

Light hybrid baryons in the constituent model of QCD

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Hybrid baryons, in which gluonic degrees of freedom play an explicit dynamical role, provide a key testing ground for nonperturbative quantum chromodynamics. In this work, we investigate the mass spectrum of light hybrid baryons composed of identical quarks within a phenomenological constituent framework, applied to a quark core-gluon approximation. In this approach, the hybrid baryon is described as a bound state of a color-octet three-quark core and a constituent gluon, allowing the original four-body problem to be reduced to a three-body calculation followed by an effective two-body treatment. The spectrum of the color-octet quark core is obtained by solving a semirelativistic three-quark Hamiltonian with linear confinement, Coulomb, and regularized hyperfine interactions using an oscillator basis expansion. Finite-size effects of the core are incorporated through the convolution of the effective core-gluon interaction with the spatial quark density. The resulting two-body problem, whose associated Hamiltonian has the same shape as the one of the core, is solved applying the helicity formalism and using the Lagrange mesh method. Our results predict the lightest hybrid baryons to occur at energies above 3 GeV, with negative-parity states generally lying below their positive-parity counterparts. The predicted spectra are compared with lattice QCD and QCD sum-rule calculations, showing qualitative agreement although the lowest-lying lattice QCD results are significantly lighter than the present ones. Possible extensions of the model and implications for future experimental searches are discussed.

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I. INTRODUCTION

The identification and characterization of hybrid hadrons, i.e., states in which gluonic degrees of freedom play an active dynamical role, constitute important challenges in contemporary hadron spectroscopy. While quantum chromodynamics (QCD) predicts the existence of nonconventional baryons with explicit gluonic excitations [1–5], their experimental confirmation remains elusive. In recent years, several experimental programs have significantly strengthened the motivation for a systematic study of hybrid baryons. The GlueX Collaboration at Jefferson Lab has initiated dedicated searches for hybrid baryons through photoproduction channels. However, to date, the most compelling experimental indications of hybrid states have been observed only in the meson sector [6,7]. Besides, analyses from CLAS12 suggest that certain excited baryons

in the light-quark sector may accommodate gluonic excitations, although current data do not yet allow a clear discrimination from conventional three-quark configurations [8].

From the theoretical point of view, hybrid baryons have been studied using a wide variety of nonperturbative approaches. Lattice QCD (LQCD) simulations predict the existence of low-lying hybrid baryons with masses in the range 2.5–3 GeV, featuring characteristic gluonic excitations with quantum numbers not compatible with simple three-quark configurations [3]. The flux-tube model [4] and QCD sum rules [2,9,10] also confirm the existence of gluon-rich baryonic excitations, though with significant spread in the predicted mass hierarchies and state multiplicities. In the constituent picture, hybrid baryons are treated as effective bound states of three quarks and one constituent gluon, interacting via a QCD-motivated potential. However, the direct four-body treatment remains technically involved due to the presence of gluon helicity degrees of freedom and the complexity of constructing properly symmetrized four-body helicity states.

In this work, we address these difficulties by employing the *quark core–gluon model* recently introduced in Ref. [11]. Instead of solving the full four-body problem, the three light quarks are first combined into an effective *quark core* with definite J^P and color-octet structure.

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The hybrid baryon then emerges as a two-body bound state of this quark core and a constituent gluon, whose quantum numbers are taken to be the natural ones, i.e., $J^P = 1^-$. This picture is analogous to the quark-diquark approximation of ordinary baryons, which recently turned out to give good results in constituent framework, in both the heavy and light sectors [12].

Besides, the core-gluon-like construction provides technical advantages. The complexity of the Hilbert space is dramatically reduced: only two-body helicity states are required, for which the associated formalism is well established [13,14]. This effectively makes the two-body system resemble a two-gluon glueball at the color level, a picture that is well described by many QCD models [15,16]. The nontrivial internal structure of the quark core is, for its part, taken into account through a convolution of the core-gluon potential with the spatial quark distribution obtained from the three-body calculation. This study mainly focuses on the prediction of mass spectrum for the lightest hybrid baryons with three identical quarks, $nnnq$, where the quark core is considered in its ground state.

The paper is organized as follows. In Sec. II, we describe the core-gluon model and its theoretical motivations. In Sec. III, we present the three-body Hamiltonian for the quark core and compute its energy spectrum. Section IV constructs the effective core-gluon potential, including the convolution with the quark density. Section V details the construction of hybrid baryon states within the helicity formalism and the numerical resolution of the two-body problem using the Lagrange mesh method (LMM). Section VI gives our hybrid baryon spectrum and presents a comparison with the literature. Finally, Sec. VII summarizes the findings and provides perspectives for future investigations.

II. THE QUARK CORE-GLUON MODEL FOR LIGHT QUARKS

In this work, the description of hybrid baryon systems is realized using a quark-constituent model: they are considered as bound states of three quarks and a constituent gluon, interacting via a QCD-inspired potential. In this framework, a Schrödinger-like equation is to be solved to get the spectrum of the particle of interest. It is shown in [17] that considering helicity degrees of freedom instead of spin while dealing with constituent gluons leads to better agreement with LQCD calculations. But constructing four-body helicity states is a very challenging task. Even for three-body systems the construction of helicity states is cumbersome (see [14] for further details). Specific approximation schemes can help mitigate this issue. The one we will use in this study, presented in [11], is called the *quark core-gluon model*. It consists of dividing the four-body problem into two subproblems which are easier to deal with. Concretely, we bring the three quarks together in a

quark core, which will then interact with the remaining constituent gluon as a two-body system. In this way, we will just have to apply the well-known two-body helicity formalism [13] to construct the physical states. It is important to note that only color interactions are considered. The principle of color confinement forces the hybrid baryon to lie in a color singlet state, **1**. Since the state associated with the gluon lies in the octet representation, **8**, of the color symmetry group $SU(3)_c$, the three-body state associated with the quark core has to lie in that same representation, as [18]

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1}_S \oplus \mathbf{8}_S \oplus \mathbf{8}_A \oplus \mathbf{10}_A \oplus \bar{\mathbf{10}}_A \oplus \mathbf{27}_S. \quad (1)$$

Octet representations of the quark core stem from the following decomposition [19]

$$\begin{aligned} \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} &= (\bar{\mathbf{3}}_A \oplus \mathbf{6}_S) \otimes \mathbf{3} \\ &= (\mathbf{1}_A \oplus \mathbf{8}_{MA}) \oplus (\mathbf{8}_{MS} \oplus \mathbf{10}_S), \end{aligned} \quad (2)$$

where the indices A , S , MA and MS reflect the symmetry of the color states under exchange of the quarks. A stands for *antisymmetric*, S for *symmetric* and M for *mixed*. This illustrates that, although quark cores seem very similar to ordinary baryons, they are not lying in the same color representation space. This raises the question of the way such an octet core should interact. In this study, this issue will be overcome by assuming the universality of color interactions. It means that the only difference between all the interactions described in the model is the color charge of the sources. In particular, the Hamiltonian associated with the quark-quark interactions in the quark core will have the same shape as for ordinary baryons, as well as for the core-gluon interaction in the hybrid baryons. The only differences lie in how the color part of that operator acts on the states, and the values of the parameters. Note that this hypothesis, sometimes called *Casimir scaling hypothesis*, has been verified using lattice computations for interactions similar to those used in the present work [20].

Note that the basic idea behind the core-gluon approximation stems from the quark-diquark picture of ordinary baryons, which was the first attempt of substructure description of multiquarks states [21,22]. Although intuitively more convenient for the description of baryons containing two heavy quarks, such as bbn or ccn [23–25], a recent study [12] showed that such an approximation holds for the lightest baryons as well. This encourages the use of the core-gluon model to describe not only heavy hybrids but also the lightest ones, which are the most likely to be detected.

In view of what has been explained above, the core-gluon approximation suggests developing the following two-step procedure that leads to the determination of the mass spectrum of hybrid baryons:

- (i) Compute the energy spectrum of the color-octet quark core by solving the quantum three-body problem. It will be done by using a semirelativistic Hamiltonian with linear confinement, Coulomb interaction, and a regularized hyperfine term. The eigenstates will be obtained via an oscillator basis expansion (OBE) method.
- (ii) Solve the resulting quark core-gluon two-body problem within the helicity formalism by treating the quark core as an extended effective particle. The interaction potential will have the same shape as for the quark core, corrected by convolution with the spatial quark density of the core to account for size effects. The mass spectrum of the light hybrid baryons will finally be determined using the LMM, suitably adapted to handle states with helicity degrees of freedom.

We will do that step by step in the next Sections.

III. THE SPECTRUM OF THE QUARK CORE

The first step when using the core-gluon approach is to calculate the mass spectrum of the quark core. The obtained eigenvalues will serve as mass parameters for the core in the subsequent two-body problem (cf. Sec. V), and the associated eigenstates will serve to encode the core spatial extent in the core-gluon interaction (cf. Sec. IV B). To do so, we need to solve a quantum three-body problem in its center of mass frame (CoMF). This task is tackled numerically using the OBE method. The considered Hamiltonian is the following:

$$\mathcal{H}_C = \sum_{i=1}^3 \sqrt{\mathbf{p}_i^2 + m_i^2} + \sum_{i<j=2}^3 \left(\sigma r_{ij} + \mathbf{F}_i \cdot \mathbf{F}_j \frac{\alpha_s}{r_{ij}} + V_0 - \mathbf{F}_i \cdot \mathbf{F}_j \frac{8\alpha_{ss}\Lambda^3}{3\sqrt{\pi}m_i m_j} e^{-\Lambda^2 r_{ij}^2} \mathbf{s}_i \cdot \mathbf{s}_j \right). \quad (3)$$

Note that m_i and $\mathbf{p}_i$ stand for the constituent mass and the momentum of the i th quark, respectively (with $\sum_{i=1}^3 \mathbf{p}_i = \mathbf{0}$), $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ for the norm of the relative distance between quarks i and j , and $\mathbf{F}_i$ and $\mathbf{s}_i$ for the color and spin operators of the i th quark. Inspired from [26], the Hamiltonian (3) has the following features:

- (i) A semirelativistic kinematic with constituent quark masses m_i , more appropriate for light quarks dynamics, is used.
- (ii) The central part is a Cornell-like potential, which contains three terms. The linear one represents the confinement of the quarks into the core, which is identical to the one for ordinary baryons [11]. The Coulomb one represents the color charge interaction, and the last one is a shift constant, justified by light hadron phenomenology [27].

- (iii) The last term is a hyperfine one, whose shape arises from a non-relativistic limit of the spin-spin part of the one-gluon exchange process [28]. However, by comparison with [28], this term has here been slightly modified in two ways: first, the mass parameters m_i are taken to be the same as in the kinetic energy. Second, in Ref. [28], the spin-spin term lets a contact Dirac delta interaction appear. Because this would lead to divergences while solving the Schrödinger equation, in (3), we have chosen to regularize it, as in Refs. [26,27], but with a Gaussian function, which makes a new independent parameter Λ appear.

The parameters of the model are fixed to the same values as in [26], except for those associated with the hyperfine term, i.e., Λ and α_{ss} . In that paper, the spin-spin interaction is regularized by a Yukawa function instead of a Gaussian. A Gaussian is used in this study because it is more suitable with respect to the OBE, leading to faster convergence, hence more accurate results. The parameters Λ and α_{ss} are therefore to be refit so that the eigenvalues of (3) agree with the experimental masses for the lightest baryons, that are the nucleon and the Δ . Such a regularization gives essentially the same results as in Ref. [26] for the light baryon spectrum, with deviations up to 50 MeV maximum. Note that the parameter α_{ss} is actually supposed to be the same as the strong coupling constant α_s from the point of view of one gluon exchange diagram. However, for the same reason as for the choice of constituent mass parameters, we allow α_{ss} to be different from the strong coupling constant α_s , as in Ref. [26]. Relaxing one more parameter in the Hamiltonian allows us to better reproduce the experimental results for the mass spectrum of the light baryons. All the parameters involved in the model are given in Table I. The comparison with the experimental results for conventional baryons is shown in Table II.

Now that the parameters are fixed, we can diagonalize (3) in the harmonic oscillator basis [29]. To do that, we first need to evaluate the action of the color and spin operators $\mathbf{F}_i \cdot \mathbf{F}_j$ and $\mathbf{s}_i \cdot \mathbf{s}_j$ on the trial states, a task that requires investigating their spin and color part. This is not as easy as for potentials associated with ordinary baryons. Indeed, since we are dealing with identical fermions, the quark core

TABLE I. Parameters associated with the quark core potential. These are fitted so that the eigenvalues of the Hamiltonian agree with the experimental masses of the nucleon and the Δ .

Parameters	Fitted values
m_n	0.337 GeV
σ	0.1215 GeV ²
α_s	0.57
α_{ss}	0.239
Λ	0.82 GeV
V_0	-0.409 GeV

TABLE II. Comparison of the light baryon spectrum with experimental masses. Model parameters are fitted to reproduce at best the N_{939} and Δ_{1232} . Resulting values are grouped in Table I. All masses are given in MeV.

Experimental masses	J^P	Model masses
N_{939}	$\frac{1}{2}^+$	941
N_{1440}	$\frac{3}{2}^+$	1618
N_{1710}	$\frac{1}{2}^+$	1879
Δ_{1232}	$\frac{3}{2}^+$	1232
Δ_{1600}	$\frac{3}{2}^+$	1866
$N_{1520} - N_{1535}$	$\frac{3}{2}^- - \frac{1}{2}^-$	1531
$N_{1650} - N_{1675} - N_{1700}$	$\frac{1}{2}^- - \frac{3}{2}^- - \frac{3}{2}^-$	1683
$\Delta_{1620} - \Delta_{1700}$	$\frac{1}{2}^- - \frac{3}{2}^-$	1648
$N_{1680} - N_{1720}$	$\frac{3}{2}^+ - \frac{3}{2}^+$	1883

must be antisymmetrized under exchange of particles. However, as indicated in (2), the quark core can lie in two different color octet representations, each of which satisfies mixed exchange symmetry properties. This renders the construction of proper basis trial states much more intricate. Their explicit construction is therefore performed in Appendix A. The present discussion aims at computing the action of the color operator $\mathbf{F}_i \cdot \mathbf{F}_j$ on the two eight-dimensional irreducible subspaces defined in (2). For the interaction between the i th and j th quarks, this operator reads

$$\mathbf{F}_i \cdot \mathbf{F}_j = \frac{1}{2} ((\mathbf{F}_{ij})^2 - \mathbf{F}_i^2 - \mathbf{F}_j^2), \quad (4)$$

where $\mathbf{F}_i^2$ stands for the quadratic Casimir of the fundamental representation of $SU_c(3)$ and $\mathbf{F}_{ij}^2$ for the quadratic Casimir of the irreducible representations supported by the tensor product of the quarks i and j . Table III lists the values of the quadratic Casimir for different irreducible representations of $SU_c(3)$.

Using the values in Table III together with the decomposition (2), we can compute the action of the color operator (4) on the mixed symmetric and antisymmetric color states, which gives

$$\langle \mathbf{F}_i \cdot \mathbf{F}_j \rangle_{MS} = \frac{1}{3} \quad \text{and} \quad \langle \mathbf{F}_i \cdot \mathbf{F}_j \rangle_{MA} = -\frac{2}{3}. \quad (5)$$

The action of the color operator on the symmetrized basis trial states is then directly obtained from both (5) and the

TABLE III. Quadratic Casimir eigenvalues for selected irreducible representations of the color group $SU_c(3)$.

Irrep of $SU(3)$	3	$\bar{3}$	6	8	1
$\mathbf{F}^2$	4/3	4/3	10/3	3	0

TABLE IV. Energy spectrum of the color-octet quark core nnn . E_0 and E_1 stand for masses of the ground and first excited states, respectively.

$J_C^{P_C}$	I_C	E_0		E_1	
		Baryon	Quark core	Baryon	Quark core
$\frac{1}{2}^+$	$\frac{1}{2}$	0.941	1.481	1.602	2.089
	$\frac{3}{2}$		1.615		2.218
	$\frac{5}{2}$		1.562		2.159
$\frac{3}{2}^+$	$\frac{1}{2}$	1.232	2.228	1.863	2.695
	$\frac{3}{2}$				

decomposition of these states in the nonsymmetrized ones given in Appendix A.

The results for the lowest-lying eigenvalues associated with the Hamiltonian (3), using the parameters of Table I, are listed in Table IV for different $I_C(J_C^{P_C})$ quantum numbers. These are compared with eigenvalues for ordinary light baryons obtained with the same parameters. Note that for this study we have assumed no global orbital excitation of the quark core, i.e., $L_C = 0$. This means that, first, the global spin quantum number S_C is equivalent to the total angular momentum, $J_C = S_C$. Second, the parity of the core is always positive. The analysis of Table IV shows that the masses of the color octet quark core are higher for all the possible quantum numbers and the mass gap increases when J_C and I_C increase. In that table, the ground states E_0 and first excited states E_1 are given even though only the ground states will be used in the next Sections. Note also that, even for $L_C = 0$, there are twice more accessible states for the quark cores compared to the associated ordinary baryons. This is due to the nontrivial symmetry of the color part of the core, which reduces the constraints on the exchange symmetry of the states, leading to more possible states (see Appendix A for an example).

Now that we have the spectrum of the quark core, let us consider each of the corresponding states as a unique effective particle whose spin, intrinsic parity, isospin and rest mass are provided by the total angular momentum J_C , parity P_C , isospin I_C and associated mass eigenvalue. These effective cores are not considered as pointlike, and their finite spatial extension will rule their interactions with the gluon. This spatial extension is considered as encoded in the obtained eigenstates. In the next Section, we will turn our attention to the description of the interaction between the quark core and the constituent gluon.

IV. DESCRIPTION OF THE QUARK CORE-GLUON INTERACTION

In this Section, we will describe the interactions between the quarks and the gluon within the core-gluon model. In this way, the quark core is considered as an effective particle with a finite spatial extension whose characteristics have been computed in the previous Section. Given the

hypothesis of universality of the color interactions, the core-gluon interaction will be treated similarly to the quark-quark one in the core, the only differences being the size effect of the core and the color charge of the sources. Note that the latter makes the entire system analogous to a two-gluon glueball and we will take advantage of that. In fact, the higher mass of the quark core could give the impression that the system looks like *gluelumps* in the constituent picture [30,31], but this possibility will not be developed in this study. The following subsections are devoted to the construction of a proper core-gluon potential, which describes their interactions taking the above comments into account.

A. The pointlike core-gluon interaction

The first step of the construction is to write down a two-body Hamiltonian taking the octet charge of the color sources into account, and to fix all the parameters of the model properly. As mentioned in the introduction of this section, the core-gluon Hamiltonian will have the same shape as the quark-quark one in (3), that is

$$\mathcal{H}_{Cg} = \sqrt{\mathbf{p}^2 + m_C^2} + \sqrt{\mathbf{p}^2 + m_g^2} + V_{Cg}(r), \quad (6)$$

with

$$V_{Cg}(r) = \sigma_{Cg} r + \mathbf{F}_C \cdot \mathbf{F}_g \frac{\alpha_{s_{Cg}}}{r} - \mathbf{F}_C \cdot \mathbf{F}_g \frac{8\alpha_{ss_{Cg}} \Lambda_{Cg}^3}{3\sqrt{\pi} m_C m_g} e^{-\Lambda_{Cg}^2 r^2} \mathbf{s}_C \cdot \mathbf{s}_g + V_{0_{Cg}}. \quad (7)$$

Above, $\mathbf{p}$ and $\mathbf{r}$ stand for the particle's momentum and relative position in the core-gluon CoMF, respectively. In Ref. [11], the constituent gluon has a vanishing mass. In the present paper, a nonzero constituent mass m_g for the gluon is used in both the kinetic energy and the spin-spin interaction, in the same way as for the core Hamiltonian (3). It is actually quite usual in phenomenological models [32–38]. It can be justified since, due to the gluon

self-interaction, these particles acquire a dynamical mass. A recent study [39] has even suggested a proper gluon mass $m_g \approx \frac{1}{2} m_p \approx 0.450$ GeV (where m_p stands for the proton mass) which could unify the results of different phenomenological models.

Contrary to the quark core case, the computation of the action of the color operator in (7) on the hybrid baryon states is straightforward since they are color singlet states. Taking back relation (4), Table III, and recalling that the gluon and the core are both color octets which combine to form a global singlet, the mean value of the color operator reads

$$\langle \mathbf{F}_C \cdot \mathbf{F}_g \rangle = \frac{1}{2} (0 - 3 - 3) = -3. \quad (8)$$

The spin-spin operator is not as easy to treat. Indeed, the hyperfine term in (7) assumes the core and the gluon carry spin degrees of freedom. However, even massive, the gluon is shown to still have helicity degrees of freedom [14,17]. This issue is overcome in Appendix B. Note also that the spin operator associated with the core, denoted $\mathbf{s}_C$, could be represented either by the total spin S_C or the total angular momentum J_C of the core, as the core is seen as a unique particle at rest. In this study, the question does not arise since $S_C = J_C$.

We chose to use a universal shape for the color interaction. However, due to the simplicity of our Hamiltonian model, we observed that it is not possible to reproduce both the baryon and glueball spectra with the same set of parameters for the potential. Therefore, parameters in (6) are adapted taking advantage of the analogy with the gluon-gluon interaction. Indeed, the Hamiltonian (6) is very similar to the one used to describe two-gluon glueballs within the helicity formalism in Refs. [14,17], with results in agreement with LQCD computations. The differences are the addition of the hyperfine term and the nonzero masses of the involved particles. The values of the string tension and the shift constant of [17] will then be taken as references for the fit. The strong coupling parameter $\alpha_{s_{Cg}}$, as well as the new parameters $\alpha_{ss_{Cg}}$

TABLE V. Comparison of the two-gluon glueball spectrum computed using the LMM with an Hamiltonian structure as (6) and the parameters in Table VI with the LQCD results of [41,40]. Notations for the states are taken from [14]. Energies are provided in GeV.

State	Present model	LQCD [41]	LQCD [40]
$ \Psi_1; S_+, 0^+\rangle$	1.596	$1.710 \pm 0.050 \pm 0.080$	$1.475 \pm 0.030 \pm 0.065$
$ \Psi_2; S_+, 0^+\rangle$	2.766	$2.670 \pm 0.180 \pm 0.130$	$2.755 \pm 0.070 \pm 0.120$
$ \Psi_1; S_-, 0^-\rangle$	2.309	$2.560 \pm 0.035 \pm 0.120$	$2.250 \pm 0.060 \pm 0.100$
$ \Psi_2; S_-, 0^-\rangle$	3.141	$3.640 \pm 0.060 \pm 0.180$	$3.370 \pm 0.150 \pm 0.150$
$ \Psi_1; D_+, 2^+\rangle$	2.366	$2.390 \pm 0.030 \pm 0.120$	$2.150 \pm 0.030 \pm 0.100$
$ \Psi_1; S_+, 2^+\rangle$	3.071	...	$2.880 \pm 0.100 \pm 0.130$
$ \Psi_1; S_-, 2^-\rangle$	3.074	$3.040 \pm 0.040 \pm 0.150$	$2.780 \pm 0.050 \pm 0.130$
$ \Psi_2; S_-, 2^-\rangle$	3.790	$3.890 \pm 0.040 \pm 0.190$	$3.480 \pm 0.140 \pm 0.160$
$ \Psi_1; D_-, 3^+\rangle$	3.367	$3.670 \pm 0.050 \pm 0.180$	$3.385 \pm 0.090 \pm 0.150$

TABLE VI. Parameters associated with the core-gluon Hamiltonian (6). These are chosen so that the two-gluon glueball mass spectrum fit the lattice results of [40,41].

Parameters	Fitted values
m_g	0.450 GeV
σ_{Cg}	0.416 GeV ²
$\alpha_{s_{Cg}}$	0.492
$\alpha_{ss_{Cg}}$	0.492
Λ_{Cg}	0.162 GeV
$V_{0_{Cg}}$	0 GeV

and Λ_{Cg} of the hyperfine term are then fitted to reproduce as much as possible the glueball spectrum obtained using LQCD in Refs. [40,41]. After the fit, the obtained glueball spectrum lies within the lattice error bars (see Table V). Table VI regroups all the parameters of the model. Note that, contrary to the fitting of the parameters done in Sec. III, $\alpha_{s_{Cg}}$ and $\alpha_{ss_{Cg}}$ are not needed to be different in order to improve the fit. For the phenomenology of two-gluon glueballs, the shifting constant V_0 is not necessary. The potential (7) finally reads

$$V_{Cg}(r) = \sigma_{Cg}r - 3\frac{\alpha_{s_{Cg}}}{r} + \frac{8\alpha_{ss_{Cg}}\Lambda_{Cg}^3}{\sqrt{\pi}m_C m_g} e^{-\Lambda_{Cg}^2 r^2} \mathbf{s}_C \cdot \mathbf{s}_g. \quad (9)$$

B. Size effect of the core

As discussed in Sec. III, the quark core has a complex internal structure, which gives a spatial extent to its color charge distribution. This size is expected to modify the core-gluon interaction. Indeed, unlike the potential (9), which depends on the norm of the relative position between the core and the gluon, the true interaction should link the gluon with all the internal quarks. This issue first appeared in the study of ordinary baryons in the quark-diquark picture. In Ref. [12], the authors suggest to overcome it by replacing the quark-diquark potential by its convolution with the color density of the diquark. This led to results with very good agreements with a full three-body description, for both light and heavy sectors. The same method will then be adopted here. Mathematically, the convolution reads

$$\tilde{V}_{Cg}(r) = \int d^3s \frac{\rho(s)}{3} V_{Cg}(|\mathbf{r} + \mathbf{s}|), \quad (10)$$

where V_{Cg} is given in (9) and $\rho/3$ is the normalized color density of the core. The integration is performed with respect

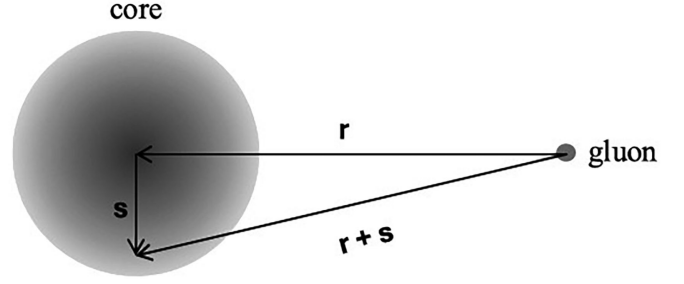


FIG. 1. Schematization of the quark core-gluon convoluted interaction, adapted from [12].

to the position relative to the CoMF of the core. Figure 1 depicts a schematic visualization of the core-gluon interaction defined by (10). In order to determine the convoluted potential, we first need to compute the color density of the core. This density is actually equal to the quark one because all quarks inside the core possess the same color charge. For three-body cores, the quark density function is defined as follows

$$\rho(\mathbf{q}) = \sum_{j=1}^3 \iiint \psi^*(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \delta^3(\mathbf{q} - \mathbf{q}_j) \times \psi(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) d^3q_1 d^3q_2 d^3q_3, \quad (11)$$

where $\psi(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ is the normalized wave function of the core in some inertial reference frame. But we want ρ to be a function of the position relative to the CoMF, $\mathbf{s}$. Thus, we will define $\mathbf{s} = \mathbf{q} - \mathbf{R}$ and $\mathbf{s}_j = \mathbf{q}_j - \mathbf{R}$, where $\mathbf{R} = \frac{\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3}{3}$ is the position of the center of mass of the three identical quarks in the inertial reference frame. In this way, by changing the $\mathbf{s}_j$ coordinates to the so-called dimensionless Jacobi coordinates $\{\mathbf{x}, \mathbf{y}\}$ (defined in Appendix A), the quark density is expressed as an integral over the internal part of the core wave function, $\psi_{\text{int}}(\mathbf{x}, \mathbf{y})$,

$$\rho(\mathbf{s}) = \sum_{j=1}^3 \iint \psi_{\text{int}}^*(\mathbf{x}, \mathbf{y}) \delta^3(\mathbf{s} - \mathbf{s}_j(\mathbf{x}, \mathbf{y})) \psi_{\text{int}}(\mathbf{x}, \mathbf{y}) d^3x d^3y, \quad (12)$$

where the integral over the center of mass coordinate has been dropped. The explicit expression of ψ_{int} is given by the resolution of the core dynamics with OBE, and is described in Appendix A. Inserting it in (12) and performing the two integrals lead to

$$\begin{aligned} \rho_{LM}(s, \theta_s, \phi_s) &= \frac{9\sqrt{3}}{a^3} \sum_{n_x, l_x, m_x} \sum_{n_y, l_y, m_y} \sum_{n'_y, l'_y, m'_y} C_{n_x l_x n_y l_y}^* C_{n_x l_x n_y l_y} (l_x m_x l'_y m'_y | LM) (l_x m_x l_y m_y | LM) \\ &\times \mathcal{R}_{n'_y l'_y} \left(\frac{\sqrt{3}}{a} s \right) \mathcal{R}_{n_y l_y} \left(\frac{\sqrt{3}}{a} s \right) Y_{l'_y m'_y}^*(\pi - \theta_s, \pi + \phi_s) Y_{l_y m_y}(\pi - \theta_s, \pi + \phi_s). \end{aligned} \quad (13)$$

In this expression, the functions $\mathcal{R}_{nl}$ and Y_{lm} are the radial solutions of the 3D harmonic oscillator [29] and the spherical harmonics, respectively, the factors $C_{n_x l_x n_y l_y}$ are expansion coefficients and a is a variational scale parameter. The values of the latter are provided by OBE. Note that the construction of the quark density presented above implicitly assumes a nonrelativistic treatment of the change of reference frame. The core wave function, and hence the resulting density $\rho(s)$, is computed in the CoMF of the core itself, whereas the convolution (10) combines it with the core-gluon potential, naturally defined in the CoMF of the core-gluon system. A fully covariant construction would require boosting the internal wave function of the core into the latter frame before performing the convolution, which considerably complicates the calculation. This approximation, reminiscent of the Bakamjian-Thomas construction of relativistic dynamics [42], is justified by the large mass of the core: its recoil velocity in the core-gluon CoMF remains small compared to the internal motion of the quarks, so relativistic corrections to the density are expected to be subdominant.

Now that we have an expression for the quark density, we can plug it into relation (10) to find the convoluted potential. Mimicking the method of computation for the convoluted potential in [12], the relation (13) has to be slightly modified in order to cancel the dependence on the magnetic quantum number of the core, present for $L \neq 0$. It is done by averaging over all possible values of M . Even though only cores with $L = 0$ are considered in this work, we give the general formula,

$$\tilde{V}_{Cg}^L(r) = \frac{1}{(2L+1)} \sum_{M=-L}^L \int d^3s \frac{\rho_{LM}(s)}{3} V_{Cg}(|\mathbf{r} + \mathbf{s}|). \quad (14)$$

Note that the dependence of $\tilde{V}_{Cg}^L$ on the orbital quantum number of the core is expected, given that this quantum number is strongly correlated with the size of the system. Now, inserting (13) in (14) and performing the integrals in spherical coordinates, we obtain

$$\tilde{V}_{Cg}^L(r) = \frac{3\sqrt{3}}{2a^3} \sum_{n_x, l_x, n_y, l_y, n'_y} C_{n_x l_x n'_y l_y} C_{n_x l_x n_y l_y} \times \int ds s^2 \mathcal{R}_{n'_y l_y} \left(\frac{\sqrt{3}}{a} s \right) \mathcal{R}_{n_y l_y} \left(\frac{\sqrt{3}}{a} s \right) I(r, s), \quad (15)$$

where

$$I(r, s) = \int_{-1}^1 d\mu V_{Cg} \left(\sqrt{r^2 + s^2 + 2rs\mu} \right). \quad (16)$$

The simplification of the Clebsh-Gordan coefficients and the spherical harmonics, leading to (15), is done using the following properties [43]:

$$\sum_{m_x, M} (l_x m_x l_y m_y | LM) (l_x m_x l'_y m'_y | LM) = \frac{2L+1}{2l_y+1} \delta_{l'_y l_y} \delta_{m'_y m_y},$$

$$\sum_m |Y_{lm}(\theta_s, \phi_s)|^2 = \frac{2l+1}{4\pi}. \quad (17)$$

Note that the function V_{Cg} in (15) is the nonconvoluted core-gluon potential, whose expression is given by (9). It can be interesting to visualize the effect of the convolution on the potential shape between the quark core and the gluon. Figure 2 compares the convoluted potential $\tilde{V}_{Cg}^L$ with its nonconvoluted version, given by (9), omitting the hyperfine term, for which the action on the hybrid baryon

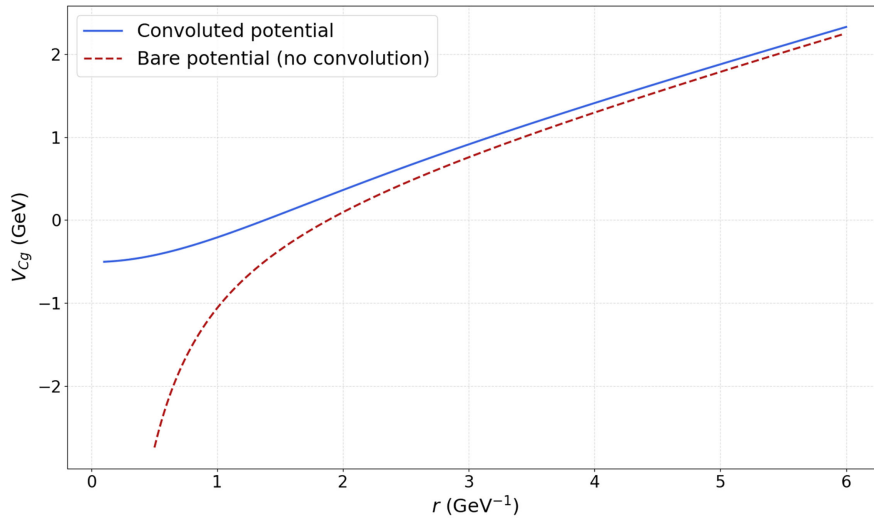


FIG. 2. Comparison of the convoluted and nonconvoluted Cornell part of the core-gluon potential (9) for $L_C = 0, I_C = 1/2, J_C = S_C = 1/2$. See text for the details.

states is not direct, for some specific quantum numbers $L_C = 0$, $I_C = 1/2$ and $J_C = S_C = 1/2$. We can see that the convolution tends to regularize the potential at short range, as expected since at this range the potential enters the sector where the quark density is large. At long range, the two potentials merge, as expected since the quark-core becomes more and more pointlike. Due to the linear long range part of the nonconvoluted potential, this merging is slow.

V. SOLVING THE CORE-GLUON DYNAMICS

In this Section, the core-gluon dynamics will be solved numerically. To do that, we first have to build the two-body states representing the hybrid baryons. Due to the presence of the gluon in the system, even though it has a constituent mass, this construction has to be realized within the helicity formalism [13]. Indeed, in Ref. [17], it is shown that the use of helicity is necessary to reproduce the LQCD glueball spectrum. Then, we will use a version of the LMM [44,45] adapted to handle helicity states [46] to obtain the associated mass spectrum.

A. Hybrid baryon states within the helicity formalism

In the helicity formalism, particle states are characterized by their helicity [13], which is a natural quantum number for massless particles [47]. In this work, as we want to predict the mass spectrum of hybrid baryons with quantum numbers J^P , the construction of such states should be done in the CoMF. In what follows, we will only focus on the spatial and spin part of the physical states, since the color part has already been treated independently in Sec. IV A. References [13,14] provide such a construction, leading to a complete set of two-body helicity states with definite total angular momentum J , M and helicities λ_C , λ_g

$$\begin{aligned} & |p, JM, \lambda_C \lambda_g; J_C\rangle \\ &= \sqrt{\frac{2J+1}{4\pi}} \int d\cos\theta d\phi D_{M\lambda_C-\lambda_g}^{J*}(\phi, \theta, 0) |p\theta\phi, \lambda_C \lambda_g; J_C\rangle. \end{aligned} \quad (18)$$

Above, the states $|p\theta\phi, \lambda_C \lambda_g; J_C\rangle$ are defined as eigenstates of the relative momentum, core total spin, core helicity and gluon helicity operators. The associated eigenvalues are given by the quantum numbers $p, \theta, \phi, J_C, \lambda_C$ and λ_g , respectively (the relative momentum is given in spherical coordinates). Note that for a massive core and a massless gluon $\lambda_C \in \llbracket -J_C, J_C \rrbracket$ and $\lambda_g \in \{-1, 1\}$. Let us recall that for this study, $J_C = S_C$ as $L_C = 0$. The states defined in (18) are by construction eigenstates of the total angular momentum operators with corresponding eigenvalues given by J, M . They have the following normalization

$$\begin{aligned} & \langle p', J'M', \lambda'_C \lambda'_g; J_C | p, JM, \lambda_C \lambda_g; J_C \rangle \\ &= \frac{4w_C(p)w_g(p)}{p^2} \delta(p' - p) \delta_{J'J} \delta_{M'M} \delta_{\lambda'_C \lambda_C} \delta_{\lambda'_g \lambda_g}, \end{aligned} \quad (19)$$

where $w_C(p) = \sqrt{p^2 + m_C^2}$ and $w_g(p) = \sqrt{p^2 + m_g^2}$ are the energies of the core and the gluon in the CoMF, respectively. The function $D_{M\lambda_C-\lambda_g}^{J*}$ in (18) is a Wigner D matrix [43]. Note that the presence of such a Wigner D matrix constraints the possible values for J so that $J \geq |\lambda_C - \lambda_g|$. Although the states (18) carry a definite total angular momentum, they do not possess a well-defined parity. Indeed, the action of the parity operator Π on these states reads [13]

$$\Pi |p, JM, \lambda_C \lambda_g; J_C\rangle = \eta_C \eta_g (-1)^{J-J_C-1} |p, JM, -\lambda_C - \lambda_g; J_C\rangle, \quad (20)$$

where η_C and η_g denote the intrinsic parities of the core and the gluon, respectively. Note that η_C is here chosen as the parity quantum number defined in Sec. III. It turns out, however, that suitable linear combinations of these states do possess definite parity. Specifically, one can define

$$\begin{aligned} |p, J^{P_i}, \lambda_C \lambda_g; J_C\rangle &= \frac{1}{\sqrt{2}} \left(|p, JM, \lambda_C \lambda_g; J_C\rangle \right. \\ &\quad \left. + (-1)^i |p, JM, -\lambda_C - \lambda_g; J_C\rangle \right), \end{aligned} \quad (21)$$

with $i = 0, 1$. The corresponding parity eigenvalues are then given by

$$P_i = (-1)^i \eta_C \eta_g (-1)^{J-J_C-1}. \quad (22)$$

The two-body states defined in (21) will be our basis states of the spatial and spin part of the Hilbert space carrying the physical hybrid baryon states. Following Ref. [14], we will define a generic hybrid baryon state with definite J^P quantum numbers as

$$|\Psi, J^{P_i}, \lambda_C \lambda_g; J_C\rangle = \int \frac{p^2 dp}{4w_C(p)w_g(p)} \Psi(p) |p, J^{P_i}, \lambda_C \lambda_g; J_C\rangle, \quad (23)$$

where the function $\Psi(p)$ should be interpreted as a radial helicity-momentum wave function, having the following normalization condition so that $\langle \Psi, J^{P_i}, \lambda_C \lambda_g; J_C | \Psi, J^{P_i}, \lambda_C \lambda_g; J_C \rangle = 1$ [14]:

$$\int \frac{p^2 dp}{4w_1(p)w_2(p)} |\Psi(p)|^2 = 1. \quad (24)$$

B. The Lagrange mesh method to solve the hybrid baryon Hamiltonian

The LMM is a numerical method to solve eigenvalue problems. It is based on a truncated basis expansion with definite trial states, whose matrix elements are computed by mean of Gauss-Laguerre quadratures [44]. This method was initially developed for problems in position space representation. The physical states of hybrid baryons (23) being constructed in momentum space representation, we should use an adapted LMM for the present problem [45]. However, the methodology employed in [45] is not suited for the Cornell potential. This LMM was recently extended in [46] for such a potential and tested on two-gluon glueballs. Let us shortly review the main calculation steps, emphasizing on the characteristics of the present system.

In the LMM adapted to our problem, we expand the generic physical states given in (23) as a linear combination of trial states,

$$|\Psi, J^{P_i}, \lambda_g; J_C\rangle = \sum_{j=1}^N \sum_{\lambda_C=1/2}^{J_C} C_{j\lambda_C} |f_j, J^{P_i}, \lambda_C \lambda_g; J_C\rangle, \quad (25)$$

where $C_{j\lambda_C}$ are the expansion coefficients to determine, and N is the number of mesh points, which is the same for each λ_C . The sum on λ_C is constrained by the condition $J \geq |\lambda'_C - \lambda_g|$ and should not cover negative values because of (21). The trial states in (25) are defined by

$$|f_j, J^{P_i}, \lambda_C \lambda_g; J_C\rangle = \int \frac{p^2 dp}{2\sqrt{w_C(p)w_g(p)}} \times \frac{f_j(p/h)}{p\sqrt{h}} |p, J^{P_i}, \lambda_C \lambda_g; J_C\rangle, \quad (26)$$

where the functions f_j are regularized Lagrange functions defined in [46], and h is a size parameter. The eigenvalue problem then becomes

$$\sum_{k, \lambda'_C} C_{k\lambda'_C} \langle f_j, J^{P_i}, \lambda_C \lambda_g; J_C | \mathcal{H}_{HB} | f_k, J^{P_i}, \lambda'_C \lambda_g; J_C \rangle = E C_{j\lambda_C}. \quad (27)$$

To solve this system, we first need to evaluate the Hamiltonian matrix of order $N \times N_{\lambda_C}$, with N_{λ_C} the number of possible helicities for the quark core. This computation is described in Appendix B. Note that the helicity of the quark core not being fixed, channels with different λ_C could be coupled with nonzero mixing coefficients $C_{j\lambda_C}$. Then, by diagonalizing the resulting Hamiltonian matrix, we find the eigenenergies E of the hybrid baryon that we want to study, and the eigencoefficients of the expansion $C_{j\lambda_C}$, which allows us to get the hybrid baryon wave function.

VI. LIGHT HYBRID BARYONS SPECTRUM

Let us now bring together all the elements that we have discussed until now, and write down explicitly the complete hybrid baryon Hamiltonian of our model. It is given by

$$\mathcal{H}_{HB} = \sqrt{p^2 + m_C^2} + \sqrt{p^2 + m_g^2} + \tilde{V}_{Cg}^L(r), \quad (28)$$

where $\tilde{V}_{Cg}^L(r)$ is the convoluted potential given by (15). This last function has to be evaluated numerically, and since we use a mesh method to solve the Hamiltonian the evaluation is done only at the mesh points. The mass parameter m_C is taken as the mass of the quark core, computed in Sec. III. All the other parameters are provided in Table VI. The results are presented in Table VII for $J_C = 1/2, 3/2$, $J^P = 1/2^\pm, 3/2^\pm$ and for the N -like and the Δ -like cores, corresponding to the total isospin $I_C = 1/2$ and $I_C = 3/2$, respectively.

We see that negative parity states are predicted to be lighter than their positive parity homologue. This means that the lightest hybrid baryon would have negative parity. Note also that the mass splitting between the hybrid N and Δ is about 120 MeV, which is two times less than for the conventional baryons. Then, even for a quark core with no orbital excitation, the hybrid baryon spectrum already looks really rich, with several states with the same radial

TABLE VII. Mass spectrum of light hybrid baryons. J_C and I_C stand for the spin and isospin quantum numbers of the core and m_C stands for its mass, computed in Sec. III. J^P stands for the quantum numbers of the hybrid baryons, and M_0 and M_1 for the mass of their associated ground and first excited states, respectively. Masses are given in GeV.

J_C	I_C	m_C	J^P	M_0	M_1
$\frac{1}{2}$	$\frac{1}{2}$	1.481	$\frac{1}{2}^-$	3.293	4.138
			$\frac{1}{2}^+$	3.444	4.219
			$\frac{3}{2}^-$	3.302	3.954
			$\frac{3}{2}^+$	3.452	3.864
	$\frac{3}{2}$	1.615	$\frac{1}{2}^-$	3.411	4.250
			$\frac{1}{2}^+$	3.561	4.330
			$\frac{3}{2}^-$	3.420	4.066
			$\frac{3}{2}^+$	3.568	3.977
$\frac{3}{2}$	$\frac{1}{2}$	1.562	$\frac{1}{2}^-$	3.590	4.162
			$\frac{1}{2}^+$	3.699	4.094
			$\frac{3}{2}^-$	3.593	4.163
			$\frac{3}{2}^+$	3.701	4.094
	$\frac{3}{2}$	2.228	$\frac{1}{2}^-$	4.202	4.748
			$\frac{1}{2}^+$	4.304	4.684
			$\frac{3}{2}^-$	4.204	4.749
			$\frac{3}{2}^+$	4.306	4.685

excitation bringing the same physical quantum numbers $I(J^P)$. This suggests that the physical hybrid baryon states could be in a quantum superposition of all these states with the same quantum numbers. Accessing to the associated mixing coefficients would require a deeper study. Finally, we have remarked that the hyperfine term in the core-gluon potential does not contribute significantly to the energy of the hybrid baryons.

In order to perform a deeper analysis of the results, let us compare them with those obtained using other theoretical methods, such as LQCD or QCD sum rules. From the lattice side, Ref. [3] focused on the positive parity hybrid spectrum for the N and the Δ (see Fig. 1 of [3]), since negative parity states turned out significantly higher in mass. For the N , hybrid states were found to have

a mass between 2.5 GeV and 3 GeV for the states with $J^P = 1/2^+, 3/2^+$ and $5/2^+$, which is about 0.9 GeV lower than our results. For the Δ , hybrid states were found to have a mass between 2.8 GeV and 3 GeV for the states with $J^P = 1/2^+$ and $J^P = 3/2^+$, which is about 0.8 GeV lower than our results. We can thus notice that, though there is a similar energy gap between states with different quantum numbers, the overall mass range of the spectrum obtained in LQCD is not in agreement with the one obtained in this study.

From the sum rule side now, both positive and negative parity hybrid spectra are determined for N and Δ (see Fig. 3 of [2]). Figure 3 shows the comparison between these spectra and the spectrum from Table VII obtained in this study. We see that for the $\Delta_{J^P} = \Delta_{1/2^+}$ and $\Delta_{3/2^+}$, which

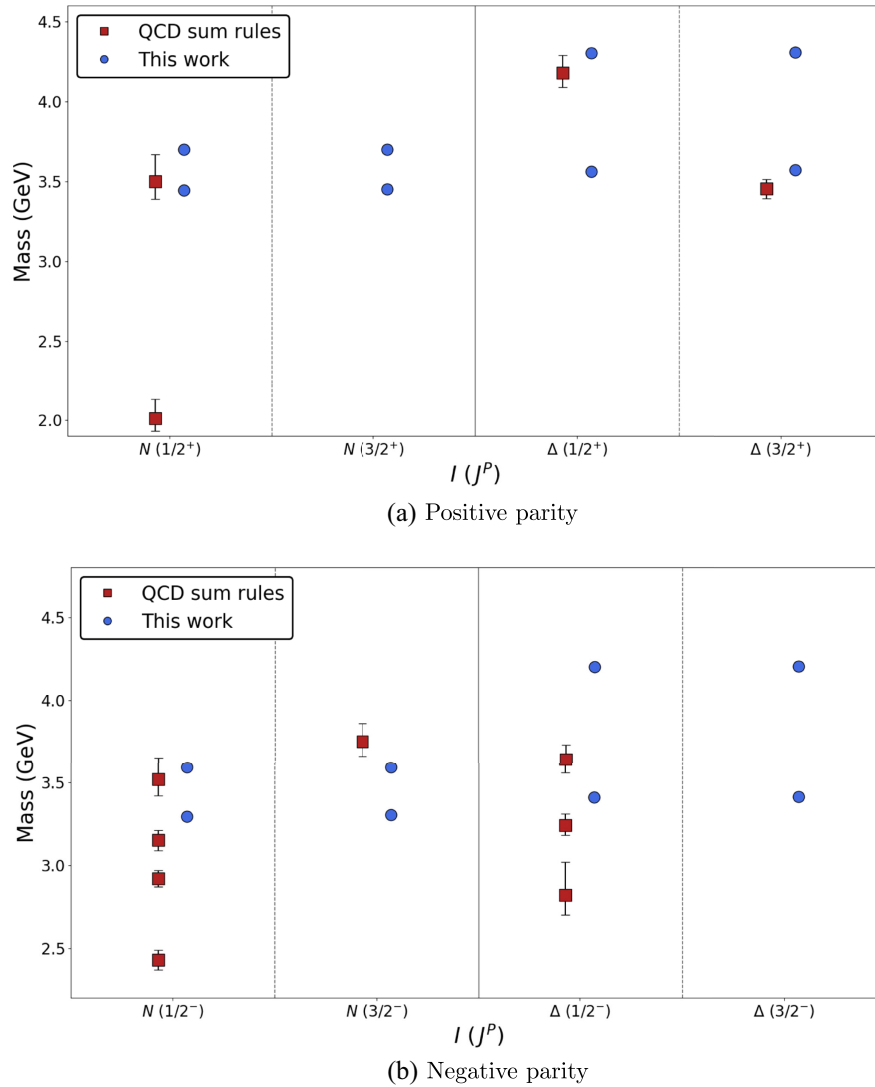
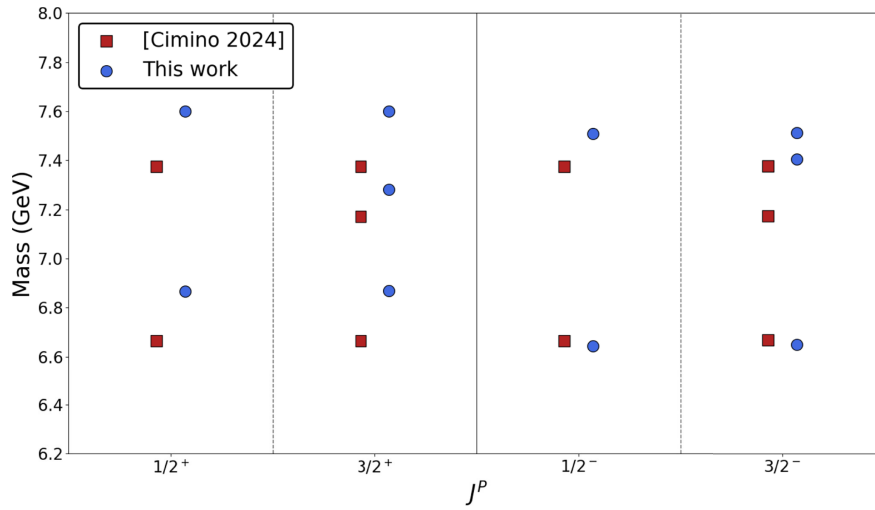


FIG. 3. Mass spectrum of light hybrid baryons. The blue circles correspond to predictions of the core-gluon model while the red squares correspond to those of the QCD sum rule method, taken from [2]. The different values of energy for same quantum numbers come from distinct reasons in the two models. For the core-gluon model, it comes from different internal structures, while for the QCD sum rule method, it comes from the different interpolating currents.

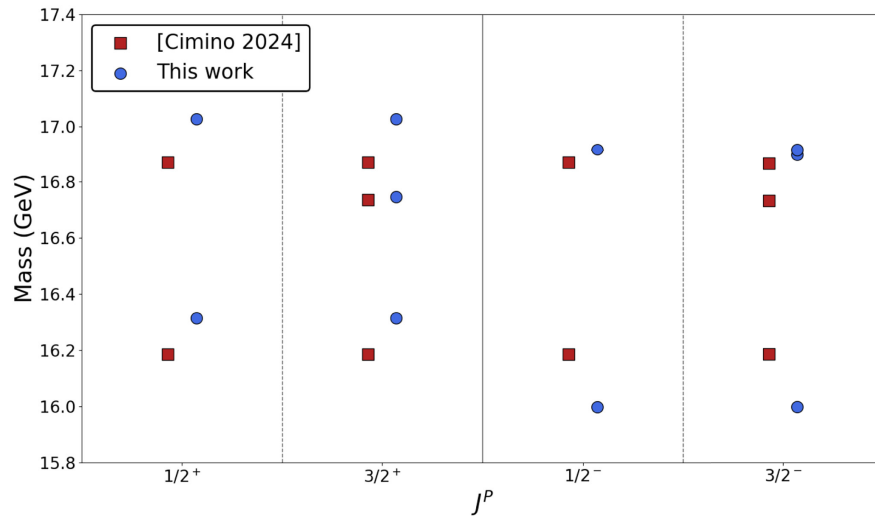
turn out to be the channels with the most stable QCD sum rules, the masses given by both methods differ by less than 100 MeV. However, for the rest of the channels, the agreement is less obvious. Indeed, the energy range spanned by the different currents is wider for the channels $N_{1/2^+}, N_{1/2^-}$ and $\Delta_{1/2^-}$, which renders the comparison difficult. What we can say is that the energy ranges predicted by our method at least overlap those predicted by the QCD sum rule method. Besides, it is important to note that, contrary to our results, those of Ref. [2] predict the lightest hybrid baryon to have a positive parity, which was also the case with the lattice results. It turns out that this disagreement was also pointed out in the study of glueballs within the quark model [31]. In fact, it is not

TABLE VIII. Same as in Table VII but for the mass spectrum of heavy hybrid baryons. For all states, $J_C = 1/2$ and $I_C = 0$.

	m_C (Gev)	J^P	M_0 (GeV)	M_1 (GeV)
$cccg$	5.369	$\frac{1}{2}^-$	6.643	7.508
		$\frac{1}{2}^+$	6.867	7.600
		$\frac{3}{2}^-$	6.646	7.402
		$\frac{3}{2}^+$	6.869	7.282
$bbbg$	15.063	$\frac{1}{2}^-$	15.998	16.918
		$\frac{1}{2}^+$	16.315	17.028
		$\frac{3}{2}^-$	15.999	16.900
		$\frac{3}{2}^+$	16.316	16.747



(a) $cccg$ mass spectrum



(b) $bbbg$ mass spectrum

FIG. 4. Mass spectrum of heavy hybrid baryons. The blue circles correspond to the two or three lowest eigenenergies computed with our model. The red squares correspond to the same eigenenergies computed with the method of Cimino *et al.* [11].

unreasonable since, as mentioned in Sec. IV, the constituent approach of gluelumps is very similar to the core-gluon model for hybrid baryons. Indeed, in Ref. [31], the author considers the gluelump as a system composed of a single transverse constituent gluon interacting with a static color-octet source via a QCD-based potential like (7). This is actually the limit of the core-gluon model for a quark core with infinite mass and 0^+ quantum numbers. It is not the case for our hybrid baryons but the core having a mass more than three times heavier than the constituent gluon, the system can be seen as approaching the one for gluelumps. In Ref. [31], the author suggested a way to overcome this disagreement by introducing a phenomenological parity splitting mass term in the potential which is justified as arising from instanton-induced interactions. This has not been done in the present work, but it could be interesting to keep in mind for future improvement of our model. Another way to overcome this parity disagreement could be to invert the intrinsic parity of the constituent gluon. Although we did not find any theoretical mechanism justifying the use of a positive parity gluon in a constituent model, other studies use a gluon field with $J^P = 1^+$ [48,49]. This modification would exchange the positive- and negative-parity spectra. It would lead to a better agreement with the LQCD and QCD sum rules results, but, given that the hybrid baryon spectrum presented in Table VII is almost parity degenerated, our results and conclusions would not fundamentally change in practice.

A. Extension of the model to heavy hybrid baryons

Although the model presented in this work is built for light hybrid baryons, it could be interesting to extend it to the study of heavy ones. The quark core-gluon model being first constructed in Ref. [11] for heavy hybrid baryons, and improved and extended in this paper for the light ones, it could be interesting to compare the mass spectrum of the heavy sector calculated with both works. The mass spectra computed for the hybrid baryons $cccg$ and $bbbg$ with the method developed in this paper are provided in Table VIII for the quark core having quantum numbers $J_C = S_C = 1/2$ and $L_C = 0$. Note that the masses of the quark core were obtained via the procedure of Sec. III, with the mass parameters of the heavy quarks c and b taken as $m_c = 1.870$ GeV and $m_b = 5.259$ GeV [50]. The comparison with the results of Ref. [11] is depicted in Fig. 4.

The results show qualitative agreement, even though the parameters of the present model are fitted for the light sector. The differences probably stem from a better treatment of the color structure of the quark core and of the computation of the convoluted potential in this study. This means that our model seems to give coherent results for all the baryonic sectors.

VII. CONCLUSION AND OUTLOOK

In this work, we have employed a phenomenological potential model, referred to as the *quark core-gluon model* and originally introduced in Ref. [11], with the aim of predicting the mass spectrum of light hybrid baryons composed of three identical quarks. Within this framework, a hybrid baryon is described as an interacting four-body system subject to the following constraints: the three quarks interact among themselves inside a quark core through QCD-inspired two-body interactions, while the remaining constituent gluon interacts with this core, forming an effective two-body system.

This description of the internal dynamics of hybrid baryons, reminiscent of the quark-diquark approximation commonly used for conventional baryons [12], allows the otherwise demanding four-body quantum problem to be reduced to a three-body problem followed by a two-body one. This reduction considerably simplifies the theoretical treatment and, in particular, enables a consistent and technically tractable implementation of the gluon helicity degrees of freedom. To account for the limitations inherent to this approximation, the interaction between the color-octet quark core and the constituent gluon has been constructed so as to incorporate finite-size effects of the core.

Given the phenomenological nature of the model, its parameters have been fixed by fitting to the experimental masses of light baryons and LQCD results for glueballs. In addition, a hyperfine contribution has been included in the interaction potentials, as such effects are known to be non-negligible for systems composed of light constituents. The results obtained within this approach have been compared with LQCD predictions [3] as well as with calculations based on QCD sum rules [2]. Our model systematically predicts the appearance of hybrid baryon states at higher energies than those found in lattice studies and sum-rule analyses, while showing overall better agreement with the latter.

Owing to certain similarities between the present framework and gluelump-like systems, it is conceivable that an improved agreement with LQCD results could be achieved by supplementing the effective two-body potential with a parity-dependent mass-splitting term, as proposed in Ref. [31]. In any case, despite quantitative differences with other existing approaches, the model presented in this work supports the general conclusion that the lightest hybrid baryonic resonances are expected to occur at energies above 2 GeV. A notable advantage of our method is that it yields a complete mass spectrum across all accessible channels.

Having established a realistic constituent description of hybrid baryons, the future directions are multiple:

- (i) Further improve the model by incorporating, for instance, polarization effects of the quark core induced by the constituent gluon, in a spirit similar

to the polarization of the electron core of atoms by its peripheral electrons [51].

- (ii) Further develop the quark core-gluon model by emphasizing its similarities with gluelump-like systems rather than with two-gluon glueball configurations.
- (iii) Investigate the decay properties of the hybrid baryons, since such observables are of primary importance for experimental efforts aimed at identifying these exotic states.
- (iv) Extend the present model to baryonic systems involving quarks of different flavors.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: BUILDING THE BASIS QUARK CORE STATES WITH GOOD SYMMETRY

The wave functions for the quark core can be decomposed in the following way:

$$|\psi_C\rangle = |\rho_L\rangle \otimes |\chi_S\rangle \otimes |\xi_F\rangle \otimes |\phi_c\rangle. \quad (\text{A1})$$

The states $|\rho_L\rangle, |\chi_S\rangle, |\xi_F\rangle, |\phi_c\rangle$ represent the spatial, spin, flavor and color wave functions, respectively. Note that, since we are dealing with three identical fermions, (A1) has to be antisymmetric under exchange of two quarks in the core due to the Pauli principle. However, it is not the case with this tensor product decomposition. Therefore, we need to find proper combinations of states that behave well under exchange symmetry. Let us start with the spatial part. In this study, the core spectrum and wave functions are computed via the OBE method [29]. In this framework, the state $|\rho_L\rangle$ is decomposed in a linear combination of the harmonic oscillator (HO) basis states in the following way:

$$\rho_{LM}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} | \rho_{LM} \rangle = \sum_{n_x l_x n_y l_y} C_{n_x l_x n_y l_y} \varphi_{n_x l_x n_y l_y}^{LM}(\mathbf{x}, \mathbf{y}), \quad (\text{A2})$$

where $\mathbf{x}$ and $\mathbf{y}$ are internal variables defined in [29] as dimensionless Jacobi coordinates,

$$a\mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2, \quad \frac{\sqrt{3}a}{2}\mathbf{y} = \frac{m_1\mathbf{r}_1 + m_1\mathbf{r}_2}{m_1 + m_2} - \mathbf{r}_3, \quad (\text{A3})$$

with a a variational parameter, and $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}_3$ the positions of the three quarks in an inertial reference frame. In (A2), the functions $\varphi_{n_x l_x n_y l_y}^{LM}(\mathbf{x}, \mathbf{y})$ are the two-body HO functions, given by

$$\begin{aligned} \varphi_{n_x l_x n_y l_y}^{LM}(\mathbf{x}, \mathbf{y}) &= [\varphi_{n_x l_x}(\mathbf{x}) \varphi_{n_y l_y}(\mathbf{y})]_{LM} \\ &= \sum_{m_x, m_y} (l_x m_x l_y m_y | LM) \varphi_{n_x l_x m_x}(\mathbf{x}) \varphi_{n_y l_y m_y}(\mathbf{y}), \end{aligned} \quad (\text{A4})$$

where

$$\varphi_{nlm}(x, \theta_x, \phi_x) = \mathcal{R}_{nl}(x) Y_{lm}(\theta_x, \phi_x) \quad (\text{A5})$$

is the solution of the three dimensional HO in spherical coordinates [29]. The constants $C_{n_x l_x n_y l_y}$ are the expansion coefficients that have to be found. The OBE method is described in detail in Ref. [29], but the only important property that we need to keep in mind here is the behavior of the HO basis states under exchange symmetry. For particles 1 and 2, it is given by

$$\mathbb{P}_{12} \varphi_{n_x l_x n_y l_y}^{LM}(\mathbf{x}, \mathbf{y}) = (-1)^{l_x} \varphi_{n_x l_x n_y l_y}^{LM}(\mathbf{x}, \mathbf{y}), \quad (\text{A6})$$

where the operator $\mathbb{P}_{12}$ encodes the permutation of particles 1 and 2. For particles 2 and 3, it is much more subtle because the action of the corresponding exchange operator, $\mathbb{P}_{23}$, on the Jacobi coordinates mixes them together. This has for consequences the mixing of the HO basis states with same quantum number L under the action of $\mathbb{P}_{23}$. The mixing coefficients can be found through the following relation [29]

$$\begin{aligned} \langle \varphi_{n'_x l'_x n'_y l'_y}^{L'M}(\mathbf{x}, \mathbf{y}) | \mathbb{P}_{23} | \varphi_{n_x l_x n_y l_y}^{LM}(\mathbf{x}, \mathbf{y}) \rangle \\ = \delta_{L'L} (-1)^{l'_x + l'_y + L} \langle n'_y l'_y n'_x l'_x; L' | n_x l_x n_y l_y; L \rangle_{\pi/6}, \end{aligned} \quad (\text{A7})$$

where $\langle n'_y l'_y n'_x l'_x; L' | n_x l_x n_y l_y; L \rangle_{\pi/6}$ is a Brody-Moshinsky coefficient [52].

Let us then turn to the study of the color wave function under exchange symmetry. If we were dealing with conventional baryons, $|\phi_c\rangle$ would have been in the color singlet representation, which is completely antisymmetric. But in this work, we are dealing with color octet quark cores. As discussed in Sec. II, there are two distinct irreducible mixed symmetric eight-dimensional subspaces [see (2)]. The first one is mixed antisymmetric, which means that its states are antisymmetric under exchange of the first two particles. On the contrary, the second one is mixed symmetric, which means that its states are symmetric under the exchange of the first two particles. As for the spatial part, the symmetry

under permutation of particles 2 and 3 is less obvious. The operator $\mathbb{P}_{23}$ actually mixes the states of the two subspaces together. The mixing coefficients are obtained by expanding the states with a fixed mixed symmetry on particles 1 and 2 in terms of those with mixed symmetry on particles 2 and 3. This leads to the following mixing matrix

$$\begin{pmatrix} \langle \phi_c^{MS} | \mathbb{P}_{23} | \phi_c^{MS} \rangle & \langle \phi_c^{MS} | \mathbb{P}_{23} | \phi_c^{MA} \rangle \\ \langle \phi_c^{MA} | \mathbb{P}_{23} | \phi_c^{MS} \rangle & \langle \phi_c^{MA} | \mathbb{P}_{23} | \phi_c^{MA} \rangle \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}. \quad (\text{A8})$$

This will have to be taken into account while building the total three-body wave function.

We now turn to the study of the spin and isospin wave functions under exchange symmetry. Since we are considering here an $SU_f(2)$ isospin symmetry for the light quarks u and d only, the treatment of these two wave functions is exactly the same. The irreducible decomposition of the tensor product of three fundamental representations of $SU(2)$ leads to [19]

$$2 \otimes 2 \otimes 2 = 4_S \oplus 2_{MA} \oplus 2_{MS}. \quad (\text{A9})$$

We see that there is, again, two distinct mixed symmetric subspaces of dimension 2. The operator $\mathbb{P}_{23}$ then mixes the states of these subspaces together, with mixing coefficients given by [43,53]

$$\begin{aligned} \mathbb{P}_{12}(|\varphi_{n_x l_x n_y l_y}^{LM}(\mathbf{x}, \mathbf{y})\rangle \otimes |\chi_S\rangle \otimes |\xi_F\rangle \otimes |\phi_c\rangle) &= -|\varphi_{n_x l_x n_y l_y}^{LM}(\mathbf{x}, \mathbf{y})\rangle \otimes |\chi_S\rangle \otimes |\xi_F\rangle \otimes |\phi_c\rangle, \\ L &= l_x + l_y = 0, \\ 2n_x + l_x + 2n_y + l_y &= 0. \end{aligned} \quad (\text{A11})$$

The two last constraints force all the HO quantum numbers to be zero, which implies the spatial part of the basis states to be symmetric through (A6) and (A7). Table IX provides all the possible basis states that we can construct so that they satisfy the first constraint of (A11). Note that, for convenience, these states are arbitrary denoted $|i\rangle$.

TABLE IX. HO basis states satisfying the symmetry constraints (A11). For convenience, states are arbitrary denoted $|i\rangle$.

State	Spin	Isospin	Color
$ 1\rangle$	MA	MA	MA
$ 2\rangle$	MA	MS	MS
$ 3\rangle$	MS	MA	MS
$ 4\rangle$	MS	MS	MA

$$\begin{aligned} \langle \chi_S^i | \mathbb{P}_{23} | \chi_S^j \rangle &= (-1)^{2s_3 + s_{12i} + s_{12j}} \sqrt{2s_{12i} + 1} \sqrt{2s_{12j} + 1} \\ &\times \begin{Bmatrix} s_1 & s_2 & s_{12i} \\ s_3 & S_C & s_{12j} \end{Bmatrix}, \end{aligned} \quad (\text{A10})$$

where s_k is the spin quantum number of the k th quark, i, j span the mixed symmetric subspaces, s_{12i} is the spin quantum number of the coupling spin states of particles 1 and 2, and S_C is the total spin of the core.

Putting all these information together, the total quark core wave function can be obtained in a two-step procedure:

- (i) Select only the basis states with good symmetry under $\mathbb{P}_{12}$ for the spatial, spin, isospin and color parts.
- (ii) Combine the latter so that they have the same good symmetry under $\mathbb{P}_{23}$ now for each of the spatial, spin, isospin and color parts again.

There is, in principle, an infinite number of basis states, which forces us to truncate the expansion. In the OBE method presented in Ref. [29], for a given total orbital quantum number L , the truncation is performed by selecting only the basis states whose sum $2n_x + l_x + 2n_y + l_y$ is smaller or equal to a fixed even number, noted $Q_{\max}$.

Let us give an example in order to render this construction clearer. Let us consider $L = 0, S = 1/2, I = 1/2$ and $Q_{\max} = 0$. The first step is to select the basis states $|\varphi_{n_x l_x n_y l_y}^{LM}(\mathbf{x}, \mathbf{y})\rangle \otimes |\chi_S\rangle \otimes |\xi_F\rangle \otimes |\phi_c\rangle$ which satisfy

Any linear combination of the states given in Table IX, denoted $|\psi_i\rangle$, will also satisfy the constraints (A11). The second step is to construct all the combinations $|\psi_i\rangle$ that are orthogonal and that satisfy the following constraint

$$\mathbb{P}_{23}|\psi_i\rangle = -|\psi_i\rangle. \quad (\text{A12})$$

In practice, we compute the matrix elements $\langle j | \mathbb{P}_{23} | k \rangle$, for $i, k \in \{1, 2, 3, 4\}$, diagonalize the resulting matrix, and select only the eigenstates with eigenvalue -1 . In this example, there is only one eigenstate, given by

$$|\psi_1\rangle = \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle - \frac{1}{2}|3\rangle - \frac{1}{2}|4\rangle. \quad (\text{A13})$$

Therefore, for the truncation $Q_{\max} = 0$, with $L = 0, S = 1/2$ and $I = 1/2$, there is only one basis state that

we can construct with good symmetry properties. This will of course lead to very approximate results. Precise results arise in general for $Q_{\max} = 8$.

APPENDIX B: COMPUTATION OF THE HAMILTONIAN MATRIX ELEMENTS FOR LIGHT HYBRID BARYONS

In this Section, the Hamiltonian matrix elements of the eigenvalue problem (27) are computed. To this end, we have to determine the action of the Hamiltonian operator (28) on the hybrid baryon states (26). This operator containing both angular and spin operators, the helicity trial states did not prove to form a suitable basis for the computation. Instead, Ref. [14] uses a so-called *canonical basis* also composed with states with definite total angular momentum J, M

$$|p, JM, ls; J_C 1\rangle = \sum_{m_l, m_s, M_C, \lambda_g} (l m_l s m_s | JM) (J_C M_C 1 \lambda_g | s m_s) \times \int d\cos\theta d\phi Y_{m_l}^l(\theta, \phi) |p\theta\phi; J_C M_C 1 \lambda_g\rangle, \quad (\text{B1})$$

where the states $|p\theta\phi, J_C M_C 1 \lambda_g\rangle$ are defined as eigenstates of the relative momentum, core total spin, its projection, the gluon total spin and its projection operators. The associated eigenvalues are given by the quantum numbers $p, \theta, \phi, J_C, M_C, 1$ and λ_g . The canonical basis defined in (B1) is built so that its states are eigenstates of the operators $\mathbf{L}^2$ and $\mathbf{S}^2$ with associated quantum numbers given by l and s , respectively. It is possible to relate the two bases, through the following relation [11,14]

$$|p, JM, \lambda_C \lambda_g; J_C\rangle = \sum_{s=|J_C-1|}^{J_C+1} \sum_{l=|J-s|}^{J+s} \sqrt{\frac{2l+1}{2J+1}} (J_C \lambda_C 1 - \lambda_g | s \lambda_C - \lambda_g) (l 0 s \lambda_C - \lambda_g | J \lambda_C - \lambda_g) |p, JM, ls; J_C 1\rangle. \quad (\text{B2})$$

Then, by plugging (B2) in (26), the evaluation of the matrix elements in (27) reduces to a coupled channel analysis with fixed coefficients,

$$\begin{aligned} \langle f_j, J^{P_i}, \lambda_C \lambda_g; J_C | \mathcal{H}_{HB} | f_k, J^{P_i}, \lambda'_C \lambda'_g; J_C \rangle &= \sum_{s=|J_C-1|}^{J_C+1} \sum_{l=|J-s|}^{J+s} \sum_{s'=|J_C-1|}^{J_C+1} \sum_{l'=|J-s'|}^{J+s'} C_{ls\lambda_C\lambda_g} C_{l's'\lambda'_C\lambda'_g} \\ &\times \int \frac{p^2 dp}{2\sqrt{w_C(p)w_g(p)}} \frac{f_j(p/h)}{p\sqrt{h}} \\ &\times \int \frac{p'^2 dp'}{2\sqrt{w_C(p')w_g(p')}} \frac{f_j(p'/h)}{p'\sqrt{h}} \langle p, JM, ls; J_C \lambda_g | \mathcal{H}_{HB} | p', JM, l's'; J_C \lambda'_g \rangle, \end{aligned} \quad (\text{B3})$$

where

$$C_{ls\lambda_C\lambda_g} = \sqrt{\frac{2l+1}{2J+1}} (J_C \lambda_C 1 - \lambda_g | s \lambda_C - \lambda_g) (l 0 s \lambda_C - \lambda_g | J \lambda_C - \lambda_g). \quad (\text{B4})$$

The computation of the second line of the equation can be performed using a procedure analogous to [46], and the generalization to coupled channel is described in [54].

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