

Effective field theory for thermal QCD with 2 + 1 flavors

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We write a long-distance effective field theory (EFT) for QCD at finite temperature just below the crossover temperature T_{co} . The low energy constants of this EFT are obtained from lattice measurements of the screening mass of pions at two temperatures for $N_f = 2 + 1$ using lattice results obtained at physical values of pion and Kaon masses, and $N_f = 2$ where the lattice simulations were performed with a heavier pion mass. The EFT gives good predictions for other static pion properties for $N_f = 2$, where lattice results are available. We show the corresponding predictions for $N_f = 2 + 1$, where they are not yet measured. We demonstrate that EFT gives excellent predictions for the phase diagram in $N_f = 2 + 1$. The predictions for the pressure are investigated, and predictions are also given for a Wick-rotated real-time quantity called the kinetic mass.

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I. THE EFFECTIVE FIELD THEORY

Extremely detailed results for thermal QCD are now available from lattice computations at finite temperature, T . However there are parts of the phase diagram of QCD which remain outside the reach of direct lattice computations. Among the outstanding problems is to compute directly the phase diagram at finite (real) baryon chemical potential. A sign also arises when trying to compute the phase diagram at finite isospin chemical potential when the light quarks are allowed to take different masses. A much bigger sign problem arises in trying to compute the real-time dynamics of thermal QCD on the lattice. This analytic continuation from the Euclidean to Minkowski metric promises to provide the answer to many questions of dynamics near equilibrium. While we are unable to answer

all the questions that a complete method would permit, we explore one avenue of systematic expansions. This is to use a low-energy effective field theory (EFT) to capture accurately the physics below a UV cutoff Λ . The effects of the UV modes are captured in the low energy constants (LECs) which appear in the Lagrangian of the EFT. The LECs are tuned using lattice computations at finite temperature, and the EFT Lagrangian is then used to extract physics below the scale Λ in domains where lattice methods are unavailable.

Notionally, an EFT is obtained by integrating out modes of a fundamental theory such as QCD down to the UV cutoff Λ . It is natural that the LECs of the EFT will depend on T since the time direction is compactified to size $1/T$. Only if $T \ll \Lambda$ can one assume that the modes being integrated out are insensitive to T . In this case the LECs may be taken to be approximately independent of T . But when $T \gg \Lambda$ then one obtains a dimensionally reduced theory, which is quite different from the EFT at $T = 0$. Here we treat the case where Λ and T are comparable. As a result, neither approximation works. In fact, the EFT should exhibit space-time symmetries which are intermediate between $T = 0$ and the dimensionally reduced theory. This is the kind of EFT that this paper deals with: one with $T/\Lambda = \mathcal{O}(1)$. The nomenclature in this paper is the

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following: $T = 0$ EFT will mean EFTs for $T \ll \Lambda$, $T = \infty$ EFT will mean EFTs for $T \gg \Lambda$, and $T > 0$ EFT will mean EFTs for the intermediate $T = \mathcal{O}(\Lambda)$. A similar nomenclature is used when we talk of $T = 0$ versus $T > 0$ LECs.

To the best of our knowledge, the nature of space time symmetry of $T > 0$ EFTs was first explored for QED [1]. For an EFT in the presence of a heat bath, it was observed that Lorentz invariance has to be given up because there exists a special frame in which the heat bath is at rest [1]. What remains is invariance under spatial rotations and the discrete groups C, P and T. A relativistic theory of matter at such T will remain Lorentz covariant, and the important issue of the counting of mass dimensions of operators will be the same as in a theory in vacuum. Two key physics issues are easily captured in such a formulation. First, that the difference between a pole and a screening mass is captured through a LEC. Second, if we are interested in a gauge theory, such as QED in matter, then gauge invariance can allow longitudinal polarization, and hence change the polarization sums in loops.

At $T = 0$ it has been argued that a finite glueball mass allows neglect of color dynamics in the low energy EFT [2]. The glue sector remains gapped and massive glueballs persist in lukewarm QCD, i.e., for $T > 0$ but in the hadron phase [3]. As a result, it may be argued that the effect of gluons may also be neglected at finite temperature. Alternatively, one may treat the neglect of color dynamics in the EFT as a working assumption, whose eventual test will be in the agreement with phenomena. As a result, we should be able to write an EFT for quarks without color dynamics. From this one can obtain the thermal version of χ PT by the usual means, albeit with space-time symmetries typical of this intermediate range of T .

In an earlier paper [4], we wrote a bottom-up EFT for two-flavor QCD at finite temperature which captured the chiral symmetry breaking and restoration involved in its phase diagram. The chiral symmetry group of relevance is determined by the texture zeroes of the quark mass matrix. With N_f flavors of chiral quarks, the global symmetry of QCD is $U_B(1) \times SU_L(N_f) \times SU_R(N_f)$. An EFT which

realizes all these symmetries below a UV cutoff Λ can be written in terms of effective quark field spinors ψ with four Dirac components, N_f flavor components, and N_c color components, resulting in a net dimension $\mathcal{N} = 4N_fN_c$. A straightforward implementation would be to use only Dirac and flavor indices on the effective quark fields. However, we would like to use the N_c counting of various quantities, so we continue to carry a color index, under which we only allow trivial transformations. All the composite operators we use are similarly color blind. The use of effectively color singlet quark fields allows us to couple the EFT to chemical potentials, and thereby extend our work to other parts of the phase diagram.

In order to build the two-flavor EFT we wrote first the mass and kinetic terms

$$L_3 = d_3 \Lambda \bar{\psi} \psi, \quad \text{and} \quad L_4 = \bar{\psi} \not{\partial}_4 \psi + d_4 \bar{\psi} \not{\nabla} \psi, \quad (1)$$

where $\mathbf{m} = d_3 \Lambda \mathcal{I}$ is the mass matrix for two degenerate quarks, each of mass $m_0 = d_3 \Lambda$. Through this paper we shall use the indices $i = 1, 2$, and 3 for the components of spatial vectors and the index 4 for the Euclidean time. The notation used in L_4 is $\not{\partial}_4 = \gamma_4 \partial_4$ and $\not{\nabla} = \gamma_i \partial_i$, where repeated dummy indices are summed. We follow the conventions of [4,5]. The first term in L_4 would define the normalization of the quark field in any future top-down attempt to derive the EFT from QCD. The appearance of the LEC d_4 in L_4 is the origin of the difference between a screening mass and a pole mass. This important aspect of thermal physics arises from the breaking of boost invariance. The subscripts on the pieces of the Lagrangian L denote the mass dimension, D , of the operators. For $D = 3$ and 4, these terms exhaust all the operators allowed by the symmetries.

There are no terms allowed by the symmetries for $D = 5$. For $D = 6$ there are two kinds of terms— L_6^0 and L_6^3 where the superscript counts the number of derivatives in the operators. For $N_f = 2$ we have

$$\begin{aligned} L_6^0 = & \frac{d_{6,1}}{\Lambda^2} [(\bar{\psi} \psi)^2 + (\bar{\psi} \tau^a (i\gamma_5) \psi)^2] + \frac{d_{6,2}}{\Lambda^2} [(\bar{\psi} (i\gamma_5) \psi)^2 + (\bar{\psi} \tau^a \psi)^2] \\ & + \frac{d_{6,3}}{\Lambda^2} (\bar{\psi} \gamma_4 \psi)^2 + \frac{d_{6,4}}{\Lambda^2} (\bar{\psi} (i\gamma_i) \psi)^2 + \frac{d_{6,5}}{\Lambda^2} (\bar{\psi} \gamma_4 \gamma_5 \psi)^2 + \frac{d_{6,6}}{\Lambda^2} (\bar{\psi} (i\gamma_i \gamma_5) \psi)^2 \\ & + \frac{d_{6,7}}{\Lambda^2} [(\bar{\psi} \tau^a \gamma_4 \psi)^2 + (\bar{\psi} \tau^a \gamma_4 \gamma_5 \psi)^2] + \frac{d_{6,8}}{\Lambda^2} [(\bar{\psi} \tau^a (i\gamma_i) \psi)^2 + (\bar{\psi} \tau^a (i\gamma_i \gamma_5) \psi)^2] \\ & + \frac{d_{6,9}}{\Lambda^2} [(\bar{\psi} (iS_{i4}) \psi)^2 + (\bar{\psi} \tau^a S_{ij} \psi)^2] + \frac{d_{6,10}}{\Lambda^2} [(\bar{\psi} \tau^a (iS_{i4}) \psi)^2 + (\bar{\psi} S_{ij} \psi)^2], \end{aligned} \quad (2)$$

where S_{ij} and S_{i4} are defined in [4]. The operators with LEC $d_{6,1}$ appear in the Nambu-Jona-Lasinio (NJL) model. The rest of the operators are all allowed by the symmetries of the problem. This is one of the drawbacks of building bottom-up EFTs: there is a proliferation of terms and LECs that have to be tamed by other means. The other piece of the $D = 6$ Lagrangian is

$$L_6^3 = \frac{d_{6,11}}{\Lambda^2} \bar{\psi} \nabla^2 \not{\nabla} \psi. \quad (3)$$

All other terms with three derivatives can be reduced to this using the equations of motion or eliminated by the symmetries. The Lagrangian of the EFT up to $D = 6$ is $L = L_3 + L_4 + L_6^0 + L_6^3$. This is a sufficient starting point for $N_f = 2$. After spontaneous symmetry breaking it gives the correct SU(2) vector symmetry from which a pion EFT can be derived.

In this paper we discuss the extension to three chiral flavors, $N_f = 3$. Famously, the Lagrangian has an emergent symmetry $U_A(1)$, so that the Goldstone bosons are a nonet of pseudoscalars instead of the octet. It is well known in the NJL model that the UV symmetry is obtained when the 't Hooft determinant term is added (see for example the review of [6]). Since this term has $D = 9$, in the EFT approach this means that one has to include all the terms allowed up to $D = 9$. In the next section, we present details. This has a possibility of a first order phase transition. We postpone an account of this to a follow-up paper.

When the texture $N_f = 3$ is broken to $N_f = 2$, the theory is called $N_f = 2 + 1$, although at sufficiently deep infrared (IR) it is clearly an $N_f = 2$ EFT. The cutoff of the $N_f = 2 + 1$ flavor theory should lie above the K meson mass and below the proton mass. We call this Λ_{2+1} . Writing the mass matrix for the $N_f = 2 + 1$ theory as $\mathbf{m} = \text{diag}\{d_3^\ell, d_3^\ell, d_3^s\} \Lambda_{2+1}$, we know that its elements follow a hierarchy $d_3^\ell \ll d_3^s \ll 1$ at $T = 0$, and assume the same for $T > 0$. In the next section we describe the structure of the $N_f = 2 + 1$ theory, and describe how the effective Lagrangian for pseudo-Goldstone bosons, L_{pGB} in $N_f = 2 + 1$ can be obtained from this. Deep in the IR, below a UV cutoff Λ_2 which lies between the π mass and the K mass, an effective Lagrangian for pions, L_π , is sufficient. In the next section we give an account of the reduction of L_{pGB} to L_π .

II. THE EFT WITH STRANGE QUARKS

In extending the quark EFT to $N_f = 3$, the first change is in the replacement of the flavor SU(2) generators τ^a by the flavor generators T^a (with $1 \leq a \leq 8$). The normalization $\text{Tr}(T^a)^2 = 2$ allows us to identify the Gell-Mann matrices λ^a with the T^a . The remaining generator of U(3) is $T^0 = (\sqrt{2/3})\mathbf{1}$. The L_3 term in the EFT changes to include three degenerate quarks. The form of the L_4 and L_6^3 terms remain unchanged. The most general L_6^0 terms have the form of Eq. (2) with $d_{6,1} = d_{6,2}$ and $d_{6,9} = d_{6,10}$. This is the origin of an emergent $U_A(1)$ symmetry.

In order to make contact with QCD one has to then add higher dimensional terms in the EFT until this extra symmetry is removed. When constructing all possible terms of $D = 5$ which are singlet under vector flavor

symmetry and under the discrete C, P, and T symmetries, we found only three terms: $\bar{\psi} \not{\partial}_4 \not{\partial}_4 \psi$, $\bar{\psi} \not{\partial}_4 \not{\nabla} \psi$, and $\bar{\psi} \not{\nabla} \not{\nabla} \psi$ [4]. Since the product of two gamma matrices commutes with γ_5 , they are not invariant under the flavor axial symmetry. As a result, there are no dimension five terms in the EFT.

One may write two kinds of $D = 7$ terms. One set which is invariant under vector flavor symmetry and C, P, and T is $\bar{\psi} \not{\partial}_4^i \not{\nabla}^j \psi$ with $i + j = 4$. They are not axially symmetric because the product of four gamma matrices commutes with γ_5 . The other kinds of terms with $D = 7$ are $\bar{\psi} \not{\nabla} \Gamma \psi \bar{\psi} \Gamma \psi$ and $\bar{\psi} \not{\partial}_4 \Gamma \psi \bar{\psi} \Gamma \psi$. Invariance under C, P, and T require Γ to be the same in both the bilinears, but Γ and $\not{\partial}_4 \Gamma$ (or $\not{\nabla} \Gamma$) have opposite anticommutation properties with γ^5 hence the product cannot be invariant under axial transformations.

For $D = 8$ we find three kinds of structures. One is bilinear in the quark fields and has five derivatives; we call this L_8^5 . One is a product of two quark bilinears, each with a first derivative operator; we call this L_8^{11} . The last one is a product of two quark bilinears, one without derivatives, the other with two; this we name L_8^2 . The number of allowed terms in L_8^{11} and L_8^2 is very large, so we do not write them down in detail. All the operators have the emergent U(3) symmetry.

Derivative terms are ruled out for $D = 9$ by a reasoning very similar to $D = 7$. One can consider two kinds of terms with derivatives. One set which is invariant under vector flavor symmetry and C, P, and T is $\bar{\psi} \not{\partial}_4^i \not{\nabla}^j \psi$ with $i + j = 6$. They are not axially symmetric because the product of four gamma matrices commutes with γ_5 . The other kinds of terms with $D = 9$ are $\bar{\psi} \not{\partial} \Gamma \psi \bar{\psi} \not{\partial} \Gamma \psi$ and $\bar{\psi} \not{\partial}_\mu \Gamma \psi \bar{\psi} \not{\partial}_\mu \Gamma \psi$ where ∂ can refer to ∂_4 or ∇ and similarly μ can run from 1 to 4. Invariance under C, P, and T require Γ to be the same in both the bilinears, but then the combination of Dirac matrices in the two fermion bilinears has opposite anticommutation properties with γ^5 . Hence the product cannot be invariant under axial transformations. Then there are exactly two terms allowed at $D = 9$, both of which restore the axial symmetry to SU(3),

$$L_9 = \epsilon_{ff'f''} \epsilon_{gg'g''} (\bar{\psi}^f P_R \psi^g) \left[\frac{d_{9,1}}{\Lambda^5} (\bar{\psi}^{f'} P_R \psi^{g'}) (\bar{\psi}^{f''} P_R \psi^{g''}) + \frac{d_{9,2}}{\Lambda^5} (\bar{\psi}^{f'} P_R S_{ij} \psi^{g'}) (\bar{\psi}^{f''} P_R S_{ij} \psi^{g''}) \right] + (L \leftrightarrow R), \quad (4)$$

where P_R is the projection operator on right-handed quarks. The first term was obtained by 't Hooft [7] and the second by Schaefer [8]. No other terms of this order are allowed by the symmetries of QCD (see [9] for details). Any nonzero values of $d_{9,i}$ lift the accidental degeneracy.

We have used the equations of motion (EOMs) systematically in this construction. Up to $D = 4$, the EOM is

$$\not{d}_4\psi = -[d_4\not{\nabla} + \mathbf{m}]\psi, \quad (5)$$

where $\mathbf{m}$ is the mass matrix. The smallness of $\mathbf{m}/\Lambda$ is used in the same way as in the vacuum theory. This has no corrections at $D = 5$ since there are no terms of this order. So it can be used in $D = 6$ to eliminate all operators except the one shown in Eq. (3). In the vacuum theory, of course, the EOM can be used to remove all derivative terms at order $D = 6$. The key difference from fermionic theories in vacuum is that the spatial and temporal derivatives are independent operators. Beyond this we need to use the EOM with all terms up to $D = 6$ to simplify the $D = 8$ terms, since there are no terms with $D = 7$. There are no derivative terms at $D = 9$, so the EOMs are not involved.

The Lagrangian $L = L_3 + L_4 + L_6^3 + L_6^0 + L_8^5 + L_8^{11} + L_8^2 + L_9$ can be treated in a Hartree-Fock (HF) approximation. This converts all terms into quadratics in the Fermion fields once the chiral condensate $\langle\bar{\psi}\psi\rangle = \Lambda^3\sigma$ is introduced. In the Hartree-Fock approximation one finds for a quartic term with two flavor-Dirac matrices Θ and Θ' ,

$$\begin{aligned} (\bar{\psi}\Theta\psi)(\bar{\psi}\Theta'\psi) &\stackrel{\text{HF}}{=} -\Lambda^6\sigma^2[(\text{Tr}\Theta)(\text{Tr}\Theta') - \text{Tr}\Theta\Theta'] \\ &\quad + \Lambda^3\sigma[(\text{Tr}\Theta)\bar{\psi}\Theta'\psi \\ &\quad + (\text{Tr}\Theta')\bar{\psi}\Theta\psi - 2\bar{\psi}\Theta\Theta'\psi]. \end{aligned} \quad (6)$$

Using this, we find

$$\begin{aligned} L_{\text{HF}} = & -\mathcal{N}\Lambda^4\left(d_6\sigma^2 + \frac{2}{3}d_9\sigma^3\right) + m\bar{\psi}\psi + \bar{\psi}\not{d}_4\psi \\ & + d_4\bar{\psi}\not{\nabla}\psi + \frac{d_{6,11}}{\Lambda^2}\bar{\psi}\nabla^2\not{\nabla}\psi + \frac{d_8}{\Lambda^4}\bar{\psi}\nabla^4\not{\nabla}\psi, \end{aligned} \quad (7)$$

where effective LECs are

$$\begin{aligned} d_6 &= \mathcal{N}d_{6,1} - d_{6,3} + 3d_{6,4} + d_{6,5} - 3d_{6,6} \quad \text{and} \\ d_9 &= (1 + N_f)(2d_{9,1}(6 + \mathcal{N}) + 9d_{9,2}). \end{aligned} \quad (8)$$

The LECs d_4 , $d_{6,11}$ and d_8 are the same as in L_4 , L_6^3 and L_8^5 . Since the vacuum has translational invariance, there are no terms in L_{HF} from L_8^{11} and L_8^2 . The terms in d_6 and d_9 which are linear in $\mathcal{N}$ can be obtained in the Hartree approximation; the remaining come from exchange (Fock) terms. In terms of the Hartree-Fock effective LECs, the quark mass

$$m = (d_3 + 2d_6\sigma + d_9\sigma^2)\Lambda. \quad (9)$$

Since L_{HF} is quadratic in quarks, a one-loop evaluation of its free energy is exact. Since the terms in $d_{6,11}$ and d_8 are down by powers of Λ , we will use the remaining terms to define quark propagators, and treat these two terms in a perturbative expansion. With the free energy we can investigate the self-consistent solutions for σ , i.e., the gap equation. We can also find the phase structure of the theory in this approximation.

The cubic term in σ certainly opens up the possibility of a first order phase transition. In a forthcoming paper we will show that when d_9 is large enough to push η' beyond the UV cutoff Λ , then it pushes the first order transition to a region where the pseudo-Goldstone masses are around an MeV.

A. $N_f = 2 + 1$

For $N_f = 2 + 1$ it is useful to group the flavor generators into three sets. We reserve the notation T^a to mean $1 \leq a \leq 3$. The notation T^m will be used with $4 \leq m \leq 7$, and the remaining generators will be always written as T^8 and T^0 . We need to introduce the projection operator on the strange quark subspace, Π_s , and the complementary operator on the light quarks, $\Pi_\ell = 1 - \Pi_s$. Note that $\Pi^\ell T^a \Pi^\ell$ corresponds to τ^a in the light quark space, and vanishes in the strange quark space. This is an example of the more general fact that every generator is either zero or a multiple of identity in the one-dimensional strange quark space. Using $\Pi_{\ell,s}$ one can decompose every quark bilinear into a sum of two terms: one for the strange quark and the other for light quarks (see [9] for more details). Clearly, the mass matrix can be decomposed as

$$\bar{\psi}\mathbf{m}\psi = d_3^\ell\Lambda\bar{\psi}_\ell\psi_\ell + d_3^s\Lambda\bar{\psi}_s\psi_s, \quad (10)$$

where Λ stands for Λ_{2+1} in this subsection. The $D = 4$ terms decompose similarly, giving two LECs d_4^ℓ and d_4^s . The same happens in L_6^3 . However, in the products of bilinears which enter into L_6^0 , each LEC $d_{6,i}$ of the $N_f = 3$ flavors decomposed into the three LECs $d_{6,i}^{\ell\ell}$ with both bilinears in the light quark space, $d_{6,i}^{ss}$ with both bilinears for the strange quark, and $d_{6,i}^{\ell s}$ which is the product of a light quark operator and a strange quark operator. As an example, the NJL-model term decomposes as

$$\begin{aligned} &\frac{d_{6,1}}{\Lambda^2}[(\bar{\psi}\psi)^2 + (\bar{\psi}(i\gamma_5 T^i)\psi)^2] \\ &\longrightarrow \frac{d_{6,1}^{\ell\ell}}{\Lambda^2}[(\bar{\psi}_\ell\psi_\ell)^2 + (\bar{\psi}_\ell(i\gamma_5\tau^a)\psi_\ell)^2] \\ &\quad + \frac{d_{6,1}^{ss}}{\Lambda^2}\left[(\bar{\psi}_s\psi_s)^2 + \frac{4}{3}(\bar{\psi}_s(i\gamma_5)\psi_s)^2\right] \\ &\quad + \frac{d_{6,1}^{\ell s}}{\Lambda^2}[\bar{\psi}_s\psi_\ell\bar{\psi}_\ell\psi_s + \bar{\psi}_s(i\gamma_5 T^m)\psi_\ell\bar{\psi}_\ell(i\gamma_5 T^m)\psi_s]. \end{aligned} \quad (11)$$

The same kind of structure is found for L_8^{11} and L_8^2 , but L_8^5 decomposes like L_6^3 . The flavor determinants in L_9 ensure that there is no multiplicity of LECs $d_{9,i}$.

Next we examine the Hartree-Fock Lagrangian for $N_f = 2 + 1$. Since the condensate has the same symmetry as the mass term in Eq. (10), we may write the theory in terms of light and heavy condensates, σ_ℓ and σ_s , respectively. The matrix of condensates

$$\langle \bar{\psi}_a \psi_b \rangle = \Lambda^3 \Sigma, \quad \text{where } \Sigma = \sigma_\ell \Pi_\ell + \sigma_s \Pi_s. \quad (12)$$

For L_6^ℓ the decomposition works as for $N_f = 2$ [4]. Since the strange quark is in a one-dimensional subspace of the flavor space, the trace over flavor is trivial, and all traces in Eq. (6) reduce to Dirac traces.

The coupling between the light and heavy quarks comes only from L_6^s . Since the flavor structure for this can only involve T^m with $4 \leq m \leq 7$, using the flavor projection operators we can simplify this in the Hartree-Fock approximation to

$$\begin{aligned} & (\bar{\psi}_\ell T^m \Gamma \psi_s) (\bar{\psi}_s T^m \Gamma \psi_\ell) \\ & \stackrel{\text{HF}}{=} \Lambda^6 \sigma_\ell \sigma_s \text{Tr}(\Gamma \Gamma) - \Lambda^3 \sigma_\ell \bar{\psi}_s \Gamma \Gamma \psi_s \\ & \quad - \Lambda^3 \sigma_s [(\delta_{m4} + \delta_{m5}) \bar{\psi}_u \Gamma \Gamma \psi_u + (\delta_{m6} + \delta_{m7}) \bar{\psi}_d \Gamma \Gamma \psi_d], \end{aligned} \quad (13)$$

where Γ is a Dirac matrix, and we have used different spinors for the u, d, and s flavor components. We can pair the Dirac matrices into the following sets

$$\begin{aligned} & 1 + (i\gamma_5)^2, \quad \gamma_4^2 + (\gamma_4 \gamma_5)^2, \\ & (i\gamma_i)^2 + (i\gamma_i \gamma_5)^2, \quad (S_{ij})^2 + (iS_{ij} \gamma_5)^2, \end{aligned} \quad (14)$$

where we have used the relation $S_{k4} = \epsilon_{ijk4} S_{ij} \gamma_5$ in the last pair. Since γ_5 anticommutes with all the γ_μ , and $\gamma_5^2 = 1$, one finds that each pair gives a vanishing contribution to L_{HF} . So the mixing terms between the light and heavy sectors vanish in the HF theory because of the emergent symmetry. The coupling between the two condensates then comes only through the $D = 9$ term.

As a result, the Hartree-Fock Hamiltonian is

$$\begin{aligned} L_{\text{HF}} = & -\frac{1}{3} \mathcal{N} \Lambda^4 (2d_6^\ell \sigma_\ell^2 + d_6^s \sigma_s^2 + 2d_9 \sigma_\ell^2 \sigma_s) + m_\ell \bar{\psi}_\ell \psi_\ell + \bar{\psi}_\ell \not{D}_4 \psi_\ell + d_4^\ell \bar{\psi}_\ell \not{\nabla} \psi_\ell + \frac{d_{6,11}^\ell}{\Lambda^2} \bar{\psi}_\ell \nabla^2 \not{\nabla} \psi_\ell + \frac{d_8^\ell}{\Lambda^4} \bar{\psi}_\ell \nabla^4 \not{\nabla} \psi_\ell \\ & + m_s \bar{\psi}_s \psi_s + \bar{\psi}_s \not{D}_4 \psi_s + d_4^s \bar{\psi}_s \not{\nabla} \psi_s + \frac{d_{6,11}^s}{\Lambda^2} \bar{\psi}_s \nabla^2 \not{\nabla} \psi_s + \frac{d_8^s}{\Lambda^4} \bar{\psi}_s \nabla^4 \not{\nabla} \psi_s, \end{aligned} \quad (15)$$

where the definition of d_6 in Eq. (8) is replaced by

$$d_6^\ell = \frac{2}{3} \mathcal{N} d_{6,1}^\ell - d_{6,3}^\ell + 3d_{6,4}^\ell + d_{6,5}^\ell - 3d_{6,6}^\ell, \quad \text{and} \quad d_6^s = \frac{1}{3} \mathcal{N} d_{6,1}^{ss} - d_{6,3}^{ss} + 3d_{6,4}^{ss} + d_{6,5}^{ss} - 3d_{6,6}^{ss}. \quad (16)$$

Finally, the two effective masses are

$$m_\ell = (d_3^\ell + 2d_6^\ell \sigma_\ell + d_9 \sigma_\ell^2) \Lambda, \quad \text{and} \quad m_s = (d_3^s + 2d_6^s \sigma_s + d_9 \sigma_s^2) \Lambda. \quad (17)$$

A detailed analysis, which will be presented in a separate paper, shows that the coupling between the light and strange sectors causes a first order transition to appear at very small pseudo-Goldstone masses. However, for values of d_9 which push the η' mass above the UV cutoff, values of d_3^ℓ and d_3^s relevant to QCD have the same phase structure that was seen in the $N_f = 2$ theory [4].

B. The EFT of pseudo-Goldstone bosons

After chiral symmetry breaking one can introduce fluctuations about the condensates [5] through

$$\psi \rightarrow \mathcal{U} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \mathcal{U} \quad \text{where } \mathcal{U} = \exp[i\gamma_5 T^i \phi_i / (2f_i)], \quad (18)$$

T^i are the generators of the remaining vector flavor symmetry, and we have allowed for the possibility that the decay constants f_i have different values for f_a, f_m, f_8

and f_0 in the case of $N_f = 2 + 1$. Instead of using $\mathcal{U}$, one can project on the left and right spinors and use the transformations

$$\psi_L \rightarrow U \psi_L, \quad \psi_R \rightarrow U^\dagger \psi_R \quad \text{where } U = \exp[iT^i \phi_i / (2f_i)]. \quad (19)$$

The transformation matrix $\mathcal{U}$ is nontrivial in Dirac space, whereas the matrix U is only a flavor transformation and is trivial in Dirac space. The partition function for the group-valued field U uses its Haar measure.

Using U in L , along with the solutions of the gap equation obtained using L_{HF} gives a Lagrangian that couples the ϕ_i to the quark fields. However, this overcounts the degrees of freedom, since the ϕ_i are just a parametrization of the most easily excited fluctuations in the quark fields. So one needs to integrate over the quarks in order to reach the target, L_{pGB} , which is the action for the pseudo-Goldstone bosons. We find that it has the form

$$\begin{aligned}
L_{\text{pGB}} = & \frac{1}{2}\Lambda^2 [c_2^a \phi_a^2 + c_2^m \phi_m^2 + c_2^8 \phi_8^2 + c_2^0 \phi_0^2] + \frac{1}{2} [\dot{\phi}_a^2 + \dot{\phi}_m^2 + \dot{\phi}_8^2 + \dot{\phi}_0^2] \\
& + \frac{1}{2} [c_4^a (\nabla \phi_a)^2 + c_4^m (\nabla \phi_m)^2 + c_4^8 (\nabla \phi_8)^2 + c_4^0 (\nabla \phi_0)^2] + \frac{1}{8} [c_{41}^a \phi_a^4 + c_{41}^m \phi_m^4 + c_{41}^8 \phi_8^4 + c_{41}^0 \phi_0^4] \\
& + \frac{1}{4} [c_{41}^{am} \phi_a^2 \phi_m^2 + c_{41}^{a8} \phi_a^2 \phi_8^2 + c_{41}^{a0} \phi_a^2 \phi_0^2 + c_{41}^{m8} \phi_m^2 \phi_8^2 + c_{41}^{m0} \phi_m^2 \phi_0^2 + c_{41}^{80} \phi_8^2 \phi_0^2] + \dots
\end{aligned} \tag{20}$$

Since this describes pGBs for $N_f = 2 + 1$, in this subsection we continue to use the notation Λ for Λ_{2+1} . The quantum numbers of ϕ_0 and ϕ_8 allow mixing, but we will show in the next subsection that the η' can be decoupled easily and this mixing will not play a role. Integrating out the quarks to one-loop order one can write expressions for the LECs of the pseudo-Goldstone bosons in terms of those for the quark. In addition, by requiring the normalization of the time derivative terms to be as shown, one obtains the constants f_i in Eq. (19), by a natural extension of the argument in [4]. Since $d_3^s \gg d_3^\ell$, one can reduce L_{pGB} to a theory involving only pions, where the power counting is more straightforward. We have organized it in the mass dimension D , and written all the terms up to $D = 4$. By matching a sufficient number of these LECs to measurements from the lattice, one can derive the LECs of the quark theory.

1. The kinetic terms for pseudo-Goldstone bosons

We consider first the contributions to the kinetic terms in L_{pGB} . These can arise from L_4 , L_8^{11} and L_8^2 . To begin with, note that $UU^\dagger = 1$ implies that the combination $U(i\partial_\mu)U^\dagger$ is Hermitian in flavor space, and an expansion of the exponential shows that it reduces to $\partial_\mu \phi_i$. Contraction of the quark field operators in L_4 then gives for each field ϕ_i

$$\begin{aligned}
L_{\text{pGB}}(4) = & -\frac{\Lambda^2}{f_i^2} [\mathcal{I}_4(\dot{\phi}_i)^2 + d_4^2 \mathcal{I}_3(\nabla \phi_i)^2] \\
= & \frac{1}{2}(\dot{\phi}_i)^2 + \frac{1}{2}c_4^i(\nabla \phi_i)^2,
\end{aligned} \tag{21}$$

where we have suppressed the light and strange quark identifiers in d_4 and the integrals $\mathcal{I}_3$ and $\mathcal{I}_4$, which are given in Appendix C (also see [9] for further details). With all this, one finds simply

$$\begin{aligned}
\frac{f_a^2}{\Lambda^2} = & -2\mathcal{I}_4^{\ell\ell}, & c_4^a = & (d_4^\ell)^2 \frac{\mathcal{I}_3^{\ell\ell}}{\mathcal{I}_4^{\ell\ell}}, & \frac{f_m^2}{\Lambda^2} = & -2\mathcal{I}_4^{\ell s}, \\
c_4^m = & d_4^\ell d_4^s \frac{\mathcal{I}_3^{\ell s}}{\mathcal{I}_4^{\ell s}}, & \frac{f_8^2}{\Lambda^2} = & -\frac{2}{3}(\mathcal{I}_4^{\ell\ell} + 2\mathcal{I}_4^{ss}), & c_4^8 = & \frac{(d_4^\ell)^2 \mathcal{I}_3^{\ell\ell} + 2(d_4^s)^2 \mathcal{I}_3^{ss}}{\mathcal{I}_4^{\ell\ell} + 2\mathcal{I}_4^{ss}}.
\end{aligned} \tag{22}$$

For the light quarks this reproduces the results of [4].

L_8^{11} and L_8^2 can clearly give additional contributions to c_4^a , c_4^m and c_4^8 through the same mechanism. However, there are four quark fields to be contracted, so there are no contributions at one-loop order. The two and three loop integrals are complicated, but by dimensional arguments it can be shown that they are down by a power of $(T/\Lambda)^4 \times N_c/(4\pi^2)$ for each added loop order. The terms L_6^3 and L_8^5 contain higher derivative terms but only two quark fields. They give contributions to these LECs, but the extra derivatives act on the quarks and give more powers of momentum in the loops, and are therefore suppressed by $(T/\Lambda)^2$ and $(T/\Lambda)^4$, respectively.

2. Pseudo-Goldstone boson masses

We have neglected the LECs for the η' in the previous subsection, in anticipation of its decoupling. In this section we will show how it does so. Formally the LECs c_2 get contributions from L_3 and L_9 . Introducing fluctuations

through Eq. (19) gives the contribution of L_3 as

$$L_{\text{pGB}}(3) = \bar{\psi} \mathbf{M} \psi,$$

$$\text{where } \mathbf{M} = (U \mathbf{m} U - \mathbf{m}) P_L + (U^\dagger \mathbf{m} U^\dagger - \mathbf{m}) P_R, \tag{23}$$

where $P_{L,R}$ are the left and right helicity projectors for the quarks. To one loop order one can organize this by the number of insertions of $\mathbf{M}$ into the quark loop, as shown in Fig. 1. Note that every insertion of $\mathbf{M}$ can be decomposed by writing it as $(\Pi_\ell + \Pi_s) \mathbf{M} (\Pi_\ell + \Pi_s)$, so each of the topologies in Fig. 1 gives rise to diagrams with zero, one, or two strange quarks. A straightforward computation then gives

$$c_2^a = -\frac{2}{3} \mathcal{N} \left(\frac{\Lambda}{f_a} \right)^2 \left[d_3^\ell \sigma_\ell + \frac{1}{N_c} (d_3^\ell)^2 \mathcal{I}_1^{\ell\ell} \right], \tag{24}$$

where the integral $\mathcal{I}_1^{\ell\ell}$ is given in Appendix C. Since it is regular in the limit $d_3^\ell \rightarrow 0$, the pion mass vanishes in the

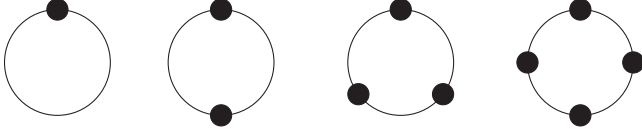


FIG. 1. First four terms organized by the number of insertions of $\mathbf{M}$ into the one-loop expression for $L_{\text{pGB}}(3)$. Each of these can be expanded in powers of the ϕ^i s. If we retain only terms up to the fourth power, then these are the only insertions which need to be considered.

chiral limit, and a thermal version of the Gell-Mann–Oakes–Renner relation is obtained from the leading term in Eq. (24). By systematically taking other flavor projections in $\mathbf{M}$ one similarly obtains c_2^m , c_2^8 and c_2^0 . It is interesting to take the $N_f = 2$ chiral limit by sending $d_3^\ell \rightarrow 0$ while holding d_3^s fixed. In this limit c_2^m , c_2^8 and c_2^0 are finite. Additionally, if d_3^s is small, then they are linear in d_3^s .

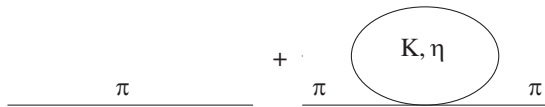
Clearly powers of ϕ_i in L_{pGB} can only come from L_3 , L_6^0 , and L_9 terms in the quark EFT, since the other terms all involve derivatives. L_6 is fully invariant under the symmetries, and hence gives no contributions in U . Since L_9 is invariant under axial SU(3) but not under the axial U(1) phase, it has a nonvanishing contribution which can be expanded in powers of ϕ_0 . Then using the fact that the expansion is made around the solution of the gap equation obtained through L_{HF} , and expanding to quadratic order in ϕ_0 , one finds only an additional contribution to c_2^0 from

$$L_{\text{pGB}}(9) = -\mathcal{N}d_9 \frac{\Lambda^4}{f_0^2} \sigma_\ell^2 \sigma_s \phi_0^2. \quad (25)$$

This is a pleasant result, since it shows that the mass of the undesired field ϕ_0 may be pushed above the UV cutoff Λ by tuning d_9 , without changing the rest of L_{pGB} since d_9 only appears explicitly here. With this, the mixing of ϕ_0 and ϕ_8 is also removed from the EFT, and the latter becomes the pure η meson state. The effect of d_9 continues to be felt in L_{pGB} since the solutions of the gap equations, namely the values of σ_ℓ and σ_s depend on d_9 . Since the η' mode can be decoupled easily, we do not consider it in the rest of this discussion.

3. The coupling terms for pseudo-Goldstone bosons

The LECs c_{41} , c_{41}^m , etc., come from the expansion of the exponentials in $\mathbf{M}$ as explained earlier. In [4] it was shown



that for $N_f = 2$ one-loop contributions to c_{41} have pieces which scale with different powers of d_3^ℓ , ranging from one to four. These come from the topologies shown in Fig. 1. In the $N_f = 2 + 1$ theory, the flavor projections are only an extra complication. Handling them is tedious but does not require new techniques. For the pion self-coupling, the result is

$$c_{41}^a = -\frac{m_\pi^2}{3f_a^2} + \frac{2}{3} \left(\frac{\Lambda}{f_a} \right)^4 (2d_3^\ell)^2 (4\mathcal{I}_1^{\ell\ell} + 3\mathcal{I}_2^{\ell\ell}) + \dots, \quad (26)$$

where the integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ are discussed in Appendix C, and we have written down the results from the first two topologies of Fig. 1. The remaining diagrams give contributions of higher order in d_3^ℓ . For the other couplings the propagators change due to the flavor projections at the vertices; leading to changes in these two integrals (see Appendix C). The IR and UV properties of the integrals are unchanged. The expansions are more generally joint expansions in d_3^ℓ and d_3^s , but the sum of the powers of the two do not exceed 4. In the light quark chiral limit, when $d_3^\ell \rightarrow 0$, holding d_3^s finite, c_{41}^a and c_{41}^{a8} vanish, but none of the others do. Nevertheless, in this limit

$$c_{41}^m = -\frac{c_2^m \Lambda^2}{3f_m^2} \quad \text{and} \quad c_{41}^8 = -\frac{4c_2^8 \Lambda^2}{9f_8^2}, \quad (27)$$

so that the couplings vanish quadratically in the corresponding Goldstone boson masses. Also, in the same limit, $c_{41}^{am} \propto d_3^s$, and close to T_{co} becomes $-m_K^2/(12f_a^2)$.

The scale factors and all the LECs of the pseudo-Goldstone bosons are directly computable in thermal QCD, and so can be used to match L_{pGB} to lattice computations. Furthermore, the integral expressions here can then be used to match them to the LECs of the quark EFT.

4. The pion EFT

A further simplification is now possible. One can integrate over the strange mesons and the hard modes of the pions, as shown diagrammatically in Fig. 2, and so get an effective pion theory at even smaller energy

$$L_\pi = \frac{1}{2} \Lambda_2^2 c_2 \phi_a^2 + \frac{1}{2} \dot{\phi}_a^2 + \frac{1}{2} c_4 (\nabla \phi_a)^2 + \frac{1}{8} c_{41} \phi_a^4 + \dots \quad (28)$$

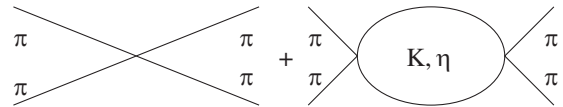


FIG. 2. Pion EFT is obtained by integrating over all hard modes in an energy shell between Λ_{2+1} and Λ_2 . The main corrections to the pion 2-point and 4-point functions are shown. There are also integration over the pions with momenta greater than Λ_2 . The one loop correction is resummed using a Dyson-Schwinger formulation.

We will also use the notation $m_\pi = \sqrt{c_2}\Lambda_2$ and $u_\pi = \sqrt{c_4}$ below. Thermal effects in the integration of kaons and η are expected to be exponentially small, and the methods of [10] may be used to accomplish this. However, when the pion mass is realistic, it is smaller than T_{co} , and one expects chiral power counting to work in L_π . In this counting every power of m_π scales in the same way as a derivative, so that the mass and kinetic terms are all of the same order (leading order, LO). The c_{41} s in L_{pGB} are only one of several new types of LECs that are obtained at the next-to-leading order (NLO) in this counting. Obtaining a consistent power counting again in the reduced theory would need to include all the other NLO terms while doing the one-loop integrations. We do not perform this higher order computation, whose only purpose would be to allow us to express the LECs of Eq. (28) to those of Eq. (20), extended fully to NLO.

Instead, only a few simple facts are needed from knowing this can be done. First, that the effect of the strange quarks is implicit in the LECs of L_π , although strangeness is not explicit in this low-energy EFT. A second useful point is that the UV cutoff of this EFT, Λ_2 , is lower than Λ_{2+1} which would be appropriate for Eq. (20). Finally, recall again that in chiral power counting the first three terms are of LO, whereas the last term is the first of several NLO terms.

C. UV insensitivity of the low-energy theory

Since the low-energy EFT of Eq. (28) which is obtained for $N_f = 2 + 1$ is the same as that obtained in [4] for $N_f = 2$, the low-energy EFT is insensitive to the UV theory. Also, QCD with $N_f = 2 + 1$ has a crossover at finite temperature, just as QCD with $N_f = 2$ does. Then it becomes convenient to treat Eq. (28) as if it descending from an $N_f = 2$ quark EFT, since this has a smaller number of LECs. Of course these LECs will be matched to pion properties in the $N_f = 2 + 1$ lattice computations, so they have implicit knowledge of the effect of the strange quark on low-energy dynamics. The bonus is that this UV insensitivity can be utilized by computing the phase diagram of $N_f = 2 + 1$ QCD from this $N_f = 2$ quark EFT. In this flavor-reduced quark EFT, one needs to take into account only the $D = 3, 4$, and 6 terms, since the correct symmetry of the continuum theory is already recovered with $D = 6$. In this section and later, whenever the LECs of the quark EFT are written without superscripts ℓ and s , they refer to the flavor-reduced EFT. As a result, in such cases we can lighten the notation from Λ_2 to Λ without ambiguity.

Since L_{HF} is quadratic in fields, the free energy can be evaluated exactly in this approximation, and turns out to be

$$\begin{aligned} \Omega = -\mathcal{N}V \left[\Lambda^4 d_6 \sigma^2 + \frac{m^4}{64\pi^2 d_4^3} \left\{ \log\left(\frac{m^2}{M^2}\right) - \frac{3}{2} \right\} + \frac{5m^6 d_{6,11}}{128\pi^2 d_4^6 \Lambda^2} \left\{ \frac{23}{30} - \log\left(\frac{m^2}{M^2}\right) \right\} \right. \\ \left. + \frac{T}{2\pi^2 d_4^3} \int_0^\infty dp p^2 \log(1 + e^{-E_p/T}) + \frac{d_{6,11}}{2\pi^2 d_4^6 \Lambda^2} \int_0^\infty \frac{dp}{E_p} \frac{p^6}{1 + e^{E_p/T}} \right], \end{aligned} \quad (29)$$

where $E_p^2 = m^2 + p^2$. The factors of d_4^3 in the kinetic term have been absorbed by the redefinition $p \rightarrow d_4 p$, and give rise to the powers of d_4 in front of the integrals. Here $d_{6,11}$ has been included to linear order. Since the corresponding operator is a correction to the kinetic term, when taken to all orders, it changes the definition of E_p and gives

$$E_p^2 = m^2 + p^2 \left(d_4 - d_{6,11} \frac{p^2}{\Lambda^2} \right)^2. \quad (30)$$

When $T \ll \Lambda$, then the thermal integrals cut off the range of momentum that is important to the problem and imply that $p \ll \Lambda$. Then clearly it is sufficient to expand the result to leading order in $d_{6,11}$ in order to get Eq. (29). The computation can be easily extended to finite baryon chemical potential, μ_B , by recalling that this results in adding the term $(\mu_B/N_c)\bar{\psi}\gamma_4\psi$ to L_{HF} for the chemical potential on the quarks.

D. The phase diagram

With this, the gap equation can be written down. There is a critical point only for $d_3 = 0$. The equation for T_c is obtained by requiring the second derivative of the free energy with respect to the condensate Σ to vanish. The integrals over the Fermi distribution can be easily performed in this limit. Using the notation $z = \mu_B^2/\Lambda^2$, and $t = T_c(\mu_B)/\Lambda$ one then obtains

$$\begin{aligned} \frac{\pi^2 d_{6,11}}{d_4^3} \left(\frac{7}{2} t^4 + \frac{5}{3\pi^2} z t^2 + \frac{5}{54\pi^4} z^2 \right) + \left(t^2 + \frac{z}{3\pi^2} \right) - t_0^2 = 0 \\ \text{where } t_0^2 = \frac{12d_4^3}{d_6}. \end{aligned} \quad (31)$$

In [4] we had considered the case with $d_{6,11} = 0$. Retaining only the positive solution of the quadratic in this limit, one has $T_c = t_0 \Lambda$. In this limit there is a line of second order transitions,

$$\left(\frac{T_c(\mu_B)}{T_c}\right)^2 = 1 - \frac{3}{N_c^2 \pi^2} \left(\frac{\mu_B}{T_c}\right)^2, \quad (32)$$

where we continue to use the lighter notation T_c for $T_c(\mu_B = 0)$. Since this is the equation of an ellipse in the phase diagram of T versus μ_B , we call this the chiral critical ellipse. This is the phase diagram of a generic NJL model, i.e., a model which has the kinetic terms and the $D = 6$ four-Fermi terms constrained by the flavor symmetry.

When $d_{6,11}$ is nonvanishing the gap equation is a disguised quadratic equation in t^2 . Negative or complex solutions are discarded, but there may still be two positive solutions if the discriminant of the quadratic is larger than zero. However, for the EFT to model QCD, there should be only one acceptable solution for t , and this should lie in the range $0 < t < 1$. We find that for $d_{6,11} \geq -d_4^3/(14\pi^2 t_0^2)$ there are two positive solutions for T_c . At exactly this critical value, the two solutions are degenerate and give $t = \sqrt{2}t_0$. With increasing $d_{6,11}$ one increases and the other decreases toward zero. There is exactly one positive solution when

$$d_{6,11} > \frac{2d_4^3}{7\pi^2} (t_0^2 - 1). \quad (33)$$

A straightforward computation using the definitions in Eqs. (A2) and (31) then gives us the curvature coefficients

$$\kappa_2 = \frac{3}{2N_c^2 \pi^2} \frac{1 + 5\epsilon}{1 + 7\epsilon}, \quad \text{and} \quad \tilde{\kappa}_4 = -\frac{3\epsilon}{N_c^4 \pi^4} \frac{1 + 20\epsilon + 70\epsilon^2}{(1 + 7\epsilon)^3}. \quad (34)$$

Notice that $\tilde{\kappa}_4$ vanishes linearly with $\epsilon = \pi^2 d_{6,11} T_c^2 / (d_4^3 \Lambda^2)$. It is also interesting to observe that in the large N_c limit taken together with the chiral limit, the second order chiral symmetry restoring transition happens at a T_c which is independent of μ_B . This is different from the first order deconfining line found in [11], and it has been conjectured that it either lies below or is coincident with it [12]. Notice, furthermore, that $\tilde{\kappa}_4$ is suppressed by two extra powers of N_c compared to κ_2 . As a result, the chiral critical ellipse may be a good approximation to the shape of the phase diagram at relatively small N_c .

Table I collects the lattice results for the curvature coefficients which were obtained by different groups using different methods. In the last decade κ_2 has begun to converge to a common value, with the most recent computations being in very good agreement with each other. The values of κ_4 are also beginning to be accessible in lattice measurements, and we quote the currently available values. From these we extract the values of $\tilde{\kappa}_4$. The present data on this quantity indicates that it is consistent with zero, which is also consistent with the

TABLE I. Recent lattice measurements of the curvature coefficients ([13] reports results using two methods, as indicated). The values of $\tilde{\kappa}_4$ are derived using Eq. (A3) and standard Gaussian error propagation.

Reference	κ_2	κ_4	$\tilde{\kappa}_4$
[15,16]	0.020(4)		
[17,18]	0.0135(20)		
[19]	0.0145(25)		
[13] ($\partial_T^2 \Sigma = 0$)	0.015(4)	-0.001(3)	-0.001(3)
($\partial_T \chi = 0$)	0.016(5)	0.002(6)	0.002(6)
[14]	0.0153(18)	0.00032(67)	0.00020(42)

large N_c power counting. Consistency of both κ_2 and κ_4 as measured on the lattice with the EFT requires a small value of $d_{6,11}$. In view of this, in the remainder of this paper we report numerical work with the version of the EFT with $d_{6,11}$ set to zero. One sees that in this case one has the prediction

$$\kappa_2 = \frac{3}{2N_c^2 \pi^2} \xrightarrow{N_c=3} 0.0169, \quad (35)$$

which is consistent with the results of [13,14] at the 68% confidence level (CL).

III. USING LATTICE COMPUTATIONS FOR $N_f = 2$

We start by setting out our procedure for determining the LECs from measurements and then using these in the EFT to produce further predictions. Since $N_f = 2$ in this section, we continue to use the notation Λ for Λ_2 . We choose the number of inputs to be the same as the number of LECs to be extracted, hence the process amounts to solving three coupled equations. However, each of the input quantities have errors, and they propagate to the LECs, and through them to the predictions of the EFT. So it is numerically easier to treat the extraction as a fitting process which minimizes χ^2 , defined in the usual way as the sum of the squares of the difference between the theory and measurement normalized by the measurement error. We check that the “best fit” value of χ^2 is the same as the machine precision; this implies that the input measurements are properly described by the model. For any other values of the LECs, the value of χ^2 can then be used as usual to define the 68%, 95%, and 99% CLs [20] on the LECs. By a bootstrap sampling within these CLs, the statistical distribution of EFT predictions can be obtained, and quoted as CLs on them. All error bars on predictions are obtained in this way.

Fits to the LECs using what is called the set C1 lattice data for $N_f = 2$ [21], and the predictions that come out of it were given in [4]. (The notation m_π in [21] corresponds to our m_π^D , u_f of [21] to our u_π , and the definition of the chiral condensate in [21] corresponds to $-\mathcal{N}\langle\bar{\psi}\psi\rangle$ in our

notation.) The method that we used to extract the LECs in [4], namely to use one value of m_π^D and one of u_π as inputs to the fits cannot be used for $N_f = 2 + 1$, since measurements of u_π have not been performed yet for $N_f = 2 + 1$. Here we explore a different scheme for extracting the LECs.

A. Extracting the LECs by matching lattice data

The method of extraction of the LECs used in [4] was geared to the choice $\Lambda = T_c$. However, the value of the UV cutoff should be flexible, and it is more instructive to take it to be large enough to include thermal pion effects in full. Here we will utilize $\Lambda = 300$ MeV. Our first extraction of the LECs uses as inputs the lattice values of m_π^D and u_π at a T below T_{co} and the value of T_{co} . Another change is that we now use the Schwinger-Dyson resummed expression of m_π^D [22] for these extractions.

The EFT can also be adapted to $T = 0$ by restoring full Lorentz invariance (i.e., $d_4 = 1$ and some of the $d_{6,A}$ are degenerate) then one has only two couplings to determine in the corresponding L_{HF} , namely d_3 and d_6 . They can be determined from the $T = 0$ values of m_π and the pion decay constant f_π . We use as input into the determination of $T = 0$ LECs the lattice data at bare couplings corresponding to those used in the finite T computations. In Fig. 3 we compare the LECs obtained at $T = 0$ with those obtained at finite temperature using the method of [4]. In both these extractions we have used $\Lambda = 300$ MeV.

Note that the best fit ranges of d_3 at $T = 0$ and $T > 0$ are completely compatible with each other. In this case the major effect of temperature is a large shift in d_6 . We will utilize this observation to extract the LECs of the finite temperature EFT from lattice data in another way. This will be done in two stages, first by using the $T = 0$ data to

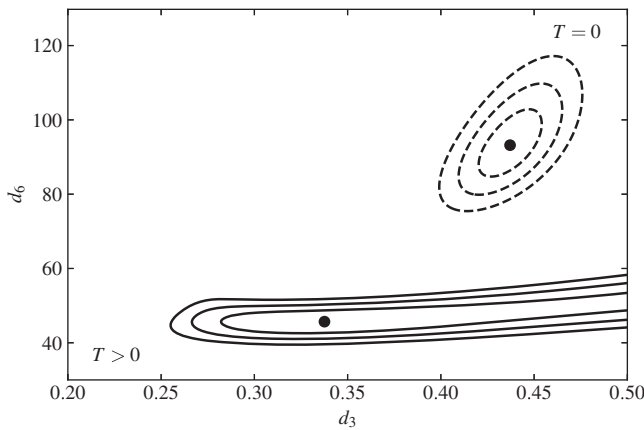


FIG. 3. We compare extractions of the LECs d_3 and d_6 at $T = 0$ and finite T . The best fit values of the LECs are indicated by a dot, and the successive contours enclose the 68%, 95%, and 99% CLs. For the finite T theory the best fit value of $d_4 = 1.21^{+0.09}_{-0.07}$.

extract the d_3 and d_6 . Next the range of d_3 obtained in this way is taken over to finite temperature where the remaining LECs, namely d_4 and d_6 , are obtained by fitting to lattice measurements of m_π^D at two nearby values of T below T_{co} . This changes the best fit values of the LECs, as is to be expected. However, the predictions of physical quantities do not change much, as we next show.

B. The phase diagram

With this scheme only static pion properties go into the determination of the LECs. The first prediction that we can get is for T_{co} (which is the temperature at which the chiral susceptibility peaks), and its limiting value in the chiral limit, T_c . The histograms for these predictions are shown in Fig. 4, when points are sampled within the $\Delta\chi^2$ range of d_4 and d_6 corresponding to the 90% CL with weight proportional to $\Delta\chi^2$. The skewness of the distributions is seen in two ways. First the upper and lower edges of the 68% CLs are not symmetric around the median. Also, due to the skewness the best-fit LECs give slightly different predictions than the median. However, these differences are mild.

The predicted value for T_c is now 144 ± 6 MeV, which is somewhat below the value of 170 MeV reported in [21]. From Fig. 4 it is clear that they are quite compatible within the 95% CL limit. However, the value of T_c obtained in the EFT and that quoted in [23] differ significantly. The latter were obtained using $O(4)$ exponents. Using mean field exponents instead would decrease T_c by 2–3 MeV, but not result in agreement with the EFT prediction. An assumption that is made in the EFT prediction is that the other LECs do not change appreciably as d_3 is taken to zero. This may not be accurate when extrapolating to the chiral limit from such large values of d_3 . Moreover, the extrapolation of [23] is also made from the same large input quark mass, and lattice results may also shift considerably when lighter quarks are used. The fits of the LECs for $N_f = 2 + 1$ (see the next section) where the quark mass is lighter show very good agreement between the EFT prediction and the direct lattice extractions, indicating that the higher quark masses here are the cause of the mild disagreement. It would be interesting, when future lattice computations of static pion properties become available, to see how the LECs change as the pion mass is tuned on the lattice.

C. Pion properties

It has been argued before that at a smooth cross over a description of matter with hadron degrees of freedom may be useful even for T slightly larger than T_{co} [22]. The failure of this picture will be gradual. In Fig. 5 one can see a remarkable ability of the EFT to predict static pion properties at temperatures about 10%–15% above T_{co} . One may conclude from this that d_3 , as shown in Fig. 3, is large

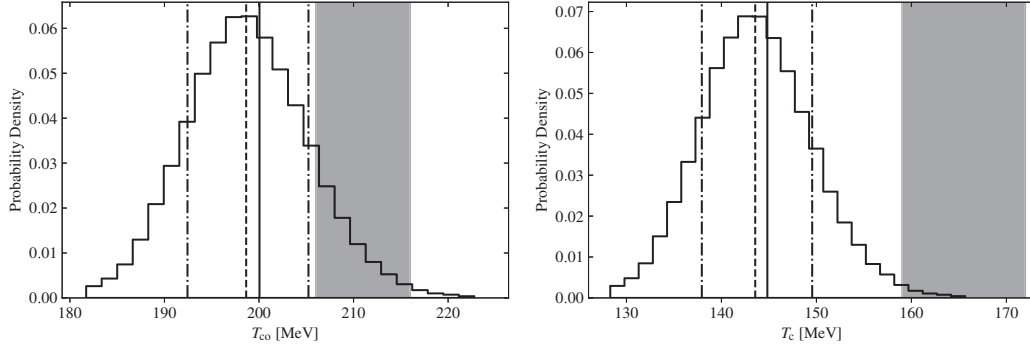


FIG. 4. Predictions for T_{co} and T_c using the fitted LECs shown as histograms obtained by sampling the 90% CLs of the fits. The median value and the limits of the 68% band for the EFT predictions are shown with broken vertical lines. The continuous vertical line shows the best fit value of the LECs, and the gray band is the lattice extraction of the corresponding quantity [21].

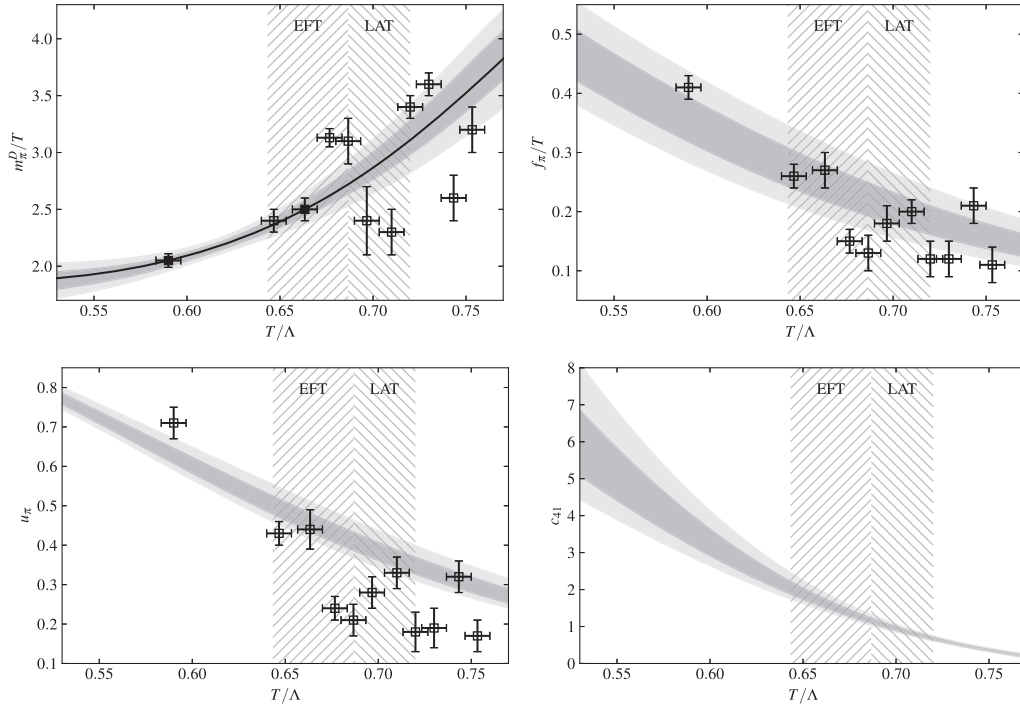


FIG. 5. Predictions for static pion properties from the EFT using the LECs determined by the new scheme. The gray bands show the 68% (darker color) and 95% (lighter color) CLs on the predictions. The vertical bands are the predicted value of T_{co} in the EFT, and the corresponding lattice determination [21]. Two values of m_π^D which are inputs to the fits are shown as the filled points (the black line shows the predictions from the best fit LECs). Clearly the prediction of the temperature dependence of m_π^D/T , f_π/T and u_π are as good as the data can support. The pion self-coupling c_{41} has not yet been measured on the lattice; we show the EFT prediction for it.

enough for such a remarkable continuity of the hadron description.

A diagnostic for this continuity is u_π . In the chiral limit it goes to zero with a critical exponent [24], and the pressure and various thermodynamic response functions have a singular behavior, signaling a true critical point. As one sees in the lattice data, u_π remains well above zero for this simulation at finite quark mass. The EFT prediction of u_π , shown in Fig. 5, is consistent with the data within its scatter. Whether this is improved by NLO corrections to the pGB

action is an investigation that we hope to address in the future.

In Fig. 5 are also shown the EFT prediction of the pion self coupling c_{41} . This has not been measured on the lattice since it requires analysis of pion 4-point functions. However, the predictions are reasonably accurate, and it seems to be worthwhile to make the effort to measure it on the lattice, since it is a completely independent test of our quantitative understanding of the universal properties underlying thermal pion physics and the phase diagram.

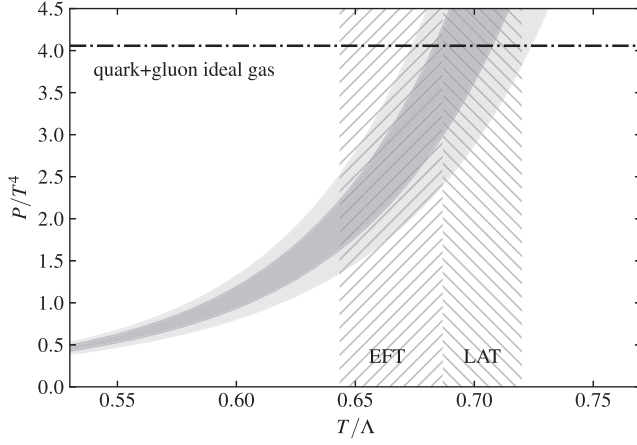


FIG. 6. EFT predictions for the pressure are shown here. The 68% (darker color) and 95% (lighter color) bands of prediction are shown. The vertical bands show the predicted range of values for T_{co} in the EFT and on the lattice [21]. The rapid rise in the predicted value of the pressure is noticeable.

Note that the predicted values of c_{41} are positive, whereas the relation of Eq. (26) predicts a negative value. This implies that the terms in higher powers of m_π are important at these large values of d_3 , and these lattice simulations for $N_f = 2$ are not very close to the chiral limit. The mismatch between the lattice and EFT predictions of T_c could be related. Nevertheless, the fact that so many predictions of the EFT are in reasonable agreement with the measurements shows that arguments based on chiral symmetry are a good guide to the essential underlying physics.

D. The pressure

The continuity of the hadron description encourages us to investigate the thermodynamics of matter using this EFT. With the choice of Λ well above T_{co} this is even quantitatively possible. In Fig. 6 we show the prediction for P/T^4 as a function of T . The rapid rise in the pressure is a generic feature of such EFTs, and due to the fact that the integral over spatial momenta of pion propagators $1/(p_0^2 + u_\pi^2 p^2 + m^2)$, can be converted to usual boson

integrals with the replacement $u_\pi p \rightarrow p$ [this is the equivalent of a similar transformation used to obtain Eq. (29)]. Through the volume element d^3p this then gives a factor of $1/u_\pi^3$ to the pressure. This is one of the sources of the singularity in the pressure in the chiral limit. In the Hartree-Fock approximation the critical exponent will have the mean field value. An epsilon expansion would be needed to recover the correct $O(4)$ exponent [25].

The pressure reaches the limit of the ideal quark-gluon gas immediately above T_{co} , indicating that there is a breakdown in the computation. One cannot rule out the possibility that the range of applicability of the EFT is different for each quantity. However, it will be discussed in the section for $N_f = 2 + 1$, where lattice measurements of P/T^4 are available that there are more subtle problems that need to be resolved.

There is not only a rapid increase in P/T^4 as T approaches T_{co} , but also a rapid increase in its uncertainty. The 95% CL band covers almost a factor of two in P/T^4 near T_{co} . One notes two possible origins for this error. One, of course, is that u_π appears to a high power in the expression for P/T^4 and therefore its uncertainty is multiplied. The other is that T_{co} is also uncertain and this may produce part of the uncertainty in P/T^4 .

In Fig. 7 we show the 68% CL on the joint distribution of T_{co} and P/T^4 at a fixed T below T_{co} . This is the distribution of uncertainties propagated from the uncertainties in the fitted LECs. There is clearly a correlation between them: as T_{co} increases the pressure decreases. But there is also a significant remnant uncertainty in P/T^4 . The second panel shows the same region when plotted as the joint distribution of T_{co} and $u_\pi^3 P/T^4$. This shows that a major component of the uncertainty in P/T^4 is due to u_π . Once this is removed, the dependence of P/T^4 on the particular combinations of the LECs which give T_{co} is much clearer. Measuring a variety of pion properties on the lattice would let us test schemes for extraction of the LECs which would best constrain the propagation of errors in EFT predictions.

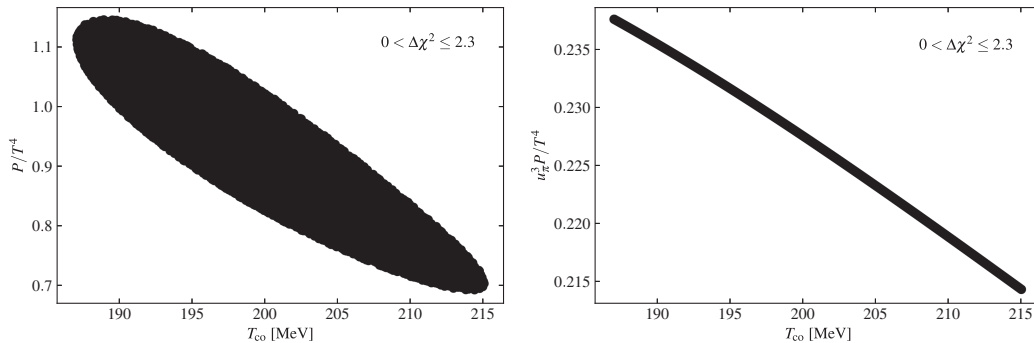


FIG. 7. 68% CL in P/T^4 at $T = 177$ MeV is plotted against that in T_{co} . Scaling P by u_π^3 reduces the uncertainty band in this direction, but the variation with T_{co} is unchanged.

IV. USING LATTICE COMPUTATIONS FOR $N_f = 2 + 1$

In this section we report the matching of the EFT to lattice data for 2 + 1 flavors. Our inputs are the lattice measurements of m_π^D [26], and the predictions are tested by T_{co} [13] and thermodynamics [27]. The lattice measurements are presented for a physical value of the strange quark mass, m_s , obtained by matching m_η to its physical value. However, two different light quark masses, m_l , are used. A choice of $m_l = m_s/27$ corresponds to the physical value of $m_\pi = 140$ MeV, and the results for m_π^D [26] and T_{co} [13] are given for this. However the thermodynamics [27] is reported for a heavier light quark mass $m_l = m_s/20$. The pion properties at $T = 0$ have also been reported only for this heavier quark mass. T_{co} has also been reported for the heavier quark mass [28].

We deal with this complication by assuming that the values of d_3 scale in proportion to m_l , so we have for the heavier quark mass a value d_3^h , and for the lighter physical quark mass we have $d_3 = (20/27)d_3^h$. We assume that both d_3 and d_3^h are light enough that the other two LECs do not change when we go from one to the other. We extract d_3^h from the fits to m_π and f_π at $T = 0$, and use the scaled lighter value d_3 along with m_π^D to extract d_4 and d_6 . These can be used to predict T_{co} as well as the other pion properties for $T > 0$. The extrapolation can be tested by using d_3^h to compare the EFT prediction with lattice extractions of T_{co}^h . The predictions for P/T^4 are also performed with d_3^h .

From $T = 0$ hadron physics we know that at these values of d_3 chiral power counting is accurate. Then the leading order chiral Lagrangian for the thermal EFT is

$$L_{LO} = \frac{1}{2}c_2\Lambda^2(\pi^a\pi^a) + \frac{1}{2}(\partial_4\pi^a)(\partial_4\pi^a) + \frac{1}{2}c_4(\nabla_i\pi^a)(\nabla_i\pi^a), \quad (36)$$

since $m_\pi^2 = c_2\Lambda^2$ has the same scaling dimension as the two derivatives in the kinetic terms. Since we use the two-flavor reduction of the $N_f = 2 + 1$ EFT in this section, every occurrence of Λ here is what we called Λ_2 in Sec. II. The term in c_{41} in Eq. (28) is one of several next-to leading order (NLO) terms that contribute to the EFT. In view of this we treat both m_π and $m_\pi^D = m_\pi/u_\pi$ without the Dyson-Schwinger resummation which was adopted for the description of the lattice measurements for $N_f = 2$ in Sec. III.

A. Extracting the LECs by matching lattice measurements

As before, we will set the LEC d_3 using the $T = 0$ lattice data which are used to set the scale of the finite temperature computations. The lattice has three bare parameters. Two

are the light and strange quark masses and these are used to tune the pion and other hadron masses. In addition, the lattice has a bare coupling which can be traded for the lattice spacing, a , which is the inverse of the UV cutoff of the lattice computation. Ideally the continuum limit of lattice measurements is taken by letting the lattice spacing go to zero (i.e., the lattice UV cutoff goes to infinity) while keeping physical quantities (such as the T and m_π) fixed.

When $\Lambda a \ll 1$, i.e., the UV cutoff of the lattice is much larger than the UV cutoff of the low energy EFT, then the process of taking continuum limits can be made shorter, since lattices with different a satisfying this condition are all equivalent as far as the low energy effects are concerned. This has a practical consequence in fixing d_3 , as we show here.

With $\Lambda = 300$ MeV as before, we extract d_3 and d_6 for the $T = 0$ EFT using lattice data with three different bare couplings fixed so that $a = 1/(TN_t)$ is a roughly constant value of T slightly below T_{co} for the three values of $N_t = 8, 12$ and 16. The three $T = 0$ simulations used correspond to bare couplings which on the $N_t = 8$ lattice give $T = 156 \pm 2$ MeV, on $N_t = 12$ to $T = 151.2 \pm 0.6$ MeV and for $N_t = 16$ to $T = 149.4 \pm 0.5$ MeV. The three temperatures are equal within 95% CLs. Note that a changes by a factor of two, while Λa remains significantly less than unity for all three simulations. The confidence limits on the extracted LECs are shown in Fig. 8. The three different sets of input data give LECs which are completely compatible with each other, as shown. Note, however, that the errors for the fit at the lattice spacing corresponding to $N_t = 8$ have much larger errors. These are a consequence of the errors in the input measurements from the lattice.

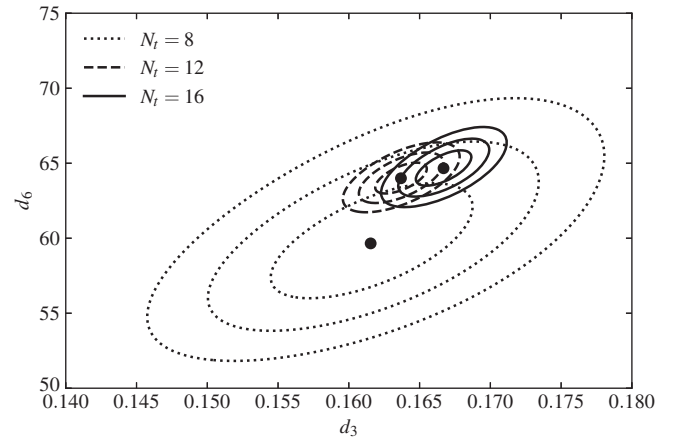


FIG. 8. LECs fitted at $T = 0$ for bare couplings which correspond to nearly the same temperature in finite temperature lattices with Euclidean time extent $N_t = 8, 12$ and 16, each at the corresponding mass. The three concentric ovals for each case are the 68%, 95%, and 99% CLs around the best fit point at the center. One sees that keeping $\Lambda = 300$ MeV for the three different fits gives compatible results within 95% confidence intervals.

TABLE II. Table contains the LECs fitted to lattice data for $N_f = 2 + 1$ for the set with $m_l = m_s/27$ using $\Lambda = 300$ MeV. Also shown are the EFT predictions for T_{co} and T_c . For the last two the 68%, 95%, and 99% confidence levels are shown along with the bootstrap median. The lattice determination is $T_{co} = 156.5 \pm 1.5$ for this value of the light quark mass [13]. The ratio T_{co}/T_c lies between 1.18 and 1.20 in all cases. By scaling d_3 in the ratio of m_l , we also find T_{co}^h corresponding to the heavier $m_l = m_s/20$ light quarks. For this case lattice measurements report $T_{co} = 155.9 \pm 8.0$ MeV [28].

N_t	$T_{\text{Lat}}^{\text{input}}$ (MeV)	d_3	d_4	d_6	T_c (MeV)	T_{co} (MeV)	T_{co}^h (MeV)
8	145, 156	0.120	$1.30^{+0.06}_{-0.05}$	118^{+25}_{-20}	$141^{+3.6,9}_{-3.5,7}$	$166^{+3.6,9}_{-3.5,7}$	$172^{+3.6,9}_{-3.5,8}$
		0.125	$1.33^{+0.06}_{-0.06}$	130^{+28}_{-22}	$140^{+3.6,9}_{-3.5,7}$	$166^{+3.6,9}_{-3.5,8}$	$171^{+3.6,9}_{-3.6,8}$
12	145, 157	0.120	$1.40^{+0.09}_{-0.07}$	169^{+47}_{-32}	$132^{+3.6,9}_{-3.7,9}$	$157^{+3.7,10}_{-3.7,10}$	$162^{+3.7,10}_{-3.7,10}$
		0.125	$1.44^{+0.09}_{-0.07}$	187^{+54}_{-37}	$131^{+3.6,9}_{-3.7,9}$	$157^{+3.7,10}_{-3.7,10}$	$162^{+3.7,10}_{-4.7,10}$
16	140, 152	0.120	$1.37^{+0.16}_{-0.11}$	156^{+95}_{-52}	$130^{+7.16,27}_{-7.15,20}$	$155^{+8.17,28}_{-8.16,21}$	$160^{+8.17,28}_{-8.16,21}$
		0.125	$1.41^{+0.17}_{-0.12}$	172^{+109}_{-58}	$129^{+8.17,27}_{-8.16,21}$	$155^{+8.17,28}_{-8.17,22}$	$160^{+8.18,29}_{-8.17,22}$

With the choice of $\Lambda = 300$ MeV we use two representative values of d_3 , namely 0.120 and 0.125, in the extraction of the two remaining LECs. We do this for one value of N_t at a time. For each N_t we find that there are three measurements of m_π^D reported in [26] for temperatures $T_1 < T_2 < T_3 \leq T_{co}$. For all three $N_t = 8, 12$ and 16, we chose to use the measurements at T_1 and T_3 (the values are given in the column marked $T_{\text{Lat}}^{\text{input}}$ in Table II). An estimate of the systematic errors is obtained by changing the pair of input measurements in the fits. In all cases we found that the extraction using $\{T_1, T_3\}$ lies between those using the other two pairs, and this source of systematic uncertainty is a little less than the 68% confidence limits shown in Table II.

Plotting the 68% CLs in the d_4 – d_6 plane shows a strong covariance of the two LECs. Since the boundary of this region is not an ellipse, we cannot use the usual second moment definition of the covariance matrix to perform a principal components analysis. Instead it turns out to be useful to rotate the axes by a numerically determined angle and plot the 68% CLs in this rotated frame, as shown in Fig. 9.

We notice several kinds of systematics here. First see that the 68% CLs of the fits for $d_3 = 0.120$ and 0.125 are compatible with each other. Furthermore, the CLs for $N_t = 12$ and 16 have very good overlap. Both have some overlap with the contours for $N_t = 8$, but the latter are clearly to one side of the rest. The reason for this is not hard to find. We find that the $T_{\text{lat}}^{\text{input}}$ s for $N_t = 8$ and 12 are close, but the corresponding values of m_π^D are quite different.

This may be due to large lattice corrections in the pion correlation function in going from $N_t = 8$ to 12 which are not suppressed as the power counting would seem to suggest. This is shown in [26] in terms of the taste-breaking of the pseudoscalar masses, where it is shown that the rms mass of the taste partners is split by a much larger amount for $N_t = 8$ than for $N_t = 12$ or 16. The large, but statistically insignificant differences between the EFT

predictions for T_{co} and T_c with $N_t = 8$ and the rest, as shown in Table II, then seem to be due to a lattice artifact rather than a shortcoming of the EFT. In view of this lattice uncertainty, in the remaining part of this section we only show the EFT predictions using LECs fitted to $N_t = 12$ and 16. These agree for all the predictions we examined.

B. The phase diagram from the EFT

The distributions of the EFT predictions for T_{co} with the LECs obtained from lattice measurements of the two largest values of N_t and with the two values of d_3 are shown in Fig. 10. Note the excellent agreement between the EFT prediction of T_{co} and the continuum extrapolated lattice determination from [13]. Recall that only pion properties have been used to determine the LECs, so this is a good test of the EFT in two ways. The first is a quantitative test of the underlying generalization of the universality argument which is that pion properties are intimately connected with the phase diagram. The second is

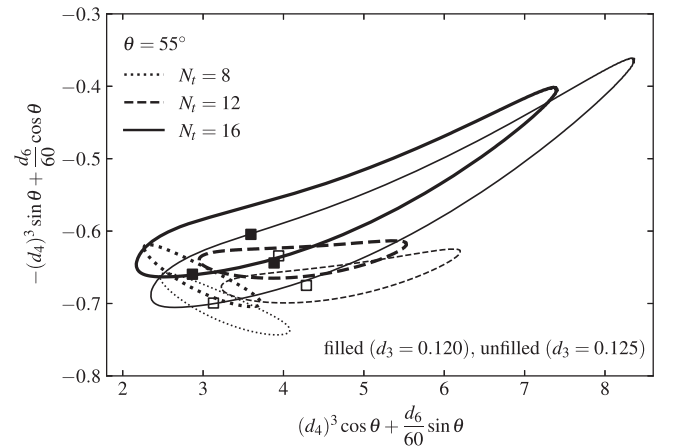


FIG. 9. LECs d_4 and d_6 extracted by fits to lattice data for m_π^D for the two values of d_3 on three different N_t . The 68% confidence regions are marked (the thicker lines are for $d_3 = 0.120$ and thinner for $d_3 = 0.125$).

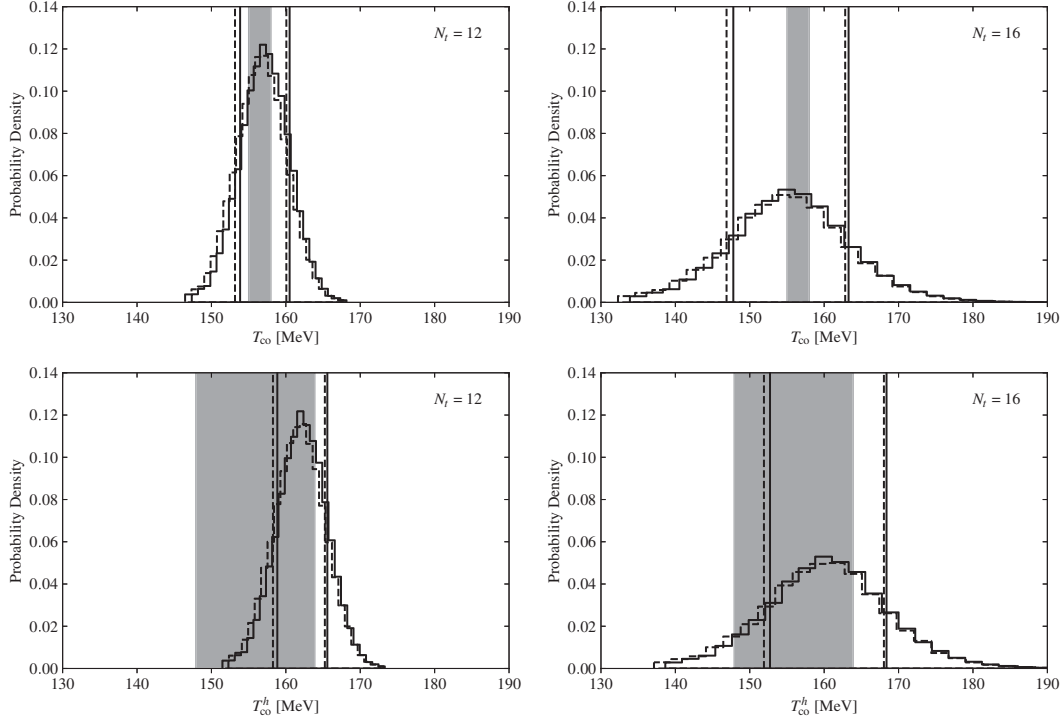


FIG. 10. Histograms for T_{co} and T_{co}^h obtained by sampling the 90% CL of d_4 and d_6 for the two values of d_3 for $N_t = 12$ and $N_t = 16$. For each N_t the full line gives the histogram and 68% confidence level of T_{co} for $d_3 = 0.120$ ($d_3 = 0.160$ for T_{co}^h) and the broken line for $d_3 = 0.125$ ($d_3 = 0.169$ for T_{co}^h). For the lighter quark the EFT predictions are in excellent agreement with the lattice extraction [13]. The predictions for T_{co}^h are also in agreement with the lattice extraction [28].

the test that a single EFT with UV cutoff Λ describes the long-distance behavior of lattice computations with different lattice UV cutoffs $1/a$, as long as pion properties do not show unusual sensitivity to the dimensionless numbers Λa .

In Fig. 10 we also show the histogram for T_{co}^h predicted by the EFT when d_3 is scaled in the ratio 27/20, which is the ratio of the light quark masses for the extraction of the remaining LECs, and the lattice extraction of T_{co} in [28]. Again in this case, there is agreement between the EFT prediction of T_{co}^h and the lattice determination. In future if the screening masses for the heavier quark mass are published then a direct fit can be used to further check this result. The agreement of the EFT and lattice determination of T_{co} in these two cases leads us to believe that the mild disagreement for $N_f = 2$ is the result of the light quark mass being significantly higher there.

The EFT also gives the prediction for T_c extrapolated to zero quark mass, $T_c = 131_{-6}^{+7}$ MeV. This can be compared to the value $T_c = 132_{-6}^{+3}$ MeV quoted in [29] obtained by an extrapolation of the lattice data to the continuum and then to chiral limit using $O(4)$ scaling. Recall that the prediction for the curvature coefficient κ_2 was given in Eq. (35) and that $\tilde{\kappa}_4 = 0$. Both these results are in good agreement with $N_f = 2 + 1$ lattice measurements collected in Table I. This completes the EFT predictions for the phase diagram.

C. Pions

With two lattice measurements as input, the primary predictions of the EFT are the values of the four LECs for the pion Lagrangian in Eq. (28), and the derived quantities $m_\pi^D = m_\pi/u_\pi$, the pion's Debye screening mass, and $f_\pi = f u_\pi$, the finite temperature pion decay constant. For $N_f = 2 + 1$, of these four quantities only m_π^D has been measured. We show the prediction for this Debye screening mass against the measurements on the lattice for $N_t = 12$ and 16 in Fig. 11. Notice that the difference between the fit uncertainties for the $N_t = 12$ and 16 lattices are simply propagated from the substantially larger error in one of the input data for the $N_t = 16$ lattices. As one can see in Fig. 11 the only effect is in the larger error bars for predictions.

Even this close to the chiral limit, there is good agreement with the screening masses above T_c . At a crossover one does not expect abrupt changes in the description of matter, but it is nevertheless surprising to see the quantitative agreement between the EFT and lattice data at temperatures more than 5% above T_{co} . This could well be due to using the Lagrangian in Eq. (36). Adding NLO terms could modify this behavior.

Whether the $N_t = 12$ lattice measurements are used as input or that from $N_t = 16$, the EFT predictions of other pion properties do not change. In Fig. 12 we show these predictions for m_π^D , f_π , u_π and c_{41} using the LECs extracted

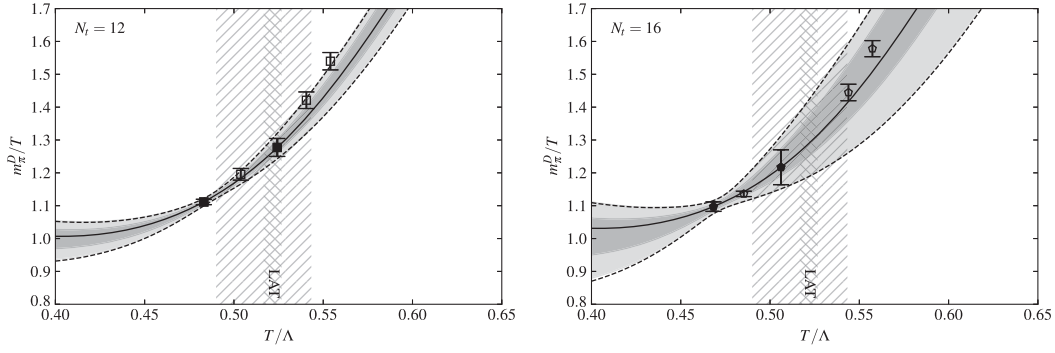


FIG. 11. EFT predictions for m_π^D at finite lattice spacing for $N_t = 12$ and 16 . The input data for extracting the LECs are shown with filled symbols. The prediction with the best fit LECs for $d_3 = 0.120$ are shown with the continuous line, and the 68% (darker shade) and 95% (lighter shade) CL bands of the EFT predictions are also shown, along with lattice measurements [26]. The limits of the 95% CL band for $d_3 = 0.125$ are shown with dashed lines. The vertical bands show the T_{co}/Λ predicted by EFT and measured on the lattice.

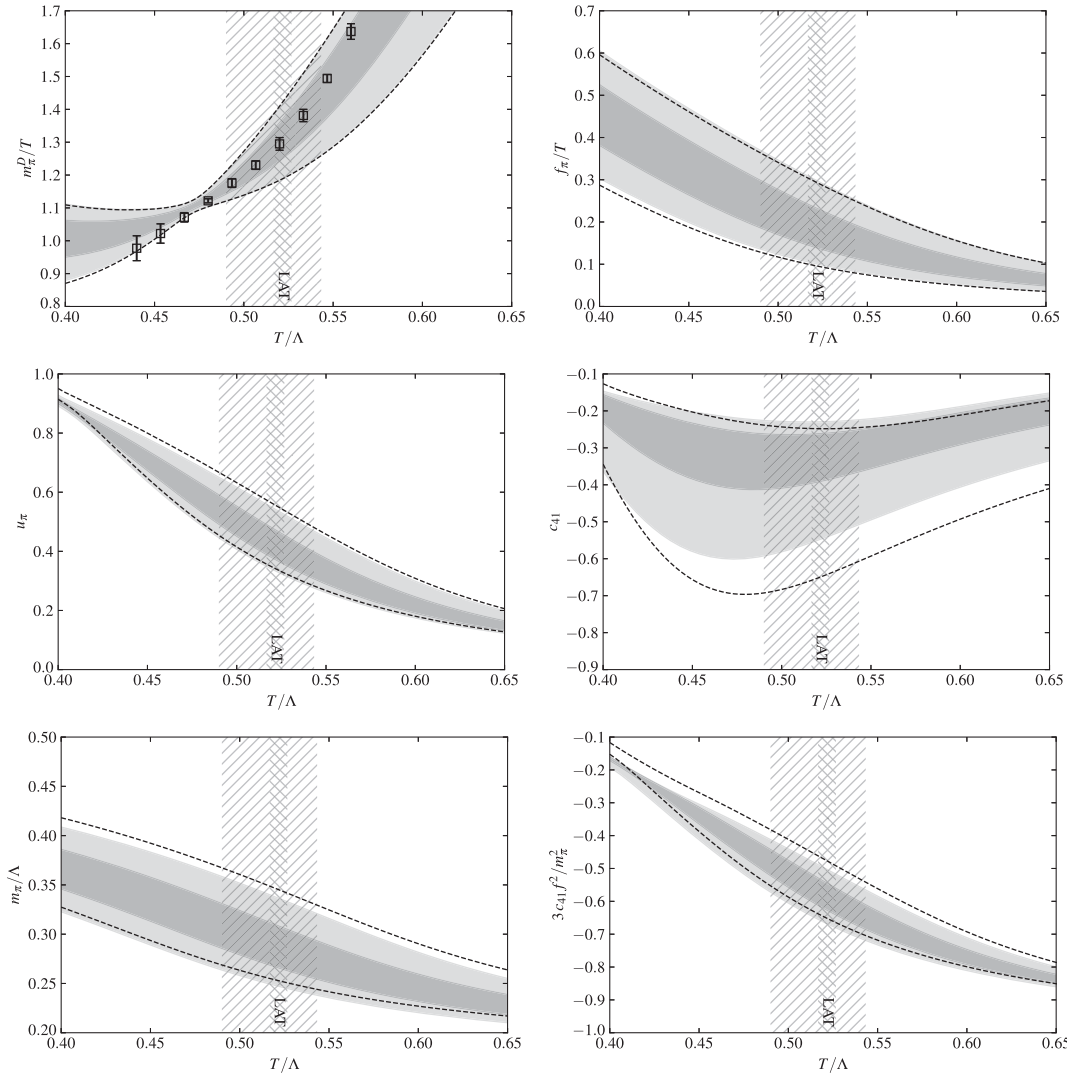


FIG. 12. EFT predictions for static pion properties are shown as the 68% (darker shade) and 95% (lighter shade) CL bands when $d_3 = 0.120$. The limits of the 95% CL band for $d_3 = 0.125$ are shown with dashed lines. The vertical bands show the T_{co}/Λ predicted by EFT and measured on the lattice [13] (the latter is entirely contained within the former). The continuum extrapolated values of m_π^D reported in [26] are shown. Note that at the lowest temperatures this continuum extrapolation used lattice spacing with $N_t = 8$ or coarser. The bottom two panels show that the pion mass is large enough that NLO terms in the thermal chiral EFT may be numerically important.

by matching to $N_t = 16$. Also shown are continuum extrapolations of lattice measurements of m_π^D as given in [13]. The minor, and statistically insignificant, mismatch at the lowest T then is a little surprising at first. However, we find that the continuum extrapolation at this temperature is obtained from a coarse lattice with $N_t = 8$ at best. Large lattice artifacts for these coarser lattice spacings have been discussed already.

In Fig. 12 we also show the pole mass, $m_\pi = \sqrt{c_2}\Lambda$. Note the falling trend as it approaches T_{co} . In physical units the EFT predicts $m_\pi \simeq 100$ MeV at T_{co} (the screening mass is about twice as large). When the temperature is 15% lower, the pole mass climbs to about 110 MeV (and the screening mass is about 135 MeV). The pole masses are significantly less than the $T = 0$ value of the mass, but they are nevertheless of order T . In the chiral limit the pole mass would vanish.

The results for c_{41} shown in Fig. 12 were computed using all terms in the expansion in d_3 , and not just the leading two terms shown in Eq. (26). Close enough to the chiral limit, i.e., when d_3 is small, Eq. (26) implies that $3c_{41}f^2/m_\pi^2$ should be close to -1 . In the final panel of Fig. 12 we show that numerically this is far from exact. In fact c_{41} is expanded in powers of the dimensionless ratio $m_\pi/f_a = u_\pi^2 m_\pi^D/f_\pi$, which turns out to be close to unity at T_{co} . So the higher order terms in the series for c_{41} are not parametrically suppressed. Much lower values of the pion mass at $T = 0$ would be required for the leading term to be numerically accurate at all temperatures. This behavior of c_{41} implies that NLO corrections may be needed in some predictions for realistic pion mass.

One knows from current algebra phenomenology at $T = 0$ that the assumption of broken chiral symmetry leads to some strikingly good results. However, in other domains these predictions were not quantitatively reliable. Today we understand that higher order terms in chiral perturbation theory are needed to reach the same level of accuracy in other predictions. The situation seems to be similar at finite temperature. A key question seems to be how small a ratio like m_π/f needs to be.

D. The pressure

The pressure of strongly interacting matter is another prediction from the EFT. The results from lattice measurements are plotted in Fig. 13 along with the prediction from L_{LO} of Eq. (36). Since this is a quadratic Lagrangian, the result is the ideal gas pressure apart from the factor of $1/u_\pi^3$ which has been discussed previously. The EFT gives a quantitatively reliable prediction of P/T^4 for $T/\Lambda < 0.5$.

Thereafter, the rapid rise in the prediction of P/T^4 visible in Fig. 13 is mainly due to the drop in u_π as one approaches T_{co} . The effect of the drop in m_π with T is subleading. In Sec. II we argued that at least a 2-loop resummation of the Dyson Schwinger equation for the

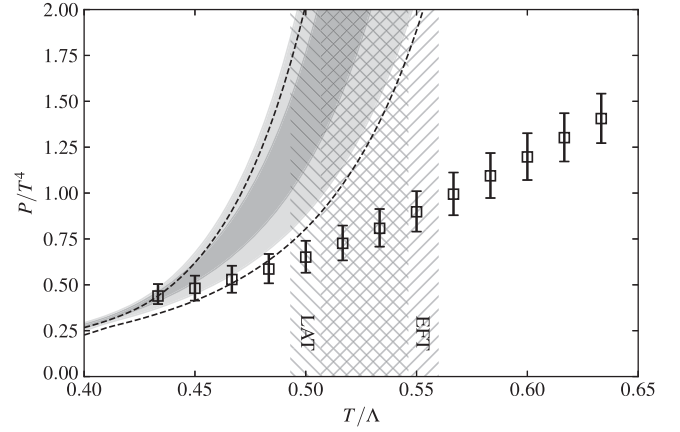


FIG. 13. EFT predictions for P/T^4 are shown as the 68% (darker shade) and 95% (lighter shade) CL bands when $d_3^h = 0.160$. The limits of the 95% CL band for $d_3^h = 0.169$ are shown with dashed lines. The vertical bands show the T_{co}/Λ predicted by EFT and measured on the lattice [28]. The continuum extrapolated values of P/T^4 reported in [27] are shown.

Lagrangian in Eq. (28) is needed to change u_π from its tree level value.

The formal argument remains valid even when the term in c_{41} has to be included with chiral power counting. However, in that case one has to account for all the other NLO terms in the thermal chiral EFT. In the $T = 0$ chiral perturbation theory the unitarized resummation of all these terms gives rise to the resonance spectrum of mesons [30]. In this sense it seems that a higher order computation of the pressure in the EFT could be formally equivalent to a computation in an interacting resonance gas described by a finite temperature chiral EFT. This is an extension of the chiral EFT approach that we leave to the future.

V. CONCLUSIONS

We described here a thermal EFT for $N_f = 2 + 1$ flavors of interacting quarks which we treated in the Hartree-Fock approximation (see Sec. II A). We then obtained (in Sec. II B) an EFT for the pseudo-Goldstone bosons, which are the small fluctuations around the solution of the resulting gap equation. This appears in the form of a thermal chiral perturbation theory ($T\chi$ PT) with an octet of pseudoscalar mesons which can be matched to lattice computations. We also argued that it can be reduced further to a $T\chi$ PT involving only pions. UV insensitivity of low-energy EFTs then allows us to treat the pion theory as descending from an effective $N_f = 2$ quark theory whose LECs contain the information of the effects of the strange quark (see Sec. II C). From this we found an expression for the shape of the phase boundary which has an interesting large N_c limit. With increasing N_c the phase boundary first approaches an elliptical shape, which then flattens out, with

T_c becoming independent of μ_B . The lattice measurements [13,14] of κ_4 can be understood in the context of the EFT from large N_c counting.

Different schemes for extracting the LECs from lattice inputs give essentially the same results. One sees this by comparing the results given for $N_f = 2$ in Sec. III with the fits given in [4], where the LECs are extracted in different ways. The comparison shows that different ways of extracting the LECs can give rise to different uncertainties in predictions.

Our main new results are the description of lattice measurements for $N_f = 2 + 1$. These are given in Sec. IV, and use the method of extracting the LECs which was tested in Sec. III. Our first interesting observation is that the thermal chiral EFT gives essentially the same predictions whether input data is taken from lattices with $N_t = 12$ or 16. This is understandable since the UV cutoff of the pion EFT, Λ_2 , is much smaller than that of either lattice. A subtlety with $N_t = 8$ is discussed in Sec. IV.

With static pion properties as input, the EFT is able to predict results for the phase diagram of QCD which are in good agreement with the direct lattice measurements (see Fig. 10). It is interesting to see that the prediction of both T_{co} and its chiral extrapolation, T_c , agree extremely well with the lattice extractions. This implies that the EFT is in the region where d_3 is small enough for chiral symmetry to be quantitatively useful already at the leading order of power counting in the EFT.

The EFT also predicts other static properties of the pion which are not currently available through lattice measurements. Among these we count the pion decay constant at finite temperature, f_π , the pion thermal “velocity,” u_π , the pole mass, m_π , and the pion self coupling, c_{41} . These quantities are defined in Sec. II and the predictions are shown in Fig. 12.

On the other hand, m_π/f is of order unity, as a result of which the leading chiral expression for c_{41} in Eq. (26) is not numerically accurate. Similar behavior has been seen in hadron phenomenology at $T = 0$ where some quantities are well described by LO chiral perturbation theory, but others require at least an NLO treatment.

The attempt to describe the pressure of strongly interacting matter is shown in Fig. 13. We note that the EFT is able to quantitatively capture the behavior of P/T^4 for $T/\Lambda_2 < 0.5$. Beyond this the EFT prediction rises much faster than that measured on the lattice. We argue that this is an NLO effect. This computation is substantial, and outside the scope of this paper. So it is left for the future.

A domain where lattice computations are unreliable is in the analytic continuation to real time. However, the analytic continuation of the EFT is straightforward [22]. For example, the dispersion relation of the pion in Eq. (36) is

$$E_p = \sqrt{m_\pi^2 + u_\pi^2 p^2} \simeq m_\pi + \frac{p^2}{2m_\pi/u_\pi^2} + \dots \quad (37)$$

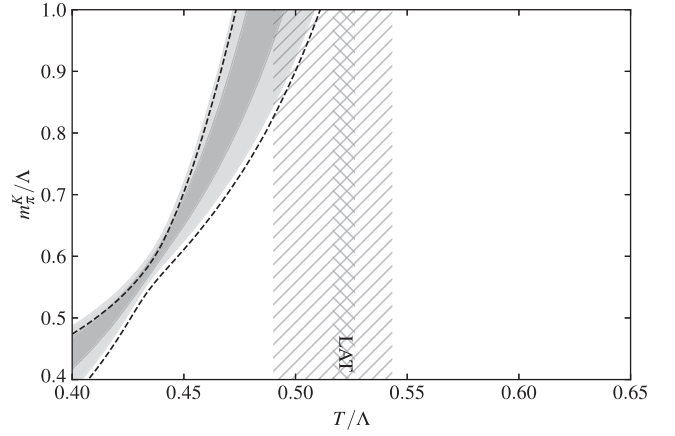


FIG. 14. Pion kinetic mass, m_π^K , in the EFT taken at leading order in chiral power counting for $N_f = 2 + 1$ QCD with realistic pion and kaon masses.

at low momenta $p \ll \Lambda_2$. This means that the kinetic energy involves a kinetic mass $m_\pi^K = m_\pi/u_\pi^2$. We show our prediction for this quantity for $N_f = 2 + 1$ with realistic $T = 0$ masses of the pion and kaon in Fig. 14. Note the very rapid rise in the kinetic mass as T increases. This rise may be moderated when NLO power counting terms are included in the EFT. Although the numbers may change, the fact that u_π would fall close to T_c means that m_π^K is bound to increase. It is interesting to note that this means that with increasing T the kinetic energy added by an increase in momentum decreases. Reactions which were possible at low temperature might be blocked due to this reason at finite temperature. The unexpected coexistence in chiral symmetry restored matter of a slow rise of the screening mass, implying the presence of pion collective excitations, and a rapid rise of the kinetic mass, implying its decoupling from the dynamics, points to a complex picture of strongly interacting matter across the crossover.

In summary then, based on the chiral symmetry of quarks we wrote a finite temperature EFT that took input from a small number of static pion properties computed for $N_f = 2 + 1$ QCD in equilibrium with realistic pion and Kaon masses (at $T = 0$). This gave predictions of the QCD phase diagram with a leading order computation, which were in excellent agreement with lattice measurements. The EFT also made predictions for other static pion properties which can be tested in future lattice computations. We noted that the errors of the EFT predictions are due to propagation of errors from the inputs. Therefore, improved measurements of m_π^D can substantially improve the test of the EFT predictions. We noted that some quantities like the pressure of strongly interacting matter and the real time quantity called the kinetic mass, defined in Eq. (37), may require an NLO computation in the EFT. This is a future research direction.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: CURVATURE COEFFICIENTS

The change in T_c with the baryon chemical potential, μ_B , has been used to define the curvature coefficients

$$T_c(\mu_B) = T_c \left[1 - \kappa_2 \left(\frac{\mu_B}{T_c} \right)^2 - \kappa_4 \left(\frac{\mu_B}{T_c} \right)^4 + \dots \right] \quad (\text{A1})$$

in agreement with the notation of [13,14]. In terms of derivatives we have

$$\begin{aligned} \kappa_2 &= -T_c \left(\frac{d}{d\mu_B^2} \right) T_c(\mu_B) \Big|_{\mu_B=0} \quad \text{and} \\ \kappa_4 &= -\frac{1}{2} T_c^3 \left(\frac{d}{d\mu_B^2} \right)^2 T_c(\mu_B) \Big|_{\mu_B=0} \end{aligned} \quad (\text{A2})$$

so that the curvature coefficients are explicitly dimensionless. Note that the derivatives are taken with respect to a variable μ_B^2 .

Comparing this with the chiral critical ellipse, which is the phase diagram of the NJL-like models, one can quantify the departure from ellipticity in terms of the parameter

$$\tilde{\kappa}_4 = \kappa_4 - \frac{1}{2} \kappa_2^2. \quad (\text{A3})$$

Lattice measurements of κ_2 began to converge to a common value following the work of [15,17]. In recent years the value of κ_4 has also been reported. In Table I we collect all the recent measurements that we are aware of.

APPENDIX B: CHANGING Λ_2

We noted that the UV cutoff Λ_2 used to define the EFT can be chosen to be anywhere between the pion and kaon masses. In this sense it is a pseudoparameter: a different

choice of Λ_2 would change the LECs but not the predictions. This is the meaning of a renormalization group flow in an EFT.

We demonstrate this in 2 + 1 QCD with the alternate choice of $\Lambda_2 = 450$ MeV. The fit to the same $T = 0$ data for pions used in the main text changes the best fit values of d_3^h . Using the scaled d_3 and the same inputs for $T > 0$ lattice data as before, we find that the best fit LECs change substantially. However, as can be seen by comparing the results in Tables II and III, the predictions for T_{co} and T_{co}^h are unchanged within errors.

There is a downward movement in the extrapolation of T_c in the limit of massless quarks, but this is also within the 95% CL of the lattice fits. In any case, such minor differences in predictions with two values of the cutoff are expected when the EFT is treated approximately. Even for perturbative QCD, changing the renormalization scheme changes the results of finite order perturbative predictions [31]; only all orders predictions are expected to be precisely unchanged.

APPENDIX C: LOOP INTEGRALS

For loop integrals in thermal EFT we follow the notation and procedure of [4]. Since we deal only with one-loop contributions, there is only a single loop momentum to integrate over, the 4-momentum $p = (p_4, \mathbf{p})$ in the following. Integrals over 4-momenta mean a sum over Matsubara modes and integral over three momenta. We will need the three basis integrals

$$\begin{aligned} J_0^{ab} &= \frac{N_c}{\Lambda^2} \int \frac{d^4 p}{(2\pi)^4} \frac{m_a m_b}{(p_a^2 + m_a^2)(p_b^2 + m_b^2)}, \\ J_1^{ab} &= \frac{N_c}{\Lambda^2} \int \frac{d^4 p}{(2\pi)^4} \frac{(p_a)_4 (p_b)_4}{(p_a^2 + m_a^2)(p_b^2 + m_b^2)}, \\ J_2^{ab} &= \frac{N_c}{\Lambda^2} \int \frac{d^4 p}{(2\pi)^4} \frac{|\mathbf{p}_a| |\mathbf{p}_b|}{(p_a^2 + m_a^2)(p_b^2 + m_b^2)} \end{aligned} \quad (\text{C1})$$

where the two quarks a and b have momenta p_a and p_b , and can be either light or strange flavors, with m_a and m_b taking the appropriate values. Furthermore, we have zero external momentum at the vertices, so we can take $|\mathbf{p}_a| = |\mathbf{p}_b|$. The integrals have been rendered dimensionless using powers of Λ . The overall factors of the number of colors, N_c , and the dimension of the Dirac spinor, N_s , come from the trace over all components of the quarks. The

TABLE III. Table contains the LECs and T_c , T_{co} and T_{co}^h with for $N_f = 2 + 1$ for UV cutoff $\Lambda_2 = 450$ MeV. The input data from lattice measurements is exactly as for $\Lambda_2 = 300$ MeV.

N_t	$T_{\text{Lat}}^{\text{input}}$ (MeV)	d_3	d_4	d_6	T_c (MeV)	T_{co} (MeV)	T_{co}^h (MeV)
12	145, 157	0.09	$1.51^{+0.12}_{-0.09}$	520^{+194}_{-119}	$125^{+4.7,11}_{-4.8,11}$	$153^{+4.8,12}_{-4.9,12}$	$159^{+4.8,12}_{-4.9,12}$
16	140, 152	0.09	$1.45^{+0.22}_{-0.14}$	455^{+363}_{-169}	$123^{+9.18,27}_{-9.18,21}$	$151^{+9.19,29}_{-10.19,22}$	$157^{+9.19,29}_{-10.19,23}$

trace over flavors is complicated because of the splitting of strange and light flavors and the factors coming from them will be written explicitly when the LECs are written.

There are possible UV divergences in the vacuum parts of the loop integrals, and they are treated in dimensional regularization (see [4]). There are no UV divergences in the thermal parts of the integrals since they are regulated by the Fermi distribution which arises from the Matsubara sum. It is also readily checked that IR divergences do not arise in any of the three integrals. The zero temperature pieces of the integrals have powers of m multiplying any $\log m$ that appears. So all of these integrals are regular in the chiral limit.

The integrals that are needed can be written in terms of these basis integrals. For examples, in order to write the scale factor and LECs we need the one-loop integrals

$$\begin{aligned}\mathcal{I}_1^{ab} &= J_0^{ab} + J_1^{ab} + J_2^{ab}, & \mathcal{I}_2^{ab} &= J_0^{ab} - J_1^{ab} - J_2^{ab}, \\ \mathcal{I}_3^{ab} &= -J_0^{ab} + J_1^{ab} + \frac{1}{3}J_2^{ab}, & \mathcal{I}_4^{ab} &= -J_0^{ab} - J_1^{ab} + J_2^{ab}.\end{aligned}\tag{C2}$$

For f_a , c_2^a , c_4^a and c_{41}^a both quarks are light in all the integrals, for the corresponding LECs for Kaons, one of the quarks is strange, and so on. The notation of [4] was $\mathcal{I}$ instead of $\mathcal{I}_1^{\ell\ell}$, $\mathcal{I}_{ii}$ for $I_3^{\ell\ell}$ and $\mathcal{I}_{44}$ instead of $I_4^{\ell\ell}$.

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- [1] H. A. Weldon, *Phys. Rev. D* **26**, 1394 (1982).
 - [2] A. Dhar and S. R. Wadia, *Phys. Rev. Lett.* **52**, 959 (1984).
 - [3] T. Arikawa, K. Sakai, and S. Sasaki, *Phys. Rev. D* **112**, 014506 (2025).
 - [4] S. Gupta and R. Sharma, *Phys. Rev. D* **97**, 036025 (2018).
 - [5] S. Weinberg, *The Quantum Theory of Fields: Volume 2, Modern Applications* (Cambridge University Press, Cambridge, England, 1996).
 - [6] S. P. Klevansky, *Rev. Mod. Phys.* **64**, 649 (1992).
 - [7] G. 't Hooft, *Phys. Rep.* **142**, 357 (1986).
 - [8] T. Schäfer, *Phys. Rev. D* **65**, 094033 (2002).
 - [9] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/q332-kkvc> for further details and derivations.
 - [10] J. Gasser and H. Leutwyler, *Nucl. Phys.* **B250**, 465 (1985).
 - [11] L. McLerran and R. D. Pisarski, *Nucl. Phys.* **B796**, 83 (2007).
 - [12] S. Datta and S. Gupta, *Phys. Rev. D* **82**, 114505 (2010).
 - [13] A. Bazavov *et al.* (HotQCD Collaboration), *Phys. Lett. B* **795**, 15 (2019).
 - [14] S. Borsanyi, Z. Fodor, J. N. Guenther, R. Kara, S. D. Katz, P. Parotto, A. Pasztor, C. Ratti, and K. K. Szabo, *Phys. Rev. Lett.* **125**, 052001 (2020).
 - [15] P. Cea, L. Cosmai, and A. Papa, *Phys. Rev. D* **89**, 074512 (2014).
 - [16] P. Cea, L. Cosmai, and A. Papa, *Phys. Rev. D* **93**, 014507 (2016).
 - [17] C. Bonati, M. D'Elia, M. Mariti, M. Mesiti, F. Negro, and F. Sanfilippo, *Phys. Rev. D* **90**, 114025 (2014).
 - [18] C. Bonati, M. D'Elia, M. Mariti, M. Mesiti, F. Negro, and F. Sanfilippo, *Phys. Rev. D* **92**, 054503 (2015).
 - [19] C. Bonati, M. D'Elia, F. Negro, F. Sanfilippo, and K. Zambello, *Phys. Rev. D* **98**, 054510 (2018).
 - [20] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, New York, 2007).
 - [21] B. B. Brandt, A. Francis, H. B. Meyer, and D. Robaina, *Phys. Rev. D* **90**, 054509 (2014).
 - [22] S. Gupta and R. Sharma, *Int. J. Mod. Phys. A* **35**, 2030021 (2020).
 - [23] B. B. Brandt, A. Francis, H. B. Meyer, O. Philipsen, and H. Wittig, *Proc. Sci. LATTICE2013* (2014) 162 [arXiv:1310.8326].
 - [24] D. T. Son and M. A. Stephanov, *Phys. Rev. D* **66**, 076011 (2002).
 - [25] S. Gupta and R. Sharma, *Proc. Sci. CPOD2014* (2015) 011 [arXiv:1503.03206].
 - [26] A. Bazavov, S. Dentinger, H. T. Ding, P. Hegde, O. Kaczmarek, F. Karsch, E. Laermann, A. Lahiri, S. Mukherjee, H. Ohno *et al.* (HotQCD Collaboration), *Phys. Rev. D* **100**, 094510 (2019).
 - [27] A. Bazavov *et al.* (HotQCD Collaboration), *Phys. Rev. D* **90**, 094503 (2014).
 - [28] A. Bazavov, T. Bhattacharya, M. Cheng, C. DeTar, H. T. Ding, S. Gottlieb, R. Gupta, P. Hegde, U. M. Heller, F. Karsch *et al.* (HotQCD Collaboration), *Phys. Rev. D* **85**, 054503 (2012).
 - [29] H. T. Ding *et al.* (HotQCD Collaboration), *Phys. Rev. Lett.* **123**, 062002 (2019).
 - [30] A. Gomez Nicola and J. R. Pelaez, *Phys. Rev. D* **65**, 054009 (2002).
 - [31] P. M. Stevenson, *Phys. Rev. D* **23**, 2916 (1981).