - \mathcal{B}_2^2 - 4(2\mathcal{C}_1\mathcal{C}_2 - L^2(S_1^2 + S_2^2))(\sigma_1 - \sigma_2)^2}{\mathcal{B}_1^2 + \mathcal{B}_2^2 - 4(2\mathcal{C}_1\mathcal{C}_2 + L^2(S_1^2 + S_2^2))(\sigma_1 - \sigma_2)^2}\right)}, \quad (6)$$

the corrected expression of the fifth action variable at the leading PN order becomes

$$\begin{aligned} \mathcal{J}_5 \sim & \frac{1}{4L(\sigma_1 - \sigma_2) \sqrt{1 + \frac{4\mathcal{C}_1\mathcal{C}_2}{-2\mathcal{C}_1\mathcal{C}_2 - L^2(S_1^2 + S_2^2) + \frac{\mathcal{B}_1^2 + \mathcal{B}_2^2}{4(\sigma_1 - \sigma_2)^2}} (\mathcal{B}_1^2 + \mathcal{B}_2^2 - 4(2\mathcal{C}_1\mathcal{C}_2 + L^2(S_1^2 + S_2^2))(\sigma_1 - \sigma_2)^2)}} \\ & \times [-\mathcal{B}_1^3(\mathcal{D} - 1) - \mathcal{B}_2^3(\mathcal{D} - 1) - \mathcal{B}_1^2\mathcal{B}_2(\mathcal{D} + 1) - \mathcal{B}_1\mathcal{B}_2^2(\mathcal{D} + 1) \\ & + 4(\sigma_1 - \sigma_2)^2(\mathcal{B}_1(2\mathcal{C}_1\mathcal{C}_2\mathcal{D} + L^2((\mathcal{D} + 1)S_1^2 + (\mathcal{D} - 1)S_2^2)) + \mathcal{B}_2(2\mathcal{C}_1\mathcal{C}_2\mathcal{D} + L^2((\mathcal{D} - 1)S_1^2 + (\mathcal{D} + 1)S_2^2)))]. \quad (7) \end{aligned}$$

Just like in the original article, because of the assumption $m_1 > m_2$, this expression is not manifestly symmetric with respect to the label exchange $1 \leftrightarrow 2$, due to the $(\sigma_1 - \sigma_2)$ term in the denominator. However, replacing $(\sigma_1 - \sigma_2)$ with $-|\sigma_1 - \sigma_2|$ restores the symmetry.

In the end, we would also like to withdraw the claim made in the first and the last paragraphs of Sec. V, that the expression of the leading PN order of the fifth action can be used to write the 1.5PN order Hamiltonian explicitly in terms of all the

action integrals. This is so because the corrected expression of $\mathcal{J}_5$ presented above is more complicated than the incorrect one presented in Eq. (47) of the original article. Hence, it is no longer apparent that a quartic equation in $S_{\text{eff}} \cdot L$ (with other action variables as parameters) can indeed be written. In the original article, this quartic equation was the bridge between $\mathcal{J}_5$ and the Hamiltonian as an explicit function of the action variables. However, this realization poses no hurdle to computing the frequencies of the system, and can be done by inverting a certain Jacobian matrix (see Sec. VI of the original article).

Other results and conclusions of the original article remain unaffected.