

Mutual information from modular flow in general conformal field theories

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(Received 14 May 2026; accepted 30 July 2026; published 25 August 2026)

The vacuum mutual information (MI) of subregion algebras provides a universal window into the data of general conformal field theories (CFTs). Exploiting the geometric nature of the modular flow associated to ball-shaped regions and the operator product expansion of twist operators implementing the replica symmetry in an n -fold version of a CFT, it is possible to construct a hierarchy of increasingly refined approximations to the full MI. In this letter, we use the two-point functions of primaries of arbitrary spin in the replicated theory to constrain the twist operators, and find their contribution to the MI of arbitrarily boosted balls in any d -dimensional CFT. When the two-point functions involve the primary with the lowest scaling dimension, our result provides the most precise approximation for the long-distance behavior of the MI, superseding all previous expansions. Building upon this result and certain universal properties of the short- and long-distance regimes, we put forward a new high-precision analytic approximation to the MI for arbitrary separations. The accuracy of our approach is validated against exact $d = 2$ and lattice $d = 3$ results. We further apply it to characterize the MI of a $d = 4$ Maxwell field, a case for which no prior results are available.

DOI: [10.1103/pwmb-pl64](https://doi.org/10.1103/pwmb-pl64)

The vacuum mutual information (MI) of operator algebras associated to spacetime regions is expected to fully characterize any given QFT. On the one hand, the MI provides a universal geometric regulator for entanglement entropy, capturing all the associated universal coefficients [1–8]. Complementarily, it has been argued that a long-distance expansion of the MI should allow for the systematic extraction of the full underlying QFT data [9–22].

The second approach relies on two key facts: (i) the geometric nature of the modular flow associated with ball regions in any conformal field theory (CFT) [23,24]; and (ii) the possibility of expressing the MI in terms of a correlator of two *twist operators*, which implement the $\mathbb{Z}_n$

orbifolding of the n copies of the theory $\text{CFT}^{\otimes n}$ [25–28]. These features allow for increasingly sophisticated and complete approximations to the MI. These arise from imposing certain consistency conditions that connect the expectation value of the twist operators probed by local operators in different copies of the CFT to the expectation value of modularly-evolved operators in the original theory [18,20,22].

Using this approach, we present the contribution to the MI of arbitrarily boosted balls arising from the two-copy sector associated to a primary operator of arbitrary spin in a general d -dimensional CFT. Our formula supersedes and generalizes all previous approximations to the MI long-distance expansion. Not only does it capture the contribution of the lowest dimensional primary in the replicated CFT [16,18], it also includes the contribution from all the two-copy primaries and their descendants. Hence, this entails a resummation of an infinite subset of the conformal blocks that provide the MI expansion [12,13,15]. Unlike the results of [22], which hold only for scalar CFTs and parallel spheres, our formula applies for completely general CFTs—primaries of arbitrary spin representation—and boosted spheres. Additionally, we use this formula to put forward a new analytic high-precision approximation

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to the MI of any CFT valid for arbitrary separations. We show that this accurately reproduces the exact MI curve for a $d = 2$ free fermion and lattice results for a $d = 3$ free scalar, and we use it to predict the result for a $d = 4$ Maxwell field—see Fig. 1.

Mutual information and modular flows: The MI between two disjoint spacetime regions A, B , can be defined in terms of the von Neumann entropies of the corresponding reduced density matrices, $I(A, B) \equiv S_A + S_B - S_{A \cup B}$. Here, $S_A \equiv -\text{Tr}[\rho_A \log \rho_A]$, where $\rho_A \equiv \text{Tr}_{\bar{A}}|\Omega\rangle\langle\Omega|$, and from now on we will consider the vacuum $|\Omega\rangle$ as our global state. While all the individual entropies appearing in the above formula suffer from UV divergences, they always cancel one another in $I(A, B)$, which is a well-defined quantity in the continuum [31,32].

A standard technique for evaluating the MI for a given CFT entails evaluating the expectation value of two properly normalized *twist operators* defined in $\text{CFT}^{\otimes n}$ [25–27]. The precise formula reads

$$I(A, B) = \lim_{n \rightarrow 1} \frac{1}{n-1} \langle \tilde{\Sigma}_A^{(n)} \tilde{\Sigma}_B^{(n)} \rangle. \quad (1)$$

Here $\Sigma_X^{(n)} \equiv \langle \Sigma_X^{(n)} \rangle (\mathbf{1} + \tilde{\Sigma}_X^{(n)})$ and $\Sigma_X^{(n)}$ implements the global cyclic symmetry transformation to operators localized in region X , namely $\Sigma_X^{(n)}: \mathcal{O}_X^i \rightarrow \mathcal{O}_X^{i+1}$, while acting trivially on operators outside X [25,27]. The index i labels the copies, subject to the cyclic identification $n+1 \equiv 1$.

In order to characterize $\tilde{\Sigma}_A^{(n)}$, we may consider its correlation function with an arbitrary set of operators. Here we will be interested in contributions arising from operators supported on two distinct copies, $1 \leq l \neq k \leq n$. As a consequence, it will suffice to invoke the following result, which holds in the $n \rightarrow 1$ limit [18,20,22,33]

$$\langle \tilde{\Sigma}_A^{(n)} \mathcal{O}_\alpha^l(x_1) \mathcal{O}_\beta^k(x_2) \rangle \stackrel{n \rightarrow 1}{=} \begin{cases} \langle \mathcal{O}_\alpha(x_1) \mathcal{O}_\beta^A[x_2, i\tau_{lk}] \rangle & (l > k) \\ \langle \mathcal{O}_\beta^A[x_2, i\tau_{kl}] \mathcal{O}_\alpha(x_1) \rangle & (l < k) \end{cases}. \quad (2)$$

This remarkable formula requires some explanation. On the one hand, $\mathcal{O}_\alpha^l(x)$ is a local operator, with a possible spin index α , defined in the l th copy of the CFT at the spacetime point x . Also, we defined $\mathcal{O}_\alpha^A[x, s] \equiv \rho_A^{-is} \mathcal{O}_\alpha(x) \rho_A^{is}$, which is the operator resulting from the modular-flow evolution associated to ρ_A , and we introduced the notation $\tau_{kl} \equiv \pi(k-l)/n$. The correlators in the right-hand side are analytic and well-defined in the region $\text{Im}(s) \in (0, 1)$, which is the case for $s = i|\tau_{lk}|$. Hence, Eq. (2) connects a correlation function in the replicated theory with a correlation function of modularly transformed operators in the seed CFT.

For spherical entangling surfaces, the transformation that implements the modular flow corresponds to a conformal

transformation [23,24],

$$x_A^\pm[\xi, s] = R_A \frac{(R_A + \xi^\pm) - e^{\mp 2\pi s}(R_A - \xi^\pm)}{(R_A + \xi^\pm) + e^{\mp 2\pi s}(R_A - \xi^\pm)}, \quad (3)$$

where R_A is the radius of the sphere A and $\xi^\pm = r \pm t$ are null coordinates with respect to its center. In that case, the relevant correlator takes the form

$$\langle \mathcal{O}_\alpha(x_1) \mathcal{O}_\beta^A[x_2, s] \rangle = \left| \frac{\partial x_{2,A}[s]}{\partial x_2} \right|^\Delta \mathcal{R}_\beta^\lambda(\Lambda_A(x_2, s)) \times G_{\alpha\lambda}(x_1, x_{2,A}[s]), \quad (4)$$

where we introduced the notation $x_{2,A}[s] \equiv x_A[x_2, s]$. Also, $\mathcal{R}_\beta^\lambda(\Lambda_A(x_2, s))$ is the local Lorentz transformation associated to the conformal transformation (3) based at x_2 in the representation $\mathcal{R}$ of $\mathcal{O}$ and $|\partial x_{2,A}[s]/\partial x_2|$ is its associated scaling factor. Finally, $G_{\alpha\lambda}$ denotes the two-point function of $\mathcal{O}$, namely, $G_{\alpha\beta}(x, y) \equiv \langle \mathcal{O}_\alpha(x) \mathcal{O}_\beta(y) \rangle = H_{\alpha\beta}(x-y) \cdot |x-y|^{-2\Delta}$, where $H_{\alpha\beta}(x)$ captures the tensor structure of the correlator and Δ is the scaling dimension of $\mathcal{O}$.

Our goal is to evaluate the contribution to the MI arising from the part of the twist operator which is responsible for making (2) hold. Analogous formulas for higher-point correlators can be derived following [22], which would allow one to capture multiple-copy contributions to the MI.

An arbitrary operator in the replica theory can be written as a linear combination of products of operators associated to the individual copies. The most general operator that can contribute to the left hand side of (2) can be written as a double integral with support in the causal completion of the entangling region that is generated from the product of two primaries in two different replicas $l \neq k$, namely,

$$\tilde{\Sigma}_A^{(n)} = \sum_{l \neq k} \iint_{D_A} \mathcal{G}_{A,\alpha\beta}^{lk}(\xi_1, \xi_2) \mathcal{O}_\alpha^l(\xi_1) \mathcal{O}_\beta^k(\xi_2), \quad (5)$$

where we denoted $\iint_{D_A} \equiv \int_{D_A} \int_{D_A} d^d \xi_1 d^d \xi_2$. Plugging the Ansatz (5) into (2) leads to

$$\langle \tilde{\Sigma}_A^{(n)} \mathcal{O}_\alpha^l(x_1) \mathcal{O}_\beta^k(x_2) \rangle \stackrel{n \rightarrow 1}{=} 2 \iint_{D_A} \mathcal{G}_{A,\alpha\beta}^{lk}(\xi_1, \xi_2) G_{\alpha\mu}(\xi_1, x_1) G_{\beta\nu}(\xi_2, x_2), \quad (6)$$

which can be inverted to find the kernel function $\mathcal{G}_{A,\alpha\beta}^{lk}(\xi_1, \xi_2)$. We define the inverse correlator as

$$\begin{aligned} \int_{D_A} d^d \xi G_{A,\alpha\beta}^{-1}(x, \xi) G_{\beta\gamma}(\xi, \chi) &= \delta_{\alpha\gamma} \delta^d(x - \chi), \\ \int_{D_A} d^d \chi G_{\alpha\beta}(\zeta, \chi) G_{A,\beta\gamma}^{-1}(\chi, \xi) &= \delta_{\alpha\gamma} \delta^d(\zeta - \xi), \end{aligned} \quad (7)$$

and similarly for B . In (7), $\{x, \chi\} \in D_{\tilde{A}}$ while $\{\zeta, \xi\} \in D_A$. We claim that such correlators exist in any CFT [22]. Once we solve for the kernel, we plug the *Ansatz* (5) into (1) and obtain a formula for the MI in terms of a discrete sum over modular parameters. Such a sum can be transformed into an integral, via

$$i\tau_{lk} \rightarrow s + \frac{i}{2}, \quad \lim_{n \rightarrow 1} \sum_{l > k} \rightarrow \frac{1}{4} \int_{-\infty}^{\infty} \frac{\pi ds}{\cosh^2 \pi s}. \quad (8)$$

This provides the appropriate analytic continuation, as discussed in [16, 18, 22]. The result reads

$$\begin{aligned} I_{\Delta} = & \int_{-\infty}^{\infty} \frac{\pi ds}{4 \cosh^2 \pi s} \int_{D_A} d^d \xi \int_{D_B} d^d \chi \left| \frac{\partial \chi_A^J[s]}{\partial \chi} \right|^{\Delta} \left| \frac{\partial \xi_B^J[-s]}{\partial \xi} \right|^{\Delta} \\ & \times \delta^d(\chi - \xi_B^J[-s]) \delta^d(\xi - \chi_A^J[s]) \\ & \times \text{tr}[\mathcal{R}(\Lambda_A(\chi^J, s) \cdot \Lambda_B(\xi^J, -s))], \end{aligned} \quad (9)$$

where J represents the action by the Tomita operator and $\chi_A^J[s] \equiv x_A[\chi, s + \frac{i}{2}]$. The above formula can be further simplified by carrying out the remaining integrals using the delta functions. The final result is thus given in terms of a single integral over the modular parameter s , as shown in Eq. (10) below. This represents the two-copy contribution to the MI associated to a primary operator in an arbitrary representation of the conformal group.

Relatively boosted spheres: The most general version of I_{Δ} is obtained by considering two disjoint spheres with a general relative boost. Without loss of generality, we consider a geometric configuration in which the spheres of radii R_A, R_B are concentric and satisfy $R_A > R_B \exp |\beta|$, where β represents the relative boost which we apply to the smaller one. This guarantees that the smaller sphere is completely contained within the larger one. Notice that the entangling region associated to the larger sphere is the complement of a ball of radius R_A . This means, in particular, that its associated modular flow is $x_A[\chi, -s]$. The flow associated to the boosted sphere can be obtained from the unboosted one via the following composition, $x_{B_{\beta}}[\xi, s] = \Lambda_{\beta} \circ x_B[\Lambda_{-\beta}(\xi), s]$, where Λ_{β} is the boost with rapidity β and B_{β} is the associated boosted sphere of radius R_B . Carrying out the remaining integrals in (9) leads to

$$\begin{aligned} I_{\Delta} = & \frac{1}{4} \int_{-\infty}^{\infty} \frac{\pi ds}{\cosh^2 \pi s} \left| \frac{\partial \chi_A^J(s)}{\partial \chi} \right|_{\chi=\chi^*}^{d-\Delta} \left| \frac{\partial \chi_{B_{\beta}}^J(-s)}{\partial \chi} \right|_{\chi=\chi^*}^{\Delta} \\ & \times \frac{\text{tr}[\mathcal{R}(\Lambda_{B_{\beta}}(\chi^J, -s) \cdot \Lambda_A^{-1}(\chi^J, s))]}{|\det(\frac{\partial f^a(\chi)}{\partial \chi^b})|_{\chi=\chi^*}}. \end{aligned} \quad (10)$$

Here χ^* is determined by the solution to the equation $f(\chi^*) = 0$ subject to the condition that $\chi^* \in D_A$, where $f(\chi) \equiv x_A^J[\chi, s] - x_{B_{\beta}}^J[\chi, -s]$. The solutions to this equation come in four branches, but the condition $\chi^* \in D_A$ singles

out the following two,

$$\frac{\chi_{\pm}^{\pm}}{R_A} = \pm \frac{(x_{\beta}^{\pm} - 1) \cosh \pi s - (1 + x_{\beta}^{\pm}) \sqrt{\cosh^2 \pi s - z^{\pm}}}{2 \sinh \pi s}, \quad (11)$$

where we define $x \equiv R_B/R_A$ and their boosted analogs, namely $x_{\beta}^{\pm} \equiv e^{\pm \beta} x$. We also introduce the notation $(z^-, z^+) \equiv (z, \bar{z})$ for the conformal ratios, which in this concentric configuration read $z^{\pm} = 4x_{\beta}^{\pm}/(1 + x_{\beta}^{\pm})^2$. Each term in (10) can be computed explicitly—see Supplemental Material [34], which includes additional references to [35–37]. A crucial ingredient of (10) is the character of the representation evaluated for a particular Lorentz transformation. This corresponds to a pure boost, $\tilde{\beta}$, which depends on the modular parameter s , the original boost β and the geometric configuration x . Indeed, we may write

$$\chi_{\mathcal{R}}(\Lambda(\tilde{\beta})) \equiv \text{tr}[\mathcal{R}(\Lambda_{B_{\beta}}(\chi^{*J}, -s) \cdot \Lambda_A^{-1}(\chi^{*J}, s))], \quad (12)$$

where $\tilde{\beta}$ satisfies

$$\cosh \frac{\tilde{\beta}}{2} = \frac{1 - \sqrt{(1 - wz)(1 - w\bar{z})}}{w\sqrt{z\bar{z}}}. \quad (13)$$

All this leads to the main result of our paper,

$$\begin{aligned} I_{\Delta} = & \frac{z^{\Delta} \bar{z}^{\Delta}}{2^{d+4}} \int_0^1 dw \frac{w^{2\Delta}}{\sqrt{1-w}} \frac{(1 + \sqrt{1-wz})^{d-2\Delta}}{\sqrt{1-w\bar{z}}} \\ & \times \frac{(1 + \sqrt{1-w\bar{z}})^{d-2\Delta}}{\sqrt{1-wz}} \\ & \times \frac{\chi_{\mathcal{R}}(\Lambda(\tilde{\beta}))}{(\sqrt{1-wz} + \sqrt{1-w\bar{z}})^{d-2}}, \end{aligned} \quad (14)$$

where the integral over w is a reparametrization of the modular integral via $w \rightarrow \frac{1}{\cosh^2 \pi s}$.

Explicit formulas for the character of the transformation $\chi_{\mathcal{R}}(\Lambda(\tilde{\beta}))$ were worked out recently in the context of MI in [18] following [38]. We give a short summary in Supplemental Material [34], along with a few relevant examples.

In the limit where the regions are very far away, $\cosh \tilde{\beta}/2 \approx \frac{z+\bar{z}}{2\sqrt{z\bar{z}}}$. This can be approximated in terms of normal vectors characterizing the geometry by $\cosh \tilde{\beta}/2 \approx \cosh \beta = 2(n_A \cdot l)(n_B \cdot l) - n_A \cdot n_B$, where n_A and n_B give the orientation of the spheres and l is the normal vector that connects the past causal tips of the regions—see Supplemental Material [34].

Another limit that is worth mentioning is the one of parallel spheres. In that case $\cosh \tilde{\beta}/2 = 0$. Therefore, $\Lambda(\tilde{\beta}) = \mathbf{1}$ and $\chi_{\mathcal{R}}(\mathbf{1}) = \dim R$. This limit provides a simple

extension of the result presented in [22] to the case of arbitrary primary operators, namely

$$I_\Delta = \dim \mathcal{R} \int_0^1 \frac{dw (zw)^{2\Delta} (1 + \sqrt{1 - wz})^{2(d-2\Delta)}}{4^{d+1} \sqrt{1-w} (1-wz)^{\frac{d}{2}}}. \quad (15)$$

Large separations and conformal block expansions: When the regions are very far from each other, the Rényi twist operators become effectively local, and the MI is dominated by correlations between arbitrary pairs of points, one in each region. The leading contribution comes from two copies of the primary with the lowest conformal dimension, $\mathcal{O}\mathcal{O}$ [18]. Subleading contributions arising from descendants such as $\partial\mathcal{O}\mathcal{O}$, from multiple copies, as well as from other primaries become relevant as the regions approach.

Interestingly, for ball regions, considering the two-point functions at coincident points in the replica manifold—corresponding to the $x_2 \rightarrow x_1$ limit of the general constraint (2)—completely fixes the form of the twist operator to be the OPE block: $\mathcal{O}\mathcal{O}$ + descendants. This full tower can be resumed under the integral representation $\tilde{\Sigma}_A^{(n)} \approx \int_{D(x_A^+, x_A^-)} g(\xi) \mathcal{O}(\xi) \mathcal{O}(\xi)$, where the leading primary is smeared over the causal completion of the sphere $D(x_A^+, x_A^-)$. In turn, $g(\xi)$ is a local kernel that represents the fusion of $\mathcal{O}(x_A^+)$ and $\mathcal{O}(x_A^-)$ in the channel of the shadow operator [39,40]. Consequently, the resulting MI is the conformal block associated to the leading primary, $I_\Delta^{\text{local}} \propto G_{2\Delta}^\ell(z, \bar{z})$, with ℓ the corresponding spin representation.

On the other hand, as explained above, the expectation value of the two copy-sector at arbitrary insertion points $x_1 \neq x_2$ leads to a bilocal kernel *Ansatz* for the twist operator—see Eq. (5). Following the analysis of [22], the correlator of this twist gives rise to a sum of conformal blocks, $I_\Delta^{\text{bi-local}} \propto \sum_i b_i G_{\tilde{\Delta}_i}^{\ell_i}(z, \bar{z})$. Each block is associated to a different conformal primary in replica space—with conformal dimension $\tilde{\Delta}_i$ and spin ℓ_i —made out of the product of two seed CFT operators, $\mathcal{O}\mathcal{O}$ being the one with lowest conformal dimension—namely, $\tilde{\Delta}_1 = 2\Delta$. This is explained in more detail in Supplemental Material [34], where we explicitly verify the agreement between the long-distance expansion of our general formula and the corresponding conformal blocks sum in the case of the scalar representation. In Fig. 1 we compare the local and bilocal *Ansätze* with the known exact result [30] for a two-dimensional free chiral scalar. As expected on general grounds, the latter provides a better approximation at long distances.

A high-precision Ansatz for arbitrary separations: An obvious limitation of (14) is that it lacks information about the multiple-copy sector. Hence, although it should approximate extremely well the exact MI at long distances

for general theories, it will also deviate very significantly from the expected result at short distances. In fact, for parallel, concentric balls of radii R and $R + \delta$, Eq. (15) behaves, in $d > 2$, as

$$\frac{I_\Delta}{\dim \mathcal{R}} = \frac{\sqrt{\pi} \Gamma(\frac{d-1}{2})}{2^{d+3} \Gamma(\frac{d}{2})} \left(\frac{R}{\delta}\right)^{d-1} - \left(\Delta - \frac{d}{2}\right) \frac{\sqrt{\pi} \Gamma(\frac{d-2}{2})}{2^{d+2} \Gamma(\frac{d-1}{2})} \left(\frac{R}{\delta}\right)^{d-2} + \dots, \quad (16)$$

when $\delta \ll R$. This is a volume law divergence that severely differs from the short-distance mutual information of an ordinary local CFT

$$I = k_d \frac{2\pi^{\frac{d-1}{2}}}{\Gamma[\frac{d-1}{2}]} \left(\frac{R}{\delta}\right)^{d-2} + c_{d-4} \left(\frac{R}{\delta}\right)^{d-4} + \dots, \quad (17)$$

where k_d is the universal coefficient appearing in the EE of a strip region [1,4,41]. Moreover, note that while (16) includes all integer powers of the expansion parameter, the exact result exclusively includes either odd or even powers of (R/δ) .

In order to find a precise approximation to the exact MI of spherical regions in an arbitrary CFT, we propose a twofold modification of (15). First, we change the number of spacetime dimensions from d to $d - 1$. Crucially, the first few leading terms in the MI long-distance expansion—see Supplemental Material [34]—do not depend on d , but only on Δ . Hence, this apparently drastic modification does not spoil the approximation to the exact result at long distances. Second, we add the contribution of an additional operator of dimension Δ^* , with σ degrees of freedom, namely,

$$I^{\text{Ansatz}} \equiv I_\Delta(d-1) + \sigma \cdot I_{\Delta^*}(d-1). \quad (18)$$

In order to exactly match the area term, we fix σ in terms of the strip coefficient [42],

$$\sigma \equiv k_d \frac{2^{d+3} \pi^{\frac{d-2}{2}}}{\dim \mathcal{R} \Gamma[\frac{d-2}{2}]} - 1, \quad (d > 2). \quad (19)$$

This works because, crucially again, the leading term in the short-distance expansion of I_Δ does not depend on Δ explicitly. On the other hand, different strategies can be considered in order to optimize Δ^* : (i) we can fix it by requiring that the spurious $(R/\delta)^{d-3}$ term vanishes, which entails setting $\Delta^* = (1 + \frac{1}{\sigma})(\frac{d-1}{2}) - \frac{\Delta}{\sigma}$, as long as $\Delta^* > \Delta$ (so we do not spoil the long-distance behavior); (ii) we can also fix it by choosing it to be equal to the second lowest primary conformal dimension of the theory. In Fig. 1 we try these approaches in the case of a $d = 2$ free fermion, a $d = 3$ free scalar—finding an excellent agreement with the available exact and numerical lattice results, respectively—and a $d = 4$ Maxwell field, for which our *Ansatz* is

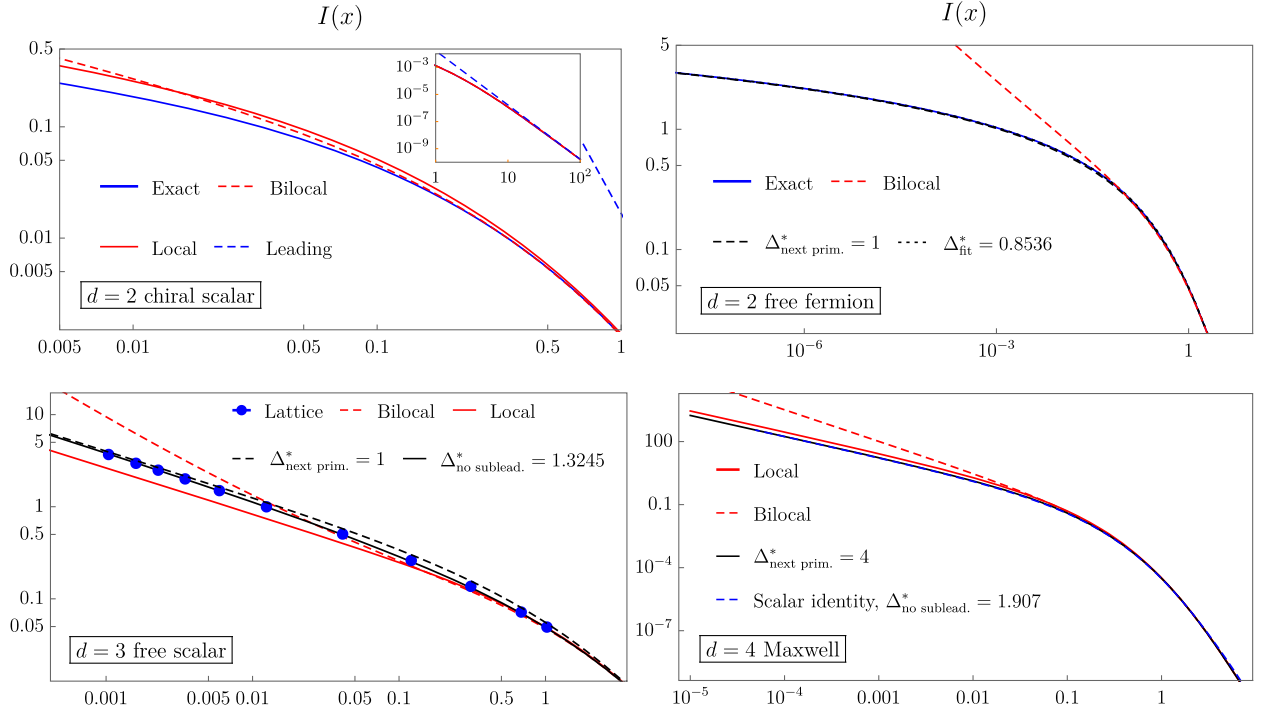


FIG. 1. Approximations to the MI of pairs of parallel disjoint balls for various free theories. In all cases, $x \equiv D/L$, where D is the distance between the regions' closest boundaries, and L is the length of both intervals in $d = 2$, and the disks/spheres radii in $d = 3, 4$. Both for the free chiral scalar and fermion in $d = 2$ (upper row) the exact results are known [28–30] and they are plotted in blue. Numerical results from the lattice—with conservative error bars corresponding to the size of the dots—are presented for the $d = 3$ free scalar. In all plots, the two red curves represent approximations arising from the modular flow of the two-copy sector, both of which become exact at long distances. While the bilocal *Ansatz* (dashed) is consistently more accurate than the local one (solid) at intermediate separations—with the exception of the $d = 2$ free fermion, for which the latter is identical to the exact result [2, 17]—it goes astray at short distances, due to its volume-law behavior. The black curves represent refinements of the bilocal approximation in which the number of spacetime dimensions is switched, $d \rightarrow d - 1$, and an additional contribution from an operator of dimension Δ^* and σ degrees of freedom is added so that the correct area law is recovered without spoiling the long-distance behavior—see Eqs. (18) and (19). In the plots, Δ_{fit}^* corresponds to a least-squares fit to the exact result, $\Delta_{\text{next prim.}}^*$ to adding the next-to-leading primary contribution and $\Delta_{\text{no sublead.}}^*$ to the choice which removes the spurious subleading term in the short-distance expansion.

expected to provide the best available approximation to the unknown exact result. As recently shown in [43], the $d = 4$ Maxwell MI is double the difference between the $d = 4$ free scalar and $d = 2$ chiral scalar results. We can therefore alternatively approximate the Maxwell MI indirectly by first applying our method to the $d = 4$ free scalar. We compare both approaches in Fig. 1—see the black and dashed blue curves—finding excellent agreement.

In sum, from the knowledge of the lowest scaling dimension of a given CFT, Δ , its spin representation, and its corresponding strip coefficient, k_d , we can construct high-precision approximations to the MI of pairs of spherical regions at arbitrary separations using (18).

Conclusions. In this Letter we have computed the general contribution of a given primary to the MI of two balls resulting from the two-copy sector of a general d -dimensional CFT. Our result is most naturally applied to the operator with the lowest scaling dimension of the theory, providing the state-of-the-art long-distance approximation

to the MI of any CFT. On the other hand, such formula leads to a pathological volume-law behavior in the short-distance regime. Interestingly, exploiting the dimension-independent form of the leading long-distance term and the Δ -independent form of the leading short-distance one, we have put forward a new *Ansatz*, which interpolates between the exact results in both regimes. This approximates very accurately the available exact free-field results for arbitrary separations.

The fact that using only partial information—the two-copy sector—leads to a volume law at short distances is reminiscent of generalized free fields (GFFs), for which the same phenomenon arises [44]. In fact, for unboosted spheres our formula (15) provides an excellent approximation to the MI of a GFF in an arbitrary spin representation in the full range of cross ratios. As happens in the case of spin-0 representations [22], this approximation becomes exact in the large- d limit. This is because both quantities differ from their scalar-CFT counterparts only by an overall multiplicative factor given by the number of degrees of

freedom. This match suggests that our formula may be capturing the contribution from some kind of coarse grained sector of the theory. Moreover, the similarity between our formula and the GFF result perfectly aligns with and further supports the modifications implemented in (18). The fact that the GFF MI corresponds to the MI of a massive scalar in AdS_{d+1} motivates the change $d \rightarrow d - 1$, while the absence of a stress tensor that characterizes a GFF suggests that some degrees of freedom might be added to compensate for that.

Additional future directions should also include incorporating the effects of operators from multiple copies. Such developments would move us even closer to a full systematic characterization of any CFT in terms of vacuum MIs.

Acknowledgments. We would like to thank Horacio Casini and Guim Planella for useful discussions. C. A. is funded by the program “Saavedra Fajardo” 22824/SF/24 from Fundación Séneca de la región de Murcia, and by the Spanish MINECO Grant No. PID2024-155685NB-C22. P.B. and GvdV were supported by a Proyecto de Consolidación Investigadora (No. CNS 2023-143822) from Spain Ministry of Science, Innovation and Universities, and by the Grant No. PID2022-136224NB-C22, funded by MCIN/AEI/10.13039/501100011033/FEDER, UE.

Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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