

Leading large N_c contributions to lepton number violating meson decays

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Lepton number violating meson decays, such as $M_1^- \rightarrow M_2^+ \ell_1^- \ell_2^-$, provide constraints on $d = 9$ $\Delta L = 2$ operators. Renormalization group equation (RGE)-improved bounds on the Wilson coefficients of these operators have been presented in the literature, taking into account perturbative quantum chromodynamics (QCD) one-loop corrections and the corresponding operator mixing. Here, we present for the first time the contribution of connected diagrams to the hadronic matrix elements $\langle M_2 | \mathcal{O}_h | M_1 \rangle$. These diagrams, usually overlooked under the assumption that $\langle M_2 | \mathcal{O}_h | M_1 \rangle \sim \langle M_2 | J_{q_3 q_4} | 0 \rangle \times \langle 0 | J_{q_1 q_2} | M_1 \rangle \gg \langle M_2 | J_{q_3 q_2} \times J_{q_1 q_4} | M_1 \rangle$, can give indeed a significant contribution to the matrix element. Including these connected diagrams is but the first step toward a full nonperturbative computation of the long-range QCD effects in these operators, that should be performed using lattice field theory techniques. However, connected diagrams represent the leading order in the $1/N_c$ expansion of the QCD nonperturbative effects and thus our work can be understood as a realistic, first approximation to a complete calculation of the long-range part of the matrix elements.

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I. INTRODUCTION

In the standard model lepton number is conserved at the perturbative level. However, in Standard Model effective field theory (SMEFT) [1–5] lepton number violation (LNV) appears in the form of nonrenormalizable operators [6,7]. The most stringent test of lepton number violation is neutrinoless double beta decay ($0\nu\beta\beta$), for reviews see for example [8–10].

However, $0\nu\beta\beta$ is sensitive only to LNV in the first generation of leptons. To test LNV for higher generation leptons one has to resort to exotic processes, such as $\tau^\pm \rightarrow \ell^\mp M_1 M_2$, and $M_1^- \rightarrow M_2^+ \ell_1^- \ell_2^-$ where ℓ stands for $\ell = e, \mu$ in case of tau decays and, in principle, any SM generation in meson decays (when kinematically allowed). Experimental constraints for all of these processes can be found in the Particle Data Book [11], where the results from

flavor factories [12–17] and flavored experiments at the LHC [18–21] are summarized.

Comparing the experimental data with the theoretical predictions in the lepton number violating sector of the SMEFT allows to put constraints on the Wilson coefficients of individual effective operators, for earlier work considering constraints from neutrino masses and/or LNV meson decays see [22–40], while results for LNV muon conversion can be found, for example, in [41–44].¹ However, the matching between the experimental data and the theoretical prediction is affected by a number of uncertainties related to the perturbative order at which the theoretical computation of the Wilson coefficients has been performed and on the specific regularization and renormalization schemes adopted. The running of the Wilson coefficients between the scale at which the theoretical (perturbative) computation of short-range effects is sound and the (lower) scale at which the experimental measurements are made must be taken into account, introducing scheme dependence. Moreover, quite generally it is found that loop computations introduce mixing of the operators, responsible for a given transition at tree-level, with other operators originally not present, thus enlarging the operator basis to be taken into account. This was shown to be also the case for $M_1^- \rightarrow M_2^+ \ell_1^- \ell_2^-$ and $\tau^- \rightarrow \ell^+ M_1^- M_2^-$

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¹We note in passing that also searches at LHC can put bounds on LNV models [45–48].

decays [49], where QCD one-loop corrections where computed and renormalization group (RG) running was properly accounted for. The inclusion of these effects was shown to have a significant impact on the bounds on the relevant Wilson coefficients.

In this paper, our aim is to go one step beyond the work done in [49], and introduce a first estimate of the effect of long-range physics. This can be done by computing the relevant hadronic matrix element at leading order in the $1/N_c$ expansion.

The hadronic matrix element $\langle M_2 | \mathcal{O}_h | M_1 \rangle$ is usually computed under the assumption that the s-channel-like contribution dominates, i.e., that $\langle M_2 | \mathcal{O}_h | M_1 \rangle \simeq \langle M_2 | J_{q_3 q_4} \times J_{q_1 q_2} | M_1 \rangle$, with J_{qq} a quark current and where the initial and final mesons sources are $M_1 = \bar{q}_1 \Gamma q_2$ and $M_2 = \bar{q}_3 \Gamma' q_4$, with Γ, Γ' some γ -matrix dependent on the mesons quantum numbers. In order to get an estimate of the hadronic matrix element, the *vacuum insertion approximation* (VIA) is usually adopted: $\langle M_2 | J_{q_3 q_4} \times J_{q_1 q_2} | M_1 \rangle \sim \langle M_2 | J_{q_3 q_4} | 0 \rangle \langle 0 | J_{q_1 q_2} | M_1 \rangle$. This approximation corresponds to neglecting all soft gluon exchanges between the initial and final mesons and, at the same time, to neglecting the production of intermediate hadronic states (such as, for example, other heavier mesons with the appropriate quantum numbers). In the language of the $1/N_c$ expansion, the resulting *disconnected* diagram is $\mathcal{O}(N_c^2)$, as it contains two fermion loops, contributing N_c each. All other possible diagrams, on the other hand, are subleading under N_c -power-counting. We will, therefore, compute here the leading N_c contribution to the hadronic matrix element, represented by the *connected* diagram $\langle M_2 | J_{q_3 q_4} \times J_{q_1 q_2} | M_1 \rangle$ in which the quarks in $\mathcal{O}_h$ connect to external quark lines with t-channel-like contractions. We will demonstrate how including this correction modifies the bounds on the Wilson coefficients, in some cases by considerable factors.

The rest of this paper is organized as follows: in Sec. II we consider the case of the $K^+ \rightarrow \pi^- \ell^+ \ell^+$ decay as an example and present the low-energy effective operators basis responsible for this process; in Sec. III we compute the decay widths and run the Wilson coefficients down at

the hadronic scale μ ; in Sec. IV we introduce the VIA and compute the leading contribution to the hadronic matrix element in $1/N_c$; in Sec. V we compute the $\mathcal{O}(1/N_c)$ correction to the hadronic matrix element; in Sec. VI we derive improved bounds to the Wilson coefficients at the hadronic scale μ at the subleading order in $1/N_c$; finally, in Sec. VII we present our conclusions. In Appendix A we summarize the anomalous dimensions at one-loop relevant for the process at hand; in Appendix B we show the numerical values of the RGE evolution matrices; in Appendix C we list the Fierz identities used in the paper; in Appendix D we give the sum of connected and disconnected diagrams in the VIA. Finally, in Appendix E, we introduce an optimal definition of the B -parameters to be computed on the lattice in order to perform a numerical computation of the hadronic matrix elements.

II. DIMENSION-9 OPERATORS FOR $\Delta L = 2$ TRANSITIONS

The process $K^+ \rightarrow \pi^- \ell^+ \ell^+$ is the paradigmatic $\Delta L = 2$ meson decay. It can occur if an operator that violates lepton number by 2 units exists, such as for example in type-I seesaw models where a heavy Majorana “right-handed” neutrino is added to the Standard Model spectrum.

A. $K^+ \rightarrow \pi^- \ell^+ \ell^+$ in type-I seesaw models

Within this particular model, that we illustrate as an example, it was shown in Refs. [22,50] that this decay may occur through two tree-level diagrams (see Fig. 1):

- (1) *s-channel*, Fig. 1 (left): in this case, the process goes as $K^+ \rightarrow W^+ \rightarrow \ell^+ \ell^+ W^- \rightarrow \pi^-$. Assuming that the $\Delta L = 2$ operator arises through the exchange of N_R heavy Majorana neutrinos of mass M_{ν_j} , we have

$$L_j^{\mu\nu}(p, p') = M_{\nu_j} \frac{1}{q^2 - M_{\nu_j}^2} \bar{v}(p) \gamma^\mu \gamma^\nu P_R v^c(p'). \quad (1)$$

Eventually, the amplitude for the s-channel diagram is (after integrating out the W's)

$$A_s = 2G_F^2 f_K f_\pi [V_{ud} V_{us}]^* \sum_{j=1}^{N_R} [U_{\ell j} U_{\ell' j}]^* p_K^\mu p_\pi^\nu [L_{\mu\nu, j}(p, p') - \delta_{\ell\ell'} L_{\mu\nu, j}(p', p)], \quad (2)$$

where the decay constants f_K, f_π come by neglecting soft gluons interchange between initial and final mesons and applying the PCAC to the hadronic matrix elements $\langle 0 | \gamma_\mu \gamma_5 | K(q) \rangle$ and $\langle 0 | \gamma_\mu \gamma_5 | \pi(q - p - p') \rangle$.

- (2) *t-channel*, Fig. 1 (right): in this case, the W's are produced from two different kaon quark lines and, therefore, the hadronic matrix element is more complicated as, strictly speaking, it is a four-point function: $\langle \pi(z) | \bar{u}(y) \gamma_L^\mu d(y) \bar{u}(x) \gamma_L^\nu s(x) | K(0) \rangle$. In Fourier transform

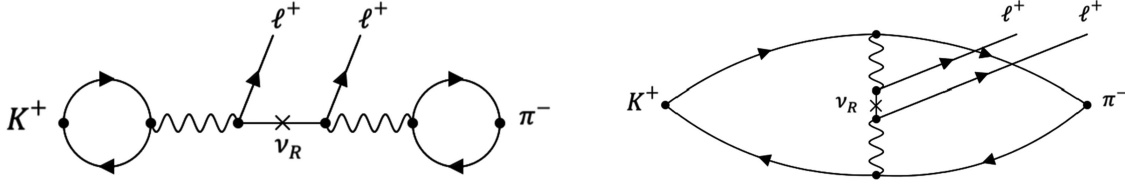


FIG. 1. Tree-level diagrams responsible for the $K^+ \rightarrow \pi^- \ell^+ \ell^+$ transition in the type-I seesaw model. Left: s-channel amplitude. Right: t-channel amplitude.

$$A_t = 2G_F^2 [V_{ud}V_{us}]^* \sum_{j=1}^{N_R} [U_{lj}U_{l'j}]^* [L_{\mu\nu,j}(p, p') - \delta_{ll'} L_{\mu\nu,j}(p', p)] \otimes \int d^4x d^4y e^{i(p_d - p_u)y} e^{i(p_s - p_{\bar{u}})x} \langle \pi(z) | \bar{u}(y) \gamma_L^\mu d(y) \bar{u}(x) \gamma_L^\nu s(x) | K(0) \rangle. \quad (3)$$

If $q^2 \ll M_{\nu_j}$ for any j , we can integrate out the heavy right-handed neutrinos and get a local lepton current

$$L_j^{\mu\nu}(p, p') \sim -\frac{1}{m_{\nu_j}} \bar{v}(p) \gamma^\mu \gamma^\nu P_R v^c(p') + \dots \quad (4)$$

After integrating out both W 's and the heavy right-handed neutrinos, the two quark currents give a local four-fermion $\Delta Q = 2$ hadronic current-current operator $\mathcal{O}_h = J_h \times J_h$ (with $J_h = \bar{q}_i \Gamma q_j$ a fermion bilinear), such that the hadronic amplitude is

$$\mathcal{M}(z, x) = \langle \pi(z) | \mathcal{O}_{\mu\nu}^{\Delta Q=2, LL} | K(0) \rangle = \langle \pi(z) | \bar{u}(x) \gamma_\mu^L d(x) \bar{u}(x) \gamma_\nu^L s(x) | K(0) \rangle. \quad (5)$$

This matrix element is a three-point function and it must be computed nonperturbatively, for example on the lattice.

As we have seen, if we are able to start from a definite fundamental theory, we have some hints on the spinor structure of the effective operators to be computed. In the case of the type-I seesaw, we have an s-channel-induced product of two two-point functions (whose nonperturbative estimate may be obtained using the PCAC), or a t-channel-induced three-point function (for which we have no estimate, in principle). However, these two amplitudes

are a quite general result independently of the original fundamental theory. Assume that we have a four-fermion local operator in the form $\mathcal{O}_h = J_h \times J_h$ responsible for a $\Delta L = 2$ transition in combination with a leptonic current, $\mathcal{O}_{\Delta L=2} = \mathcal{O}_h \times J_{\ell\ell}$. When computing the hadronic contribution to the process, we must sandwich the local four-quark operator between hadronic states to get a (nonperturbative) hadronic matrix element. Once contracting over the external states the hadronic local operator $\mathcal{O}_h$, in general two contributions arise from the T -product of the quark fields: one connected “eightlike” diagram (Fig. 2, right), or two disconnected loops (Fig. 2, left). It is easy to relate the two diagrams with the two different amplitudes in the type-I seesaw model: the disconnected diagram arises integrating out heavy fields in the s-channel, whereas the connected diagram arises from the t-channel. From a bottom-up approach such as the one used in SMEFT [1–5], however, we will not be able to determine the specific structure of the diagrams in the fundamental theory, as we have in principle no idea of what the fundamental theory is.

B. $K^+ \rightarrow \pi^- \ell^+ \ell^+$ in the OPE

If we have no fundamental theory to derive the low-energy operators relevant for the process $K^+ \rightarrow \pi^- \ell^+ \ell^+$, we can still write a complete basis of operators that may be responsible for the transition, and then use the experimental

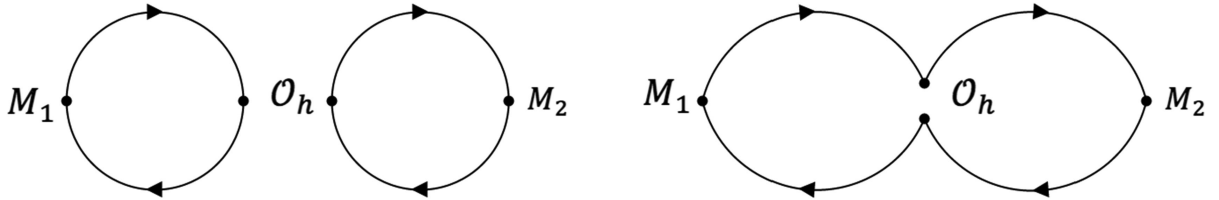


FIG. 2. Disconnected (left) and connected (right) diagrams of the hadronic matrix element for the decay $M_1^+ \rightarrow M_2^- \ell^+ \ell^+$, with effective operator $\mathcal{O} = \mathcal{O}_h \times J_{\ell\ell}$. Left diagram is $\mathcal{O}(N_c^2)$ and right diagram is $\mathcal{O}(N_c)$, as they contain two and one fermion loops, respectively.

data to constrain the corresponding Wilson coefficients. This is the approach followed in Refs. [49,51–54].

The effective Lagrangian for a particular process that violates ΔL by two units is

$$\mathcal{L}_{\text{eff}}^{\Delta L=2} = \frac{1}{\Lambda_{\text{EW}}^5} \sum_i C_i(\Lambda) \mathcal{O}_i^{\Delta L=2}(\Lambda), \quad (6)$$

where $C_i(\Lambda)$ is the *tree-level* Wilson coefficients of the bare operator $\mathcal{O}_i^{\Delta L=2}$ and i runs over the operator basis. We have normalized the Wilson coefficients with a common factor $1/\Lambda_{\text{EW}}^5$, in order to obtain² adimensional coefficients. If we know the fundamental theory, the Wilson coefficients can be computed directly by matching the theory where heavy d.o.f.'s have been integrated out with the fundamental theory at some high-energy scale Λ (as it is done in the case of the $\Delta S = 2$ transitions in the SM, see Ref. [55] and references therein). The coefficients C_i can be then constrained by experiments that look for a given process. If we do not know the fundamental theory, we can write down all possible operators that mediate the process at hand compatible with the symmetries of the SM.

For the process $K^+ \rightarrow \pi^- \ell_1^+ \ell_2^+$, it was found that the most general effective Lagrangian contains 7 operators in the case $\ell_1 = \ell_2$, each of them with a $\mathcal{O}_h \times J_{\ell\ell}$ structure

$$\begin{aligned} \mathcal{O}_1^{XYZ} &= 8(\bar{u}P_X d \bar{u}P_Y s) \otimes j_{\ell\ell}^Z, \\ \mathcal{O}_2^{XXX} &= 8(\bar{u}\sigma^{\mu\nu}P_X d \bar{u}\sigma_{\mu\nu}P_X s) \otimes j_{\ell\ell}^X, \\ \mathcal{O}_3^{XYZ} &= 8(\bar{u}\gamma^\mu P_X d \bar{u}\gamma_\mu P_Y s) \otimes j_{\ell\ell}^Z, \\ \mathcal{O}_4^{XYZ} &= 8(\bar{u}\gamma^\mu P_X d \bar{u}\sigma_\mu^\nu P_Y s) \otimes j_{\ell\ell,\nu}^Z, \\ \mathcal{O}_4^{XYZ} &= 8(\bar{u}\sigma_\mu^\nu P_X d \bar{u}\gamma^\mu P_Y s) \otimes j_{\ell\ell,\nu}^Z, \\ \mathcal{O}_5^{XYZ} &= 8(\bar{u}\gamma^\mu P_X d \bar{u}P_Y s) \otimes j_{\ell\ell,\mu}^Z, \\ \mathcal{O}_5^{XYZ} &= 8(\bar{u}P_X d \bar{u}\gamma^\mu P_Y s) \otimes j_{\ell\ell,\mu}^Z, \end{aligned} \quad (7)$$

where in these expressions P_X are the chirality projectors, $P_{L,R} = 1/2(1 \mp \gamma_5)$. The lepton bilinears j for two generic leptons ℓ_1 and ℓ_2 are

$$\begin{aligned} j_{\ell_1\ell_2}^Z &= \bar{\ell}_1 P_Z \ell_2^c, \\ j_{\ell_1\ell_2,\mu}^Z &= \bar{\ell}_1 \gamma_\mu P_Z \ell_2^c, \\ j_{\ell_1\ell_2,\mu\nu}^Z &= \bar{\ell}_1 \sigma_{\mu\nu} P_Z \ell_2^c. \end{aligned} \quad (8)$$

Notice that the tensor lepton current $j_{\ell_1\ell_2,\mu\nu}^Z$ vanishes when $\ell_1 = \ell_2$. This means that, when $\ell_1 \neq \ell_2$ (differently from the example that we were studying above, $K^+ \rightarrow \pi^- \ell^+ \ell^+$), new operators should be added to our

basis. Eventually, for $\ell_1 \neq \ell_2$, we should include four new operators

$$\begin{aligned} \mathcal{O}_6^{XYZ} &= 8(\bar{u}\gamma^\mu P_X d \bar{u}\gamma^\nu P_Y s) \otimes j_{\ell_1\ell_2,\mu\nu}^Z, \\ \mathcal{O}_7^{XZZ} &= 8(\bar{u}P_X d \bar{u}\sigma^{\mu\nu}P_Z s) \otimes j_{\ell_1\ell_2,\mu\nu}^Z, \\ \mathcal{O}_7^{ZXX} &= 8(\bar{u}\sigma^{\mu\nu}P_Z d \bar{u}P_X s) \otimes j_{\ell_1\ell_2,\mu\nu}^Z, \\ \mathcal{O}_8^{XXX} &= 8(\bar{u}\sigma^{\mu\alpha}P_X d \bar{u}\sigma_\alpha^\nu P_X s) \otimes j_{\ell_1\ell_2,\mu\nu}^X. \end{aligned} \quad (9)$$

The primed operators $\mathcal{O}_6^{XYZ}$ and $\mathcal{O}_8^{XXX}$ are not included, since $\mathcal{O}_6^{XYZ} = -\mathcal{O}_6^{YXZ}$ and $\mathcal{O}_8^{XXX} = -\mathcal{O}_8^{XXX}$. Notice that only particular combinations of chiral projectors are allowed for these additional operators.

We now briefly review the operator counting in our basis. For the $K \rightarrow \pi \ell \ell$ decay, there are 36 independent dimension-9 operators when the final-state leptons are identical, and 70 independent operators when the leptons are different. This counting follows from the chiral structure of the operators and from symmetry relations among leptonic bilinears. For identical leptons, only the operator structures in Eq. (7) need to be considered. If no symmetry relations or vanishing operators are taken into account, each operator structure would contribute $2^3 = 8$ independent operators, corresponding to the choices $X, Y, Z = L, R$. However, for operators involving vector currents with identical leptons, the relation $\bar{\ell}\gamma_\mu P_{L,R}\ell^c = -\bar{\ell}\gamma_\mu P_{R,L}\ell^c$ reduces the number of independent operators by a factor of two, whereas for operators containing two tensor bilinears the vanishing of mixed-chirality structures also reduces the number of independent operators by a factor of two. As a consequence, the operator structures $\mathcal{O}_1$ and $\mathcal{O}_3$ each contribute eight operators, while the remaining five structures contribute only four operators each, leading to a total of 36. For the case of different leptons, taking $\ell_1 = e$ and $\ell_2 = \mu$ as a representative example, the relations $\bar{e}P_Z\mu^c = \bar{\mu}P_Z e^c$, $\bar{e}\gamma_\mu P_{R,L}\mu^c = -\bar{\mu}\gamma_\mu P_{L,R}e^c$, and $\bar{e}\sigma_{\mu\nu}P_Z\mu^c = -\bar{\mu}\sigma_{\mu\nu}P_Z e^c$ have been taken into account in the definition of the basis. Each of the operator structures in Eq. (7) contributes eight independent operators, with the exception of $\mathcal{O}_2$, which contributes only four, leading to a total of 52 operators. In addition, further contributions arise from the operator structures in Eq. (9), which contribute 18 additional independent operators. Specifically, $\mathcal{O}_6$ contributes $2^3 = 8$ operators, $\mathcal{O}_7$ and $\mathcal{O}_7'$ contribute four each (since the two tensor currents must have the same chirality), and $\mathcal{O}_8$ contributes only two operators, corresponding to the purely left-handed (*LLL*) and purely right-handed (*RRR*) cases. Consequently, the total number of independent operators for different leptons is 70.

Once we have introduced the full basis that may be responsible for the transition $K^+ \rightarrow \pi^- \ell_1^+ \ell_2^+$, it is easy to see that the operator that we have found in the case of the type-I seesaw is, in fact, $\mathcal{O}_3^{XZZ}$ for two identical final leptons, whereas for processes with leptons from two

²The normalization scale is, of course, arbitrary. We find it convenient to use the scale of the electroweak symmetry breaking, $\Lambda_{\text{EW}} = 246$ GeV, to compare directly with the case of the type-I seesaw model, in which the heavy mediators are W bosons.

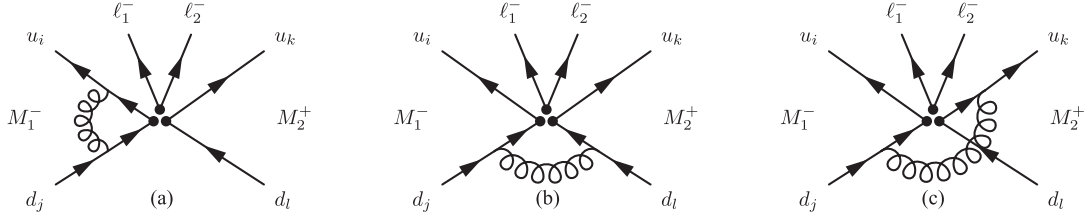


FIG. 3. Effective $d = 9$ operator description of the short-range mechanisms (SRM) of the meson decay $M_1^- \rightarrow M_2^+ \ell_1^- \ell_2^-$. The tree-level diagram can be obtained, e.g., by integrating out gauge bosons and heavy fermions in Fig. 1. On the other hand, diagrams (a)–(c) give some representative one-loop perturbative QCD corrections to the amplitude arising by the interchange of a hard gluon.

different generations there will also be a contribution from $\mathcal{O}_6^{XZ}$.

III. FROM THE EFFECTIVE LAGRANGIAN $\mathcal{L}_{\text{eff}}$ TO BOUNDS ON C_i

The effective Lagrangian in Eq. (6) must be computed within definite initial and final states, in order to use the experimental data to constrain the Wilson coefficients.

However, this must be done at some hadronic scale μ much lower than the scale Λ where the fundamental theory is replaced by the effective one.

A. Hadronic decay widths

The tree-level amplitudes for the process $K^+ \rightarrow \pi^- \ell^+ \ell^+$ mediated by the effective Lagrangian in Eq. (6) are given by

$$\mathcal{M}(K^+ \rightarrow \pi^- \ell_1^+ \ell_2^+) = \langle \pi^- \ell_1^+ \ell_2^+ | \mathcal{L}_{\text{eff}}^{\Delta L=2} | K^+ \rangle = \frac{1}{\Lambda_{\text{EW}}^5} \sum_n \sum_{XYZ} C_n^{XYZ}(\Lambda) \mathcal{A}_n^{XYZ}(\Lambda), \quad (10)$$

where $\mathcal{A}_n^{XYZ}(\Lambda) = \langle \pi^- \ell_1^+ \ell_2^+ | \mathcal{O}_n^{XYZ}(\Lambda) | K^+ \rangle = \langle \pi^- | \mathcal{O}_n^{XYZ}(\Lambda) | K^+ \rangle \otimes J_{\ell_1 \ell_2}$. The amplitude $\mathcal{A}_n^{XYZ}$ depends on the scale Λ since the hadronic matrix elements are dressed by $\mathcal{O}(\Lambda)$ gluons. The partial decay widths for the meson decays are then given by [29]

$$\Gamma(K^+ \rightarrow \pi^- \ell_1^+ \ell_2^+) = \frac{1}{64(2\pi)^3 m_K^3 \Lambda_{\text{EW}}^{10}} \int_{s^-}^{s^+} ds \int_{t^-}^{t^+} dt \left| \sum_n \sum_{XYZ} C_n^{XYZ} \bar{\mathcal{A}}_n^{XYZ} \right|^2, \quad (11)$$

where $\bar{\mathcal{A}}_n$ are spin-averaged amplitudes and the kinematical variables $s \equiv (p_{\ell_1} + p_{\ell_2})^2$ and $t \equiv (p_{\ell_2} + p_{M_2})^2$, have integration limits given by

$$s^- = (m_{\ell_1} + m_{\ell_2})^2, \quad s^+ = (m_{M_1} - m_{M_2})^2, \quad (12)$$

$$t^\pm = m_{M_1}^2 + m_{\ell_1}^2 - \frac{1}{2s} \left[(s + m_{M_1}^2 - m_{M_2}^2)(s + m_{\ell_1}^2 - m_{\ell_2}^2) \mp \lambda^{1/2}(s, m_{\ell_1}^2, m_{\ell_2}^2) \lambda^{1/2}(s, m_{M_1}^2, m_{M_2}^2) \right], \quad (13)$$

with $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$. The quark flavor indices (i, j, k, l) are fixed by the quark content of the mesons: $M_1^- = (\bar{u}_i d_j)$, $M_2^+ = (u_k \bar{d}_l)$.

The hadronic matrix element $\langle \pi^- | \mathcal{O}_n^{XYZ}(\Lambda) | K^+ \rangle$ should be in principle computed *nonperturbatively* at the same scale Λ where the matching between the fundamental theory and the effective theory takes place. However, it is not possible to perform a nonperturbative QCD computation at that scale. What is instead done, is to compute the matrix elements nonperturbatively at some low-energy scale $\mu \ll \Lambda$ where dynamical d.o.f.'s are mesons and baryons (i.e., at a scale where quarks are confined and

only long-distance soft-gluons should be taken into account).

B. One-loop QCD running of the Wilson coefficients

The main goal of the RGE running of the Wilson coefficients is to connect the high-energy scale Λ , at which the matching between the fundamental theory and the effective one is supposed to be performed, with some low-energy scale μ where experimental data are measured. The high-energy scale is typically $\Lambda = \mathcal{O}(100)$ GeV in LEFT, i.e., a scale large enough for perturbative QCD to be reliable. On the other hand, the low-energy scale μ should

be low enough for nonperturbative computations of the hadronic matrix element $\langle M_2 | \mathcal{O}_i^{\Delta L=2}(\mu) | M_1 \rangle$ to be carried out. Depending on the particular nonperturbative approach used, this scale ranges³ from 1 to 2 GeV. The RGE running of the Wilson coefficients gives

$$\mathcal{L}_{\text{eff}}^{\Delta L=2} = \frac{1}{\Lambda_{\text{EW}}^5} \sum_i C_i(\mu) \mathcal{O}_i^{\Delta L=2}(\mu), \quad (14)$$

where the RG evolution, in a given regularization and renormalization scheme, is given by the solution of the following equation:

$$\frac{dC_i(\mu)}{d \ln(\mu/\Lambda)} = \gamma_{ij}(\alpha_s) C_j(\mu), \quad (15)$$

where γ_{ij} is an element of the anomalous dimension matrix, $\hat{\gamma}$. The anomalous dimensions of the operators in Eqs. (7) and (9) have been computed in dimensional regularization in the $\overline{\text{MS}}$ renormalization scheme in Ref. [49]. At leading order in α_s , the anomalous dimension matrix takes the form

$$\gamma_{ij}(\alpha_s) = \frac{\alpha_s}{4\pi} \gamma_{ij}, \quad \text{with} \quad \gamma_{ij} = -2(b_{ij} - 2C_F \delta_{ij}), \quad (16)$$

where $C_F = (N_c^2 - 1)/(2N_c)$ and b_{ij} are determined from the one-loop QCD corrections. The perturbative results of Ref. [49] are summarized in Appendix A. In order to give a feeling of the one-loop computations to be performed, some representative one-loop QCD diagrams contributing to the renormalization of the effective operators in the basis defined above are shown in Fig. 3 (additional diagrams obtained via left-right or up-down reflections have also been taken into account).

The solution to the RGE in Eq. (15) can be expressed in terms of an evolution matrix $\hat{U}(\mu, \Lambda)$ connecting a high-energy scale Λ to a lower scale μ

$$C_i(\mu) = U_{ij}(\mu, \Lambda) C_j(\Lambda). \quad (17)$$

At leading logarithmic accuracy, this evolution matrix is given by

$$\hat{U}(\mu, \Lambda) = \hat{V} \left[\left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)} \right)^{\hat{\gamma}_{\text{diag}}/(2\beta_0)} \right] \hat{V}^{-1}, \quad (18)$$

where $\hat{V}$ diagonalizes the matrix $\hat{\gamma}$

³The higher the low-energy scale, the more we can trust the perturbative running of the Wilson coefficient. In lattice computation, typically $\mu = 2$ GeV. For other NP method, sometimes $\mu = 1$ GeV is also chosen. Quite in general, though, to reach lower scales one-loop corrections are not enough and higher-order perturbative computations are needed [56].

$$\hat{\gamma}_{\text{diag}} = \hat{V}^{-1} \hat{\gamma} \hat{V}. \quad (19)$$

The RGE-improved amplitudes for the process $K^+ \rightarrow \pi^- \ell_1^+ \ell_2^+$ are

$$\mathcal{M}(K^+ \rightarrow \pi^- \ell_1^+ \ell_2^+) = \frac{1}{\Lambda_{\text{EW}}^5} \sum_n \sum_{XYZ} C_n^{XYZ}(\mu) \mathcal{A}_n^{XYZ}(\mu), \quad (20)$$

and the corresponding partial decay widths are, thus

$$\begin{aligned} \Gamma(K^+ \rightarrow \pi^- \ell_1^+ \ell_2^+) &= \frac{1}{64(2\pi)^3 m_K^3 \Lambda_{\text{EW}}^{10}} \int_{s^-}^{s^+} ds \int_{t^-}^{t^+} dt \\ &\times dt \left| \sum_n \sum_{XYZ} C_n^{XYZ}(\mu) \bar{\mathcal{A}}_n^{XYZ}(\mu) \right|^2, \end{aligned} \quad (21)$$

where the spin-averaged amplitudes $\bar{\mathcal{A}}_n^{XYZ}(\mu)$ are the matrix elements of the renormalized operators at the low-energy scale μ

$$\bar{\mathcal{A}}_n^{XYZ}(\mu) = \langle \pi^- | \hat{\mathcal{O}}_n(\mu) | K^+ \rangle \otimes J_{\ell_1 \ell_2}^{XYZ}. \quad (22)$$

Once the hadronic matrix element of the renormalized operators is computed at the scale μ in the same regularization and renormalization scheme as the RGE-evolved Wilson coefficients, the effective Lagrangian can be shown to be scale-independent,

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \frac{1}{\Lambda_{\text{EW}}^5} \sum_n C_n(\Lambda) \langle \pi^- \ell_1^+ \ell_2^+ | \mathcal{O}_n(\Lambda) | K^+ \rangle \\ &= \frac{1}{\Lambda_{\text{EW}}^5} \sum_n C_n(\mu) \langle \pi^- \ell_1^+ \ell_2^+ | \hat{\mathcal{O}}_n(\mu) | K^+ \rangle, \end{aligned} \quad (23)$$

where the renormalized operator is

$$\hat{\mathcal{O}}_n(\mu) = U_{nm}(\mu, \Lambda) \mathcal{O}_m(\Lambda), \quad (24)$$

being $\hat{U}$ the same matrix that gives the RGE evolution of the Wilson coefficients.

For completeness, we recall that the QCD coupling constant at leading log evolves with the scale as

$$\alpha_s(\mu) = \frac{\alpha_s(\Lambda)}{1 - \frac{\beta_0}{2\pi} \alpha_s(\Lambda) \ln(\Lambda/\mu)}, \quad (25)$$

with $\beta_0 = (33 - 2N_f)/3$ and N_f the number of active quark flavors at the scale μ (either 3 or 4, depending on the mass of the decaying particle). As a starting point we took $\alpha_s(M_Z) = 0.1180(9)$ at the Z-boson mass [11]. The numerical values of the evolution matrix $\hat{U}(\mu, \Lambda)$ in the $\overline{\text{MS}}$ for $\mu = 2$ GeV and $\Lambda = M_Z$ are given in Appendix B.

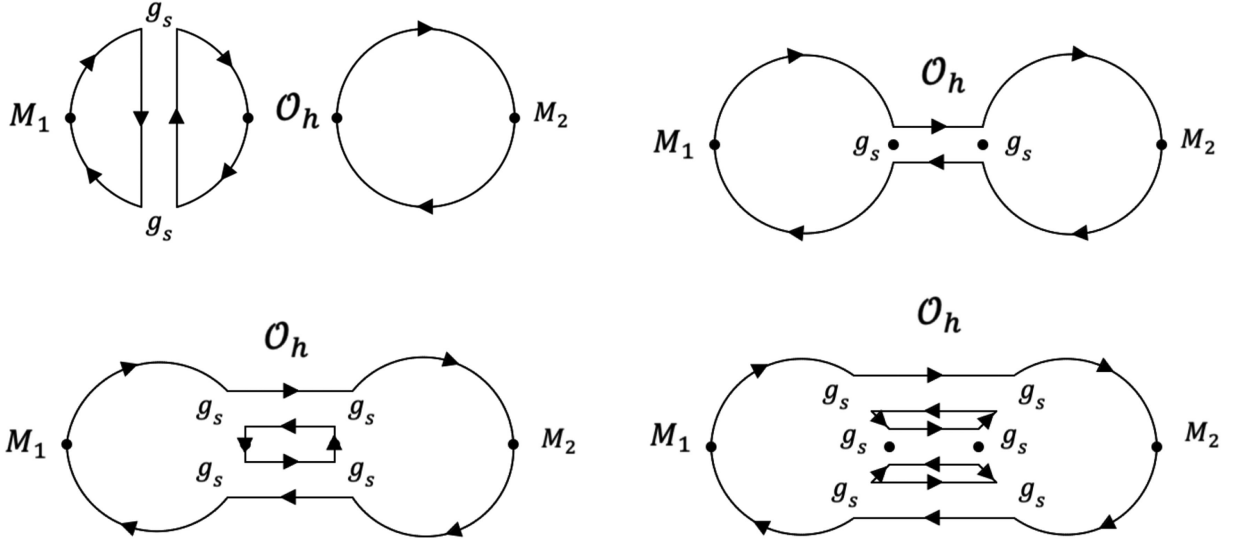


FIG. 4. Disconnected diagrams with one gluon emission between initial quark lines (top left) or one (top right), two (bottom left) or three (bottom right) gluons emission between initial and final quark lines for the decay $M_1^+ \rightarrow M_2^- \ell^+ \ell^+$, with effective operator $\mathcal{O} = \mathcal{O}_h \times J_{\ell\ell}$. The top left diagram is $\mathcal{O}(N_c^2)$, as it contains three loops and two couplings, $g_s^2 \propto 1/N_c$. On the other hand, the other three diagrams are $\mathcal{O}(1)$, as they contain one, two and three loops and g_s^2, g_s^4, g_s^6 couplings, respectively.

IV. HADRONIC MATRIX ELEMENTS AT LARGE N_c

In order to explain clearly how we may estimate the hadronic matrix element under study at the next-to-leading order in large N_c , we must first introduce with the same degree of clarity how the leading contribution is computed. This will be done by shortly reviewing some basic rules of N_c power-counting and, then, introducing the *vacuum insertion approximation*.

A. Planar diagrams with soft gluon emissions at large N_c

It is well known that a quark loop (of flavor q_i) contributes at $\mathcal{O}(N_c)$ to a QCD amplitude (since quarks belong to the fundamental representation of the gauge group), whereas gluon loops (being in the adjoint representation) count as $\mathcal{O}(N_c^2)$. However, the coupling constant g_s and the number of colors satisfy the 't Hooft scaling in that $g_s^2 N_c = \lambda$, with λ the 't Hooft coupling [57]. This means that dressing the diagrams in Fig. 2 with internal loops introduces an N_c -dependence that counts both the number and type of particles in a loop and the number of QCD vertices. In order to understand better this point, in Fig. 4 we show diagrams in which gluons are emitted between initial quark lines or one initial and one final quark line. Gluon lines are represented as two parallel quark lines with opposite fermion number, as it is standard in the large N_c expansion. The number of QCD vertices is also depicted. In Fig. 4 (top left), we see a gluon emitted (and absorbed) by two initial quark lines. Since a gluon is represented by two (anti)parallel quark lines, we see that

the initial loop splits into two loops and the final loop is untouched. As a result of the gluon emission, three loops are present, that therefore count as $\mathcal{O}(N_c^3)$. However, two QCD couplings are also present, and thus $g_s^2 \propto 1/N_c$ at fixed 't Hooft coupling λ . Eventually, emission of a gluon between initial quark lines give a diagram that is $\mathcal{O}(N_c^2)$, the same as the original disconnected tree-level diagram. The same would happen if the gluon is emitted between two final quarks lines. On the other hand, we show in Fig. 4 (top right) the emission of a gluon between one initial and one final quark line. In this case, only one closed loop is present, as the two disconnected ones become color-connected. Therefore, the power-counting gives N_c (one closed loop) times $1/N_c$ (due to two couplings g_s^2), such that this diagram is $\mathcal{O}(1)$. It can be seen that, at large N_c , the second diagram is $\mathcal{O}(1/N_c^2)$ with respect to the first one (and to the tree-level one). The same result is obtained for Fig. 4 (bottom left) and Fig. 4 (bottom right), where the emission of two and three gluons between initial and final quark lines is shown. Also in these two cases, the number of color loops is the same as the number of g_s^2 insertions and, thus, all of these diagrams are $\mathcal{O}(1)$.

Consider now the tree-level diagram in Fig. 2 (right). In this case, we have a single loop and no QCD couplings. Therefore, at large N_c this diagram counts as $\mathcal{O}(N_c)$. In Fig. 5 we show gluon emissions between initial (or final) quark lines (left) or between one initial and one final quark line (right) in the case of the connected diagram. It is easy to see that both diagrams are $\mathcal{O}(N_c)$, as they have two loops and two QCD couplings, $g_s^2 \propto 1/N_c$. Therefore, we can classify the different diagrams as follows:

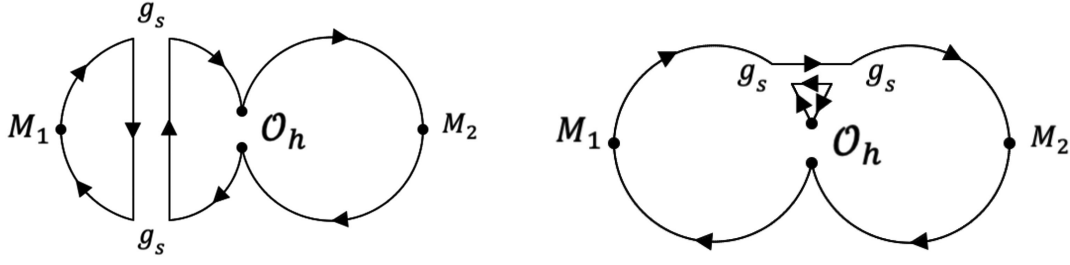


FIG. 5. Connected diagrams with one gluon emission between initial quark lines (left) or between initial and final quark lines (right) for the decay $M_1^+ \rightarrow M_2^- \ell^+ \ell^+$, with effective operator $\mathcal{O} = \mathcal{O}_h \times J_{\ell\ell}$. Both diagrams are $\mathcal{O}(N_c)$, as they contain two loops and two couplings, $g_s^2 \propto 1/N_c$.

- (i) *Leading*: Disconnected diagrams (tree-level and with gluons between either initial or final quarks), $\mathcal{O}(N_c^2)$;
- (ii) *Subleading*: Connected diagrams (tree-level and dressed), $\mathcal{O}(N_c)$;
- (iii) *Subsubleading*: Disconnected diagrams with gluons between initial and final quarks, $\mathcal{O}(1)$.

B. The vacuum insertion approximation

Since carrying out the nonperturbative computation is usually a hard task, if we consider only the leading contribution at large N_c , we may get a first estimate of the hadronic matrix element. At leading order in N_c , for the $K^+ \rightarrow \pi^- \ell^+ \ell^+$ process we have

$$\begin{aligned} & \langle \pi(z) | \bar{u}(x) \Gamma d(x) \bar{u}(x) \Gamma' s(x) | K(0) \rangle \\ &= \sum_n \langle \pi(z) | \bar{u}(x) \Gamma d(x) | n \rangle \langle n | \bar{u}(x) \Gamma' s(x) | K(0) \rangle + \mathcal{O}(1/N_c), \end{aligned} \quad (26)$$

where Γ, Γ' are some of the γ -matrices in the operator basis of Eq. (7). As the leading contribution at large N_c is represented by disconnected diagrams, we have inserted a complete set of intermediate states $|n\rangle$ and summed over them. The set of intermediate states takes into account that, nonperturbatively, any possible state compatible with the quantum numbers of $J_h |K^+\rangle$ and $J'_h |\pi^-\rangle$ should be produced in principle. Of all the possible states that can be produced, the lightest is of course $|0\rangle$. On the other hand, if we produce a meson state $|M\rangle \langle M|$, we should recall that at large N_c any source of a meson field counts as $1/\sqrt{N_c}$ [58,59]. This means that producing and annihilating an intermediate meson state $|n\rangle$, different from the vacuum, corresponds to a subleading contribution at large N_c .

As a consequence, at large N_c the leading contribution to the hadronic matrix element comes from the disconnected diagram in which we only consider the vacuum as a possible intermediate state. The so-called *vacuum insertion approximation* (VIA) states, then, that

$$\begin{aligned} & \langle \pi(z) | \bar{u}(x) \Gamma d(x) \bar{u}(x) \Gamma' s(x) | K(0) \rangle \\ &= \langle \pi(z) | \bar{u}(x) \Gamma d(x) | 0 \rangle \langle 0 | \bar{u}(x) \Gamma' s(x) | K(0) \rangle + \mathcal{O}(1/N_c). \end{aligned} \quad (27)$$

The two resulting two-point correlation functions, $\langle \pi^- | \bar{u}(x) \Gamma d(x) | 0 \rangle$ and $\langle 0 | \bar{u}(x) \Gamma' s(x) | K(0) \rangle$, can be estimated nonperturbatively using soft-pion theorems, as is reminded in Ref. [49]

$$\begin{aligned} \langle 0 | \bar{q}_1 \gamma_5 q_2 | M \rangle &= i \xi_M f_M, \\ \langle 0 | \bar{q}_1 \gamma_\mu \gamma_5 q_2 | M \rangle &= i p_\mu f_M, \end{aligned} \quad (28)$$

with $\xi_M = m_M^2 / (m_1 + m_2)$ the pseudoscalar density of the meson M (with m_1, m_2 the masses of the corresponding constituent quarks), f_M the meson decay constant and p_μ the meson momentum. On the other hand, being the vacuum a singlet under the Lorentz group, the matrix element of the tensor current vanishes

$$\langle 0 | \bar{q}_1 \sigma^{\mu\nu} P_X q_2 | M \rangle = 0 \quad (29)$$

due to Lorentz invariance (see also Ref. [24]).

In our computations, we have used $f_{K^\pm} = 155.7 \pm 0.7$ MeV and $f_{\pi^\pm} = 130.2 \pm 0.8$ MeV (from the most recent lattice results, see Ref. [60]) and the PDG values of the quarks running masses in the $\overline{\text{MS}}$ scheme at $\mu = 2$ GeV (see Ref. [61]), $m_u^{\overline{\text{MS}}}(2 \text{ GeV}) = 2.16^{+0.49}_{-0.26}$ MeV, $m_d^{\overline{\text{MS}}}(2 \text{ GeV}) = 4.67^{+0.48}_{-0.17}$ MeV and $m_s^{\overline{\text{MS}}}(2 \text{ GeV}) = 93.4^{+8.6}_{-3.4}$ MeV. For the operator $\mathcal{O}_h = \mathcal{O}_3^{\text{XZX}}$ in the type-I seesaw model, this is how we obtained the expression in Eq. (2). Notice that applying soft-pion theorems to the two-point correlation functions above gives a nonperturbative computation of each disconnected loop, which is encoded in a nonperturbative parameter to be measured experimentally, as in the case of f_π and f_K . In this approximation, the only scale dependence of the hadronic matrix elements comes through the quark running masses (as, due to the PCAC, the decay constants do not run). For the general case at hand, we may derive an estimate for the amplitudes $\mathcal{A}_i^{\text{XYZ}}$ valid at

leading order in $1/N_c$

$$\mathcal{A}_{1d}^{XYZ} = \pm 2f_{M_1}f_{M_2}\xi_{M_1}\xi_{M_2} \otimes [\bar{u}(p_{\ell_1})P^Z v(p_{\ell_2})] + \mathcal{O}(1/N_c), \quad (30)$$

$$\mathcal{A}_{3d}^{XYZ} = \pm 2f_{M_1}f_{M_2}(p_M \cdot p'_M) \otimes [\bar{u}(p_{\ell_1})P^Z v(p_{\ell_2})] + \mathcal{O}(1/N_c), \quad (31)$$

$$\mathcal{A}_{5d}^{XYZ}, \mathcal{A}'_{5d}^{XYZ} = \pm 2f_{M_1}f_{M_2}\xi_{M_1}p_M^\mu \otimes [\bar{u}(p_{\ell_1})\gamma_\mu P^Z v(p_{\ell_2})] + \mathcal{O}(1/N_c), \quad (32)$$

$$\mathcal{A}_{6d}^{XYZ} = \pm 2f_{M_1}f_{M_2}p_{M_1}^\mu p_{M_2}^\nu \otimes [\bar{u}(p_{\ell_1})\sigma_{\mu\nu} P^Z v(p_{\ell_2})] + \mathcal{O}(1/N_c), \quad (33)$$

where $+$ and $-$ correspond to cases $X = Y$ and $X \neq Y$, respectively, and the label d reminds us that the amplitudes only take into account the contribution of the disconnected diagram. Due to Eq. (29), the amplitudes $\mathcal{A}_{2d}$, $\mathcal{A}_{4d}$, $\mathcal{A}_{7d}$ and $\mathcal{A}_{8d}$ vanish in this approximation for any choice of the chiralities X, Y, Z , both for identical or different final leptons.

C. Bounds on $C_n(\mu)$ at leading order in $1/N_c$

Once we have an estimate of the hadronic matrix elements at leading order in $1/N_c$, we may use Eq. (11) to find tree-level bounds on the Wilson coefficients $C_{n(q_1 q_2)(q_3 q_4)}^{XYZ}$. The bounds on the tree-level Wilson coefficients are shown in Sec. VI, in the second column of Tables I and II (for $\ell_1 = \ell_2 = e$ and μ , respectively) and III

(for $\ell_1 = e$ and $\ell_2 = \mu$). Notice that no tree-level bound can be put on the Wilson coefficients C_2, C_4, C_7 and C_8 , as the corresponding amplitudes, $\mathcal{A}_{2d}$, $\mathcal{A}_{4d}$, $\mathcal{A}_{7d}$, and $\mathcal{A}_{8d}$, vanish at the leading order in $1/N_c$, since meson decay modes mediated by tensor currents are suppressed [24,29].

Dressing the relevant operators with hard gluons and including the corresponding one-loop QCD corrections in our amplitudes, we can use the RG evolution matrices $\hat{U}(\mu, \Lambda)$, computed from Eq. (18), to derive RGE-improved limits on the Wilson coefficients.

If, using Eq. (11), we were able to put an experimental bound on the tree-level Wilson coefficient $C_n \lesssim C_n^{\text{exp}}$ then, due to radiative corrections, we must compute the RGE-evolved Wilson coefficient $C_n(\mu)$ and apply the constraint as follows:

$$C_n \rightarrow C_n(\mu) = \sum_m \hat{U}_{nm}(\mu, \Lambda) C_m(\Lambda) \lesssim C_n^{\text{exp}}, \quad (34)$$

where the indices n, m run over the operators that are mixed by the evolution matrix $\hat{U}(\mu, \Lambda)$. In the absence of a

TABLE I. Bounds on Wilson coefficients at $\mu = 2$ GeV from the process $K^+ \rightarrow \pi^- e^+ e^+$: at tree level, including RGE, and with the effect of connected diagrams. The labels X, Y stand for the chiralities of the two quark bilinears, whereas Z refers to the chirality of the lepton one, respectively. When the two quark-related labels are identical (XX), the bound applies to both LL, RR operators. When they differ (XY), the bound applies to LR, RL . On the other hand, the chirality of the lepton bilinear is irrelevant but for C_2^{XXX} , for which the bound applies only for the combinations LLL, RRR .

Coefficient	Tree	RGE-Imp [49]	RGE + connected diag.
C_1^{XXZ}	2.5×10^2	1.5×10^2	1.2×10^2
C_1^{XYZ}	2.5×10^2	1.1×10^2	1.1×10^2
C_2^{XXX}	Not applicable	2.8×10^4	4.5×10^2
C_3^{XXZ}	1.9×10^4	2.3×10^4	2.0×10^4
C_3^{XYZ}	1.9×10^4	2.1×10^4	8.5×10^2
C_4^{XXZ}	Not applicable	3.0×10^4	9.2×10^3
C_4^{XXZ}	Not applicable	3.3×10^4	1.1×10^4
C_4^{XYZ}	Not applicable	1.3×10^5	1.3×10^4
C_4^{XYZ}	Not applicable	1.2×10^5	1.1×10^4
C_5^{XXZ}	2.3×10^3	1.5×10^3	1.7×10^3
C_5^{XXZ}	2.1×10^3	1.4×10^3	1.5×10^3
C_5^{XYZ}	2.3×10^3	1.8×10^3	1.9×10^3
C_5^{XYZ}	2.1×10^3	1.6×10^3	1.7×10^3

TABLE II. Bounds on Wilson coefficients at $\mu = 2$ GeV from the process $K^+ \rightarrow \pi^- \mu^+ \mu^+$: at tree level, including RGE, and with the effect of connected diagrams. Notice that in this case the bound apply to all lepton bilinear chiralities.

Coefficient	Tree	RGE-Imp [49]	RGE + connected diag.
C_1^{XXZ}	3.7×10^2	2.2×10^2	1.8×10^2
C_1^{XYZ}	3.7×10^2	1.6×10^2	1.6×10^2
C_2^{XXZ}	Not applicable	4.2×10^4	6.7×10^2
C_3^{XXZ}	3.1×10^4	3.8×10^4	3.2×10^4
C_3^{XYZ}	3.1×10^4	3.4×10^4	1.3×10^3
C_4^{XXZ}	Not applicable	5.6×10^4	1.8×10^4
C_4^{XXZ}	Not applicable	6.3×10^4	$2. \times 10^4$
C_4^{XYZ}	Not applicable	2.5×10^5	2.3×10^4
C_4^{XYZ}	Not applicable	2.3×10^5	2.1×10^4
C_5^{XXZ}	4.3×10^3	2.8×10^3	3.1×10^3
C_5^{XXZ}	3.8×10^3	2.5×10^3	2.7×10^3
C_5^{XYZ}	4.3×10^3	3.3×10^3	3.6×10^3
C_5^{XYZ}	3.8×10^3	3.0×10^3	3.2×10^3

TABLE III. Bounds on Wilson coefficients from the process $K^+ \rightarrow \pi^- e^+ \mu^+$ at $\mu = 2$ GeV: at tree level, including RGE, and with the effect of connected diagrams. Notice that for different final leptons, the bounds on the Wilson coefficients C_1, C_3, C_4, C'_4, C_5 , and C'_5 do not depend on the chirality of the lepton bilinear, whereas for C_2, C_6, C_7, C'_7 , and C_8 they do.

Coefficient	Tree	RGE-Imp [49]	RGE + connected diag.
C_1^{XXZ}	2.0×10^2	1.2×10^2	9.1×10^1
C_1^{XYZ}	2.0×10^2	8.3×10^1	8.3×10^1
C_2^{XXX}	Not applicable	2.2×10^4	3.6×10^2
C_3^{XXZ}	1.5×10^4	1.9×10^4	1.6×10^4
C_3^{XYZ}	1.5×10^4	1.7×10^4	6.7×10^2
C_4^{XXZ}	Not applicable	2.6×10^4	7.9×10^3
C_4^{XXZ}	Not applicable	2.9×10^4	8.9×10^3
C_4^{XYZ}	Not applicable	1.2×10^5	1.1×10^4
C_4^{XYZ}	Not applicable	1.0×10^5	9.4×10^3
C_5^{XXZ}	2.0×10^3	1.3×10^3	1.4×10^3
C_5^{XXX}	1.8×10^3	1.2×10^3	1.3×10^3
C_5^{XYZ}	2.0×10^3	1.5×10^3	1.7×10^3
C_5^{XYZ}	1.8×10^3	1.4×10^3	1.5×10^3
C_6^{XXX}	3.2×10^4	3.5×10^4	3.0×10^4
C_6^{XXY}	3.2×10^4	2.8×10^4	3.4×10^4
C_6^{XYX}	3.2×10^4	3.4×10^4	3.4×10^4
C_6^{YYX}	3.2×10^4	3.4×10^4	3.4×10^4
C_7^{XXX}	Not applicable	Not applicable	Not applicable
C_7^{XXY}	Not applicable	5.4×10^5	7.3×10^4
C_7^{XXX}	Not applicable	Not applicable	Not applicable
C_7^{XYX}	Not applicable	5.4×10^5	7.3×10^4
C_8^{XXX}	Not applicable	Not applicable	Not applicable

nonperturbative computation of the hadronic matrix elements, we may again make use of the VIA at the leading order in $1/N_c$ to get an estimate of the experimental bounds on the RGE-evolved Wilson coefficient. This is done using Eq. (21). We give the RGE-improved bounds for the Wilson coefficients in Sec. VI, in the third column of Tables I, II (for identical final leptons), and III (for different final leptons).

The main impact of the RGE-running of the Wilson coefficients from the high-energy scale Λ to the low-energy

scale μ is that we are able to put constraints on the Wilson coefficients $C_2, C_4, C'_4, C_6, C_7, C'_7$, something impossible using the tree-level expressions. This is because, through the evolution matrix $\hat{U}$, the matrix element of the renormalized operator $\hat{\mathcal{O}}(\mu)$ is

$$[\mathcal{A}_n^{XYZ}(\mu)]_d = \sum_m U_{nm}(\mu, \Lambda) [\mathcal{A}_m^{XYZ}(\Lambda)]_d, \quad (35)$$

where the right-hand side can be computed in the VIA using Eqs. (30)–(31).

Notice that the VIA constitutes a robust recipe to obtain an estimate of the hadronic matrix elements, as it represents the leading order in their $1/N_c$ expansion. In the absence of a full nonperturbative computation, this is indeed the only way to get a reasonable quantitative estimate. In chiral perturbation theory, as used for example in Ref. [37], the effective Lagrangian is written in terms of a set of operators. In this approach the nonperturbative information is encoded in the low-energy constants (LECs), to be either computed by lattice gauge theory or to be taken from experimental data. In the absence of such additional information from lattice or experiment, any values for the relevant LECs are allowed. Thus, in order to derive quantitative limits on Wilson coefficients, chiral perturbation theory relies on outside information.⁴

V. CONNECTED DIAGRAMS CONTRIBUTION TO $K^+ \rightarrow \pi^- l^+ l^+ \Delta L = 2$ TRANSITIONS

After computing the perturbative one-loop renormalization of the operators in the basis shown in Eq. (7) (Sec. III) and giving a nonperturbative estimate of the hadronic matrix elements of those operators applying the VIA at the leading order in $1/N_c$, we may now compute the long-range subleading contribution at large N_c to the hadronic matrix elements. Recall that the amplitude corresponding to inserting a quark four-fermion operator $\mathcal{O}_h$ between the initial and final meson states $|K^+\rangle$ and $|\pi^-\rangle$ can be obtained by the following $\mathcal{T}$ -product:

$$\begin{aligned}
 \langle \pi^-(z) | \mathcal{O}_i^{XYZ}(x) | K^+(0) \rangle &= \mathcal{T} \left[\langle 0 | (\bar{d}\gamma_5 u)_z (\bar{u}\Gamma_X d)_x (\bar{u}\Gamma_Y s)_x (\bar{s}\gamma_5 u)_0 | 0 \rangle \right] \\
 &= \langle 0 | \overbrace{\bar{d}(z)\gamma_5 u(z)} \overbrace{\bar{u}(x)\Gamma_X d(x)} \overbrace{\bar{u}(z)\Gamma_Y s(z)} \overbrace{\bar{s}(0)\gamma_5 u(0)} | 0 \rangle \\
 &\quad - \langle 0 | \overbrace{\bar{d}(z)\gamma_5 u(z)} \overbrace{\bar{u}(x)\Gamma_X d(x)} \overbrace{\bar{u}(z)\Gamma_Y s(z)} \overbrace{\bar{s}(0)\gamma_5 u(0)} | 0 \rangle.
 \end{aligned} \quad (36)$$

⁴We note in passing that in Ref. [37] only limits on a particular subset of three operators are discussed. The authors take the corresponding LECs from lattice QCD.

The first amplitude corresponds to the disconnected diagram, and it can be directly approximated inserting a complete set of intermediate states $|n\rangle\langle n|$ and then applying the VIA (see Sec. IV). On the other hand, the second amplitude corresponds to the connected diagram and the VIA cannot be applied directly. The relative sign between the two comes from the different number of closed fermion loops.

In order to get a VIA-consistent estimate of the hadronic matrix element up to subleading order in $1/N_c$, we must relate the connected diagram (unaccounted for up to now) to the disconnected diagram, whose computation in the VIA gives the $N_c \rightarrow \infty$ limit and was studied in the previous section. This is usually done by means of Fierz-rearranging the fields in the local effective operator, so as to relate one kind of contractions with the external states (that give rise to the connected diagram) to the other one (that give rise to the disconnected diagram). By doing this, operators belonging to the basis in Eq. (7) may mix among themselves. Quite generally, $1/N_c$ factors arise due to the corresponding rearrangement of color indices. This is a straightforward consequence of the fact that the connected diagram is $\mathcal{O}(N_c)$ in the large N_c expansion, whereas the disconnected diagram is $\mathcal{O}(N_c^2)$. The understanding of the VIA in the language of the $1/N_c$ expansion was first described in the case of $\Delta S = 2$ and $\Delta S = 1$ kaon processes in Refs. [62,63] and for charmed mesons decays in Ref. [64].

The Fierz-rearranging rules of spinor indices in Minkowski space-time for the four-fermion operators relevant for the $K^+ \rightarrow \pi^- \ell^+ \ell^+$ transition at hand are given in Appendix C. Notice that the anticommutation of fermion fields in the $\mathcal{T}$ -product is already considered. After

performing spin-indices rearrangement, a color-indices rearrangement must be applied, too, using the relation

$$\delta_{AB} \otimes \delta_{CD} = -\frac{1}{2N_c} \delta_{AD} \otimes \delta_{CB} + \frac{1}{2} \sum_a t_{AD}^a \otimes t_{CB}^a, \quad (37)$$

with $a = 1, \dots, N_c^2 - 1$ color indices in the adjoint representation and $A, B, C, D = 1, \dots, N_c$ color indices in the fundamental representation, respectively. Eventually, after rearranging spinor and color indices, we can give an estimate of the nonperturbative amplitudes in terms of the $\mathcal{O}(1/N_c)$ mixing matrix $\hat{B}$

$$[\vec{\mathcal{A}}^{XYZ}]_{\text{VIA}} = \hat{B} \left(\frac{1}{N_c} \right) [\vec{\mathcal{A}}^{XYZ}]_d \quad (38)$$

being the disconnected amplitudes $\vec{\mathcal{A}}_d^{XYZ}$ given in Eqs. (30)–(31), for the two cases of identical chiralities in the hadronic currents (XXZ), or different chiralities (XYZ , with $X \neq Y$).

The computation of the sum of connected and disconnected diagrams in the VIA is given in Appendix D. Using, then, the relations in Eqs. (28) and (29), it is possible to reduce the nonperturbative amplitude $\langle M_2 | \mathcal{O}_i^{XYZ} | M_1 \rangle$ to simple expressions that depend on the meson decay constants and on the pseudoscalar densities, only (such as those in Eqs. (13)–(16) of Ref. [49]), albeit including $\mathcal{O}(1/N_c)$ corrections induced by the connected diagrams. As we have seen above, the connected diagrams may induce mixing between different operators belonging to the basis in Eqs. (7) and (9). In the case of identical chiralities for the two hadronic currents, we have

$$\begin{pmatrix} \mathcal{A}_1^{XXZ} \\ \mathcal{A}_2^{XXZ} \\ \mathcal{A}_3^{XXZ} \\ \mathcal{A}_4^{XXZ} \\ \mathcal{A}_4'^{XXZ} \\ \mathcal{A}_5^{XXZ} \\ \mathcal{A}_5'^{XXZ} \end{pmatrix}_{\text{VIA}} = \begin{pmatrix} 1 - \frac{1}{4N_c} & 0 & 0 & 0 & 0 & 0 & 0 \\ \left(-\frac{3}{N_c}\right)^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 + \frac{1}{2N_c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3i}{4N_c} \\ 0 & 0 & 0 & 0 & 0 & -\frac{3i}{4N_c} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4N_c} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4N_c} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1^{XXZ} \\ \mathcal{A}_2^{XXZ} \\ \mathcal{A}_3^{XXZ} \\ \mathcal{A}_4^{XXZ} \\ \mathcal{A}_4'^{XXZ} \\ \mathcal{A}_5^{XXZ} \\ \mathcal{A}_5'^{XXZ} \end{pmatrix}_d, \quad (39)$$

where the $*$ near the matrix element B_{21} , that gives the relation between $\mathcal{A}_2^{XXZ}$ in the VIA with $\mathcal{A}_{1d}^{XXZ}$, indicates that it only differs from zero for $Z = X$. In the case of operators with different chiralities, we have

$$\begin{pmatrix} \mathcal{A}_1^{XYZ} \\ \mathcal{A}_2^{XYZ} \\ \mathcal{A}_3^{XYZ} \\ \mathcal{A}_4^{XYZ} \\ \mathcal{A}_4'^{XYZ} \\ \mathcal{A}_5^{XYZ} \\ \mathcal{A}_5'^{XYZ} \end{pmatrix}_{\text{VIA}} = \begin{pmatrix} 1 & 0 & -\frac{1}{4N_c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{N_c} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3i}{4N_c} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3i}{4N_c} \\ 0 & 0 & 0 & 0 & 0 & 1 - \frac{1}{4N_c} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 - \frac{1}{4N_c} \end{pmatrix} \begin{pmatrix} \mathcal{A}_1^{XYZ} \\ \mathcal{A}_2^{XYZ} \\ \mathcal{A}_3^{XYZ} \\ \mathcal{A}_4^{XYZ} \\ \mathcal{A}_4'^{XYZ} \\ \mathcal{A}_5^{XYZ} \\ \mathcal{A}_5'^{XYZ} \end{pmatrix}_d. \quad (40)$$

The amplitudes of the operators $\mathcal{O}_6^{XYZ}$, $\mathcal{O}_7^{XYZ}$, $\mathcal{O}_7'^{XYZ}$, and $\mathcal{A}_8^{XYZ}$ mostly do not mix under Fierzing. We have

$$\begin{aligned} [\mathcal{A}_6^{XXX}]_{\text{VIA}} &= \left(1 + \frac{1}{2N_c}\right) \mathcal{A}_{6d}^{XXX}, \\ [\mathcal{A}_6^{XXZ}]_{\text{VIA}} &= \left(1 - \frac{1}{2N_c}\right) \mathcal{A}_{6d}^{XXZ}, \\ [\mathcal{A}_6^{XYX}]_{\text{VIA}} &= \mathcal{A}_{6d}^{XYX}, \end{aligned} \quad (41)$$

and

$$\begin{aligned} [\mathcal{A}_7^{XXX}]_{\text{VIA}} &= 0, \\ [\mathcal{A}_7'^{XXX}]_{\text{VIA}} &= 0, \\ [\mathcal{A}_8^{XXX}]_{\text{VIA}} &= 0. \end{aligned} \quad (42)$$

The only exception is the case of operators with XYX chiralities, for which

$$\begin{pmatrix} \mathcal{A}_6^{XYX} \\ \mathcal{A}_7^{XYX} \\ \mathcal{A}_7'^{XYX} \end{pmatrix}_{\text{VIA}} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{i}{N_c} & 0 & 0 \\ \frac{i}{N_c} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{A}_6^{XYX} \\ \mathcal{A}_7^{XYX} \\ \mathcal{A}_7'^{XYX} \end{pmatrix}_d \quad (43)$$

The main result in Eqs. (39)–(43) is that we may improve the bounds on the Wilson coefficients $C_2(\mu)$, $C_4(\mu)$, $C_7(\mu)$ and $C_8(\mu)$ using the $\mathcal{O}(1/N_c)$ -improved VIA, due to the mixing of the corresponding operators with those for which the amplitudes $\mathcal{A}_d^{XYZ}$ are nonvanishing.

VI. IMPROVED BOUNDS ON THE WILSON COEFFICIENTS

In this section, we eventually present our $\mathcal{O}(1/N_c)$ improved bounds on the Wilson coefficients corresponding to the operator basis defined in Eqs. (7)–(9), by taking into account the contributions from the connected diagrams in the hadronic matrix elements in the *vacuum insertion approximation*. In order to derive the bounds, we consider the effect of one operator at a time. The bounds are obtained from the most recent experimental limits [12,13,16]

$$\begin{aligned} \text{Br}(K^+ \rightarrow \pi^- e^+ e^+) &\leq 5.3 \times 10^{-11}, \\ \text{Br}(K^+ \rightarrow \pi^- \mu^+ \mu^+) &\leq 4.2 \times 10^{-11}, \\ \text{Br}(K^+ \rightarrow \pi^- e^+ \mu^+) &\leq 4.2 \times 10^{-11}, \end{aligned} \quad (44)$$

and are shown in Tables I and II, for identical final state leptons ($e^+ e^+$ and $\mu^+ \mu^+$, respectively) and (III) for different final state leptons. In each Table, the first column lists the considered Wilson coefficient, numbered as their corresponding operator, with an explicit reference to the chiralities of the two quark currents (in the case of different final leptons, Table III, when needed also the chirality of the lepton current is given); the second column presents the tree-level bounds obtained from the expression for the decay width, Eq. (11); the third column includes the effect of (perturbative) operator mixing under RGE evolution, as developed in Ref. [49] and reviewed in Secs. III and IV; and, finally, the fourth column shows the bounds incorporating both RGE evolution effects and the (nonperturbative) contribution of connected diagrams to the hadronic matrix elements, presented in Sec. V.

We will use Table I to explain in detail our results. First of all notice that, at tree-level, only some Wilson coefficients can be bounded. This is because some of the hadronic matrix elements vanish, when approximated by their disconnected diagram contribution neglecting soft gluon interchange between the initial and final meson (i.e., in the VIA at leading order in $1/N_c$). This is indeed the case of the operators $\mathcal{O}_2$, $\mathcal{O}_4$, and $\mathcal{O}_4'$ (for identical final leptons), together with $\mathcal{O}_7$, $\mathcal{O}_7'$, and $\mathcal{O}_8$ (for different final leptons). The tree-level bounds at the high-energy scale Λ (that in this work is taken to be M_Z) are rather loose: $\mathcal{O}_1$ has a Wilson coefficient $C_1(M_Z) \leq 2.5 \times 10^2$; $\mathcal{O}_5$ has a Wilson coefficient $C_5(M_Z) \leq 2.3 \times 10^3$; $\mathcal{O}_5'$ has a Wilson coefficient $C_5'(M_Z) \leq 2.1 \times 10^3$; and, eventually, $\mathcal{O}_3$ has a Wilson coefficient $C_3(M_Z) \leq 1.9 \times 10^4$. In all cases, the chiralities of the two quark currents are irrelevant: the same bounds are obtained for LL, RR, LR or RL (irrespectively of the chirality of the lepton current). Notice that the operator $\mathcal{O}_3$ (the one we would get in a theory whose new physics only involves left-handed vector currents, as in the SM) has the loosest bound.

After running the Wilson coefficients down to the scale where the hadronic matrix elements should be computed (for which we choose $\mu = 2$ GeV), we can see that the effect of perturbative mixing induced by one-loop RGE is extremely significant: due to mixing, we can indeed constrain operators whose tree-level Wilson coefficients were unbounded. This is the case of $\mathcal{O}_2$, for which we get $C_2(2 \text{ GeV}) \leq 2.8 \times 10^4$; $\mathcal{O}_4$, for which we get⁵ $C_4(2 \text{ GeV}) \leq 3.0 \times 10^4$ (1.3×10^5); and $\mathcal{O}'_4$, for which we get $C'_4(2 \text{ GeV}) \leq 3.3 \times 10^4$ (1.2×10^5). The other operators are also significantly modified. RGE-improved bounds on $\mathcal{O}_1$, $\mathcal{O}_3$, $\mathcal{O}_5$ and $\mathcal{O}'_5$ are: $C_1(2 \text{ GeV}) \leq 1.5 \times 10^2$ (1.1×10^2); $C_5(2 \text{ GeV}) \leq 1.5 \times 10^3$ (1.8×10^3); and $C'_5(2 \text{ GeV}) \leq 1.4 \times 10^3$ (1.6×10^3) for identical (different) quark bilinear chiralities. We can see that the difference ranges from 50% to 300% (in the case of C_1^{XYZ}). The only operator for which the RGE from M_Z down to $\mu = 2$ GeV loosen the tree-level bound is $\mathcal{O}_3$, for which we get $C_3(2 \text{ GeV}) \leq 2.3 \times 10^4$ (2.1×10^4), depending of the chiralities of the quark currents (XX or XY , with $X \neq Y$, respectively).

Eventually, in the last column we show the impact on the Wilson coefficients of the contribution of the connected diagrams. These diagrams, although subleading in $1/N_c$, must be added to the disconnected ones in order to reconstruct consistently the full hadronic amplitude in a controlled approximation (the VIA, i.e., neglecting soft-gluon emissions and the creation of mesonic intermediate states between initial and final mesons). The impact of the connected diagrams is of two kinds: for the operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, and $\mathcal{O}'_4$, we get a significant improvement on the bounds on the corresponding Wilson coefficients (but for C_1^{XYZ} and C_3^{XXZ} , for which the improvement is rather small); or, for $\mathcal{O}_5$ and $\mathcal{O}'_5$, we find that the bounds are a little looser than after the perturbative running (but still tighter than the tree-level ones). These behaviors can be easily traced back to the mixing induced by the inclusion of connected diagrams: for the first set of operators, we can see that operators with loose bounds (such as $\mathcal{O}_2$) get mixed with operators with a tighter bound (in the case of $\mathcal{O}_2$, with $\mathcal{O}_1$). As a consequence, tighter constraints apply and we get that $C_2^{XXX}(2 \text{ GeV})$ indeed go from 2.8×10^4 to 4.5×10^2 (i.e., a reduction of a factor 62), or $C_3^{XYZ}(2 \text{ GeV})$ go from 2.1×10^4 to 8.5×10^2 (i.e., a reduction of a factor 25). On the other hand, $\mathcal{O}_5$ and $\mathcal{O}'_5$ mix between themselves with a negative contribution of the connected diagram to the disconnected one, thus making the bound looser by $\mathcal{O}(1/N_c)$.

When we move to Table II, we can see that for two muons in the final state the situation is not much different: also in this case the biggest improvements with respect

to the one-loop RG-evolved bounds can be found for C_2^{XXZ} , that goes from 4.2×10^4 down to 6.7×10^2 , and for C_3^{XYZ} , that goes from 3.4×10^4 down to 1.3×10^3 (in both cases, the improvement is identical to that found for ee in the final state).

If we eventually focus on Table III, where our results for two different lepton in the final state are shown, we see that for the operators $\mathcal{O}_1 - \mathcal{O}_5$ there are no big differences with respect to the case of two identical leptons. On the other hand, for the new set of operators that are only present in this case, $\mathcal{O}_6$, $\mathcal{O}_7$, and $\mathcal{O}_8$, we find no improvement for the bounds on the Wilson coefficients C_6 , while at the same we get a significant change in the bounds for C_7^{XYX} , C'_7^{XYX} , for which we go from 5.4×10^5 for the one-loop RG-evolved coefficients down to 7.3×10^4 for the full VIA result (a reduction of a factor 7). Notice that, as it was the case for both the tree-level and the RGE-improved results, we are not able to constrain the Wilson coefficient of the operator $\mathcal{O}_8$.

VII. CONCLUSIONS

In this paper we have considered lepton number violating meson decays. Building on previous work [49], which included QCD one-loop corrections as well as renormalization group (RG) running in the matching of the $d = 9$ LNV operators describing these decays, we have calculated, for the first time in the literature, $\mathcal{O}(1/N_c)$ corrections to the hadronic matrix element.

Previous works on LNV meson decays have all estimated the hadronic matrix element by considering the disconnected diagram in the so-called vacuum insertion approximation, only. We first have discussed in detail, how this procedure describes the leading term of the hadronic matrix element in the limit $N_c \rightarrow \infty$. Connected diagrams, previously neglected, should be also included to have a consistent VIA estimate, though. These diagrams, after Fierz rearrangement of spinor and color indices, give $\mathcal{O}(1/N_c)$ corrections to the previous estimates and induce mixing between different operators, which has to be rediagonalized in order to extract limits on Wilson coefficients.

Naive power counting in $(1/N_c)$ would lead to the expectation that the inclusion of the connected $\mathcal{O}(1/N_c)$ diagrams, that we have considered in this work, would change limits on the Wilson coefficients by a factor roughly given as $(1 + 1/N_c)$, i.e., a naive 30% effect. Our main numerical results, shown in Tables I–III, however, show that this is not always the case. In some particular cases bounds on Wilson coefficients change by nearly two orders of magnitude. This can be traced back to the fact that the naive expectation neglects that after the inclusion of connected diagrams one needs to rediagonalize the operator basis. The effect is similar to that found when the tree-level

⁵For the operators $\mathcal{O}_4$ and $\mathcal{O}'_4$, chirality matters: we get different bounds depending on whether we have identical (LL or RR) or different (LR or RL) chiralities for the quark currents.

bounds are improved by taking into account RG running. Operator mixing can lead, in some cases, to bounds on Wilson coefficients, which were completely absent at tree-level. That long-range contributions to the hadronic matrix elements can have such an important effect on the Wilson coefficients has, to the best of our knowledge, not been discussed in the literature before for $\Delta L = 2$ operators.

We should, however, also add a disclaimer here. While our calculation gives improved bounds for some Wilson coefficients, these bounds still do not provide competitive limits. This fact is easy to understand. Any of the $d = 9$ $\Delta L = 2$ operators we consider can be generated from integrating out some beyond-the-SM particles in the UV. We can convert the bounds on the Wilson coefficients into a bound on the (geometric) mean mass of those BSM states. With even the best limits in our tables giving numbers much larger than one, the constraint on the mean mass will be below the mass of the W-boson. (Charged) BSM particles with such low masses should have already been found by LEP. Thus, in order for the lepton number violating meson and tau decays to become competitive the experimental limits need to be improved by at least four order of magnitudes.

Improving the experimental limits by such large factors seems a difficult task, but not impossible. The HIKE experiment at CERN [65] was supposed to produce roughly 8×10^{13} K^+ mesons in the four years of running in its phase-I. And, according to the experimental collaboration, this would have allowed to probe exotic LNV decay modes of the kaon down to branching ratios of order 10^{-13} . The beam dump of the DUNE experiment [66], on the other hand, is estimated to produce K^+ 's in excess of 5×10^{21} in 10 years of running [67,68]. One could expect, therefore, that installing a HIKE like detector in the decay tunnel of DUNE's beam dump could probe rare kaon decays several orders of magnitude better than HIKE. Of course, no such proposal exists to the best of our knowledge.

We close this paper by reiterating that the inclusion of the connected $\mathcal{O}(1/N_c)$ diagrams is just the beginning of a full, nonperturbative computation of the long-range QCD effects in the matching of the LNV operators. A complete realistic calculation can only be performed using lattice field theory techniques. However, connected diagrams represent the leading order in the $1/N_c$ expansion of the QCD nonperturbative effects and thus we are fairly confident that our work presents a valid first step toward a complete calculation of the long-range part of the matrix elements.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ANOMALOUS DIMENSION MATRICES

We present in this appendix the anomalous dimensions for the operators in Eqs. (7) and (9), computed in dimensional regularization in the $\overline{\text{MS}}$ scheme at one-loop.

For identical or different final leptons we have⁶:

$$\begin{aligned}\hat{\gamma}_{(12)}^{XXZ} &= -2 \begin{pmatrix} 6C_F - 3 & \frac{N-2}{4N} \\ -\frac{12(2+N)}{N} & -3 - 2C_F \end{pmatrix}, \\ \hat{\gamma}_{(13)}^{XYZ} &= -2 \begin{pmatrix} 6C_F & 0 \\ -6 & -\frac{3}{N} \end{pmatrix}, \\ \hat{\gamma}_{(3)}^{XXZ} &= -2 \left(-3 + \frac{3}{N} \right), \\ \hat{\gamma}_{(4'45'5)}^{XXZ} &= -2 \begin{pmatrix} -C_F & -\frac{3}{2} & -\frac{3i}{N} & -\frac{3i}{2} \\ -\frac{3}{2} & -C_F & -\frac{3i}{2} & -\frac{3i}{N} \\ \frac{i}{N} & -\frac{i}{2} & 3C_F & -\frac{3}{2} \\ -\frac{i}{2} & \frac{i}{N} & -\frac{3}{2} & 3C_F \end{pmatrix}, \\ \hat{\gamma}_{(4'5')}^{XYZ} &= -2 \begin{pmatrix} -\frac{3}{2} - C_F & \frac{3i(N+2)}{2N} \\ \frac{i(N-2)}{2N} & -\frac{3}{2} + 3C_F \end{pmatrix},\end{aligned}\quad (\text{A1})$$

On the other hand, for different final leptons, we also have

⁶Notice that these matrices are not identical to those in Ref. [49], since in this work we are using a slightly different convention for the operator basis.

$$\begin{aligned}
\hat{\gamma}_{(6)}^{XXX} &= -2 \left(-1 + \frac{1}{N} \right), \\
\hat{\gamma}_{(6)}^{XXY} &= -2 \left(1 + \frac{1}{N} \right), \\
\hat{\gamma}_{(67')}^{XYX} &= -2 \begin{pmatrix} -\frac{1}{N} & -\frac{i}{2} \\ 0 & 2C_F \end{pmatrix}, \\
\hat{\gamma}_{(7'78)}^{XXX} &= -2 \begin{pmatrix} 2C_F - \frac{1}{N} - 2 & -\frac{1}{N} & \frac{i(8+N)}{8N} \\ -\frac{1}{N} & 2C_F - \frac{1}{N} - 2 & -\frac{i(8+N)}{8N} \\ 0 & 0 & -2C_F \end{pmatrix}. \quad (\text{A2})
\end{aligned}$$

APPENDIX B: μ -EVOLUTION MATRICES

For completeness, in this appendix we list the evolution matrices $\hat{U}(\mu, \Lambda)$, relevant for calculating the values in the third column (RGE-Imp) in Tables I–III. These matrices were calculated in Ref. [49] from the anomalous dimension matrix in Appendix A, for $\mu = 2$ GeV and $\Lambda = m_Z$.

For identical or different final leptons we have⁷

$$\begin{aligned}
\hat{U}_{(12)}^{XXZ} &= \begin{pmatrix} 1.68 & 0.01 \\ -2.12 & 0.55 \end{pmatrix}, \\
\hat{U}_{(13)}^{XYZ} &= \begin{pmatrix} 2.31 & 0. \\ -0.94 & 0.90 \end{pmatrix}, \\
\hat{U}_{(3)}^{XXZ}(iji\ell) &= 0.81, \\
\hat{U}_{(4'45'5)}^{XXZ} &= \begin{pmatrix} 0.88 & -0.14 & -0.09i & -0.17i \\ -0.14 & 0.88 & -0.17i & -0.09i \\ 0.05i & -0.07i & 1.54 & -0.24 \\ -0.07i & 0.05i & -0.24 & 1.54 \end{pmatrix}, \\
\hat{U}_{(4'5')}^{XYZ} &= \begin{pmatrix} 0.74 & 0.26i \\ 0.02i & 1.30 \end{pmatrix}. \quad (\text{B1})
\end{aligned}$$

On the other hand, for different final leptons we also have

⁷Notice that these matrices are not identical to those in Ref. [49], since in this work we are using a slightly different convention for the operator basis and we compute them at $\mu = 2$ GeV.

$$\begin{aligned}
\hat{U}_{(6)}^{XXX} &= 0.93, \\
\hat{U}_{(6)}^{XXY} &= 1.15, \\
\hat{U}_{(67')}^{XYX} &= \begin{pmatrix} 0.97 & -0.06i \\ 0 & 1.32 \end{pmatrix}, \\
\hat{U}_{(7'78)}^{XXX} &= \begin{pmatrix} 1.04 & -0.04 & 0.04i \\ -0.04 & 1.04 & -0.04i \\ 0 & 0 & 0.76 \end{pmatrix}. \quad (\text{B2})
\end{aligned}$$

APPENDIX C: FIERZ IDENTITIES

We write in this appendix the Fierz identities used in Sec. V. They have been obtained in Minkowski space-time. The chiral projectors are defined as follow

$$P_{L,R} = \frac{1}{2}(1 \mp \gamma_5) \quad (\text{C1})$$

and the tensor Dirac matrix is

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]. \quad (\text{C2})$$

No specific convention is needed for γ_5 . We list below the relevant Fierz identities needed to rearrange spinor indices of the four-fermion operators in Eqs. (7) and (9).

$$(1) \quad \mathcal{O}_1^{XYZ} = 8(\bar{u}^A P_X d^B) \otimes (\bar{u}^C P_Y s^D) \otimes j^Z \otimes (\delta_{AB} \delta_{CD})$$

$$\begin{aligned}
&(\bar{a}_\alpha^A P_X^{\alpha\beta} b_\beta^B) \otimes (\bar{c}_\gamma^C P_X^{\gamma\delta} d_\delta^D) \\
&= -\frac{1}{2}(\bar{a}_\alpha^A P_X^{\alpha\delta} d_\delta^D) \otimes (\bar{c}_\gamma^C P_X^{\gamma\beta} b_\beta^B) \\
&\quad -\frac{1}{8}(\bar{a}_\alpha^A [\sigma_{\mu\nu} P_X]^{\alpha\delta} d_\delta^D) \otimes (\bar{c}_\gamma^C [\sigma^{\mu\nu} P_X]^{\gamma\beta} b_\beta^B), \quad (\text{C3})
\end{aligned}$$

$$\begin{aligned}
&(\bar{a}_\alpha^A P_X^{\alpha\beta} b_\beta^B) \otimes (\bar{c}_\gamma^C P_Y^{\gamma\delta} d_\delta^D) \\
&= -\frac{1}{2}(\bar{a}_\alpha^A [\gamma_\mu P_Y]^{\alpha\delta} d_\delta^D) \otimes (\bar{c}_\gamma^C [\gamma^\mu P_X]^{\gamma\beta} b_\beta^B), \quad (\text{C4})
\end{aligned}$$

where $P_X = P_{L,R}$ and $P_Y = P_{R,L}$. In order to be extremely clear, we have explicitly written color indices (A, B, C, D) and spinor indices ($\alpha, \beta, \gamma, \delta$) of the four-fermion operator. It can be seen that, whereas spinor indices of the Dirac matrices on the right-hand side are correctly contracted over the spinor indices of the corresponding fermion fields, color indices do not match. In Sec. V, the rearrangement of color indices will be performed explicitly so to relate the connected diagrams with the disconnected ones.

$$(2) \quad \mathcal{O}_2^{XXX} = 8(\bar{u}^A \sigma_{\mu\nu} P_X d^B) \otimes (\bar{u}^C \sigma^{\mu\nu} P_X s^D) \otimes j^X \otimes (\delta_{AB} \delta_{CD})$$

$$\left(\bar{a}_\alpha^A [\sigma_{\mu\nu} P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C [\sigma^{\mu\nu} P_X]^{\gamma\delta} d_\delta^D \right) = -6 \left(\bar{a}_\alpha^A P_X^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C P_X^{\gamma\beta} b_\beta^B \right) + \frac{1}{2} \left(\bar{a}_\alpha^A [\sigma_{\mu\nu} P_X]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\sigma^{\mu\nu} P_X]^{\gamma\beta} b_\beta^B \right), \quad (C5)$$

where $P_X = P_{L,R}$.

$$(3) \quad \mathcal{O}_3^{XYZ} = 8(\bar{u}^A \gamma^\mu P_X d^B) \otimes (\bar{u}^C \gamma_\mu P_Y s^D) \otimes j^Z \otimes (\delta_{AB} \delta_{CD})$$

$$\left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C [\gamma^\mu P_X]^{\gamma\delta} d_\delta^D \right) = \left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\gamma^\mu P_X]^{\gamma\beta} b_\beta^B \right), \quad (C6)$$

$$\left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C [\gamma_\mu P_Y]^{\gamma\delta} d_\delta^D \right) = -2 \left(\bar{a}_\alpha^A P_Y^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C P_X^{\gamma\beta} b_\beta^B \right), \quad (C7)$$

where $P_X = P_{L,R}$ and $P_Y = P_{R,L}$.

$$(4) \quad \mathcal{O}_4^{XYZ} = 8(\bar{u} \gamma^\mu P_X d) \otimes (\bar{u} \sigma_\mu^\nu P_Y s) \otimes j_\nu^Z \otimes (\delta_{AB} \delta_{CD})$$

$$\left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C [\sigma^{\mu\nu} P_X]^{\gamma\delta} d_\delta^D \right) = \frac{1}{2} \left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\sigma^{\mu\nu} P_X]^{\gamma\beta} b_\beta^B \right) - \frac{3i}{2} \left(\bar{a}_\alpha^A [\gamma^\nu P_X]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C P_X^{\gamma\beta} b_\beta^B \right), \quad (C8)$$

$$\left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C [\sigma^{\mu\nu} P_Y]^{\gamma\delta} d_\delta^D \right) = \frac{1}{2} \left(\bar{a}_\alpha^A [\sigma^{\mu\nu} P_Y]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\gamma_\mu P_X]^{\gamma\beta} b_\beta^B \right) + \frac{3i}{2} \left(\bar{a}_\alpha^A P_Y^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\gamma^\nu P_X]^{\gamma\beta} b_\beta^B \right), \quad (C9)$$

where $P_X = P_{L,R}$ and $P_Y = P_{R,L}$.

$$(5) \quad \mathcal{O}_4^{XYZ} = 8(\bar{u} \sigma_\mu^\nu P_X d) \otimes (\bar{u} \gamma^\mu P_Y s) \otimes j_\nu^Z \otimes (\delta_{AB} \delta_{CD})$$

These Fierz identities can be derived from those for $\mathcal{O}_4^{XYZ}$ by changing the quark flavors, $(\bar{a}, b) \leftrightarrow (\bar{c}, d)$.

$$(6) \quad \mathcal{O}_5^{XYZ} = 8(\bar{u} \gamma^\mu P_X d) \otimes (\bar{u} P_Y s) \otimes j_\mu^Z \otimes (\delta_{AB} \delta_{CD})$$

$$\left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C P_X^{\gamma\delta} d_\delta^D \right) = -\frac{1}{2} \left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C P_X^{\gamma\beta} b_\beta^B \right) + \frac{i}{2} \left(\bar{a}_\alpha^A [\gamma_\nu P_X]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\sigma^{\nu\mu} P_X]^{\gamma\beta} b_\beta^B \right), \quad (C10)$$

$$\left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C P_Y^{\gamma\delta} d_\delta^D \right) = -\frac{1}{2} \left(\bar{a}_\alpha^A P_Y^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\gamma_\mu P_X]^{\gamma\beta} b_\beta^B \right) + \frac{i}{2} \left(\bar{a}_\alpha^A [\sigma^{\mu\nu} P_Y]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\gamma_\nu P_X]^{\gamma\beta} b_\beta^B \right). \quad (C11)$$

$$(7) \quad \mathcal{O}_5^{XYZ} = 8(\bar{u} P_X d) \otimes (\bar{u} \gamma^\mu P_Y s) \otimes j_\mu^Z \otimes (\delta_{AB} \delta_{CD})$$

These Fierz identities can be derived from those for $\mathcal{O}_5^{XYZ}$ by changing the quark flavors, $(\bar{a}, b) \leftrightarrow (\bar{c}, d)$.

$$(8) \quad \mathcal{O}_6^{XYZ} = 8(\bar{u} \gamma^\mu P_X d) \otimes (\bar{u} \gamma^\nu P_Y s) \otimes j_{\mu\nu}^Z \otimes (\delta_{AB} \delta_{CD})$$

For this operator we have several choices, depending on the chirality of the lepton current

$$\left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C [\gamma_\nu P_X]^{\gamma\delta} d_\delta^D \right) \otimes j_X^{\mu\nu} = - \left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\gamma_\nu P_X]^{\gamma\beta} b_\beta^B \right) \otimes j_X^{\mu\nu}, \quad (C12)$$

$$\left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C [\gamma_\nu P_X]^{\gamma\delta} d_\delta^D \right) \otimes j_Y^{\mu\nu} = \left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\gamma_\nu P_X]^{\gamma\beta} b_\beta^B \right) \otimes j_Y^{\mu\nu}, \quad (C13)$$

$$\left(\bar{a}_\alpha^A [\gamma_\mu P_X]^{\alpha\beta} b_\beta^B \right) \otimes \left(\bar{c}_\gamma^C [\gamma_\nu P_Y]^{\gamma\delta} d_\delta^D \right) \otimes j_X^{\mu\nu} = -\frac{i}{2} \left(\bar{a}_\alpha^A P_Y^{\alpha\delta} d_\delta^D \right) \otimes \left(\bar{c}_\gamma^C [\sigma_{\mu\nu} P_X]^{\gamma\beta} b_\beta^B \right) \otimes j_X^{\mu\nu}, \quad (C14)$$

$$\left(\bar{a}_\alpha^A[\gamma_\mu P_X]^{\alpha\beta} b_\beta^B\right) \otimes \left(\bar{c}_\gamma^C[\gamma_\nu P_Y]^{\gamma\delta} d_\delta^D\right) \otimes j_Y^{\mu\nu} = \frac{i}{2} \left(\bar{a}_\alpha^A[\sigma_{\mu\nu} P_Y]^{\alpha\delta} d_\delta^D\right) \otimes \left(\bar{c}_\gamma^C P_X^{\gamma\beta} b_\beta^B\right) \otimes j_Y^{\mu\nu}. \quad (\text{C15})$$

$$(9) \quad \mathcal{O}_7^{XZZ} = 8(\bar{u}P_X d) \otimes (\bar{u}\sigma^{\mu\nu}P_Z s) \otimes j_{\mu\nu}^Z \otimes (\delta_{AB}\delta_{CD})$$

For this operator we have several choices, depending on the chirality of the lepton current

$$\begin{aligned} \left(\bar{a}_\alpha^A P_X^{\alpha\beta} b_\beta^B\right) \otimes \left(\bar{c}_\gamma^C[\sigma^{\mu\nu}P_X]^{\gamma\delta} d_\delta^D\right) \otimes j_{\mu\nu}^X &= -\frac{1}{2} \left(\bar{a}_\alpha^A[\sigma^{\mu\nu}P_X]^{\alpha\delta} d_\delta^D\right) \otimes \left(\bar{c}_\gamma^C P_X^{\gamma\beta} b_\beta^B\right) \otimes j_{\mu\nu}^X \\ &\quad -\frac{1}{2} \left(\bar{a}_\alpha^A P_X^{\alpha\delta} d_\delta^D\right) \otimes \left(\bar{c}_\gamma^C[\sigma^{\mu\nu}P_X]^{\gamma\beta} b_\beta^B\right) \otimes j_{\mu\nu}^X \\ &\quad +\frac{i}{2} \left(\bar{a}_\alpha^A[\sigma_{\mu\rho}P_X]^{\alpha\delta} d_\delta^D\right) \otimes \left(\bar{c}_\gamma^C[\sigma_\nu^\rho P_X]^{\gamma\beta} b_\beta^B\right) \otimes j_{\mu\nu}^X, \end{aligned} \quad (\text{C16})$$

$$\left(\bar{a}_\alpha^A P_X^{\alpha\beta} b_\beta^B\right) \otimes \left(\bar{c}_\gamma^C[\sigma^{\mu\nu}P_Z]^{\gamma\delta} d_\delta^D\right) \otimes j_{\mu\nu}^Z = 2i \left(\bar{a}_\alpha^A[\gamma^\mu P_Z]^{\alpha\delta} d_\delta^D\right) \otimes \left(\bar{c}_\gamma^C[\gamma^\nu P_X]^{\gamma\beta} b_\beta^B\right) \otimes j_{\mu\nu}^Z. \quad (\text{C17})$$

$$(10) \quad \mathcal{O}_7^{XZZ} = 8(\bar{u}\sigma^{\mu\nu}P_X d) \otimes (\bar{u}P_Z s) \otimes j_{\mu\nu}^Z \otimes (\delta_{AB}\delta_{CD})$$

These Fierz identities can be derived from those for $\mathcal{O}_7^{XZZ}$ by changing the quark flavors, $(\bar{a}, b) \leftrightarrow (\bar{c}, d)$.

$$(11) \quad \mathcal{O}_8^{XXX} = 8(\bar{u}\sigma^{\mu\rho}P_X d) \otimes (\bar{u}\sigma_\rho^\nu P_X s) \otimes j_{\mu\nu}^X \otimes (\delta_{AB}\delta_{CD})$$

In this last case only the four-quark operator with chiral structure XXX is nonvanishing

$$\begin{aligned} \left(\bar{a}_\alpha^A[\sigma^{\mu\rho}P_X]^{\alpha\beta} b_\beta^B\right) \otimes \left(\bar{c}_\gamma^C[\sigma_\rho^\nu P_X]^{\gamma\delta} d_\delta^D\right) \otimes j_{\mu\nu}^X &= \frac{1}{8} \left(\bar{a}_\alpha^A[\sigma^{\mu\rho}P_X]^{\alpha\delta} d_\delta^D\right) \otimes \left(\bar{c}_\gamma^C[\sigma_\rho^\nu P_X]^{\gamma\beta} b_\beta^B\right) \otimes j_{\mu\nu}^X \\ &\quad + i \left(\bar{a}_\alpha^A[\sigma^{\mu\nu}P_X]^{\alpha\delta} d_\delta^D\right) \otimes \left(\bar{c}_\gamma^C P_X^{\gamma\beta} b_\beta^B\right) \otimes j_{\mu\nu}^X \\ &\quad - i \left(\bar{a}_\alpha^A P_X^{\alpha\delta} d_\delta^D\right) \otimes \left(\bar{c}_\gamma^C[\sigma^{\mu\nu}P_X]^{\gamma\beta} b_\beta^B\right) \otimes j_{\mu\nu}^X. \end{aligned} \quad (\text{C18})$$

APPENDIX D: SUM OF CONNECTED AND DISCONNECTED DIAGRAMS IN THE VIA

We compute here the sum of connected and disconnected diagrams in the VIA, using reorganization of spinor indices (according to the Fierz rules given in Appendix C) and reorganization of color indices following Eq. (37).

We have

$$(1) \quad \mathcal{O}_1^{XYZ}$$

$$\begin{aligned} 8\langle\pi^-|\bar{u}P_X d\bar{u}P_X s|K^+\rangle \otimes j_{\ell\ell}^Z &= 8\left\{\left(1 - \frac{1}{4N_c}\right)\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}P_X s|K^+\rangle\right. \\ &\quad \left.- \frac{1}{16N_c}\langle\pi^-|\bar{u}\sigma_{\mu\nu}P_X d|0\rangle\langle 0|\bar{u}\sigma^{\mu\nu}P_X s|K^+\rangle\right\} \otimes j_{\ell\ell}^Z + \dots, \\ 8\langle\pi^-|\bar{u}P_X d\bar{u}P_Y s|K^+\rangle \otimes j_{\ell\ell}^Z &= 8\left\{\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}P_Y s|K^+\rangle\right. \\ &\quad \left.- \frac{1}{4N_c}\langle\pi^-|\bar{u}\gamma_\mu P_X d|0\rangle\langle 0|\bar{u}\gamma^\mu P_Y s|K^+\rangle\right\} \otimes j_{\ell\ell}^Z + \dots, \end{aligned} \quad (\text{D1})$$

where the dots stand for the octet-octet contribution coming from the $t^a \otimes t^a$ term in the color-rearrangement rule of Eq. (37). This contribution (that is subleading in $1/N_c$) vanishes in the VIA, since the vacuum is a color singlet. As we can see, the operator $P_{L,R} \otimes P_{L,R}$ mixes with $\sigma_{\mu\nu}P_{L,R} \otimes \sigma_{L,R}^{\mu\nu}$ under Fierzing. Since in the VIA the matrix element of the tensor current vanishes, the operator $P_{L,R} \otimes P_{R,L}$ only mixes with the $\mathcal{O}_3^{XYZ}$ operator.

(2) $\mathcal{O}_2^{XXX}$

$$8\langle\pi^-|\bar{u}\sigma^{\mu\nu}P_X d\bar{u}\sigma_{\mu\nu}P_X s|K^+\rangle\otimes j_{\ell\ell}^X = 8\left\{\left(1+\frac{1}{4N_c}\right)\langle\pi^-|\bar{u}\sigma^{\mu\nu}P_X d|0\rangle\langle 0|\bar{u}\sigma_{\mu\nu}P_X s|K^+\rangle - \frac{3}{N_c}\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}P_X s|K^+\rangle\right\}\otimes j_{\ell\ell}^X + \dots \quad (\text{D2})$$

Notice that, once projected onto the vacuum, the leading contribution proportional to the operator $\sigma_{\mu\nu}P_X \otimes \sigma^{\mu\nu}P_X$ vanishes in the VIA due to Eq. (29). Therefore, the dominant contribution to the matrix element of $\mathcal{O}_2$ is the $1/N_c$ term proportional to $P_X \otimes P_X$ that comes from the connected diagram.

(3) $\mathcal{O}_3^{XYZ}$

$$\begin{aligned} 8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}\gamma_\mu P_X s|K^+\rangle\otimes j_{\ell\ell}^Z &= 8\left\{\left(1+\frac{1}{2N_c}\right)\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\gamma_\mu P_X s|K^+\rangle\right\}\otimes j_{\ell\ell}^Z + \dots, \\ 8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}\gamma_\mu P_Y s|K^+\rangle\otimes j_{\ell\ell}^Z &= 8\left\{\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\gamma_\mu P_Y s|K^+\rangle - \frac{1}{N_c}\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}P_Y s|K^+\rangle\right\}\otimes j_{\ell\ell}^Z + \dots. \end{aligned} \quad (\text{D3})$$

Notice that, as is well-known, four-fermion quark operators with a spinor structure as $\gamma_\mu P_{L,R} \otimes \gamma^\mu P_{L,R}$ are self-Fierzing, and no mixing with other operators arise. The connected diagram just contributes a $1/N_c$ correction to the disconnected one. On the other hand, for $X \neq Y$ we get that $\gamma_\mu P_{L,R} \otimes \gamma^\mu P_{R,L}$ mixes with $\mathcal{O}_1^{XYZ}$.

(4) $\mathcal{O}_4^{XYZ}$

In this case, we have two options, for identical chiral projectors within the two quark currents, $\mathcal{O}_4^{XXZ}$, or different, $\mathcal{O}_4^{XYZ}$.

$$\begin{aligned} 8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}\sigma_\mu^\nu P_X s|K^+\rangle\otimes j_{\ell\ell,\nu}^Z &= 8\left\{\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\sigma_\mu^\nu P_X s|K^+\rangle + \frac{1}{4N_c}\langle\pi^-|\bar{u}\sigma_\mu^\nu P_X d|0\rangle\langle 0|\bar{u}\gamma^\mu P_X s|K^+\rangle - \frac{3i}{4N_c}\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\gamma_\nu P_X s|K^+\rangle\right\}\otimes j_{\ell\ell,\nu}^Z + \dots, \\ 8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}\sigma_\mu^\nu P_Y s|K^+\rangle\otimes j_{\ell\ell,\nu}^Z &= 8\left\{\left(1+\frac{1}{4N_c}\right)\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\sigma_\mu^\nu P_Y s|K^+\rangle + \frac{3i}{4N_c}\langle\pi^-|\bar{u}\gamma_\nu P_X d|0\rangle\langle 0|\bar{u}P_Y s|K^+\rangle\right\}\otimes j_{\ell\ell,\nu}^Z + \dots. \end{aligned} \quad (\text{D4})$$

Also in this case, as it was for $\mathcal{O}_2$, the dominant contribution in the VIA comes from the connected diagram, since the leading disconnected one vanishes due to Eq. (29).

(5) $\mathcal{O}_4^{XYZ}$

We have

$$\begin{aligned}
8\langle\pi^-|\bar{u}\sigma_\mu^\nu P_X d\bar{u}\gamma^\mu P_X s|K^+\rangle \otimes j_{\ell\ell,\nu}^Z &= 8\left\{\langle\pi^-|\bar{u}\sigma_\mu^\nu P_X d|0\rangle\langle 0|\bar{u}\gamma^\mu P_X s|K^+\rangle\right. \\
&\quad + \frac{1}{4N_c}\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\sigma_\mu^\nu P_X s|K^+\rangle \\
&\quad \left.- \frac{3i}{4N_c}\langle\pi^-|\bar{u}\gamma_\nu P_X d|0\rangle\langle 0|\bar{u}P_X s|K^+\rangle\right\} \otimes j_{\ell\ell,\nu}^Z + \dots, \\
8\langle\pi^-|\bar{u}\sigma_\mu^\nu P_X d\bar{u}\gamma^\mu P_Y s|K^+\rangle \otimes j_{\ell\ell,\nu}^Z &= 8\left\{\left(1 + \frac{1}{4N_c}\right)\langle\pi^-|\bar{u}\sigma_\mu^\nu P_X d|0\rangle\langle 0|\bar{u}\gamma^\mu P_Y s|K^+\rangle\right. \\
&\quad \left.+ \frac{3i}{4N_c}\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\gamma_\nu P_Y s|K^+\rangle\right\} \otimes j_{\ell\ell,\nu}^Z + \dots. \tag{D5}
\end{aligned}$$

(6) $\mathcal{O}_5^{XYZ}$

Also in this case, we have two options

$$\begin{aligned}
8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}P_X s|K^+\rangle \otimes j_{\ell\ell,\mu}^Z &= 8\left\{\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}P_X s|K^+\rangle\right. \\
&\quad - \frac{1}{4N_c}\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\gamma^\mu P_X s|K^+\rangle \\
&\quad \left.+ \frac{i}{4N_c}\langle\pi^-|\bar{u}\sigma^{\alpha\mu} P_X d|0\rangle\langle 0|\bar{u}\gamma_\alpha P_X s|K^+\rangle\right\} \otimes j_{\ell\ell,\mu}^Z + \dots, \\
8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}P_Y s|K^+\rangle \otimes j_{\ell\ell,\mu}^Z &= 8\left\{\left(1 - \frac{1}{4N_c}\right)\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}P_Y s|K^+\rangle\right. \\
&\quad \left.- \frac{i}{4N_c}\langle\pi^-|\bar{u}\gamma_\alpha P_X d|0\rangle\langle 0|\bar{u}\sigma^{\alpha\mu} P_Y s|K^+\rangle\right\} \otimes j_{\ell\ell,\mu}^Z + \dots, \tag{D6}
\end{aligned}$$

We see that mixing arises with the tensor operator $\sigma P_{L,R}$. However, as the corresponding matrix element between a meson and the vacuum vanishes in the VIA, the only mixing that is induced by the Fierz-rearrangement is that between $\mathcal{O}_5$ and $\mathcal{O}'_5$.

(7) $\mathcal{O}_5^{XYZ}$

$$\begin{aligned}
8\langle\pi^-|\bar{u}P_X d\bar{u}\gamma^\mu P_X s|K^+\rangle \otimes j_{\ell\ell,\mu}^Z &= 8\left\{\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\gamma^\mu P_X s|K^+\rangle\right. \\
&\quad - \frac{1}{4N_c}\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}P_X s|K^+\rangle \\
&\quad \left.+ \frac{i}{4N_c}\langle\pi^-|\bar{u}\gamma_\alpha P_X d|0\rangle\langle 0|\bar{u}\sigma^{\alpha\mu} P_X s|K^+\rangle\right\} \otimes j_{\ell\ell,\mu}^Z + \dots, \\
8\langle\pi^-|\bar{u}P_X d\bar{u}\gamma^\mu P_Y s|K^+\rangle \otimes j_{\ell\ell,\mu}^Z &= 8\left\{\left(1 - \frac{1}{4N_c}\right)\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\gamma^\mu P_Y s|K^+\rangle\right. \\
&\quad \left.+ \frac{i}{4N_c}\langle\pi^-|\bar{u}\sigma^{\mu\alpha} P_X d|0\rangle\langle 0|\bar{u}\gamma_\alpha P_Y s|K^+\rangle\right\} \otimes j_{\ell\ell,\mu}^Z + \dots, \tag{D7}
\end{aligned}$$

We have the same results as for $\mathcal{O}_5^{XYZ}$.

(8) $\mathcal{O}_6^{XYZ}$

$$\begin{aligned}
8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}\gamma^\nu P_X s|K^+\rangle \otimes j_{\ell\ell,\mu\nu}^X &= 8\left(1 + \frac{1}{2N_c}\right)\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\gamma^\nu P_X s|K^+\rangle \otimes j_{\ell\ell,\mu\nu}^X + \dots, \\
8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}\gamma^\nu P_X s|K^+\rangle \otimes j_{\ell\ell,\mu\nu}^Z &= 8\left(1 - \frac{1}{2N_c}\right)\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\gamma^\nu P_X s|K^+\rangle \otimes j_{\ell\ell,\mu\nu}^Z + \dots,
\end{aligned} \tag{D8}$$

with $Z \neq X$ in the last equation. In order to get the second line of each Fierz identity we have used anticommutation of Lorentz indices in the lepton current $j_{\ell\ell,\mu\nu}^Z$.

The last relation is

$$\begin{aligned}
8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}\gamma^\nu P_Y s|K^+\rangle \otimes j_{\ell\ell,\mu\nu}^X &= 8\left\{\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\gamma^\nu P_Y s|K^+\rangle\right. \\
&\quad \left.- \frac{i}{4N_c}\langle\pi^-|\bar{u}\sigma^{\mu\nu} P_X d|0\rangle\langle 0|\bar{u}P_Y s|K^+\rangle\right\} \otimes j_{\ell\ell,\mu\nu}^X + \dots, \\
8\langle\pi^-|\bar{u}\gamma^\mu P_X d\bar{u}\gamma^\nu P_Y s|K^+\rangle \otimes j_{\ell\ell,\mu\nu}^Y &= 8\left\{\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\gamma^\nu P_Y s|K^+\rangle\right. \\
&\quad \left.+ \frac{i}{4N_c}\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\sigma^{\mu\nu} P_Y s|K^+\rangle\right\} \otimes j_{\ell\ell,\mu\nu}^Y + \dots.
\end{aligned} \tag{D9}$$

(9) $\mathcal{O}_7^{XZZ}$

We have in this case

$$\begin{aligned}
8\langle\pi^-|\bar{u}P_X d\bar{u}\sigma^{\mu\nu} P_X s|K^+\rangle \otimes j_{\ell\ell',\mu\nu}^X &= 8\left\{\left(1 - \frac{1}{4N_c}\right)\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\sigma^{\mu\nu} P_X s|K^+\rangle\right. \\
&\quad \left.- \frac{1}{4N_c}\langle\pi^-|\bar{u}\sigma^{\mu\nu} P_X d|0\rangle\langle 0|\bar{u}P_X s|K^+\rangle\right. \\
&\quad \left.- \frac{i}{4N_c}\langle\pi^-|\bar{u}\sigma^{\mu\rho} P_X d|0\rangle\langle 0|\bar{u}\sigma_\rho^\nu P_X s|K^+\rangle\right\} \otimes j_{\ell\ell',\mu\nu}^X + \dots, \\
8\langle\pi^-|\bar{u}P_X d\bar{u}\sigma^{\mu\nu} P_Z s|K^+\rangle \otimes j_{\ell\ell',\mu\nu}^Z &= 8\left\{\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\sigma^{\mu\nu} P_Z s|K^+\rangle\right. \\
&\quad \left.- \frac{i}{N_c}\langle\pi^-|\bar{u}\gamma^\mu P_X d|0\rangle\langle 0|\bar{u}\gamma^\nu P_Z s|K^+\rangle\right\} \otimes j_{\ell\ell',\mu\nu}^Z + \dots.
\end{aligned}$$

(10) $\mathcal{O}_7^{XZZ}$

$$\begin{aligned}
8\langle\pi^-|\bar{u}\sigma^{\mu\nu} P_X d\bar{u}P_X s|K^+\rangle \otimes j_{\ell\ell',\mu\nu}^X &= 8\left\{\left(1 - \frac{1}{4N_c}\right)\langle\pi^-|\bar{u}\sigma^{\mu\nu} P_X d|0\rangle\langle 0|\bar{u}P_X s|K^+\rangle\right. \\
&\quad \left.- \frac{1}{4N_c}\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\sigma^{\mu\nu} P_X s|K^+\rangle\right. \\
&\quad \left.+ \frac{i}{4N_c}\langle\pi^-|\bar{u}\sigma_\rho^\mu P_X d|0\rangle\langle 0|\bar{u}\sigma^{\nu\rho} P_X s|K^+\rangle\right\} \otimes j_{\ell\ell',\mu\nu}^X + \dots, \\
8\langle\pi^-|\bar{u}\sigma^{\mu\nu} P_Z d\bar{u}P_X s|K^+\rangle \otimes j_{\ell\ell',\mu\nu}^Z &= 8\left\{\langle\pi^-|\bar{u}\sigma^{\mu\nu} P_Z d|0\rangle\langle 0|\bar{u}P_X s|K^+\rangle\right. \\
&\quad \left.+ \frac{i}{N_c}\langle\pi^-|\bar{u}\gamma^\mu P_Z d|0\rangle\langle 0|\bar{u}\gamma^\nu P_X s|K^+\rangle\right\} \otimes j_{\ell\ell',\mu\nu}^Z + \dots.
\end{aligned}$$

(11) $\mathcal{O}_8^{XXX}$

$$\begin{aligned}
8\langle\pi^-|\bar{u}\sigma^{\mu\rho}P_X d\bar{u}\sigma_\rho^\nu P_X s|K^+\rangle \otimes j_{\ell\ell',\mu\nu}^X &= 8\left\{\left(1 - \frac{1}{16N_c}\right)\langle\pi^-|\bar{u}\sigma^{\mu\rho}P_X d|0\rangle\langle 0|\bar{u}\sigma_\rho^\nu P_X s|K^+\rangle\right. \\
&+ \frac{i}{2N_c}\langle\pi^-|\bar{u}P_X d|0\rangle\langle 0|\bar{u}\sigma^{\mu\nu}P_X s|K^+\rangle \\
&\left.- \frac{i}{2N_c}\langle\pi^-|\bar{u}\sigma^{\mu\nu}P_X d|0\rangle\langle 0|\bar{u}P_X s|K^+\rangle\right\} \otimes j_{\ell\ell',\mu\nu}^X + \dots,
\end{aligned}$$

APPENDIX E: B-PARAMETERS FOR $K^+ \rightarrow \pi^- \ell \ell'$ TRANSITIONS AT LARGE N_c

Without a nonperturbative computation of the hadronic matrix elements $\langle\pi^-|\mathcal{O}_n^{XYZ}(\mu)|K^+\rangle$, the best we can do is what is shown in the fourth column of Tables I–III. We will, however, sketch hereafter the procedure that should be carried on in order to go one step further and match the present precision in the computation of Wilson coefficients of $\Delta F = 2$ Lagrangians (with F a flavor-related quantum number, such as the strangeness S , [69–71]). For definiteness, we consider a would-be lattice computation of the hadronic matrix elements. Our procedure, however, does not depend on the nonperturbative method adopted and is completely general.

1. The standard definition of the B -parameters

In Eq. (21) we introduced the meson decay width at the scale μ , summing over all the spin-averaged amplitudes $\bar{\mathcal{A}}_n^{XYZ}$ defined in Eq. (35). The hadronic amplitudes are, on the other hand, computed nonperturbatively as follows:

$$\begin{aligned}
\mathcal{A}_n^{XYZ}(\mu) &= \left(\frac{\langle\pi^-|\mathcal{O}_n^{XYZ}(\mu)|K^+\rangle}{[\mathcal{A}_n^{XYZ}]_{\text{VIA}}} \right) [\mathcal{A}_n^{XYZ}]_{\text{VIA}} \\
&= \mathcal{B}_n^{XYZ}(\mu) [\mathcal{A}_n^{XYZ}]_{\text{VIA}}.
\end{aligned} \tag{E1}$$

The B -parameter $\mathcal{B}_n^{XYZ}(\mu)$ represents, therefore, the nonperturbative correction to the VIA that we are using to estimate the hadronic matrix elements. From a numerical point of view, it is much better to compute nonperturbatively the ratio of the three-point correlation function in the numerator normalized with the product of two two-point correlation functions in the denominator, as in the ratio the statistical fluctuations over the gauge configurations wash out and we may get a more stable signal. Some systematics cancel in the ratio, too. For this reason, numerical computation on the lattice of hadronic matrix elements is usually given in terms of B -parameters and not of the matrix elements themselves.

2. Going from μ to the NP scale

As we have stressed above, the matrix elements of these operators between initial and final meson states should be

computed nonperturbatively at the scale μ . However, quite in general, these matrix elements are computed at yet another scale, Λ_{NP} . In the case of a lattice QCD computation, the scale at which the nonperturbative computation takes place is a^{-1} , where the lattice spacing is fixed by measuring some external quantity (such as the mass of the pion on that lattice, for example) with, in general, $a^{-1} < \mu$. We have, then, to run upward the results using RGE from a^{-1} up to μ . This is done as follows:

$$\mathcal{A}_n^{XYZ}(\mu) = Z_{nm}(\mu, a^{-1}) \mathcal{B}_m^{XYZ}(a^{-1}) [\mathcal{A}_m^{XYZ}]_{\text{VIA}}, \tag{E2}$$

where the B -parameter in the rhs takes into account the exchange of soft, long-distance, $\mathcal{O}(\Lambda_{\text{QCD}})$ gluons and the renormalization constant matrix $\hat{Z}$ is computed in some convenient regularization and renormalization scheme. If, for example, the hadronic matrix element is computed on the lattice, $\hat{Z}$ must be computed either perturbatively through lattice perturbation theory [72,73] or nonperturbatively in the RI-MOM scheme [74–76] on the lattice. In order to remove the scheme dependence introduced by the mismatch between Wilson coefficient and hadronic matrix element regularization and renormalization, we should still (perturbatively) match the nonperturbative matrix element and the Wilson coefficient. This is done through yet another matching matrix $\hat{R}$, such that

$$\begin{aligned}
\mathcal{L}_{\text{eff}}^{\Delta L=2} &= \frac{1}{\Lambda_{\text{EW}}^5} \sum_i C_i(\mu) R_{ij}^{\text{lattice} \rightarrow \overline{\text{MS}}}(\mu) Z_{jk}(\mu, a^{-1}) \\
&\quad \mathcal{B}_k^{XYZ}(a^{-1}) [\mathcal{A}_k^{XYZ}]_{\text{VIA}},
\end{aligned} \tag{E3}$$

where $\hat{R}^{\text{lattice} \rightarrow \overline{\text{MS}}}(\mu)$ is the matching matrix that relates the matrix elements computed in a given lattice regularization and renormalization scheme (here summarized as “lattice”) with the Wilson coefficients computed in dimensional regularization in the $\overline{\text{MS}}$ scheme, at the scale μ . After performing this procedure, the resulting effective Lagrangian is μ -independent up to higher-orders in α_s .

In this paper, in the absence of a nonperturbative computation of the hadronic matrix element, we have made the following approximation:

$$\mathcal{A}_n^{XYZ}(\mu) \equiv [\mathcal{A}_n^{XYZ}]_{\text{VIA}} \quad (\text{E4})$$

that, after the inclusion of the connected diagrams, represent indeed a consistent nonperturbative computation up to $\mathcal{O}(1/N_c)$ in the $1/N_c$ expansion. However, this approximation implicitly implies that

$$\vec{\mathcal{B}}^{XYZ}(\mu) \Big|_{\overline{\text{MS}}} = \hat{R}^{\text{lattice} \rightarrow \overline{\text{MS}}} \hat{Z}(\mu, a^{-1}) \vec{\mathcal{B}}^{XYZ}(a^{-1}) = \mathbf{I} + \dots \quad (\text{E5})$$

This is not always the case. For example, let us think about the matrix element of the operator $\mathcal{O}_8^{XX}(\mu)$. In the VIA, this matrix element vanishes, as it is proportional to the two-point correlation function of the tensor current between the vacuum and one meson state. This means that we have no (nonperturbative) estimate of the value of this matrix element, as our only guideline (the VIA) states that this matrix element is zero. For this reason, we are not able to constrain the corresponding Wilson coefficient, that is therefore unbounded.

3. A better definition of the B -parameters

We have introduced in Sect. E 1 the standard definition of the B -parameters used to compute the nonperturbative contribution of long-range $\mathcal{O}(\Lambda_{\text{QCD}})$ soft gluons to the hadronic matrix element at hand. However, as stressed above, this definition is unsuitable in the case the VIA estimate of the matrix element vanishes, since the denominator of the observable to be computed is ill-defined. This is not the only problem of the standard definition of the B -parameters, though. Consider, for example, the amplitudes corresponding to the matrix elements of the operators $\mathcal{O}_1^{XYZ}$ and $\mathcal{O}_3^{XYZ}$ for $X \neq Y$. In both cases, the VIA estimate of the amplitude, that includes $\mathcal{O}(1/N_c)$ corrections due to the connected diagrams, mixes between themselves the disconnected contributions of the two operators that, in the VIA, are proportional to the product of two pseudoscalar densities or two axial currents, respectively. Now, as we stressed in Sec. IV B, the meson decay constants do not run (as their renormalization are finite, due to the PCAC), whereas the pseudoscalar densities do (as they depend on the quark running masses, computed in a given renormalization scheme at a given renormalization scale). Also in this case, therefore, the denominator of the B -parameter is not optimal, as it makes more complicated the

renormalization of the B -parameter. In Ref. [77] it was proposed to normalize operators whose VIA mixed two-point correlation functions of axial currents and pseudoscalar densities with the corresponding disconnected diagrams, only

$$\begin{aligned} \mathcal{A}_n^{XYZ}(\mu) &= \left(\frac{\langle \pi^- | \mathcal{O}_n^{XYZ}(\mu) | K^+ \rangle}{[\mathcal{A}_n^{XYZ}]_d} \right) [\mathcal{A}_n^{XYZ}]_d \\ &= \tilde{\mathcal{B}}_n^{XYZ}(\mu) [\mathcal{A}_n^{XYZ}]_d. \end{aligned} \quad (\text{E6})$$

The RGE for the B -parameter defined as above is

$$\mu \frac{d\tilde{\mathcal{B}}_n^{XYZ}(\mu)}{d\mu} = (\gamma_{\mathcal{O}_n} - 2\gamma_{\mathcal{P}}) \tilde{\mathcal{B}}_n^{XYZ}(\mu), \quad (\text{E7})$$

where $\gamma_{\mathcal{O}_n}$ is the anomalous dimension of the operator $\mathcal{O}_n$ and $\gamma_{\mathcal{P}}$ is that of the pseudoscalar density. Since the disconnected diagram in the VIA renormalize with the anomalous dimension $2\gamma_{\mathcal{P}}$ (as only the quark masses run with the scale), the renormalized B -parameter times the amplitude of the disconnected diagram in the VIA renormalize with the anomalous dimension of the operator, only, thus reducing the systematic errors.

This definition of the B -parameter, though, does not resolve the problem of those matrix elements of operators whose disconnected diagram in the VIA vanish, such as $\mathcal{O}_8$. For this reason, therefore, it is better to normalize the three-point correlation function with the two-point correlation function of the axial current that, as we already stressed, gets only a finite renormalization constant on the lattice

$$\begin{aligned} \mathcal{A}_n^{XYZ}(\mu) &= \left(\frac{\langle \pi^- | \mathcal{O}_n^{XYZ}(\mu) | K^+ \rangle}{[\mathcal{A}_3^{XYZ}]_d} \right) [\mathcal{A}_3^{XYZ}]_d \\ &= \tilde{\mathcal{B}}_n^{XYZ}(\mu) [\mathcal{A}_3^{XYZ}]_d. \end{aligned} \quad (\text{E8})$$

This proposal was advanced for the first time in Ref. [78] in order to simplify the matching between the nonperturbative computation of the hadronic matrix elements of $\Delta S = 2$ operators on the lattice with the $\overline{\text{MS}}$ perturbative computation of the corresponding Wilson coefficients. In our case, however, it is even more suited, as it permits a proper normalization of matrix elements of operators whose estimate in the VIA would vanish, while retaining the useful reduction of statistical and systematic errors in the ratio of three- to two-point functions.

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