

Quantum gravity and matter fields in a general background gauge

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We analyze the gauge dependence of the effective action in an interacting quantum theory of gravitational and matter fields. An explicit off-shell result is obtained in a general background gauge at one-loop order, which reduces in a particular gauge to the effective action found by 't Hooft and Veltman [Ann. Inst. Henri Poincaré Phys. Theor. A **20**, 9 (1974)]. We confirm the validity of DeWitt-Kallosh theorem, which implies that the on-shell effective action should be independent of the gauge-fixing parameter. We employ this theorem to expose the nonrenormalizability of the theory in a general background gauge.

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I. INTRODUCTION

Classical general relativity is a successful theory which provides a very good description of the gravitational interactions that occur at low energies. There have been many attempts to quantize gravity along the lines used in other gauge theories, and it was recognized that Einstein's theory of gravity is generally not renormalizable in the usual sense. The contributions generated by the Feynman loops to all orders require an infinite number of counterterms to cancel the ultraviolet divergences present in this theory. The current point of view is that general relativity could be considered as a low energy effective field theory of some fundamental theory [1,2].

Pure quantum gravity is on-shell renormalizable at one-loop order, so that its divergences can be absorbed by a redefinition of the fields. On the other hand, it was shown in Ref. [3] that when a scalar field is coupled to gravity the theory is no longer on-shell renormalizable at one-loop order, in the sense that its divergences cannot be removed by field redefinitions. This outcome has been found in a particular background-field gauge.

Many of the perturbative computations done in quantum gravity are used the background-field method, which preserves gauge invariance at all stages of the calculations [3–6].

It was shown that the S matrix calculated in this approach is equivalent to the conventional S matrix [7]. Thus, one expects the on-shell counterterms to be independent of the gauge-fixing parameters and of the parametrization of graviton fields. This statement was proved by DeWitt and Kallosh through a formal theorem, which asserts that the loop counterterms on mass shell should be independent of the gauge fixing and of the parametrization [8,9]. Their arguments have been subsequently improved in several papers [10–15].

However, some questions have been raised on the general validity of this theorem in the nonrenormalizable theory of quantum gravity [16–18]. Nevertheless, the gauge-fixing independence of the on-shell effective action was explicitly checked, in a rather general gauge, at one-loop order in quantum gravity with a cosmological constant [19]. Moreover, the on-shell independence of the effective action in this model has been verified for a very general parametrization of the metric field [20].

It may be useful to verify through explicit calculations the DeWitt-Kallosh theorem for quantum gravity in the presence of matter fields. A wide class of such theories has drawn considerable attention in the areas of quantum [21] and inflationary [22] cosmology, where the scalar fields are nonminimally coupled to gravity in a very involved way. For these reasons, we consider in this paper a simple coupled scalar-gravity theory. We use the background-field method with a general nonminimal gauge-fixing Lagrangian,¹ that is characterized by two generic gauge parameters: ξ and ζ . We

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¹A systematic approach to handle nonminimal gauges in the Schwinger-DeWitt technique was derived in [23] for pure quantum gravity.

show that, in the limit $\xi \rightarrow 0$, there appear in individual Feynman diagrams singularities which could hinder the general validity of this theorem. However, it turns out that such singularities cancel out when adding all diagrams.

We outline the Becchi-Rouet-Stora-Tyutin (BRST) [24] quantization of this theory in Sec. II, where we clarify some aspects of the DeWitt-Kallosh theorem within the context of this theory. In Sec. III, we present the calculation of the off-shell effective action at one-loop order, which explicitly depends on the generic gauge parameters ξ and ζ . In particular, we recover the result found in Ref. [10] in the limit $\xi = 1$ and reproduce the 't Hooft and Veltman result in the limit $\xi = \zeta = 1$. We confirm this theorem by showing that the on-shell effective action at one-loop is independent of the gauge parameters. As shown in Sec. IV, the DeWitt-Kallosh theorem enables one to prove in a simple way that this theory remains nonrenormalizable in a general background-field gauge. A summary of the results is presented in Sec. V. Some useful details of the calculations are given in the Appendixes.

II. BRST SYMMETRY AND DEWITT-KALLOSH THEOREM

Let us consider, for definiteness, the interaction between a scalar field $\bar{\phi}$ and a gravitational field $\bar{g}^{\mu\nu}$ described by the Einstein-Hilbert Lagrangian [3],

$$\mathcal{L}^{\text{inv}}(\bar{g}, \bar{\phi}) = -\frac{1}{\kappa^2} \sqrt{\bar{g}} \left[\bar{R} + \frac{\kappa^2}{2} \bar{g}^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} \right], \quad (2.1)$$

where $\kappa^2 = 16\pi G_N$ and $\bar{R}$ is the Ricci scalar.

In order to preserve the basic diffeomorphism symmetry of the theory, we use the background-field method [4], where the fields are split into a background and a quantum part as

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu}; \quad \bar{\phi} = \phi + \varphi. \quad (2.2)$$

Here, $g_{\mu\nu}$ and ϕ are background fields, and $h_{\mu\nu}$ and φ are quantum fields. The Lagrangian (2.1), with the substitution (2.2), is invariant under two types of transformations. The first type is a background-field transformation given by

$$\begin{aligned} \frac{1}{\kappa} \bar{\delta} g_{\mu\nu} &= \epsilon^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu \epsilon^\alpha + g_{\nu\alpha} \partial_\mu \epsilon^\alpha \equiv E_{\mu\nu\alpha}(g) \epsilon^\alpha, \\ \frac{1}{\kappa} \bar{\delta} \phi &= \epsilon^\alpha \partial_\alpha \phi, \end{aligned} \quad (2.3a)$$

$$\begin{aligned} \frac{1}{\kappa} \bar{\delta} h_{\mu\nu} &= \epsilon^\alpha \partial_\alpha h_{\mu\nu} + h_{\mu\alpha} \partial_\nu \epsilon^\alpha + h_{\nu\alpha} \partial_\mu \epsilon^\alpha, \\ \frac{1}{\kappa} \bar{\delta} \varphi &= \epsilon^\alpha \partial_\alpha \varphi. \end{aligned} \quad (2.3b)$$

where $E_{\mu\nu\alpha}(g)$ is covariant with respect to the background coordinate transformation.

The second type is a quantum field transformation given by

$$\delta g_{\mu\nu} = 0, \quad \delta \phi = 0, \quad (2.4a)$$

$$\delta h_{\mu\nu} = E_{\mu\nu\alpha}(g_{\mu\nu} + \kappa h_{\mu\nu}) \epsilon^\alpha, \quad \delta \varphi = \kappa \epsilon^\alpha \partial_\alpha (\phi + \varphi). \quad (2.4b)$$

In order to maintain the background symmetry (2.3), but break the quantum symmetry (2.4), we introduce a gauge-fixing term given by

$$\mathcal{L}_{\text{gf}}(g, \phi, h, \varphi) = -\frac{1}{2\xi} \sqrt{g} \left(h_{\mu\nu}^{\mu\nu} - \frac{1}{2} h_{\alpha}^{\alpha\mu} - \zeta \kappa \varphi \partial^\mu \phi \right)^2, \quad (2.5)$$

where ∂_μ denotes covariant derivatives with respect to the background field $g_{\mu\nu}$, and ξ, ζ are two arbitrary gauge parameters. The corresponding Faddeev-Popov ghost Lagrangian [25] is given by

$$\mathcal{L}_{\text{gh}} = \sqrt{g} \bar{\eta}^\mu [\eta_{\mu;\alpha}^\alpha - R_{\mu\nu} \eta^\nu - \zeta \kappa \partial_\mu \phi \partial_\nu \phi \eta^\nu], \quad (2.6)$$

where $R_{\mu\nu}$ is the Ricci tensor, and $\bar{\eta}^\mu$ and η^ν are vector ghost fields, treated as quantum fields. At this point, it turns out to be more convenient to use, instead of the gauge-fixing term (2.5), an equivalent Nakanishi-Lautrup Lagrangian [26,27] defined as

$$\mathcal{L}'_{\text{gf}} = \sqrt{g} \left[\frac{\xi}{2} B_\mu B^\mu + B_\mu \left(h^{\mu\nu}_{;\nu} - \frac{1}{2} h_{\alpha}^{\alpha\mu} - \zeta \kappa \varphi \partial^\mu \phi \right) \right], \quad (2.7)$$

where B^μ is a commuting quantum field. By functionally integrating over B^μ , one can easily check the equivalence of Eq. (2.7) to the gauge-fixing term Eq. (2.5).

Proceeding in this way, one can show that the total Lagrangian

$$\mathcal{L}_{\text{tot}}(g, \phi, h, \varphi, B, \bar{\eta}, \eta) = \mathcal{L}^{\text{inv}}(g + \kappa h, \phi + \varphi) + \mathcal{L}'_{\text{gf}} + \mathcal{L}_{\text{gh}} \quad (2.8)$$

is invariant under the quantum BRST transformations [28]

$$\delta g_{\mu\nu} = 0, \quad \delta \phi = 0, \quad (2.9a)$$

$$\delta h_{\mu\nu} = \omega E_{\mu\nu\tau}(\bar{g}) \eta^\tau \equiv \omega S h_{\mu\nu}, \quad \delta \varphi = \kappa \omega (\partial_\tau \bar{\phi}) \eta^\tau \equiv \omega S \varphi, \quad (2.9b)$$

$$\delta \eta^\mu = \kappa \omega \eta^\nu \eta_{\nu;\mu}^\mu \equiv \omega S \eta^\mu, \quad \delta \bar{\eta}_\mu = -\omega B_\mu \equiv \omega S \bar{\eta}_\mu, \quad (2.9c)$$

$$\delta B_\mu \equiv \omega S B_\mu = 0, \quad (2.9d)$$

where ω is an infinitesimal constant Grassmann quantity, with inverse mass dimension.

To this end, one first verifies the important relation

$$\mathcal{L}'_{\text{gf}} + \mathcal{L}_{\text{gh}} = \mathbf{s} \left\{ \sqrt{g} \left[-\frac{\xi}{2} \bar{\eta}_\mu B^\mu - \bar{\eta}_\mu \left(h^{\mu\nu}_{;\nu} - \frac{1}{2} h^\alpha_{\alpha;\mu} - \zeta \kappa \varphi \partial^\mu \phi \right) \right] \right\}, \quad (2.10)$$

which, together with the nilpotency property of the $\mathbf{s}$ operator,

$$\mathbf{s}^2 h^{\mu\nu} = \mathbf{s}^2 \varphi = \mathbf{s}^2 \eta^\mu = \mathbf{s}^2 \bar{\eta}^\mu = \mathbf{s}^2 B_\mu = 0, \quad (2.11)$$

manifestly shows the BRST invariance of the total Lagrangian Eq. (2.8).

Next, one defines the effective action $\Gamma_{\text{eff}}(g, \phi)$ as

$$\begin{aligned} \exp[i\Gamma_{\text{eff}}(g, \phi)] &= \int \mathcal{D}h \mathcal{D}\varphi \mathcal{D}B \mathcal{D}\bar{\eta} \mathcal{D}\eta \\ &\times \exp i \int d^4x \left[\mathcal{L}_{\text{tot}}(g, \phi, h, \varphi, B, \bar{\eta}, \eta) \right. \\ &\quad \left. - \mathcal{L}^{\text{inv}}(g, \phi) - \kappa \frac{\delta \mathcal{L}^{\text{inv}}}{\delta g_{\mu\nu}} h_{\mu\nu} - \frac{\delta \mathcal{L}^{\text{inv}}}{\delta \phi} \varphi \right], \end{aligned} \quad (2.12)$$

where the expression the square bracket involves terms quadratic and of higher degree in the quantum fields. To one-loop order, it is sufficient to keep only the quadratic terms.

Similarly, one defines the background expectation value of an operator O as

$$\begin{aligned} \langle O \rangle &\equiv \int \mathcal{D}h \mathcal{D}\varphi \mathcal{D}B \mathcal{D}\bar{\eta} \mathcal{D}\eta O(g, \phi, h, \varphi, B, \bar{\eta}, \eta) \\ &\times \exp i \int d^4x \left[\mathcal{L}_{\text{tot}}(g, \phi, h, \varphi, B, \bar{\eta}, \eta) \right. \\ &\quad \left. - \mathcal{L}^{\text{inv}}(g, \phi) - \kappa \frac{\delta \mathcal{L}^{\text{inv}}}{\delta g_{\mu\nu}} h_{\mu\nu} - \frac{\delta \mathcal{L}^{\text{inv}}}{\delta \phi} \varphi \right]. \end{aligned} \quad (2.13)$$

This is invariant under BRST transformations of the fields, as these can be absorbed by a change of field variables since the Jacobian of the transformation is 1.

Let us now examine the ξ dependence of the effective action, under an infinitesimal change $\xi \rightarrow \xi + \Delta\xi$ (a systematic general approach is provided in [15]). Then, using Eqs. (2.7), (2.12), and (2.13), one can see that

$$\delta_\xi \exp i\Gamma_{\text{eff}}(g, \phi) = \frac{i\Delta\xi}{2} \int d^4x \sqrt{g} \langle B_\mu B^\mu \rangle. \quad (2.14)$$

We next consider the operator $O = \bar{\eta}_\mu B^\mu$ which, due to Eq. (2.9), satisfies the relation

$$\langle \mathbf{s}(\bar{\eta}_\mu B^\mu) \rangle = -\langle B_\mu B^\mu \rangle. \quad (2.15)$$

Using the invariance of $\langle \bar{\eta}_\mu B^\mu \rangle$ under the BRST transformations of all fields in the integrand $\mathbf{s}\langle \bar{\eta}_\mu B^\mu \rangle = 0$, one gets from Eq. (2.13) the Ward identity,

$$\langle \mathbf{s}(\bar{\eta}_\mu B^\mu) \rangle = i \left\langle \bar{\eta}_\mu B^\mu \int d^4x' \left[\kappa \frac{\delta \mathcal{L}^{\text{inv}}}{\delta g_{\mu\nu}} \mathbf{s} h_{\mu\nu} + \frac{\delta \mathcal{L}^{\text{inv}}}{\delta \phi} \mathbf{s} \varphi \right]_{x'} \right\rangle. \quad (2.16)$$

Combining Eqs. (2.15) and (2.16), one finds from Eq. (2.14) the equation

$$\begin{aligned} \delta_\xi \exp i\Gamma_{\text{eff}}(g, \phi) &= \frac{\Delta\xi}{2} \int d^4x \sqrt{g(x)} \left\langle (\bar{\eta}_\mu B^\mu)(x) \right. \\ &\quad \left. \times \int d^4x' \left[\kappa \frac{\delta \mathcal{L}^{\text{inv}}}{\delta g_{\mu\nu}} \mathbf{s} h_{\mu\nu} + \frac{\delta \mathcal{L}^{\text{inv}}}{\delta \phi} \mathbf{s} \varphi \right]_{x'} \right\rangle. \end{aligned} \quad (2.17)$$

Proceeding in a similar way, one finds that under an infinitesimal change $\zeta \rightarrow \zeta + \Delta\zeta$,

$$\begin{aligned} \delta_\zeta \exp i\Gamma_{\text{eff}}(g, \phi) &= -\kappa \Delta\zeta \int d^4x \sqrt{g(x)} \left\langle (\bar{\eta}^\mu \partial_\mu \phi \varphi)(x) \right. \\ &\quad \left. \times \int d^4x' \left[\kappa \frac{\delta \mathcal{L}^{\text{inv}}}{\delta g_{\mu\nu}} \mathbf{s} h_{\mu\nu} + \frac{\delta \mathcal{L}^{\text{inv}}}{\delta \phi} \mathbf{s} \varphi \right]_{x'} \right\rangle. \end{aligned} \quad (2.18)$$

When the background fields $g_{\mu\nu}$ and ϕ are on shell, each functional derivative in Eq. (2.17) vanishes, so that one obtains the relation

$$\delta_\xi \exp i\Gamma_{\text{eff}}(g, \phi)|_{\text{on-shell}} = \delta_\zeta \exp i\Gamma_{\text{eff}}(g, \phi)|_{\text{on-shell}} = 0, \quad (2.19)$$

which confirms the DeWitt-Kallosh theorem in the theory described by the Einstein Lagrangian Eq. (2.1). In the next section, this result is explicitly verified at one-loop order for general values of ξ and ζ .

III. GAUGE-DEPENDENCE OF THE ONE-LOOP EFFECTIVE ACTION

On dimensional and symmetry grounds, the background counterterm Lagrangian at one-loop order may be expressed in terms of the six independent structures,

$$\begin{aligned} \mathcal{L}_{\text{CT}} &= \frac{\sqrt{g}}{16\pi^2 \epsilon} [c_1(\xi, \zeta) R^2 + c_2(\xi, \zeta) R_{\mu\nu} R^{\mu\nu} \\ &\quad + \kappa^2 c_3(\xi, \zeta) (\mathbf{D}_\mu \mathbf{D}^\mu \phi)^2 \\ &\quad + \kappa^2 c_4(\xi, \zeta) R \partial_\mu \phi \partial^\mu \phi + \kappa^2 c_5(\xi, \zeta) R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \\ &\quad + \kappa^4 c_6(\xi, \zeta) (\partial_\mu \phi \partial^\mu \phi)^2], \end{aligned} \quad (3.1)$$

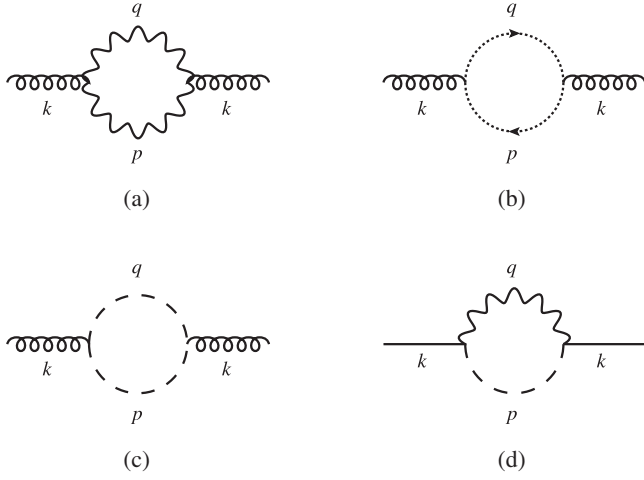


FIG. 1. One-loop contributions to $\langle g^{\mu\nu} g^{\alpha\beta} \rangle$ [(a)–(c)] and to $\langle \phi\phi \rangle$ (d). The wavy, curly, dotted, dashed, and solid lines denote the quantum graviton, background graviton, ghost, quantum scalar, and the background scalar, respectively. The momenta satisfies the relation: $q = p + k$.

where $\epsilon = (4 - D)/2$, D is the spacetime dimension, and $c_i(\xi, \zeta)$ are dimensionless coefficients, and we have taken into account the identity

$$\int d^4x \sqrt{g} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2) = 0. \quad (3.2)$$

In principle, three more independent structures, $\kappa^2 R \phi \square \phi$, $\kappa^2 R^{\mu\nu} \phi D_\mu D_\nu \phi$, $\kappa^4 \phi \partial^\mu \phi \partial^\nu \phi D_\mu D_\nu \phi$, could appear in theories more general than that in Eq. (2.1). Altogether, these nine structures (or an equivalent set) exhaust all possible independent terms expected from dimensional analysis, symmetry, and the form of the interaction terms. Such

improved gravitational theories are commonly used in cosmology [29,30]. In our model, characterized by the gauge fixing term (2.5), only the six structures shown in Eq. (3.1) actually occur.

One could also consider more general invariants, such as $\kappa^3 R \phi^3$ or $\kappa R_{\mu\nu} \phi \partial^\mu \phi \partial^\nu \phi$, which are odd in ϕ . However, our simple scalar model (2.1) is invariant under the discrete symmetry $\phi \rightarrow -\phi$. Such counterterms are therefore forbidden, as the background-field method ensures that the effective action preserves the symmetries of the classical action (2.1). Indeed, from the Feynman rules in Appendix A, no diagram leads to an odd counterterm in ϕ at one-loop order. Nevertheless, spontaneous symmetry breaking can still occur in more general theories [31].

The Feynman diagrams contributing at one-loop order to the background graviton self-energy $\Pi^{\mu\nu\alpha\beta}(k)$ and the scalar self-energies $\Pi(k)$ are shown in Fig. 1.

As shown in Appendix B [see Eq. (B5)], the divergent parts of the diagrams shown in Figs. 1(a)–1(c) yield, in momentum space, the following contribution to the graviton self-energy:

$$\Pi^{\mu\nu\alpha\beta}(k) = \frac{\kappa^2}{16\pi^2 \epsilon} k^4 \left\{ 4c_1(\xi, \zeta) L^{\mu\nu} L^{\alpha\beta} + c_2(\xi, \zeta) \times \left[L^{\mu\nu} L^{\alpha\beta} + \frac{1}{2} (L^{\mu\alpha} L^{\nu\beta} + L^{\mu\beta} L^{\nu\alpha}) \right] \right\}, \quad (3.3)$$

where $L^{\mu\nu} = k^\mu k^\nu / k^2 - \eta^{\mu\nu}$ and $c_1(\xi, \zeta)$ and $c_2(\xi, \zeta)$ are gauge-dependent coefficients given by

$$c_1(\xi, \zeta) = \frac{1}{80} + \frac{1}{6}(\xi - 1)^2, \quad c_2(\xi, \zeta) = \frac{43}{120} + \frac{\xi(\xi - 1)}{3}. \quad (3.4)$$

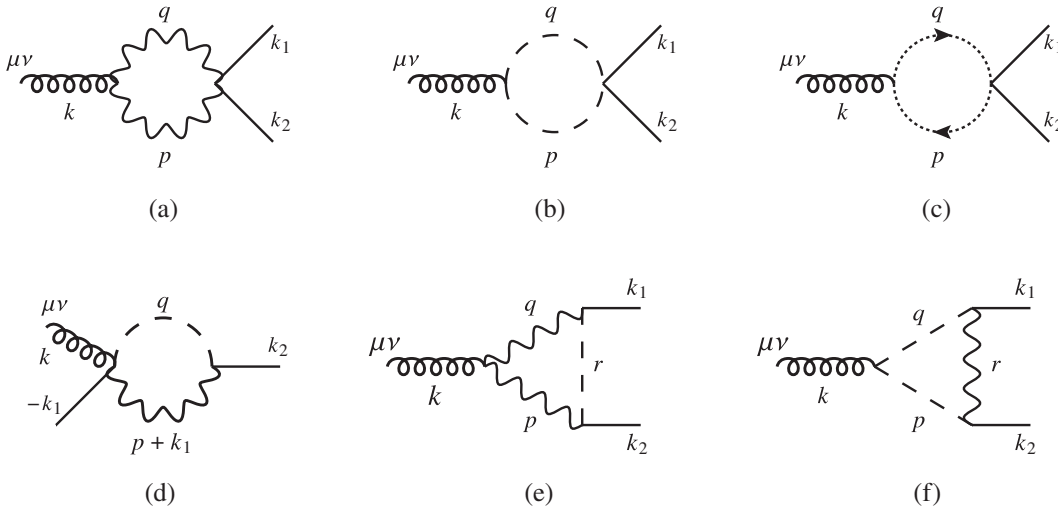


FIG. 2. One-loop contributions (permutations have been omitted) to the background 3-point function $\langle g^{\mu\nu}(k) \phi(k_1) \phi(k_2) \rangle$ [(a)–(f)]. We have that $q = p + k$, $r = p + k - k_1$ and $k = k_1 + k_2$.

We note here that the Eq. (3.3) is transverse to the momentum k , which is a consequence of the diffeomorphism invariance of the theory.

We can connect the above results with the counterterm Lagrangian (3.1) by using the relations

$$R(k) = \kappa k^2 L^{\alpha\beta} g_{\alpha\beta} + O(\kappa^2), \quad (3.5)$$

$$R^{\mu\nu}(k) = \frac{\kappa}{2} \left[L^{\alpha\beta} k^\mu k^\nu - \frac{k^2}{2} (L^{\mu\alpha} L^{\nu\beta} + L^{\mu\beta} L^{\nu\alpha}) \right] g_{\alpha\beta} + O(\kappa^2). \quad (3.6)$$

Then, one may verify that Eq. (3.4) fixes the coefficients $c_1(\xi, \zeta)$ and $c_2(\xi, \zeta)$ which appear in the Eq. (3.1). Likewise, the divergent part of the scalar self-energy graph depicted in Fig. 1(d) [see Eq. (B8)] fixes the coefficient $c_3(\xi, \zeta)$ to be

$$c_3(\xi, \zeta) = \frac{1}{2} + \frac{\zeta - 1}{2}. \quad (3.7)$$

In order to fix the coefficients $c_4(\xi, \zeta)$ and $c_5(\xi, \zeta)$ in Eq. (3.1), we consider the diagrams shown in Fig. 2. As shown in Appendix B [see Eq. (B17)], evaluating the divergent contributions of the one-loop diagrams in Fig. 2 and using the relations given in Eqs. (3.5) and (3.6), fixes the coefficients c_4 and c_5 as

$$c_4(\xi, \zeta) = -\frac{1}{12} - (\xi - 1) \frac{\xi + 5}{12} + \frac{(\xi - \zeta)^2}{4},$$

$$c_5(\xi, \zeta) = \frac{(\xi - 1)(2\xi + 5)}{6}. \quad (3.8)$$

Next, we consider the divergent parts of the diagrams shown in Fig. 3, which fix the coefficient $c_6(\xi, \zeta)$. As

displayed in Appendix B [see Eq. (B25)], these contributions fix the coefficient $c_6(\xi, \zeta)$ to be

$$c_6(\xi, \zeta) = \frac{1}{2} + \frac{\xi - 1}{4} + \frac{(\xi - \zeta)^2}{8}. \quad (3.9)$$

Substituting the coefficients (3.4), (3.7), (3.8), and (3.9) in Eq. (3.1), we obtain the result

$$\begin{aligned} \mathcal{L}_{\text{CT}} = & \frac{\sqrt{g}}{16\pi^2 \epsilon} \left[\frac{1}{240} [3 + 40(\xi - 1)^2] R^2 \right. \\ & + \frac{1}{120} [43 + 40\xi(\xi - 1)] R_{\mu\nu} R^{\mu\nu} + \frac{\kappa^2}{2} \zeta (D_\mu D^\mu \phi)^2 \\ & - \frac{\kappa^2}{12} [1 + (\xi - 1)(\xi + 5) - 3(\xi - \zeta)^2] R \partial_\mu \phi \partial^\mu \phi \\ & + \frac{\kappa^2}{6} (\xi - 1)(2\xi + 5) R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \\ & \left. + \frac{\kappa^4}{8} [4 + 2(\xi - 1) + (\xi - \zeta)^2] (\partial_\mu \phi \partial^\mu \phi)^2 \right]. \quad (3.10) \end{aligned}$$

The above expression shows that the off-shell background counterterm Lagrangian (3.10) is a gauge-dependent quantity, which reduces to that obtained in Ref. [10] for $\xi = 1$. We note here that the coefficient of $R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ vanishes when $\xi = 1$, which explains why this structure does not appear in the 't Hooft and Veltman counterterm Lagrangian. A comparison of our result in Eq. (3.10) with the one-loop counterterm found in the improved gravitational gauge theories which are used in quantum cosmology [29,30] is presented in Appendix C.

To determine the on-shell value of this quantity, we use Einstein's equations of motion,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu}, \quad D_\mu D^\mu \phi = 0, \quad (3.11)$$

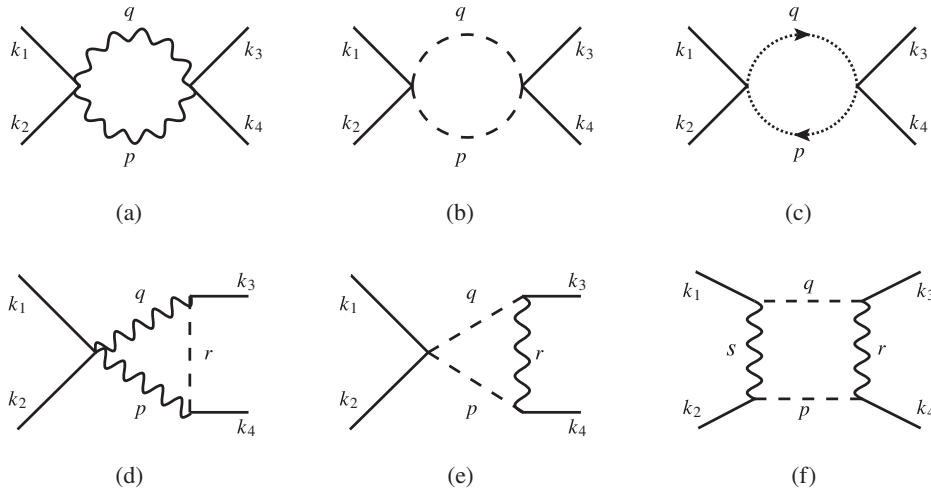


FIG. 3. One-loop contributions (permutations have been omitted) to the background 4-point function $\langle \phi(k_1) \phi(k_2) \phi(k_3) \phi(k_4) \rangle$ [(a)–(f)]. Here, $q = p + k_1 + k_2$, $r = p + k_4$ and $s = p + k_2$.

where the energy-momentum tensor $T^{\mu\nu}$ corresponding to the Lagrangian (2.1) is given by

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi - \partial_\mu \phi \partial_\nu \phi. \quad (3.12)$$

From the above equations, we obtain the on-shell relations

$$R_{\mu\nu} = -\frac{\kappa^2}{2} \partial_\mu \phi \partial_\nu \phi, \quad R = -\frac{\kappa^2}{2} \partial_\mu \phi \partial^\mu \phi. \quad (3.13)$$

Using these relations and Eqs. (3.1) and (3.10), we obtain for the on-shell counterterm Lagrangian the gauge-independent result (see Appendix D),

$$\begin{aligned} \mathcal{L}_{\text{CT}}|_{\text{on-shell}} &= \frac{\sqrt{g}\kappa^4}{16\pi^2\epsilon} \frac{1}{4} [c_1(\xi, \zeta) + c_2(\xi, \zeta) - 2c_4(\xi, \zeta) \\ &\quad - 2c_5(\xi, \zeta) + 4c_6(\xi, \zeta)] [\partial_\mu \phi \partial^\mu \phi]^2 \\ &= \frac{\sqrt{g}\kappa^4}{16\pi^2\epsilon} \frac{203}{320} [\partial_\mu \phi \partial^\mu \phi]^2, \end{aligned} \quad (3.14)$$

which is in agreement with the DeWitt-Kallosh theorem.

IV. NONRENORMALIZABILITY OF THE THEORY

In some gauge theories (like those with spontaneous symmetry breaking) there are nonrenormalizable gauges, but there also exist gauges ('t Hooft gauges) where the renormalizability is restored. One may ask if a similar feature may also occur in the coupled scalar-gravity theory, which was proved to be, in a particular background gauge, nonrenormalizable at one-loop order [3]. In contrast, this theory remains nonrenormalizable in a general background gauge, as may be shown in a simple way by using the DeWitt-Kallosh theorem.

Suppose that there may exist gauges where this theory is renormalizable in the sense that its divergences could be absorbed by field redefinitions. This property would require the relation

$$\mathcal{L}^{\text{inv}}(g + \Delta g, \phi + \Delta \phi) = \mathcal{L}^{\text{inv}}(g, \phi) + \mathcal{L}_{\text{CT}}, \quad (4.1)$$

where $\mathcal{L}_{\text{CT}}$ is the counterterm Lagrangian (3.1). Using Eq. (2.1), this relation would imply that

$$\begin{aligned} &\frac{\delta \mathcal{L}^{\text{inv}}}{\delta g_{\mu\nu}} \Delta g_{\mu\nu} + \frac{\delta \mathcal{L}^{\text{inv}}}{\delta \phi} \Delta \phi \\ &= \sqrt{g} \left\{ \left[\frac{1}{\kappa^2} \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right) - \frac{1}{2} \partial^\mu \phi \partial^\nu \phi \right. \right. \\ &\quad \left. \left. + \frac{1}{4} g^{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi \right] \Delta g_{\mu\nu} + D_\mu D^\mu \phi \Delta \phi \right\} = \mathcal{L}_{\text{CT}}. \end{aligned} \quad (4.2)$$

On symmetry grounds and taking into account the structure of the interaction Lagrangian, one can write the variations at one-loop order in the form

$$\begin{aligned} \Delta g_{\mu\nu} &= \frac{\kappa^2}{16\pi^2\epsilon} [C_1 R g_{\mu\nu} + C_2 R_{\mu\nu} + \kappa (C_3 \partial_\mu \phi \partial_\nu \phi \\ &\quad + C_4 g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi)], \end{aligned} \quad (4.3)$$

$$\Delta \phi = \frac{\kappa^2}{16\pi^2\epsilon} C_5 D_\mu D^\mu \phi, \quad (4.4)$$

where C_i are some dimensionless parameters.

Equation (4.2) leads to a system of six equations expressing the five functions C_i in terms of the six coefficients c_i occurring in Eq. (3.1), namely,

$$2C_1 + C_2 = -2c_1(\xi, \zeta), \quad (4.5a)$$

$$C_2 = c_2(\xi, \zeta), \quad (4.5b)$$

$$C_5 = c_3(\xi, \zeta), \quad (4.5c)$$

$$2C_1 + C_2 + 2C_3 + 4C_4 = -4c_4(\xi, \zeta), \quad (4.5d)$$

$$C_2 + 2C_3 = 2c_5(\xi, \zeta), \quad (4.5e)$$

$$C_3 - 2C_4 = 4c_6(\xi, \zeta). \quad (4.5f)$$

Solving the first five equations, yields the solution

$$\begin{aligned} C_1 &= -c_1(\xi, \zeta) + \frac{c_2(\xi, \zeta)}{2}, \quad C_2 = c_2(\xi, \zeta), \\ C_3 &= -\frac{c_2(\xi, \zeta)}{2} + c_5(\xi, \zeta), \end{aligned} \quad (4.6a)$$

$$\begin{aligned} C_4 &= \frac{1}{4} [2c_1(\xi, \zeta) + c_2(\xi, \zeta) - 4c_4(\xi, \zeta) - 2c_5(\xi, \zeta)] \quad \text{and} \\ C_5 &= c_3(\xi, \zeta). \end{aligned} \quad (4.6b)$$

Substituting these results in Eq. (4.5f) leads to the consistency condition for renormalizability,

$$c_1(\xi, \zeta) + c_2(\xi, \zeta) - 2c_4(\xi, \zeta) - 2c_5(\xi, \zeta) + 4c_6(\xi, \zeta) = 0. \quad (4.7)$$

If this condition holds, the on-shell counterterm Lagrangian (3.14) would vanish,

$$\mathcal{L}_{\text{CT}}|_{\text{on-shell}} = 0. \quad (4.8)$$

The condition (4.7), which implies the relation (4.8), is necessary for renormalizability since the counterterm Lagrangian has a different structure than that of the original Lagrangian. Furthermore, the above argument shows that

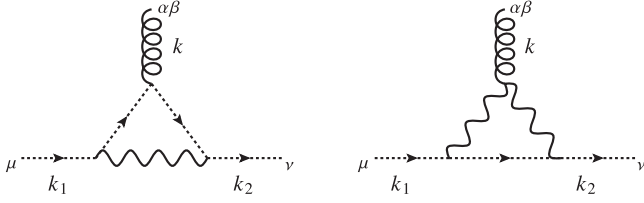


FIG. 4. One-loop contributions to the ghost-background graviton-ghost vertex.

the Eq. (4.7) is also a sufficient condition, because it follows in consequence of renormalizability.

It is worth noting that if Eq. (3.14) were gauge dependent one could find a gauge such that the renormalizability condition (4.7) would be satisfied. However, according to the DeWitt-Kallosh theorem, the on-shell counterterm Lagrangian should be a gauge-independent quantity, which turns out to be nonzero. Hence, it follows that the coupled scalar-gravity theory is not renormalizable in the above sense. However, as shown in Ref. [15], this theory is renormalizable in the sense that all divergences can be canceled by introducing the most general counterterms compatible with its gauge symmetry. This is achieved by using the Batalin–Vilkovisky technique [32] in conjunction with the background-field method.

An interesting outcome of this nonrenormalizability is the behavior of the ghost-background graviton-ghost vertex in quantum gravity in the Landau-DeWitt gauge. This vertex is illustrated in Fig. 4.

The Landau-DeWitt gauge, which is obtained in the limit $\xi \rightarrow 0$ and $\zeta \rightarrow 0$ of the gauge-fixing term (2.5), considerably simplifies the calculations and is very useful for the understanding of many physical phenomena [33,34]. In Yang-Mills theory, the ghost-background gluon-ghost vertex is finite in consequence of the Ward identities in the Landau background gauge. On the other hand, as shown in Appendix D, this no longer happens in quantum gravity, where the ghost-background graviton-ghost vertex is ultraviolet divergent in the Landau-DeWitt gauge.

V. DISCUSSION

We have examined, using the background-field method, the BRST symmetry of an interacting quantum theory of gravity and scalar fields. This symmetry leads to an important result, known as the DeWitt-Kallosh theorem. It asserts that, on shell, the counterterm Lagrangian ought to be independent of the gauge-fixing parameters and of the parametrization of the fields. It may be useful to verify such general statements by direct calculations, which could expose some aspects that might be overlooked in formal proofs. To this end, using a general background gauge characterized by two generic gauge parameters ξ and ζ , we have explicitly calculated the ultraviolet divergences at one-loop order.

We have shown that the off-shell counterterm Lagrangian (3.10) is a gauge-dependent quantity, which reduces to that obtained earlier by 't Hooft and Veltman in the particular background gauge $\xi = \zeta = 1$. Moreover, Eq. (3.12) is also consistent, for $\xi = 1$ and $\zeta = 0$, with the results obtained more recently in [29,30]. When $\xi = 1$, the graviton propagator is of the Feynman type, so that one can apply the same algorithm as that used in [3]. On the other hand, for a general ξ , this propagator is more involved, in which case one must employ a more elaborated procedure (see Appendixes). We note here that in the limit $\xi \rightarrow 0$ the contributions from some individual diagrams become singular, a feature that could pose a problem for the validity of the DeWitt-Kallosh theorem. But, when adding all terms, such singularities cancel out in consequence of the Ward identities in the Landau background gauge, which ensure that these singularities must be absent when $\xi = 0$. As a result, we obtain a polynomial contribution which is quadratic in the gauge parameters ξ and ζ .

On the other hand, the on-shell counterterm Lagrangian (3.14) becomes gauge independent, yielding a result which is in agreement with the DeWitt-Kallosh theorem. We applied this theorem to establish that in a general background gauge this theory is nonrenormalizable in the sense that its divergences cannot be absorbed by field redefinitions. However, the theory is renormalizable in the broader sense that the ultraviolet divergences can be canceled by introducing all possible counterterms allowed by its gauge symmetry.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: FEYNMAN RULES

In this appendix, we show and derive some of the Feynman rules required to compute the one-loop counterterms in the Einstein-Hilbert theory coupled to a scalar field in the background-field method. Here, we follow the usual definitions and conventions used in Ref. [3],

$$R^\mu{}_{\nu\alpha\beta} = -\partial_\beta \Gamma^\mu_{\nu\alpha} + \partial_\alpha \Gamma^\mu_{\nu\beta} + \Gamma^\mu_{\gamma\alpha} \Gamma^\gamma_{\nu\beta} - \Gamma^\mu_{\gamma\beta} \Gamma^\gamma_{\nu\alpha}, \quad (\text{A1})$$

$$R_{\mu\nu\alpha\beta} = g_{\mu\gamma} R^\gamma{}_{\nu\alpha\beta}, \quad R_{\nu\beta} = R^\mu{}_{\nu\beta\mu}, \quad R = R_{\nu\beta} g^{\nu\beta}. \quad (\text{A2})$$

Moreover, we use that

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa \mathbf{g}_{\mu\nu}, \quad (\text{A3})$$

where $\eta_{\mu\nu}$ is a flat background, and $\mathbf{g}_{\mu\nu}$ is the background graviton field.

In order to derive the Feynman rules, we use the definitions (2.2) and (A3),

$$\begin{aligned} \bar{g}^{\mu\nu} &= \eta^{\mu\nu} - \kappa \mathbf{g}^{\mu\nu} - \kappa \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta} + \kappa^2 \eta^{\mu\alpha} \mathbf{g}^{\nu\beta} h_{\alpha\beta} \\ &\quad + \kappa^2 \mathbf{g}^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta} + \kappa^2 h_{\alpha\beta}^{\mu\alpha} h^{\alpha\nu}, \end{aligned} \quad (\text{A4a})$$

$$\begin{aligned} \sqrt{\bar{g}} &= 1 + \frac{\kappa}{2} \mathbf{g}_{\mu}^{\mu} + \frac{\kappa}{2} \eta^{\mu\nu} h_{\mu\nu} - \frac{\kappa^2}{2} \mathbf{g}^{\mu\nu} h_{\mu\nu} + \frac{\kappa^2}{4} \mathbf{g}_{\mu}^{\mu} \eta^{\alpha\beta} h_{\alpha\beta} \\ &\quad + \frac{\kappa^2}{8} (h_{\alpha}^{\alpha 2} - 2 h_{\beta}^{\alpha} h_{\alpha}^{\beta}), \end{aligned} \quad (\text{A4b})$$

$$\Gamma_{\mu\nu}^{\lambda} = \frac{\kappa}{2} \eta^{\lambda\rho} (\partial_{\nu} \mathbf{g}_{\rho\mu} + \partial_{\mu} \mathbf{g}_{\rho\nu} - \partial_{\rho} \mathbf{g}_{\mu\nu}), \quad (\text{A4c})$$

$$\begin{aligned} \mathbf{D}_{\sigma} h_{\mu\nu} &= \partial_{\sigma} h_{\mu\nu} - \frac{\kappa}{2} \eta^{\alpha\rho} [(\partial_{\sigma} \mathbf{g}_{\rho\mu} + \partial_{\mu} \mathbf{g}_{\rho\sigma} - \partial_{\rho} \mathbf{g}_{\mu\sigma}) h_{\alpha\nu} \\ &\quad + (\partial_{\sigma} \mathbf{g}_{\rho\nu} + \partial_{\nu} \mathbf{g}_{\rho\sigma} - \partial_{\rho} \mathbf{g}_{\nu\sigma}) h_{\mu\alpha}], \end{aligned} \quad (\text{A4d})$$

in the effective EH action,

$$iS = i \int d^4x (\mathcal{L}^{\text{inv}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{gh}}), \quad (\text{A5})$$

where $\mathcal{L}^{\text{inv}}$, $\mathcal{L}_{\text{gf}}$, $\mathcal{L}_{\text{gh}}$ are, respectively, the invariant (2.1), gauge-fixing (2.5), and ghost (2.6) Lagrangians.

Consider the perturbation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa \tilde{\mathbf{g}}_{\mu\nu}$. One can show that, in momentum space,

$$\sqrt{\tilde{g}} \tilde{R} = \kappa q^2 L^{\alpha\beta} \tilde{\mathbf{g}}_{\alpha\beta} + \mathcal{O}(\kappa^2), \quad (\text{A6a})$$

$$\begin{aligned} \sqrt{\tilde{g}} \tilde{R}^{\mu\nu} &= \frac{\kappa}{2} \left[L^{\rho\sigma}(q) q^{\mu} q^{\nu} - \frac{q^2}{2} [L^{\mu\sigma}(q) L^{\rho\nu}(q) \right. \\ &\quad \left. + L^{\mu\rho}(q) L^{\sigma\nu}(q)] \right] \tilde{\mathbf{g}}_{\rho\sigma}(q) + \mathcal{O}(\kappa^2), \end{aligned} \quad (\text{A6b})$$

where

$$L^{\mu\nu}(q) = \frac{q^{\mu} q^{\nu}}{q^2} - \eta^{\mu\nu}. \quad (\text{A7})$$

We also have that

$$\sqrt{\tilde{g}} \tilde{R}^2 = \kappa^2 q^4 L^{\alpha\beta}(q) L^{\mu\nu}(q) \tilde{\mathbf{g}}_{\mu\nu} \tilde{\mathbf{g}}_{\alpha\beta} + \mathcal{O}(\kappa^3), \quad (\text{A8a})$$

$$\begin{aligned} \sqrt{\tilde{g}} \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} &= \kappa^2 q^4 \left[\frac{1}{4} L^{\alpha\beta}(q) L^{\mu\nu}(q) + \frac{1}{8} L^{\nu\alpha}(q) L^{\mu\beta}(q) \right. \\ &\quad \left. + \frac{1}{8} L^{\mu\alpha}(q) L^{\nu\beta}(q) \right] \tilde{\mathbf{g}}_{\mu\nu} \tilde{\mathbf{g}}_{\alpha\beta} + \mathcal{O}(\kappa^3). \end{aligned} \quad (\text{A8b})$$

1. Propagators for the quantum fields: $h_{\mu\nu}$ and $\bar{\eta}_{\mu}$, η_{μ}

The quadratic form in the quantum graviton field is given by

$$\int d^4x h_{\mu\nu} B^{\mu\nu\alpha\beta} h_{\alpha\beta}, \quad (\text{A9})$$

where

$$\begin{aligned} B^{\mu\nu\alpha\beta} &= - \left[\frac{1}{\xi} \left(\frac{1}{2} \partial^{\mu} \partial^{\nu} \eta^{\alpha\beta} - \partial^{\beta} \partial^{\nu} \eta^{\alpha\mu} \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \partial^{\alpha} \partial^{\beta} \eta^{\mu\nu} - \frac{1}{4} \partial^2 \eta^{\alpha\beta} \eta^{\mu\nu} \right) + \partial^{\beta} \partial^{\nu} \eta^{\alpha\mu} - \partial^{\alpha} \partial^{\beta} \eta^{\mu\nu} \right. \\ &\quad \left. - \frac{1}{2} \partial^2 \eta^{\alpha\mu} \eta^{\beta\nu} + \frac{1}{2} \partial^2 \eta^{\alpha\beta} \eta^{\mu\nu} \right] \end{aligned} \quad (\text{A10})$$

+ symmetrizations $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$

+ permutation $(\mu, \nu) \leftrightarrow (\alpha, \beta)$. (A11)

Going to momentum space ($\partial \rightarrow ip$) and computing $i[B^{\mu\nu\alpha\beta}(p)]^{-1}$ with computer algebra (throughout this work we have used Package- X [35]), we obtain

$$\begin{aligned} \mathcal{D}_{\mu\nu\alpha\beta}(p) &= - \frac{i}{p^2 + i\epsilon} \left[\eta_{\alpha\nu} \eta_{\beta\mu} + \eta_{\alpha\mu} \eta_{\beta\nu} - \eta_{\alpha\beta} \eta_{\mu\nu} \right. \\ &\quad \left. + \frac{\xi - 1}{p^2} (p_{\beta} p_{\nu} \eta_{\alpha\mu} + p_{\beta} p_{\mu} \eta_{\alpha\nu} \right. \\ &\quad \left. + p_{\alpha} p_{\nu} \eta_{\beta\mu} + p_{\alpha} p_{\mu} \eta_{\beta\nu}) \right]. \end{aligned} \quad (\text{A12})$$

Similarly, from Eq. (2.6), we obtain the following expression for the ghost propagator:

$$\mathcal{D}_{\mu\nu}(p) = -i \frac{\eta^{\mu\nu}}{p^2 + i\epsilon}. \quad (\text{A13})$$

2. Vertex $\mathbf{g}hh$

The interactions terms $\mathbf{g}hh$ arises from [36]

$$\begin{aligned} \mathcal{L}^{(2)} &= - \frac{1}{2} T^{\alpha\mu\nu\beta\gamma\delta} \sqrt{g} \mathbf{D}_{\alpha} h_{\mu\nu} \mathbf{D}_{\beta} h_{\gamma\delta} \\ &\quad - \frac{\sqrt{g}}{2} \left[R \left(\frac{1}{4} h^2 - \frac{1}{2} h_{\mu\nu} h^{\mu\nu} \right) \right. \\ &\quad \left. + R^{\mu\nu} (2 h_{\mu}^{\alpha} h_{\nu\alpha} - h h_{\mu\nu}) \right], \end{aligned} \quad (\text{A14})$$

where $h \equiv h_{\mu\nu} \eta^{\mu\nu}$ and

$$T^{\alpha\mu\nu\beta\gamma\delta} = H^{(0)\alpha\mu\nu\beta\gamma\delta} + \frac{1}{2}H^{(1)\alpha\mu\nu\beta\gamma\delta} - \frac{1}{2}H^{(1)\alpha\mu\gamma\beta\nu\delta} + H^{(1)\alpha\mu\beta\gamma\nu\delta} - H^{(1)\alpha\mu\nu\delta\gamma\beta} \\ + \frac{1}{\xi} \left(H^{(1)\alpha\mu\beta\nu\gamma\delta} - \frac{1}{2}H^{(1)\alpha\mu\gamma\nu\beta\delta} - \frac{1}{2}H^{(1)\alpha\mu\beta\gamma\nu\delta} + \frac{1}{4}H^{(1)\alpha\mu\gamma\beta\nu\delta} \right) + \mathcal{O}(\kappa^2), \quad (\text{A15})$$

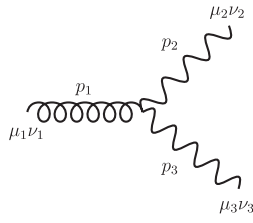
with

$$H^{(0)\alpha\mu\nu\beta\gamma\delta} \equiv \frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\gamma}\eta^{\nu\delta} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}\eta^{\gamma\delta} + \eta^{\alpha\gamma}\eta^{\mu\nu}\eta^{\beta\delta} - \eta^{\alpha\delta}\eta^{\mu\gamma}\eta^{\nu\beta} \\ + \frac{1}{\xi} \left(\eta^{\alpha\nu}\eta^{\mu\gamma}\eta^{\beta\delta} - \frac{1}{2}\eta^{\alpha\nu}\eta^{\mu\beta}\eta^{\gamma\delta} - \frac{1}{2}\eta^{\alpha\gamma}\eta^{\mu\nu}\eta^{\beta\delta} + \frac{1}{4}\eta^{\alpha\beta}\eta^{\mu\nu}\eta^{\gamma\delta} \right), \quad (\text{A16})$$

and

$$H^{(1)\alpha\mu\nu\beta\gamma\delta} = \kappa(-\delta_{\mu_1}^\alpha\delta_{\nu_1}^\beta\eta^{\mu_2\gamma}\eta^{\nu_2\delta} - \eta^{\alpha\beta}\delta_{\mu_1}^\mu\delta_{\nu_1}^\nu\eta^{\mu_2\gamma}\eta^{\nu_2\delta} - \eta^{\alpha\beta}\eta^{\mu_2\gamma}\delta_{\mu_1}^\nu\delta_{\nu_1}^\delta)\mathfrak{g}^{\mu_1\nu_1} \equiv \kappa G^{\mu_1\nu_1\alpha\mu\nu\beta\gamma\delta}\mathfrak{g}_{\mu_1\nu_1}. \quad (\text{A17})$$

The total result for the $\mathfrak{g}_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}$ interactions vertex is given by



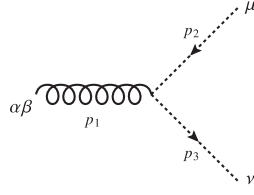
$$(-1)i\frac{\kappa}{2} \left\{ \delta_{\mu}^{\mu_2}\delta_{\nu}^{\nu_2}\delta_{\gamma}^{\mu_3}\delta_{\delta}^{\nu_3}p_{2\alpha}p_{3\beta} \left[\frac{1}{2}\eta^{\mu_1\nu_1}H^{(0)\alpha\mu\nu\beta\gamma\delta} + \frac{1}{2}G^{\mu_1\nu_1\alpha\mu\nu\beta\gamma\delta} \right. \right. \\ \left. \left. - \frac{1}{2}G^{\mu_1\nu_1\alpha\mu\gamma\beta\nu\delta} + G^{\mu_1\nu_1\alpha\mu\beta\gamma\nu\delta} - G^{\mu_1\nu_1\alpha\mu\nu\delta\gamma\beta} \right. \right. \\ \left. \left. + \frac{1}{\xi} \left(G^{\mu_1\nu_1\alpha\mu\beta\nu\gamma\delta} - \frac{1}{2}G^{\mu_1\nu_1\alpha\mu\gamma\nu\beta\delta} - \frac{1}{2}G^{\mu_1\nu_1\alpha\mu\beta\gamma\nu\delta} + \frac{1}{4}G^{\mu_1\nu_1\alpha\mu\gamma\beta\nu\delta} \right) \right] \right. \\ \left. + H^{(0)\alpha\mu\nu\beta\gamma\delta} \left[\left(\frac{1}{2}(\delta_{\mu}^{\mu_3}\delta_{\nu}^{\nu_3}\bar{p}_{1\gamma}p_{3\alpha}\delta_{\beta}^{\nu_1}\delta_{\delta}^{\mu_2}\eta^{\mu_1\nu_2} + \delta_{\mu}^{\mu_3}\eta^{\nu_1\nu_2}\delta_{\nu}^{\nu_3}\bar{p}_{1\beta}p_{3\alpha}\delta_{\gamma}^{\mu_1}\delta_{\delta}^{\mu_2} \right. \right. \right. \\ \left. \left. - \delta_{\mu}^{\mu_3}\delta_{\nu}^{\nu_3}\bar{p}_1^{\nu_2}p_{3\alpha}\delta_{\beta}^{\mu_1}\delta_{\gamma}^{\mu_2}\delta_{\delta}^{\nu_1}) + \gamma \leftrightarrow \delta \right) + (\alpha, \mu, \nu) \leftrightarrow (\beta, \gamma, \delta) \right] \right. \\ \left. + \bar{p}_1^2 L^{\mu_1\nu_1}(\bar{p}_1) \left(\frac{1}{4}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3} - \frac{1}{2}\eta^{\mu_2\mu_3}\eta^{\nu_2\nu_3} \right) \right. \\ \left. + \frac{1}{2} \left[L^{\mu_1\nu_1}(\bar{p}_1)\bar{p}_1^{\mu}\bar{p}_1^{\nu} - \frac{\bar{p}_1^2}{2} (L^{\mu\mu_1}(\bar{p}_1)L^{\nu\nu_1}(\bar{p}_1) + L^{\nu\mu_1}(\bar{p}_1)L^{\mu\nu_1}(\bar{p}_1)) \right] \right. \\ \left. (2\eta^{\nu_3\mu_2}\delta_{\mu}^{\nu_2}\delta_{\nu}^{\mu_3} - \eta^{\mu_2\nu_2}\delta_{\mu}^{\mu_3}\delta_{\nu}^{\nu_3}) \right\} + \text{symmetrizations } \mu_i \leftrightarrow \nu_i \\ + \text{permutation } (\mu_2, \nu_2, p_2) \leftrightarrow (\mu_3, \nu_3, p_3).$$

3. The vertex $\bar{\eta}g\eta$

The interaction terms coming from the ghost Lagrangian given in Eq. (2.6) read

$$\kappa \left\{ \frac{1}{2}\mathfrak{g}_{\lambda}^{\lambda}\bar{\eta}_{\mu}\partial^2\eta^{\mu} - \bar{\eta}_{\mu}(\mathfrak{g}^{\beta\lambda}\eta^{\mu\nu} + \mathfrak{g}^{\mu\nu}\eta^{\beta\lambda})\partial_{\beta}\partial_{\lambda}\eta_{\nu} + \frac{1}{2}\eta^{\mu\nu}\eta^{\beta\lambda}[(\partial_{\beta}\bar{\eta}_{\mu})\eta^{\rho\sigma}(\partial_{\lambda}\mathfrak{g}_{\sigma\nu} + \partial_{\nu}\mathfrak{g}_{\sigma\lambda} - \partial_{\sigma}\mathfrak{g}_{\nu\lambda})\eta_{\rho} \right. \\ \left. - \bar{\eta}_{\mu}\eta^{\rho\sigma}(\partial_{\beta}\mathfrak{g}_{\sigma\lambda} + \partial_{\lambda}\mathfrak{g}_{\sigma\beta} - \partial_{\sigma}\mathfrak{g}_{\lambda\beta})\partial_{\rho}\eta_{\nu} - \bar{\eta}_{\mu}\eta^{\rho\sigma}(\partial_{\beta}\mathfrak{g}_{\sigma\nu} + \partial_{\nu}\mathfrak{g}_{\sigma\beta} - \partial_{\sigma}\mathfrak{g}_{\nu\beta})\partial_{\lambda}\eta_{\rho}] \right\} - \bar{\eta}_{\mu}R^{\mu\nu}\eta_{\nu}. \quad (\text{A18})$$

From (A18), we obtain the following momentum space interaction vertex $h_{\alpha\beta}\bar{\eta}_{\mu}\eta_{\nu}$:



$$\begin{aligned}
 & i\kappa \left\{ -\frac{1}{2}p_3^2\eta^{\alpha\beta}\eta^{\mu\nu} + p_3^\alpha p_3^\beta\eta^{\mu\nu} + p_3^2\eta^{\mu\alpha}\eta^{\nu\beta} \right. \\
 & + \frac{1}{2} \left(\bar{p}_1^\nu p_2^\beta\eta^{\mu\alpha} - \bar{p}_1^\mu p_2^\beta\eta^{\nu\alpha} - \bar{p}_1 \cdot p_3\eta^{\mu\beta}\eta^{\nu\alpha} \right) \\
 & + \frac{1}{2} \left(2p_3^\alpha \bar{p}_1^\beta\eta^{\mu\nu} - \bar{p}_1 \cdot p_3\eta^{\mu\nu}\eta^{\alpha\beta} \right) \\
 & - \frac{1}{2} \left(\bar{p}_1^\nu p_2^\beta\eta^{\mu\alpha} - \bar{p}_1^\mu p_2^\alpha\eta^{\nu\beta} - \bar{p}_1 \cdot p_3\eta^{\mu\beta}\eta^{\nu\alpha} \right) \\
 & \left. - \frac{1}{2} [\bar{p}_1^\alpha (\bar{p}_1^\mu\eta^{\nu\beta} + \bar{p}_1^\nu\eta^{\mu\beta}) - \bar{p}_1^\mu \bar{p}_1^\nu\eta^{\alpha\beta} - \bar{p}_1^2\eta^{\mu\alpha}\eta^{\nu\beta}] \right\} \\
 & + \text{symmetrizations } \alpha \leftrightarrow \beta,
 \end{aligned}$$

where we have used Eq. (A6b).

4. Gravity coupled to a scalar field

Now, we consider the coupling between the graviton and a scalar field. From the classical Lagrangian

$$-\sqrt{g}\frac{\bar{g}^{\mu\nu}}{2}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}, \quad (\text{A19})$$

we obtain the scalar propagator in the momentum space,

$$-i\frac{1}{k^2}. \quad (\text{A20})$$

The interactions terms arise from the Lagrangian (2.12),

$$\begin{aligned}
 & -\sqrt{g}\frac{\bar{g}^{\mu\nu}}{2}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + \kappa\frac{\zeta}{\xi}\sqrt{g}\left(h_{\nu}^{\mu\nu} - \frac{1}{2}h_{\alpha}^{\alpha\mu}\right)\varphi\partial_\mu\phi \\
 & -\kappa^2\frac{\zeta^2}{2\xi}\sqrt{g}(\varphi\partial^\mu\phi)^2 - \zeta\kappa^2\sqrt{g}\eta^\mu\partial_\mu\phi\partial_\nu\phi\eta^\nu. \quad (\text{A21})
 \end{aligned}$$

Consider the interaction terms $g h \phi \varphi$. Arising from the first term of Eq. (A21), we have

$$\begin{aligned}
 & -\frac{\kappa^2}{4}[4\eta^{\mu\alpha}g^{\nu\beta}h_{\alpha\beta} + 4g^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta} + g_{\pi}^{\mu\alpha}\eta^{\alpha\beta}\eta^{\mu\nu}h_{\alpha\beta} \\
 & - 2(g^{\alpha\beta}h_{\alpha\beta}\eta^{\mu\nu} + \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}g_{\pi}^{\mu\nu} + g^{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta})]\partial_\mu\phi\partial_\nu\varphi. \quad (\text{A22})
 \end{aligned}$$

The remaining terms comes from the second term of Eq. (A21), which can be written as

$$\begin{aligned}
 & \kappa\frac{\zeta}{\xi}\sqrt{g}T^{\mu\nu\alpha\beta}h_{\alpha\beta;\nu}\varphi\partial_\mu\phi \\
 & \equiv \kappa\frac{\zeta}{\xi}\sqrt{g}\left(g^{\alpha\mu}g^{\beta\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)h_{\alpha\beta;\nu}\varphi\partial_\mu\phi. \quad (\text{A23})
 \end{aligned}$$

Using Eq. (A4), we obtain from the determinant the contribution

$$+\frac{\kappa^2\zeta}{2\xi}g^{\beta\gamma}\left(\partial_\nu h^{\mu\nu} - \frac{1}{2}\partial^\mu h_{\alpha\beta}\eta^{\alpha\beta}\right)\varphi\partial_\mu\phi, \quad (\text{A24})$$

and coming from the tensor $T^{\mu\nu\alpha\beta}$, we obtain

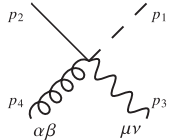
$$-\kappa^2\frac{\zeta}{\xi}\left[\eta^{\mu\alpha}g^{\lambda\beta} + g^{\mu\alpha}\eta^{\lambda\beta} - \frac{1}{2}(\eta^{\mu\lambda}g^{\alpha\beta} + g^{\mu\lambda}\eta^{\alpha\beta})\right]\partial_\lambda h_{\alpha\beta}\varphi\partial_\mu\phi, \quad (\text{A25})$$

and finally, from the covariant derivative,

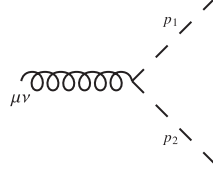
$$\begin{aligned}
 & -\frac{\kappa^2\zeta}{2\xi}\left[\eta^{\mu\alpha}\eta^{\lambda\beta} - \frac{1}{2}\eta^{\mu\lambda}\eta^{\alpha\beta}\right]\eta^{\sigma\rho}[(\partial_\lambda g_{\rho\alpha} + \partial_\alpha g_{\rho\lambda} \\
 & - \partial_\rho g_{\alpha\lambda})h_{\sigma\beta} + (\partial_\lambda g_{\rho\beta} + \partial_\beta g_{\rho\lambda} - \partial_\rho g_{\beta\lambda})h_{\alpha\sigma}]\varphi\partial_\mu\phi. \quad (\text{A26})
 \end{aligned}$$

Note that we have introduced the notation $T^{\dot{\mu}_1\cdots\dot{\mu}_n} \equiv \eta^{\mu_1\nu_2}\cdots\eta^{\mu_n\nu_n}T_{\nu_1\nu_2\cdots\nu_n}$. Adding all contributions, the vertex $g_{\alpha\beta}(p_4)h_{\mu\nu}(p_3)\varphi(p_1)\phi(p_2)$ follows in momenta space as

$$\begin{aligned}
 & -i\frac{\kappa^2}{4\xi}\left[-\xi p_1^\beta p_2^\nu\eta^{\alpha\mu} - \xi p_1^\beta p_2^\mu\eta^{\alpha\nu} - \xi p_1^\alpha p_2^\nu\eta^{\beta\mu} + \xi p_1^\nu(p_2^\mu\eta^{\alpha\beta} - p_2^\beta\eta^{\alpha\mu} - p_2^\alpha\eta^{\beta\mu}) \right. \\
 & - \xi p_1^\alpha p_2^\mu\eta^{\beta\nu} + \xi p_1^\mu(p_2^\nu\eta^{\alpha\beta} - p_2^\beta\eta^{\alpha\nu} - p_2^\alpha\eta^{\beta\nu}) + \xi p_2^\alpha p_1^\beta\eta^{\mu\nu} + \xi p_1^\alpha p_2^\beta\eta^{\mu\nu} \\
 & - \xi p_1 \cdot p_2\eta^{\alpha\beta}\eta^{\mu\nu} + \xi p_1 \cdot p_2\eta^{\alpha\nu}\eta^{\beta\mu} + \xi p_1 \cdot p_2\eta^{\alpha\mu}\eta^{\beta\nu} + \zeta p_3^\mu p_2^\nu\eta^{\alpha\beta} + \zeta p_2^\mu p_3^\nu\eta^{\alpha\beta} \\
 & - \zeta p_3^\beta p_2^\nu\eta^{\alpha\mu} - \zeta p_2^\beta p_3^\nu\eta^{\alpha\mu} - \zeta p_3^\beta p_2^\mu\eta^{\alpha\nu} - \zeta p_2^\beta p_3^\mu\eta^{\alpha\nu} - \zeta p_3^\alpha p_2^\nu\eta^{\beta\mu} \\
 & - \zeta p_2^\alpha p_3^\nu\eta^{\beta\mu} - \zeta p_3^\alpha p_2^\mu\eta^{\beta\nu} - \zeta p_2^\alpha p_3^\mu\eta^{\beta\nu} + \zeta p_3^\alpha p_2^\beta\eta^{\mu\nu} + \zeta p_2^\alpha p_3^\beta\eta^{\mu\nu} \\
 & + \zeta p_4^\mu p_2^\nu\eta^{\alpha\beta} + \zeta p_2^\mu p_4^\nu\eta^{\alpha\beta} - \zeta p_4^\beta p_2^\nu\eta^{\alpha\mu} - \zeta p_4^\beta p_2^\mu\eta^{\alpha\nu} - \zeta p_4^\alpha p_2^\nu\eta^{\beta\mu} \\
 & \left. - \zeta p_4^\alpha p_2^\mu\eta^{\beta\nu} - \zeta p_2 \cdot p_3\eta^{\alpha\beta}\eta^{\mu\nu} + \zeta p_2 \cdot p_3\eta^{\alpha\nu}\eta^{\beta\mu} + \zeta p_2 \cdot p_3\eta^{\alpha\mu}\eta^{\beta\nu}\right]. \quad (\text{A27})
 \end{aligned}$$



From the first term of Eq. (A21), we obtain the interaction terms $\mathbf{g}\varphi\varphi$, which leads to the vertex $\mathbf{g}_{\mu\nu}\varphi(p_1)\varphi(p_2)$,

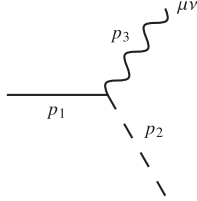


$$i\frac{\kappa}{2}(\eta_{\mu\nu}p_1 \cdot p_2 - p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu}). \quad (\text{A28})$$

This following interaction terms from Eq. (A21),

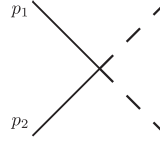
$$\kappa\frac{\zeta}{\xi}\left(h_{\nu}^{\nu\mu} - \frac{1}{2}h_{\nu}^{\nu\mu}\right)\varphi\partial_{\mu}\phi - \frac{\kappa}{2}\left(\frac{1}{2}\eta^{\mu\nu}h_{\alpha}^{\alpha} - h^{\mu\nu}\right)(\partial_{\mu}\varphi\partial_{\nu}\phi + \partial_{\mu}\phi\partial_{\nu}\varphi), \quad (\text{A29})$$

leads to the vertex $\phi\varphi h$, which, in momentum space, reads

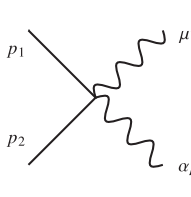


$$\frac{1}{2}i\kappa\left[\eta^{\mu\nu}p_1 \cdot p_2 - p_2^{\mu}p_1^{\nu} - p_1^{\mu}p_2^{\nu} + \frac{\zeta}{\xi}(\eta^{\mu\nu}p_1 \cdot p_3 - p_3^{\mu}p_1^{\nu} - p_1^{\mu}p_3^{\nu})\right]. \quad (\text{A30})$$

Similarly, we obtain other relevant vertices for the computation of the counterterm Lagrangian at one-loop order,

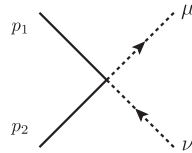


$$\frac{2i\zeta^2}{\xi}\kappa^2 p_{1\mu}p_2^{\mu}, \quad (\text{A31})$$



$$\begin{aligned} & -i\frac{\kappa^2}{4}\left[p_2^{\mu}p_1^{\nu}\eta^{\alpha\beta} + p_1^{\mu}p_2^{\nu}\eta^{\alpha\beta} - p_2^{\beta}p_1^{\nu}\eta^{\alpha\mu} - p_1^{\beta}p_2^{\nu}\eta^{\alpha\mu} \right. \\ & \quad - p_2^{\beta}p_1^{\mu}\eta^{\alpha\nu} - p_1^{\beta}p_2^{\mu}\eta^{\alpha\nu} - p_2^{\alpha}p_1^{\nu}\eta^{\beta\mu} - p_1^{\alpha}p_2^{\nu}\eta^{\beta\mu} \\ & \quad - p_2^{\alpha}p_1^{\mu}\eta^{\beta\nu} - p_1^{\alpha}p_2^{\mu}\eta^{\beta\nu} + p_2^{\alpha}p_1^{\beta}\eta^{\mu\nu} + p_1^{\alpha}p_2^{\beta}\eta^{\mu\nu} \\ & \quad \left. + \eta^{\alpha\nu}\eta^{\beta\mu}p_1 \cdot p_2 + \eta^{\alpha\mu}\eta^{\beta\nu}p_1 \cdot p_2 - \eta^{\alpha\beta}\eta^{\mu\nu}p_1 \cdot p_2\right], \end{aligned} \quad (\text{A32})$$

and the ghost vertex,



$$i\zeta\kappa^2(p_1^{\mu}p_2^{\nu} + p_1^{\nu}p_2^{\mu}). \quad (\text{A33})$$

APPENDIX B: ONE-LOOP COUNTERTERMS

Here, we compute the diagrams in Figs. 1–3 in a general background gauge using the Feynman rules derived in Appendix A. We use dimensional regularization with $D = 4 - 2\epsilon$, where D is the spacetime dimension. With these results, we derive the corresponding one-loop counterterm Lagrangians.

1. 2-point diagrams

Here, we present the divergent part of the diagrams shown in Fig. 1. We introduce the following tensor basis:

$$\mathcal{T}_{(1)}^{\mu\nu\alpha\beta} = k^\mu k^\alpha k^\nu k^\beta, \quad (\text{B1a})$$

$$\mathcal{T}_{(2)}^{\mu\nu\alpha\beta} = \eta^{\mu\nu} \eta^{\alpha\beta}, \quad (\text{B1b})$$

$$\mathcal{T}_{(3)}^{\mu\nu\alpha\beta} = \eta^{\mu\beta} \eta^{\alpha\nu} + \eta^{\mu\alpha} \eta^{\nu\beta}, \quad (\text{B1c})$$

$$\mathcal{T}_{(4)}^{\mu\nu\alpha\beta} = k^\alpha k^\beta \eta^{\mu\nu} + k^\mu k^\nu \eta^{\alpha\beta}, \quad (\text{B1d})$$

$$\mathcal{T}_{(5)}^{\mu\nu\alpha\beta} = k^\nu k^\beta \eta^{\mu\alpha} + k^\alpha k^\nu \eta^{\mu\beta} + k^\mu k^\beta \eta^{\alpha\nu} + k^\mu k^\alpha \eta^{\nu\beta}. \quad (\text{B1e})$$

a. Graviton loop

First, we compute the graviton loop, which corresponds to the graphs depicted in Figs. 1(a) and 1(b). As result, we obtained the same result found in [11,23,36] for a general background gauge ξ . These diagrams lead to the following counterterms:

$$\begin{aligned} \mathcal{L}_{2a+2b} = & \frac{\sqrt{g}}{16\pi^2\epsilon} \left[\left(\frac{1}{120} + \frac{(\xi-1)^2}{6} \right) R^2 \right. \\ & \left. + \left(\frac{7}{20} + \frac{\xi(\xi-1)}{3} \right) R_{\mu\nu}^2 \right]. \end{aligned} \quad (\text{B2})$$

b. Scalar loop

Next, we consider the scalar loop diagram shown in Fig. 1(c). In the basis B 1, we obtain that the scalar loop diagram is equal to

$$\begin{aligned} & \frac{\kappa^2}{16\pi^2\epsilon} \left(\frac{1}{60} \mathcal{T}_1^{\mu\nu\alpha\beta} + \frac{1}{80} \mathcal{T}_2^{\mu\nu\alpha\beta} k^4 + \frac{1}{480} \mathcal{T}_3^{\mu\nu\alpha\beta} k^4 \right. \\ & \left. - \frac{1}{80} \mathcal{T}_4^{\mu\nu\alpha\beta} k^2 - \frac{1}{480} \mathcal{T}_5 k^2 \right). \end{aligned} \quad (\text{B3})$$

This result leads to the following counterterms:

$$\mathcal{L}_{2c} = \frac{\sqrt{g}}{16\pi^2\epsilon} \left(\frac{1}{120} R_{\mu\nu}^2 + \frac{1}{240} R^2 \right). \quad (\text{B4})$$

c. Graviton self-energy

One can find the graviton self-energy in a general gauge with a scalar field by summing the diagrams in Figs. 1(a)–1(c). The divergent part of the graviton self-energy leads to the counterterm Lagrangian,

$$\begin{aligned} \mathcal{L}_{2a+2b+2c} = & \frac{\sqrt{g}}{16\pi^2\epsilon} \left[\left(\frac{1}{80} + \frac{1}{6}(\xi-1)^2 \right) R^2 \right. \\ & \left. + \left(\frac{43}{120} + \frac{\xi(\xi-1)}{3} \right) R_{\mu\nu}^2 \right], \end{aligned} \quad (\text{B5})$$

which is the result found in [3].

d. Scalar self-energy

The scalar self-energy can be computed in a general gauge ξ . The diagram is shown in Fig. 1(d). The divergent part of this diagram reads

$$\kappa^2 \frac{\zeta}{16\pi^2\epsilon} k^4. \quad (\text{B6})$$

Using the prescription,

$$k^4 \rightarrow \frac{1}{2!} (D_\mu D^\mu \phi)^2, \quad (\text{B7})$$

one finds the counterterms,

$$\mathcal{L}_{2d} = \frac{\kappa^2 \zeta}{2} \frac{\sqrt{g}}{16\pi^2\epsilon} (D_\mu D^\mu \phi)^2, \quad (\text{B8})$$

which correspond to one quarter of the 't Hooft and Veltman result (when $\xi = \zeta = 1$). This result also agrees with Ref. [10] for $\xi = 1$.

2. 3-point function

The one-loop diagrams that contribute to the 3-point function $\mathbf{g}_{\mu\nu}(k)\phi(k_1)\phi(k_2)$ are shown in Fig. 2. In order to simplify our results, we introduce the tensor basis,

$$\mathcal{T}_R^{\mu\nu}(k, k_1, k_2) = -2k_1 \cdot k_2 (k^\mu k^\nu - k^2 \eta^{\mu\nu}), \quad (\text{B9a})$$

$$\begin{aligned} \mathcal{T}_{R_{\alpha\beta}}^{\mu\nu}(k, k_1, k_2) = & (k \cdot k_1)(k \cdot k_2) \eta^{\mu\nu} \\ & + \frac{k^2}{2} (k_2^\mu k_1^\nu + k_1^\mu k_2^\nu) - \frac{1}{2} [k_2^\mu k^\nu (k \cdot k_1) \\ & + k^\mu k_2^\nu (k \cdot k_1) + k_1^\mu k^\nu (k \cdot k_2) \\ & + k^\mu k_1^\nu (k \cdot k_2)]. \end{aligned} \quad (\text{B9b})$$

Note that the tensor structures above corresponds in momentum space to the invariants: $R \partial_\mu \phi(k_1) \partial^\mu \phi(k_2)$ and $R_{\mu\nu}(k) \partial^\mu \phi(k_1) \partial^\nu \phi(k_2)$, respectively.

Using this basis, one can show that the divergent parts of the diagrams in Figs. 2(a)–2(c) are, respectively, given by

$$-\frac{1}{3} \kappa^3 (\xi-1)(\xi+4) \left[\frac{1}{4} \mathcal{T}_R^{\mu\nu}(k, k_1, k_2) - \mathcal{T}_{R_{\alpha\beta}}^{\mu\nu}(k, k_1, k_2) \right], \quad (\text{B10})$$

$$+ \frac{1}{12\xi} \kappa^3 \zeta^2 \mathcal{T}_R^{\mu\nu}(k, k_1, k_2), \quad (\text{B11})$$

$$-\kappa^3 \zeta \left[\frac{1}{6} \mathcal{T}_R^{\mu\nu}(k, k_1, k_2) + \mathcal{T}_{R_{\alpha\beta}}^{\mu\nu}(k, k_1, k_2) \right]. \quad (\text{B12})$$

These results can be rewritten in position space as

$$-\frac{1}{3} \kappa^2 (\xi - 1)(\xi + 4) \left[\frac{1}{4} R \partial^\mu \phi \partial_\mu \phi - R_{\mu\nu} \partial^\mu \phi \partial^\nu \phi \right], \quad (\text{B13})$$

$$+ \frac{1}{12\xi} \kappa^2 \zeta^2 R \partial_\mu \phi \partial^\mu \phi, \quad (\text{B14})$$

and

$$-\frac{1}{6} \kappa^2 \zeta R \partial^\mu \phi \partial_\mu \phi - \kappa^2 \zeta R_{\mu\nu} \partial^\mu \phi \partial^\nu \phi. \quad (\text{B15})$$

Now, we consider the divergent parts of the diagrams in Figs. 2(d)–2(f). Summing these diagrams and subtracting the induced 3-point contribution from the counterterm (B8), we obtain a transverse result that can be written in terms of the basis (B9),

$$\kappa^3 \left[\frac{(\xi - \zeta)^2}{\xi} \frac{3\xi - 1}{12} \mathcal{T}_R^{\mu\nu}(k_1 + k_2, k_1, k_2) - \frac{\xi - 1 - 2\zeta}{2} \mathcal{T}_{R_{\alpha\beta}}^{\mu\nu}(k_1 + k_2, k_1, k_2) \right]. \quad (\text{B16})$$

The total result, using Eqs. (B13) and (B16), leads to the counterterms,

$$\mathcal{L}_{3P} = \kappa^2 \frac{\sqrt{g}}{16\pi^2 \epsilon} \left[\frac{2(\xi - 1)^2 - 6\zeta(\xi - 1) + 2 + 3\zeta(\zeta - 2)}{12} \right. \\ \left. \times R \partial_\mu \phi \partial^\mu \phi + \frac{\xi - 1}{6} (2\xi + 5) R_{\mu\nu} \partial^\mu \phi \partial^\nu \phi \right]. \quad (\text{B17})$$

When $\xi = \zeta = 1$, it reduces to the result found in [3]. When $\xi = 1$, we find that

$$\mathcal{L}_{3P}|_{\xi=1} = \kappa^2 \frac{\sqrt{g}}{16\pi^2 \epsilon} \left[-\frac{1}{12} + \frac{(\zeta - 1)^2}{4} \right] R \partial_\mu \phi \partial^\mu \phi, \quad (\text{B18})$$

which is the same result found in Ref. [10].

3. 4-point scalar function

Next, let us consider the divergent part of the 4-point scalar function. We remember the reader that permutations are implied.

The diagram shown in Fig. 3(a) (summing all the permutations) leads to

$$2(\xi^2 + \xi + 1) \kappa^4 [(k_1 \cdot k_4)(k_2 \cdot k_3) + (k_1 \cdot k_3)(k_2 \cdot k_4) + (k_1 \cdot k_2)(k_3 \cdot k_4)]. \quad (\text{B19})$$

Computing the diagram in Fig. 3(b), we find

$$\frac{2\zeta^4}{\xi^2} \kappa^4 [(k_1 \cdot k_4)(k_2 \cdot k_3) + (k_1 \cdot k_3)(k_2 \cdot k_4) + (k_1 \cdot k_2)(k_3 \cdot k_4)]. \quad (\text{B20})$$

The ghost diagram in Fig. 3(c) yields

$$-4\zeta^2 \kappa^4 [(k_1 \cdot k_4)(k_2 \cdot k_3) + (k_1 \cdot k_3)(k_2 \cdot k_4) + (k_1 \cdot k_2)(k_3 \cdot k_4)]. \quad (\text{B21})$$

Now, we compute the triangle and box diagram shown in Figs. 3(d)–3(f). We find, respectively, that

$$-3(\xi - \zeta)^2 \kappa^4 [(k_1 \cdot k_4)(k_2 \cdot k_3) + (k_1 \cdot k_3)(k_2 \cdot k_4) + (k_1 \cdot k_2)(k_3 \cdot k_4)], \quad (\text{B22})$$

$$-4 \frac{\xi^2}{\xi^2} (\xi - \zeta)^2 \kappa^4 [(k_1 \cdot k_4)(k_2 \cdot k_3) + (k_1 \cdot k_3)(k_2 \cdot k_4) + (k_1 \cdot k_2)(k_3 \cdot k_4)], \quad (\text{B23})$$

and

$$\frac{2}{\xi^2} (\xi - \zeta)^4 \kappa^4 [(k_1 \cdot k_4)(k_2 \cdot k_3) + (k_1 \cdot k_3)(k_2 \cdot k_4) + (k_1 \cdot k_2)(k_3 \cdot k_4)]. \quad (\text{B24})$$

The counterterm Lagrangian in a general gauge induced by the 4-point function is given by

$$\mathcal{L}_{4P} = \frac{1}{8} \kappa^4 [4 + 2(\xi - 1) + (\xi - \zeta)^2] \frac{\sqrt{g}}{16\pi^2 \epsilon} (\partial_\mu \phi g^{\mu\nu} \partial_\nu \phi)^2. \quad (\text{B25})$$

When $\xi = 1$, one finds the result obtained in Ref. [10], and now setting $\zeta = 1$, it reduces to the result in t' Hooft and Veltman [3].

APPENDIX C: COUNTERTERMS IN GENERAL SCALAR-GRAVITY MODELS

In Refs. [29,30], a more general scalar-gravity model is considered. It is interesting both for quantum and inflationary cosmology. The action of this improved gravitational theory reads

$$S = \int_{\mathcal{M}} d^4x \sqrt{\bar{g}} \left(U(\phi) R - \frac{1}{2} G(\phi) \bar{g}^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} - V(\phi) \right). \quad (\text{C1})$$

In principle, the field-dependent couplings $U(\phi)$, $G(\phi)$ and $V(\phi)$ are general.

A general calculation showing the quantum inequivalence of the Jordan and Einstein frames is provided in [30]. Meanwhile, in [29], the one-loop counterterms in the De Donder gauge ($\xi = 1$ and $\zeta = 0$) are derived with $U = 1/\kappa^2$ (or equivalently $U = -1/\kappa^2$ and $R \rightarrow -R$) and $G = 1$. The counterterms Lagrangian, in our notation, reads

$$\begin{aligned} \frac{\sqrt{g}}{32\pi^2\epsilon} & \left[\frac{43}{60} R_{\alpha\beta}^2 + \frac{1}{40} R^2 + \frac{5}{4} \kappa^4 (g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta})^2 \right. \\ & + \kappa^2 g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \left(-\frac{1}{3} R + \kappa^2 V - 2 \frac{\partial^2 V}{\partial \phi^2} \right) \\ & + R \left(-\frac{13}{3} \kappa^2 V - \frac{1}{6} \frac{\partial^2 V}{\partial \phi^2} \right) + \frac{5}{2} \kappa^4 V^2 \\ & \left. - 2\kappa^2 \left(\frac{\partial V}{\partial \phi} \right)^2 + \frac{1}{2} \left(\frac{\partial^2 V}{\partial \phi^2} \right)^2 \right]. \end{aligned} \quad (\text{C2})$$

For our simple scalar-gravity model (2.1), we have to set $R \rightarrow -R$, $V = 0$ in Eq. (C2) which leads to Eq. (3.10).

APPENDIX D: ON-SHELL COUNTERTERM LAGRANGIAN

The counterterm Lagrangian of quantum gravity and scalar matter fields in a general gauge is obtained by adding all the contributions coming from the 2-, 3-, and 4-point functions obtained in Eqs. (B2), (B4), (B8), (B17), and (B25). This Lagrangian reads [the explicit form is shown Eq. (3.10)]

$$\mathcal{L}_{\text{CT}} = \mathcal{L}_{2\text{P}} + \mathcal{L}_{3\text{P}} + \mathcal{L}_{4\text{P}}, \quad (\text{D1})$$

where $\mathcal{L}_{2\text{P}} = \mathcal{L}_{2a+2b} + \mathcal{L}_{2c} + \mathcal{L}_{2d}$.

Using the on-shell condition (3.13), we have that

$$\begin{aligned} R_{\mu\nu} \partial^\mu \phi \partial^\nu \phi &= R (\partial^\mu \phi)^2, \quad R^2 = R_{\mu\nu}^2 = -\frac{\kappa^2}{2} R (\partial_\mu \phi)^2 \quad \text{and} \\ \kappa^2 \partial_\mu \phi \partial^\mu \phi &= -2R (\partial_\mu \phi)^2. \end{aligned} \quad (\text{D2})$$

To obtain the on-shell counterterm Lagrangian, we consider each term of Eq. (D1),

$$\mathcal{L}_{2\text{P}}|_{\text{on-shell}} = -\frac{\kappa^2 \sqrt{g}}{16\pi^2 \epsilon} \left[\frac{43}{160} + \frac{\xi^2}{4} - \frac{\xi}{3} \right] R \partial_\mu \phi \partial^\mu \phi, \quad (\text{D3a})$$

$$\mathcal{L}_{3\text{P}}|_{\text{on-shell}} = \frac{\kappa^2 \sqrt{g}}{16\pi^2 \epsilon} \left[-\frac{1}{2} + \frac{\xi^2}{2} + \frac{\xi}{6} + \frac{\zeta^2 - 2\xi\zeta}{4} \right] R \partial_\mu \phi \partial^\mu \phi, \quad (\text{D3b})$$

$$\mathcal{L}_{4\text{P}}|_{\text{on-shell}} = -\frac{\kappa^2 \sqrt{g}}{16\pi^2 \epsilon} \left[\frac{1}{2} + \frac{\xi^2}{4} + \frac{\xi}{2} + \frac{\zeta^2 - 2\xi\zeta}{4} \right] R \partial_\mu \phi \partial^\mu \phi. \quad (\text{D3c})$$

Remember that $D_\mu D^\mu \phi = 0$, and this implies that $\mathcal{L}_{2d}$ vanishes on shell.

Clearly, summing all contribution in the left-hand side of Eq. (D1), the dependence on ξ and ζ vanish. Then, we arrive to the well-known result,

$$\begin{aligned} \mathcal{L}_{\text{CT}}|_{\text{on-shell}} &= -\frac{\sqrt{g} \kappa^2}{16\pi^2 \epsilon} \frac{203}{160} R \partial_\mu \phi \partial^\mu \phi \\ &= \frac{\sqrt{g} \kappa^4}{16\pi^2 \epsilon} \frac{203}{320} [\partial_\mu \phi \partial^\mu \phi]^2, \end{aligned} \quad (\text{D4})$$

where we have used (D2).

APPENDIX E: LANDAU-DEWITT BACKGROUND GAUGE

We consider here, for simplicity, the pure quantum gravity, where this gauge is characterized by the condition

$$h_{;\nu}^{\mu\nu} - \frac{1}{2} h_{\alpha}^{\alpha;\mu} = 0, \quad (\text{E1})$$

where $h_{\alpha\beta} = \bar{g}_{\alpha\beta} - g_{\alpha\beta} = \bar{g}_{\alpha\beta} - \eta_{\alpha\beta} - \kappa g_{\alpha\beta}$. In order to quantize this theory in the Landau-DeWitt gauge, we introduce external sources (antifields) $G^{*\alpha\beta}$ and $C^{*\mu}$ coupled, respectively, to the variations of the graviton field $g_{\alpha\beta}$ and the ghost field η_μ , under the BRST transformations (2.9).

The full Lagrangian is obtained by the usual BRST procedure,

$$\begin{aligned} \mathcal{L}^{\text{full}} &= \frac{\sqrt{\bar{g}}}{\kappa^2} \bar{R} - \mathbf{s} \left[-\sqrt{\bar{g}} \bar{\eta}_\mu \left(h_{;\nu}^{\mu\nu} - \frac{1}{2} h_{\alpha}^{\alpha;\mu} \right) \right. \\ & \left. + \frac{1}{\kappa} G^{*\alpha\beta} \bar{g}_{\alpha\beta} - C^{*\mu} \eta_\mu \right], \end{aligned} \quad (\text{E2})$$

where we have used Eq. (2.10) in the limit $\xi \rightarrow 0$, and omitted the matter fields.

When applying the BRST operator $\mathbf{s}$ to the background graviton field $\bar{g}_{\alpha\beta}$ in Eq. (E2), it is useful to introduce a source $W_{\alpha\beta}$ defined as

$$\mathbf{s} \bar{g}_{\alpha\beta} = W_{\alpha\beta}. \quad (\text{E3})$$

We note that the mass dimensions of $G^{*\alpha\beta}$, $C^{*\mu}$, and $W_{\alpha\beta}$ are 2, while their ghost numbers are, respectively, -1 , -2 , 1 . These sources are invariant under BRST transformations, so that

$$\mathbf{s} G^{*\alpha\beta} = \mathbf{s} C^{*\mu} = \mathbf{s} W_{\alpha\beta} = 0. \quad (\text{E4})$$

Using the BRST symmetry of the theory, one obtains the Slavnov-Taylor identity for the action $S = \int d^4x \mathcal{L}^{\text{full}}$,

$$\begin{aligned} & \int d^4x \left[\frac{\delta S}{\delta g_{\mu\nu}} (s g_{\mu\nu}) + \frac{\delta S}{\delta \eta_\mu} (s \eta_\mu) + \frac{\delta S}{\delta \bar{\eta}_\mu} (s \bar{\eta}_\mu) + \frac{\delta S}{\delta h_{\mu\nu}} (s h_{\mu\nu}) \right] \\ &= \int d^4x \left[\frac{\delta S}{\delta g_{\mu\nu}} \frac{\delta S}{\delta G^{*\mu\nu}} + \frac{\delta S}{\delta \eta_\mu} \frac{\delta S}{\delta C^{*\mu}} - \frac{\delta S}{\delta \bar{\eta}_\mu} B_\mu + \frac{\delta S}{\delta h_{\mu\nu}} W_{\mu\nu} \right] = 0. \end{aligned} \quad (E5)$$

Moreover, using Eq. (2.7) in the limit $\xi \rightarrow 0$ and $\zeta \rightarrow 0$, one deduces the Nakanishi identity,

$$\frac{\delta S}{\delta B_\mu} = h_{\mu\nu}^{\alpha\mu} - \frac{1}{2} h_{\alpha}^{\alpha\mu}. \quad (E6)$$

Employing the above relations, extended to the quantum effective action Γ , one can derive the Ward identities which relate the one-particle irreducible Green functions. We do not go into the details of this derivation here, which involve rather lengthy calculations. For our purpose, it is sufficient to point out that one such Ward identity relates

the ghost-background graviton-ghost vertex $\Gamma_{\eta\bar{\eta}g}$ to the 1PI Green function Γ_{WG^*g} . In momentum space, one may represent schematically this relation as

$$\Gamma_{\eta\mu\bar{\eta}^\nu g_{\alpha\beta}}(k_1, k_2) \sim k^\rho k_{1\sigma} \Gamma_{W_{\mu\rho} G^{\nu\sigma} g_{\alpha\beta}}(k_1, k_2). \quad (E7)$$

We note that in the Yang-Mills theory a similar relation involves the ghost-background gluon-ghost vertex $\Gamma_{\eta\bar{\eta}\bar{A}}$ and the Green function $\Gamma_{\Omega A^* \bar{A}}$. In this case, by power counting, one can see that this Green function is finite, which implies the finiteness of ghost-background gluon-ghost vertex in the Landau gauge. On the other hand, one can see by power counting that in quantum gravity the Green function Γ_{WG^*g} is quadratically divergent, which implies that the ghost-background graviton-ghost vertex is ultraviolet divergent in the Landau-DeWitt background gauge. This is confirmed by an explicit evaluation of the diagrams shown in Fig. 4, which yields a very involved divergent result.

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