

Collapse scenario and final state of evaporation for Schwarzschild black hole in dimensionally reduced model of dilaton gravity

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We study a model of $(1 + 1)$ -dimensional dilaton gravity derived from the four-dimensional Einstein-Hilbert action by dimensional reduction in a semiclassical approximation including backreaction. The reduced action involves the cosmological constant and admits black hole solutions; among these, the solutions of interest are the evaporating black holes. We solve the equations of motion perturbatively by demanding that the initial state geometry is a Minkowski space-time. When the infalling matter intersects the space-time boundary, the black hole forms and begins to evaporate. We find that as the black hole evaporates, its horizon shrinks and at a finite space-time point, it meets the singularity. Along this hypersurface, the metric can be continuously matched to a static end-state geometry. This end-state geometry is a quantum-corrected Minkowski space-time within the first order of perturbation theory.

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I. INTRODUCTION

Various models of $(1 + 1)$ -dimensional dilaton gravity, such as Jackiw-Teitelboim (JT) [1,2] and Callan-Giddings-Harvey-Strominger (CGHS) [3], have proved to be very useful for analytical investigation of the black hole formation and evaporation, as well as the eternal black holes. The utility of these models is that after integration of the fluctuations of matter fields and including one-loop quantum corrections, field equations can be made exactly solvable by introducing suitable correction terms as in Russo-Susskind-Thorlacius (RST), Bose-Parker-Peleg (BPP), and CGHS model [4–7]. Even when not completely solvable, the two-dimensional equations are less complicated and can be solved perturbatively, as in the case of the dimensionally reduced Einstein-Hilbert (DREH) model. The DREH model was studied in [8,9] in the case of the eternal black hole solution. Dimensional reduction of the Einstein-Hilbert action coupled with electrodynamics has also been analyzed in [10–12]. A more comprehensive account of dilaton gravity models can be found in [13,14].

Two-dimensional dilaton gravity models are mostly used as toy models for the purpose of resolving the information loss paradox [15]. It is widely believed that the resolution of this paradox could be an important milestone at understanding the quantum nature of gravity, as well as the black hole entropy [16–18]. Hawking’s calculation [19] predicts

that the entanglement entropy of the Hawking radiation monotonically increases with time, even beyond the entropy limit set up by the Bekenstein-Hawking (BH) formula: $S_{\text{BH}} = A(\text{horizon})/4G_{\text{N}}^{(4)}$ [we set $c = \hbar = k_{\text{B}} = 1$, and $G_{\text{N}}^{(4)}$ stands for the Newton’s constant in four space-time dimensions]. This behavior of the entanglement entropy leads to the violation of the unitarity principle of quantum mechanics, and, in turn, to the loss of information, hence the name—information loss paradox.

The results derived from many of the two-dimensional gravity models indeed suggest that information does get lost in the process of black hole evaporation [4–7]. Recently, a new approach has been studied in which the so-called “island formula” for calculation of the entanglement entropy in gravitational systems is used to reproduce the Page curve [20,21]; the curve that the entanglement entropy of the radiation should follow so that the unitary evolution is achieved.

The Page curve has recently been reproduced in a variety of two-dimensional dilaton gravity models [8,22–30]. Another important step in the analysis of unitarity is to find the end-state geometry, after the evaporation of the black hole. This is the problem that we shall address in this paper.

This paper is organized in the following manner. In the subsequent section we study a model of $(1 + 1)$ -dimensional dilaton gravity derived from Einstein-Hilbert action by dimensional reduction. The main result of this section is a classical gravitational collapse scenario. Section III analyzes the contribution of one-loop quantum corrections to the energy-momentum tensor of matter fields. The metric for an evaporating black hole is

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computed in Sec. IV, up to first order in $\hbar$, with the initial condition being that the space-time before the formation of the black hole is Minkowski space-time. Finally, in Sec. V, the end state of the radiation is discussed. A conclusion and some proposals for the future work are given in Sec. VI.

II. DREH MODEL

The DREH model is a $(1+1)$ -dimensional model of dilaton gravity obtained from the usual four-dimensional Einstein-Hilbert (EH) action by using a spherically symmetric ansatz and integrating out the angles. It is similar to the CGHS model of dilaton gravity [3]. In particular, it admits a Schwarzschild black hole solution.

Dimensional reduction is a well-known procedure, and we only give a brief overview. More technical details can be found in [11]. We begin with the EH action in four dimensions,

$$S_{\text{EH}} = \frac{1}{16\pi G_N^{(4)}} \int d^4x \sqrt{-g^{(4)}} R^{(4)}, \quad (1)$$

where $G_N^{(4)}$ is the four-dimensional Newton's constant, $g_{AB}^{(4)}$ ($A, B = 0, 1, 2, 3$) is the four-dimensional metric, and $R^{(4)}$ is the four-dimensional curvature scalar.

Consider the following spherically symmetric ansatz for the metric,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \lambda^{-2} e^{-2\phi} [d\theta^2 + \sin^2\theta d\varphi^2], \quad (2)$$

where $\mu, \nu = 0, 1$, and $g_{\mu\nu}$ depends only on x^0 and x^1 , the dilaton field ϕ is related to the radial coordinate $r = \lambda^{-1} e^{-\phi}$, and λ^2 is a constant that plays the role of the cosmological constant in the reduced theory. Using the

ansatz (2), we can derive the relation between the curvature scalar R of the reduced $(1+1)$ -dimensional theory, and the curvature scalar $R^{(4)}$ of the four-dimensional theory,

$$R^{(4)} = R + 2(\nabla\phi)^2 + 2\lambda^2 e^{2\phi} - 2e^{2\phi} \square e^{-2\phi}. \quad (3)$$

Also, we have

$$d^4x \sqrt{-g^{(4)}} = d^2x d\theta d\varphi \sqrt{-g} \frac{e^{-2\phi}}{\lambda^2} \sin^2\theta. \quad (4)$$

The EH gravity action reduces to the dilaton gravity action S_ϕ (up to a surface term):

$$S_\phi = \frac{1}{4G} \int d^2x \sqrt{-g} [e^{-2\phi} (R + 2(\nabla\phi)^2) + 2\lambda^2], \quad (5)$$

where $G \equiv \lambda^2 G_N^{(4)}$ is the Newton's constant of the reduced theory.

To take the quantum corrections into account, we introduce a massless scalar field f minimally coupled to gravity. The corresponding action is denoted by S_m , so the classical DREH action is given by

$$\begin{aligned} S_{\text{DREH}} &= S_\phi + S_m \\ &= \frac{1}{4G} \int d^2x \sqrt{-g} [e^{-2\phi} (R + 2(\nabla\phi)^2) + 2\lambda^2] \\ &\quad - \frac{1}{2} \int d^2x \sqrt{-g} (\nabla f)^2. \end{aligned} \quad (6)$$

The classical field equations are obtained by varying S_{DREH} with respect to $g_{\mu\nu}$, ϕ , and f , respectively, resulting in

$$[2\nabla_\mu \nabla_\nu \phi - 2\nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu} (3(\nabla\phi)^2 - 2\square\phi - \lambda^2 e^{2\phi})] e^{-2\phi} = 2GT_{\mu\nu, \text{class}}^{(f)}, \quad (7)$$

$$(\nabla\phi)^2 - \square\phi = \frac{R}{2}, \quad (8)$$

$$\square f = 0, \quad (9)$$

with a classical energy-momentum tensor of field f , given by

$$T_{\mu\nu, \text{class}}^{(f)} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} = \nabla_\mu f \nabla_\nu f - \frac{1}{2} g_{\mu\nu} (\nabla f)^2. \quad (10)$$

Equations (7) and (8) can be simplified. By contracting equation (7) with the metric tensor $g^{\mu\nu}$ we obtain $\square e^{-2\phi} = 2\lambda^2 + 2GT$, where $T = g^{\mu\nu} T_{\mu\nu}$. Introducing a

new field $\varphi = e^{-\phi}$, after a few manipulations we obtain

$$\nabla_\mu \nabla_\nu \varphi - \frac{1}{2} R_{\mu\nu} \varphi = -G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \varphi^{-1}, \quad (11)$$

$$\square \varphi^2 = 2\lambda^2 + 2GT. \quad (12)$$

Equations (11) and (12), along with (9), is the set of equations that we will solve.

A. Vacuum solution of classical equations of motion

Now we will find the general solution to the classical vacuum equations of motion in the conformal gauge: $ds^2 = -e^{2\rho} dx^+ dx^-$. In this gauge, the nonvanishing components of the connection, Ricci curvature, and scalar curvature are

$$\Gamma_{\pm\pm}^{\pm} = 2\partial_{\pm}\rho, \quad R_{+-} = -2\partial_+\partial_-\rho, \quad R = 8e^{-2\rho}\partial_+\partial_-\rho. \quad (13)$$

Equations of motion (11) and (12) become

$$\partial_{\pm}(e^{-2\rho}\partial_{\pm}\varphi) = -GT_{\pm\pm}\varphi^{-1}e^{-2\rho}, \quad (14)$$

$$\partial_+\partial_-\varphi + \varphi\partial_+\partial_-\rho = 0, \quad (15)$$

$$\partial_+\partial_-\varphi^2 + \frac{\lambda^2}{2}e^{2\rho} = 0. \quad (16)$$

We first solve these equations in the vacuum case $T_{\pm\pm} = 0$. By integrating Eq. (14), we get

$$\partial_{\pm}\varphi = \frac{\lambda}{2}F^{\mp}(x^{\mp})e^{2\rho}, \quad (17)$$

where $F^{\mp}(x^{\mp})$ are arbitrary functions; we can think of them as components of a vector field. Functions $\partial_{\pm}\varphi$ satisfy the following equation $\partial_+\partial_-\varphi = \partial_-\partial_+\varphi$, which gives

$$\nabla_+F^+ = \nabla_-F^-, \quad (18)$$

$$\partial_+\partial_-\varphi = \frac{\lambda}{2}\nabla_+F^+e^{2\rho}. \quad (19)$$

Combining this result with Eqs. (16) and (17) we obtain φ in terms of $F^{\pm}$ and ρ :

$$\varphi = -\frac{\lambda}{2} \frac{1 + F^+F^-e^{2\rho}}{\nabla_+F^+}. \quad (20)$$

Next, we calculate $\partial_+\partial_-\rho$ by substituting Eqs. (20) and (19) into Eq. (15):

$$\partial_+\partial_-\rho = \frac{(\nabla_+F^+)^2e^{2\rho}}{1 + F^+F^-e^{2\rho}}. \quad (21)$$

It is easy to see that $\partial_-(1 + F^+F^-e^{2\rho}) = F^+\nabla_+F^+e^{2\rho}$ and $\partial_+\partial_-\rho = \frac{\partial_-(\nabla_+F^+)}{2F^+}$. This, in combination with equation (21), yields

$$\nabla_+F^+ = G^+(x^+)(1 + F^+F^-e^{2\rho})^2, \quad (22)$$

where $G^+(x^+)$ is an arbitrary function. Differentiating with respect to x^+ we get the same equation, which means we also have

$$\nabla_+F^+ = G^-(x^-)(1 + F^+F^-e^{2\rho})^2, \quad (23)$$

where $G^-(x^-)$ is also an arbitrary function. Since these two equations are the same, the functions $G^{\pm}$ must be the same constant: $G^+(x^+) = G^-(x^-) = -\frac{1}{2a}$, where the meaning of a will soon become apparent. Thus, with the help of Eq. (20) we have the following result:

$$\nabla_+F^+ = -\frac{1}{2a}(1 + F^+F^-e^{2\rho})^2 = -\frac{\lambda}{2}\frac{\lambda a}{\varphi^2}. \quad (24)$$

We can verify that ∇_+F^+ also satisfies $\nabla_+F^+ = F^{\pm}\partial_{\pm}\ln(F^+F^-e^{2\rho})$. Together with Eqs. (24) and (20) the derivatives of φ may be expressed as:

$$\partial_{\pm}\varphi = -\frac{\lambda}{2F^{\pm}}\frac{\varphi - \lambda a}{\varphi}. \quad (25)$$

Now, we can integrate this equation. The final result is given by the following two equations:

$$\varphi + \lambda a \ln\left(\frac{\varphi}{\lambda a} - 1\right) = -\frac{\lambda}{2}\left(\int \frac{dx^+}{F^+} + \int \frac{dx^-}{F^-}\right), \quad (26)$$

$$F^+F^-e^{2\rho} = \frac{\lambda a}{\varphi} - 1. \quad (27)$$

The solution to the classical vacuum equations is defined by one constant and two functions. We write these as (F^+, F^-, a) . The meaning of arbitrary functions $F^{\pm}$ is now obvious; they represent the conformal transformations of the coordinates. We can define new coordinates:

$$\frac{d\sigma^{\pm}}{dx^{\pm}} = \mp \frac{1}{F^{\pm}}. \quad (28)$$

These $\sigma^{\pm}$ coordinates are interpreted as Eddington-Finkelstein coordinates. Then Eq. (26) represents the formula for the tortoise coordinate:

$$r_* \equiv \sigma = \frac{1}{2}(\sigma^+ - \sigma^-). \quad (29)$$

Equation (27) is the equation for the metric tensor in the Eddington-Finkelstein coordinates. We can directly check this by calculating the interval:

$$ds^2 = -e^{2\rho}dx^+dx^- = F^+F^-e^{2\rho}d\sigma^+d\sigma^-. \quad (30)$$

We conclude that the solution defined by $(-1, 1, a)$ amounts to choosing Eddington-Finkelstein coordinates. This choice we call the Eddington-Finkelstein gauge. The constant a is the Schwarzschild radius. The dilation can be chosen to be proportional to the radial coordinate: $\varphi = e^{-\phi} = \lambda r$. So we see that we have reproduced the Schwarzschild solution in the same way as in [8].

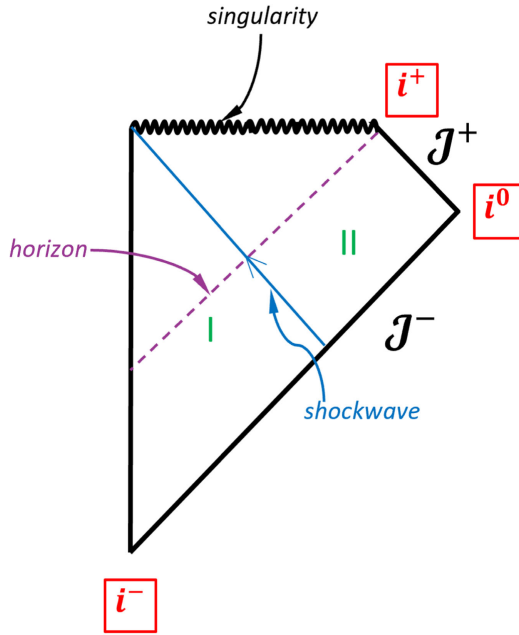


FIG. 1. Penrose diagram of a gravitational collapse.

B. Classical gravitational collapse

To formulate the classical gravitational collapse, we need to introduce the matter in the model. The easiest way to do this is to define the energy-momentum tensor as an incoming shockwave: $T_{++} = s\delta(x^+ - x_0^+)$ and $T_{--} = 0$. The corresponding Penrose diagram is shown in Fig. 1. There are two parts of space-time (I and II in Fig. 1), which we need to connect continuously. Metric and dilaton need to be continuous functions of x^+ and x^- at the $x^+ = x_0^+$ hypersurface:

$$e^{2\rho_I}\Big|_{x^+=x_0^+-\epsilon} = e^{2\rho_{II}}\Big|_{x^+=x_0^++\epsilon}, \quad (31)$$

$$\varphi_I\Big|_{x^+=x_0^+-\epsilon} = \varphi_{II}\Big|_{x^+=x_0^++\epsilon}, \quad (32)$$

when $\epsilon \rightarrow 0$, and indices I and II correspond to part I and part II of space-time, respectively (see Fig. 1). In part I, we have the vacuum solution defined by (F_I^+, F_I^-, a_I) , while in part II, the solution is defined by $(F_{II}^+, F_{II}^-, a_{II})$. Integrating Eq. (14), we obtain

$$\begin{aligned} e^{-2\rho_{II}}\partial_+\varphi_{II}\Big|_{x^+=x_0^++\epsilon} - e^{-2\rho_I}\partial_+\varphi_I\Big|_{x^+=x_0^+-\epsilon} \\ = -Gs e^{-2\rho}\varphi^{-1}\Big|_{x^+=x_0^+}. \end{aligned} \quad (33)$$

Equation (33), together with Eqs. (17) and (31), implies the jump in the derivatives of the dilaton field, which, in turn, implies the jump in F^- given by

$$F_{II}^- = F_I^- - \frac{2sG}{\lambda} e^{-2\rho}\varphi^{-1}\Big|_{x^+=x_0^+}. \quad (34)$$

The derivative $\partial_-\varphi$ is continuous, implying $F_{II}^+ = F_I^+$. Using Eq. (31) and solution (27) together with Eq. (32) we get that the constant a changes at $x^+ = x_0^+$, as well:

$$a_{II} = a_I - \frac{2sG}{\lambda^2} F_I^+(x_0^+). \quad (35)$$

For the correct interpretation, we have to connect the constant s with the mass of the shockwave. This can be done with the help of the time-translation Killing vector ξ (on part I of space-time):

$$M = \int_{\Sigma} d\sigma \sqrt{\gamma} n^\mu \xi^\nu T_{\mu\nu}, \quad (36)$$

where Σ is a space-like hypersurface, n is an orthogonal vector to the surface Σ normalized as $n_\mu n^\mu = -1$, γ is the induced metric on Σ , and $T_{\mu\nu}$ is the energy-momentum tensor. Solving the Killing equation, we find that the time-translation Killing vector is given by

$$\xi = F^- \partial_- - F^+ \partial_+, \quad (37)$$

i.e., $F^\pm$ are components of ξ . Since ξ is orthogonal to Σ , we can choose n to be proportional to ξ : $n^\mu = \frac{\xi^\mu}{\sqrt{-\xi_\nu \xi^\nu}}$. The induced metric on the hypersurface Σ is then $ds^2 = -F^+ F^- e^{2\rho} d\sigma^2$. Placing all of this into Eq. (36) we obtain

$$M = -s F^+(x_0^+). \quad (38)$$

If the solution in part I of space-time is given by (F_I^+, F_I^-, a_I) , then the solution in part II of space-time is given by

$$\left(F_I^+, F_I^- \left(1 + \frac{2MG/\lambda}{\lambda a_I - \varphi(x_0^+, x^-)} \right), a_I + \frac{2MG}{\lambda^2} \right). \quad (39)$$

Equation (39) tells us what form the solution in part II of space-time takes if we already have a black hole solution in part I of space-time and that black hole consumes matter of mass M . Without loss of generality, we can choose the Eddington-Finkelstein gauge in part I of space-time, which amounts to $F_I^+ = -1$ and $F_I^- = 1$. To analyze gravitational collapse, we can further choose that the metric of part I is a Minkowski space-time by setting $a_I = 0$. The corresponding dilaton is given by $\varphi_I = \frac{\lambda}{2}(\sigma^+ - \sigma^-)$. Then, Eq. (39) implies that part II is a black hole of mass M :

$$\left(-1, 1 - \frac{4MG/\lambda^2}{\sigma_0^+ - \sigma^-}, \frac{2MG}{\lambda^2} \right). \quad (40)$$

Let us define the notation that we will use from now on. First, we assume that the solution in part I of space-time is the Minkowski vacuum solution, defined by $(-1, 1, 0)$. Next, we define $\hat{\varphi}(\sigma^-) = \frac{\lambda}{2}(\sigma_0^+ - \sigma^-)$. Using this function, we can express the coordinate transformations in part II as

$$F^+ = -1, \quad F^- = \frac{\hat{\varphi} - \lambda a}{\hat{\varphi}}. \quad (41)$$

Notice that we have dropped the index II on functions $F_{\text{II}}^\pm$ and constant a as well, so we have $a = \frac{2MG}{\lambda^2}$. In part II of space-time the relation (26) still holds, so we can introduce new Eddington-Finkelstein coordinates $\hat{\sigma}^\pm$. In terms of $\sigma^\pm$, they are given by

$$\hat{\sigma}^+ = \sigma^+, \quad (42)$$

$$\hat{\sigma}^- = \sigma_0^+ - \frac{2}{\lambda} \left[\hat{\varphi} + \lambda a \ln \left(\frac{\hat{\varphi}}{\lambda a} - 1 \right) \right]. \quad (43)$$

Additionally, we define a new coordinate δ :

$$\delta(\sigma^+) = e^{\frac{\sigma^+ - \sigma_0^+}{2a}}, \quad (44)$$

which satisfies $\delta < 1$ in part II of space-time ($\sigma^+ > \sigma_0^+$). This property will be used extensively later on. Solutions (26) and (27) in part II can now be written in the following form (in terms of functions δ and $\hat{\varphi}$):

$$\begin{aligned} \varphi + \lambda a \ln \left(\frac{\varphi}{\lambda a} - 1 \right) &= -\lambda a \ln \delta + \hat{\varphi} + \lambda a \ln \left(\frac{\hat{\varphi}}{\lambda a} - 1 \right), \\ F^+ F^- e^{2\rho} &= \frac{\lambda a}{\varphi} - 1. \end{aligned} \quad (45)$$

Now we can analyze solution (45). The equation for the metric tells us that we have a black hole with the singularity at $\varphi = 0$ and the horizon at $\varphi = \lambda a$. Keep in mind that $\varphi = \lambda \hat{r}$, where $\hat{r}$ is the radial coordinate in part II of space-time. Looking again at solution (45), we see that $\varphi = \lambda a$ corresponds to $\hat{\varphi} = \lambda a$. This means that the horizon and the singularity are respectively given by

$$\sigma_{\text{H}}^- = \sigma_0^+ - 2a, \quad (46)$$

$$\ln \delta_s = \frac{\hat{\varphi}_s}{\lambda a} + \ln \left| \frac{\hat{\varphi}_s}{\lambda a} - 1 \right|. \quad (47)$$

When $t \rightarrow \infty$, we have $\sigma^+ \rightarrow \infty$, which is equivalent to $\delta \rightarrow 0$. Using Eq. (47) we deduce that the horizon and the singularity intersect at future time infinity i^+ , which is the expected result. Both lines are represented in the conformal diagram (see Fig. 1).

III. QUANTUM CORRECTIONS IN THE DREH MODEL

Having established the classical DREH model, we consider quantization of matter fields (a single massless scalar field) on the classical background of the Schwarzschild black hole. Quantum corrections come in the form of a Polyakov-Liouville (PL) action [31],

$$\begin{aligned} S_{\text{PL}} &= -\frac{\hbar}{96\pi} \int d^2x \int d^2x' \sqrt{-g(x)} \sqrt{-g(x')} \\ &\quad \times R(x) G(x-x') R(x'), \end{aligned} \quad (48)$$

where $G(x-x')$ stands for the Green's function for the massless Klein-Gordon equation in curved $(1+1)$ -dimensional space-time. This action represents the one-loop effective action obtained by integrating out fluctuations of the massless scalar field,

$$e^{\frac{i}{\hbar} S_{\text{PL}}} = \int \mathcal{D}\chi e^{\frac{i}{\hbar} \int d^2x \sqrt{-g} [-\frac{1}{2}(\nabla\chi)^2]}, \quad (49)$$

and it can be converted into a local form by introducing an auxiliary field ψ ,

$$S_{\text{PL}} = -\frac{\hbar}{96\pi} \int d^2x \sqrt{-g} [2R\psi + (\nabla\psi)^2]. \quad (50)$$

Action (50) is on-shell equivalent to (48). The field equation for the auxiliary field is

$$\square\psi = R. \quad (51)$$

The full action for the one-loop quantum DREH model is given by

$$S = S_{\text{DREH}} + S_{\text{PL}}. \quad (52)$$

Variation of S_{PL} with respect to $g_{\mu\nu}$ gives the quantum correction to the energy-momentum tensor for the scalar field f ,

$$\begin{aligned} \langle \Delta T_{\mu\nu}^{(f)} \rangle &= \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{PL}}}{\delta g^{\mu\nu}} = \frac{\hbar}{48\pi} \left[-2\nabla_\mu \nabla_\nu \psi \right. \\ &\quad \left. + \nabla_\mu \psi \nabla_\nu \psi + g_{\mu\nu} \left(2\square\psi - \frac{1}{2}(\nabla\psi)^2 \right) \right]. \end{aligned} \quad (53)$$

To define $\langle \Delta T_{\mu\nu}^{(f)} \rangle = \langle \Psi | \Delta T_{\mu\nu}^{(f)} | \Psi \rangle$, we also need to specify the quantum state $|\Psi\rangle$ that we are considering. The full energy-momentum tensor consists of the classical part and the one-loop quantum correction coming from the PL effective action,

$$T_{\mu\nu}^{(f)} = T_{\mu\nu, \text{class}}^{(f)} + \langle \Delta T_{\mu\nu}^{(f)} \rangle. \quad (54)$$

The metric equation (7) changes only due to this quantum correction of the energy-momentum tensor and reads

$$[2\nabla_\mu \nabla_\nu \phi - 2\nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu}(3(\nabla\phi)^2 - 2\Box\phi - \lambda^2 e^{2\phi})]e^{-2\phi} = 2G_N \left\{ \nabla_\mu f \nabla_\nu f - \frac{1}{2} g_{\mu\nu} (\nabla f)^2 + \frac{\hbar}{48\pi} \left[-2\nabla_\mu \nabla_\nu \psi + \nabla_\mu \psi \nabla_\nu \psi + g_{\mu\nu} \left(2\Box\psi - \frac{1}{2} (\nabla\psi)^2 \right) \right] \right\}. \quad (55)$$

Since the quantum correction to the energy-momentum tensor (53) depends only on the auxiliary field, we need to solve Eq. (51). In the conformal gauge (13), this equation reads $\partial_+ \partial_- (\psi + 2\rho) = 0$. Its solution in terms of the metric ρ , is given by

$$\psi = -2\rho + f_+(x^+) + f_-(x^-), \quad (56)$$

where $f_\pm(x^\pm)$ are arbitrary functions. Then, the quantum correction to the energy-momentum tensor defined in Eq. (53), calculated in the conformal gauge (13), in terms

of the metric ρ reads

$$\langle \Delta T_{\pm\pm}^{(f)} \rangle = \frac{\hbar}{12\pi} [\partial_\pm^2 \rho - (\partial_\pm \rho)^2 - t_\pm(x^\pm)], \quad (57)$$

$$\langle \Delta T_{+-}^{(f)} \rangle = -\frac{\hbar}{12\pi} \partial_+ \partial_- \rho, \quad (58)$$

where functions $t_\pm(x^\pm)$ are given by $t_\pm(x^\pm) = \frac{1}{2} \partial_\pm^2 f_\pm - \frac{1}{4} (\partial_\pm f_\pm)^2$. After adding quantum corrections, the equations of motion become as follows:

$$\partial_\pm (e^{-2\rho} \partial_\pm \varphi) = -Gs\delta(x^+ - x_0^+) \varphi^{-1} e^{-2\rho} + \frac{\hbar G}{12\pi} [(\partial_\pm \rho)^2 - \partial_\pm^2 \rho + t_\pm(x^\pm)] \varphi^{-1} e^{-2\rho}, \quad (59)$$

$$\partial_+ \partial_- \varphi + \varphi \partial_+ \partial_- \rho = 0, \quad (60)$$

$$\partial_+ \partial_- \varphi^2 + \frac{\lambda^2}{2} e^{2\rho} + \frac{\hbar G}{6\pi} \partial_+ \partial_- \rho = 0. \quad (61)$$

Equation (60) is the same as Eq. (15). This is because the right-hand side of Eq. (11) is zero, even after adding the quantum correction. The reason is the gravitational trace anomaly. The trace of the quantum correction to the energy-momentum tensor is given by $\langle \Delta T^{(f)} \rangle = \frac{\hbar}{3} e^{-2\rho} \partial_+ \partial_- \rho$. The same reason drives the change of Eq. (16) to Eq. (61).

These quantum-corrected equations cannot be exactly solved. We will solve them perturbatively to the first order

in the perturbative parameter $\varepsilon = \frac{\hbar G}{12\pi}$. To demonstrate how our perturbative approach works, we have included Appendix B, where the same perturbation scheme is applied within the framework of a BPP model of dilaton gravity. We expand the dilaton field as $\varphi \mapsto \varphi + \varepsilon\theta$ and the metric as $\rho \mapsto \rho + \varepsilon\alpha$. The quantum correction to Eqs. (59)–(61), in terms of θ and α , reads

$$\partial_+ [e^{-2\rho} \partial_+ \theta - 2\alpha e^{-2\rho} \partial_+ \varphi] = Gs \left(\frac{\theta}{\varphi} + 2\alpha \right) e^{-2\rho} \varphi^{-1} \delta(x^+ - x_0^+) + [(\partial_+ \rho)^2 - \partial_+^2 \rho + t_+(x^+)] \varphi^{-1} e^{-2\rho}, \quad (62)$$

$$\partial_- [e^{-2\rho} \partial_- \theta - 2\alpha e^{-2\rho} \partial_- \varphi] = [(\partial_- \rho)^2 - \partial_-^2 \rho + t_-(x^-)] \varphi^{-1} e^{-2\rho}, \quad (63)$$

$$\partial_+ \partial_- \theta + \varphi \partial_+ \partial_- \alpha + \theta \partial_+ \partial_- \rho = 0, \quad (64)$$

$$\partial_+ \partial_- (\theta \varphi + \rho) + \frac{\lambda^2}{2} \alpha e^{2\rho} = 0. \quad (65)$$

A. Vacuum solution to the quantum-corrected equations of motion

As in the case of the classical gravitational collapse, we will drop the δ -function term in Eq. (62) and find the general solution to the quantum-corrected equations of motion in vacuum. Then, we will patch up the solution over the $x = x_0^+$ hypersurface by integrating Eq. (62). Expressing ρ from (27) and substituting it in $(\partial_\pm \rho)^2 - \partial_\pm^2 \rho + t_\pm(x^\pm)$ yields

$$(\partial_{\pm}\rho)^2 - \partial_{\pm}^2\rho + t_{\pm}(x^{\pm}) = \frac{1}{4}(\partial_{\pm}\ln F^{\pm})^2 + \frac{1}{2}\partial_{\pm}^2\ln F^{\pm} + t_{\pm}(x^{\pm}) + \left(\frac{\lambda}{2F^{\pm}}\right)^2 \lambda a \frac{\varphi - \frac{3}{4}\lambda a}{\varphi^4}. \quad (66)$$

Using coordinate transformation (28) we can easily see that the first two terms on the right-hand side reduce to the Schwarzian derivative (defined in Appendix A):

$$\frac{1}{4}(\partial_{\pm}\ln F^{\pm})^2 + \frac{1}{2}\partial_{\pm}^2\ln F^{\pm}(x^{\pm}) = -\frac{1}{2}D_{x^{\pm}}[\sigma^{\pm}]. \quad (67)$$

Together with the term $t_{\pm}(x^{\pm})$, we get the transformation of this variable, given by Eq. (A6) in Appendix A. Placing this back into Eqs. (62) and (63) we arrive at

$$\begin{aligned} \partial_{\pm}(e^{-2\rho}\partial_{\pm}\theta - \lambda a F^{\mp}) &= \frac{1}{F^{\pm 2}} t_{\pm}(\sigma^{\pm}) e^{-2\rho} \varphi^{-1} \\ &+ \frac{\lambda}{2} F^{\mp} \lambda a \frac{\varphi - \frac{3}{4}\lambda a}{\varphi^3(\varphi - \lambda a)^2} \partial_{\pm}\varphi. \end{aligned} \quad (68)$$

Next we define the quantity $\mathcal{D}(\varphi)$ through its derivative:

$$\frac{d\mathcal{D}}{d\varphi} = \lambda a \frac{\varphi - \frac{3}{4}\lambda a}{\varphi^3(\varphi - \lambda a)^2}. \quad (69)$$

By integrating Eq. (68), we obtain the derivatives of θ :

$$\partial_+\theta = \frac{\lambda}{2}[F^-(\mathcal{D} + 2\alpha) + G^-]e^{2\rho} - F^-e^{2\rho} \int \frac{dx^+}{F^+} \frac{t_+(\sigma^+)}{\varphi - \lambda a}, \quad (70)$$

$$\partial_-\theta = \frac{\lambda}{2}[F^+(\mathcal{D} + 2\alpha) + G^+]e^{2\rho} - F^+e^{2\rho} \int \frac{dx^-}{F^-} \frac{t_-(\sigma^-)}{\varphi - \lambda a}, \quad (71)$$

where $G^{\pm}(x^{\pm})$ are a new set of arbitrary functions. Later we will see that these functions represent quantum correction to the coordinate transformations. We have not yet found the exact form of the derivatives $\partial_{\pm}\theta$ since Eqs. (70) and (71) have unevaluated integrals. The problem lies in the functional dependence $\varphi = \varphi(x^+, x^-)$ given through the transcendental Eq. (26). For now, the solution will be written in terms of these integrals. To integrate Eq. (70) once more, we need to eliminate the unknown function α . This can be done by employing Eq. (65). In conjunction with Eqs. (17) and (27) we find

$$\theta + \frac{2F^-}{\lambda}(\varphi\partial_-\theta + \theta\partial_-\varphi + \partial_-\rho) = \int \mathcal{D}d\varphi + \frac{G^-}{F^-}\varphi - \int \frac{dx^+}{F^+} \left(\frac{\lambda a}{\varphi} - 1\right) \int \frac{dx^+}{F^+} \frac{t_+(\sigma^+)}{\varphi - \lambda a} + H^-, \quad (72)$$

where we have introduced another arbitrary function $H^-(x^-)$. Eliminating the derivatives $\partial_-\varphi$ and $\partial_-\theta$ via Eqs. (25) and (71) respectively, and applying the same procedure to Eq. (71) we obtain the following equations:

$$\begin{aligned} \theta &= \frac{\varphi(\varphi - \lambda a)}{\lambda a} \left(\mathcal{D} + 2\alpha + \frac{G^+}{F^+}\right) + \frac{\varphi^2 G^-}{\lambda a F^-} + \frac{1}{2\varphi} \\ &+ \frac{\varphi}{\lambda a} \left[\int \mathcal{D}d\varphi - \frac{2}{\lambda}(\varphi - \lambda a) \int \frac{dx^-}{F^-} \frac{t_-(\sigma^-)}{\varphi - \lambda a} - \int \frac{dx^+}{F^+} \left(\frac{\lambda a}{\varphi} - 1\right) \int \frac{dx^+}{F^+} \frac{t_+(\sigma^+)}{\varphi - \lambda a} - H^- \right], \end{aligned} \quad (73)$$

$$\begin{aligned} \theta &= \frac{\varphi(\varphi - \lambda a)}{\lambda a} \left(\mathcal{D} + 2\alpha + \frac{G^-}{F^-}\right) + \frac{\varphi^2 G^+}{\lambda a F^+} + \frac{1}{2\varphi} \\ &+ \frac{\varphi}{\lambda a} \left[\int \mathcal{D}d\varphi - \frac{2}{\lambda}(\varphi - \lambda a) \int \frac{dx^+}{F^+} \frac{t_+(\sigma^+)}{\varphi - \lambda a} - \int \frac{dx^-}{F^-} \left(\frac{\lambda a}{\varphi} - 1\right) \int \frac{dx^-}{F^-} \frac{t_-(\sigma^-)}{\varphi - \lambda a} - H^+ \right]. \end{aligned} \quad (74)$$

Equating expressions (73) and (74) we get the following equation:

$$H^- + \lambda a \frac{G^-}{F^-} - \frac{2}{\lambda} \int \frac{dx^-}{F^-} t_-(\sigma^-) = H^+ + \lambda a \frac{G^+}{F^+} - \frac{2}{\lambda} \int \frac{dx^+}{F^+} t_+(\sigma^+) = C, \quad (75)$$

where C is an arbitrary constant. Adding Eqs. (73) and (74), and using Eq. (75) we get the following expression for θ in terms of φ and α :

$$\theta = \frac{\varphi(\varphi - \lambda a)}{\lambda a} \left[\mathcal{D} + 2\alpha + \frac{G^+}{F^+} + \frac{G^-}{F^-} \right] + \frac{1}{2\varphi} + \frac{\varphi}{\lambda a} \int \mathcal{D} d\varphi - \frac{\varphi}{\lambda a} \left[\int \frac{dx^+}{F^+} \left(\frac{\lambda a}{\varphi} - 1 \right) \int \frac{dx^+}{F^+} \frac{t_+(\sigma^+)}{\varphi - \lambda a} + \int \frac{dx^-}{F^-} \left(\frac{\lambda a}{\varphi} - 1 \right) \int \frac{dx^-}{F^-} \frac{t_-(\sigma^-)}{\varphi - \lambda a} \right]. \quad (76)$$

Once again, this is not the final expression for the field θ , since it still depends on the unknown function α . We need to somehow eliminate it. An easy way to do this is to use Eq. (70) or (71). In this way, we get another differential equation of first order in θ , which means that we need to perform another integration. Note that this constant C , defined through Eq. (75), does not appear in Eq. (76); it is

the integration constant of the integral $\int \mathcal{D} d\varphi$. Later, we will see that it is a quantum correction to the constant a . We can arrive at the same result using either of the two Eqs. (71) or (70). This will result in two expressions for θ , with another set of arbitrary functions, which can be eliminated by equating these expressions. The final result is then given by

$$\theta = \frac{\varphi - \lambda a}{\varphi} \left[\frac{\lambda}{2} \left(\int dx^+ \frac{G^+}{F^{+2}} + \int dx^- \frac{G^-}{F^{-2}} \right) - \int d\varphi \left(\frac{\lambda a}{2\varphi(\varphi - \lambda a)^2} + \frac{\varphi}{(\varphi - \lambda a)^2} \int \mathcal{D} d\varphi \right) + \int \frac{dx^+}{F^+} \frac{1}{\varphi - \lambda a} \int \frac{dx^+}{F^+} t_+(\sigma^+) + \int \frac{dx^-}{F^-} \frac{1}{\varphi - \lambda a} \int \frac{dx^-}{F^-} t_-(\sigma^-) \right], \quad (77)$$

in terms of x^+ , x^- , $t_+(\sigma^+(x^+))$, $t_-(\sigma^-(x^-))$ and $\varphi(x^+, x^-)$. The next step would be to find α . One way to do this would be by using Eq. (70); it is easy to derive an expression for α in terms of x^+ and x^- . We will not present that calculation here. Rather, we will calculate it in a special case of the quantum-corrected collapse scenario.

Before evaluating the integrals in (77), we simplify the expressions by introducing the reduced fields $x = \frac{\varphi}{\lambda a}$ and $\hat{x} = \frac{\hat{\varphi}}{\lambda a}$. In terms of x , the function $\mathcal{D}(x)$ and its integral are given by

$$\mathcal{D} = \frac{1}{4(\lambda a)^2} \left[\ln \left(1 - \frac{1}{x} \right) - \frac{1}{x-1} + \frac{3}{2} \frac{1}{x^2} + \frac{2}{x} \right], \quad (78)$$

$$\int \mathcal{D} d\varphi = \frac{1}{4\lambda a} \left[(x-2) \ln \left(1 - \frac{1}{x} \right) - \frac{3}{2} \frac{1}{x} \right] + C, \quad (79)$$

where C is a constant defined in Eq. (75). For future simplifications, we introduce the function

$$\theta_0(\varphi) = - \int d\varphi \left(\frac{\lambda a}{2\varphi(\varphi - \lambda a)^2} + \frac{\varphi}{(\varphi - \lambda a)^2} \int \mathcal{D} d\varphi \right), \quad (80)$$

which, by utilizing Eq. (79), becomes

$$\theta_0(x) = - \frac{1}{4\lambda a} \frac{1}{x-1} \left[\frac{1}{2} + (x^2 - 3x + 3) \ln(x-1) - (x^2 - 2x + 2) \ln x \right] + C \left(\frac{1}{x-1} - \ln(x-1) \right). \quad (81)$$

We also define integrals:

$$I_{\pm} = \int \frac{dx^{\pm}}{F^{\pm}} \frac{1}{\varphi - \lambda a} \int \frac{dx^{\pm}}{F^{\pm}} t_{\pm}(\sigma^{\pm}). \quad (82)$$

With the help of all this newly defined notation, Eq. (77) can be written in a much simpler form:

$$\theta = \frac{x-1}{x} \left[\frac{\lambda}{2} \left(\int dx^+ \frac{G^+}{F^{+2}} + \int dx^- \frac{G^-}{F^{-2}} \right) + \theta_0 + I \right], \quad (83)$$

where $I = I_+ + I_-$. Let us now show that $G^{\pm}$ are the quantum corrections of the coordinate transformations,

by showing how Eq. (26) changes when the quantum corrections are added. From Eq. (83) we extract the part connected to the functions $G^{\pm}$, multiply it by ε , and add it to the right-hand side of Eq. (26):

$$\begin{aligned} & \varphi + \lambda a \ln \left(\frac{\varphi}{\lambda a} - 1 \right) + \varepsilon \frac{\varphi \theta}{\varphi - \lambda a} - \varepsilon \theta_0 - \varepsilon I \\ &= - \frac{\lambda}{2} \left[\int \frac{dx^+}{F^+} \left(1 - \varepsilon \frac{G^+}{F^+} \right) + \int \frac{dx^-}{F^-} \left(1 - \varepsilon \frac{G^-}{F^-} \right) \right]. \end{aligned} \quad (84)$$

Now, we revert to the quantum-corrected fields $\varphi + \varepsilon \theta \mapsto \varphi$. If we expand Eq. (26) to the first order in ε ,

we find exactly the term linear in θ appearing in Eq. (84). On the right-hand side of Eq. (84) we have the expansion of $(F^\pm + \varepsilon G^\pm)^{-1}$ to the first order in ε , which means that $F^\pm + \varepsilon G^\pm \mapsto F^\pm$ is the quantum-corrected coordinate transformation. Also, note that the constant C appearing in Eq. (81) is the quantum correction $\lambda a + \varepsilon C \mapsto \lambda a$. This is easy to check by expanding the term $(\lambda a + \varepsilon C) \ln(\varphi/(\lambda a + \varepsilon C) - 1)$ to the first order in ε . We are also interested in the effect of quantum corrections on

Eq. (27). Let us use the newly defined quantum-corrected coordinate transformations:

$$F^+ F^- e^{2\rho} \mapsto F^+ F^- e^{2\rho} \left[1 + \varepsilon \left(2a + \frac{G^+}{F^+} + \frac{G^-}{F^-} \right) \right]. \quad (85)$$

The quantity in brackets of Eq. (85) appears in Eq. (76), which means that we can extract it from that equation. Before we do that, we define another set of integrals:

$$I'_\pm = \int \frac{dx^\pm}{F^\pm} \left(\frac{\lambda a}{\varphi} - 1 \right) \int \frac{dx^\pm}{F^\pm} \frac{t_\pm(\sigma^\pm)}{\varphi - \lambda a}. \quad (86)$$

Also, we define $I' = I'_+ + I'_-$, substitute it into Eq. (85), and use Eq. (27) to arrive at the expression

$$F^+ F^- e^{2\rho} \mapsto \frac{\lambda a}{\varphi} - 1 - \varepsilon \frac{\lambda a \theta}{\varphi^2} + \varepsilon \left[\frac{\varphi - \lambda a}{\varphi} \mathcal{D} + \frac{\lambda a}{2\varphi^3} + \frac{1}{\varphi} \int \mathcal{D} d\varphi + \frac{1}{\varphi} I' \right]. \quad (87)$$

The first three terms represent the expansion of $\lambda a(\varphi + \varepsilon \theta)^{-1} - 1$. Since C is the integration constant of $\int \mathcal{D} d\varphi$, the only term containing this constant is $\varepsilon C/\varphi$, which demonstrates that it is in fact the quantum correction of the constant λa . Rewriting Eqs. (84) and (87) in terms of the quantum-corrected reduced field x and the quantum-corrected constant λa , we get the following final result:

$$x + \ln(x - 1) + \frac{\varepsilon}{4(\lambda a)^2} \frac{1}{x - 1} \left[\frac{1}{2} + (x^2 - 3x + 3) \ln(x - 1) - (x^2 - 2x + 2) \ln x \right] - \frac{\varepsilon}{\lambda a} I = -\frac{1}{2a} \left(\int \frac{dx^+}{F^+} + \int \frac{dx^-}{F^-} \right), \quad (88)$$

$$F^+ F^- e^{2\rho} = \frac{1}{x} - 1 + \frac{\varepsilon}{4(\lambda a)^2} \left[\frac{2x - 3}{x} \ln \left(1 - \frac{1}{x} \right) + \frac{1}{x} - \frac{2}{x^2} + \frac{1}{2} \frac{1}{x^3} \right] + \frac{\varepsilon}{\lambda a} \frac{I'}{x}. \quad (89)$$

Equations (88) and (89) are the general solution to the vacuum equations (62)–(65). So, the solution of the quantum-corrected equations of motion is given by (F^+, F^-, a, t_+, t_-) . Keep in mind that $F^\pm$ and a are quantum-corrected quantities. A concrete solution can be written when the integrals I and I' are evaluated. These integrals are dependent on each other, connected through Eqs. (70) and (71), which means that only one of them needs to be calculated. Since $t_\pm(x^\pm) = 0$ defines the vacuum we are choosing (see Appendix A), the integrals I and I' are closely related to the choice of the vacuum state.

B. Solution without energy flux at the asymptotic infinity

Here, we consider the solution that does not have energy flux in either of the two asymptotic infinities of space-time. Note that in our case it is different from [6]. There exists an exact Minkowski vacuum, which is the reason we are calling this a solution without the energy flux at infinity. Since the Eddington-Finkelstein coordinates $\sigma^\pm$ are asymptotically flat, we define the vacuum state by demanding

$t_\pm(\sigma^\pm) = 0$ (see Appendix A). This corresponds to the Boulware vacuum state $|B\rangle$. Since the integrals $I_\pm$ [Eq. (82)] and $I'_\pm$ [Eq. (86)] are proportional to $t_\pm(\sigma^\pm)$ they vanish, that is, $I = 0$ and $I' = 0$.

The solution is given by Eqs. (88) and (89), without the integrals I and I' . Note that the limit $\lambda a \rightarrow 0$ leads to the Minkowski vacuum, which is expected. This limit is apparently divergent, but one should not forget that the form of the λa constant is given by $\lambda a \mapsto \lambda a + \varepsilon C(\lambda a)$. The divergence can be canceled with an appropriate choice of the constant C as a function of λa . This choice is given by $C(\lambda a) = \frac{1}{4\lambda a}$.

Since these configurations correspond to solutions without energy flux at infinity, one expects them to be horizonless; in other words, they should not resemble black hole solutions. It has been demonstrated in [32,33] that the classical horizon is removed once quantum corrections are taken into account. The horizon is defined as the hypersurface along which the Killing vector (37) becomes null, i.e. where $F^+ F^- e^{2\rho} = 0$. In a perturbative approach, if a horizon exists, its location should be expressed as

$x_H = x_H^{(0)} + \varepsilon x_H^{(1)}$. From the expression (89), we directly infer that $x_H^{(0)} = 1$, which corresponds to the classical horizon. The first-order correction is then obtained by taking the limit $x \rightarrow 1$ of the first-order term in (89). This limit diverges, indicating that perturbative approach is not applicable in the space-time region close to the classical horizon. We therefore arrive at the same conclusion as in [32,33].

Another important feature of solution (88) and (89) with vanishing energy flux at infinity is that it contains a curvature singularity whenever the mass is nonzero. This can be verified explicitly by computing the scalar curvature, which in the first order of the perturbative expansion takes the form

$$R = \frac{\lambda^2}{2} \frac{\lambda a}{(\varphi^2 - \varepsilon)^{3/2}}. \quad (90)$$

Note that this expression indeed vanishes in the massless case, i.e., for $a = 0$. This reproduces the same conclusion as in the RST and BPP models of dilaton gravity [4–6]. There is, however, a subtle distinction between the present model and the BPP model. In the BPP case the linear dilaton vacuum is characterized by a nonzero energy flux at asymptotic infinity. In contrast, in our model the limit $a \rightarrow 0$ yields an exact Minkowski vacuum with no energy flux at infinity. On the other hand, this behavior coincides with that of the RST model.

Furthermore, we observe that when the central charge of the conformal fields satisfies $\varepsilon < 0$, the singularity is removed from this Boulware state within the first order of the perturbation theory. This feature has been examined in detail in the framework of the RST model [34,35]. A negative central charge may arise through the inclusion of nonphysical, ghostlike fields. In addition, numerical studies within the DREH model [36] also indicate the emergence of this kind of behavior.

IV. EVAPORATING BLACK HOLE SCENARIO

The evaporating black hole scenario encapsulates the process of gravitational collapse into a black hole, and its subsequent evaporation through Hawking radiation. The classical gravitational collapse has been studied in Sec. II. Without quantum backreaction to the metric, the process of evaporation cannot be studied. We have already seen that the black hole does not start to evaporate in the classical case, since the horizon and the singularity do not intersect in finite time. At the beginning, the metric of space-time is that of Minkowski space-time. This is true even when the quantum corrections are included, since the Minkowski metric solves Eqs. (59)–(61), as we have commented at the end of the previous section. This means that the quantum state is the Minkowski vacuum state. But because the black

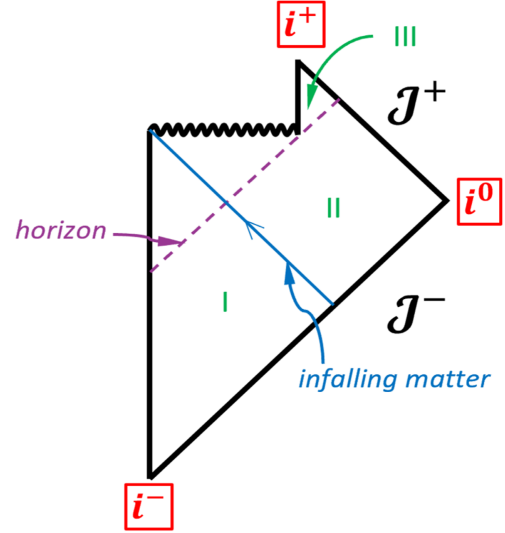


FIG. 2. Penrose diagram of an evaporating black hole.

hole is created later on, it is actually the Unruh state $|U\rangle = |\sigma, 0\rangle$, defined as $t_{\pm}(\sigma^{\pm}) = 0$. After the creation of the black hole, the asymptotically flat coordinates change to $\hat{\sigma}^{\pm}$, whose classical parts are defined by Eqs. (42) and (43). In these new coordinates, the energy flux at future infinity is nonzero, since the vacuum state is not $|\hat{\sigma}, 0\rangle$, so the black hole starts to evaporate.

Before the creation of the black hole (part I of the space-time shown in Fig. 2), we choose to use the Eddington-Finkelstein gauge, which implies that coordinates $x^{\pm}$ in solutions (88) and (89) are actually $\sigma^{\pm}$ coordinates. The solution is then given by $(-1, 1, a, 0, 0)$. In part II of space-time there exists an evaporating black hole. The Eddington-Finkelstein coordinates in this part of space-time are $\hat{\sigma}^{\pm}$, which implies that we need to calculate $t_{\pm}(\hat{\sigma}^{\pm})$. This can be easily done using the transformation law (A6) of the functions $t_{\pm}$, from the coordinates $\sigma^{\pm}$ to the coordinates $\hat{\sigma}^{\pm}$. Since that coordinate transformation (28) is given in terms of functions $F^{\pm}$, defined in (41) the computation comes down to calculating the Schwarzian derivative. The result is given in terms of the reduced field $\hat{x}$:

$$t_{+}(\hat{\sigma}^{+}) = 0 \quad \text{and} \quad t_{-}(\hat{\sigma}^{-}) = -\frac{\lambda^2}{4(\lambda a)^2} \frac{\hat{x} - \frac{3}{4}}{\hat{x}^4}. \quad (91)$$

Notice that space-time in Fig. 2 differs from that of Fig. 1 by the existence of a third region (III), which is space-time left after the black hole evaporates completely. The next step would be to calculate θ [from Eq. (83)] in part II of space-time, since it holds all the necessary information regarding the metric and dilaton field. The solution is defined up to the integral I defined by Eq. (82). Since $t_{+} = 0$, the integral I_{+} vanishes, and we are only left with the integral I_{-} :

$$I = \int \frac{d\sigma^-}{F^-} \frac{1}{\varphi - \lambda a} \int \frac{d\sigma^-}{F^-} t_-(\hat{\sigma}^-). \quad (92)$$

The inner integral is given in terms of reduced field $\hat{x}(\sigma^-)$ by the following expression:

$$\begin{aligned} \int \frac{d\sigma^-}{F^-} t_-(\hat{\sigma}^-) &= \frac{1}{8a} F(\hat{x}) \\ &= \frac{1}{8a} \left[\ln \left(1 - \frac{1}{\hat{x}} \right) + \frac{1}{\hat{x}} - \frac{3}{2\hat{x}^2} \right], \end{aligned} \quad (93)$$

where we have defined another function F . Now, the integral I can be rewritten as

$$I = -\frac{1}{4\lambda a} \int \frac{\hat{x} d\hat{x}}{\hat{x} - 1} \frac{F(\hat{x})}{x - 1}. \quad (94)$$

A. Calculating the integral I

The main problem arising here is the fact that we have to calculate the integral with respect to $\hat{x}$, but the integrand also depends on the field x , and those two are connected to each other through the transcendental equation (45), which can be rewritten in the form

$$e^{\hat{x}}(\hat{x} - 1) = \delta e^x(x - 1). \quad (95)$$

To proceed, we need to solve Eq. (95) with respect to the variable $\hat{x} = \hat{x}(x, \delta)$. Using the fact that $\delta < 1$ in

part II of space-time, Eq. (95) can be solved analytically. The detailed solution is given in the Appendix B (Theorems C.1 and C.4). Here we give the final answer:

$$\hat{x} = \begin{cases} 1 - \sum_{n=1}^{\infty} \frac{\delta^n}{n!} e^{n(x-1)} (1-x)^n P_{n-1}(1), & x \leq 1 \\ x - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} \left(1 - \frac{1}{x}\right)^n P_{n-1}\left(\frac{1}{x}\right), & x > 1 \end{cases}, \quad (96)$$

where $P_n(x)$ are the polynomials defined in Theorem C.1. For this calculation, the function $\ln \hat{x} = \ln \hat{x}(x, \delta)$ will also be needed. It is given by (Theorems C.7 and C.9)

$$\ln \hat{x} = \begin{cases} -\sum_{n=1}^{\infty} \frac{\delta^n}{n!} e^{n(x-1)} (1-x)^n Q_{n-1}(1), & x \leq 1 \\ \ln x - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} \left(1 - \frac{1}{x}\right)^n Q_{n-1}\left(\frac{1}{x}\right), & x > 1 \end{cases}, \quad (97)$$

where $Q_n(x)$ are the polynomials defined in Corollary C.7.1. Now, using the expansions (96) and (97) and the uniform convergence (see Appendix B), it is easy to show that I takes the form

$$I(x, \delta) = \frac{1}{8\lambda a} \left\{ \begin{aligned} &I_0^<(x) + I_{-1}^<(x) \ln \delta - \sum_{n=1}^{\infty} \frac{\delta^n}{n!} I_n^<(x) \\ &I_0^>(x) + I_{-1}^>(x) \ln \delta - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} I_n^>(x) \end{aligned} \right\}, \quad (98)$$

where the newly defined functions I_n are given by

$$\begin{aligned} I_0^<(x) &= \frac{1}{x-1} - \frac{x-3}{x-1} \ln(1-x) - \ln^2(1-x) - 2x + c_0, \\ I_0^>(x) &= \frac{1}{x-1} - \frac{x-3}{x-1} \ln\left(1 - \frac{1}{x}\right) - \ln^2(x-1) - 2L(x) + c_0, \\ I_{-1}^>(x) &= I_{-1}^<(x) = 2\left(\frac{1}{x-1} - \ln|x-1| + c_{-1}\right), \\ I_n^>(x) &= \int \frac{(2x-1)dx}{(x-1)^2} \left(1 - \frac{1}{x}\right)^n P_{n-1}\left(\frac{1}{x}\right) + \int \frac{(2x-3)dx}{(x-1)^2} \left(1 - \frac{1}{x}\right)^n Q_{n-1}\left(\frac{1}{x}\right) - \left(1 - \frac{1}{x}\right)^n \frac{P_{n-1}\left(\frac{1}{x}\right) + 3Q_{n-1}\left(\frac{1}{x}\right)}{x-1} + c_n, \\ I_n^<(x) &= P_{n-1}(1) \int \frac{(2x-1)dx}{(x-1)^2} e^{n(x-1)} (1-x)^n + Q_{n-1}(1) \int \frac{(2x-3)dx}{(x-1)^2} e^{n(x-1)} (1-x)^n \\ &\quad + e^{n(x-1)} (1-x)^{n-1} (P_{n-1}(1) + 3Q_{n-1}(1)) + \tilde{c}_n, \end{aligned} \quad (99)$$

where function $L(x) = \frac{\pi^2}{6} - \int_0^x \frac{ds}{s-1} \ln s$, so that $L(1) = 0$. The constants c_n are arbitrary; comparing them with (88), we can interpret them as the choice of coordinate σ^+ , since the part of I that depends on them is only a function of $\delta = \delta(\sigma^+)$. The constants $\tilde{c}_n$ are not arbitrary since we have

to impose the condition of continuity on the functions I_n , meaning: $I_n^>(1) = I_n^<(1)$.

The integral (94) has been calculated up to the simple integrals appearing in (99). This means that we can write an expression for the function $\theta(x, \delta)$ from Eq. (83):

$$\theta(x, \delta) = \frac{x-1}{x} \left[\frac{\lambda}{2} \left(\int d\sigma^+ \frac{G^+}{F^{+2}} + \int d\sigma^- \frac{G^-}{F^{-2}} \right) + \frac{1}{8\lambda a} \left(S_0(x) + S_{-1}(x) \ln \delta - \sum_{n=1}^{\infty} \frac{\tilde{\delta}^n}{n!} S_n(x) \right) \right], \quad (100)$$

where $S_0(x) = I_0(x) + (8\lambda a)\theta_0(x)$, $S_{-1} = I_{-1}$, and $S_n = I_n$. The function $\theta_0(x)$ is defined in Eq. (81). Note that we have given both cases $x < 1$ and $x > 1$ in Eq. (100), using $\tilde{\delta} = \delta$ for the case of $x < 1$ and $\tilde{\delta} = 1 - \delta$ for the case of $x > 1$. $S_0(x)$ is given by Eq. (102). Other functions $S_n(x)$ are the same as the corresponding functions $I_n(x)$, so they will not be given here again. It is simple to check that the functions $S_n(x)$ satisfy a recurrence relation:

$$n \frac{dS_n}{dx} - \frac{1}{x-1} \frac{d}{dx} \left(\frac{(x-1)^2}{x} \frac{dS_n}{dx} \right) = \begin{cases} \frac{dS_{n+1}}{dx} & x > 1 \\ 0 & x < 1 \end{cases}. \quad (101)$$

$$S_0(x) = \begin{cases} 2 \frac{x^2-2x+2}{x-1} \ln x - (2x-3) \ln(1-x) - \ln^2(1-x) - 2x + 8\lambda a C \left(\frac{1}{x-1} - \ln(1-x) \right) + c_0 & ; x \leq 1 \\ (2x-1) \ln x - (2x-3) \ln(x-1) - \ln^2(x-1) - 2L(x) + 8\lambda a C \left(\frac{1}{x-1} - \ln(x-1) \right) + c_0 & ; x \geq 1 \end{cases}. \quad (102)$$

Using Eq. (70), $\alpha(x, \delta)$ can be calculated. Substituting the solution for θ and α in the equations of motion (62)–(65) we see that (100) is indeed the solution.

B. Evaporating black hole solution

In the previous sections we found a solution in part II of space-time (see Fig. 2) where the black hole has already

been created. It depends on many constants: c_0 , c_{-1} , c_n , $\tilde{c}_n$, and C . Since we can choose constants c_n and $\tilde{c}_n$ as we want (they correspond to the choice of the function G^+) we define them such that $S_n(1) = 0$ and $c_0 = 0$. With the help of Eq. (70) it is easy to derive $\alpha(x, \delta)$. Then, applying the same procedure as before, $\rho + \varepsilon a \mapsto \rho$, $x + \frac{\varepsilon}{\lambda a} \theta \mapsto x$ and $F^\pm + \varepsilon G^\pm \mapsto F^\pm$ [see Eqs. (84) and (85)] we arrive at the following equations:

$$x + \ln|x-1| - \frac{\varepsilon}{8(\lambda a)^2} \left[S_0(x) + S_{-1}(x) \ln \delta - \sum_{n=1}^{\infty} \frac{\tilde{\delta}^n}{n!} S_n(x) \right] = -\frac{\lambda}{2\lambda a} \left[\int \frac{d\sigma^+}{F^+} + \int \frac{d\sigma^-}{F^-} \right] \equiv \frac{\hat{\sigma}^+ - \hat{\sigma}^-}{2a}, \quad (103)$$

$$\begin{aligned} F^+ F^- e^{2\rho} &= \left(\frac{1}{x} - 1 \right) \left\{ 1 + \frac{\varepsilon}{8(\lambda a)^2} \left[\frac{x-1}{x} \frac{dS_0(x)}{dx} - 2c_{-1} - c_1 + \frac{x-1}{x} \frac{dS_{-1}(x)}{dx} \ln \delta \right. \right. \\ &\quad \left. \left. - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} \left(\frac{x-1}{x} \frac{dS_n(x)}{dx} + S_{n+1}(x) - nS_n(x) \right) \right] \right\}, \quad \text{when } x \geq 1, \\ F^+ F^- e^{2\rho} &= \left(\frac{1}{x} - 1 \right) \left\{ 1 + \frac{\varepsilon}{8(\lambda a)^2} \left[\frac{x-1}{x} \frac{dS_0(x)}{dx} - S_{-1}(x) - 8(\lambda a)^2 \mathcal{D}(x) + \frac{x-1}{x} \frac{dS_{-1}(x)}{dx} \ln \delta \right. \right. \\ &\quad \left. \left. - \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \left(\frac{x-1}{x} \frac{dS_n(x)}{dx} - nS_n(x) \right) \right] \right\}, \quad \text{when } x \leq 1. \end{aligned} \quad (104)$$

The only remaining arbitrary constants are c_{-1} and C . The solution in part II of space-time needs to be continuously connected to the solution in part I of space-time over the hypersurface $\sigma^+ = \sigma_0^+$ or $\delta = 1$ (Fig. 1). First, let us examine the “++” equation from (59):

$$\begin{aligned} \partial_+(e^{-2\rho} \partial_+ \varphi) &= \frac{\lambda \lambda a}{2 F^+} \delta(\sigma^+ - \sigma_0^+) \varphi^{-1} e^{-2\rho} \\ &\quad + \varepsilon((\partial_+ \rho)^2 - \partial_+^2 \rho) \varphi^{-1} e^{-2\rho}. \end{aligned} \quad (105)$$

Integrating this equation, we get

$$\begin{aligned} e^{-2\rho} \partial_+ \varphi_{>} - e^{-2\rho} \partial_+ \varphi_{<} \\ = -\frac{\lambda}{2} \lambda a \varphi^{-1} e^{-2\rho} - \varepsilon(\partial_+ \rho_{>} - \partial_+ \rho_{<}) \varphi^{-1} e^{-2\rho}, \end{aligned} \quad (106)$$

where “>” stands for the region where $\sigma^+ > \sigma_0^+$ and “<” stands for the region where $\sigma^+ < \sigma_0^+$. The last term in Eq. (106) warrants explanation; it comes from the last term of Eq. (105). Since $\partial_+ \rho = \frac{\lambda \lambda a}{4 \varphi^2} \eta(\sigma^+ - \sigma_0^+) + o(\varepsilon)$, where $\eta(\sigma^+ - \sigma_0^+)$ represents the step function, we can deduce that the second derivative $\partial_+^2 \rho$ has a term that behaves as a Dirac

δ -function. From Eq. (70) one can show that

$$\partial_+ \varphi_> = \frac{\lambda}{2} F_>^- e^{2\rho} (1 + \varepsilon \mathcal{D}(x)). \quad (107)$$

Since when $\delta = 1$, we have $x = \hat{x}$, $e^{2\rho} = 1$, and $\partial_+ \varphi_< = \lambda/2$. After a brief calculation, we arrive at the following expression for the function $F^-(\hat{x})$:

$$F^- = \left(1 - \frac{1}{\hat{x}}\right) \left[1 - \varepsilon \mathcal{D}(\hat{x}) - \frac{\varepsilon}{2(\lambda a)^2} \frac{1}{\hat{x}^2(\hat{x} - 1)}\right]. \quad (108)$$

We may now impose the continuity condition on ρ and θ . This will be done in two separate cases $x > 1$ and $x < 1$. First, we will consider the case where $x > 1$. Inserting $\delta = 1$ into Eq. (104) and using Eq. (108) we arrive at

$$\frac{8\lambda a C - 2}{\hat{x} - 1} + 2c_{-1} + c_1 = 0. \quad (109)$$

Equation (109) implies that $C = \frac{1}{4\lambda a}$ and $c_{-1} = -c_1/2$. Since $c_1 = 7$ has already been determined from $S_1(1) = 0$, where $S_1(x) = 2 \ln x - \frac{4}{x} - \frac{3}{x^2} + c_1$, it follows $c_{-1} = -\frac{7}{2}$. The other continuity condition is $\theta(\hat{x}, 1) = 0$. Using Eq. (103), this condition becomes

$$\frac{\lambda}{2} \int d\sigma^+ \frac{G^+}{F^{+2}} \Big|_{\delta=1} + \frac{\lambda}{2} \int d\sigma^- \frac{G^-}{F^{-2}} = -\frac{S_0(\hat{x})}{8\lambda a}. \quad (110)$$

For a completely determined solution, we choose $G^+ = 0$. This gives the following coordinate transformations:

$$F^- = \left(1 - \frac{1}{\hat{x}}\right) \left[1 + \frac{\varepsilon}{8(\lambda a)^2} \frac{\hat{x} - 1}{\hat{x}} \frac{dS_0(\hat{x})}{d\hat{x}}\right], \quad (111)$$

$$F^+ = -1, \quad (112)$$

$$\hat{\sigma}^- = \sigma_0^+ - 2a \left[\hat{x} + \ln |\hat{x} - 1| - \frac{\varepsilon}{8(\lambda a)^2} S_0(\hat{x}) \right], \quad (113)$$

$$\hat{\sigma}^+ = \sigma^+. \quad (114)$$

Next, we consider the $x < 1$ case. When $\delta = 1$ is inserted into Eq. (104), it reduces to the condition

$$\begin{aligned} 2 \ln \hat{x} - 2 \frac{\ln \hat{x}}{\hat{x} - 1} - \frac{3}{\hat{x}} - 2 - 2c_{-1} + 2 \frac{1 - 4\lambda a C}{\hat{x} - 1} \\ = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{\hat{x} - 1}{\hat{x}} \frac{dS_n(\hat{x})}{d\hat{x}} - n S_n(\hat{x}) \right]. \end{aligned} \quad (115)$$

Inserting $\delta = 1$ into Eqs. (96) and (97), we find

$$\hat{x} = 1 - \sum_{n=1}^{\infty} \frac{P_{n-1}(1)}{n!} e^{n(\hat{x}-1)} (1 - \hat{x})^n, \quad (116)$$

$$\ln \hat{x} = - \sum_{n=1}^{\infty} \frac{Q_{n-1}(1)}{n!} e^{n(\hat{x}-1)} (1 - \hat{x})^n. \quad (117)$$

Using Eq. (99) we obtain the following derivatives of $S_n(x)$:

$$\begin{aligned} \frac{dS_n^<(\hat{x})}{d\hat{x}} &= \hat{x}(1 - \hat{x})^{n-2} e^{n(\hat{x}-1)} \\ &\times [(2 - n)P_{n-1}(1) + (2 - 3n)Q_{n-1}(1)]. \end{aligned} \quad (118)$$

Changing Eq. (118) into the first sum in Eq. (115) and using (116) and (117) it is easy to calculate the corresponding sum. We get

$$\frac{\hat{x} - 1}{\hat{x}} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{dS_n^<(\hat{x})}{d\hat{x}} = -2 - 2 \frac{\ln \hat{x}}{\hat{x} - 1} + \frac{1}{\hat{x}} + \frac{3}{\hat{x}^2}. \quad (119)$$

The second sum appearing in Eq. (115) can be easily calculated after taking the derivative, using the same method as in the case of the previous sum, and integrating the result with the initial condition that the sum is equal to zero when $\hat{x} = 1$. The final result is given by

$$- \sum_{n=1}^{\infty} \frac{S_n^<(\hat{x})}{(n-1)!} = 2 \ln \hat{x} - \frac{4}{\hat{x}} - \frac{3}{\hat{x}^2} + 7 \equiv S_1^>(\hat{x}). \quad (120)$$

Substituting the sums (119) and (120) into Eq. (115) returns condition (109), as expected. The last condition we need to impose is the continuity of the function θ . Using Eq. (103), this continuity condition may be recast in the following form:

$$S_0^>(\hat{x}) = S_0^<(\hat{x}) - \sum_{n=1}^{\infty} \frac{S_n^<(\hat{x})}{n!}. \quad (121)$$

Again, with the help of Eqs. (116) and (117) this sum can be calculated. The result is given by

$$\sum_{n=1}^{\infty} \frac{S_n^<(\hat{x})}{n!} = 2L(\hat{x}) - 2\hat{x} - \frac{\hat{x} - 3}{\hat{x} - 1} \ln \hat{x}. \quad (122)$$

Substituting Eq. (122) into Eq. (121), one can show that the continuity condition is met.

C. Asymptotically flat solution

Our final task is to check if the solution given by Eqs. (103) and (104) is asymptotically flat. To do this, it is instructive to better understand the form of $S_n^>(x)$ as $x \rightarrow \infty$. We show that the following equation holds when $n > 1$:

$$S_n^>(x) = (n-1)!S_1^>(x) + \frac{6}{x^4}Z_{2(n-2)}\left(\frac{1}{x}\right) - 6Z_{2(n-2)}(1), \quad (123)$$

where $Z_{2(n-2)}(x)$ are polynomials of degree $2(n-2)$. The expression (123) is written with the condition $S_n(1) = 0$ in mind, since $S_1(1) = 0$. The last two terms in Eq. (123) will be denoted by $R_n(x)$. Using the recurrence relation (101), the following recurrence relation for the functions R_n can be derived:

$$\frac{dR_{n+1}}{dx} - n\frac{dR_n}{dx} = -(n-1)!\frac{24}{x^5} - \frac{1}{x-1}\frac{d}{dx}\left(\frac{(x-1)^2}{x}\frac{dR_n}{dx}\right). \quad (124)$$

Now we can substitute $\frac{dR_n}{dx} = -\frac{24}{x^5}M_n(x)$ into Eq. (124), which yields

$$M_{n+1} - nM_n = (n-1)! + 2\left(\frac{2}{x} - \frac{3}{x^2}\right)M_n - \frac{x-1}{x}\frac{dM_n}{dx}. \quad (125)$$

Since $M_2(x) = 1$, Eq. (125) implies that the functions $M_n(x)$ are polynomials of $1/x$ and that their degree increases by two when n increases by one. This means that the degree of n th polynomial is $2(n-2)$. After integration of the functions M_n/x^5 , we end up with the form (123) for the functions $S_n^>(x)$. To simplify future expressions, we will denote $6Z_{2(n-2)}(1) = z_n$. To see if space-time is asymptotically flat, let us check the behavior of the metric when $x \rightarrow \infty$. This will be done term by term using Eq. (104). The term that does not depend on δ is given by

$$\left(\frac{x-1}{x}\right)^2 \frac{dS_0^>}{dx} = -2\frac{x-1}{x} \ln\left(1 - \frac{1}{x}\right) - \frac{2}{x} + \frac{1}{x^2} - \frac{1}{x^3}, \quad (126)$$

which implies that this term tends to zero when $x \rightarrow \infty$. The $\ln \delta/x$ term also tends to zero when $x \rightarrow \infty$. Since $\frac{dS_n^>}{dx} \sim (n-1)!\frac{2}{x}$ when $x \rightarrow \infty$, this term also vanishes. The last remaining term in the sum is

$$S_{n+1}^> - nS_n^> = -(z_{n+1} - nz_n) + \mathcal{O}\left(\frac{1}{x^4}\right). \quad (127)$$

This implies that we are left with the following expression:

$$\lim_{x \rightarrow \infty} F^+ F^- e^{2\rho} = -1 - \frac{\varepsilon}{8(\lambda a)^2} \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} (z_{n+1} - nz_n). \quad (128)$$

From Eq. (128) we conclude that in $\hat{\sigma}^\pm$ coordinates, defined in (113) and (114), the metric is not asymptotically flat. Since the metric at infinity depends only on σ^+ , we can try to find a coordinate transformation that makes the metric flat at infinity: $ds^2 = e^{2\rho} d\sigma^+ d\sigma^- = -F^+ F^- e^{2\rho} d\hat{\sigma}^+ d\hat{\sigma}^- = -F^+ F'^+ F^- e^{2\rho} d\hat{\sigma}'^+ d\hat{\sigma}^-$. Demanding that the metric be flat at infinity in the coordinates $\hat{\sigma}^-$ and $\hat{\sigma}'^+$ boils down to choosing

$$F'^+ = 1 - \frac{\varepsilon}{8(\lambda a)^2} \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} (z_{n+1} - nz_n). \quad (129)$$

To obtain the coordinate $\hat{\sigma}'^+$, the integral $\int \frac{d\hat{\sigma}^+}{F'^+}$ needs to be calculated. The result is given by

$$\hat{\sigma}'^+ = -2a \ln \delta + \frac{a\varepsilon}{4(\lambda a)^2} \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} z_n + 2aD, \quad (130)$$

where D is an undetermined constant. Next, we check what happens with Eq. (103) for the dilaton field, at infinity. Expressing $\ln \delta$ from Eq. (130) and substituting it into Eq. (103) yields

$$x + \ln(x-1) - \frac{\varepsilon}{8(\lambda a)^2} \left[S_0^>(x) + S_{-1}(x) \ln \delta - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} (S_n^>(x) + z_n) \right] = \frac{\hat{\sigma}'^+ - \hat{\sigma}^-}{2a} - D. \quad (131)$$

First we will work out the δ -dependent terms in Eq. (131). At infinity $S_n(x) \sim (n-1)!(2 \ln x + 7) - z_n$, allowing us to express the sum appearing in Eq. (131) as

$$-\sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} (S_n^>(x) + z_n) \sim \ln \delta (2 \ln x + 7). \quad (132)$$

This term (132) cancels exactly with the term $S_{-1}(x) \ln \delta$ when $x \rightarrow \infty$. The $S_0^>(x)$ term in (131) behaves as

$$\lim_{x \rightarrow \infty} S_0^>(x) = 2 + \frac{\pi^2}{3}, \quad (133)$$

as $x \rightarrow \infty$. The important step in the derivation of the formula (133) is that the behavior of $L(x)$ at infinity is given by $L(x) \sim -\frac{1}{2} \ln^2(x-1) - \frac{\pi^2}{6}$. At infinity, Eq. (131) reduces to

$$x + \ln(x-1) - \frac{\varepsilon}{4(\lambda a)^2} \left(\frac{\pi^2}{6} + 1 \right) = \frac{\hat{\sigma}'^+ - \hat{\sigma}^-}{2a} - D. \quad (134)$$

Demanding that the asymptotic behavior of the dilaton field (at infinity) be unaffected by quantum corrections leads to the choice of the constant $D = \frac{\varepsilon}{4(\lambda a)^2} \left(\frac{\pi^2}{6} + 1 \right)$. For simplicity, we may redefine the coordinate transformation

of σ^+ by choosing $F^+F'^+ \mapsto F^+$ and $\delta'^+ \mapsto \hat{\sigma}^+$. In this new notation, the coordinate transformations are given by

$$F^- = \left(1 - \frac{1}{\hat{x}}\right) \left[1 + \frac{\varepsilon}{8(\lambda a)^2} \frac{\hat{x} - 1}{\hat{x}} \frac{dS_0(\hat{x})}{d\hat{x}}\right], \quad (135)$$

$$F^+ = -1 + \frac{\varepsilon}{8(\lambda a)^2} \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} (z_{n+1} - nz_n). \quad (136)$$

The new asymptotically flat coordinates are given by

$$\hat{\sigma}^- = \sigma_0^+ - 2a \left[\hat{x} + \ln |\hat{x} - 1| - \frac{\varepsilon}{8(\lambda a)^2} S_0(\hat{x}) \right], \quad (137)$$

$$\hat{\sigma}^+ = \sigma^+ + \frac{2a\varepsilon}{8(\lambda a)^2} \left[\frac{\pi^2}{3} + 2 + \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} z_n \right]. \quad (138)$$

We define a new $\hat{\delta}$ coordinate in terms of $\hat{\sigma}^+$:

$$\hat{\delta} = \exp \left(-\frac{\hat{\sigma}^+ - \sigma_0^+}{2a} \right). \quad (139)$$

Now we can write down the final expression of the metric and the dilaton field in part II of the space-time (see Fig. 1),

$$F^+F^-e^{2\rho} = \left(\frac{1}{x} - 1 \right) \left\{ 1 + \frac{\varepsilon}{8(\lambda a)^2} \left[\frac{x-1}{x} \frac{dS_0^>(x)}{dx} - \frac{2\ln \delta}{x-1} - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} \left(\frac{x-1}{x} \frac{dS_n^>(x)}{dx} + S_{n+1}^>(x) - nS_n^>(x) \right) \right] \right\} \\ \times \left[1 - \frac{\varepsilon}{8(\lambda a)^2} \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} (z_{n+1} - nz_n) \right], \quad (140)$$

$$F^+F^-e^{2\rho} = \left(\frac{1}{x} - 1 \right) \left\{ 1 + \frac{\varepsilon}{8(\lambda a)^2} \left[\frac{x-1}{x} \frac{dS_0^<(x)}{dx} + S_1^>(x) - \frac{2\ln \delta}{x-1} - \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \left(\frac{x-1}{x} \frac{dS_n^<(x)}{dx} - nS_n^<(x) \right) \right] \right\} \\ \times \left[1 - \frac{\varepsilon}{8(\lambda a)^2} \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} (z_{n+1} - nz_n) \right] \quad (141)$$

$$x + \ln(x-1) - \frac{\varepsilon}{8(\lambda a)^2} \left[S_0^>(x) - \frac{\pi^2}{3} - 2 + \ln \delta S_{-1}(x) - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} (S_n^>(x) + z_n) \right] \\ = -\ln \hat{\delta} + \hat{x} + \ln(\hat{x}-1) - \frac{\varepsilon}{8(\lambda a)^2} S_0^>(\hat{x}) \equiv \frac{\hat{\sigma}^+ - \hat{\sigma}^-}{2a} \equiv \frac{\hat{\sigma}}{a}, \quad (142)$$

$$x + \ln(1-x) - \frac{\varepsilon}{8(\lambda a)^2} \left[S_0^<(x) - \frac{\pi^2}{3} - 2 + \ln \delta S_{-1}(x) - \sum_{n=1}^{\infty} \frac{\delta^n}{n!} S_n^<(x) - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} z_n \right] \\ = -\ln \hat{\delta} + \hat{x} + \ln(1-\hat{x}) - \frac{\varepsilon}{8(\lambda a)^2} S_0^<(\hat{x}) \equiv \frac{\hat{\sigma}^+ - \hat{\sigma}^-}{2a} \equiv \frac{\hat{\sigma}}{a}. \quad (143)$$

Equation (140) represents the solution for the metric when $x \geq 1$, while Eq. (141) represents the solution for the metric when $x \leq 1$. Equations (142) and (143) represent the transcendental equations for the dilaton field $x = \frac{1}{\lambda a} e^{-\phi}$ when $x \geq 1$ and $x \leq 1$, respectively, in terms of the coordinates σ^+ (through the dependence of δ) and σ^- (through the dependence of $\hat{x}$). The functions $S_0^{>/<}$ and $S_n^{>/<}(x)$ that appear in Eqs. (140)–(143) are given by Eqs. (102) and (99), respectively. Notice that when $\delta = 1$ both Eqs. (140) and (141) are reduced to Eq. (135) for the coordinate transformation F^- . This implies that $e^{2\rho} = 1$ when $\delta = 1$, which is the continuity condition for the metric. In addition, Eqs. (142) and (143) imply that

$x = \hat{x}$ when $\delta = 1$, which is the continuity condition of the dilaton field. In Eqs. (142) and (143) we can see new asymptotically flat coordinates $\hat{\sigma}^\pm$, defined in Eqs. (137) and (138).

V. THE END-STATE OF BLACK HOLE EVOLUTION

In the previous section we have found an evaporating black hole solution created by collapsing matter. In this section, we investigate the final state of black hole evolution. First, let us calculate the apparent horizon. It is defined by $\partial_+ e^{-2\phi} = 0$, which is equivalent to $x \partial_+ x = 0$. Using formula (107), the direct calculation yields

$$x = 1 + \frac{\varepsilon}{8(\lambda a)^2} \left[2 \ln \delta + 8 - 2(x-2) \ln x - \frac{3}{x} + \frac{4}{x^2} - 5x + (x-1) \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \left(\frac{x-1}{x} \frac{dS_n(x)}{dx} - nS_n(x) \right) \right]. \quad (144)$$

Solving Eq. (144) perturbatively, by choosing ansatz $x_{AH} = x_{AH}^{(0)} + \varepsilon x_{AH}^{(1)}$, we obtain

$$x_{AH} = 1 + \frac{\varepsilon}{4(\lambda a)^2} (2 + \ln \delta). \quad (145)$$

As time elapses, δ gets smaller and smaller, which in turn reduces the apparent horizon. That is, in fact, the expected result. Note that we have used expression (141) in calculating $\partial_+ x$. The same result for the apparent horizon can be derived using the formula (140) since $x_{AH}^{(0)} = 1$. It is not possible to find the dependence $\hat{x}_{AH}(\delta)$ using Eqs. (142) or (143) since they diverge when $x = 1 + \mathcal{O}(\varepsilon)$. This is an artifact of the perturbative expansion. We need to rewrite these equations in a more suitable form. The details are given in Appendix D. Using the new notation defined in Appendix D, and equating expression (D3) to zero, the apparent horizon is given by

$$y_{AH} = \sqrt{1 + \frac{\varepsilon}{(\lambda a)^2}}. \quad (146)$$

It is easy to check that this equation reproduces the same result (145). Using formula (D23) the $\hat{x}(\delta)$ dependence of the apparent horizon is given by

$$\hat{x}_{AH} = 1 + \frac{\varepsilon}{4(\lambda a)^2} \left[1 + \left(1 + \frac{\varepsilon}{4(\lambda a)^2} \ln \delta \right)^{\frac{4(\lambda a)^2}{\varepsilon}} \right]. \quad (147)$$

After taking the limit $\varepsilon \rightarrow 0$ of the term within the square brackets, since that term is already of $\mathcal{O}(\varepsilon)$ order, we get the following simplified expression for the apparent horizon in $(\hat{x}, \delta)$ coordinates:

$$\hat{x}_{AH}(\delta) = 1 + \frac{\varepsilon}{4(\lambda a)^2} (1 + \delta). \quad (148)$$

The next step is to calculate the line of the singularity. In order to do that, we need to calculate the scalar curvature. It is given by $R = 8e^{-2\rho} \partial_+ \partial_- \rho$. With the help of the equations of motion (60) and (61) the curvature can be expressed in terms of the first derivatives of the dilaton field:

$$R = \frac{2}{a^2} \frac{1 + 4a^2 e^{-2\rho} \partial_+ x \partial_- x}{x^2 - \frac{\varepsilon}{(\lambda a)^2}}. \quad (149)$$

Note that Eq. (149) naively implies a change in the position of the singularity from $x_S = 0$ to $x_S = \frac{\sqrt{\varepsilon}}{\lambda a}$ in the presence of quantum corrections. If this holds true, it might lead to an

issue within the perturbative approach. Using the expression (149) and the recurrence formula (101), the scalar curvature becomes

$$R = \frac{2}{a^2 x^3} \left\{ 1 + \frac{\varepsilon}{8(\lambda a)^2} \left[2 \ln x - 3 - \frac{2}{x} + \frac{15}{x^2} + 2x + 2 \ln \delta + \frac{(x-1)^2}{x} \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \frac{dS_n(x)}{dx} \right] \right\}. \quad (150)$$

Let us first consider the beginning of the evaporation $\delta = 1$. Using the formula (119), we arrive at

$$R = \frac{2}{a^2 x^3} \left(1 + \frac{3\varepsilon}{2(\lambda a)^2} \frac{1}{x^2} \right). \quad (151)$$

Equation (151) yields two solutions for the line of singularity:

$$x_S = 0 \quad \text{and} \quad x_S \approx \sqrt{\frac{3\varepsilon}{2(\lambda a)^2}}. \quad (152)$$

Neither of these two solutions corresponds to the expected result $x_S = \frac{\sqrt{\varepsilon}}{\lambda a}$. This could be a consequence of the perturbative approach. Fortunately, it is possible to find a non-perturbative expression for the scalar curvature at the beginning of the evaporation. We will solve the equations of motion along the $\sigma^+ = \sigma_0^+ + \varepsilon$, $\varepsilon \rightarrow 0$ hypersurface. Here, the metric is fixed: $e^{2\rho} = 1$, and for the dilaton field we have $x = \frac{\sigma_0^+ - \sigma^-}{2a}$. Equation (106) yields

$$x \partial_+ x + \frac{\varepsilon}{(\lambda a)^2} \partial_+ \rho = \frac{x-1}{2a}. \quad (153)$$

Using (153) it is easy to show that Eq. (61) is satisfied. Combining Eqs. (60) and (153) results in the following differential equation for $\partial_+ \rho = f(x(\sigma^-))$:

$$\frac{df}{dx} = -\frac{1}{2a} \frac{1 + 2a \frac{\varepsilon}{(\lambda a)^2} f}{x \left(x^2 - \frac{\varepsilon}{(\lambda a)^2} \right)}. \quad (154)$$

The solution to Eq. (154) is given by

$$f(x) = \frac{\lambda^2 a}{2\varepsilon} \left(\frac{Qx}{\sqrt{x^2 - \frac{\varepsilon}{(\lambda a)^2}}} - 1 \right), \quad (155)$$

where Q is a constant. By direct comparison between Eq. (155) and the solution for $\partial_+ \rho$ derived from (141) when

$\delta = 1$ we conclude that $Q = 1$. Then, the solution for $\partial_{\pm}\rho$ and $\partial_{\pm}x$, along with $\sigma^+ = \sigma_0^+$, is given by

$$\partial_+x = \frac{1}{2a} \left(1 - \frac{1}{\sqrt{x^2 - \frac{\varepsilon}{(\lambda a)^2}}} \right), \quad \partial_-x = -\frac{1}{2a}, \quad (156)$$

$$\partial_+\rho = \frac{\lambda^2 a}{2\varepsilon} \left(\frac{x}{\sqrt{x^2 - \frac{\varepsilon}{(\lambda a)^2}}} - 1 \right), \quad \partial_-\rho = 0. \quad (157)$$

Finally, the scalar curvature reads

$$R = \frac{2}{a^2} \frac{1}{\left(x^2 - \frac{\varepsilon}{(\lambda a)^2} \right)^{\frac{3}{2}}}. \quad (158)$$

We deduce that at the beginning of the evaporation, the singularity appears at

$$x_S = \frac{\sqrt{\varepsilon}}{\lambda a}, \quad (159)$$

as expected. When all the constants are returned, we get $\varphi_S = \sqrt{\frac{\hbar G}{12\pi c^3}}$, which is of Planck length order of magnitude.

Next, we find the quantum-corrected formula for the line of singularity (47). In the y coordinates (introduced in Appendix D), the singularity is given by

$$y_S = \frac{\sqrt{\varepsilon}}{\lambda a} \frac{1}{1 + \frac{\varepsilon}{4(\lambda a)^2} \ln \delta_S}. \quad (160)$$

Placing the expression (160) in Eq. (D20) and expanding to the $\mathcal{O}(\varepsilon)$ term leads to the following equation for the singularity:

$$\begin{aligned} & \left(1 + \frac{9\varepsilon}{8(\lambda a)^2} \right) \ln \delta_S + \frac{\varepsilon}{8(\lambda a)^2} \sum_{n=1}^{\infty} \frac{\delta_S^n}{n!} S_n^<(0) \\ &= \hat{x}_S + \ln |\hat{x}_S - 1| - \frac{\varepsilon}{8(\lambda a)^2} S_0^>(\hat{x}_S). \end{aligned} \quad (161)$$

Equation (161) tells us that the singularity (47) gets a quantum correction of $\mathcal{O}(\varepsilon)$ order.

A. The end point of the evaporation

By the cosmic censorship conjecture, a naked singularity cannot exist. On the other hand, the two hypersurfaces $y_{AH}(\delta)$ [given by Eq. (146)] and $y_S(\delta)$ (160) cross at a finite value of the σ^+ coordinate (or δ coordinate), that is $\tilde{a}_E = \frac{\sqrt{\varepsilon}}{\lambda a}$. The point defined by this value is the end point of the evaporation. It is given by

$$\sigma_E^+ = \sigma_0^+ + \frac{8a(\lambda a)^2}{\varepsilon} \left(1 - \frac{\sqrt{\varepsilon}}{\lambda a} \right). \quad (162)$$

Replacing all the constants, Eq. (162) becomes

$$\sigma_E^+ = \sigma_0^+ + \frac{768\pi M^3 G^2}{\hbar \lambda^4} \left(1 - \frac{\lambda}{4M} \sqrt{\frac{\hbar}{3\pi G}} \right). \quad (163)$$

The formula (163) tells us that the dominant term for the time of evaporation of the black hole scales as M^3 , which is the expected result from a thermodynamics standpoint, as well as the result derived in [9]. It will be useful to express the end point of the evaporation in terms of the δ coordinate:

$$\delta_E = e^{-\frac{4(\lambda a)^2}{\varepsilon} \left(1 - \frac{\sqrt{\varepsilon}}{\lambda a} \right)}. \quad (164)$$

The quantity (164) is very small $\delta_E \ll \frac{\varepsilon}{4(\lambda a)^2}$. On the other hand $\ln \delta_E$ is very large, as it scales as $1/\varepsilon$. This shows that the perturbative approach breaks down at this point. We can still get some useful new insights if we tread carefully. To obtain the exact position of the end point of the evaporation, in addition to δ_E , we need to calculate $\hat{x}_E$. From $y_E = y_{AH} = 1 + \frac{\varepsilon}{2(\lambda a)^2}$, it follows that $\hat{x}_E = 1 + \mathcal{O}(\varepsilon)$. Using the expression (D24) the other coordinate of the end point of the evaporation is

$$\hat{x}_E = 1 + \frac{\varepsilon}{4(\lambda a)^2} \left[1 + e^{-\frac{4(\lambda a)^2}{\varepsilon} \left(1 - \frac{\sqrt{\varepsilon}}{\lambda a} \right)} \right]. \quad (165)$$

Note that formula (165) can be directly derived using (148) by placing $\delta = \delta_E$.

Finally, we also comment on the position of the event horizon of the black hole. It is given by $\hat{x} = \hat{x}_E$. We expect it to appear when $x < 1$ but very close to the $x = 1$ hypersurface at the beginning of the evaporation. Then, it would gradually decrease until it intersects with the singularity. To find the line of the horizon $y_H(\delta)$ let us revisit Eq. (D23). The expression is given by

$$y_H = 1 + \frac{\varepsilon}{4(\lambda a)^2} \left[1 + \left(\frac{\tilde{a}_E}{\tilde{a}_H} \right)^{\frac{4(\lambda a)^2}{\varepsilon}} \right]. \quad (166)$$

Notice that in the limit $\tilde{a}_H \rightarrow \tilde{a}_E$ the expression (166) reduces to $y_H = y_{AH}$, which is the expected result. Now, it is easy to calculate x_H . It is given by the following equation:

$$x_H = \left(1 + \frac{\varepsilon}{4(\lambda a)^2} \ln \delta_H \right) \left[1 + \frac{\varepsilon}{4(\lambda a)^2} \left(1 + \frac{\delta_E}{\delta_H} \right) \right]. \quad (167)$$

From Eq. (149) along the apparent horizon hypersurface, the scalar curvature is the following:

$$R_{AH} = \frac{2}{a^2} \frac{1}{x^2 - \frac{\epsilon}{(\lambda a)^2}}. \quad (168)$$

This expression can also be derived using (150), which can be rewritten as

$$R = \frac{2}{a^2} \frac{1}{x^2 - \frac{\epsilon}{(\lambda a)^2}} \frac{1}{\sqrt{y^2 - \frac{\epsilon}{(\lambda a)^2}}} \left\{ 1 + \frac{\epsilon}{8(\lambda a)^2} \left[2 \ln y - 3 - \frac{2}{y} + \frac{3}{y^2} + 2y + \frac{(y-1)^2}{y} \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \frac{dS_n^<}{dy} \right] \right\}, \quad (169)$$

then, by applying (146) it becomes equivalent to (168), as expected.

B. The radiated energy

In this part, we will determine the amount of energy that was emitted by the black hole throughout its lifespan. It will be calculated as the energy flux at future null infinity from the beginning of the evaporation, $\hat{\sigma}^- \rightarrow -\infty$, until the end point of the evaporation, $\hat{\sigma}^- = \hat{\sigma}_E^-$,

$$E_{\text{rad}} = \int_{-\infty}^{\hat{\sigma}_E^-} d\hat{\sigma}^- \langle \Delta T_{--}^{(f)}(\hat{\sigma}^-) \rangle \Big|_{\mathcal{I}^+}. \quad (170)$$

Considering the fact that the metric is flat along the $\mathcal{I}^+$ hypersurface, the previous integral (170) reduces to

$$E_{\text{rad}} = -\frac{\epsilon}{G} \int_{-\infty}^{\hat{\sigma}_E^-} \frac{d\sigma^-}{F^-} t_-(\hat{\sigma}^-) = -\frac{\epsilon}{8aG} F(\hat{x}_E). \quad (171)$$

Since the Schwarzschild radius λa is directly proportional to the mass, we can use it as a measure of energy. In this regard, we can associate a value of $\lambda a_{\text{rad}} = \frac{2E_{\text{rad}}G}{\lambda}$ to the radiated energy. Then the value of λa_{rad} will be given by

$$\lambda a_{\text{rad}} = -\frac{\epsilon}{4\lambda a} F(\hat{x}_E). \quad (172)$$

The function $F(\hat{x})$ is not well defined when $\hat{x} \rightarrow 1$. This is the same type of problem that we address in Appendix D. One should calculate the first-order correction to the function $F(\hat{x})$:

$$F(\hat{x}) = -2 \frac{dF^-}{d\hat{x}} + \int \frac{d\hat{x}}{F^-} \left(\frac{dF^-}{d\hat{x}} \right)^2. \quad (173)$$

Using Eq. (D20) we arrive at the following equations:

$$\frac{1}{F^-} = \frac{d\mathcal{J}}{d\hat{x}} - \frac{\epsilon}{8(\lambda a)^2} \left(\frac{2\hat{x}}{\hat{x}-1} \ln \hat{x} + \frac{3}{\hat{x}} + 7 \right), \quad (174)$$

$$\frac{dF^-}{d\hat{x}} = -F^{-2} \left[\frac{d^2 \mathcal{J}}{d\hat{x}^2} + \frac{\epsilon}{8(\lambda a)^2} \left(\frac{3}{\hat{x}^2} + 2 \frac{\ln \hat{x} - 1}{\hat{x} - 1} \right) \right]. \quad (175)$$

Since terms of order $\mathcal{O}(\epsilon)$ behave well when $\hat{x} \rightarrow 1$, they can be neglected. After performing substitution to z [defined in Eq. (D10)], the upper limit of integration becomes 1, while the lower limit of integration changes to $z_E = z_1^+ + (\frac{\sqrt{\epsilon}}{\lambda a})^3 \frac{\delta_E}{4}$, where z_1^+ is given by Eq. (D15). Now, the integral is performed over a narrow interval around the point $z = 1$ which allows us to substitute $z = 1$ wherever it is well defined within the integrand function. The integral becomes

$$\int_{\infty}^{\hat{x}_E} \frac{d\hat{x}}{F^-} \left(\frac{dF^-}{d\hat{x}} \right)^2 = -\frac{4(\lambda a)^4}{\epsilon^2} \int_{z_E}^1 \frac{(1-z)^4 dz}{(z-z_1^+)(z-z_2^+)^2}. \quad (176)$$

Direct calculation up to $\mathcal{O}(1)$ order leads to the following expression for the function $F(\hat{x}_E)$:

$$F(\hat{x}_E) = -\frac{4(\lambda a)^2}{\epsilon} + \frac{8(\lambda a)}{\sqrt{\epsilon}} + \ln \frac{\epsilon}{4(\lambda a)^2} + \frac{3}{2}. \quad (177)$$

The radiated energy (171) is given by

$$E_{\text{rad}} = Mc^2 \left[1 - \sqrt{\frac{\hbar c}{12\pi M^2 G_N^{(4)}}} - \frac{\hbar c}{192\pi M^2 G_N^{(4)}} \ln \frac{\hbar c}{192\pi M^2 G_N^{(4)}} - \frac{3\hbar c}{384\pi M^2 G_N^{(4)}} + o\left(\frac{\hbar c}{M^2 G_N^{(4)}}\right) \right], \quad (178)$$

where all of the constants have been returned. Equation (178) implies that during the evaporation process, almost the entire initial mass of the black hole evaporates,

up to a small portion proportional to $\sqrt{\epsilon}$. This result coincides with the results within the RST and BPP models of dilaton gravity [4,6].

C. The end-state geometry

Previously, we calculated the end point of the evaporation. Now we may ask “what is the metric of space-time after the black hole evaporates?” In the perturbative framework, it may be hard to answer this question since it breaks near the end point of the evaporation. However, we may try to find some useful information.

First, let us rewrite the equations for the metric (140) and the dilaton field (142) in a more suitable form. We begin by calculating $F(\hat{x})$ defined by formula (93) in terms of x and δ . Using the same idea as in Theorem C.1,

$$F(\hat{x}) = F(x) + \sum_{n=1}^{\infty} \frac{(\delta-1)^n}{n!} \frac{\partial^n F(\hat{x})}{\partial \delta^n} \Big|_{\delta=1}. \quad (179)$$

Mirroring the method used in the Theorem C.7, we find the expression

$$\frac{\partial^n F(\hat{x})}{\partial \delta^n} \Big|_{\delta=1} = e^{nx} (x-1)^n \left(\frac{e^{-x}}{x} \frac{d}{dx} \right)^{n-1} \frac{e^{-x}}{x} \frac{dF(x)}{dx}. \quad (180)$$

On the other hand, one can demonstrate, by induction, that the following expression is satisfied:

$$\frac{\partial^n F(\hat{x})}{\partial \delta^n} \Big|_{\delta=1} = (-1)^n \left[\frac{1}{2} \frac{(x-1)^2}{x} \frac{dS_n^>(x)}{dx} - (n-1)! \right]. \quad (181)$$

The inductive step reduces to the recurrence formula for the $S_n^>(x)$ functions (101). Upon replacing this expression for $\frac{\partial^n F(\hat{x})}{\partial \delta^n} \Big|_{\delta=1}$ into expansion (179), we get the equation

$$F(\hat{x}) = F(x) + \ln \delta + \frac{(x-1)^2}{2x} \sum_{n=1}^{\infty} \frac{(1-\delta)^2}{n!} \frac{dS_n^>(x)}{dx}. \quad (182)$$

Eliminating the $\ln \delta$ term in the formulas for the metric (140) and the tortoise coordinate $\hat{\sigma}$ (142), with the help of Eq. (182), we arrive at

$$F^+ F^- e^{2\rho} = \frac{1 + \frac{\epsilon}{4(\lambda a)^2} F(\hat{x})}{x} - 1 + \frac{\epsilon}{4(\lambda a)^2} \left[\frac{x-2}{x} \ln \frac{x-1}{x} + \frac{1}{x} - \frac{3}{2x^2} + \frac{2}{x^3} + \frac{x-1}{2x} \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} (S_{n+1}^>(x) - nS_n^>(x) + z_{n+1} - nz_n) \right], \quad (183)$$

$$\begin{aligned} \frac{\hat{\sigma}}{a} = x + \left(1 + \frac{\epsilon}{4(\lambda a)^2} F(\hat{x}) \right) \ln \left(\frac{x}{1 + \frac{\epsilon}{4(\lambda a)^2} F(\hat{x})} - 1 \right) - \frac{\epsilon}{8(\lambda a)^2} \left[S_0^>(x) - \frac{\pi^2}{3} - 2 - 2 \left(\frac{x}{x-1} - \ln(x-1) \right) F(x) \right. \\ \left. - 9 \ln \delta - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} \left(S_n^>(x) + z_n + \frac{(x-1)^2}{x} \left(\frac{x}{x-1} - \ln(x-1) \right) \frac{dS_n^>(x)}{dx} \right) \right]. \end{aligned} \quad (184)$$

In Eqs. (183) and (184) the term $\varphi/\lambda a$ has been replaced by the term $\varphi/(\lambda a + \frac{\epsilon}{4\lambda a} F(\hat{x}))$, in zeroth order in ϵ . Let us define a new, running, Schwarzschild radius, dependent on the σ^- coordinate:

$$\lambda \hat{a}(\hat{x}) = \lambda a + \frac{\epsilon}{4\lambda a} F(\hat{x}). \quad (185)$$

From the perspective of the first-order terms in ϵ , $\lambda \hat{a}$ is equal to λa , so we can exchange λa in the equations for the metric and the dilaton field for $\lambda \hat{a}$.

To understand the final state of evaporation, we first must examine what happens along the $\hat{x} = \hat{x}_E$ hypersurface when $\delta < \delta_E$. Along this hypersurface $\lambda \hat{a}(\hat{x}_E)$ is constant. Since $F(\hat{x}_E) \sim \ln \delta_E$, Eq. (177) implies $\lambda \hat{a} = \mathcal{O}(\sqrt{\epsilon})$, which within the first order of perturbation theory gives $\lambda \hat{a} = 0$. Taking this limit is akin to taking the $x \rightarrow \infty$ limit discussed in the previous section. The form of $S_n^>(x)$ from Eq. (123) implies that

$$S_n^>(x) = (n-1)! S_1^>(x) - z_n + \mathcal{O}((\lambda \hat{a})^4). \quad (186)$$

Taking the limit $\lambda \hat{a} \rightarrow 0$ in the first-order term of the metric equation (183) reduces that term to zero, since the sum behaves as $\mathcal{O}((\lambda \hat{a})^2)$ and the δ -independent term is $\mathcal{O}(\lambda \hat{a})$. Next, we take the limit $\lambda \hat{a} \rightarrow 0$ of the first-order term in ϵ of the dilaton equation (184). In this limit, the δ -dependent term becomes

$$\frac{\lambda \hat{a} \ln \delta}{(\lambda \hat{a})^2} \left[S_1^> + \frac{(x-1)^2}{x} \left(\frac{x}{x-1} - \ln(x-1) \right) \frac{dS_1^>}{dx} - 9 \right]. \quad (187)$$

This term vanishes as can be seen by expanding in terms of $\lambda \hat{a}$ and then taking the $\lambda \hat{a} \rightarrow 0$ limit. An important observation to make is the fact that $\ln \delta$ behaves as $1/\lambda \hat{a}$, which means that $\lambda \hat{a} \ln \delta$ is constant. The δ -independent terms vanish almost in the same way that they vanish when taking

the $x \rightarrow \infty$ limit. By taking this limit, we get the following results:

$$F^+ F^- e^{2\rho} = \frac{\lambda \hat{a}_E}{\varphi} - 1, \quad (188)$$

$$\varphi + \lambda \hat{a}_E \ln \left(\frac{\varphi}{\lambda \hat{a}_E} \right) = \lambda \hat{\sigma}. \quad (189)$$

Since $\lambda \hat{a}_E = \lambda \hat{a}(\hat{x}_E)$ is of order $\mathcal{O}(\sqrt{\varepsilon})$, Eqs. (188) and (189) define a quantum-corrected Minkowski space-time, discussed in Sec. III B. This is a good sign since we are expecting that the final state should be one which does not exhibit any energy fluxes at infinity. The next step is to smoothly extend this solution into the vacuum solution along the $\hat{x} = \hat{x}_E$. This extension must be carried out so that no additional discontinuity appears in the coordinates, implying that the final solution takes the form (F^+, F^-, a_f) , where $F^\pm$ coincides with the coordinate transformations of the evaporating black hole, and a_f denotes a new constant.

Comparing to BPP or RST models of dilaton gravity [4–6], one could expect a shockwave at the end point of the evaporation known as a thunderpop. To check if this occurs in the DREH model, one needs to continuously connect regions II and III (Fig. 2) along the $\sigma^- = \sigma_E^-$ hypersurface by integrating the “—” equation from (59):

$$e^{-2\rho} \partial_- \varphi_> - e^{-2\rho} \partial_- \varphi_< = -\frac{\lambda}{2F^-} \lambda a_0 \varphi^{-1} e^{-2\rho} - \varepsilon (\partial_- \rho_> - \partial_- \rho_<) \varphi^{-1} e^{-2\rho}, \quad (190)$$

where $\lambda a_0 = \frac{2\Delta MG}{\lambda}$ represents the mass of the shockwave at the end point of the evaporation, “<” represents the part of space-time where $\sigma^- < \sigma_E^-$ and “>” represents the part of space-time where $\sigma^- > \sigma_E^-$. The derivatives appearing in Eq. (190) can be easily calculated using Eqs. (140) and (142) in a $\sigma^+ < \sigma_0^+$ region of space-time. The results are given by

$$\partial_- \varphi_< = -\frac{\lambda}{2F^-} \left\{ 1 - \frac{\lambda \hat{a}(\hat{x})}{\sqrt{\varphi^2 - \varepsilon}} \times \left[1 - \frac{\varepsilon}{4(\lambda a)^2} \left((2-x)F(x) - \frac{2}{x} + \frac{3}{x^2} \right) \right] \right\}, \quad (191)$$

$$\partial_- \rho_< = -\frac{1}{2} \partial_- \ln F^- - \frac{\lambda}{4F^-} \frac{\lambda \hat{a}(\hat{x})}{\varphi^2} \left[1 - \frac{\varepsilon}{2(\lambda a)^2} F(x) \right]. \quad (192)$$

Employing expressions (88) and (89) for the static solution discussed in Sec. III B, we calculate the derivatives in the region $\sigma^- > \sigma_E^-$. The resulting derivatives coincide with (191) and (192) after exchanging $\hat{a}$ with a_f . This further suggests that the final state is indeed a static solution of

Sec. III B. Analogs to the case of Eqs. (184) and (183) when $\hat{x} = \hat{x}_E$ we take the limit $\lambda \hat{a} \rightarrow 0$ within the first order of the perturbation theory. Now, Eqs. (191) and (192) reduce to

$$\partial_- \varphi_< = -\frac{\lambda}{2F^-} \left(1 - \frac{\lambda \hat{a}_E}{\varphi} \right), \quad (193)$$

$$\partial_- \rho_< = -\frac{1}{2} \partial_- \ln F^- - \frac{\lambda}{4F^-} \frac{\lambda \hat{a}_E}{\varphi^2}, \quad (194)$$

while in the region $\sigma^- > \sigma_E^-$ we have

$$\partial_- \varphi_< = -\frac{\lambda}{2F^-} \left(1 - \frac{\lambda a_f}{\varphi} \right), \quad (195)$$

$$\partial_- \rho_< = -\frac{1}{2} \partial_- \ln F^- - \frac{\lambda}{4F^-} \frac{\lambda a_f}{\varphi^2}. \quad (196)$$

By substituting expressions (193)–(196) into Eq. (190), we obtain the following form of the energy balance:

$$\lambda a_{\text{rad}} = \lambda(a - a_f) - \lambda a_0, \quad (197)$$

where we have identified $\lambda a_{\text{rad}} = \frac{2E_{\text{rad}}G}{\lambda} = -\frac{\varepsilon}{4\lambda a} F(\hat{x}_E)$ through Eq. (171). Now it is clear that the energy is conserved, i.e. the initial mass of the infalling matter λa is equal to the sum of the radiated energy λa_{rad} , the mass of the final space-time λa_f and the energy of the thunderpop λa_0 .

Since no additional coordinate transformation is allowed as the $\hat{x} = \hat{x}_E$ hypersurface is crossed, the continuity of the metric and dilaton field, through Eqs. (188) and (189), requires that the mass of the final geometry satisfies $\lambda a_f = \lambda \hat{a}_E$. Combined with the energy balance relation (197), this indicates that there is no thunderpop at the end point of the evaporation, i.e. $\lambda a_0 = 0$. This conclusion holds within the perturbative treatment to first order in ε .

Another key point that needs to be discussed is that the asymptotically flat solution presented in Sec. III B contains a curvature singularity within the first order of perturbative expansion. This feature is analogous to what appears in the BPP and RST models [4–6]. As in those cases, we are required to impose reflective boundary conditions at the singularity hypersurface.

To summarize, the final geometry corresponds to one of the static, quantum-corrected Minkowski vacuum solutions of Sec. III B, characterized by $t_\pm(\hat{\sigma}^\pm) = 0$. Consequently, although the asymptotically flat coordinates remain continuous across the $\hat{x} = \hat{x}_E$ hypersurface and no thunderpop appears at the end of evaporation, the quantum state nonetheless changes from $|0, \sigma\rangle$ to $|0, \hat{\sigma}\rangle$.

It is important to note that we cannot be sure of the precise form of the ε correction to the metric in the $\lambda \hat{a}_E$ term in (188), since at this point the perturbation theory breaks

and higher-order terms may influence the first-order terms. There may exist an additional contribution beyond $-\frac{\epsilon}{4\lambda a}F(\hat{x}_E)$. Such a term would lead to a nonvanishing thunderpop. To investigate this, one needs either a non-perturbative solution describing the end point of evaporation, or a numerical treatment of the equations of motion (59)–(61). On the other hand, this fact does not influence the derivation of the energy balance relation (197).

VI. CONCLUSION

In this paper, we studied a model of two-dimensional dilaton gravity related to the four-dimensional Einstein-Hilbert action by dimensional reduction. The gravitational collapse scenario has been constructed within this model. Initially, we explored the classical scenario, after which we analyzed the backreaction of quantum matter fields on the geometry by incorporating the Polyakov-Liouville action into the classical DREH framework. At past time infinity the metric was that of Minkowski space-time. Then, collapsing matter was formed as an infalling shockwave, which created the singularity when it touched the boundary of space-time. The metric of space-time with an evaporating black hole has been calculated within a perturbative framework. It has been found that, due to quantum corrections, the singularity changes position from $r = 0$ to $r = \sqrt{\frac{\hbar G_N^{(4)}}{12\pi c^3}}$, which is of Planck length order of magnitude. The end point of evaporation has been identified and it coincides with the thermodynamic result. It has been shown that, at this point, the perturbative approach breaks, but still there have been insights that suggest that the final state of the evaporation is the quantum-corrected Minkowski vacuum.

The groundwork for this project was laid in our previous work [8], in which we explored the case of an eternal black hole scenario and successfully derived the corresponding Page curve. The following stage focuses on generating the Page curve for a black hole undergoing evaporation [37], utilizing the solution presented in the current study. It is important to note that there are relatively few instances where a Page curve has been reproduced for an evaporating black hole, particularly in the context of a Schwarzschild-like solution.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: TRANSFORMATION LAWS OF THE ENERGY-MOMENTUM TENSOR

The quantum correction to the energy-momentum tensor is defined by Eq. (53). In the conformal gauge $ds^2 = e^{2\rho}dx^+dx^-$, it can be rewritten as

$$\langle \Delta T_{\pm\pm}^{(f)} \rangle = \frac{\epsilon}{G} [\partial_{\pm}^2 \rho - (\partial_{\pm} \rho)^2 - t_{\pm}(x^{\pm})], \quad (A1)$$

$$\langle \Delta T_{+-}^{(f)} \rangle = -\frac{\epsilon}{G} \partial_+ \partial_- \rho, \quad (A2)$$

where $t_{\pm}(x^{\pm}) = \frac{1}{2} \partial_{\pm}^2 f_{\pm} - \frac{1}{4} (\partial_{\pm} f_{\pm})^2$ is a function related to the state of the quantum fields [Eq. (56)]. Under a conformal coordinate transformation $y^{\pm} = y^{\pm}(x^{\pm})$ the energy-momentum tensor changes according to (we only consider the quantum correction)

$$\langle \Delta T_{\pm\pm}^{(f)}(y) \rangle = \left(\frac{dx^{\pm}}{dy^{\pm}} \right)^2 \langle \Delta T_{\pm\pm}^{(f)}(x) \rangle. \quad (A3)$$

For the conformal factor ρ , on the other hand, we have the following transformation law:

$$\rho(y) = \rho(x) + \frac{1}{2} \ln \frac{dy^+ dy^-}{dx^+ dx^-}. \quad (A4)$$

These give the transformation law for $t_{\pm}$:

$$t_{\pm}(y^{\pm}) = \left(\frac{dx^{\pm}}{dy^{\pm}} \right)^2 \left[t_{\pm}(x^{\pm}) - \frac{1}{2} D_{x^{\pm}}[y^{\pm}] \right], \quad (A5)$$

with the Schwartz derivative defined by

$$D_{x^{\pm}}[y^{\pm}] = \frac{(y^{\pm})'''}{(y^{\pm})'} - \frac{3}{2} \left(\frac{(y^{\pm})''}{(y^{\pm})'} \right)^2, \quad (A6)$$

where the derivatives are with respect to $x^{\pm}$.

The vacuum state of the quantum fields and the corresponding set of creation/annihilation operators depend on the reference frame, i.e. on the coordinate system. If we introduce normal ordering of the energy-momentum operator for one choice of the vacuum state $|0; x\rangle$, say in the coordinate system $x^{\pm}$, the energy-momentum operator can be decomposed as

$$\hat{T}_{\pm\pm}^{(f)}(x^{\pm}) = \hat{T}_{\pm\pm}^{(f)}(x^{\pm}) : + \langle 0; x | \hat{T}_{\pm\pm}^{(f)}(x^{\pm}) | 0; x \rangle. \quad (A7)$$

If we make a transition to another coordinate system $y^{\pm}$, the energy-momentum tensor will not be ordered normally in general. The transformation law for the normally ordered part is given by

$$:\hat{T}_{\pm\pm}^{(f)}(y^\pm) := \left(\frac{dx^\pm}{dy^\pm}\right)^2 \left[:\hat{T}_{\pm\pm}^{(f)}(x^\pm): + \frac{\hbar}{24\pi} D_{x^\pm}[y^\pm] \right]. \quad (\text{A8})$$

Comparing (A5), (A7), and (A8), we can establish the following relationship:

$$\langle 0, x | : \hat{T}_{\pm\pm}^{(f)}(x^\pm) : | 0, x \rangle = -\frac{\varepsilon}{G} t_\pm(x^\pm), \quad (\text{A9})$$

which provides an interpretation for the quantity $t_\pm$. If $t_\pm(x^\pm) = 0$ in some coordinates $x^\pm$, the energy-momentum tensor is normally ordered in those coordinates, i.e. its expectation value is zero in the vacuum defined in the coordinates $x^\pm$.

APPENDIX B: REVIEW OF THE BPP MODEL

In this section, we give a brief review of the BPP model [6] of 2D dilaton gravity. The main advantage of this model is that it is exactly solvable even when the quantum corrections are included. We will use this fact to demonstrate how the perturbative approach used throughout this paper works and to compare the results to the exact solutions. The model is defined by the following action:

$$\begin{aligned} S = & \frac{1}{2\pi G} \int d^2x \sqrt{-g} e^{-2\phi} [\mathcal{R} + 4(\nabla\phi)^2 + 4\lambda^2] \\ & - \frac{1}{2} \sum_{i=1}^N \int d^2x \sqrt{-g} (\nabla f_i)^2 + N S_{\text{PL}} \\ & + \frac{N\hbar G}{24\pi} \int d^2x \sqrt{-g} [(\nabla\phi)^2 - \phi\mathcal{R}]. \end{aligned} \quad (\text{B1})$$

In this expression, the first line denotes the dilaton gravity action. The second line is the matter action, where S_{PL} corresponds to the Polyakov–Liouville action (50) for the quantum corrections. Lastly, the third line consists of the counterterms introduced to render the equations of motion exactly solvable. After varying the action, and rewriting the subsequent equations of motion using the conformal gauge for the metric, i.e. $ds^2 = -e^{-2\rho} dx^+ dx^-$ we arrive at

$$\begin{aligned} \partial_+ \partial_- e^{-2\phi} + \lambda^2 e^{2\rho} e^{-2\phi} &= 0, \\ \partial_+ \partial_- (\phi - \rho) &= 0, \\ \partial_\pm (e^{-2\rho} \partial_\pm \phi) &= \frac{\pi G}{2} T_{\pm\pm}^{f,cl} e^{-2\rho} e^{2\phi} \\ &\quad - \frac{\varepsilon}{2} [e^{\phi-\rho} \partial_\pm^2 e^{\phi-\rho} + e^{2(\phi-\rho)} t_\pm(x^\pm)], \end{aligned} \quad (\text{B2})$$

where we have introduced the perturbative parameter $\varepsilon = \frac{N\hbar G}{12}$.

1. Exact solution to the BPP EoM

The second equation of motion in (B2) allows us to choose $\phi = \rho$, a choice known as the Kruskal gauge. The remaining equations become directly integrable resulting in the following solution:

$$\begin{aligned} e^{-2\phi} = & -\lambda^2 x^+ x^- - \pi G \int dx^+ \int dx^- (T_{++}^{f,cl} - \varepsilon t_+) \\ & - \pi G \int dx^- \int dx^+ (T_{--}^{f,cl} - \varepsilon t_-). \end{aligned} \quad (\text{B3})$$

To go back to the Eddington–Finkelstein coordinates $\sigma^\pm$, one has to make a coordinate transformation to the asymptotically flat coordinates:

$$\lambda x^\pm = \pm e^{\lambda \sigma^\pm}. \quad (\text{B4})$$

The eternal black hole solution is defined by choosing the Hartle–Hawking vacuum, i.e. $t_\pm(x^\pm) = 0$ and $T_{\mu\nu}^{f,cl} = 0$, where $x^\pm$ are Kruskal coordinates:

$$e^{-2\phi} = -\lambda^2 x^+ x^- + \frac{\pi M G}{\lambda}. \quad (\text{B5})$$

The linear dilaton vacuum solution is obtained by taking the limit of a vanishing mass $M \rightarrow 0$. The metric becomes flat in $\sigma^\pm$ coordinates, while the dilaton is a linear function $\phi = -\lambda\sigma$, hence the name. It is easy to show that this solution has nonzero energy flux at infinity, meaning that it is not really a vacuum solution. To derive a solution without energy flux at infinity, we need to apply the Boulware vacuum condition $t(\sigma^\pm) = 0$. Using the transformation law of the $t_\pm$ (A5) we get $t_\pm(x^\pm) = \frac{1}{4(x^\pm)^2}$. Now we get the following solution known as Minkowski vacuum solution:

$$e^{-2\phi} = -\lambda^2 x^+ x^- - \frac{\varepsilon}{4} \ln(-\lambda^2 x^+ x^-) + C, \quad (\text{B6})$$

where C is the integration constant that defines the zero mass case.

Of greater importance is the evaporating black hole solution. It is obtained by choosing $t_\pm(\sigma^\pm) = 0$ and modeling the classical part of the energy-momentum tensor as an infalling null shockwave of mass M $T_{++}^{f,cl} = \frac{M}{\lambda x_0^+} \delta(x^+ - x_0^+)$. In the part of space-time where $x^+ < x_0^+$ the solution is one of the Minkowski vacuums given by (B6). Then, the solution in the region $x^+ > x_0^+$ is as follows:

$$\begin{aligned} e^{-2\phi} = & -\lambda^2 x^+ (x^- + \Delta) - \frac{\varepsilon}{4} \ln(-\lambda^2 x^+ x^-) \\ & + \frac{\pi G M}{\lambda} + C, \end{aligned} \quad (\text{B7})$$

where $\Delta = \frac{\pi GM}{\lambda^2 x_0^+}$. This expression represents an evaporating black hole with apparent horizon and singularity given by

$$\lambda^2 x_s^+(x_s^- + \Delta) + \frac{\varepsilon}{4} \ln(-\lambda^2 x_s^+ x_s^-) = \frac{\pi GM}{\lambda} + C, \quad (\text{B8})$$

$$-\lambda^2 x_h^+(x_h^- + \Delta) = \frac{\varepsilon}{4}. \quad (\text{B9})$$

The new asymptotically flat coordinates are given by

$$\lambda x^+ = e^{\lambda \hat{\sigma}^+} \wedge \lambda(x^- + \Delta) = -e^{-\lambda \hat{\sigma}^-}, \quad (\text{B10})$$

which implies that $\hat{\sigma}^+ = \sigma^+$, i.e. the plus coordinate does not change. The intersection of these two hypersurfaces occurs at a finite space-time point known as the end point of the evaporation. It can be calculated exactly and is given by

$$x_E^+ = \frac{1}{\lambda^2 \Delta} \left[e^{1 + \frac{4(\pi GM + \lambda C)}{\varepsilon \lambda}} - \frac{\varepsilon}{4} \right],$$

$$x_E^- = -\frac{\Delta}{1 - \frac{\varepsilon}{4} e^{-1 - \frac{4(\pi GM + \lambda C)}{\varepsilon \lambda}}}. \quad (\text{B11})$$

The event horizon is $x^- = x_E^-$. Until this point almost all of the black hole mass has evaporated. The exact amount that has evaporated is expressed as

$$E_{\text{rad}} = M + \frac{\lambda}{\pi G} (C - C^*) - \frac{\varepsilon \lambda \Delta}{4\pi G x_E^-}, \quad (\text{B12})$$

where $C^* = \frac{\varepsilon}{4} (\ln \frac{\varepsilon}{4} - 1)$. At this point, the singularity becomes naked, which is prohibited by cosmic censorship conjecture. It is easy to show that the black hole solution (B7) can be continuously connected to a Minkowski vacuum solution (B6) associated with the new flat coordinates (B10) given by

$$e^{-2\phi} = -\lambda^2 x^+(x^- + \Delta) - \frac{\varepsilon}{4} \ln(-\lambda^2 x^+(x^- + \Delta)) + C^*, \quad (\text{B13})$$

with $C = C^*$. From (B12), we observe that the initial mass $M + \frac{\lambda}{\pi G} C$ is distributed among the mass of the final space-time $\frac{\lambda}{\pi G} C^*$, the radiated energy E_{rad} , and a small amount of negative energy $\frac{\varepsilon \lambda \Delta}{4\pi G x_E^-}$. This remaining portion of the initial mass must be emitted as an outgoing shockwave that escapes the black hole at the end point of evaporation, propagating along the null hypersurface $x^- = x_E^-$. A careful evaluation of the $--$ component of (B2) then yields

$$\partial_-^2 e^{-2\phi} = \frac{\varepsilon}{4} \left[\frac{\eta(x_E^- - x^-)}{(x^-)^2} + \frac{\eta(x^- - x_E^-)}{(x^- + \Delta)^2} \right]$$

$$+ \frac{\varepsilon \Delta}{4x_E^-(x_E^- + \Delta)} \delta(x^- - x_E^-). \quad (\text{B14})$$

In this expression, the term proportional to the δ -function takes the form $\frac{\pi G \Delta M}{x_E^- + \Delta} \delta(x^- - x_E^-)$, where ΔM coincides precisely with the mass of the shockwave. In contrast, the contributions involving the Heaviside functions η correspond to the normally ordered component $t_-(x^-)$ of the energy-momentum tensor. These latter terms indicate that the quantum vacuum transitions from $|0, \sigma\rangle$ for $x^- < x_E^-$ to $|0, \hat{\sigma}\rangle$ for $x^- > x_E^-$.

2. Perturbative approach to the BPP model

We now apply the method presented in this paper to analyze the BPP model. We begin by determining the exact classical collapse scenario similar to the one described in Sec. II B. Next, we incorporate quantum corrections and solve Eq. (B2) perturbatively to first order in ε . Finally, we demonstrate how the energy balance Eq. (B12) and the resulting final space-time configuration (B13) emerge within this perturbative framework.

a. Classical gravitational collapse

Here we closely follow the derivation presented in Sec. II B. We begin by finding the vacuum solution at zeroth order in ε . From the second equation in (B2), we obtain

$$\partial_{\pm} \phi = -\frac{\lambda}{2} F^{\mp} (x^{\mp}) e^{2\rho}. \quad (\text{B15})$$

Substituting this result into the first equation yields

$$\rho - \phi = -\frac{1}{2} \ln F^+ - \frac{1}{2} \ln F^- - \frac{\lambda}{2} \left(\int \frac{dx^+}{F^+} + \int \frac{dx^-}{F^-} \right), \quad (\text{B16})$$

using these two results together with the third equation gives the following general solution for the metric and the dilaton field:

$$F^+ F^- e^{2\rho} = b e^{2\phi} - 1,$$

$$\ln |b - e^{-2\phi}| = -\lambda \left(\int \frac{dx^+}{F^+} + \int \frac{dx^-}{F^-} \right), \quad (\text{B17})$$

where $F^{\pm}$ are arbitrary coordinate transformations leading to the asymptotically flat coordinates $\sigma^{\pm}$ via Eq. (28), while b is a constant related to the mass of space-time. Hence, the most general solution is expressed as (F^+, F^-, b) . The linear dilaton vacuum is given by $(-1, 1, 0)$. It is easy to see that the Kruskal coordinates are indeed given by Eq. (B4) in terms of the flat coordinates.

Next we introduce the infalling matter along the $x^+ = x_0^+$ hypersurface in a standard way $T_{++} = -\frac{M}{F^+} \delta(x^+ - x_0^+)$ and continuously connect the linear dilaton vacuum in the region $x^+ < x_0^+$ to another vacuum solution in the region

$x^+ > x_0^+$. The new vacuum solution when $x^+ > x_0^+$ is as follows:

$$\left(-1, 1 - \frac{\pi MG}{\lambda} e^{-\lambda(\sigma_0^+ - \sigma^-)}, \frac{\pi MG}{\lambda}\right). \quad (\text{B18})$$

This is a black hole solution with mass M . Then we introduce the δ and $\hat{x}$ coordinates in the same manner as in Sec. II B. The solution can be rewritten as

$$F^+ F^- e^{2\rho} = b e^{2\phi} - 1, \quad \ln |b - e^{-2\phi}| = -2 \ln \delta + \ln |b - e^{2\hat{x}}|, \quad (\text{B19})$$

where the constant $b = \frac{\pi MG}{\lambda}$. Notice that the equation for the dilaton field is not a transcendental equation in the case of the BPP model. It is easy to express $\hat{x}$ in terms of ϕ and δ .

b. Quantum-corrected collapse scenario

The Unruh state is specified by setting $t_{\pm}(\sigma^{\pm}) = 0$. We proceed to work in Eddington-Finkelstein coordinates. Unlike in the DREH model, in the BPP model both regions of space-time require quantum corrections. This is consistent with the fact that the linear dilaton vacuum exhibits a nonzero energy flux at infinity. In solving the equations, it is not necessary to handle the two space-time regions separately, because the Minkowski vacuum solution must arise as the limit $b \rightarrow 0$ of the solution in the space-time region $x^+ > x_0^+$, where the black hole is present.

We begin by expanding to first order in ε , $\rho \mapsto \rho + \varepsilon\alpha$, and $\phi \mapsto \phi + \varepsilon\theta$. Equation (B2) in first order in ε read

$$\begin{aligned} \lambda\alpha e^{2\rho} &= \frac{1}{\lambda} \partial_+ \partial_- \theta + e^{2\rho} (F^+ \partial_+ \theta + F^- \partial_- \theta), \\ \partial_{\pm} (e^{-2\rho} (\partial_{\pm} \theta + \lambda\alpha F^{\pm} e^{2\rho})) &= -\frac{1}{2} e^{\phi-\rho} \partial_{\pm}^2 e^{\phi-\rho}, \\ \partial_+ \partial_- \theta &= \partial_+ \partial_- \alpha. \end{aligned} \quad (\text{B20})$$

Since the right-hand side of the second equation in (B2) is already of ε order, a simple substitution of $\phi - \rho$ (B16) in zeroth order yields

$$\partial_{\pm} (e^{-2\rho} (\partial_{\pm} \theta + \lambda\alpha F^{\mp} e^{2\rho})) = \frac{e^{2(\phi-\rho)}}{F^{\pm 2}} \left[t_{\pm}(\hat{\sigma}^{\pm}) + \frac{\lambda^2}{4} \right], \quad (\text{B21})$$

where the normally ordered components of energy-momentum tensor are given by

$$t_+(\hat{\sigma}^+) = 0 \wedge t_-(\hat{\sigma}^-) = \frac{\lambda^2}{4} (F^{-2} - 1). \quad (\text{B22})$$

Expressing F^- in terms of ϕ and δ with the help of (B19) and substituting into Eq. (B21) yields

$$\begin{aligned} \partial_+ \theta &= -\frac{\lambda}{2} F^- e^{2\rho} \left[2\alpha + \frac{G^-}{F^-} + \frac{1}{4} \frac{1}{b - e^{-2\phi}} \right], \\ \partial_- \theta &= -\frac{\lambda}{2} F^+ e^{2\rho} \left[2\alpha + \frac{G^+}{F^+} + \frac{1}{4} \frac{\delta^2}{\delta^2(b - e^{-2\phi}) - b} \right], \end{aligned} \quad (\text{B23})$$

where another set of arbitrary functions $G^{\pm}(\sigma^{\pm})$ is introduced. Expressing α from the first equation in (B20) and substituting it into these equations, followed by integration gives

$$\begin{aligned} \lambda b \theta e^{2\phi} &= \frac{\lambda G^-}{2 F^-} - F^- \partial_- \theta + \frac{\lambda}{8} e^{2\phi} (\ln |b - e^{-2\phi}| + H^-), \\ \lambda b \theta e^{2\phi} &= \frac{\lambda G^+}{2 F^+} - F^+ \partial_+ \theta \\ &\quad + \frac{\lambda}{8} e^{2\phi} (\ln |b - \delta^2(b - e^{-2\phi})| + H^+), \end{aligned} \quad (\text{B24})$$

with an additional pair of functions $H^{\pm}(x^{\pm})$. Requiring these two equations to coincide determines the form of the functions $H^{\pm}$ in terms of another constant D , giving the expression

$$\begin{aligned} \lambda b \theta e^{2\phi} &= \frac{\lambda}{2} (b e^{2\phi} - 1) \left[2\alpha + \frac{G^+}{F^+} + \frac{G^-}{F^-} \right] \\ &\quad + \frac{\lambda}{8} e^{2\phi} [-2 \ln \delta - D + 1 + \ln |b - \delta^2(b - e^{-2\phi})|]. \end{aligned} \quad (\text{B25})$$

Repeating the same procedure once more gives the final form of the θ function:

$$\begin{aligned} \theta &= \frac{\lambda}{2} (b e^{2\phi} - 1) \left(\int dx^+ \frac{G^+}{F^{+2}} + \int dx^- \frac{G^-}{F^{-2}} \right) \\ &\quad + \frac{1}{8} e^{2\phi} (1 - D - 2 \ln \delta + \ln |b - \delta^2(b - e^{-2\phi})|) \end{aligned} \quad (\text{B26})$$

We now switch back to the quantum-corrected fields, $\phi + \varepsilon\theta \mapsto \phi$ and $\rho + \varepsilon\alpha \mapsto \rho$, by substituting the solutions for α (B25) and θ (B26). In this way, we obtain the quantum-corrected counterparts of the expressions in (B19):

$$\begin{aligned} F^+ F^- e^{2\rho} &= b e^{2\phi} - 1 + \frac{\varepsilon}{4} e^{2\phi} [2 \ln \delta + D - 1 - \ln |b - \delta^2(b - e^{-2\phi})|], \\ \ln |b - e^{-2\phi}| + \frac{\varepsilon}{4} e^{2\phi} \frac{2 \ln \delta + D - 1 - \ln |b - \delta^2(b - e^{-2\phi})|}{b e^{2\phi} - 1} &= -\lambda \left(\int \frac{dx^+}{F^+} + \int \frac{dx^-}{F^-} \right) \equiv 2\lambda\hat{\sigma}, \end{aligned} \quad (\text{B27})$$

where $F^\pm + \varepsilon G^\pm \mapsto F^\pm$. Taking the limit $b \rightarrow 0$ gives the following quantum-corrected Minkowski vacuum solution,

$$F^+ F^- e^{2\rho} = -1 + \frac{\varepsilon}{2} (\phi + \phi_0) e^{2\phi},$$

$$\phi + \frac{\varepsilon}{4} (\phi + \phi_0) e^{2\phi} = \frac{\lambda}{2} \left(\int \frac{dx^+}{F^+} + \int \frac{dx^-}{F^-} \right), \quad (\text{B28})$$

where ϕ_0 is the integration constant. Selecting the Eddington–Finkelstein coordinates as the initial coordinate system corresponds to setting $F^+ = -1$ and $F^- = 1$. By subsequently carrying out a further coordinate transformation to Kruskal coordinates (B4), one can readily verify that the solution (B28) is, to first order in perturbation theory, equivalent to the exact Minkowski vacuum solution (B6). The constant C appearing in (B6) is mapped to the constant ϕ_0 as follows $C = \frac{\varepsilon}{2} \phi_0$.

Next, we continuously connect solution (B27) to the initial Minkowski vacuum solution (B28) by requiring that $e^{2\rho}$ and ϕ remain continuous functions, while their derivatives $\partial_+ \rho$ and $\partial_+ \phi$ are allowed to be discontinuous across the hypersurface $x^+ = x_0^+$. The resulting evaporating black hole solution is determined by the following values of constants b , D and coordinate transformations F^+ and F^- appearing in the solution (B27):

$$b = \frac{\pi G M}{\lambda}, \quad D = 1 + 2\phi_0, \quad F^+ = -1, \quad F^- = 1 - b e^{-2\hat{x}}. \quad (\text{B29})$$

As we can see, the coordinate transformations do not gain any quantum corrections in this procedure, and the coordinates are the same as in the classical case:

$$2\lambda\hat{\sigma} = -2\ln\delta + \ln|b - e^{2\hat{x}}|. \quad (\text{B30})$$

This is contrary to the DREH model, but it is nonetheless expected, because the exact solution displays the same behavior. A direct check shows that solution (B29) coincides with (B7) to first order in ε .

c. The end state of the evaporation

The apparent horizon is defined by $\partial_+ e^{-2\phi} = 0$. Using the solution (B27) the apparent horizon hypersurface to the first order in ε is given by

$$e^{2\hat{x}} = b + \frac{\varepsilon}{4} \delta^2. \quad (\text{B31})$$

The curvature $R = 8e^{-2\rho} \partial_+ \partial_- \rho$ is as follows:

$$R = \frac{4\lambda^2 \tilde{b} \delta^2}{\tilde{b} \delta^2 - b + e^{2\hat{x}}}, \quad (\text{B32})$$

where we have introduced a quantum-corrected constant b as $\tilde{b} = b - \frac{\varepsilon}{2} (\hat{x} - \ln\delta - \phi_0)$. For the line of singularity we have

$$e^{2\hat{x}} = b - \tilde{b} \delta^2, \quad (\text{B33})$$

since this is the hypersurface along which the curvature diverges. To compute the end point of the evaporation, one has to find the intersection of the apparent horizon and singularity hypersurfaces. It can be calculated using Eqs. (B31) and (B33) up to the first order in ε . The result is

$$\delta_E^2 = b e^{-\frac{4b}{\varepsilon} - 1 - 2\phi_0}, \quad (\text{B34})$$

$$e^{2\hat{x}_E} = b \left[1 + \frac{\varepsilon}{4} e^{-\frac{4b}{\varepsilon} - 1 - 2\phi_0} \right]. \quad (\text{B35})$$

By expressing the solution in Kruskal coordinates and using $\delta_E^2 = \frac{x_0^+}{x_E^-}$ together with $e^{2\hat{x}_E} = -\lambda^2 x_0^+ x_E^-$, we find that these formulas reproduce the exact end point (B11) to first order in ε .

Next, we calculate the total energy radiated by the black hole over the entire course of its evaporation:

$$\begin{aligned} E_{\text{rad}} &= \frac{\varepsilon}{\pi G} \int_{-\infty}^{\sigma_E^-} \frac{d\sigma^-}{F^-} t_-(\hat{\sigma}^-) \\ &= \frac{\varepsilon \lambda}{4\pi G} [b e^{-2\hat{x}_E} - \ln|b e^{-2\hat{x}_E} - 1|] \\ &= M + \frac{\varepsilon \lambda}{4\pi G} \left[2\phi_0 - \ln \frac{\varepsilon}{4} + 1 - \frac{\Delta}{x_E^-} \right], \end{aligned} \quad (\text{B36})$$

where we have employed the expression (B22) for $t_-(\hat{\sigma}^-)$. Notice that this expression coincides exactly with the exact result (B12). This agreement arises because the coordinate transformation F^- , which is used to compute the radiated energy, does not receive any quantum corrections in the BPP model.

Finally, we demonstrate how to determine the end-state geometry. The first step is to rewrite the solution in terms of the radial coordinate ϕ and the $\hat{x}$ coordinate, because the hypersurface separating the evaporating black hole from the end-state geometry is specified by this coordinate, namely by the condition $\hat{x} = \hat{x}_E$. To that end, we introduce the running b constant:

$$\hat{b} = b + \frac{\varepsilon}{2} \phi_0 + \frac{\varepsilon}{4} \ln|b e^{-2\hat{x}} - 1|. \quad (\text{B37})$$

Since the difference from constant b is of ε order we can exchange all appearances of b within the first order of perturbation theory with $\hat{b}$. Then Eq. (B27) change in the following way:

$$F^+ F^- e^{2\rho} = \hat{b} e^{2\phi} - 1 - \frac{\varepsilon}{4} e^{2\phi} \ln |\hat{b} - e^{-2\phi}|,$$

$$\ln |\hat{b} - e^{-2\phi}| - \frac{\varepsilon \ln |\hat{b} - e^{-2\phi}|}{4 \hat{b} - e^{-2\phi}} = 2\lambda \hat{\sigma}. \quad (\text{B38})$$

These two relations hold for all values of the coordinate $\hat{x}$. As we approach the event horizon at $\hat{x} = \hat{x}_E$, running constant $\hat{b}$ decreases continuously and takes the value

$$\hat{b}_E = \frac{\varepsilon}{4} \left(\ln \frac{\varepsilon}{4} - 1 \right) \quad (\text{B39})$$

at $\hat{x} = \hat{x}_E$. Since $\hat{b}_E$ is of order ε at this stage, we can set $\hat{b} \rightarrow 0$ in the first-order perturbative expansion. Once this limit is taken, Eq. (B38) reduce to the Minkowski vacuum solution (B28) with constant $\phi'_0 = \frac{2}{\varepsilon} \hat{b}_E$. By comparing with the exact solution (B13), we see that, after transforming to Kruskal coordinates, the two solutions agree to first order.

To check if there is another outgoing shockwave from the end point of the evaporation, we integrate the $--$ equation from (B2) from $\sigma_E^- - \varepsilon$ to $\sigma_E^- + \varepsilon$, and taking $\varepsilon \rightarrow 0$ gives

$$\lim_{\varepsilon \rightarrow 0} e^{-2\rho} \left[\partial_- \phi + \frac{\varepsilon}{2} e^{2\phi} \partial_- (\phi - \rho) \right] \Big|_{\sigma_E^- - \varepsilon}^{\sigma_E^- + \varepsilon} = \frac{\lambda b_0}{2F^-} e^{2(\phi - \rho)}, \quad (\text{B40})$$

where $b_0 = \frac{\pi G \Delta M}{\lambda}$; and ΔM is the mass of the shockwave. The direct substitution of the derivatives gives

$$-\frac{b_0}{F^-} e^{2(\phi - \rho)} = \frac{\varepsilon}{4} e^{2\phi} + \frac{\varepsilon}{2} \frac{1}{b - e^{-2\phi}} + \frac{\varepsilon}{4} \frac{\delta^2}{b - \delta^2(b - e^{-2\phi})}. \quad (\text{B41})$$

A careful calculation of this expression yields

$$b_0 = -\frac{\varepsilon}{4} b e^{-2\hat{x}_E} - \frac{\varepsilon}{4} b e^{-2\phi}. \quad (\text{B42})$$

Since $e^{-2\hat{x}_E} \propto 1/b$, the first term does not vanish in the limit $b \rightarrow 0$, while the second does, giving the following for the mass of the thunderpop:

$$\Delta M = -\frac{\varepsilon \lambda}{4\pi G} b e^{-2x_E} = \frac{\varepsilon \lambda \Delta}{4\pi G x_E^-}, \quad (\text{B43})$$

which coincides with the exact result. Finally, the energy balance can be rewritten in terms of the b parameters:

$$b_{\text{rad}} = b + \frac{\varepsilon}{2} \phi_0 - \frac{\varepsilon}{2} \phi'_0 - b_0. \quad (\text{B44})$$

Here, $\frac{\varepsilon}{2} \phi_0$ corresponds to the mass of the space-time prior to collapse, while $\frac{\varepsilon}{2} \phi'_0$ is related to the mass of the final

geometry. The quantity $b = \frac{\pi G M}{\lambda}$ denotes the b parameter associated with the incoming shockwave, b_0 represents the b parameter of the outgoing shockwave, and $b_{\text{rad}} = \frac{\pi G E_{\text{rad}}}{\lambda}$ is the b parameter characterizing the Hawking radiation.

APPENDIX C: TRANSCENDENTAL EQUATION SOLUTION

Here we solve the transcendental equation (95), that is

$$e^{\hat{x}}(\hat{x} - 1) = \delta e^x(x - 1), \quad (\text{C1})$$

when $\delta \leq 1$, $\hat{x} \geq 0$ and $x \geq 0$. Notice that when $\delta = 1$, we have a simple solution $\hat{x} = x$. This is the only solution, since both functions are monotonically increasing, which implies that they intersect at most once. This is true simply because when $\sigma^+ = \sigma_0^+$, the metric becomes that of the Minkowski space-time (see Sec. II B). This fact will be extensively used as well as the fact that $\delta < 1$ when $\sigma^+ > \sigma_0^+$. Two cases will be studied, since the function on the right-hand side of Eq. (C1) behaves differently depending on whether $x \geq 1$ or $x \leq 1$.

It is important to further discuss the condition $\hat{x} \geq 0$. Figure 1 shows space-time. In part I of space-time the boundary is defined by $x = 0$. The formation of the singularity happens when the collapsing matter's world-line $\sigma^+ = \sigma_0^+$ hits the boundary of space-time. This corresponds to $\hat{x} = 0$, which implies that $\hat{x} \geq 0$.

Theorem C.1. The solution to Eq. (C1) $\hat{x} = \hat{x}(x, \delta)$, when $\delta \leq 1$, $\hat{x} \geq 0$ and $x \geq 1$ is given by the following functional series:

$$\hat{x}(x, \delta) = x - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} \left(1 - \frac{1}{x} \right)^n P_{n-1} \left(\frac{1}{x} \right), \quad (\text{C2})$$

where $P_n(x)$ are polynomials of the n th degree, defined by the following recurrence relation,

$$P_n(x) = \left[n(x+1) + x^2 \frac{d}{dx} \right] P_{n-1}(x), \quad (\text{C3})$$

and the initial condition $P_0(x) = 1$.

Proof. We expand the function $\hat{x} = \hat{x}(x, \delta)$ in a Taylor series around $\delta = 1$:

$$\hat{x} = x + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n \hat{x}}{\partial \delta^n} \Big|_{\delta=1} (\delta - 1)^n, \quad (\text{C4})$$

where we have used the fact that $\hat{x}(x, 1) = x$. The differential of the function $\hat{x} = \hat{x}(x, \delta)$ is given by

$$d\hat{x} = \frac{e^x(x-1)}{\hat{x}e^{\hat{x}}} d\delta + \delta \frac{x e^x}{\hat{x} e^{\hat{x}}} dx. \quad (\text{C5})$$

The n th derivative can be written as

$$\frac{\partial^n \hat{x}}{\partial \delta^n} = e^{nx} (x-1)^n \left(\frac{e^{-\hat{x}}}{\hat{x}} \frac{d}{d\hat{x}} \right)^{n-1} \frac{e^{-\hat{x}}}{\hat{x}}. \quad (C6)$$

Demanding that $\delta = 1$ is equivalent to replacing $\hat{x} = x$ on the right-hand side of Eq. (C6). We define the array of polynomials $P_n(\frac{1}{x})$ by

$$P_{n-1}\left(\frac{1}{x}\right) = (-1)^{n-1} x^n e^{nx} \left(\frac{e^{-x}}{x} \frac{d}{dx} \right)^{n-1} \frac{e^{-x}}{x}. \quad (C7)$$

Using mathematical induction, it is easy to show that the Rodrigues formula [Eq. (C7)] really defines polynomials of degree $n-1$. With the help of Eq. (C7) it is straightforward to derive a recurrence relation for the $P_n(1/x)$ polynomials, which is given by

$$P_n\left(\frac{1}{x}\right) = \left[n \left(1 + \frac{1}{x} \right) - \frac{d}{dx} \right] P_{n-1}\left(\frac{1}{x}\right). \quad (C8)$$

From this relation, the recurrence relation (C3) is directly derived by the substitution $x = 1/x$. Replacing $n = 1$ in Eq. (C7), the initial condition reads $P_0(x) = 1$. ■

To properly use Eq. (C2), it is important to show that this series is uniformly convergent. If so, then the solution of Eq. (C1) when $x \geq 1$ is given by Eq. (C2).

Theorem C.2. The series $\sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} f_n(x)$ is uniformly convergent when $x \geq 1$. The functional array $f_n(x)$ is defined by the following equation:

$$f_n(x) = \left(1 - \frac{1}{x} \right)^n P_{n-1}\left(\frac{1}{x}\right). \quad (C9)$$

Proof. To prove this theorem, we need to prove the following lemma first.

Lemma C.3. The functions $f_n(x)$ are monotonically increasing functions of x when $x \geq 1$.

Proof. We will calculate the derivative of the function $f_n(x)$, and show that it is positive when $x > 1$. Using Eq. (C8) the derivative of the polynomial $P_{n-1}(x)$ can be eliminated, resulting in

$$\begin{aligned} \frac{df_n(x)}{dx} &= \left(1 - \frac{1}{x} \right)^{n-1} \\ &\times \left[n P_{n-1}\left(\frac{1}{x}\right) - \left(1 - \frac{1}{x} \right) P_n\left(\frac{1}{x}\right) \right]. \end{aligned} \quad (C10)$$

Since $x > 1$, $f'_n(x) > 0$ is equivalent to proving that

$$n P_{n-1}\left(\frac{1}{x}\right) > \left(1 - \frac{1}{x} \right) P_n\left(\frac{1}{x}\right), \quad \forall n \geq 1. \quad (C11)$$

We write the polynomials in terms of their coefficients as $P_n(x) = \sum_{k=0}^n a_n^{(k)} x^k$. Equation (C11) now takes the form

$$\begin{aligned} n a_{n-1}^{(0)} - a_n^{(0)} + \sum_{k=1}^{n-1} \left[n a_{n-1}^{(k)} + a_n^{(k-1)} - a_n^{(k)} \right] \frac{1}{x^k} \\ + \left[a_n^{(n-1)} - a_n^{(n)} \right] \frac{1}{x^n} + a_n^{(n)} \frac{1}{x^{n+1}} > 0. \end{aligned} \quad (C12)$$

We will show that the coefficient of every order in Eq. (C12) is greater than 0. To do this we need the recurrence relation between the coefficients $a_n^{(k)}$, which is easily derived using recurrence relation (C3). The result is given by

$$k = 0: a_n^{(0)} = n a_{n-1}^{(0)}, \quad (C13)$$

$$k = 1: a_n^{(1)} = n a_{n-1}^{(1)} + n a_{n-1}^{(0)}, \quad (C14)$$

$$1 < k < n: a_n^{(k)} = n a_{n-1}^{(k)} + (n+k-1) a_{n-1}^{(k-1)}, \quad (C15)$$

$$k = n: a_n^{(n)} = (2n-1) a_{n-1}^{(n-1)}. \quad (C16)$$

Together with the initial condition $a_0^{(0)} = 1$, Eqs. (C13) through (C16) give $a_n^{(0)} = n!$, $a_n^{(1)} = nn!$, and $a_n^{(n)} = (2n-1)!!$ respectively. Now, we return to proving the inequality (C12). From Eq. (C13) it follows that the $k = 0$ term in the (C12) vanishes. The first-order term vanishes because $a_n^{(1)} = nn!$. Now, the inequality (C12) may be recast as

$$(\forall k \in \{2, 3, \dots, n-1\}) n a_{n-1}^{(k)} + a_n^{(k-1)} - a_n^{(k)} > 0. \quad (C17)$$

With the help of the relation (C15), the inequality (C17) can be rewritten in the form

$$(n+k-2) a_{n-1}^{(k-2)} > (k-1) a_{n-1}^{(k-1)}. \quad (C18)$$

Shifting $k-1$ to k , and $n-1$ to n , we arrive at

$$(\forall k \in \{1, 2, \dots, n-1\}) (n+k) a_n^{(k-1)} > k a_n^{(k)}. \quad (C19)$$

Now, we apply mathematical induction to prove this inequality. First, we prove the relation for $n = 3$ (the base case of the induction). The coefficients of the polynomial $P_3(x) = 15x^3 + 25x^2 + 18x + 6$ clearly satisfy relations (C19). We need to prove that the relation holds for $n+1$ if relation (C19) is satisfied for n (induction hypothesis):

$$(\forall k \in \{1, 2, \dots, n\}) (n+k+1) a_{n+1}^{(k-1)} > k a_{n+1}^{(k)}. \quad (C20)$$

Using relation (C15), the inequality (C20) can be rewritten as

$$\begin{aligned}
& (\forall k \in \{2, \dots, n\})(n+1)[(n+k)a_n^{(k-1)} - ka_n^{(k)}] \\
& + (n+k+1)[(n+k-1)a_n^{(k-2)} - (k-1)a_n^{(k-1)}] > 0.
\end{aligned} \quad (C21)$$

Following from the hypothesis, both terms are greater than 0. Only two cases remain to be proven: $k=1$ and $k=n$. In the case of $k=1$, the relation is given by $(n+2)a_{n+1}^{(0)} > a_{n+1}^{(1)}$. Using the closed forms for these two coefficients, $(n+2)(n+1)! > (n+1)(n+1)!$, which is clearly satisfied. In the case of $k=n$, we are allowed to use formula (C21), which takes the following form: $(n+1)n[2a_n^{(n-1)} - a_n^{(n)}] + (2n+1)[(2n-1)a_n^{(n-2)} - (n-1)a_n^{(n-1)}] > 0$. The second term is greater than zero (from the hypothesis). Using relation (C15) $2a_n^{(n-1)} > 2na_{n-1}^{(n-1)} = 2n(2n-3)!! > (2n-1)!! = a_n^{(n)}$. With this, the mathematical induction is finished and we have proven inequality (C19), and subsequently the inequality (C17).

The last remaining part of inequality (C12) that needs to be proven is the relation $(\forall n \geq 1)a_n^{(n-1)} > a_n^{(n)}$. Using the relation (C15) [and the result $a_n^{(n)} = (2n-1)!!$] repeatedly, it is easy to arrive at the following expression for the closed form of $a_n^{(n-1)}$:

$$a_n^{(n-1)} = \sum_{k=0}^{n-1} \frac{2^k(n-k)(n-1)!(2(n-k)-3)!!}{(n-k-1)!}. \quad (C22)$$

The zeroth term in this sum is equal to $n(2n-3)!!$, and every other term is greater than $(2n-3)!!$, which means that $a_n^{(n-1)} > (2n-1)(2n-3)!! = a_n^{(n)}$. This is what we needed to prove.

With this, we have proven inequality (C12) and subsequently inequality (C11), thus proving the lemma. ■

To prove Theorem C.2, the Weierstrass criterion will be used. Lemma C.3 implies that

$$(\forall x \geq 1)f_n(x) < \lim_{x \rightarrow \infty} f_n(x) = a_{n-1}^{(0)} = (n-1)!, \quad (C23)$$

$\forall n \geq 1$. The inequality (C23) implies the following inequality for the coefficients of the series $\sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} f_n(x)$:

$$(\forall n \geq 1)(\forall x \geq 1) \left| \frac{(1-\delta)^n}{n!} f_n(x) \right| < \frac{(1-\delta)^n}{n}. \quad (C24)$$

The next step is to prove that the series $\sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n}$ converges. By the comparison test, this is true for $|1-\delta| < 1$, which is satisfied. Then, according to the Weierstrass criterion, the series $\sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} f_n(x)$ converges uniformly.

This result is important because the infinite sum may exchange places with integrals and limits with respect to x ,

which will be used in the main text. Now we move on to the case when $x < 1$.

Theorem C.4. The solution to Eq. (C1) $\hat{x} = \hat{x}(x, \delta)$, when $\delta \leq 1$, $\hat{x} \geq 0$ and $0 \leq x \leq 1$ is given by the following functional series:

$$\hat{x}(x, \delta) = 1 - \sum_{n=1}^{\infty} \frac{\delta^n}{n!} P_{n-1}(1) e^{n(x-1)} (1-x)^n, \quad (C25)$$

where $P_n(x)$ are the polynomials defined in Theorem C.1.

Proof. If $0 \leq x \leq 1$, then $0 \leq e^x(1-x) \leq 1$, which means that we can choose $\tilde{\delta} = \delta e^x(1-x)$ as a small parameter, and expand the solution as a Taylor series with respect to $\tilde{\delta}$ around $\tilde{\delta} = 0$,

$$\hat{x}(\tilde{\delta}) = \hat{x}(0) + \sum_{n=1}^{\infty} \frac{\tilde{\delta}^n}{n!} \frac{d^n \hat{x}}{d\tilde{\delta}^n} \Big|_{\tilde{\delta}=0}. \quad (C26)$$

The equation we are solving, in terms of $\hat{x} = \hat{x}(\tilde{\delta})$, is given by

$$e^{\hat{x}(\tilde{\delta})} (1 - \hat{x}(\tilde{\delta})) = \tilde{\delta}. \quad (C27)$$

With the help of Eq. (C27) the n th derivative of the function $\hat{x}(\tilde{\delta})$ is given by

$$\frac{d^n \hat{x}}{d\tilde{\delta}^n} = (-1)^n \left(\frac{e^{-\hat{x}}}{\hat{x}} \frac{d}{d\hat{x}} \right)^{n-1} \frac{e^{-\hat{x}}}{\hat{x}} = -\frac{e^{-n\hat{x}}}{\hat{x}^n} P_{n-1} \left(\frac{1}{\hat{x}} \right), \quad (C28)$$

where Eq. (C7) has been used. Using the fact that $\hat{x} = 1$ when $\tilde{\delta} = 0$ we arrive at

$$\frac{d^n \hat{x}}{d\tilde{\delta}^n} = -e^{-n} P_{n-1}(1). \quad (C29)$$

Changing this result in formula (C26) we find $\tilde{\delta} = \delta e^x(1-x)$ and we have derived formula (25). ■

To be able to use this result (C25), once again we need to prove that the series appearing in Eq. (C25) is uniformly convergent.

Theorem C.5. The series $\sum_{n=1}^{\infty} \frac{\delta^n}{n!} P_{n-1}(1) e^{n(x-1)} (1-x)^n$ is uniformly convergent when $0 \leq x \leq 1$.

Proof. Since $0 \leq x \leq 1$, we have $e^{nx}(1-x)^n < 1$, which means

$$\left| \frac{\delta^n}{n!} P_{n-1}(1) e^{n(x-1)} (1-x)^n \right| < \frac{\delta^n}{n!} P_{n-1}(1) e^{-n}, \quad (C30)$$

$\forall 0 \leq x \leq 1$ and $\forall n \geq 1$. The next step is to prove that the series $\sum_{n=1}^{\infty} \frac{\delta^n}{n!} P_{n-1}(1) e^{-n}$ is convergent. To be able to do this, we need a closed form for $P_{n-1}(1)$, which will be derived in the following lemma, without proof.

Lemma C.6. The polynomials defined in Theorem C.1 have the following property: $P_n(1) = (n+1)^n$, $\forall n \geq 0$.

Now we can use the comparison test to check if the series $\sum_{n=1}^{\infty} \frac{\delta^n}{n!} P_{n-1}(1) e^{-n}$ is convergent. Using Lemma C.6 we get

$$\frac{\frac{\delta^{n+1}}{(n+1)!} (n+1)^n e^{-(n+1)}}{\frac{\delta^n}{n!} n^{n-1} e^{-n}} = \frac{\delta}{e} \left(1 + \frac{1}{n}\right)^{n-1} \xrightarrow{n \rightarrow \infty} \delta, \quad (\text{C31})$$

which means that the series is convergent when $\delta < 1$. This requirement is satisfied by the theorem statement. Then, according to the Weierstrass criterion, the series $\sum_{n=1}^{\infty} \frac{\delta^n}{n!} P_{n-1}(1) e^{n(x-1)} (1-x)^n$ converges uniformly.

Sometimes functions of $\hat{x}$ will appear in our calculations (for example, $\ln \hat{x}$). Similar expressions to those of Theorems C.1 and C.4 are needed for these functions.

Theorem C.7. The differentiable function $F(\hat{x}(x, \delta))$, where $\hat{x}$ is a solution of Eq. (C1), when $\delta \leq 1$, $\hat{x} \geq 0$ and $x \geq 1$, is given by the following functional series:

$$F(\hat{x}) = F(x) - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} \left(1 - \frac{1}{x}\right)^n F_{n-1}\left(\frac{1}{x}\right), \quad (\text{C32})$$

where $F_n(x)$ is an array of functions, defined by the recurrence relation,

$$F_n(x) = \left[n(x+1) + x^2 \frac{d}{dx} \right] F_{n-1}(x), \quad (\text{C33})$$

and the initial condition $F_0(x) = -x^2 \frac{dF(1/x)}{dx}$.

Proof. The proof is analogous to that of Theorem C.1. We start by writing the Taylor series of the function $F(x, \delta)$ around the point $\delta = 1$:

$$F(\hat{x}) = F(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n F(\hat{x})}{\partial \delta^n} \Big|_{\delta=1} (\delta - 1)^n. \quad (\text{C34})$$

The n th derivative is now given by the following expression:

$$\frac{\partial^n F(\hat{x})}{\partial \delta^n} = e^{nx} (x-1)^n \left(\frac{e^{-\hat{x}}}{\hat{x}} \frac{d}{d\hat{x}} \right)^{n-1} \frac{e^{-\hat{x}}}{\hat{x}} \frac{dF(\hat{x})}{d\hat{x}}. \quad (\text{C35})$$

The same way as in the Theorem C.1 we define an array of functions:

$$F_{n-1}\left(\frac{1}{x}\right) = (-1)^{n-1} x^n e^{nx} \left(\frac{e^{-x}}{x} \frac{d}{dx} \right)^{n-1} \frac{e^{-x}}{x} \frac{dF(x)}{dx}. \quad (\text{C36})$$

It is now obvious that the recurrence relation for the functions F_n would be the same as for the polynomials defined in the Theorem C.1. The only difference would be the initial condition, which is now given by $F_0(1/x) = \frac{dF(x)}{dx}$.

Applying the substitution $x \mapsto 1/x$, we arrive at the expression given in the statement of the theorem. ■

Once again, the question of uniform convergence must be asked. This time, it depends on the form of the function $F(\hat{x})$. The first requirement is that F does not diverge when $\hat{x} \rightarrow \infty$, since the radius of convergence would depend on x , and the discussion of uniform convergence would become meaningless. From the proof of Theorem C.2 we can see that the necessary condition is

$$(\forall x \geq 1)(\exists n_0)(\forall n \geq n_0) \left| \left(1 - \frac{1}{x}\right)^n F_{n-1}\left(\frac{1}{x}\right) \right| < n!. \quad (\text{C37})$$

The function $\ln \hat{x}$ is the only one directly involved in the calculation. Since this is the case, we will take extra care of this function.

Corollary C.7.1. The function $\ln \hat{x}(x, \delta)$ when $\delta \leq 1$, $\hat{x} \geq 0$ and $x \geq 1$, is given by the following functional series:

$$\ln \hat{x} = \ln x - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} \left(1 - \frac{1}{x}\right)^n Q_{n-1}\left(\frac{1}{x}\right), \quad (\text{C38})$$

where $Q_n(x)$ are polynomials of degree $n+1$, defined by the following recurrence relation:

$$Q_n(x) = \left[n(x+1) + x^2 \frac{d}{dx} \right] Q_{n-1}(x), \quad (\text{C39})$$

and the initial condition $Q_0(x) = x$.

Proof. This corollary is a direct consequence of Theorem C.7. The initial condition is obvious since $-x^2 (\ln(1/x))' = x$. We only need to prove that Q_n are polynomials of degree $n+1$, which follows directly from the recurrence relation (C39) since it increases the degree of the polynomial by one, and the degree of the zeroth polynomial is 1. This concludes the proof. ■

Theorem C.8. The series $\sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} f_n(x)$ is uniformly convergent when $x \geq 1$. The functional array $f_n(x)$ is defined by the following equation:

$$f_n(x) = x \left(1 - \frac{1}{x}\right)^{n+1} Q_{n-1}\left(\frac{1}{x}\right). \quad (\text{C40})$$

Proof. This proof will not be as detailed as the previous proofs since it is based on the same ideas. By mathematical induction, it is easy to prove that the functions $f_n(x)$ are monotonically increasing, the same way as in Lemma C.3. This fact implies that the functions $f_n(x)$ take their maximum value when $x \rightarrow \infty$. Then, the recurrence relations (C39) tell us that the maximal value is $(n-1)!$. The rest of the proof is exactly the same as in Theorem C.2. ■

Now we examine the $x < 1$ case.

Theorem C.9. The function $\ln \hat{x}(x, \delta)$, when $\delta \leq 1$, $\hat{x} \geq 0$ and $0 \leq x \leq 1$ is given by the following functional series:

$$\ln \hat{x} = - \sum_{n=1}^{\infty} \frac{\delta^n}{n!} Q_{n-1}(1) e^{n(x-1)} (1-x)^n, \quad (\text{C41})$$

where $Q_n(x)$ are the polynomials defined in Corollary C.7.1.

Proof. The proof of this theorem is essentially the same as the proof of Theorem C.4. The only difference is the appearance of the polynomials Q_n instead of the polynomials P_n . ■

Finally, we need to prove the uniform convergence of the series appearing in Eq. (C41).

Theorem C.10. When $0 \leq x \leq 1$, the functional series $\sum_{n=1}^{\infty} \frac{\delta^n}{n!} Q_{n-1}(1) e^{n(x-1)} (1-x)^n$ is uniformly convergent.

Proof. As in Theorem C.5, we can write

$$\left| \frac{\delta^n}{n!} Q_{n-1}(1) e^{n(x-1)} (1-x)^n \right| < \frac{\delta^n}{n!} Q_{n-1}(1) e^{-n}. \quad (\text{C42})$$

Now we need the closed form of $Q_{n-1}(1)$, which will be given by the following lemma without proof.

Lemma C.11. The polynomials defined in Corollary C.7.1 have the following property: $(\forall n \geq 1) Q_{n-1}(1) = (n-1)! \sum_{k=0}^{n-1} \frac{n^k}{k!}$.

Using Lemma C.11, the following inequality holds:

$$Q_{n-1}(1) = (n-1)! \sum_{k=1}^{n-1} \frac{n^k}{k!} < (n-1)! e^n. \quad (\text{C43})$$

Using (C43), the inequality (C42) becomes

$$\left| \frac{\delta^n}{n!} Q_{n-1}(1) e^{n(x-1)} (1-x)^n \right| < \frac{\delta^n}{n}. \quad (\text{C44})$$

The next step would be to prove the convergence of the series $\sum_{n=1}^{\infty} \frac{\delta^n}{n}$. This series is convergent by the comparison test when $\delta < 1$, which is satisfied.

APPENDIX D: MORE SUITABLE FORM OF THE SOLUTION

Here we will rewrite Eqs. (142) and (143) in a more suitable form so that they are well defined when $x = 1 + \mathcal{O}(\tilde{\varepsilon})$ or $\hat{x} = 1 + \mathcal{O}(\tilde{\varepsilon})$, where we have defined $\tilde{\varepsilon} = \frac{\varepsilon}{(\lambda a)^2}$. This is important since many hypersurfaces of interest are given by expressions of this kind (apparent horizon, horizon, island, QES, etc.). Looking at the equations of motion (59)–(61) and the way they are solved (62)–(77) one can expect that the expressions for $\partial_{\pm} x$ (70) and (71) are well defined when $x = 1 + \mathcal{O}(\tilde{\varepsilon})$ since they are derived by simple integration of the original equations of motion. Indeed, this is the case:

$$\partial_+ x = \frac{1}{2a} \left\{ 1 - \frac{1 + \frac{\tilde{\varepsilon}}{4} \ln \delta}{\sqrt{x^2 - \tilde{\varepsilon}}} \left[1 - \frac{\tilde{\varepsilon}}{8} \left(2(x-2) \ln x - 8 + 5x + \frac{3}{x} - (x-1) \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \left(\frac{x-1}{x} \frac{dS_n^<}{dx} - n S_n^< \right) \right) \right] \right\}, \quad (\text{D1})$$

$$\partial_- x = -\frac{1}{2aF^-} \left\{ 1 - \frac{1 + \frac{\tilde{\varepsilon}}{4} \ln \delta}{\sqrt{x^2 - \tilde{\varepsilon}}} \left[1 - \frac{\tilde{\varepsilon}}{8} \left(2(x-2) \ln x - 2(x-1) \ln(1-x) + 1 - 2x + \frac{3}{x} - \frac{(x-1)^2}{x} \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \frac{dS_n^<}{dx} \right) \right] \right\}. \quad (\text{D2})$$

This suggests that the issue arises during the execution of the final integration (76) and (77). Note that when taking the limit $\delta \rightarrow 1$, both Eqs. (D1) and (D2) reduce to the exact solutions (156). Now we define a new coordinate $y = \frac{x}{\tilde{a}}$, where $\tilde{a} = 1 + \frac{\tilde{\varepsilon}}{4} \ln \delta$, so that $a\tilde{a}$ can be seen as a quantum-corrected constant a . Applying this, Eqs. (D1) and (D2) become

$$\partial_+ x = \frac{1}{2a} \left\{ 1 - \frac{1}{\sqrt{y^2 - \tilde{\varepsilon}}} \left[1 - \frac{\tilde{\varepsilon}}{8} \left(2(y-2) \ln y - 8 + 5y + \frac{3}{y} - (y-1) \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \left(\frac{y-1}{y} \frac{dS_n^<}{dy} - n S_n^< \right) \right) \right] \right\}, \quad (\text{D3})$$

$$\partial_- x = -\frac{1}{2aF^-} \left\{ 1 - \frac{1}{\sqrt{y^2 - \tilde{\varepsilon}}} \left[1 - \frac{\tilde{\varepsilon}}{8} \left(2(y-2) \ln y - 2(y-1) \ln(1-y) + 1 - 2y + \frac{3}{y} - \frac{(y-1)^2}{y} \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \frac{dS_n^<}{dy} \right) \right] \right\}. \quad (\text{D4})$$

For a moment let us exclude the $\mathcal{O}(\tilde{\varepsilon})$ part from the square brackets within (D3) and define

$$\partial_+ x^{(0)} = \frac{1}{2a} \left(1 - \frac{1}{\sqrt{y^2 - \tilde{\varepsilon}}} \right). \quad (\text{D5})$$

And then, let us define a function $\mathcal{J}(x)$ so that it satisfies

$$\tilde{a}\mathcal{J}(y) = \frac{\sigma^+}{2a}. \quad (\text{D6})$$

After taking the derivative of (D6) with respect to σ^+ and substituting the expression (D5) one arrives at the differential equation for $\mathcal{J}$:

$$\frac{d\mathcal{J}}{dy} - \frac{\tilde{\epsilon}}{4} \frac{\sqrt{y^2 - \tilde{\epsilon}}}{(1 + \frac{\tilde{\epsilon}}{4}y)\sqrt{y^2 - \tilde{\epsilon}} - 1} \mathcal{J} = \frac{\sqrt{y^2 - \tilde{\epsilon}}}{(1 + \frac{\tilde{\epsilon}}{4}y)\sqrt{y^2 - \tilde{\epsilon}} - 1}. \quad (\text{D7})$$

The solution to Eq. (D7) is then given by

$$\mathcal{J}(y) = -\frac{4}{\tilde{\epsilon}} + S \exp\left(\frac{\tilde{\epsilon}}{4}\tilde{f}(y)\right), \quad (\text{D8})$$

where S is an unknown constant that needs to be determined, and $\tilde{f}(y)$ is given by

$$\tilde{f}(y) = \int \frac{\sqrt{y^2 - \tilde{\epsilon}}}{(1 + \frac{\tilde{\epsilon}}{4}y)\sqrt{y^2 - \tilde{\epsilon}} - 1} dy. \quad (\text{D9})$$

After making a substitution $z = \frac{\sqrt{y - \sqrt{\tilde{\epsilon}}}}{y + \sqrt{\tilde{\epsilon}}}$, the integral (D9) becomes

$$\tilde{f} = 8\tilde{\epsilon} \int \frac{z^2 dz}{(z^2 - 1)R(z)}, \quad (\text{D10})$$

where $R(z) = z^4 + 2\sqrt{\tilde{\epsilon}}(1 - \frac{\tilde{\epsilon}^{3/2}}{4})z^3 - 2z^2 - 2\sqrt{\tilde{\epsilon}}(1 + \frac{\tilde{\epsilon}^{3/2}}{4})z + 1$. This polynomial can be factorized as $R(z) = (z^2 + \alpha_1 z + \beta_1)(z^2 + \alpha_2 z + \beta_2)$. The constants $\alpha_{1,2}$ and $\beta_{1,2}$ satisfy the following equations:

$$\begin{aligned} \alpha_1 + \alpha_2 &= 2\sqrt{\tilde{\epsilon}}\left(1 - \frac{\tilde{\epsilon}^{3/2}}{4}\right), \\ \beta_1 + \beta_2 &= -2 - \alpha_1\alpha_2, \\ \alpha_1\beta_2 + \alpha_2\beta_1 &= -2\sqrt{\tilde{\epsilon}}\left(1 + \frac{\tilde{\epsilon}^{3/2}}{4}\right), \\ \beta_1\beta_2 &= 1. \end{aligned} \quad (\text{D11})$$

The system of Eq. (D11) can be exactly solved, but we will solve it perturbatively up to $\mathcal{O}(\tilde{\epsilon}^4)$. The solution is given by

$$\begin{aligned} \alpha_1 &= 2\sqrt{\tilde{\epsilon}}\left(1 - \frac{\tilde{\epsilon}^{3/2}}{4}\right)\left[1 - \frac{\tilde{\epsilon}^2}{16}\left(1 + \frac{\tilde{\epsilon}^3}{2}\right)\right], \\ \alpha_2 &= \frac{\tilde{\epsilon}^{5/2}}{8}\left(1 + \frac{\tilde{\epsilon}^3}{4}\right), \\ \beta_1 &= -1 + \frac{\tilde{\epsilon}^{3/2}}{2}\left(1 - \frac{\tilde{\epsilon}^2}{4} + \frac{\tilde{\epsilon}^2}{8}\right), \\ \beta_2 &= -1 - \frac{\tilde{\epsilon}^{3/2}}{2}\left(1 + \frac{\tilde{\epsilon}^3}{4} + \frac{\tilde{\epsilon}^2}{8}\right). \end{aligned} \quad (\text{D12})$$

The expressions for the zeros of the polynomial $R(z)$ are

$$z_2^+ = 1 + \frac{\tilde{\epsilon}^{3/2}}{4} - \frac{\tilde{\epsilon}^{5/2}}{16} + \frac{\tilde{\epsilon}^3}{32} + \frac{\tilde{\epsilon}^2}{32} - \frac{\tilde{\epsilon}^4}{64}, \quad (\text{D13})$$

$$z_2^- = -1 - \frac{\tilde{\epsilon}^{3/2}}{4} - \frac{\tilde{\epsilon}^{5/2}}{16} - \frac{\tilde{\epsilon}^3}{32} - \frac{\tilde{\epsilon}^2}{32} - \frac{\tilde{\epsilon}^4}{64}, \quad (\text{D14})$$

$$z_1^+ = 1 - \tilde{\epsilon}^{1/2} + \frac{\tilde{\epsilon}}{2} - \frac{\tilde{\epsilon}^{3/2}}{4} + \frac{\tilde{\epsilon}^2}{8} - \frac{\tilde{\epsilon}^{5/2}}{16} + \frac{\tilde{\epsilon}^3}{32} - \frac{\tilde{\epsilon}^4}{128}, \quad (\text{D15})$$

$$z_1^- = -1 - \tilde{\epsilon}^{1/2} - \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}^{3/2}}{4} + \frac{3\tilde{\epsilon}^2}{8} + \frac{3\tilde{\epsilon}^{5/2}}{16} - \frac{\tilde{\epsilon}^3}{32} + \frac{5\tilde{\epsilon}^4}{128}. \quad (\text{D16})$$

Now, the integral (D10) can be calculated directly. The result is given by the expression

$$\begin{aligned} \tilde{f}(z) &= \frac{4}{\tilde{\epsilon}} \ln \left| \frac{z^2 + \alpha_2 z + \beta_2}{1 - z^2} \right| + \left(1 - \frac{5}{4}\tilde{\epsilon}\right) \ln |z - z_1^+| \\ &\quad - \left(1 - \frac{1}{4}\tilde{\epsilon}\right) \ln |z - z_1^-| - \left(1 - \frac{3}{4}\tilde{\epsilon}\right) \ln |z - z_2^+| \\ &\quad + \left(1 + \frac{3}{4}\tilde{\epsilon}\right) \ln |z - z_2^-| + \frac{\tilde{\epsilon}}{4} \ln \frac{\tilde{\epsilon}}{16}. \end{aligned} \quad (\text{D17})$$

The appropriate choice for the constant S appearing in (D8) is $S = \frac{4}{\tilde{\epsilon}} + \frac{9}{2}$, which guarantees that the divergent part of Eqs. (142) and (143) can be adsorbed within the function $\mathcal{J}(y)$. Expanding the function $\mathcal{J}$, we arrive at

$$\begin{aligned} \mathcal{J}(y) &= y + \ln |y - 1| - \frac{\tilde{\epsilon}}{8} \left[-4 \ln y - (2y - 1) \ln |y - 1| \right. \\ &\quad \left. - \ln^2 |y - 1| + \frac{2}{y - 1} - 5y \right] + \frac{9}{2}. \end{aligned} \quad (\text{D18})$$

From (D18) we can conclude that the function $\mathcal{J}$ has absorbed all the divergent terms, as well as $\ln \delta$ (after substituting $y = \frac{x}{a}$) within Eqs. (142) and (143). Those equations can now be rewritten as

$$\begin{aligned} & \left(1 + \frac{\varepsilon}{4(\lambda a)^2} \ln \delta\right) \mathcal{J}(y) - \frac{\varepsilon}{8(\lambda a)^2} \left[(2y+3) \ln y + 5y - 2L(y) - \sum_{n=1}^{\infty} \frac{(1-\delta)^n}{n!} S_n^>(y) \right] \\ &= -\ln \delta + \mathcal{J}(\hat{x}) - \frac{\varepsilon}{8(\lambda a)^2} [(2\hat{x}+3) \ln \hat{x} + 5\hat{x} - 2L(\hat{x})], \quad \text{when } y > 1 \end{aligned} \quad (\text{D19})$$

$$\begin{aligned} & \left(1 + \frac{\varepsilon}{4(\lambda a)^2} \ln \delta\right) \mathcal{J}(y) - \frac{\varepsilon}{8(\lambda a)^2} \left[\frac{2y^2}{y-1} \ln y + 3y - \sum_{n=1}^{\infty} \frac{\delta^n}{n!} S_n^<(y) \right] \\ &= -\ln \delta + \mathcal{J}(\hat{x}) - \frac{\varepsilon}{8(\lambda a)^2} [(2\hat{x}+3) \ln \hat{x} + 5\hat{x} - 2L(\hat{x})], \quad \text{when } y < 1. \end{aligned} \quad (\text{D20})$$

Note that both equations are well defined around $y = 1 + \mathcal{O}(\tilde{\varepsilon})$, which has been the main goal of this section. Let us find the general solution when $y = 1 + \frac{\varepsilon}{(\lambda a)^2} \eta$ and $\tilde{a} \gg 0$. This implies $\hat{x} = 1 + \frac{\varepsilon}{(\lambda a)^2} \mu$. Both Eqs. (D19) and (D20) become

$$\tilde{a} \mathcal{J}(1 + \tilde{\varepsilon} \eta) = -\ln \delta + \mathcal{J}(1 + \tilde{\varepsilon} \mu). \quad (\text{D21})$$

This formula (D21) is further reduced to

$$\tilde{a} \exp\left(\frac{\tilde{\varepsilon}}{4} \tilde{f}(1 + \tilde{\varepsilon} \eta)\right) = \exp\left(\frac{\tilde{\varepsilon}}{4} \tilde{f}(1 + \tilde{\varepsilon} \mu)\right). \quad (\text{D22})$$

Careful examination of the $\tilde{f}(z)$ function leads to the following:

$$\tilde{a}^{\frac{4}{\tilde{\varepsilon}}} = \left| \frac{4\mu - 1}{4\eta - 1} \right|. \quad (\text{D23})$$

Since the solution is given up to the first order of perturbation theory, we have $\tilde{a}^{4/\tilde{\varepsilon}} = \delta$. We expect that this relation holds in higher orders of perturbation theory as well. Then, Eq. (D23) simplifies to

$$\delta = \left| \frac{4\mu - 1}{4\eta - 1} \right|. \quad (\text{D24})$$

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- [1] C. Teitelboim, Gravitation and Hamiltonian structure in two space-time dimensions, *Phys. Lett.* **126B**, 41 (1983).
 - [2] R. Jackiw, Lower dimensional gravity, *Nucl. Phys.* **B252**, 343 (1985).
 - [3] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Evanescent black holes, *Phys. Rev. D* **45**, R1005 (1992).
 - [4] J. G. Russo, L. Susskind, and L. Thorlacius, End point of Hawking radiation, *Phys. Rev. D* **46**, 3444 (1992).
 - [5] J. G. Russo, L. Susskind, and L. Thorlacius, Cosmic censorship in two-dimensional gravity, *Phys. Rev. D* **47**, 533 (1993).
 - [6] S. Bose, L. Parker, and Y. Peleg, Semi-infinite throat as the end-state geometry of two-dimensional black hole evaporation, *Phys. Rev. D* **52**, 3512 (1995).
 - [7] T. M. Fiola, J. Preskill, A. Strominger, and S. P. Trivedi, Black hole thermodynamics and information loss in two dimensions, *Phys. Rev. D* **50**, 3987 (1994).
 - [8] S. Djordjević, A. Gočanin, D. Gočanin, and V. Radovanović, Page curve for an eternal Schwarzschild black hole in a dimensionally reduced model of dilaton gravity, *Phys. Rev. D* **106**, 105015 (2022).
 - [9] M. Burić and V. Radovanović, Quantum corrections for an (anti)evaporating black hole, *Phys. Rev. D* **63**, 044020 (2001).
 - [10] V. P. Frolov, W. Israel, and S. N. Solodukhin, On one loop quantum corrections to the thermodynamics of charged black holes, *Phys. Rev. D* **54**, 2732 (1996).
 - [11] M. Buric, V. Radovanovic, and A. R. Mikovic, One loop correction for Schwarzschild black hole via 2-D dilaton gravity, *Phys. Rev. D* **59**, 084002 (1999).
 - [12] M. Buric and V. Radovanovic, Quantum corrections for the Reissner-Nordström charged black hole, *Classical Quantum Gravity* **16**, 3937 (1999).
 - [13] A. Fabbri and J. Navarro-Salas, *Modeling Black Hole Evaporation* (World Scientific / Imperial College Press, London, 2005).
 - [14] D. Grumiller, W. Kummer, and D. V. Vassilevich, Dilaton gravity in two-dimensions, *Phys. Rep.* **369**, 327 (2002).
 - [15] S. W. Hawking, Breakdown of predictability in gravitational collapse, *Phys. Rev. D* **14**, 2460 (1976).
 - [16] J. Bekenstein, Black holes and the second law, *Lett. Nuovo Cimento* **4**, 737 (1972).
 - [17] J. D. Bekenstein, Black holes and entropy, *Phys. Rev. D* **7**, 2333 (1973).
 - [18] S. W. Hawking, Black holes and thermodynamics, *Phys. Rev. D* **13**, 191 (1976).
 - [19] S. W. Hawking, Particle creation by black holes, *Commun. Math. Phys.* **43**, 199 (1975); **46**, 206(E) (1976).

- [20] D. N. Page, Information in black hole radiation, *Phys. Rev. Lett.* **71**, 3743 (1993).
- [21] D. N. Page, Time dependence of hawking radiation entropy, *J. Cosmol. Astropart. Phys.* **09** (2013) 028.
- [22] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, Replica wormholes and the entropy of hawking radiation, *J. High Energy Phys.* **05** (2020) 013.
- [23] T. Hartman, E. Shaghoulian, and A. Strominger, Islands in asymptotically flat 2d gravity, *J. High Energy Phys.* **07** (2020) 022.
- [24] X. Wang, R. Li, and J. Wang, Page curves for a family of exactly solvable evaporating black holes, *Phys. Rev. D* **103**, 126026 (2021).
- [25] K. Hashimoto, N. Iizuka, and Y. Matsuo, Islands in Schwarzschild black holes, *J. High Energy Phys.* **06** (2020) 085.
- [26] T. Anegawa and N. Iizuka, Notes on islands in asymptotically flat 2d dilaton black holes, *J. High Energy Phys.* **07** (2020) 036.
- [27] M.-H. Yu and X.-H. Ge, Islands and Page curves in charged dilaton black holes, *Eur. Phys. J. C* **82**, 14 (2022).
- [28] X. Wang, R. Li, and J. Wang, Islands and Page curves of Reissner-Nordström black holes, *J. High Energy Phys.* **04** (2021) 103.
- [29] A. Almheiri, R. Mahajan, and J. E. Santos, Entanglement islands in higher dimensions, *SciPost Phys.* **9**, 001 (2020).
- [30] F. F. Gautason, L. Schneiderbauer, W. Sybesma, and L. Thorlacius, Page curve for an evaporating black hole, *J. High Energy Phys.* **05** (2020) 091.
- [31] A. M. Polyakov, Quantum geometry of bosonic strings, *Phys. Lett.* **103B**, 207 (1981).
- [32] A. Fabbri, S. Farese, J. Navarro-Salas, G. J. Olmo, and H. Sanchis-Alepuz, Semiclassical zero-temperature corrections to Schwarzschild spacetime and holography, *Phys. Rev. D* **73**, 104023 (2006).
- [33] P.-M. Ho, Y. Matsuo, and Y. Yokokura, Analytic description of semiclassical black-hole geometry, *Phys. Rev. D* **102**, 024090 (2020).
- [34] Y. Potaux, S. N. Solodukhin, and D. Sarkar, Spacetime structure, asymptotic radiation, and information recovery for a quantum hybrid state, *Phys. Rev. Lett.* **130**, 261501 (2023).
- [35] Y. Potaux, D. Sarkar, and S. N. Solodukhin, Hybrid quantum states in 2d dilaton gravity, *Phys. Rev. D* **108**, 125012 (2023).
- [36] A. del Río, F. J. Marañón-González, and J. Navarro-Salas, Singularity resolution in spherically reduced 2D semiclassical gravity with negative central charge, *Phys. Rev. D* **111**, 045025 (2025).
- [37] S. Djordjević and V. Radovanović, Page curve for an evaporating Schwarzschild black hole in dimensionally reduced model of dilaton gravity, [arXiv:2507.17855](https://arxiv.org/abs/2507.17855).