

Gauged extended field theory and generalized Cartan geometry

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Cartan geometry provides a unifying algebraic construction of curvature and torsion, based on an underlying model Lie algebra; a viewpoint that can be extended naturally to the higher algebraic structures underlying supergravity. We present a Cartan-geometric framework for generalized geometries governed by a differential graded Lie algebra, extending previous results. The extended tangent bundle admits the action of both a global duality group $\mathcal{G}$ and a local gauge group H . This algebraic structure is implemented via a brane current algebra; the phase space Poisson structure of p -branes. Within this Cartan-inspired framework, we define a hierarchy of generalized connections and compute their linearized torsion and curvature tensors, including the higher curvatures required by the tensor hierarchy. This provides a systematic construction of curvature and torsion tensors in generic generalized geometries.

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I. INTRODUCTION

Geometry has always been central to our understanding of the physical world—from the curvature of spacetime in Einstein’s general relativity to the intricate gauge symmetries of particle physics. A clear manifestation of this connection appears in (nonlinear) σ -models, which are theories whose fields are maps $X: \Sigma \rightarrow M$ from a p -dimensional world-volume Σ to a d -dimensional target space M , typically equipped with a metric g . In their various incarnations, σ -models capture a wide range of phenomena, from fundamental interactions in particle physics to effective descriptions in condensed matter systems.

In the Hamiltonian formulation of σ -models, the phase space often carries a global symmetry group $\mathcal{G}$, known as the duality group. For example, the bosonic string σ -model enjoys $\mathcal{G} = O(d, d)$ T-duality symmetry [1–19]. Also for several higher-dimensional σ -models, reorganizing the degrees of freedom so that a $\mathcal{G}$ -covariance, typically $\mathcal{G} = E_{d(d)}$, is manifest—either at the level of the Lagrangian [20–25] or in the Hamiltonian framework [26–41]—has proven to be a powerful organizational principle. This reformulation not only streamlines the analysis of symmetries and dynamics, but also reveals deep connections to extensions

of classical geometry, such as (exceptional) generalized geometry [28,42–51], see [52–54] for recent reviews.

In many situations, the σ -model also possesses a local gauge symmetry H , leading to gauged σ -models. The presence of this gauge symmetry reduces the physical phase space; only H -equivariant configurations are physical. This reduction naturally fits into the language of constrained Hamiltonian systems and symplectic reduction [55], where the physical phase space is obtained as a quotient by the H -action. In this work, we seek a unifying geometric framework that makes *both* $\mathcal{G}$ -covariance and H -gauge symmetry manifest.

A natural source of inspiration for such a framework is Cartan geometry. Originally developed to generalize Riemannian geometry, Cartan geometry describes a manifold M by modelling its tangent spaces as homogeneous spaces G/H , where G is a Lie group and H a closed subgroup [56,57]. This perspective encompasses a wide range of geometric structures, including Riemannian, conformal, and projective geometries, and is inherently tied to the theory of Lie algebroids [58–61].

Generalized geometry provides another, orthogonal, extension of the classical picture; instead of working solely with the tangent bundle TM , one considers an extended tangent bundle, such as $TM \oplus T^*M$ or more generally $TM \oplus \wedge^p T^*M \oplus \dots$, to incorporate gauge transformations of form fields into the geometric framework. This can be formulated in terms of $\mathcal{G}$ -algebroids—such as Courant or Leibniz algebroids [62–64]—which generalize the Lie algebroid structure underlying Cartan geometry.

Our main motivation is to extend Cartan geometry beyond Lie algebroids, replacing them by more general algebroid structures appropriate to generalized or exceptional geometry.

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Such an extension not only implements the symmetries and dualities of string and M-theory into the Cartan geometric language, but also provides a unified description of the metric, B -field, and other higher-form fields. This broader framework—bridging Cartan geometry and generalized geometry—promises new insights into the geometric foundations of duality-symmetric theories and opens new directions for both physical applications and pure mathematical developments. In the following, we will review the state of the art in the relevant subfields.

A. Curvature in generalized geometries

In classical differential geometry, the curvature of an affine connection plays a central role. It allows one to define the Ricci scalar and tensor, which appear in the Einstein-Hilbert action and its equations of motion. Requiring the connection to be torsion-free and metric-compatible uniquely determines it as the Levi-Civita connection. In generalized geometry, however, this uniqueness is lost; the natural analogs of these two conditions fail to fix the generalized connection uniquely, leading instead to a family of *generalized Levi-Civita connections* [5,45,65–68]. While this is not by itself problematic, a naïve substitution of the Lie bracket by the Dorfman or Courant bracket in the usual definitions of torsion and curvature produces nontensorial objects [69,70]. For the torsion, this issue can be resolved by introducing the *Gualtieri torsion*, which contains an additional term precisely canceling the nontensorial contribution [69].

Curvature in generalized geometry can likewise be redefined so that it is genuinely tensorial, though it still depends on the choice of generalized connection [5,66,70,71]. This construction crucially relies on the existence of the $O(d, d)$ metric on the generalized tangent bundle and therefore does not directly extend to arbitrary generalized geometries. Moreover, in this setting the curvature can no longer be interpreted as the commutator of covariant derivatives. An alternative approach is to restrict attention to certain subbundles of the generalized tangent bundle on which the noncovariance vanishes [65,72]. Remarkably, one can still define a *generalized Ricci scalar* and a *generalized Ricci tensor* which are covariant and independent of the particular choice for the connection [45,65,73]. This enables the construction of an action principle analogous to the Einstein-Hilbert action, reproducing the bosonic Neveu-Schwarz–Neveu-Schwarz sector of 10d type II or heterotic supergravity [for $O(d, d)$ and $O(d, d + n)$] and the internal d -dimensional sector of 10d type II supergravity and 11d supergravity (for $E_{d(d)}$ with $d \leq 7$) [45,65,72,74]. For incorporating α' corrections, however, an analog of the Riemann tensor in generalized geometry is required [66].

An elegant, alternative construction of the generalized curvature tensor was given by Poláček and Siegel [75], who computed it by commuting covariant derivatives on an *extended space* including coordinates for the local

symmetry. As we will explain below, this mirrors the structure of *Cartan geometry*, where the Cartan connection unifies the vielbein and spin connection, whose curvature contains both torsion and Riemann components [56,57,76]. To generalize this construction beyond $O(d, d)$, one requires a *hierarchy* of connections, naturally organized in the framework of tensor hierarchies [77]. Related perspectives on torsion and curvature also arise from the graded geometry approach [78–82].

B. Cartan geometry

Cartan geometry provides a unifying framework encompassing Riemannian, projective, conformal, and more general *Klein geometries* [56,57]. It replaces the flat tangent space of Riemannian geometry with a homogeneous space G/H , where G is a Lie group and H is a closed subgroup. Nonhomogeneous spaces are then viewed as *infinitesimally* Kleinian; they are modeled locally on G/H , but their curvature breaks the global homogeneity. For instance, in general relativity the model is Minkowski space $ISO(1, d-1)/SO(1, d-1) \cong \mathbb{R}^{1, d-1}$, and the Cartan connection unifies the translational vielbein and rotational spin connection into a single geometric object.

Formally, let M be a d -dimensional manifold, and $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G and H with $\dim(\mathfrak{g}/\mathfrak{h}) = d$. A *Cartan geometry* modelled on G/H is a principal H -bundle $\pi: P \rightarrow M$ with a $\mathfrak{g}$ -valued one-form θ on P (the *Cartan connection*) satisfying the following properties:

- (1) *Absolute parallelism*: $\theta|_p: T_p P \rightarrow \mathfrak{g}$ is a linear isomorphism for every $p \in P$;
- (2) *H -equivariance*: $R_h^* \theta = \text{Ad}_{h^{-1}} \theta$ for all $h \in H$; and
- (3) *Reproduction of fundamental fields*: $\theta(X_\xi) = \xi$ for $\xi \in \mathfrak{h}$, where X_ξ is the corresponding fundamental vector field.

The Cartan curvature is the $\mathfrak{g}$ -valued two-form

$$\Theta = -d\theta + \frac{1}{2}[\theta, \theta]. \quad (1.1)$$

For *reductive* Cartan geometries—those with a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ invariant under $\text{Ad}(H)$ —the connection decomposes as $\theta = \omega + e$, where ω is an Ehresmann connection (e.g., spin connection) and e a coframe. The Cartan curvature similarly splits into an $\mathfrak{h}$ -valued curvature R^ω and a $\mathfrak{g}/\mathfrak{h}$ -valued torsion T^ω for the connection ω .

C. Generalized Cartan geometry

Motivated by generalized geometry, the notion of Cartan geometry can itself be *generalized* [76], building on earlier work by Poláček and Siegel [75]. There, techniques inspired by Cartan geometry were used to systematically construct covariant torsion and curvature tensors for generalized geometry and double field theory—previously obtained only via *ad hoc* methods.

In this setting, absolute parallelism is replaced by a linear isomorphism,

$$\theta|_p: T_p P \oplus T_p^* P \longrightarrow \mathfrak{d}, \quad (1.2)$$

where $\mathfrak{d}$ is a $2(d + \dim \mathfrak{h})$ -dimensional Lie algebra with a nondegenerate split symmetric ad-invariant bilinear form η , and $\mathfrak{h} \subset \mathfrak{d}$ is isotropic. The connection preserves η , and $\mathfrak{d}$ is assumed to admit an $\text{Ad}(H)$ -invariant decomposition,

$$\mathfrak{d} \cong \mathfrak{h} \oplus \tilde{\mathfrak{d}} \oplus \mathfrak{h}^*,$$

with $\tilde{\mathfrak{d}}$ modeling the generalized tangent space $T_x M \oplus T_x^* M$. The generalized Cartan connection then takes a block form involving (Ω, E) —the analogs of (ω, e) —together with an additional field $\rho \in \wedge^2 \mathfrak{h}$, which acts as a compensating gauge field for the dual $\mathfrak{h}^*$.

Its curvature decomposes into generalized torsion T^Ω , curvature R^Ω of the generalized spin connection Ω , and a higher curvature R^ρ for ρ . The structure mirrors that of ordinary Cartan geometry, but with an extra layer; just as the torsion constructed from e alone is made covariant by introducing ω , the curvature of Ω becomes covariant only upon introducing ρ , whose own curvature is then covariant on its own, without the need for additional connections. This reflects the inherent *tensor hierarchy* structure of extended geometries [77].

Based on these ideas, we introduce a *minimal* Cartan-geometric framework adapted to extended geometries, generalizing the standard theory by replacing the underlying Lie algebroid of TP with more general algebroids over the principal bundle P governed by a differential graded Lie algebra. This yields a covariant and systematic formulation of generalized connections, torsion, and curvature—naturally fitting into a tensor hierarchy—relevant to both double and exceptional field theories.

Section II defines a class of generalized geometries based on $H \times \mathcal{G}$, with H the local frame group and $\mathcal{G}$ a global duality group. Section III realizes these geometries in the phase space of branes, showing how their current algebras encode the extended tangent bundle. Section IV presents our main geometric results; explicit linearized curvatures for the full generalized Cartan connection, including higher gauge fields like ρ , and their organization into a tensor hierarchy. Section V concludes with applications and open directions.

II. $H \times \mathcal{G}$ -GENERALIZED GEOMETRY

In this section, we introduce an extended notion of generalized geometry, which geometrizes the action of a gauge group H in addition to a duality group $\mathcal{G}$. To this end, we define an extended notion (in the Cartan geometry sense) of the generalized tangent bundle. Moreover, we show which and how notions of generalized geometry, such as the tensor hierarchy and generalized Lie derivative, can be defined on this bundle. In Sec. III, we demonstrate how

this geometry is naturally realized on the phase space of p -branes.

A. Input from $\mathcal{G}$ -generalized geometry

We assume a generalized geometry associated with a duality group $\mathcal{G}$, the typical examples with applications in supergravity and string theory being $\mathcal{G} = \text{O}(d, d)$ or $E_{d(d)}$. Let us introduce the central objects and conventions used in this article. A key role is played by the tensor hierarchy (algebra) associated with $\mathcal{G}$. This is understood here as the graded vector space,

$$\mathcal{T} = \bigoplus_{p \geq 1} \mathcal{R}_p, \quad (2.1)$$

where $\mathcal{R}_p$ denote representations of $\mathcal{G}$ or their associated bundles. Elements of $\mathcal{R}_p$ have degree $p - 1$, and we will use $K_p, L_p, M_p, \dots$ to denote their indices. $\mathcal{T}$ can be understood as an infinity-enhanced Leibniz algebra, or alternatively (after degree shift) a differential graded Lie algebra¹ [83–92]. The two ingredients for this are the $\bullet$ -product (corresponding to the algebra bracket) $\bullet: \mathcal{R}_p \otimes \mathcal{R}_q \rightarrow \mathcal{R}_{p+q}$, and the differential $\partial: \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$,

$$(V \bullet W)^{L_{p+q}} = \eta^{L_{p+q}}_{M_p N_q} V^{M_p} W^{N_q}, \quad (2.2)$$

$$(\partial V)^{L_{p-1}} = D_{K_p}^{L_{p-1} M_1} \partial_{M_1} V^{K_p}, \quad (2.3)$$

for $V \in \mathcal{R}_p$, $W \in \mathcal{R}_q$. The partial derivative ∂_{M_1} is understood with respect to generalized coordinates X^{M_1} . These algebraic objects are characterized by structure constants, called η - and D -symbols in the following. Explicit expressions for $\mathcal{G} = E_{d(d)}$ can be found in [93]. The η -symbols are graded symmetric² in the sense,

$$\eta^{L_{p+q}}_{M_p N_q} = (-1)^{(p-1)(q-1)} \eta^{L_{p+q}}_{N_q M_p}. \quad (2.4)$$

Let us note that the D -symbols are often (but not always) the dual objects to the η -symbols [41]. The following conditions on the η - and D -symbols are needed here,

$$D_{K_2}^{L_1 M_1} \partial_{L_1} \otimes \partial_{M_1} = 0, \quad (2.5)$$

$$D_{K_{p+1}}^{L_p N_1} D_{M_{p+2}}^{K_{p+1} P_1} \partial_{(N_1} \otimes \partial_{P_1)} = 0, \quad (2.6)$$

¹It becomes a differential graded Lie algebra only when including $\mathcal{R}_0$. Without it, we need to have a Leibniz product on $\mathcal{R}_1$ as an additional algebraic input.

²For the application to exceptional generalized geometry in arbitrary dimensions or arbitrary high representations in the tensor hierarchy, this condition has to be relaxed as some of the η -symbols do not have fixed symmetry, starting from $\mathcal{R}_{9-d}$ for $E_{d(d)}$ generalized geometry.

in addition to relations that can be understood from graded Jacobi (Leibniz) identities,

$$\eta^{N_{p+q}} K_p L_q \eta^{P_{p+q+r}} N_{p+q} M_r + (-1)^{(p-1)} \eta^{N_{q+r}} L_q M_r \eta^{P_{p+q+r}} K_p N_{q+r} + (-1)^{(q-1)(r-1)} \eta^{N_{p+r}} K_p M_r \eta^{P_{p+q+r}} N_{p+r} L_q = 0, \quad (2.7)$$

when understanding the tensor hierarchy as a differential graded Lie (Leibniz) algebra [88–90,92]. Furthermore, compatibility between the differential ∂ and the $\bullet$ -product implies the following equations relating the η - and D -symbols:

$$\begin{aligned} & (\eta^{M_{p+q}} L_q K_p D_{P_{p+1}}^{K_p N_1} + D_{K_{p+q+1}}^{M_{p+q} N_1} \eta^{K_{p+q+1}} L_q P_{p+1}) \partial_{N_1} \\ &= \begin{cases} \delta_{P_{p+1}}^{M_{p+1}} \partial_{L_1} & \text{for } q = 1 \\ 0 & \text{for } q > 1 \end{cases}. \end{aligned} \quad (2.8)$$

Moreover, let us comment on the relation between D - and η -symbols. In many relevant cases, one can identify the D -symbols with the (canonical) duals of the η -symbols. For instance, for $\mathcal{G} = E_{d(d)}$ this holds for the representations $\mathcal{R}_p$ up to $p < 9 - d$. Beyond this point, the relevant representations are typically reducible, e.g., $\mathcal{R}_{9-d} = \mathbf{1} \oplus \mathbf{adj}$, in which case the D -symbol is a linear combination of the relevant dual η -symbols.

The first condition (2.5) is the *section condition* that restricts physical functions to only depend on a subset x^m of the coordinates X^{M_1} ,

$$\partial_{M_1} = (\partial_m, 0, \dots), \quad (2.9)$$

while the second condition (2.6) is a consequence of nilpotency of the differential $\partial^2 = 0$. As usual after solving the section condition, the representations $\mathcal{R}_p$ correspond to bundles over some underlying manifold M with coordinates x^m . Their concrete form will not be relevant in the following. The $\mathcal{R}_1$ -representation corresponds to the so-called *generalized tangent bundle* $\mathcal{R}_1 = TM \oplus \dots$, where “...” corresponds to various p -forms and mixed-symmetry tensors, depending on the concrete example. On this generalized tangent bundle, one can define a *generalized Lie derivative*,

$$\begin{aligned} \mathcal{L}_V W^{N_1} &= V^{M_1} \partial_{M_1} W^{N_1} - W^{M_1} \partial_{M_1} V^{N_1} \\ &+ Y^{M_1 N_1}_{K_1 L_1} \partial_{M_1} V^{K_1} W^{L_1}, \end{aligned} \quad (2.10)$$

for $V, W \in \mathcal{R}_1$. The Y -tensor³ is defined in terms of the η - and D -symbols as

$$Y^{M_1 N_1}_{K_1 L_1} = D_{P_2}^{M_1 N_1} \eta^{P_2}_{K_1 L_1}. \quad (2.11)$$

The algebra of *generalized diffeomorphisms*, generated by the generalized Lie derivative (2.10) closes into the

³As explained in [41], the Y -tensors of $E_{7(7)}$ and $E_{8(8)}$ can also be written in this form.

so-called C -bracket [46],

$$\begin{aligned} [\mathcal{L}_{V_1}, \mathcal{L}_{V_2}] W^{N_1} &= \mathcal{L}_{[V_1, V_2]_C} W^{N_1}, \\ [V_1, V_2]_C &= \frac{1}{2} (\mathcal{L}_{V_1} V_2 - \mathcal{L}_{V_2} V_1). \end{aligned} \quad (2.12)$$

Similarly, there is an action of generalized vectors $\Lambda \in \mathcal{R}_1$ on sections Φ of an $\mathcal{R}_p$ -bundle for $p > 1$, defined in terms of the tensor hierarchy differential ∂ and $\bullet$ -product [74,94,95],

$$\mathcal{L}_\Lambda \Phi = \Lambda \bullet (\partial \Phi) + \partial (\Lambda \bullet \Phi). \quad (2.13)$$

Again this closes into the C -bracket, $[\mathcal{L}_{\Lambda_1}, \mathcal{L}_{\Lambda_2}] \Phi = \mathcal{L}_{[\Lambda_1, \Lambda_2]_C} \Phi$.

B. Extended generalized geometry

Generalized geometry geometrizes the action of the duality group $\mathcal{G}$ in the sense that all quantities appearing are phrased as $\mathcal{G}$ -representations. The goal now is to geometrize an *additional* gauge symmetry $\mathfrak{h} = \text{Lie}(H)$. When comparing to general relativity, $\mathcal{G}$ would correspond to $\text{GL}(d)$, and H could be taken as the Lorentz group $\text{O}(1, d-1)$. Alternatively, $\mathfrak{h}$ could correspond to an arbitrary algebra of generalized Killing vectors.

1. Action of an additional symmetry group H

The Lie group H with Lie algebra $\mathfrak{h}$ (with structure constants $f_{\alpha\beta}{}^\gamma$) describes an additional gauge symmetry that is associated with an action of a generator $t_\alpha \in \mathfrak{h}$ on an object $V \in \mathcal{R}_p$, defined as $t_\alpha \cdot V^{M_p} := f_{\alpha N_p}^{M_p} V^{N_p}$ such that

$$f_{\alpha\beta}{}^\gamma f_{\gamma M_p}^{N_p} = -2f_{[\alpha M_p}^{K_p} f_{\beta] K_p}^{N_p}. \quad (2.14)$$

Furthermore, we assume that $\mathfrak{h}$ leaves the η and D -symbols invariant:

$$f_{\alpha M_p}^{K_p} \eta^{L_{p+q}}_{K_p N_q} + f_{\alpha N_q}^{K_q} \eta^{L_{p+q}}_{M_p K_q} = f_{\alpha K_{p+q}}^{L_{p+q}} \eta^{K_{p+q}}_{M_p N_q}, \quad (2.15)$$

$$\begin{aligned} & f_{\alpha K_p}^{M_p} D_{L_{p+1}}^{K_p N_1} + f_{\alpha K_1}^{N_1} D_{L_{p+1}}^{M_p K_1} \\ &= f_{\alpha L_{p+1}}^{K_{p+1}} D_{K_{p+1}}^{M_p N_1}. \end{aligned} \quad (2.16)$$

A typical example of H will be the maximal compact subgroup of $\mathcal{G}$.

2. Representations

As a first step, we define an extended $\mathfrak{R}_1$ -representation as the vector space,

$$\begin{aligned} \mathfrak{R}_1 &= \mathfrak{h} \oplus \mathcal{R}_1 \oplus (\mathcal{R}_2 \otimes \mathfrak{h}^*) \oplus (\mathcal{R}_3 \otimes \mathfrak{h}^* \wedge \mathfrak{h}^*) \oplus \dots \\ &= \mathfrak{h} \oplus \bigoplus_{q \geq 0} (\mathcal{R}_{1+q} \otimes \bigwedge^q \mathfrak{h}^*), \end{aligned} \quad (2.17)$$

that geometrizes both the action of $\mathcal{G}$ via the appearance of $\mathcal{G}$ -representations $\mathcal{R}_p$ and the action of H with $\mathfrak{h}$ and $\mathfrak{h}^*$ transforming in the adjoint and coadjoint representation, and $\mathcal{R}_p$ transforming as above (2.14). For an object V in this representation, we introduce the notation

$$V^{\mathcal{M}_1} = (V^\mu, V^{M_1}, V_\mu^{M_2}, V_{\mu_1\mu_2}^{M_3}, \dots, V_{\mu_1\dots\mu_{q-1}}^{M_q}, \dots) \quad (2.18)$$

for its index. For $p > 1$, this can be extended to a hierarchy of representations $\mathfrak{R}_p$,

$$\begin{aligned} \mathfrak{R}_p &= \mathcal{R}_p \oplus (\mathcal{R}_{p+1} \otimes \mathfrak{h}^*) \oplus (\mathcal{R}_{p+2} \otimes \mathfrak{h}^* \wedge \mathfrak{h}^*) \oplus \dots \\ &= \bigoplus_{q \geq 0} (\mathcal{R}_{p+q} \otimes \bigwedge^q \mathfrak{h}^*), \end{aligned} \quad (2.19)$$

with indices

$$V^{\mathcal{M}_p} = (V^{M_p}, V_\mu^{M_{p+1}}, \dots, V_{\mu_1\dots\mu_q}^{M_{p+q}}, \dots). \quad (2.20)$$

Note that despite being seemingly infinite-dimensional, these representations are finite-dimensional (given that all representations $\mathcal{R}_p$ in the original tensor hierarchy are finite-dimensional). They are restricted by the dimensionality n of the gauge algebra $\mathfrak{h}$, which cuts-off all form indices beyond $\mu_1\dots\mu_n$. From the point of view of supergravity, the ordinary tensor hierarchy of $E_{d(d)}$ typically ends at $\mathcal{R}_{9-d}$ [96], whereas from the point of view of branes, the ordinary tensor hierarchy $E_{d(d)}$ is restricted by the P -brane dimension to $\mathcal{R}_{p+1}$.

For $\mathcal{G} = E_{d(d)}$, the approach here is closely related to the “megaspaces” exceptional field theory in [77]. There, an extended geometry is proposed that naturally corresponds to generalized geometry associated to $\mathcal{G} = E_{d+n(d+n)}$ where $n = \dim(\mathfrak{h})$. The $\mathfrak{R}_1$ -representation (2.17) proposed here, could be understood as a subset of the $\mathcal{R}_1$ of $E_{d+n(d+n)}$,

$$\mathfrak{R}_1(E_{d(d)}, \mathfrak{h}) \subset \mathcal{R}_1(E_{d+n(d+n)}), \quad (2.21)$$

in both the finite- and infinite-dimensional cases. $\mathfrak{R}_1(E_{d(d)}, \mathfrak{h})$ appears as leading contribution in the level decomposition of $\mathcal{R}_1(E_{d+n(d+n)})$. In particular, $\mathfrak{R}_1$ ignores the mixed-symmetry tensors that appear in the $\text{GL}(d)$ -decomposition of $E_{d+n(d+n)}$ for $\mathcal{R}_p$ with $p > 9 - (d + n)$.

The approach in [77] has the effect that it geometrizes not only duality transformations by elements in $\mathcal{G} = E_{d(d)}$ and the gauge group H , but full duality transformations by $E_{d+n(d+n)}$. This larger group allows to capture generalized dualities [97], but the trade-off is that for generic choices of $d < 9$ and n one ends up with irregular, infinite-dimensional Lie algebras⁴ $\mathfrak{e}_{d(d)}^{++\dots+}$. To circumvent

this problem, we propose here a *minimal* extended symmetry algebra that is determined *solely* by the global $\mathcal{G}$ - and local H -covariance.

3. Extending $\mathfrak{R}_p$ to $p=0$

In Sec. IV we will introduce the Cartan connection. To deal with it in an effective way, it is practical to define $\mathfrak{R}_0$ as

$$\begin{aligned} \mathfrak{R}_0 &= \bigoplus_{q \geq 1} (\mathcal{R}_q^* \otimes \bigwedge^q \mathfrak{h}) \oplus ((\mathfrak{h} \otimes \mathfrak{h}^*) \oplus \mathcal{R}_0) \\ &\quad \oplus \bigoplus_{q \geq 1} (\mathcal{R}_q \otimes \bigwedge^q \mathfrak{h}^*). \end{aligned} \quad (2.22)$$

This is motivated by the existence and similar structure of an $\mathcal{R}_0$ -representation of the duality group $\mathcal{G}$, which is simply the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. Here, we introduce it by hand. It is clearly a representation of $H \times \mathcal{G}$. A special role will be played by the subalgebra

$$\tilde{\mathfrak{R}}_0 = \bigoplus_{q \geq 1} (\mathcal{R}_q^* \otimes \bigwedge^q \mathfrak{h}), \quad (2.23)$$

in which the physical fields, in our cases the generalized spin connection and higher versions, are contained. We focus on computing torsion, curvatures and the corresponding Bianchi identities for these fields specifically. Therefore, it will be practical to limit the discussion to this subalgebra. We can define the following Lie algebra structure of $\tilde{\mathfrak{R}}_0$,

$$\left[R_{\alpha_1\dots\alpha_p}^{A_p}, R_{\beta_1\dots\beta_q}^{B_q} \right] = \frac{1}{(p+q)!} \eta_{C_{p+q}}^{A_p B_q} \delta_{\alpha_1\dots\alpha_p \beta_1\dots\beta_q}^{\gamma_1\dots\gamma_{p+q}} R_{\gamma_1\dots\gamma_{p+q}}^{C_{p+q}}, \quad (2.24)$$

in terms of the canonical duals of the η -symbols (2.2) subject to the normalization

$$\eta_{C_{p+q}}^{A_p B_q} \eta^{D_{p+q}}_{A_p B_q} = \delta_{C_{p+q}}^{D_{p+q}}, \quad (2.25)$$

with the convention that the generalized Kronecker delta takes values ± 1 for its nonzero components. Here $R_{\alpha_1\dots\alpha_p}^{A_p}$ denote the generators of $\mathcal{R}_p^* \otimes \bigwedge^p \mathfrak{h}$. The commutator in (2.24) should be viewed as an *ad hoc* definition of a Lie algebra structure; nevertheless, consistency is ensured by the graded Jacobi identity satisfied by the dual η -symbols.

4. η - and D -symbols

In principle, these are simply inherited from the η - and D -symbols of $\mathcal{G}$ -generalized geometry. The extended

⁴Though see [98–104] for recent progress in extended geometries based on infinite-dimensional (affine/Kac-Moody) Lie algebras.

η -symbol $\eta^{\mathcal{L}_2}_{\mathcal{M}_1\mathcal{N}_1} : \mathfrak{R}_1 \otimes \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$ has the following nonvanishing components for $p, q, r \geq 1$:

$$\begin{aligned} \eta^{\lambda_1 \dots \lambda_{p-2}}_{L_p} \mu, \nu_1 \dots \nu_{q-1} N_q &= \eta^{\lambda_1 \dots \lambda_{p-2}}_{L_p} \nu_1 \dots \nu_{q-1} \mu N_q = \begin{cases} -(-1)^q \delta^{\nu_1 \dots \nu_{q-1}}_{\mu \lambda_1 \dots \lambda_{q-2}} \delta^L_{N_q}, & \text{for } p = q \geq 2 \\ 0 & \text{else} \end{cases} \\ \eta^{\lambda_1 \dots \lambda_{r-2}}_{L_r} \mu_1 \dots \mu_{p-1} M_p, \nu_1 \dots \nu_{q-1} N_q &= \begin{cases} -(-1)^{pq} \eta^{L_{p+q}}_{M_p N_q} \delta^{\mu_1 \dots \mu_{p-1} \nu_1 \dots \nu_{q-1}}_{\lambda_1 \dots \lambda_{p+q-2}}, & \text{for } r = p + q \\ 0 & \text{else} \end{cases} \end{aligned} \quad (2.26)$$

In particular, we note that $\eta^{\mathcal{L}_2}_{\mu\nu} = 0$ and $\eta^{\mathcal{L}_2}_{\mu N_1} = \eta^{\mathcal{L}_2}_{N_1 \mu} = 0$. We adopt conventions in which contractions of form indices include the standard combinatorial factors. For example,

$$X^{\mathcal{L}_2} Y_{\mathcal{L}_2} = X^{L_2} Y_{L_2} + X^{\lambda_3} Y_{L_3}^\lambda + \dots + \frac{1}{(p-2)!} X^{\lambda_1 \dots \lambda_{p-2}}_{L_p} Y^{\lambda_1 \dots \lambda_{p-2}}_{L_p} + \dots$$

The $H \times \mathcal{G}$ D -symbols are defined similarly in terms of the $\mathcal{G}$ D -symbols,

$$\begin{aligned} D^{\lambda_1 \dots \lambda_{p-2}}_{L_p} \mu, \nu_1 \dots \nu_{q-1} N_q &= D^{\lambda_1 \dots \lambda_{p-2}}_{L_p} \nu_1 \dots \nu_{q-1} \mu N_q = \begin{cases} -(-1)^q \delta^{\mu \lambda_1 \dots \lambda_{q-2}}_{\nu_1 \dots \nu_{q-1}} \delta^N_{L_q}, & \text{for } p = q \geq 2 \\ 0 & \text{otherwise} \end{cases} \\ D^{\lambda_1 \dots \lambda_{r-2}}_{L_r} \mu_1 \dots \mu_{p-1} M_p, \nu_1 \dots \nu_{q-1} N_q &= \begin{cases} -(-1)^{pq} D^{L_{p+q}}_{M_p N_q} \delta^{\lambda_1 \dots \lambda_{p+q-2}}_{\mu_1 \dots \mu_{p-1} \nu_1 \dots \nu_{q-1}}, & \text{for } r = p + q \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for $p, q \geq 1$ and $r \geq 2$. Here, we have formally introduced a D -symbol $D^{M_p N_q}_{L_{p+q}}$ for $q > 1$. Such objects do not usually appear in the exceptional field theory literature. However, they are not needed in practice; in all relevant expressions, the D -symbols are contracted with derivatives. Hence, these hypothetical components will not appear explicitly, due to the section condition which is explained below.

The extended η - and D -symbols are used to define the extension of the Y -tensor,

$$Y^{\mathcal{K}_1 \mathcal{L}_1}_{\mathcal{M}_1 \mathcal{N}_1} = D_{\mathcal{P}_2}^{\mathcal{K}_1 \mathcal{L}_1} \eta^{\mathcal{P}_2}_{\mathcal{M}_1 \mathcal{N}_1}. \quad (2.27)$$

In a similar vein, one can introduce η - and D -symbols that map between different representations $\mathfrak{R}_p$.

5. Section condition

As usual, we associate generalized coordinates $X^{\mathcal{M}_1}$ to the representation $\mathfrak{R}_1$. A section is defined by the constraint on the coordinate dependence of functions, namely,

$$D^{\mathcal{M}_1 \mathcal{N}_1}_{\mathcal{L}_2} \partial_{\mathcal{M}_1} \otimes \partial_{\mathcal{N}_1} = 0. \quad (2.28)$$

The form of the η -symbols (2.26) makes it clear that the only solution to the section condition at this point is one with coordinates y^μ associated to $\mathfrak{h}$ and coordinates $X^{\mathcal{M}_1}$ associated to the usual $\mathcal{R}_1$ -representation, implying

$$\partial_{\mathcal{M}_1} = (\partial_\mu, \partial_{M_1}, 0, \dots), \quad (2.29)$$

where ∂_{M_1} is subject to the usual $\mathcal{G}$ -section condition,

$$D^{M_1 N_1}_{L_2} \partial_{M_1} \otimes \partial_{N_1} = 0. \quad (2.30)$$

Hence, an appropriate setup for a generalized notion of Cartan geometry uses the coordinates,

$$X^{\mathcal{M}_1} = (y^\mu, X^{M_1}, 0, \dots), \quad (2.31)$$

on section, where y^μ corresponds to the coordinates of the gauge group H and X^{M_1} to the generalized coordinates of $\mathcal{G}$ -generalized geometry. In the context of generalized Cartan geometry, the former become the fiber coordinates of the principal H -bundle $P \rightarrow M$, and the latter would be extended coordinates associated with the generalized tangent bundle of M .

C. Generalized Lie derivative

The natural generalized Lie derivative, or Dorfman bracket, of sections $\mathcal{V}, \mathcal{W}$ of the bundle associated to $\mathfrak{R}_1$ would be⁵

⁵Depending on the context we will also write $\mathcal{M} \equiv \mathcal{M}_1$, when there is no reason for confusion.

$$\mathcal{L}_{\mathcal{V}}\mathcal{W}^{\mathcal{M}} = \mathcal{V}^{\mathcal{N}}\partial_{\mathcal{N}}\mathcal{W}^{\mathcal{M}} - \mathcal{W}^{\mathcal{N}}\partial_{\mathcal{N}}\mathcal{V}^{\mathcal{M}} + Y^{\mathcal{M}\mathcal{N}}_{\mathcal{K}\mathcal{L}}\partial_{\mathcal{N}}\mathcal{V}^{\mathcal{K}}\mathcal{W}^{\mathcal{L}}. \quad (2.32)$$

Closure of this generalized Lie derivative requires the following identities for the Y -tensor [46]:

$$0 = \partial_P\partial_{\mathcal{N}}U^{\mathcal{R}}V^{\mathcal{S}}W^{\mathcal{Q}}(Y^{\mathcal{M}\mathcal{P}}_{\mathcal{K}\mathcal{Q}}Y^{\mathcal{K}\mathcal{N}}_{\mathcal{R}\mathcal{S}} - Y^{\mathcal{M}\mathcal{N}}_{\mathcal{R}\mathcal{S}}\delta^{\mathcal{P}}_{\mathcal{Q}}), \quad (2.33)$$

$$\begin{aligned} 0 = & (\partial_P U^{\mathcal{R}}\partial_{\mathcal{N}}V^{\mathcal{S}} - \partial_{\mathcal{N}}U^{\mathcal{S}}\partial_P V^{\mathcal{R}})W^{\mathcal{Q}} \\ & \times (2Y^{\mathcal{M}\mathcal{N}}_{\mathcal{S}\mathcal{K}}Y^{\mathcal{K}\mathcal{P}}_{\mathcal{R}\mathcal{Q}} + Y^{\mathcal{M}\mathcal{N}}_{\mathcal{K}\mathcal{Q}}Y^{\mathcal{K}\mathcal{P}}_{\mathcal{R}\mathcal{S}} \\ & + 2Y^{\mathcal{M}\mathcal{N}}_{\mathcal{R}\mathcal{Q}}\delta^{\mathcal{P}}_{\mathcal{S}} + Y^{\mathcal{M}\mathcal{N}}_{\mathcal{S}\mathcal{R}}\delta^{\mathcal{P}}_{\mathcal{Q}}). \end{aligned} \quad (2.34)$$

While the first identity holds because the Y -tensor comes from a consistent tensor hierarchy algebra, the second one does not hold in general because terms proportional to y -derivatives could cause problems. A simple proof of (2.33) for $\mathcal{G}$ -generalized geometry goes as follows. We know from (2.8) that

$$(\eta^{M_2}_{L_1 K_1} D_{P_2}^{K_1 N_1} + D_{K_3}^{M_2 N_1} \eta^{K_3}_{P_2 L_1}) \partial_{N_1} = \delta^{M_2}_{P_2} \delta^{N_1}_{L_1} \partial_{N_1} \quad (2.35)$$

holds. By tensoring this relation with $D_{M_2}^{R_1 S_1} \partial_{S_1}$ and symmetrizing the derivatives, we see that the second term vanishes due to (2.6) and this eventually leaves us with

$$Y^{R_1 S_1}_{L_1 K_1} D_{P_2}^{K_1 N_1} \partial_{(N_1} \otimes \partial_{S_1)} = D_{P_2}^{R_1 S_1} \delta^{N_1}_{L_1} \partial_{(N_1} \otimes \partial_{S_1)}. \quad (2.36)$$

Further contracting this with $\eta^{P_2}_{M_1 Q_1}$ gives us our desired result (2.33).

To deal with (2.34), we will introduce two additional assumptions on the generalized Lie derivative, that will lead to a consistent gauge structure. By committing to a larger

duality group (in particular one which would rotate to a different choice of $\mathfrak{h}$) they could be lifted but this is not the aim of the present, *minimal*, setup. It should be understood in contrast to the *maximal extensions* of [77] where the generalized Lie derivative is defined for the full $\mathfrak{R}_1$ representation. For $\mathcal{G} = E_{d(d)}$ the latter leads to $\mathfrak{R}_1$ being a representation of $E_{d+n(d+n)}$ for $n = \dim(\mathfrak{h})$. The fact that for $\mathcal{G} = O(d, d)$ our definition of the $\mathfrak{R}_1$ -structure coincides with $O(d+n, d+n)$, as shown in [75, 76, 105–107], is a peculiarity of this specific duality group.

Assumptions:

- (1) Parameters of the generalized diffeomorphism are required to be of the form

$$\mathcal{V} = v + V_1 \in \mathfrak{h} \oplus \mathcal{R}_1, \quad (2.37)$$

because they describe the action of the original duality group $\mathcal{G}$ and the action of a gauge algebra $\mathfrak{h}$.

- (2) We assume a special ansatz for the dependence on y . It only enters via a twist, depending on the representation of $\mathfrak{h}$ through

$$T^{\mu \dots}_{M_p \dots}(y, X) = e^{\mu}_{\alpha}(y) \dots e^{A_p}_{M_p}(y) T^{\alpha \dots}_{A_p \dots}(X), \quad (2.38)$$

where $e^{\alpha}_{\mu}(y)$ is associated to the right-invariant Maurer-Cartan form on H . Moreover, $e^{A_p}_{M_p}(y)$

mediates the action of H on the representations $\mathcal{R}_p$.

As hinted already by (2.15) and (2.16), the H -action has to preserve the symbols η and D of $\mathcal{G}$, restricting the respective Lie algebra $\mathfrak{h}$ to be a subalgebra of $\mathfrak{g}$. The role of this twist becomes clearer in the current algebra picture where it is explicitly derived in Sec. III B.

Acting with a $\mathcal{V}$ from (2.37) on a general

$$\mathcal{W} = w + W_1 + W_2 + \dots \sim (w^{\mu}, W_1^{M_1}, (W_2)_{\mu}^{M_2}, \dots) \in \mathfrak{R}_1,$$

the generalized Lie derivative in (2.32) simplifies to

$$\begin{aligned} \mathcal{L}_{v+V_1}\mathcal{W} = & ([v, w]_{\mathfrak{h}}^{\mu} + \mathcal{L}_{V_1} w^{\mu} - \mathcal{L}_{W_1} v^{\mu}) \partial_{\mu} \\ & + (\mathcal{L}_{V_1} W_1^{M_1} + \partial_{N_1} v^{\nu} (W_2)_{\nu}^{L_2} D_{L_2}^{M_1 N_1}) \partial_{M_1} \\ & + (\mathcal{L}_{V_1} (W_2)_{\mu}^{M_2} + 2 \partial_{N_1} v^{\nu} (W_3)_{\mu\nu}^{L_3} D_{L_3}^{M_2 N_1} - v^{\kappa} f_{\kappa\mu}^{\nu} (W_2)_{\nu}^{M_2}) \partial_{M_2} \\ & + \dots \\ & + \frac{1}{(p-1)!} (\mathcal{L}_{V_1} (W_p)_{\mu_1 \dots \mu_{p-1}}^{M_p} + p \partial_{N_1} v^{\nu} (W_{p+1})_{\mu_1 \dots \mu_{p-1} \nu}^{L_{p+1}} D_{L_{p+1}}^{M_p N_1} \\ & - (p-1) v^{\kappa} f_{\kappa[\mu_{p-1}}^{\nu} (W_p)_{\mu_1 \dots \mu_{p-2}] \nu}^{M_p}) \partial_{M_p}^{\mu_1 \dots \mu_{p-1}} \\ & + \dots, \end{aligned} \quad (2.39)$$

with $[v, w]_{\mathfrak{h}}^{\mu} = f_{\kappa\lambda}^{\mu} v^{\kappa} w^{\lambda}$. Remarkably, the algebra of such extended generalized diffeomorphisms (2.37) closes on arbitrary sections $\mathcal{W}$ of the $\mathfrak{R}_1$ -bundle,

$$\mathcal{L}_{u+U_1} \mathcal{L}_{v+V_1} \mathcal{W} - \mathcal{L}_{v+V_1} \mathcal{L}_{u+U_1} \mathcal{W} = \mathcal{L}_{[u+U_1, v+V_1]_C} \mathcal{W}, \quad (2.40)$$

with the extended Courant bracket

$$[u + U_1, v + V_1]_C = ([u, v]_{\mathfrak{h}} + \mathcal{L}_{U_1} v - \mathcal{L}_{V_1} u) + [U_1, V_1]_C, \quad (2.41)$$

where the last term is the ordinary C -bracket associated to the duality group $\mathcal{G}$. The proof of this closure crucially depends on the identities (2.6) and (2.8) and is presented in Appendix A.

III. DERIVATION FROM BRANE CURRENT ALGEBRA

In the previous section, we introduced a new generalized geometry that makes a duality group $\mathcal{G}$ and a gauge symmetry algebra $\mathfrak{h}$ manifest. Here, we will show that this setting has a very natural origin in the realization of the phase space of branes and its parametrization in terms of $\mathcal{G}$ -covariant currents.

A. Review: Brane currents in $\mathcal{G}$ -generalized geometry

For the realization of $\mathcal{G}$ -generalized geometry on the phase space of $\frac{1}{2}$ -BPS P -branes, the following dictionary has been established in [41]:

- (i) We consider the Hamiltonian formulation of a P -brane in d -dimensional space.⁶ The phase space variables can be put into the form of a hierarchy of currents, associated to the tensor hierarchy of $\mathcal{G}$ (usually this will be $E_{d(d)}$) for $P > 1$, namely,

$$(P - p + 1)\text{-forms } t_{M_p} \in \mathcal{R}_p. \quad (3.1)$$

Typically, these t_{M_p} are constructed from the canonical momentum density of the P -brane, the embedding coordinate fields and world-volume gauge fields. Examples of explicit realizations of the brane currents for the Hamiltonian formulation of D -branes, M -branes or Kaluza-Klein monopole world-volume theories have been presented in [37, 41]. For example, for M2-branes, the typical currents are

⁶The extension to $(11 = d + n)$ - or $(10 = d + n - 1)$ -dimensional supergravity backgrounds is straightforward and has been discussed in [41].

$$\begin{aligned} t_{M_1} &= (p_m, dx^m \wedge dx^{m'}, 0, \dots), \\ t_{M_2} &= (dx^m, 0, \dots), \\ t_{M_3} &= (1, 0, \dots). \end{aligned} \quad (3.2)$$

- (ii) The Poisson brackets of these P -form currents, from now on referred to as *current algebra*, take the form

$$\{t_{A_p}(\sigma), t_{B_q}(\sigma')\} = -\eta^{C_{p+q}}_{A_p B_q} t_{C_{p+q}}(\sigma') \wedge d'\delta(\sigma - \sigma'). \quad (3.3)$$

Additionally, there can be also spatial world-volume boundary contributions $\int d(\dots)$. They have been discussed in detail in [37, 108] for the string and the membrane. Here, we will always neglect them. There are two underlying gradings for a current t_{M_p} —one from its nature as a spatial $(P - p + 1)$ -form (hence depending on P) and as an element of $\mathcal{R}_p$ in the tensor hierarchy/infinity-enhanced Leibniz algebroid. With respect to the former, the current algebra bracket of $(P - p + 1)$ -form currents is in fact graded skew-symmetric (again up to world-volume boundary terms) with degree $-P$

$$\{t_{A_p}(\sigma), t_{B_q}(\sigma')\} = -(-1)^{(p-1)(q-1)} \{t_{B_q}(\sigma'), t_{A_p}(\sigma)\}. \quad (3.4)$$

This grading is consistent with the graded symmetry of the η -symbols (2.4);

- (iii) A correspondence between generalized geometry and this current algebra structure comes from a section Φ of the generalized tangent bundle or a general $\mathcal{R}_p$ -bundle, which can be associated to the object

$$\Phi = \int \Phi^{A_p}(X(\sigma)) t_{A_p}(\sigma), \quad (3.5)$$

where the integral is to be understood over the spatial part of the world-volume. In general, the Poisson bracket of two such sections, for *generic currents* satisfying (3.3), does *not* automatically reproduce the generalized Lie derivative $\mathcal{L}$,

$$\{\Phi, \Lambda\} = \mathcal{L}_{\Lambda} \Phi, \quad (3.6)$$

for a section $\Lambda = \int \Lambda^{A_1}(X(\sigma)) t_{A_1}(\sigma) \in \mathcal{R}_1$. In order to obtain this correspondence between Poisson structure and generalized Lie derivative (up to total derivatives), the hierarchy of currents additionally

has to satisfy the so-called *brane charge constraints*,

$$t_{M_p} \wedge dX^{N_1} \partial_{N_1} = D_{M_p}^{K_{p-1} N_1} t_{K_{p-1}} \partial_{N_1}, \quad (3.7)$$

together with the assumption $t_{M_1} = (p_m, \dots)$, where p_m is the canonical momentum of the brane. Remarkably, solutions to the constraints (3.7) are in one-to-one correspondence with a Hamiltonian formulation of $\frac{1}{2}$ -BPS branes;

- (iv) Hamiltonian and spatial diffeomorphism constraints are characterized by a generalized metric $\mathcal{H}^{A_1 B_1}$, in which also the coupling to a background metric and p -form gauge fields is encoded,

$$H = \mathcal{H}^{A_1 B_1} t_{A_1} \wedge \star t_{B_1} \approx 0, \quad (3.8)$$

and the D -symbols

$$D_{C_2}^{A_1 B_1} t_{A_1} \wedge \star t_{B_1} \approx 0. \quad (3.9)$$

In principle, this is just a reformulation of $\mathcal{G}$ -generalized geometry, but in the following we use it as a computational device. Moreover, if we neglect world-volume boundary contributions, the algebraic structure becomes simpler; it gives rise to a Lie algebroid. In particular, the Jacobi

identity of the current algebra (3.3) corresponds to the graded Jacobi identity of the η -symbols (2.7).

B. Adding $\mathfrak{h}$ -gauge symmetry

Following the idea of [75], pursued further in [76,109], we supplement the above brane current algebra with the local action of a gauge group $H \subset \mathcal{G}$. It is generated on the brane phase space by the currents s_α with the Poisson brackets,

$$\begin{aligned} \{s_\alpha(\sigma), s_\beta(\sigma')\} &= f_{\alpha\beta}{}^\gamma s_\gamma(\sigma) \delta(\sigma - \sigma'), \\ \{s_\alpha(\sigma), t_{M_p}(\sigma')\} &= f_{\alpha M_p}{}^{N_p} t_{N_p}(\sigma) \delta(\sigma - \sigma'), \end{aligned} \quad (3.10)$$

which are governed by the structure constants $f_{\alpha\beta}{}^\gamma$ and the $\mathfrak{h}$ -action $f_{\alpha M_p}{}^{N_p}$ from Sec. II B. However, the extended algebra of the currents $(s_\alpha, t_{A_1}, t_{A_2}, \dots)$ does not close into a Poisson algebra any more. Fortunately, the authors of [75] noticed that the Jacobi identity can be restored after introducing dual gauge symmetry generators Σ^α into the current algebra in a specific way. Here, we do the same and introduce dual generators such that the extended current algebra becomes an (infinite-dimensional) Lie algebra,

$$\begin{aligned} \{s_\alpha(\sigma), s_\beta(\sigma')\} &= f_{\alpha\beta}{}^\gamma s_\gamma(\sigma) \delta(\sigma - \sigma'), \\ \{s_\alpha(\sigma), t_{M_p}(\sigma')\} &= f_{\alpha M_p}{}^{N_p} t_{N_p}(\sigma) \delta(\sigma - \sigma'), \\ \{s_\alpha(\sigma), \Sigma^\beta(\sigma')\} &= \delta_\alpha^\beta d\delta(\sigma - \sigma') + f_{\gamma\alpha}{}^\beta \Sigma^\gamma(\sigma) \delta(\sigma - \sigma'), \\ \{\Sigma^\beta(\sigma), t_{M_p}(\sigma')\} &= 0 = \{\Sigma^\alpha(\sigma), \Sigma^\beta(\sigma')\}, \\ \{t_{M_p}(\sigma), t_{N_q}(\sigma')\} &= -\eta^{L_{p+q}}{}_{M_p N_q} t_{L_{p+q}}(\sigma') \wedge d'\delta(\sigma - \sigma') + f_{\alpha M_p}{}^{K_p} \eta^{L_{p+q}}{}_{K_p N_q} t_{L_{p+q}} \wedge \Sigma^\alpha(\sigma) \delta(\sigma - \sigma'). \end{aligned} \quad (3.11)$$

Written in this way, the last equation is not manifestly skew-symmetric. But it turns out (up to neglected boundary terms) that it is, after assuming

$$dt_{M_p} = -f_{\alpha M_p}{}^{N_p} \Sigma^\alpha \wedge t_{N_p}. \quad (3.12)$$

Due to the Maurer-Cartan equation $d\Sigma^\alpha = \frac{1}{2} f_{\alpha\beta}{}^\gamma \Sigma^\beta \wedge \Sigma^\gamma$, similar identities hold for

$$(-1)^{P-p+1} dt_{M_p}^\alpha = -f_{\beta M_p}{}^{N_p} t_{N_p}^{\beta\alpha} + \frac{1}{2} f_{\beta\gamma}{}^\alpha t_{M_p}^{\beta\gamma}, \quad (3.13)$$

where the prefactor $(-1)^{P-p+1}$ is due to the form degree of t_{M_p} .

Combined with this relation, the current algebra (3.11) should be seen as a *minimal extension* of (3.10) such that it is a Poisson algebra, which satisfies the Jacobi identity as is shown in Appendix B 2. It can be easily understood as a

twisted version of a trivial extension of the current algebra (3.3) associated to $\mathcal{G}$ -generalized geometry, as shown in Appendix B 1.

From the currents in (3.11), one can construct

$$t_{M_p}^{\alpha_1 \dots \alpha_q} := t_{M_p} \wedge \Sigma^{\alpha_1} \wedge \dots \wedge \Sigma^{\alpha_q}, \quad (3.14)$$

where $\Sigma^\alpha = e_\mu{}^\alpha(y) dy^\mu$ is the right-invariant Maurer-Cartan form and the coordinates y^μ are canonically conjugate to s_α , defined by

$$\{e_\mu{}^\alpha s_\alpha(\sigma), y^\nu(\sigma')\} = -\delta_\mu^\nu \delta(\sigma - \sigma'). \quad (3.15)$$

Moreover, we define the dual frame fields $e_\alpha{}^\mu(y)$ with $e_\alpha{}^\mu(y) e_\mu{}^\beta(y) = \delta_\alpha^\beta$. At this point, we have found a higher-dimensional motivation for the second assumption (2.38) from Sec. II C. The twist described there arises after combining (3.12) with (3.14)—the first of these two

equations gives rise to the e_α^μ factors, while the second results in the $e_{M_p}^{A_p}$ twists.

From the P -brane current point of view, $\mathfrak{R}_1$ will consist of all the possible spatial P -forms,

$$t_{\mathcal{M}_1} = (s_\mu, t_{M_1}, t_{M_2}^\mu, t_{M_3}^{\mu_1\mu_2}, \dots). \quad (3.16)$$

Similarly, $\mathfrak{R}_{k>1}$ will consist of all possible $(P - k + 1)$ -forms,

$$t_{\mathcal{M}_k} = (t_{M_k}, t_{M_{k+1}}^\mu, t_{M_{k+2}}^{\mu_1\mu_2}, \dots). \quad (3.17)$$

The relevant bracket for these currents in the $\mathfrak{R}_1$ -representation⁷ $t_{M_p}^{\alpha_1 \dots \alpha_{p-1}}$ can be obtained as

$$\begin{aligned} \{t_{M_p}^{\alpha_1 \dots \alpha_{p-1}}(\sigma), t_{N_q}^{\beta_1 \dots \beta_{q-1}}(\sigma')\} &= (-1)^{pq} \left(\eta^{L_{p+q} M_p N_q} t_{L_{p+q}}^{\alpha_1 \dots \alpha_{p-1} \beta_1 \dots \beta_{q-1}}(\sigma') \wedge d'\delta(\sigma - \sigma') \right. \\ &\quad - f_{\gamma M_p}{}^{K_p} \eta^{L_{p+q} K_p N_q} t_{L_{p+q}}^{\alpha_1 \dots \alpha_{p-1} \beta_1 \dots \beta_{q-1} \gamma}(\sigma) \delta(\sigma - \sigma') \\ &\quad \left. - (-1)^{(p+q)} \eta^{L_{p+q} M_p N_q} \frac{(p-1)}{2!} f_{\gamma \delta}^{[\alpha_1} t_{L_{p+q}}^{\alpha_2 \dots \alpha_{p-1}] \beta_1 \dots \beta_{q-1} \gamma \delta}(\sigma) \delta(\sigma - \sigma') \right) \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} \{s_\alpha(\sigma), t_{M_p}^{\beta_1 \dots \beta_{p-1}}(\sigma')\} &= (p-1) \delta_\alpha^{[\beta_{p-1}} t_{M_p}^{\beta_1 \dots \beta_{p-2}]}(\sigma') \wedge d'\delta(\sigma - \sigma') \\ &\quad + f_{\gamma \alpha}^{[\beta_1} t_{M_p}^{\beta_2 \dots \beta_{p-1}]}(\sigma) \delta(\sigma - \sigma') \\ &\quad + f_{\alpha M_p}{}^{N_p} t_{N_p}^{\beta_1 \dots \beta_{p-1}}(\sigma) \delta(\sigma - \sigma'). \end{aligned} \quad (3.19)$$

In front of the $d\delta$ -terms, we recognize the “extended” η -symbols defined in (2.26). Making use of them, we are eventually left with the compact current algebra

$$\begin{aligned} \{t_{\mathcal{M}_1}(\sigma), t_{\mathcal{N}_1}(\sigma')\} &= -\eta^{\mathcal{L}_2 \mathcal{M}_1 \mathcal{N}_1} t_{\mathcal{L}_2}(\sigma') \wedge d'\delta(\sigma - \sigma') \\ &\quad + f_{\mathcal{M}_1 \mathcal{N}_1}{}^{\mathcal{L}_1} t_{\mathcal{L}_1}(\sigma) \delta(\sigma - \sigma'). \end{aligned} \quad (3.20)$$

Similarly to above one can define representations of $\mathfrak{R}_1[P]$ as brane world-volume *general* sections of

$$\mathcal{V} = \int (v^\mu s_\mu(\sigma) + V_1^{M_1} t_{M_1}(\sigma) + V_2^{M_2} t_{M_2}^a(\sigma) + \dots).$$

In direct analogy with the $\mathcal{G}$ -generalized geometry case, one can reproduce the extended generalized Lie bracket between such objects as

$$\{\mathcal{W}, \mathcal{V}\} = \mathcal{L}_{\mathcal{V}} \mathcal{W} \quad (3.21)$$

if an *extended brane charge condition* holds,

$$t_{\mathcal{M}_p} \wedge dX^{\mathcal{N}_1} \partial_{\mathcal{N}_1} = D_{\mathcal{M}_p}{}^{\mathcal{L}_{p-1} \mathcal{N}_1} t_{\mathcal{L}_{p-1}} \partial_{\mathcal{N}_1} \quad (3.22)$$

for the currents $t_{\mathcal{M}_p}$. Decomposing it into $\mathfrak{h}$ - and $\mathcal{R}_p$ -indices, this simply reduces to the brane charge condition in $\mathcal{G}$ -generalized geometry (3.7) and the condition (3.14).

Let us note that the brane current algebra always closes, not only when (2.37) is satisfied. Only the following identity holds:

$$\begin{aligned} \{\mathcal{U}, \{\mathcal{V}, \mathcal{W}\}\} + c.p. &= \int ([\mathcal{L}_{\mathcal{V}}, \mathcal{L}_{\mathcal{W}}] \mathcal{U} - \mathcal{L}_{[\mathcal{V}, \mathcal{W}]_c} \mathcal{U} + \text{boundary terms}) = 0. \end{aligned} \quad (3.23)$$

Hence, in general closure of the extended generalized Lie bracket (2.32) is only guaranteed *up to boundary terms*.⁸ As shown in Sec. II B, closure without such boundary terms will only be possible if (2.37) satisfied.

C. The zero-mode algebra

The structure constants $f_{\mathcal{M}_1 \mathcal{N}_1}{}^{\mathcal{L}_1}$ in the current algebra (3.20) correspond to a Leibniz algebra (in particular they are not manifestly skew-symmetric). In the context of generalized Cartan geometry, this algebra plays the role of the *model algebra* as explained below in Sec. IV. In particular, we are going to show that the structure constants in (3.20) naturally correspond to a Leibniz algebra associated with a differential graded Lie algebra

⁷For the generic case one obtains,

$$\begin{aligned} \{t_{M_p}^{\alpha_1 \dots \alpha_r}(\sigma), t_{N_q}^{\beta_1 \dots \beta_s}(\sigma')\} &= (-1)^{(r+s)} \left(-\eta^{L_{p+q} M_p N_q} t_{L_{p+q}}^{\alpha_1 \dots \alpha_r \beta_1 \dots \beta_s}(\sigma') \wedge d'\delta(\sigma - \sigma') \right. \\ &\quad + f_{\gamma M_p}{}^{K_p} \eta^{L_{p+q} K_p N_q} t_{L_{p+q}}^{\alpha_1 \dots \alpha_r \beta_1 \dots \beta_s \gamma}(\sigma) \delta(\sigma - \sigma') \\ &\quad \left. + (-1)^{(r+s)} \frac{r}{2} \eta^{L_{p+q} M_p N_q} f_{\gamma \delta}^{[\alpha_1} t_{L_{p+q}}^{\alpha_2 \dots \alpha_r] \beta_1 \dots \beta_s \gamma \delta}(\sigma) \delta(\sigma - \sigma') \right). \end{aligned}$$

⁸A side result of this is that the extended generalized Lie bracket closes with requiring (2.37), up to total derivative terms.

structure. The latter comes from the canonical Poisson structure of spatial p -forms and the canonical momentum. Crucially, this analogy is only valid *up to world-volume boundary terms* (associated with the world-volume de Rham differential).

Starting from the currents $t_{A_p}^{\alpha_r} \equiv t_{A_p} \wedge \Sigma^{\alpha_1} \wedge \dots \wedge \Sigma^{\alpha_r}$, we define their zero-modes $T_{A_p}^{\alpha_r} := \int t_{A_p}^{\alpha_r}(\sigma) \in \mathfrak{R}_{p-r}$ for $r < p$, and a differential $Q: \mathfrak{R}_i \rightarrow \mathfrak{R}_{i-1}$ acting on these by

$$\begin{aligned} QT_{A_p}^{\alpha_r} &:= (-1)^{P+1} \int dt_{A_p}^{\alpha_r}(\sigma) \\ &= (-1)^{1+p+r} f_{\gamma A_p}{}^{B_p} T_{B_p}^{\alpha_r \gamma} + (-1)^p \frac{r}{2} f_{\beta \gamma}{}^{[\alpha_1} T_{A_p}^{\alpha_2 \dots \alpha_r] \beta \gamma}. \end{aligned} \quad (3.24)$$

This operator squares to zero thanks to (2.14) and the Jacobi identity of $\mathfrak{h}$. We also define a graded symmetric product $\bullet: \mathfrak{R}_i \otimes \mathfrak{R}_j \rightarrow \mathfrak{R}_{i+j}$ for the zero-modes, explicitly acting on these as

$$T_{A_p}^{\alpha_r} \bullet T_{B_q}^{\beta_s} = (-1)^{r(q-1)} \eta^{C_{p+q}}{}_{A_p B_q} T_{C_{p+q}}^{\alpha_r \beta_s}. \quad (3.25)$$

From this formula we read off the degree $|T_{A_p}^{\alpha_r}| = p - r - 1$, which agrees with the fact that $T_{A_p}^{\alpha_r}$ is an object of the vector space $\mathfrak{R}_{p-r}$.

Using (2.15), we check that the differential Q is a derivation of the product $\bullet$,

$$Q(T_{A_p}^{\alpha_r} \bullet T_{B_q}^{\beta_s}) + QT_{A_p}^{\alpha_r} \bullet T_{B_q}^{\beta_s} + (-1)^{|T_{A_p}^{\alpha_r}|} T_{A_p}^{\alpha_r} \bullet QT_{B_q}^{\beta_s} = 0. \quad (3.26)$$

From the definition of $\bullet$ and the properties of the η -symbols (2.7), the product $\bullet$ satisfies the graded Jacobi identity,

$$\begin{aligned} (T_{A_p}^{\alpha_r} \bullet T_{B_q}^{\beta_s}) \bullet T_{C_l}^{\gamma_k} + (-1)^{|T_{A_p}^{\alpha_r}|} T_{A_p}^{\alpha_r} \bullet (T_{B_q}^{\beta_s} \bullet T_{C_l}^{\gamma_k}) \\ + (-1)^{|T_{B_q}^{\beta_s}| |T_{C_l}^{\gamma_k}|} (T_{A_p}^{\alpha_r} \bullet T_{C_l}^{\gamma_k}) \bullet T_{B_q}^{\beta_s} = 0. \end{aligned} \quad (3.27)$$

The differential $Q: \mathfrak{R}_i \rightarrow \mathfrak{R}_{i-1}$ is defined for $i \geq 2$, but extending the graded vector space to include a further space $\mathfrak{R}_0$, we can also extend the differential to $Q: \mathfrak{R}_1 \rightarrow \mathfrak{R}_0$. As was discussed in [88], this extension allows us to take the differential Q and the product $\bullet$ as the fundamental structures, forming a differential graded Lie algebra. In particular, we can define the Leibniz product $\circ: \mathfrak{R}_1 \otimes \mathfrak{R}_1 \rightarrow \mathfrak{R}_1$ as

$$\begin{aligned} T_{A_p}^{\alpha_{p-1}} \circ T_{B_q}^{\beta_{q-1}} &:= -T_{B_q}^{\beta_{q-1}} \bullet QT_{A_p}^{\alpha_{p-1}} \equiv -QT_{A_p}^{\alpha_{p-1}} \bullet T_{B_q}^{\beta_{q-1}} \\ &= -(-1)^{(p-1)(q-1)} f_{\gamma A_p}{}^{D_p} \eta^{C_{p+q}}{}_{D_p B_q} T_{C_{p+q}}^{\alpha_{p-1} \beta_{q-1} \gamma} \\ &\quad - (-1)^{pq} \left(\frac{p-1}{2} \right) \eta^{C_{p+q}}{}_{A_p B_q} f_{\gamma \delta}{}^{[\alpha_1} T_{C_{p+q}}^{\alpha_2 \dots \alpha_{p-1}] \beta_1 \dots \beta_{q-1} \gamma \delta} \\ &= f_{\substack{A_p \quad B_q \\ \alpha_{p-1} \quad \beta_{q-1}}}{}^{\substack{C_{p+q} \\ \gamma_{p+q-1}}} T_{C_{p+q}}^{\gamma_{p+q-1}}, \end{aligned} \quad (3.28)$$

where $T_{A_p}^{\alpha_{p-1}}$ and $T_{B_q}^{\beta_{q-1}}$ are zero-modes in $\mathfrak{R}_1$. The graded Jacobi identity of $\bullet$ (3.27) and the compatibility of Q with $\bullet$ (3.26) ensure the Leibniz identity for the product $\circ$,

$$\begin{aligned} T_{A_p}^{\alpha_{p-1}} \circ (T_{B_q}^{\beta_{q-1}} \circ T_{C_l}^{\gamma_{l-1}}) &= -T_{A_p}^{\alpha_{p-1}} \circ (T_{C_l}^{\gamma_{l-1}} \bullet QT_{B_q}^{\beta_{q-1}}) = (T_{C_l}^{\gamma_{l-1}} \bullet QT_{B_q}^{\beta_{q-1}}) \bullet QT_{A_p}^{\alpha_{p-1}} \\ &= -T_{C_l}^{\gamma_{l-1}} \bullet (QT_{B_q}^{\beta_{q-1}} \bullet QT_{A_p}^{\alpha_{p-1}}) + (T_{C_l}^{\gamma_{l-1}} \bullet QT_{A_p}^{\alpha_{p-1}}) \bullet QT_{B_q}^{\beta_{q-1}} \\ &= (T_{A_p}^{\alpha_{p-1}} \circ T_{B_q}^{\beta_{q-1}}) \circ T_{C_l}^{\gamma_{l-1}} + T_{B_q}^{\beta_{q-1}} \circ (T_{A_p}^{\alpha_{p-1}} \circ T_{C_l}^{\gamma_{l-1}}). \end{aligned}$$

If we take the maximal compact subgroup of the duality group $\mathcal{G}$, the model algebras we construct here can be understood as a generalization of the Euclidean or Poincaré group; hence, in the context of $O(d, d)$ in [76] this was dubbed *generalized Poincaré algebra*. To see why, let us consider the simplest possible duality group $\mathcal{G} = GL(d)$. It is special because its tensor hierarchy only has a nontrivial $\mathcal{R}_1$ representation, the fundamental of $GL(d)$. After identifying $H = O(d)$ with its maximal compact subgroup, the structure constants in (3.28) are those of the Euclidean group's Lie algebra.

IV. CURVATURES AND TORSIONS IN THE LINEARIZED THEORY

A. Extended generalized Cartan geometry

Given a manifold M , the aim is to define curvature tensors that are both covariant under generalized diffeomorphisms, i.e., admit an action of a duality group $\mathcal{G}$, and under (local) H -gauge transformations. Let us define the setting step-by-step, almost in full analogy with [76] where the same steps are discussed for $O(d, d)$ generalized geometry. The final curvature tensors agree with [77], as far as they have been calculated there.

- (1) The underlying algebraic structure is constituted by
 - (i) A Lie group H , called *gauge group*, with Lie algebra $\mathfrak{h}$. Given the underlying duality group $\mathcal{G}$, H is typically the maximal compact subgroup of $\mathcal{G}$ —i.e., for $O(d, d)$ it is the double Lorentz group $O(1, d-1) \times O(d-1, 1)$, or for $E_{4(4)} = SL(5)$ it would be $SO(5)$. We assume the differential graded Lie algebra structure for the tensor hierarchy of $\mathcal{G}$, as described in Sec. II. $H \subset \mathcal{G}$ ensures that the identities in (2.15) hold. In standard Cartan geometry, the tangent bundle TP of the principal H -bundle $P \rightarrow M$ plays a central role. Its generalization in this context will be a bundle over P , associated to the $\mathfrak{R}_1$ -representation of $H \times \mathcal{G}$ generalized geometry introduced in Sec. II,

$$\mathfrak{R}_1[P] \sim TH \oplus \mathcal{R}_1[M] \oplus (\mathcal{R}_2[M] \otimes T^*H) \oplus \dots, \quad (4.1)$$

where $\mathcal{R}_p[M]$ represents the usual $\mathcal{R}_p$ -bundles of $\mathcal{G}$ -generalized geometry. Locally, which is all that we are interested in, this agrees with the result [76] that for $O(d, d)$ the extended generalized tangent bundle is $\mathfrak{R}_1[P] = TP \oplus T^*P$. Crucially, as shown in Secs. II and III, it is possible to define a generalized Lie derivative on this bundle. For sections $\mathcal{V}^{\mathcal{M}_1} = (v^\mu, V_1^{\mathcal{M}_1}, 0, \dots)$ of $\mathfrak{R}_1[P]$ acting on general sections $\mathcal{W}$ of $\mathfrak{R}_p[P]$, the bracket (2.39) closes. On the other hand, it turned out that on the brane world-volume *general* sections of $\mathfrak{R}_1[P]$ have a well-defined algebra defined by the current algebra (3.18) and (3.19) *up to world-volume boundary terms*. It is the latter structure that we will employ below in order to obtain curvature tensors;

- (ii) The *model algebra* $\mathfrak{Q}$ is a differential graded Lie algebra [88,89] with an isotropic subalgebra $\mathfrak{h} \subset \mathfrak{I}_1$. In particular, we assume the existence of a graded vector space,

$$\mathfrak{Q} = \mathfrak{I}_0 + \mathfrak{I}_1 + \mathfrak{I}_2 + \dots = \bigoplus_{p \geq 0} \mathfrak{I}_p, \quad (4.2)$$

with a graded symmetric product $\bullet: \mathfrak{I}_p \times \mathfrak{I}_q \rightarrow \mathfrak{I}_{p+q}$ and a differential $Q: \mathfrak{I}_p \rightarrow \mathfrak{I}_{p-1}$, subject to the compatibility conditions in [88,89]. From this differential graded Lie algebra, one can derive a Leibniz bracket on $\mathfrak{I}_1$ via the derived bracket

$$a \circ b = -Qa \bullet b, \quad \text{for } a, b \in \mathfrak{I}_1. \quad (4.3)$$

As an example, we had demonstrated above in Sec. III C that the $H \times \mathcal{G}$ tensor hierarchy introduces exactly such a differential graded Lie algebra as zero-mode part of the current algebra (up to world-volume boundary terms). For the extended generalized Cartan geometry setting, we also assume $\mathfrak{I}_1$ to contain $\mathfrak{h}$ as a (nonmaximal) *isotropic Lie subalgebra*,

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad \text{and} \quad \eta(\mathfrak{h}, \mathfrak{h}) = 0. \quad (4.4)$$

Similar to the $\mathfrak{R}_1$ -bundle, $\mathfrak{I}_1$ can be decomposed as a direct sum of vector spaces (but *not* of Lie/Leibniz algebras),

$$\mathfrak{I}_1 = \mathfrak{h} \oplus \mathcal{R}_1 \oplus (\mathfrak{h}^* \otimes \mathcal{R}_2) \oplus \dots \quad (4.5)$$

In particular, since $\dim(\mathfrak{I}_1) = \dim(\mathfrak{R}_1)$, we will label its elements with $\mathfrak{R}_1$ -indices $\mathcal{K}, \mathcal{L}$. These satisfy

$$T_{\mathcal{K}} \circ T_{\mathcal{L}} = f_{\mathcal{KL}}^{\mathcal{M}} T_{\mathcal{M}}. \quad (4.6)$$

In this article, we consider the above minimal setting of a model algebra $\mathfrak{I}_1$, which is defined alone by the action of $\mathfrak{h}$ on the tensor hierarchy $\{\mathcal{R}_p\}$ of $\mathcal{G}$, and the tensor hierarchy structure itself. Nevertheless, a more general Leibniz algebra structure (with $\mathfrak{h}$ an isotropic Lie subalgebra) should be easy to incorporate in what is to follow.

- (2) A *generalized Cartan connection* θ is a pointwise isomorphism between the two algebraic settings introduced above. It is defined for $p \in P$ as

$$\theta^{(1)}|_p: \mathfrak{R}_1[P]_p \rightarrow \mathfrak{I}_1, \quad (4.7)$$

and preserves the differential graded Lie algebra structure which is governed by the η - and D -symbols in (2.26). This implies that this isomorphism—the generalized Cartan connection—is in fact a (pointwise) isomorphism of differential graded Lie algebras

$$\theta|_p: \bigoplus_q \mathfrak{R}_q[P]_p \rightarrow \mathfrak{Q} \quad (4.8)$$

consisting of a collection of maps $\theta^{(q)}|_p: \mathfrak{R}_q[P]_p \rightarrow \mathfrak{I}_q$. Nevertheless, all additional connections $\theta^{(q)}$ besides $\theta^{(1)}$ will not introduce new degrees of freedom in the definition of the generalized Cartan connection, as it is shown below in Sec. IV B. Hence, we will typically refer to $\theta \equiv \theta^{(1)}$ as the generalized Cartan connection, although one should understand it as acting on the full hierarchy.

The further standard assumptions on θ are the same as in standard Cartan geometry and generalized Cartan geometry [76]:

- (i) Left-invariant vector fields of TH are identified with the Lie subalgebra $\mathfrak{h} \subset \mathfrak{I}_1$; and
- (ii) *Equivariance*. The right-action R_h of $h \in H$ has to match the adjoint action of h^{-1} , through

$$R_h^* \theta = \text{Ad}_{h^{-1}} \theta. \quad (4.9)$$

Again, we will interpret this relation as gauge transformations for the components of θ .

It will turn out that, under these assumptions, a natural choice is to take θ generated by the subalgebra $\tilde{\mathfrak{R}}_0$ introduced in (2.23),

$$\theta = \exp \left(\Omega_{A_1}^\alpha R_{A_1}^{A_1} + \frac{1}{2} \rho_{A_2}^{\alpha_1 \alpha_2} R_{A_2}^{A_2} + \dots \right). \quad (4.10)$$

This will be explained in detail in the next section where we will also construct representations $(R_{\alpha_1 \dots \alpha_p}^{A_p})_{B_1}^{\mathcal{M}_1}$ of the generators of $\tilde{\mathfrak{R}}_0$.

- (3) For a definition of the *generalized Cartan curvature* Θ , we employ the brane current algebra (3.18) and (3.19). Consider the generalized Cartan connection θ represented as a brane current,

$$\theta_{A_p}^{(p)}(\sigma) = \theta_{A_p}^{(p), \mathcal{M}_p}(\sigma) t_{\mathcal{M}_p}(\sigma). \quad (4.11)$$

As explained in the next section, we will only be concerned with $p = 1$, as the other components do not contain any new information compared to $\theta^{(1)}$. The generalized Cartan curvature will then be given by the nontrivial components $\Theta_{A_1 B_1}^{C_1}$ of the current algebra,⁹

$$\begin{aligned} \{\theta_{A_1}^{(1)}(\sigma), \theta_{B_1}^{(1)}(\sigma')\} &= -\eta^{C_2}_{A_1 B_1} \theta_{C_2}^{(2)}(\sigma') \wedge d'\delta(\sigma - \sigma') \\ &\quad + \Theta_{A_1 B_1}^{C_1} \theta_{C_1}^{(1)}(\sigma) \delta(\sigma - \sigma'). \end{aligned} \quad (4.12)$$

Due to the additional twist by the generalized Cartan connection, it contains

- (i) The model algebra (3.20) including the action of $\mathfrak{h}$ on the representations $\mathcal{R}_p$ of $\mathcal{G}$, because it is already contained in (3.20), and, additionally; and
- (ii) *Generalized torsions and curvatures*, which are covariant under generalized diffeomorphisms and $\mathfrak{h}$ -gauge transformations.

This data can be extracted from a decomposition of the $\mathfrak{R}_1$ -indices of $\Theta_{A_1 B_1}^{C_1}$ into $H \times \mathcal{G}$ -indices. All required steps are detailed below in Sec. IV C.

The definition (4.12) generalizes the one originating from string currents in the $O(d, d)$ case in [76]. But it is also closely related to the one in [77], which is based on a generalized Lie derivative or Dorfman bracket $[\cdot, \cdot]_{\mathfrak{d}, D}$ (without referring to the string current algebra), twisted by the generalized model algebra $\mathfrak{d}$: $\Theta_{AB} = -[\theta_A, \theta_B]_{\mathfrak{d}, \mathfrak{d}} \in \mathfrak{d}$.

As the connections $\theta^{(p)}$ in general, do not only contain components in $\mathfrak{h}$ and $\mathfrak{R}_1$, a similar definition for the extended generalized Cartan curvature is not possible in the framework of $H \times \mathcal{G}$ -generalized geometry, because the generalized Lie derivative (2.39) only closes after imposing the additional constraint (2.37). Therefore, we employ the brane current algebra in order to define the generalized Cartan curvature in (4.12), which is well-defined up to boundary terms. In contrast [77] is able to make use of the generalized Lie derivative directly because they are using the maximal extension to $E_{d+n(d+n)}$ for which the generalized Lie derivative closes without additional constraints.

B. Generalized Cartan connection

As an ansatz for the parametrization of the $\mathfrak{R}_1$ -connection $\theta \equiv \theta^{(1)}|_p: \mathfrak{R}_1[P]_p \rightarrow \mathfrak{I}_1$ consistent with our assumptions, we propose

$$\theta_{A_1}^{\mathcal{M}_1} = \begin{pmatrix} \delta_\alpha^\mu & 0 & 0 & 0 & \dots & 0 \\ \Omega_{A_1}^\mu & E_{A_1}^{M_1} & 0 & 0 & \dots & 0 \\ \Omega_{A_2}^{\alpha, \mu} & \delta_\nu^\alpha \Omega_{A_2}^{M_1, \nu} & \delta_\mu^\alpha E_{A_2}^{M_2} & 0 & \dots & 0 \\ \Omega_{A_3}^{\alpha_1 \alpha_2, \mu} & \delta_{\nu_1 \nu_2}^{\alpha_1 \alpha_2} \Omega_{A_3}^{M_1, \nu_1 \nu_2} & \delta_{\mu \nu}^{\alpha_1 \alpha_2} \Omega_{A_3}^{M_2, \nu} & \delta_{\mu_1 \mu_2}^{\alpha_1 \alpha_2} E_{A_3}^{M_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}. \quad (4.13)$$

⁹The fact that the $\theta^{(p)}$ leave the η -symbols invariant, guarantees this form of the current algebra.

This form of the generalized Cartan connection extends both the *megavielbein* from [110] and the generalized Cartan connection in $O(d, d)$ generalized geometry [76].¹⁰ Its components organize themselves into a tower of *higher connections* Ω , beginning with the spin connection $\Omega_{A_1}{}^\mu$. In particular, for $q < p$, we have

$$\delta\theta_{A_p, \mu_1 \dots \mu_{q-1}}^{(1)\alpha_1 \dots \alpha_{p-1}, M_q} = \delta_{\mu_1 \dots \mu_{q-1} \nu_1 \dots \nu_{p-q}}^{\alpha_1 \dots \alpha_{p-1}} \Omega_{A_p}^{M_q, \nu_1 \dots \nu_{p-q}}, \quad (4.14)$$

and a dependence on the generalized frames $E_{A_p}{}^{M_p}$ of the $\mathcal{R}_p$ -bundles of $\mathcal{G}$ -generalized geometry. Similar to the Cartan connection, the physically relevant input is given by $E_{A_1}{}^{M_1}$, while all the other generalized frames can be derived from this quantity. They represent a natural generalization of the solder form familiar from ordinary Cartan geometry.

1. Linearized constraints from compatibility with η

Not all the connection components are independent. There are relations between the Ω 's because the connections $\theta^{(p)}$ are required to preserve the graded structure, and in particular the η -symbols

$$\eta^{C_2}{}_{\mathcal{A}_1 \mathcal{B}_1} = \theta_{\mathcal{A}_1}^{(1) \mathcal{M}_1} \theta_{\mathcal{B}_1}^{(1) \mathcal{N}_1} \theta_{\mathcal{C}_2}^{(2) C_2} \eta^{\mathcal{L}_2}{}_{\mathcal{M}_1 \mathcal{N}_1}. \quad (4.15)$$

As such, the generalized Cartan connections give rise to a Lie group. But working with this Lie group directly is

cumbersome, one could choose different parametrizations and a preferred one is not immediately obvious. Instead, we focus here on the underlying Lie algebra. It arises when we work at the *linearized level* in Ω . Moreover, for the sake of brevity of the final expressions, we work in the metric formalism and assume $E_{A_p}{}^{M_p} = \delta_{A_p}^{M_p}$ for the generalized frame fields of $\mathcal{G}$ -generalized geometry. After performing the variation, we are left with

$$0 = \delta\theta_{\mathcal{A}_1}^{(1) \mathcal{M}_1} \eta^{C_2}{}_{\mathcal{M}_1 \mathcal{B}_1} + \delta\theta_{\mathcal{B}_1}^{(1) \mathcal{M}_1} \eta^{C_2}{}_{\mathcal{A}_1 \mathcal{M}_1} + \delta\theta_{\mathcal{M}_2}^{(2) C_2} \eta^{\mathcal{M}_2}{}_{\mathcal{A}_1 \mathcal{B}_1} \quad (4.16)$$

to solve. For the sake of brevity, one might ignore changes of $E_{A_p}{}^{M_p}$ to circumvent the Lie algebra of the duality group $\mathcal{G}$ and instead focus on the generators for the connections that arise as generalizations of the affine connection. These assumptions are sufficient to analyze the independent components of the generalized Cartan connection, and to gain insights into the construction and consistency of generalized torsions and curvatures. As a consequence of the compatibility condition (4.16), we conclude:

- (i) The components of the connections $\theta^{(p)}$ for $p > 1$ are fixed by the components of $\theta \equiv \theta^{(1)}$. Assuming $\mathcal{A}_1 = \alpha_1 \dots \alpha_{p-1}$ for $p > 1$, $\mathcal{B}_1 = \beta$ in (4.16) can be used to iteratively determine $\theta^{(p)}$. For example, for $\theta^{(2)}$ one finds the identity for $q < p$

$$\delta\theta_{A_p, \mu_1 \dots \mu_{q-2}}^{(2)\alpha_1 \dots \alpha_{p-2}, M_q} = (-1)^{(p-q)} \Omega_{A_p}^{M_q, \nu_1 \dots \nu_{p-q}} \delta_{\mu_1 \dots \mu_{q-2} \nu_1 \dots \nu_{p-q}}^{\alpha_1 \dots \alpha_{p-2}}. \quad (4.17)$$

- (ii) All independent components of the Cartan connection are contained in the first column of (4.13), which is formed by $\delta\theta_{\mathcal{A}_1}^\mu$. There are further restrictions which eventually only keep

$$\Omega_{M_1}^\mu, \rho_{M_2}^{\mu_1 \mu_2}, \rho_{M_3}^{\mu_1 \mu_2 \mu_3}, \dots \quad (4.18)$$

with

$$\rho_{A_p}^{\beta \alpha_1 \dots \alpha_{p-1}} := \Omega_{A_p}^{[\beta, \alpha_1 \dots \alpha_{p-1}]} = \Omega_{A_p}^{\beta, \alpha_1 \dots \alpha_{p-1}} \quad (4.19)$$

as independent components. To understand the derivation of this tensor structure, put $\mathcal{A}_1 =$

$\alpha_1 \dots \alpha_{p-1}$ for $p > 1$, $\mathcal{B}_1 = \beta$ in (4.15), and project

¹⁰These special cases arise from truncating the differential graded Lie algebra; the generalized Cartan connection of [76] corresponds to the case where the tower terminates at $\mathcal{R}_2 = \mathbf{1}$, while the ordinary Cartan connection is recovered when the tower terminates already at $\mathcal{R}_1$.

onto the $\mathcal{C}_2 = \gamma_1 \dots \gamma_{q-2}$ -component for $q > 1$, then for $(p, q) = (2, 2)$,

$$\Omega_{A_2}^{\beta, \alpha} \delta_{B_2}^{C_2} + \Omega_{B_2}^{\alpha, \beta} \delta_{A_2}^{C_2} = \eta^{D_4}{}_{A_2 B_2} \Omega_{D_4}^{C_2 \alpha \beta}. \quad (4.20)$$

After contraction with of B_2 and C_2 , for example, one notices that

$$\Omega_{A_2}^{(\alpha, \beta)} = 0, \quad \Omega_{A_2}^{[\alpha, \beta]} \sim \eta^{D_4}{}_{A_2 B_2} \Omega_{D_4}^{B_2 \alpha \beta} \quad (4.21)$$

showing that $\Omega_{A_2}^{\alpha, \beta}$ only has a skew-symmetric contribution. Moreover, we see that parts of the $\Omega_{D_4}^{A_2}$ -component are fixed by this choice.

For $p > q = 2$ we have that

$$\Omega_{A_p}^{\beta, \alpha_1 \dots \alpha_{p-1}} \delta_{B_2}^{C_2} = (-1)^p \eta^{D_{p+2}}{}_{A_p B_2} \Omega_{D_{p+2}}^{C_2, \alpha_1 \dots \alpha_{p-1} \beta} \quad (4.22)$$

showing that $\Omega_{A_p}^{\beta, \alpha_1 \dots \alpha_{p-1}} = \Omega_{A_p}^{[\beta, \alpha_1 \dots \alpha_{p-1}]}$,

- (iii) Components in the remaining columns are connected to fundamental ones (4.18) in the first column via identities like

$$\begin{aligned}\Omega_{D_{p+1}}^{C_{p+1}\nu}\eta^{D_{p+1}A_pB_1} &= \Omega_{A_p}^{D_{p+1}\nu}\eta^{C_pD_{p-1}B_1} - (-1)^p\Omega_{B_1}^\nu\delta_{A_p}^{C_p}, \\ \Omega_{D_{p+1}}^{C_r\nu_1\cdots\nu_{p-r+1}}\eta^{D_{p+1}A_pB_1} &= \Omega_{A_p}^{D_{r-1}\nu_1\cdots\nu_{p-r+1}}\eta^{C_rD_{r-1}B_1}, \quad 2 \leq r < p.\end{aligned}\quad (4.23)$$

This essentially determines the first subdiagonal component of the Cartan connection, expressed in terms of $\Omega_{A_1}^{\nu_1}$;

- (iv) These results suggest that the components of the generalized Cartan connection arise from the vector space $\mathfrak{R}_0$ dual to the one defined in (2.23). We do not see the fact that $\mathcal{R}_0 = \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the duality group, because we keep the physical frame fixed. Hence, from here on, we take the Cartan connection to be generated by $\mathfrak{R}_0$. This is in agreement with the known results for the duality group $O(d, d)$ [75,76] and the alternative approach for exceptional duality groups presented in [77]; and
- (v) Nonlinearity in the connection will *not* impact the counting of the degrees of freedom of the generalized Cartan connection θ . The only difference to what we have seen here is that contractions and $\bullet$ -products of the independent components will appear.

2. Solving the constraints

Still, we should check if the constraints (4.23) indeed admit solutions for all columns beyond the first. Taking a look at the first relation there, we find the solution,

$$\Omega_{L_{p+1}}^{K_p\mu} = -(-1)^p D_{L_{p+1}}^{K_p M_1} \Omega_{M_1}^\mu, \quad (4.24)$$

assuming that

$$(D_{E_{p+1}}^{C_p B_1} \eta^{E_{p+1} A_p D_1} + \eta^{C_p E_{p-1} D_1} D_{A_p}^{E_{p-1} B_1}) \Omega_{B_1}^\nu = \Omega_{D_1}^\nu \delta_{A_p}^{C_p} \quad (4.25)$$

holds. Remarkably, this relation is nothing else than the $\eta - D$ identity (2.8) where the partial derivative is replaced by $\Omega_{A_1}^\nu$. It constrains the $\mathcal{R}_1^*$ part of the generalized connection, while the $\mathfrak{h}$ factor is a mere spectator. But due to the $\mathcal{G}$ -covariance of (2.8), either this full $\mathcal{R}_1^*$ irreducible representation is compatible with it or nothing at all. Hence, we conclude that (4.25) holds generally.

Assume for a moment that $\rho_{M_p}^{\mu_1\cdots\mu_p} = 0$ and only $\Omega_{M_1}^\mu$ contributes to $\delta\theta_{A_1}^{\mathcal{M}_1}$ through

$$\delta\theta_{A_1}^{\mathcal{M}_1} = \Omega_{B_1}^\nu (R_\nu^{B_1})_{A_1}^{\mathcal{M}_1} \quad (4.26)$$

with the generators $R_\nu^{B_1}$. As matrices, the latter only have entries on the first subdiagonal, namely,

$$\begin{aligned}(R_\nu^{B_1})_{A_1}^\mu &= \delta_\nu^\mu \delta_{A_1}^{B_1}, \quad \text{and} \\ (R_\nu^{B_1})_{A_{p+1}}^{\alpha_1\cdots\alpha_p M_p} &= \delta_{\mu_1\cdots\mu_{p-1}\nu}^{\alpha_1\cdots\alpha_p} D_{A_{p+1}}^{M_p B_1} \quad \text{for } p \geq 1.\end{aligned}\quad (4.27)$$

From this set of generators, one can derive the generators which have to be contracted with the various ρ 's we encountered. The first of them is

$$R_{\nu_1\nu_2}^{B_2} := \eta^{B_2 C_1 D_1} [R_{\nu_1}^{C_1}, R_{\nu_2}^{D_1}], \quad (4.28)$$

which can be derived from (2.24). It comes with only the second subdiagonal populated by

$$(R_{\nu_1\nu_2}^{B_2})_{A_2}^{\alpha\mu} = \delta_{\nu_1\nu_2}^{\alpha\mu} \delta_{A_2}^{B_2}, \quad \text{and} \quad (R_{\nu_1\nu_2}^{B_2})_{A_{p+2}}^{\alpha_1\cdots\alpha_{p+1} M_p} = \delta_{\mu_1\cdots\mu_{p-1}\nu_1\nu_2}^{\alpha_1\cdots\alpha_{p+1}} \eta^{B_2 C_1 D_1} D_{A_{p+2}}^{E_{p+1} C_1} D_{E_{p+1}}^{M_p D_1} \quad \text{for } p \geq 1. \quad (4.29)$$

This pattern can be continued recursively to obtain the explicit form of the higher-order generators,

$$R_{\mu_1\cdots\mu_p}^{B_p} = \eta^{B_p C_{p-1} D_1} [R_{[\mu_1\cdots\mu_{p-1}}^{C_{p-1}}, R_{\mu_p]}^{D_1}], \quad (4.30)$$

where the normalization factor is fixed by first row of the resulting matrix as

$$(R_{\nu_1\cdots\nu_p}^{B_p})_{A_p}^{\alpha_1\cdots\alpha_{p-1}\mu} = \delta_{\nu_1\cdots\nu_p}^{\alpha_1\cdots\alpha_{p-1}\mu} \delta_{A_p}^{B_p}. \quad (4.31)$$

¹¹From Eq. (4.15), these can be obtained by the choice $\mathcal{A}_1 = \alpha_1\cdots\alpha_{p-1}$ for $p > 1$ and $\mathcal{B}_1 = B_1$. Similar identities follow for other choices of $\mathcal{B}_1$.

As argued in Sec. II B these generators form the Lie algebra $\tilde{\mathfrak{R}}_0$ if complemented with the generators of $\mathcal{G}$. At this point, there is no guarantee that $\tilde{\mathfrak{R}}_0$ is the maximal subalgebra, under the assumptions of the generalized Cartan connection, that gives rise to (4.15).

Eventually, they find their use in writing

$$\delta\theta = \Omega_{A_1}^\mu R_\mu^{A_1} + \sum_{p \geq 2} \frac{1}{p!} \rho_{A_p}^{\mu_1 \dots \mu_p} R_{\mu_1 \dots \mu_p}^{A_p}. \quad (4.32)$$

To continue with the computation of curvatures in the next section, the component

$$\Omega_{L_{p+1}}^{K_1 \mu_1 \dots \mu_p} = -(-1)^p D_{L_{p+1}}^{M_p K_1} \rho_{M_p}^{\mu_1 \dots \mu_p} \quad (4.33)$$

is needed. It can be either obtained by getting the relevant components of the generators $R_{\mu_1 \dots \mu_p}^{A_p}$ and combining them with (4.32) or by employing the conditions (4.23).

C. A hierarchy of curvatures

As the generalized Cartan curvature, we understand the structure functions $\Theta_{A_1 B_1}^{C_1}$ in Eq. (4.12). Here, we calculated these components using the brane current algebra. Alternatively, one could obtain the same result using the extended generalized Lie derivative (2.39); the generalized Cartan connection θ_{A_1} can be interpreted as a $\mathfrak{I}_1$ -valued section of the $\mathfrak{R}_1^*$ -bundle. But remember that due to (2.37), the extended Lie derivative only closes for sections $\theta_{A_1} \in \mathfrak{h} \oplus \mathcal{R}_1$. However, as we will propose below, this is enough because the independent components of the curvature and torsion are completely contained in:

- (i) $\{\theta_\alpha(\sigma), \theta_{B_p}^{\beta_1 \dots \beta_{p-1}}(\sigma')\}$, which captures the model algebra, and
- (ii) $\{\theta_{A_1}(\sigma), \theta_{B_p}^{\beta_1 \dots \beta_{p-1}}(\sigma')\}$, which unifies the generalized torsions and curvatures of the physical space.

Let us have a look at the latter. Using the solution (4.24), its components can be written as

$$\begin{aligned} \{\theta_{A_1}(\sigma), \theta_{B_p}^{\beta_1 \dots \beta_{p-1}}(\sigma')\} &= (-1)^p (\eta^{C_{p+1}}_{A_1 B_p} \theta_{C_{p+1}}^{\beta_1 \dots \beta_{p-1}}(\sigma') \wedge d'\delta(\sigma - \sigma') + f_{\alpha A_1}^{D_1} \eta^{C_{p+1}}_{D_1 B_p} \theta_{C_{p+1}}^{\beta_1 \dots \beta_{p-1} \alpha}(\sigma) \delta(\sigma - \sigma')) \\ &+ \frac{1}{(p-1)!} \Theta_{A_1 B_p}^{C_p} \beta_{\gamma_1 \dots \gamma_{p-1}}^{\beta_1 \dots \beta_{p-1}}(\sigma) \theta_{C_p}^{\gamma_1 \dots \gamma_{p-1}}(\sigma) \delta(\sigma - \sigma') \quad (\mathcal{R}_p\text{-torsion}) \\ &+ \frac{1}{(p-2)!} \Theta_{A_1 B_p}^{C_{p-1}} \beta_{\gamma_1 \dots \gamma_{p-2}}^{\beta_1 \dots \beta_{p-1}}(\sigma) \theta_{C_{p-1}}^{\gamma_1 \dots \gamma_{p-2}}(\sigma) \delta(\sigma - \sigma') \quad (\mathcal{R}_1\text{-curvature}) \\ &\vdots \\ &+ \Theta_{A_1 B_p}^{C_1} \beta_{\gamma_1 \dots \gamma_{p-1}}^{\beta_1 \dots \beta_{p-1}}(\sigma) \theta_{C_1}(\sigma) \delta(\sigma - \sigma') \quad (\mathcal{R}_{p-1}\text{-curvature}) \\ &+ \Theta_{A_1 B_p}^{\beta_1 \dots \beta_{p-1} \gamma}(\sigma) \theta_\gamma(\sigma) \delta(\sigma - \sigma') \quad (\mathcal{R}_p\text{-curvature}) \end{aligned} \quad (4.34)$$

From this expression, we extract several fundamental quantities. All of them can be expressed in terms of the independent components of the generalized Cartan connection. We take a generalized coordinate basis with the generalized frame $E_{A_1}^{M_1} = \delta_{A_1}^{M_1}$ to obtain

(i) $\mathcal{R}_p$ -torsion:

$$\begin{aligned} \Theta_{A_1 B_p}^{C_p} \beta_{\gamma_1 \dots \gamma_{p-1}}^{\beta_1 \dots \beta_{p-1}} &= f_{\alpha B_p}^{C_p} \Omega_{A_1}^\alpha \delta_{\gamma_1 \dots \gamma_{p-1}}^{\beta_1 \dots \beta_{p-1}} + f_{\alpha A_1}^{D_1} \eta^{L_{p+1}}_{D_1 B_p} D_{L_{p+1}}^{C_p E_1} \Omega_{E_1}^\alpha \delta_{\gamma_1 \dots \gamma_{p-1}}^{\beta_1 \dots \beta_{p-1}} \\ &- (p-1) f_{[\gamma_{p-1} A_1}^{D_1} \delta_{B_p}^{C_p} \Omega_{D_1}^\alpha \delta_{\gamma_1 \dots \gamma_{p-2}}^{\beta_1 \dots \beta_{p-1}}] \alpha}. \end{aligned} \quad (4.35)$$

We appreciate that this result nicely generalizes the standard expression for $p = 1$

$$\Theta_{A_1 B_1}^{C_1} = -2f_{\alpha[A_1}^{C_1} \Omega_{B_1]}^\alpha + f_{\alpha A_1}^{D_1} \eta^{L_2}_{D_1 B_1} D_{L_2}^{C_1 E_1} \Omega_{E_1}^\alpha$$

of the torsion in exceptional generalized geometry [74]. As in [76], we naturally identify $f_{\alpha B_1}^{C_1} \Omega_{A_1}^\alpha$ with a connection $\Gamma_{A_1 B_1}^{C_1}$ on the generalized tangent bundle $\mathcal{R}_1[M]$. Therefore, we call it $\mathcal{R}_p$ -torsion, but in the frame formalism one might also understand it as a curvature for the generalized frame;

(ii) $\mathcal{R}_p$ -curvature:

$$p = 1: \Theta_{A_1 B_1}^\beta = -2\partial_{[A_1} \Omega_{B_1]}^\beta + f_{\alpha A_1} C_1 \eta^{D_2}{}_{C_1 B_1} \rho_{D_2}^{\beta\alpha}, \quad (4.36)$$

$$p > 1: \Theta_{A_1 B_p}^{\beta_1 \dots \beta_p} = -\partial_{A_1} \rho_{B_p}^{\beta_1 \dots \beta_p} + f_{\alpha A_1} C_1 \eta^{D_{p+1}}{}_{C_1 B_p} \rho_{D_{p+1}}^{\beta_1 \dots \beta_p \alpha}. \quad (4.37)$$

These formulas reduce to those of [77] after linearization. As suggested in (4.34), we claim that the $\mathcal{R}_q$ -curvatures for $q < p$ are also contained in the generalized Cartan curvature. Let us exemplify this by looking at the $\mathcal{R}_1$ -curvature. The relevant component takes the form

$$\Theta_{A_1 B_p}^{C_{p-1} \beta_1 \dots \beta_{p-1} \gamma_1 \dots \gamma_{p-2}} = -(-1)^p D_{B_p}^{C_{p-1} D_1} \Theta_{A_1 D_1}^{\alpha \delta \beta_1 \dots \beta_{p-1} \gamma_1 \dots \gamma_{p-2} \alpha}. \quad (4.38)$$

Hence, under our assumptions many components of the generalized Cartan curvature are described by the $\mathcal{R}_1$ -curvature.

Although we do not present a general proof that all components of the generalized Cartan curvature are fixed this way, it seems obvious that at least all dynamical contributions (which contain a derivative of the Cartan connection) are. Derivatives can only carry $\mathcal{R}_1$ -indices. Moreover, we cover all independent contributions to the Cartan connection.

As for the connection, these are only the *linearized* expressions. At the nonlinear level, all possible contractions of the spin connections Ω_{M_1}, ρ_{M_p} with η -symbols and structure constants $f_{\alpha\beta}^\gamma, f_{\alpha M_p}^{N_p}$ are expected to contribute. Nevertheless, already at the linear level, the key characteristic of these connections—their hierarchical structure—and their interplay with the constraints of the generalized Cartan connection become evident.

For the generalized Cartan connection, it was possible to organize all its components into the Lie algebra $\mathfrak{R}_0$. One might ask if something along those lines works here, too. Looking at the representations which contribute to (4.35), (4.36), and (4.37), it is suggestive to define the vector space

$$\mathfrak{R}_{-1} = \mathcal{R}_{-1} \oplus \bigoplus_{q \geq 1} (\mathcal{R}_1 \otimes \mathcal{R}_q \otimes \bigwedge^q \mathfrak{h}^*), \quad (4.39)$$

with $\mathcal{R}_{-1}$ denoting the representation of the embedding tensor for the duality group $\mathcal{G}$. It hosts the $\mathcal{R}_p$ -torsion and all the $\mathcal{R}_p$ -curvatures. While the $\mathfrak{h}$ factors are very similar to the other $\mathfrak{R}_p$ spaces we have already encountered, the respective representations of the duality group $\mathcal{G}$ do not fit into the tensor hierarchy any more. A notable exception is the duality group $O(d, d)$ where one finds

$$\mathfrak{R}_{-1}^{O(d,d)} = \mathcal{R}_{-1} \oplus \bigoplus_{q \geq 1} (\mathcal{R}_{q-1} \otimes \bigwedge^q \mathfrak{h}^*). \quad (4.40)$$

1. An ambiguity

The form of the Cartan curvature, as presented in (4.35), (4.36), and (4.37) is not unique. Given our definition via the current algebra (4.34), additional redundant components of the curvature arise, which do not appear in the desired form. Take for example¹²

$$\Theta_{E_2 C_1}^{\gamma, \alpha} = -\partial_{C_1} \rho_{E_2}^{\gamma \alpha} - f_{\beta E_2}^{D_2} \eta^{H_3}{}_{D_2 C_1} \rho_{H_3}^{\alpha \gamma \beta} + \frac{1}{2} f_{\delta \epsilon}^{\gamma} \eta^{H_3}{}_{E_2 C_1} \rho_{H_3}^{\alpha \delta \epsilon}, \quad (4.41)$$

in the conventions of (3.18). It differs from the expected (4.37) by world-volume boundary terms or “total derivative” terms.¹³ Up to a total sign, one can conveniently put both in the form,

$$p = 1: \Theta_{A_1 B_1}^\beta = -2\partial_{[A_1} \Omega_{B_1]}^\beta + f_{\alpha A_1} C_1 \eta^{D_2}{}_{C_1 B_1} \rho_{D_2}^{\beta\alpha} + \alpha \eta^{C_2}{}_{A_1 B_1} \left(f_{\alpha C_2}^{D_2} \rho_{D_2}^{\beta\alpha} + \frac{1}{2} f_{\gamma \delta}^{\beta} \rho_{C_2}^{\gamma \delta} \right), \quad (4.42)$$

$$p > 1: \Theta_{A_1 B_p}^{\beta_1 \dots \beta_{p-1} \beta} = -\partial_{A_1} \rho_{B_p}^{\beta_1 \dots \beta_{p-1} \beta} + f_{\alpha A_1} C_1 \eta^{D_{p+1}}{}_{C_1 B_p} \rho_{D_{p+1}}^{\beta_1 \dots \beta_{p-1} \beta \alpha} + \alpha \eta^{C_{p+1}}{}_{A_1 B_p} \left(f_{\alpha C_{p+1}}^{D_{p+1}} \rho_{D_{p+1}}^{\beta_1 \dots \beta_{p-1} \beta \alpha} + (-1)^{p+1} \frac{1}{2} f_{\gamma \delta}^{\beta} \rho_{C_{p+1}}^{\gamma \delta \beta_1 \dots \beta_{p-1}} \right). \quad (4.43)$$

In general, any real number α should give a reasonable definition of a curvature. The form presented in (4.36) and (4.37) (for $\alpha = 0$) is chosen such that it reproduces the expressions from [77] and that the curvatures fit nicely into $\mathfrak{R}_{-1}$ defined above. One interpretation of the difference is, that we could use (4.37) together with (4.41) (corresponding to $\alpha = -1$) as the

¹²This is defined as the coefficient in $\{\theta_{E_2}^\gamma(\sigma), \theta_{C_1}(\sigma')\} = \dots + \Theta_{E_2 C_1}^{\gamma, \alpha}(\sigma) \theta_\alpha(\sigma) \delta(\sigma - \sigma')$.

¹³Because up to world-volume boundary terms, $\{\theta_{A_1}(\sigma), \theta_{B_p}^{\beta_1 \dots \beta_{p-1}}(\sigma')\} = -\{\theta_{B_p}^{\beta_1 \dots \beta_{p-1}}(\sigma'), \theta_{A_1}(\sigma)\}$.

structure “functions” of a Leibniz algebroid which arises from a *Dorfman* bracket. Alternatively one could take (4.43) with $\alpha = -\frac{1}{2}$ to obtain a “*Courant* bracket” analog.

2. Bianchi identities

In analogy to the differential and algebraic Bianchi identities for torsion and curvature in Riemannian geometry, we expect to derive consistency conditions relating the different components of Θ . A complete collection of such identities is beyond the scope of this paper, so we content ourselves with a sketch of two ways of how to approach Bianchi identities for the curvatures, and what types of relations one can expect. Based on the ambiguity mentioned above, we have to decide whether we interpret the generalized Cartan curvature as structure functions of the Courant or Dorfman bracket (resulting in a Leibniz algebra). In both cases, they are derived from the Jacobi identity of the current algebra. Let us demonstrate this with a few crucial examples. The calculations are similar to the ones presented in Appendix B 2 for the “flat” current algebra (i.e., the one that is not twisted by the generalized Cartan connection). An additional complication is that the Jacobi identity requires “generalized Cartan curvature in higher representations,” namely the structure constants in

$$\{\theta_{A_1}(\sigma), \theta_{B_2}^{(2)}(\sigma')\} \sim \dots + \Theta_{A_1 B_2}^{(2)} \theta_{C_2}^{(2)}(\sigma) \delta(\sigma - \sigma'), \quad (4.44)$$

where ... refer to the $d\delta$ -terms that are not included explicitly here. We will not present the technical details as they do not offer new insights but only additional complications in comparison to Appendix B 2. In principle,

the nontrivial¹⁴ part of the Jacobi identities can be decomposed into two parts; a Jacobi identity of the type

$$\{\theta_\alpha(\sigma_1), \{\theta_{A_1}(\sigma_2), \theta_{B_p}(\sigma_3)\}\} + c.p. = 0 \quad (4.45)$$

captures the fact that the curvatures are $\mathfrak{h}$ -tensors. They transform covariantly as

$$\nabla_\alpha T_{A_p \dots}^{\beta \dots} = f_{\alpha A_p}{}^{B_p} T_{B_p \dots}^\beta + f_{\gamma \alpha}{}^\beta T_{A_p \dots}^{\gamma \dots}. \quad (4.46)$$

Moreover, this Jacobi identity also identifies components of Θ with the ones of Θ ,

$$(p-1) \delta_\alpha^{\beta_{p-1}} \Theta_{A_1 B_p, \gamma_1 \dots \gamma_{j-2}}^{(2) \beta_1 \dots \beta_{p-2}, C_j} = \Theta_{A_1 B_p, \gamma_1 \dots \gamma_{j-2} \alpha}^{(1) \beta_1 \dots \beta_{p-1}, C_j}, \quad (4.47)$$

for $2 \leq j \leq p$. For example, for $j = p = 2$ we get

$$\delta_\gamma^{\beta} \Theta_{A_1 B_2}^{(2) C_2} = \Theta_{A_1 B_2, \gamma}^{(1) \beta, C_2}. \quad (4.48)$$

For the differential Bianchi identities, we obtain different versions, or interpretations. Depending on α in (4.42) and (4.43), we are dealing with

- (i) “*Courant bracket*”-type *Bianchi identity*: Looking at the $\mathfrak{h}$ -component of the Jacobi identity of type

$$\{\theta_{A_1}(\sigma_1), \{\theta_{B_1}(\sigma_2), \theta_{C_p}(\sigma_3)\}\} + c.p. = 0 \quad (4.49)$$

for $p \geq 2$, one finds at the linearized level in connections the Bianchi identity

$$\begin{aligned} 0 = & \nabla_{[A_1} \Theta_{B_1] C_p}^{\gamma_1 \dots \gamma_{p-1}, \delta} + f_{\alpha [A_1}{}^{F_1} (\Theta_{E_2 C_p}^{\alpha \gamma_1 \dots \gamma_{p-1}, \delta} \eta^{E_2}{}_{F_1 B_1}] + \Theta_{[B_1] E_{p+1}}^{\alpha \gamma_1 \dots \gamma_{p-1}, \delta} \eta^{E_{p+1}}{}_{F_1 C_p}) \\ & + f_{\alpha [A_1}{}^{F_1} (f_{\beta C_p}{}^{G_p} \eta^{E_{p+1}}{}_{F_1 G_p} \eta^{D_{p+2}}{}_{[B_1] E_{p+1}} + f_{\beta [B_1]}{}^{G_1} \eta^{E_{p+1}}{}_{F_1 C_p} \eta^{D_{p+2}}{}_{G_1 E_{p+1}}) \rho_{D_{p+2}}^{\alpha \beta \gamma_1 \dots \gamma_{p-1}, \delta} \\ & + \left(\frac{p-1}{2} \right) f_{\alpha \beta [\gamma_1} \left(f_{\epsilon [A_1}{}^{F_1} \eta^{E_{p+1}}{}_{F_1 C_p} \eta^{D_{p+2}}{}_{[B_1] E_{p+1}} + \frac{1}{2} f_{\epsilon F_{p+2}}{}^{D_{p+2}} \eta^{F_{p+2}}{}_{C_p E_2} \eta^{E_2}{}_{A_1 B_1} \right) \rho_{D_{p+2}}^{\alpha \beta [\gamma_2 \dots \gamma_{p-1}], \delta \epsilon}, \end{aligned} \quad (4.50)$$

where $\nabla_{A_1} = \partial_{A_1} + \Omega_{A_1}{}^\mu \nabla_\mu$.¹⁵ Note that this relation contains naked connection components which are not part of the covariant derivative. They originate from derivatives of δ -functions in the brane current algebra. For $p = 1$, (4.50) becomes

$$\nabla_{[A_1} \Theta_{B_1 C_1]}^\gamma - 2 f_{\alpha [A_1}{}^{F_1} (\eta^{E_2}{}_{F_1 B_1} \Theta_{C_1] E_2}^{\alpha \gamma} + f_{\beta B_1}{}^{G_1} \eta^{E_2}{}_{F_1 G_1} \eta^{D_3}{}_{[C_1] E_2} \rho_{D_3}^{\alpha \beta \gamma}) = 0, \quad (4.51)$$

and, as expected, it can be checked that this identity is satisfied by the $\mathcal{R}_1$ -curvature (4.36). Usually, naked connections are not expected in the consistency condition for the curvature. Therefore, our result here gives rise to a new interpretation of the tensor hierarchy of connections: the $\mathcal{R}_1$ connection is introduced to correct the noncovariance of the partial derivative. It has itself a curvature, which is noncovariant unless one introduces a higher $\mathcal{R}_2$ connection. Moreover, as shown in [76], the $\mathcal{R}_2$ connection takes the role of a higher torsion. Now, the Bianchi identity of the $\mathcal{R}_1$ curvature is also corrected by the

¹⁴Other parts give rise to the Jacobi identity of $\mathfrak{h}$ or the identity (2.14).

¹⁵Of course, in the linearized framework the connection part does not contribute when acting on physical fields, $\nabla_{A_1} \Theta = \partial_{A_1} \Theta$.

introduction of an $\mathcal{R}_3$ connection (which also corrects the curvature of the $\mathcal{R}_2$ connection). As a consistency check, let us note that the last term does not exist for $\mathcal{G} = \mathcal{O}(d, d)$ because its tensor hierarchy ends with $\mathcal{R}_2$. Also in this case, there is no ambiguity in the definition of the generalized curvature and torsion because Courant and Dorfman brackets give the same results. From this Jacobi identity we also have the following identity relating higher curvatures to lower ones (similar to the above case) for $1 \leq j \leq p$:

$$\Theta_{A_1 E_{p+1}, \delta_{j-1}}^{\gamma_1 \dots \gamma_{p-1}, D_{j+1}} \eta^{E_{p+1}}_{B_1 C_p} = (-1)^j \Theta_{A_1 C_p, \delta_{j-1}}^{\gamma_1 \dots \gamma_{p-1}, E_j} \eta^{D_{j+1}}_{B_1 E_j}. \quad (4.52)$$

There is a plethora of such Bianchi identities for the other components of the Jacobi identity. We will not consider them here explicitly for the sake of brevity; typically, they will correspond to algebraic identities, relating different components of the generalized Cartan curvature; and

(ii) “Dorfman bracket”-type Bianchi identity. In contrast to the above construction that only refers to curvatures of the form $\Theta_{A_1 B_p}^{\gamma_1 \dots \gamma_p}$, one can derive an identity that resembles more the Leibniz identity of a Dorfman bracket,

$$2\partial_{[A_1} \Theta_{B_1]C_1}^{\gamma} + \partial_{C_1} \Theta_{A_1 B_1}^{\gamma} + 2f_{\alpha[A_1}^{F_1} \Theta_{B_1]E_2}^{\alpha\gamma} \eta^{E_2}_{F_1 C_1} - f_{\alpha A_1}^{F_1} \Theta_{E_2 C_1}^{\gamma, \alpha} \eta^{E_2}_{F_1 B_1} = 0, \quad (4.53)$$

where $\Theta_{E_2 C_1}^{\gamma, \alpha}$ is defined in (4.41). It can be easily verified by substituting the known curvatures and using the relations (2.7) and (2.14). An advantage of this formulation in comparison to the “Courant bracket”-type is that no naked connections appear in the Bianchi identity. On the other hand, one needs to introduce a curvature (4.41) in addition to (4.37). Similarly, one can get the Bianchi identity for the $\mathcal{R}_2$ -curvature,

$$2\partial_{[A_1} \Theta_{B_1]C_2}^{\delta} + 2f_{\alpha[A_1}^{F_1} \Theta_{B_1]E_3}^{\gamma\delta\alpha} \eta^{E_3}_{F_1 C_2} - f_{\alpha A_1}^{F_1} \Theta_{E_2 C_2}^{\gamma, \delta, \alpha} \eta^{E_2}_{F_1 B_1} = 0, \quad (4.54)$$

with the algebraic curvature $^{16}\Theta_{E_2 C_2}^{\gamma, \delta, \alpha} = -f_{\beta E_2}^{D_2} \eta^{H_4}_{D_2 C_2} \rho_{H_4}^{\alpha\gamma\delta\beta} - \frac{1}{2} \eta^{H_4}_{E_2 C_2} f_{\rho\sigma}^{\gamma} \rho_{H_4}^{\delta\rho\sigma\alpha}$. The Bianchi identity for the general $\mathcal{R}_p$ -curvature for $p \geq 2$ is

$$2\partial_{[A_1} \Theta_{B_1]C_p}^{\gamma_1 \dots \gamma_p} + 2f_{\alpha[A_1}^{F_1} \Theta_{B_1]E_{p+1}}^{\gamma_1 \dots \gamma_p, \alpha} \eta^{E_{p+1}}_{F_1 C_p} - f_{\alpha A_1}^{F_1} \Theta_{E_2 C_p}^{\gamma_1 \dots \gamma_p, \alpha} \eta^{E_2}_{F_1 B_1} = 0 \quad (4.55)$$

with the curvature

$$\Theta_{E_2 C_p}^{\epsilon, \gamma_1 \dots \gamma_{p-1}, \alpha} = -f_{\beta E_2}^{D_2} \eta^{H_{p+2}}_{D_2 C_p} \rho_{H_{p+2}}^{\epsilon\gamma_1 \dots \gamma_{p-1}, \alpha\beta} - (-1)^p \frac{1}{2} f_{\rho\sigma}^{\epsilon} \eta^{H_{p+2}}_{E_2 C_p} \rho_{H_{p+2}}^{\rho\sigma\gamma_1 \dots \gamma_{p-1}, \alpha}, \quad (4.56)$$

which can be obtained from the current algebra.

V. OUTLOOK

In this article, we constructed a *minimal setting* for gauged exceptional geometry, i.e., an extended geometry that simultaneously geometrizes the action of generalized diffeomorphisms (and their associated tensor hierarchy) and an underlying compatible gauge symmetry. Building on this and generalizing previous work on extensions of Cartan geometry [75–77, 107], we proposed a setting for computing curvatures that are covariant under both symmetries. As a proof of concept, we derived in Sec. IV C expressions for a hierarchy of torsion and curvatures at the linearized level. In particular, we derived the $\mathcal{R}_p$ -curvatures (4.36), (4.37), and the $\mathcal{R}_p$ -torsion (4.35), which are the only independent components which contribute to the generalized Cartan curvature. While the linearized analysis

is sufficient in order to obtain insights into the hierarchical structure, with remarkable new features like Dorfman or Courant-like curvatures and Bianchi identities, it would be desirable to eventually find the full nonlinear expressions for the curvatures.

Extending our analysis further will eventually require a complete collection of explicit expressions of the tensor hierarchy—in the language of this article: all η - and D -symbols for all representations $\mathcal{R}_p$. Moreover, the present construction prominently uses the language of differential graded Lie algebras and thereby suggests an index-free version of generalized Cartan geometry which is more in line with the mathematical treatment of Cartan geometry in the language of differential forms. Since differential graded Lie algebras are strict L_∞ algebras, a natural extension of our framework would be to allow for generic L_∞ algebras with nontrivial higher brackets [111–113]. From supergravity it is known that in “realistic” tensor hierarchies of $E_{d(d)}$, the graded symmetry structure breaks down at high degree. This constitutes one of the major challenges of the *maximal* approach proposed in [77], where the symmetries

¹⁶Meaning that it does not contain derivatives at the linearized level.

$H \times \mathcal{G}$ are embedded into an exceptional group of suitable dimension. In practice, this puts severe restrictions on the dimension of the gauge group H . Our construction circumvents this problem, albeit at the price of losing the ability to capture generalized U-dualities as explored in [97]. On the other hand, as we are now able to deal with gauge groups H of arbitrary dimension, it is possible to address two further questions that are fundamental from a geometric point of view.

Firstly, we focussed solely on the construction of covariant curvatures. But in a physical setting requiring metric-compatibility of the connections (here in particular, what was called the $\mathcal{R}_1$ -connection $\Omega_{A_1}^\mu$) is central in order to identify the propagating degrees of freedom of the theory. As is well-known already in $O(d, d)$, vanishing torsion and compatibility with a generalized metric (a symmetric $\mathcal{R}_1^* \otimes \mathcal{R}_1^*$ -tensor) does not uniquely fix the generalized affine connection [66]. The same problem still exists for the generalized Cartan connection. One potential solution is to introduce a new gauge symmetry which shifts the nonfixed components as was suggested in [110]. But on its own, this new symmetry does not close; it must be extended, which—at least for $O(d, d)$ —leads to a graded, infinite-dimensional symmetry group H . Performing a similar construction in M-theory will only be possible with the tools developed here. Moreover, infinite-dimensional gauge groups H are central in the construction of higher-derivative corrections though the generalized Bergshoeff-de Roo mechanism [114,115] as has been shown recently in [116]. Secondly, our hierarchy of connections and curvatures strongly resembles the structures of higher-gauge theory [113,117–120]. Phrasing our setting in such a language seems feasible and desirable. In particular, our construction might hint at a gravitational analog of higher-gauge theory, consistent with the success of Cartan geometry in formulating gravity as a gauge theory.

Finally, the present construction made essential use of brane current algebras [27,29–32,37,38,41,109,121],

primarily as a computational tool. As suggested in [41], the gauged version of exceptional geometry developed here may find applications in the description of exotic branes (such as the Kaluza-Klein monopole) where one or more transverse directions carry gauge symmetries. First results in that direction were already proposed in [22]. Potentially, the world-volume theory of such exotic branes could be constructed from first principles using gauged exceptional geometry and the construction outlined in [41]. This would not only provide a systematic approach to classify and analyze their dynamics, but also offer a setting in which dualities (U-dualities and their generalizations [122,123]) can be studied explicitly.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: CLOSURE OF EXTENDED GENERALIZED LIE DERIVATIVE

Assuming closure of the ordinary generalized Lie derivative $\mathcal{L}_{U_1}$, we only need to check the extra terms involving the D -symbols and the structure constants $f_{\alpha\beta}^\gamma$ in (2.39). Looking only at the M_1 component of the left hand side (lhs) of (2.40), we get

$$\begin{aligned} ([\mathcal{L}_U, \mathcal{L}_V]\mathcal{W})^{M_1} &= \mathcal{L}_{U_1}(D_{L_2}^{M_1 N_1} \partial_{N_1} v^\nu (W_2)_{\nu}^{L_2}) - \mathcal{L}_{V_1}(D_{L_2}^{M_1 N_1} \partial_{N_1} u^\nu (W_2)_{\nu}^{L_2}) \\ &\quad + D_{L_2}^{M_1 N_1} \partial_{N_1} u^\mu (\mathcal{L}_{V_1}(W_2)_\mu^{L_2} + 2D_{K_3}^{L_2 P_1} \partial_{P_1} v^\nu (W_3)_{\mu\nu}^{K_3} + f_{\mu\nu}^\sigma v^\nu (W_2)_\sigma^{L_2}) \\ &\quad - D_{L_2}^{M_1 N_1} \partial_{N_1} v^\mu (\mathcal{L}_{U_1}(W_2)_\mu^{L_2} + 2D_{K_3}^{L_2 P_1} \partial_{P_1} u^\nu (W_3)_{\mu\nu}^{K_3} + f_{\mu\nu}^\sigma u^\nu (W_2)_\sigma^{L_2}) \\ &= D_{L_2}^{M_1 N_1} (W_2)_\mu^{L_2} (U_1^{P_1} \partial_{P_1} \partial_{N_1} v^\mu - \partial_{P_1} \partial_{N_1} u^\mu V_1^{P_1} + f_{\nu\sigma}^\mu \partial_{N_1} u^\nu v^\sigma - f_{\nu\sigma}^\mu u^\sigma \partial_{N_1} v^\nu) \\ &\quad + (Y^{M_1 P_1}{}_{R_1 S_1} D_{L_2}^{S_1 N_1} - Y^{K_2 P_1}{}_{L_2 R_1} D_{K_2}^{M_1 N_1}) (\partial_{P_1} U_1^{R_1} \partial_{N_1} v^\mu - \partial_{N_1} u^\mu \partial_{P_1} V_1^{R_1}) (W_2)_\mu^{L_2} \\ &\quad + 2D_{K_3}^{L_2 P_1} D_{L_2}^{M_1 N_1} (\partial_{N_1} u^\mu \partial_{P_1} v^\nu (W_3)_{\mu\nu}^{K_3} - \partial_{P_1} u^\nu \partial_{N_1} v^\mu (W_3)_{\mu\nu}^{K_3}). \end{aligned}$$

Going from the first to the second equality, we expanded the generalized Lie derivatives and grouped similar terms. Using the skew-symmetry of $(W_3)_{\mu\nu}^{K_3}$, we observe that the last line combines to give

$$4D_{K_3}^{L_2 P_1} D_{L_2}^{M_1 N_1} \partial_{(N_1} u^\mu \partial_{P_1)} v^\nu (W_3)_{\mu\nu}^{K_3} = 0,$$

which vanishes thanks to the property (2.6). The terms involving the Y -tensor simplify using (2.8) and (2.36) to

$$\begin{aligned}
& (Y^{M_1 P_1}_{R_1 S_1} D_{L_2}^{S_1 N_1} - Y^{K_2 P_1}_{L_2 R_1} D_{K_2}^{M_1 N_1})(\partial_{P_1} U_1^{R_1} \partial_{N_1} v^\mu - \partial_{N_1} u^\mu \partial_{P_1} V_1^{R_1})(W_2)_{\mu}^{L_2} \\
& \stackrel{(2.8)}{=} (Y^{M_1 P_1}_{R_1 S_1} D_{L_2}^{S_1 N_1} + Y^{M_1 N_1}_{R_1 S_1} D_{L_2}^{S_1 P_1})(\partial_{P_1} U_1^{R_1} \partial_{N_1} v^\mu - \partial_{N_1} u^\mu \partial_{P_1} V_1^{R_1})(W_2)_{\mu}^{L_2} \\
& \quad - D_{L_2}^{M_1 N_1}(\partial_{R_1} U_1^{R_1} \partial_{N_1} v^\mu - \partial_{N_1} u^\mu \partial_{R_1} V_1^{R_1})(W_2)_{\mu}^{L_2} \\
& \stackrel{(2.36)}{=} D_{L_2}^{M_1 N_1}(2\partial_{(R_1} U_1^{R_1} \partial_{N_1)} v^\mu - 2\partial_{(N_1} u^\mu \partial_{R_1)} V_1^{R_1} - \partial_{R_1} U_1^{R_1} \partial_{N_1} v^\mu + \partial_{N_1} u^\mu \partial_{R_1} V_1^{R_1})(W_2)_{\mu}^{L_2} \\
& = D_{L_2}^{M_1 N_1}(\partial_{N_1} U_1^{R_1} \partial_{R_1} v^\mu - \partial_{R_1} u^\mu \partial_{N_1} V_1^{R_1})(W_2)_{\mu}^{L_2}.
\end{aligned}$$

This leaves us with the following terms:

$$([\mathcal{L}_U, \mathcal{L}_V] \mathcal{W})^{M_1} = D_{L_2}^{M_1 N_1} (W_2)_{\mu}^{L_2} (U_1^{P_1} \partial_{P_1} \partial_{N_1} v^\mu - \partial_{P_1} \partial_{N_1} u^\mu V_1^{P_1} + f_{\nu\sigma}{}^\mu \partial_{N_1} u^\nu v^\sigma + f_{\nu\sigma}{}^\mu u^\nu \partial_{N_1} v^\sigma + \partial_{N_1} U_1^{P_1} \partial_{P_1} v^\mu - \partial_{P_1} u^\mu \partial_{N_1} V_1^{P_1}),$$

which is exactly the M_1 component of the right hand side (rhs) of (2.40), namely,

$$\begin{aligned}
(\mathcal{L}_{[u+U_1, v+V_1]_C} \mathcal{W})^{M_1} &= D_{L_2}^{M_1 N_1} \partial_{N_1} ([u, v]_{\mathfrak{h}}^\mu + \mathcal{L}_{U_1} v^\mu - \mathcal{L}_{V_1} u^\mu) (W_2)_{\mu}^{L_2} \\
&= D_{L_2}^{M_1 N_1} (W_2)_{\mu}^{L_2} (f_{\nu\sigma}{}^\mu \partial_{N_1} u^\nu v^\sigma + f_{\nu\sigma}{}^\mu u^\nu \partial_{N_1} v^\sigma + \partial_{N_1} U_1^{P_1} \partial_{P_1} v^\mu + U_1^{P_1} \partial_{P_1} \partial_{N_1} v^\mu \\
&\quad - \partial_{P_1} u^\mu \partial_{N_1} V_1^{P_1} - \partial_{P_1} \partial_{N_1} u^\mu V_1^{P_1}),
\end{aligned}$$

and matches the terms above from the lhs, establishing closure.

One can with a bit more work also show closure for all components of $\mathfrak{R}_1$. The lhs is something like,

$$\begin{aligned}
([\mathcal{L}_U, \mathcal{L}_V] \mathcal{W})_{\mu_1 \dots \mu_{p-1}}^{M_p} &= \mathcal{L}_{U_1} (\mathcal{L}_{V_1} (W_p)_{\mu_1 \dots \mu_{p-1}}^{M_p} + p D_{L_{p+1}}^{M_p N_1} \partial_{N_1} v^\nu (W_{p+1})_{\mu_1 \dots \mu_{p-1} \nu}^{L_{p+1}} - (p-1) v^\nu f_{\nu[\mu_{p-1}}{}^\lambda (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p} \\
&\quad + p D_{L_{p+1}}^{M_p P_1} \partial_{P_1} u^\rho (\mathcal{L}_{V_1} (W_{p+1})_{\mu_1 \dots \mu_{p-1} \rho}^{L_{p+1}} + (p+1) D_{Q_{p+2}}^{L_{p+1} N_1} \partial_{N_1} v^\nu (W_{p+2})_{\mu_1 \dots \mu_{p-1} \rho \nu}^{Q_{p+2}} \\
&\quad - p v^\nu f_{\nu[\rho}{}^\lambda (W_{p+1})_{\mu_1 \dots \mu_{p-1}] \lambda}^{L_{p+1}}) \\
&\quad - (p-1) u^\rho f_{\rho[\mu_{p-1}}{}^\lambda (\mathcal{L}_{V_1} (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p} + p D_{L_{p+1}}^{M_p N_1} \partial_{N_1} v^\nu (W_{p+1})_{\mu_1 \dots \mu_{p-2} \lambda \nu}^{L_{p+1}} \\
&\quad - (p-1) v^\nu f_{\nu \lambda}{}^\sigma (W_p)_{\mu_1 \dots \mu_{p-2} \sigma}^{M_p}) - ((u, U_1) \leftrightarrow (v, V_1)).
\end{aligned}$$

One can immediately see that the term involving W_{p+2} cancels out, thanks to the nilpotency of the D -symbols (2.6). The very first term above already closes, by assumption. Expanding the remaining terms, this becomes

$$\begin{aligned}
([\mathcal{L}_U, \mathcal{L}_V] \mathcal{W})_{\mu_1 \dots \mu_{p-1}}^{M_p} &= p D_{L_{p+1}}^{M_p N_1} [(U_1 \partial_{P_1} \partial_{N_1} v^\nu - \partial_{P_1} \partial_{N_1} u^\nu V_1^{P_1})(W_{p+1})_{\mu_1 \dots \mu_{p-1} \nu}^{L_{p+1}} \\
&\quad + (U_1^{P_1} \partial_{N_1} v^\nu - \partial_{N_1} u^\nu V_1^{P_1}) \partial_{P_1} (W_{p+1})_{\mu_1 \dots \mu_{p-1} \nu}^{L_{p+1}}] \\
&\quad + p D_{L_{p+1}}^{P_p N_1} Y^{M_p L_1}_{P_p Q_1} (\partial_{L_1} U_1^{Q_1} \partial_{N_1} v^\nu - \partial_{N_1} u^\nu \partial_{L_1} V_1^{Q_1})(W_{p+1})_{\mu_1 \dots \mu_{p-1} \nu}^{L_{p+1}} \\
&\quad - (p-1) f_{\nu[\mu_{p-1}}{}^\lambda (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p} (U_1^{P_1} \partial_{P_1} v^\nu - \partial_{P_1} u^\nu V_1^{P_1}) \\
&\quad - (p-1) f_{\nu[\mu_{p-1}}{}^\lambda \partial_{P_1} (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p} (U_1^{P_1} v^\nu - u^\nu V_1^{P_1}) \\
&\quad - (p-1) Y^{M_p N_1}_{P_p Q_1} f_{\nu[\mu_{p-1}}{}^\lambda (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p} (\partial_{N_1} U_1^{Q_1} v^\nu - u^\nu \partial_{N_1} V_1^{Q_1}) \\
&\quad + p D_{L_{p+1}}^{M_p P_1} [(\partial_{N_1} u^\nu V_1^{P_1} - U_1^{P_1} \partial_{N_1} v^\nu) \partial_{P_1} (W_{p+1})_{\mu_1 \dots \mu_{p-1} \nu}^{L_{p+1}} \\
&\quad + Y^{L_{p+1} P_1}_{R_{p+1} Q_1} (\partial_{N_1} u^\nu \partial_{P_1} V_1^{Q_1} - \partial_{P_1} U_1^{Q_1} \partial_{N_1} v^\nu) (W_{p+1})_{\mu_1 \dots \mu_{p-1} \nu}^{R_{p+1}} \\
&\quad - p f_{\nu[\rho}{}^\lambda (W_{p+1})_{\mu_1 \dots \mu_{p-1}] \lambda}^{L_{p+1}} (\partial_{N_1} u^\rho v^\nu - u^\rho \partial_{N_1} v^\rho)] \\
&\quad - (p-1) f_{\nu[\mu_{p-1}}{}^\lambda \partial_{N_1} (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p} (u^\nu V_1^{N_1} - U_1^{N_1} v^\nu) \\
&\quad - (p-1) Y^{M_p N_1}_{L_p P_1} (u^\rho \partial_{N_1} V_1^{P_1} - \partial_{N_1} U_1^{P_1} v^\rho) (W_p)_{\mu_1 \dots \mu_{p-2} \lambda}^{L_p} \\
&\quad - p(p-1) D_{L_{p+1}}^{M_p N_1} f_{\rho[\mu_{p-1}}{}^\lambda (W_{p+1})_{\mu_1 \dots \mu_{p-2}] \lambda \nu}^{L_{p+1}} (u^\rho \partial_{N_1} V_1^{P_1} - \partial_{N_1} U_1^{P_1} v^\rho) \\
&\quad + (p-1)^2 (u^\rho v^\nu - u^\nu v^\rho) f_{\rho[\mu_{p-1}}{}^\lambda f_{\nu \gamma}{}^\lambda (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p}.
\end{aligned}$$

This should be equal to the rhs, which evaluates to

$$\begin{aligned}
(\mathcal{L}_{[u+U_1, v+V_1]_C} \mathcal{W})_{\mu_1 \dots \mu_{p-1}}^{M_p} &= \mathcal{L}_{[U_1, V_1]_C} (W_p)_{\mu_1 \dots \mu_{p-1}}^{M_p} \\
&\quad + p D_{L_{p+1}}^{M_p N_1} \partial_{N_1} ([u, v]_{\mathfrak{h}}^\nu + \mathcal{L}_{U_1} v^\nu - \mathcal{L}_{V_1} u^\nu) (W_{p+1})_{\mu_1 \dots \mu_{p-1}}^{L_{p+1}} \\
&\quad - (p-1) ([u, v]_{\mathfrak{h}}^\nu + \mathcal{L}_{U_1} v^\nu - \mathcal{L}_{V_1} u^\nu) f_{\nu[\mu_{p-1}}^\lambda (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p} \\
&= p D_{L_{p+1}}^{M_p N_1} (W_{p+1})_{\mu_1 \dots \mu_{p-1}}^{L_{p+1}} (f_{\alpha\beta}^\nu (\partial_{N_1} u^\alpha v^\beta + u^\alpha \partial_{N_1} v^\beta) + U_1^{P_1} \partial_{N_1} \partial_{P_1} v^\nu \\
&\quad - \partial_{N_1} \partial_{P_1} u^\nu V_1^{P_1} + \partial_{N_1} U_1^{P_1} \partial_{P_1} v^\nu - \partial_{P_1} u^\nu \partial_{N_1} V_1^{P_1}) \\
&\quad - (p-1) (U_1^{P_1} \partial_{P_1} v^\nu - \partial_{P_1} u^\nu V_1^{P_1}) f_{\nu[\mu_{p-1}}^\lambda (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p} \\
&\quad - (p-1) f_{\alpha\beta}^\nu f_{\nu[\mu_{p-1}}^\lambda u^\alpha v^\beta (W_p)_{\mu_1 \dots \mu_{p-2}] \lambda}^{M_p}.
\end{aligned}$$

When comparing this to the lhs, we observe that all the terms simplify thanks to the identities (2.6), (2.8), (2.15), (2.16), and the Jacobi identity for $\mathfrak{h}$, establishing closure in much the same way as for the $\mathcal{R}_1$ component.

APPENDIX B: PROPERTIES OF THE EXTENDED CURRENT ALGEBRA

1. Derivation from underlying Poisson algebra

The fact that the above current algebra is indeed a Poisson algebra can be checked explicitly (below). But it can be understood from a relation to an underlying Poisson algebra,

$$\{s_\mu(\sigma), y^\nu(\sigma')\} = -\delta_\mu^\nu \delta(\sigma - \sigma'), \quad (\text{B1})$$

$$\{\tilde{t}_{M_p}(\sigma), \tilde{t}_{M_q}(\sigma')\} = -\eta^{L_{p+q}}_{M_p N_q} \tilde{t}_{L_{p+q}}(\sigma') \wedge d\delta(\sigma - \sigma'), \quad (\text{B2})$$

$$\begin{aligned}
\{s_\mu(\sigma), \tilde{t}_{M_p}(\sigma')\} &= 0 = \{y^\mu(\sigma), \tilde{t}_{M_p}(\sigma')\} \\
&= \{s_\mu(\sigma), s_\nu(\sigma')\} = \{y^\mu(\sigma), y^\nu(\sigma')\}.
\end{aligned} \quad (\text{B3})$$

The coordinates y^μ are canonically conjugate to the gauge generators s_α . This is obviously a Poisson algebra, assuming that the $\tilde{t}^{(p)}$ form a tensor hierarchy algebra [i.e., assuming that the η -symbols satisfy the graded Jacobi identity (2.7)]. As usual for the brane currents, it is assumed that $d\tilde{t}^{(p)} = 0$.

The quantities in (3.11) are related by

$$s_\alpha = e_\alpha^\mu(y) s_\mu, \quad t_{M_p}^{(p)} = e_{M_p}^{N_p}(y) \tilde{t}_{N_p}^{(p)}, \quad \Sigma^\alpha = e^\alpha_\mu(y) dy^\mu.$$

$e_\alpha^\mu(y)$ is a right-invariant frame on H , with dual coframe $e^\alpha_\mu(y)$. $e_{M_p}^{N_p}(y)$ is an element of the duality group $\mathcal{G}$ in the representation $\mathcal{R}_p \otimes \mathcal{R}_p^*$.

For this derivation, these “vielbeine” are chosen such that the structure functions are constant and given by

$$f_{\alpha\beta}^\gamma = -2e_{[\alpha}^\mu (\partial_\mu e_{\beta]}^\nu) e^\gamma_\nu, \quad (\text{B4})$$

$$f_{\alpha M_p}^{N_p} = -e_\alpha^\mu (\partial_\mu e_{M_p}^{L_p}) e_{L_p}^{-1 N_p}. \quad (\text{B5})$$

2. Jacobi identities

$$\begin{aligned}
&\{s_\alpha(\sigma_1), \{t_{M_p}^{(p)}(\sigma_2), t_{N_q}^{(q)}(\sigma_3)\}\} + \{t_{N_q}^{(q)}(\sigma_3), \{s_\alpha(\sigma_1), t_{M_p}^{(p)}(\sigma_2)\}\} - \{t_{M_p}^{(p)}(\sigma_2), \{s_\alpha(\sigma_1), t_{N_q}^{(q)}(\sigma_3)\}\} \\
&= -(-f_{\alpha K}^{L_{p+q}} \eta^{K_{p+q}}_{M_p N_q} + f_{\alpha M_p}^{K_p} \eta^{L_{p+q}}_{K_p N_q} + f_{\alpha N_q}^{K_q} \eta^{L_{p+q}}_{M_p K_q}) t_{L_{p+q}}^{(p+q)}(\sigma_3) \wedge d\delta(\sigma_2 - \sigma_3) \delta(\sigma_1 - \sigma_3) \\
&\quad + t_{P_{p+q}}^{(p+q)} \wedge \Sigma^\beta \delta(\sigma_1 - \sigma_2) \delta(\sigma_2 - \sigma_3) (-f_{\alpha M_p}^{K_p} f_{\beta L_{p+q}}^{P_{p+q}} \eta^{L_{p+q}}_{K_p N_q} + f_{\alpha L_{p+q}}^{P_{p+q}} f_{\beta M_p}^{K_q} \eta^{L_{p+q}}_{K_p N_q} \\
&\quad + f_{\alpha M_p}^{K_p} f_{\beta N_q}^{P_q} \eta^{L_{p+q}}_{K_p P_q} - f_{\alpha N_q}^{K_q} f_{\beta M_p}^{L_p} \eta^{P_{p+q}}_{L_p N_q} + f_{\beta\alpha}^\gamma f_{\gamma M_p}^{K_p} \eta^{P_{p+q}}_{K_p N_q}) \\
&= 0.
\end{aligned}$$

Here, manipulations of the $d\delta$ -terms as distributions up to world-volume boundary terms were performed. The first of the δ -terms arises due to (3.12).

The first term vanishes due to the $\mathfrak{h}$ -invariance of the η -symbols (2.15). The second term vanishes due to a combination of $\mathfrak{h}$ -invariance of the η -symbols (first step) and the fact that the $f_{\alpha M_p}^{N_p}$ are a representation of $\mathfrak{h}$ (second step), (2.14),

$$\begin{aligned}
& -f_{\alpha M_p}^{K_p} (f_{\beta L_{p+q}}^{P_{p+q}} \eta^{L_{p+q}} K_{pN_q} - f_{\beta N_q}^{P_q} \eta^{L_{p+q}} K_{pP_q}) \\
& + f_{\beta M_p}^{K_q} (f_{\alpha L_{p+q}}^{P_{p+q}} \eta^{L_{p+q}} K_{pN_q} - f_{\alpha N_q}^{K_q} \eta^{P_{p+q}} L_{pN_q}) \\
& + f_{\beta \alpha'} \gamma f_{\gamma M_p}^{K_p} \eta^{P_{p+q}} K_{pN_q} \\
& = 2f_{[\beta M_p}^{K_p} f_{\alpha K_p}^{L_p} \eta^{P_{p+q}} L_{pN_q} + f_{\beta \alpha'} \gamma f_{\gamma M_p}^{K_p} \eta^{P_{p+q}} K_{pN_q} \\
& = 0.
\end{aligned}$$

Similarly, one can also show the Jacobi identity for $\{t^{(1)}, t^{(1)}, t^{(1)}\}$, which should generalize easily to the generic case $\{t^{(k)}, t^{(l)}, t^{(m)}\}$. This gives

$$\begin{aligned}
& \{t_{A_1}(\sigma), \{t_{B_1}(\sigma'), t_{C_1}(\sigma'')\}\} + \text{cyc}((A_1, \sigma), (B_1, \sigma'), (C_1, \sigma'')) \\
& = \eta^{E_3}_{B_1 D_2} \eta^{D_2}_{C_1 A_1} t_{E_3}(\sigma) \wedge d\delta(\sigma - \sigma') \wedge d\delta(\sigma - \sigma'') \quad (\text{B6})
\end{aligned}$$

$$+ (f_{\alpha A_1}^{F_1} \eta^{E_3}_{C_1 D_2} \eta^{D_2}_{F_1 B_1} - f_{\alpha B_1}^{F_1} \eta^{E_3}_{F_1 D_2} \eta^{D_2}_{C_1 A_1}) t_{E_3}^\alpha(\sigma) \wedge d\delta(\sigma - \sigma'') \delta(\sigma - \sigma') \quad (\text{B7})$$

$$\begin{aligned}
& + f_{\alpha A_1}^{F_1} f_{\beta B_1}^{G_1} \eta^{E_3}_{F_1 D_2} \eta^{D_2}_{G_1 C_1} t_{E_3}^{\alpha\beta}(\sigma) \delta(\sigma - \sigma') \delta(\sigma' - \sigma'') \\
& + \text{cyc}((A_1, \sigma), (B_1, \sigma'), (C_1, \sigma'')). \quad (\text{B8})
\end{aligned}$$

By consistency, this should give zero as a distributional identity, i.e., integrated with a suitable test function $\phi(\sigma, \sigma', \sigma'')$.

Let us first look at (B6), i.e., the terms proportional to $d\delta \wedge d\delta$. Dropping boundary terms after integrating by parts twice to get rid of the delta functions, these terms look like,

$$\begin{aligned}
& \eta^{E_3}_{B_1 D_2} \eta^{D_2}_{C_1 A_1} \int_{\sigma} \int_{\sigma'} \int_{\sigma''} \phi(\sigma, \sigma', \sigma'') t_{E_3}(\sigma) \wedge d\delta(\sigma - \sigma') \wedge d\delta(\sigma - \sigma'') + \text{cyc}((A_1, \sigma), (B_1, \sigma'), (C_1, \sigma'')) \\
& = \eta^{E_3}_{B_1 D_2} \eta^{D_2}_{C_1 A_1} \int \partial_2 \partial_3 \phi(\sigma, \sigma, \sigma) \wedge t_{E_3}(\sigma) + \eta^{E_3}_{C_1 D_2} \eta^{D_2}_{A_1 B_1} \int \partial_3 \partial_1 \phi(\sigma, \sigma, \sigma) \wedge t_{E_3}(\sigma) \\
& + \eta^{E_3}_{A_1 D_2} \eta^{D_2}_{B_1 C_1} \int \partial_1 \partial_2 \phi(\sigma, \sigma, \sigma) \wedge t_{E_3}(\sigma),
\end{aligned}$$

where $\partial_i \phi(\sigma, \sigma, \sigma)$ for $i = 1, 2, 3$ is short for an exterior derivative with respect to the i th argument of the test function ϕ , e.g., $\partial_1 \phi(\sigma, \sigma, \sigma) = \int_{\sigma} \int_{\sigma'} d\phi(\sigma, \sigma', \sigma'') \delta(\sigma - \sigma') \delta(\sigma' - \sigma'')$. In particular, after integrating the delta functions, we have $d = \partial_1 + \partial_2 + \partial_3$. This fact, along with an integration by parts and (3.12) allows us to write these three integrals as

$$\begin{aligned}
& (\eta^{E_3}_{B_1 D_2} \eta^{D_2}_{C_1 A_1} + \eta^{E_3}_{C_1 D_2} \eta^{D_2}_{A_1 B_1} + \eta^{E_3}_{A_1 D_2} \eta^{D_2}_{B_1 C_1}) \int \partial_1 \partial_2 \phi(\sigma, \sigma, \sigma) \wedge t_{E_3}(\sigma) \\
& + f_{\alpha F_3}^{E_3} \eta^{F_3}_{C_1 D_2} \eta^{D_2}_{A_1 B_1} \int t_{E_3}^\alpha(\sigma) \wedge \partial_1 \phi(\sigma, \sigma, \sigma) - f_{\alpha F_3}^{E_3} \eta^{F_3}_{B_1 D_2} \eta^{D_2}_{C_1 A_1} \int t_{E_3}^\alpha(\sigma) \wedge \partial_2 \phi(\sigma, \sigma, \sigma).
\end{aligned}$$

The first line above vanishes due to (2.7), while the remaining terms will combine with the terms (B7) proportional to $\delta d\delta$, which are now of the form

$$\begin{aligned}
& (f_{\alpha B_1}^{F_1} \eta^{E_3}_{A_1 D_2} \eta^{D_2}_{F_1 C_1} - f_{\alpha C_1}^{F_1} \eta^{E_3}_{F_1 D_2} \eta^{D_2}_{A_1 B_1} + f_{\alpha F_3}^{E_3} \eta^{F_3}_{C_1 D_2} \eta^{D_2}_{A_1 B_1}) \int t_{E_3}^\alpha(\sigma) \wedge \partial_1 \phi(\sigma, \sigma, \sigma) \\
& + (f_{\alpha C_1}^{F_1} \eta^{E_3}_{B_1 D_2} \eta^{D_2}_{F_1 A_1} - f_{\alpha A_1}^{F_1} \eta^{E_3}_{F_1 D_2} \eta^{D_2}_{B_1 C_1} - f_{\alpha F_3}^{E_3} \eta^{F_3}_{B_1 D_2} \eta^{D_2}_{C_1 A_1}) \int t_{E_3}^\alpha(\sigma) \wedge \partial_2 \phi(\sigma, \sigma, \sigma) \\
& + (f_{\alpha A_1}^{F_1} \eta^{E_3}_{C_1 D_2} \eta^{D_2}_{F_1 B_1} - f_{\alpha B_1}^{F_1} \eta^{E_3}_{F_1 D_2} \eta^{D_2}_{C_1 A_1}) \int t_{E_3}^\alpha(\sigma) \wedge \partial_3 \phi(\sigma, \sigma, \sigma).
\end{aligned}$$

The strategy is to use the identities (2.7) and (2.15) to combine the partial derivatives $\partial_i \phi$ to an exterior derivative $d\phi$. This gives us

$$\begin{aligned} & - (f_{\alpha A_1}{}^{F_1} \eta^{D_2}{}_{B_1 C_1} + f_{\alpha B_1}{}^{F_1} \eta^{D_2}{}_{C_1 A_1} + f_{\alpha C_1}{}^{F_1} \eta^{D_2}{}_{A_1 B_1}) \eta^{E_3}{}_{F_1 D_2} \int t_{E_3}^\alpha(\sigma) \wedge d\phi(\sigma, \sigma, \sigma) \\ & + (-f_{\alpha F_2}{}^{D_2} \eta^{E_3}{}_{C_1 D_2} \eta^{F_2}{}_{A_1 B_1} - f_{\alpha A_1}{}^{F_1} \eta^{E_3}{}_{B_1 D_2} \eta^{F_2}{}_{C_1 F_1} + f_{\alpha F_3}{}^{E_3} \eta^{F_3}{}_{C_1 D_2} \eta^{D_2}{}_{A_1 B_1}) \int t_{E_3}^\alpha(\sigma) \wedge \partial_1 \phi(\sigma, \sigma, \sigma) \\ & + (-f_{\alpha F_2}{}^{D_2} \eta^{E_3}{}_{A_1 D_2} \eta^{F_2}{}_{B_1 C_1} - f_{\alpha B_1}{}^{F_1} \eta^{E_3}{}_{C_1 D_2} \eta^{F_2}{}_{A_1 F_1} - f_{\alpha F_3}{}^{E_3} \eta^{F_3}{}_{B_1 D_2} \eta^{D_2}{}_{C_1 A_1}) \int t_{E_3}^\alpha(\sigma) \wedge \partial_2 \phi(\sigma, \sigma, \sigma) \\ & + (-f_{\alpha F_2}{}^{D_2} \eta^{E_3}{}_{B_1 D_2} \eta^{F_2}{}_{C_1 A_1} - f_{\alpha C_1}{}^{F_1} \eta^{E_3}{}_{A_1 D_2} \eta^{F_2}{}_{B_1 F_1}) \int t_{E_3}^\alpha(\sigma) \wedge \partial_3 \phi(\sigma, \sigma, \sigma). \end{aligned}$$

Using the same aforementioned identities, it turns out that the $t \wedge \partial_i \phi$ integrals all share the same prefactor, namely the one in front of the $\partial_3 \phi$ integral. So we can also complete the exterior derivative, and then integrate by parts and use the identity (3.13), resulting in

$$[-(f_{\alpha A_1}{}^{F_1} \eta^{D_2}{}_{B_1 C_1} + f_{\alpha B_1}{}^{F_1} \eta^{D_2}{}_{C_1 A_1}) \eta^{E_3}{}_{F_1 D_2} - f_{\alpha A_1}{}^{F_1} \eta^{E_3}{}_{B_1 D_2} \eta^{D_2}{}_{C_1 F_1}] \int \left(f_{\alpha E_3}{}^{F_3} t_{F_3}^\alpha(\sigma) + \frac{1}{2} f_{\beta\gamma}^\alpha t_{E_3}^{\beta\gamma}(\sigma) \right) \phi.$$

Simplifying these and grouping terms gives us

$$-f_{\alpha A_1}{}^{F_1} f_{\beta B_1}{}^{G_1} \eta^{E_3}{}_{F_1 D_2} \eta^{D_2}{}_{G_1 C_1} \int t_{E_3}^{\alpha\beta}(\sigma) \phi(\sigma, \sigma, \sigma) + \text{cyc}(A_1, B_1, C_1),$$

which precisely cancels out the contribution coming from the $\delta\delta$ terms (B8). This proves that the Jacobi identity $\{t^{(1)}, t^{(1)}, t^{(1)}\} = 0$ is satisfied.

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