

Energy correlators in semi-inclusive electron-positron annihilation

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We investigate energy correlators in semi-inclusive electron-positron annihilation as precision probes of parton hadronization dynamics. Using soft-collinear effective theory, we analyze the correlation patterns between the examined hadron and the rest of QCD radiations in both large-angle and small-angle limits, which establishes a direct correspondence with transverse-momentum-dependent fragmentation functions and fragmentation energy correlators. The two complementary regimes encode the transition from perturbative parton branching to nonperturbative confinement, enabling a unified description of hadron formation across all kinematic regimes. Using renormalization group evolution, we obtain joint $N^3\text{LL}/\text{NNLL}$ quantitative predictions for energy correlations both in the sudakov and jet fragmentation region. Our results demonstrate that semi-inclusive energy correlators provide direct, theoretically controlled access to QCD dynamics underlying hadronization, opening new avenues for precision studies at future lepton colliders as well as through reanalyses of archival LEP data.

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I. INTRODUCTION

In high-energy collisions, quarks and gluons produced at short distances cannot be directly observed due to color confinement. Instead, they undergo a nonperturbative transition process known as hadronization, in which colored partons transform into color-neutral hadrons. This transition, governed by the long-distance dynamics of QCD, bridges the calculable partonic cross sections and the experimentally accessible hadronic final states. Among various processes, the single-inclusive hadron production in electron-positron annihilation (SIA), $\ell + \ell' \rightarrow h + X$, offers the cleanest probe of parton-to-hadron transition. The nonperturbative aspects of this transition are encoded in fragmentation functions, which describe the probability for a parton to produce a specific hadron carrying a given fraction of its momentum, and enter the cross section through a convolution with perturbatively calculable coefficient functions [1–7], which are known up to next-to-next-to-next-to-leading order ($N^3\text{LO}$) in QCD [8–10], including threshold resummation effects [11].

In addition to the conventional SIA cross section, the energy correlators (ECs) have emerged as a complementary probe of hadronization, mapping partonic correlations in

the microscopic energy flux onto the observed macroscopic correlations among hadrons, and introducing new non-perturbative correlation functions [12–16].

The energy-energy correlators were first introduced in Refs. [17–20] as tests of strong-interaction dynamics in the early development of QCD. They can be expressed as correlation functions of energy-flow operators,

$$\langle \mathcal{E}(\hat{n}_1) \mathcal{E}(\hat{n}_2) \rangle = \int d^4x e^{iq \cdot x} \langle 0 | J(x) \mathcal{E}(\hat{n}_1) \mathcal{E}(\hat{n}_2) J(0) | 0 \rangle, \quad (1)$$

where J denotes the source current, e.g., the electromagnetic current $J^\mu = \bar{\psi} \gamma^\mu \psi$. ECs have since been generalized to diverse collision systems, enabling the study of transverse-momentum-dependent dynamics via back-to-back jets [21–37] and jet substructure via the collinear limit [38–56]. They have been further applied to probe quarkonium production [57], the nucleon spin structures [58,59], and small- x dynamics [60–64]. Moreover, the rigorous operator definition of the energy correlators offers a clear path toward a first-principle understanding of underlying nonperturbative dynamics. For instance, universality in nonperturbative corrections has been investigated both theoretically and through global fits [65–74], and recent measurements across e^+e^- and hadronic collisions [75–78] have opened new opportunities to test these predictions. See Ref. [79] for a comprehensive review.

Motivated by the need to access hadronization dynamics, we introduce the semi-inclusive energy correlator as a variant of the inclusive energy-energy correlator (EEC) in Eq. (1),

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$$\langle \mathcal{E}(\hat{n}) \rangle_h = \int d^4x e^{iq \cdot x} \langle 0 | J(x) \mathcal{E}(\hat{n}) a_h^\dagger a_h J(0) | 0 \rangle, \quad (2)$$

where h denotes an identified hadron created by $a_h^\dagger$, and $\hat{n}$ is the generic radial unit vector where energy weight is applied. We study its dependence on the relative polar angle χ between $\hat{n}$ and $\hat{n}_h$ both in the small- and wide-angle limits, where effective field theory is applied to reveal the corresponding factorization structures, and obtain precision predictions via renormalization group (RG) evolution.

II. ECS IN SINGLE INCLUSIVE HADRON PRODUCTION

In this paper, we consider polar angle χ distribution for single inclusive hadron production in electron-positron annihilation,

$$\ell(p_\ell) + \ell'(p_{\ell'}) \rightarrow \gamma^*(q) \rightarrow \pi(P_\pi) + X(P_X), \quad (3)$$

where the identified hadron is taken as a pion. We work in the overall center-of-mass frame, in which the virtual photon is at rest $q = \frac{Q}{2}(\bar{n} + n) = Q(1, 0, 0, 0)$. We will parametrize the events by angular variable χ , defined to be the polar angle between the detected particles (jets) with respect to pion velocity.

We are interested in energy-weighted angular χ correlations,

$$E_\pi \frac{d\sigma^{\text{EC}}}{d^3\vec{P}_\pi} = \frac{2\alpha_e^2}{Q^6} L_{\mu\nu} W_{\text{EC}}^{\mu\nu}. \quad (4)$$

In this case, the conventional hadronic tensor [1] is replaced by a correlator in which the energy-flow operator is inserted between the electromagnetic currents,

$$\begin{aligned} W_{\text{EC}}^{\mu\nu}(q, P_\pi) &= \frac{1}{4\pi} \sum_X \int d^4z e^{iq \cdot z} \\ &\times \langle 0 | j^\mu(z/2) \hat{\mathcal{E}}(\chi) | P_\pi, X \rangle \langle P_\pi, X | j^\nu(-z/2) | 0 \rangle, \end{aligned} \quad (5)$$

where the cumulant EC operator quantifies the energy deposited in the detector for radiation within a radial angle bounded above by χ ,

$$\hat{\mathcal{E}}(\chi) | X \rangle = \sum_{h \in X} \frac{E_h}{E_\pi} \Theta(\chi - \chi_{h-\pi}) | X \rangle. \quad (6)$$

The spin-averaged lepton current $L^{\mu\nu}$ for the decay of a virtual photon is given by

$$L^{\mu\nu} = p_\ell^\mu p_{\ell'}^\nu + p_\ell^\nu p_{\ell'}^\mu - \frac{Q^2}{2} g^{\mu\nu}. \quad (7)$$

Using SO(3) global rotation symmetry, we integrate over all polar directions, this leaves behind a single kinematic invariant

$$x = \frac{2P_\pi \cdot q}{Q^2}, \quad (8)$$

which is related to the deep-inelastic scattering Bjorken x_B by inversion $x = 1/x_B$, x is interpreted as the center-of-mass energy of the detected pion relative to its maximum value $Q/2$. The energy-weighted cross section receives combined contribution from longitudinal and transverse polarization modes,

$$d\sigma^{\text{EC}}(x, Q, \chi) = d\sigma_L^{\text{EC}}(x, Q, \chi) + d\sigma_T^{\text{EC}}(x, Q, \chi). \quad (9)$$

We will also investigate Bjorken x weighted ECs, defined as follows:

$$\begin{aligned} \sigma_\pi^{\text{EC}}(N, Q, \chi) &= \sum_h \int x^{N-1} d\sigma(x, Q, P_h) \frac{E_h}{E_\pi} \Theta(\chi - \chi_{h-\pi}) \\ &= \int x^{N-1} d\sigma_\pi^{\text{EC}}(x, Q, \chi). \end{aligned} \quad (10)$$

III. BACK-TO-BACK LIMIT AND THE TMD FRAGMENTATION FUNCTIONS

In this section, we are concerned with ECs in the back-to-back limit $\chi \rightarrow \pi$, see Fig. 1. The angular χ correlation is closely related to the conventional q_T spectrum, the latter measures the transverse momentum of the detected hadrons in the photon-pion frame, or the photon transverse momentum in hadron-pion frame. The explicit relation is

$$\sin \chi = \frac{q_T}{Q/2}. \quad (11)$$

This kinematic region is understood as the crossed counterpart of current-fragmentation region (CFR) in semi-inclusive deep-inelastic scattering (SIDIS), and the appropriate theoretical framework for describing it is transverse momentum dependent (TMD) factorization. To derive the factorization formulas, it is useful to identify a hadron inside the leading jet in the first place. To this end, we consider differential cross

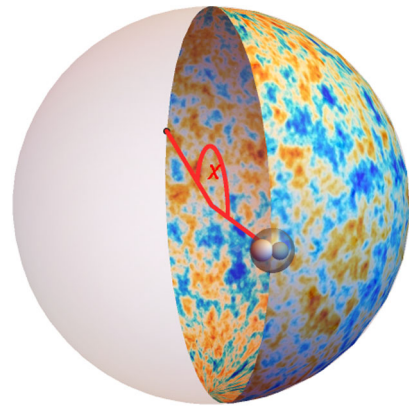


FIG. 1. ECs back-to-back limit.

section with two identified hadrons,

$$d\sigma_{\ell+\ell'\rightarrow h_A+h_B+X} = \frac{1}{2Q^2} \frac{(4\pi\alpha_e)^2}{Q^4} N_c \prod_{i=A,B} \frac{d^3P_i}{(2\pi)^3 2E_i} \times L_{\mu\nu}(P_\ell, P_{\ell'}) W^{\mu\nu}(q, P_A, P_B). \quad (12)$$

The hadronic tensor with two resolved hadrons is

$$W^{\mu\nu}(q, P_A, P_B) = \prod_X \frac{d^3P_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^{(4)}(P_A + P_B + P_X - q) \times \langle 0 | j^\mu | P_A, P_B, X \rangle \langle P_A, P_B, X | j^\nu | 0 \rangle. \quad (13)$$

Kinematic constraints in the back-to-back require that all final-state radiation in X be collimated, with momenta collinear to either of the observed hadrons, and the hadronic amplitude is factorized through [80]

$$\langle h_A h_B; X_{AB}; X_s | J^\mu | 0 \rangle \simeq \mathcal{C}(Q^2, \mu) \langle X_s | T[Y_n^\dagger Y_n] | 0 \rangle \times \langle h_B; X_B | \bar{\chi}_{\bar{n}} | 0 \rangle \gamma^\mu \langle h_A; X_A | \chi_n | 0 \rangle, \quad (14)$$

where we absorb vacuum expectation values of zero-bin Wilson lines into the normalization of the collinear fields, this is equivalent to the zero-bin subtracted soft-collinear effective theory (SCET) Lagrangian [70,80]. The hard amplitude $\mathcal{C}(Q^2, \mu)$ is insensitive to the infrared (IR) dynamics of soft and collinear degrees of freedom, and thus can be computed at partonic level. It is advantageous to consider $\gamma^* \rightarrow q\bar{q}$, with no soft radiation present,

$$\langle q\bar{q} | J^\mu | 0 \rangle = \mathcal{C}(Q^2, \mu) \langle q | \bar{\chi}_{\bar{n}} | 0 \rangle \gamma^\mu \langle \bar{q} | \chi_n | 0 \rangle. \quad (15)$$

In this case, the leading-power collinear expansion is exact ($=$ instead of $\simeq$), as each collinear trajectory contains just one particle—taking the collinear limit simply does nothing. By construction, the IR divergences of the partonic amplitude $\langle q\bar{q} | J^\mu | 0 \rangle$ is reproduced by the UV divergences of the hard scattering operators in SCET [81–84]. Although a thorough treatment would also require Glauber operators to reproduce the $i\pi$ terms in the low-energy part of the amplitude $\gamma^* \rightarrow q\bar{q}$, the Cheshire nature of the Glauber exchange ensures that the hard matching coefficients remain unchanged without including the Glauber interactions [85]. Indeed, the right-hand side of Eq. (15) is

$$\langle q\bar{q} | \mathcal{O}_{\text{sct}}^\lambda | 0 \rangle = \langle q\bar{q} | \mathcal{O}_{\text{sct}}^\lambda | 0 \rangle_{\text{tree}} (1 + \delta_{Z_\lambda}(\epsilon_{\text{IR}})), \quad (16)$$

where the helicity basis is $\mathcal{O}_{\text{sct}}^\lambda = \bar{\chi}_n^\pm \gamma^\mu \chi_n^\pm$, $\lambda = \pm$. The result is proportional to tree-level 't Hooft-Veltman [86] (tHV) bases because pure virtual-loop corrections in SCET are scaleless and vanish identically. As a result, the IR poles of a helicity amplitude is reproduced by UV renormalization counterterm diagrams provided by $\delta Z_\lambda = Z_\lambda - 1$ [87].

Once the hard and soft-collinear degrees of freedom are disentangled, the next step is to enumerate all allowed hard events, and within each hard label, zoom in to the soft-collinear subprocesses and count over particles in a proper way: the classes of momenta entering into the soft-collinear phase space differ only by their typical size of energy or rapidity, as a result, the phase space should be integrated over with a rapidity cutoff ν , here we employ exponential cutoff scheme [88]. Additionally, a zero-bin subtraction is performed to remove double counting between collinear and soft sectors, yielding genuine TMD fragmentation functions [89]. Consequently, we have factorization formulas [1],

$$\begin{aligned} \frac{d\sigma_{\ell+\ell'\rightarrow h_A+h_B+X}}{dz_A dz_B d\cos\theta d\vec{q}_\perp} &\simeq \frac{N_c}{2} \frac{4\pi\alpha_e^2}{Q^4} L_{\mu\nu} \sum_f H_f(Q^2, \mu) \int \frac{d\vec{b}_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{q}_\perp} \mathcal{S}(b_\perp, \mu, \nu) \text{Tr}[\mathcal{F}_{h_A/f} \gamma^\mu \mathcal{F}_{h_B/f} \gamma^\nu] \\ &= \frac{4\pi\alpha_e^2}{Q^4} N_c \frac{1 + \cos^2\theta}{8} \sum_f H_f(Q^2, \mu) \int \frac{d\vec{b}_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{q}_\perp} \mathcal{S}(b_\perp, \mu, \nu) \mathcal{F}_{h_A/f} \left(z_A, \frac{b_\perp}{z_A}, E_n, \mu, \nu \right) \\ &\quad \times \mathcal{F}_{h_B/f} \left(z_B, \frac{b_\perp}{z_B}, E_{\bar{n}}, \mu, \nu \right), \end{aligned} \quad (17)$$

where $H_f(Q^2, \mu)$ is square of crossings of the SIDIS hard matching coefficient [90–92], z_i are momentum fractions, and E_n ($E_{\bar{n}}$) are the energies of fragmented partons. $\vec{q}_\perp$ is the transverse momentum of the virtual photon in hadron-hadron frame, or is a parametrization of the net transverse

momentum of the detected dijets $\vec{q}_\perp = -\sum_{i=A,B} \vec{P}_i^\perp / z_i$ as a result of soft recoil [1]. θ is the angle between the back-to-back jets and the lepton beam, which we choose to integrate over. Consequently,

$$\frac{d\sigma_{\ell+\ell' \rightarrow h_A+h_B+X}}{dz_A dz_B d\vec{q}_\perp} \simeq \frac{4\pi\alpha_e^2}{3Q^2} N_c \sum_q H_q(Q^2, \mu) \int \frac{d\vec{b}_\perp}{(2\pi)^2} D_1^{\bar{q}}\left(z_A, \frac{b_\perp}{z_A}, \xi^n, \mu\right) D_1^q\left(z_B, \frac{b_\perp}{z_B}, \xi^{\bar{n}}, \mu\right), \quad (18)$$

where the physical TMDs are obtained by dressing the genuine collinear functions with the cloud of soft gluons,

$$D_1^q\left(z, \frac{b_\perp}{z}, \xi, \mu\right) = \mathcal{F}_{h/q}\left(z, \frac{b_\perp}{z}, E, \mu, \nu\right) \sqrt{\mathcal{S}(b_\perp, \mu, \nu)}. \quad (19)$$

As a result, the rapidity divergences $\ln(4E_i^2/\nu^2)$ cancels between collinear and soft sectors, leaving behind physical rapidity logarithms $\ln(\xi^i/\mu_b^2)$ with $\mu_b = b_0/b_T$, $b_0 = 2e^{-\gamma_E}$. ξ^i are the Collins-Soper (CS) scales [1,93,94]. These rapidity scales are subject to the constraint $\xi^n \xi^{\bar{n}} = Q^4$. For dihadron production, a canonical choice is $\xi^n = \xi^{\bar{n}} = Q^2$. In this work, we are concerned with energy correlations between a single identified hadron and the jet in the opposite direction. By summing over nonordered tuples $\sum_{\{A,B\}} = \sum_B 1/2 \sum_A$, we recover a factorization formula given by the convolution of TMDs with EEC TMD jet function,

$$z \frac{d\sigma_{\ell+\ell' \rightarrow h+X}^{\text{EEC}}}{dz d\vec{q}_\perp} \simeq \frac{4\pi\alpha_e^2}{3Q^2} N_c \frac{1}{2} \sum_q H_q(Q^2, \mu) \int \frac{d\vec{b}_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{q}_\perp} J_{\bar{q}}(b_\perp, \xi^n, \mu) D_1^q\left(z, \frac{b_\perp}{z}, \xi^{\bar{n}}, \mu\right), \quad (20)$$

where we perform twist-2 matchings of the TMD fragmentation function,

$$D_1^q\left(z, \frac{b_\perp}{z}, \xi, \mu\right) = \sum_i d_{h/i} \otimes \mathcal{C}_{iq} + \mathcal{O}(b_T^2 \Lambda_{\text{QCD}}^2), \quad (21)$$

and use fragmentation function (FF) sum rules to obtain the EEC TMD jet function [23]

$$J_q(b_\perp, \xi, \mu) \equiv \sum_i \int_0^1 d\hat{z} \hat{z} \mathcal{C}_{iq}\left(\hat{z}, \frac{b_\perp}{\hat{z}}, \xi, \mu\right). \quad (22)$$

The factor z in the front of Eq. (20) arises due to $E_h/E_\pi = 1/x E_h/(Q/2)$, and $x = z$ in the back-to-back limit. Furthermore, the factorization formula for energy-weighted back-to-back jets reads,

$$\begin{aligned} \frac{d\sigma_{\ell+\ell' \rightarrow X}^{\text{EEC}}}{d\vec{q}_\perp} &\simeq \sum_{\{A,B\}} \int \prod_{i=A,B} z_i dz_i \frac{d\sigma_{\ell+\ell' \rightarrow h_A+h_B+X}}{dz_A dz_B d\vec{q}_\perp} = \sum_h \int z^2 dz \frac{d\sigma_{\ell+\ell' \rightarrow h+X}^{\text{EEC}}}{dz d\vec{q}_\perp} \\ &= \frac{4\pi\alpha_e^2}{3Q^2} N_c \frac{1}{2} \sum_q H_q(Q^2, \mu) \int \frac{d\vec{b}_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{q}_\perp} J_{\bar{q}}(b_\perp, \xi^n, \mu) J_q(b_\perp, \xi^{\bar{n}}, \mu). \end{aligned} \quad (23)$$

The above analysis is consistent with Ref. [23] and reproduces the fixed-order result of Ref. [95] in dijet limit. Nonperturbative corrections to back-to-back EEC is recently analyzed in Refs. [33,74].

The μ -evolution of the hard matrix and the TMDs is linear in $\ln \mu$ to all-loop orders, as can be proved by RG consistency,

$$\frac{d}{d \ln \mu} \ln D_1^q(z, b_\perp/z, \xi, \mu) = -\gamma_{\text{cusp}}^q(\alpha_s(\mu)) \ln \frac{\xi}{\mu^2} - \gamma_H^q. \quad (24)$$

The μ -evolution is solved by two conventional Sudakov integrals,

$$\begin{aligned} K_\Gamma(\mu_0, \mu) &= - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \frac{\Gamma(\alpha_s)}{\beta(\alpha_s)} \int_{\alpha_s(\mu_0)}^{\alpha_s} d\alpha'_s \frac{1}{\beta(\alpha'_s)}, \\ A_\gamma(\mu_0, \mu) &= - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \frac{\gamma(\alpha_s)}{\beta(\alpha_s)}. \end{aligned} \quad (25)$$

their perturbative expansions can be found, e.g., in Ref. [90]. The rapidity evolution of the TMDs is governed by the CS equation,

$$K(b_\perp, \mu) \equiv \frac{d}{d \ln \sqrt{\xi}} \ln D_1^q(z, b_\perp/z, \xi, \mu), \quad (26)$$

where $K(b_\perp, \mu)$ is referred to as the CS kernel. Again, by RG consistency, the μ -evolution of $K(b_\perp, \mu)$ is controlled

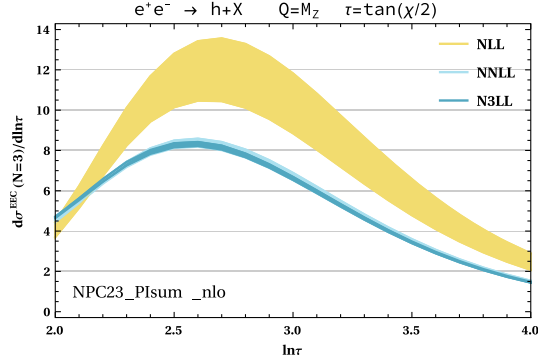


FIG. 2. Semi-inclusive ECs without b_* prescription [109] in the back-to-back limit, with LHAPDF6 [110] PDF dataset NPC23-PISum-nlo [111,112].

by the cusp anomalous dimension,

$$\frac{d}{d \ln \mu} K(b_\perp, \mu) = 2\gamma_{\text{cusp}}^q(\alpha_s(\mu)). \quad (27)$$

From above we can solve the RG equation to give

$$K(b_\perp, \mu) = -2A_{\text{cusp}}^q(\mu, \mu_b) + \gamma_R^q(\mu_b), \quad (28)$$

the solution has a evolution factor $A_{\text{cusp}}^q(\mu, \mu_b)$ controlled by the cusp anomalous dimension γ_{cusp}^q , as given by Eq. (25), it also leaves a residual boundary term $\gamma_R^q(\mu_b)$ known as QCD rapidity anomalous dimension. While the cusp anomalous dimension γ_{cusp}^q is a purely perturbative object which originates from the UV renormalization of the cusped Wilson loops [96,97], the rapidity anomalous dimension $\gamma_R^q(\mu_b)$, on the other hand, has both perturbative part as $b_T \rightarrow 0$ and genuine nonperturbative part as $b_T \rightarrow \infty$. The perturbative accuracy for the CS kernel has reached to up to N⁴LL [24,98,99], and there has been

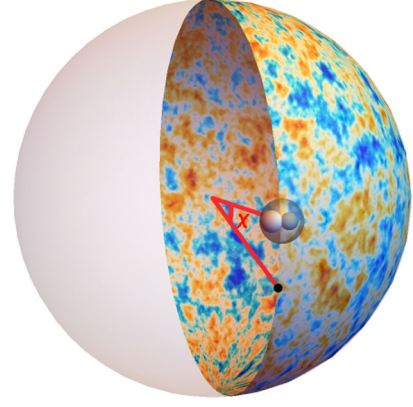


FIG. 3. ECs collinear limit.

tremendous progress towards lattice calculation for the nonperturbative part as well [100–106]. Equations (24), (26), and (28) are referred to as Collins-Soper equations [1,93,94], they are equivalent to the modern language of SCET rapidity renormalization groups (RRGs) [107,108]. Figure 2 shows the RG-improved prediction for the Mellin space ECs according to the definition of Eq. (10).

IV. COLLINEAR LIMIT AND THE FRAGMENTATION ENERGY CORRELATORS

In this section, we are concerned with collinear limit $\chi \rightarrow 0$ where correlation between the examined hadron and the surrounding radiations is measured, see Fig. 3. This kinematic region is understood as the crossed counterpart of target-fragmentation region (TFR) in SIDIS, and the appropriate theoretical framework for describing it is through fragmentation energy correlators (FECs) [13], whose operator definitions are given below in a gauge invariant manner,

$$\begin{aligned} [D_{h/q}^{\text{FEC}}]^{ij} \left(z, \ln \frac{E_h \sin \chi}{\mu}, \mu \right) &\equiv z \int \frac{dt^-}{2\pi} e^{it^- \frac{p_h^+}{z}} \langle 0 | W(\infty, t^-) \Psi_i(t^-) \hat{\mathcal{E}}(\chi) | h; X \rangle \langle h; X | \bar{\Psi}_j W(0, \infty) | 0 \rangle, \\ [D_{h/g}^{\text{FEC}}]^{\mu\nu} \left(z, \ln \frac{E_h \sin \chi}{\mu}, \mu \right) &\equiv \frac{z^2}{2P_h^+} \int \frac{dt^-}{2\pi} e^{it^- \frac{p_h^+}{z}} \langle 0 | \mathcal{W}(\infty, t^-) F^{+\mu}(t^-) \hat{\mathcal{E}}(\chi) | h; X \rangle \langle h; X | F^{+\nu} \mathcal{W}(0, \infty) | 0 \rangle. \end{aligned} \quad (29)$$

The FECs provide a joint description of fragmentation dynamics into the examined hadron and of its energy correlation pattern with the surrounding QCD jets. In addition, they encode the spin flow in the formation of the observed hadron, analogous to the conventional FFs [113]. In this work, we focus on

chiral-even spin projectors of helicity sums and asymmetries.

To leading power in the collinear limit, the EC structure function in Eq. (9) is dominated by twist-2 operators with energy flow operators inserted [12,114], the matching condition reads,

$$\frac{d\sigma^{\text{EC}}(x, Q, \chi)}{dx} \simeq \frac{1}{2} \frac{4\pi\alpha_e^2}{3Q^2} N_c \left\{ \sum_{i=1}^{N_f} Q_i^2 D_{h/i}^{\text{FEC,NS}}(\chi, \mu) \otimes \mathcal{C}_i^{\text{NS}}(Q, \mu) + \left(\sum_{i=1}^{N_f} Q_i^2 \right) \right. \\ \left. \times \left[\frac{1}{N_f} D_{h/q}^{\text{FEC}}(\chi, \mu) \otimes \mathcal{C}_q^{\text{S}}(Q, \mu) + D_{h/g}^{\text{FEC}}(\chi, \mu) \otimes 2\mathcal{C}_g^{\text{S}}(Q, \mu) \right] \right\}, \quad (30)$$

where the nonsinglet and singlet combinations of the FECs are defined as

$$D_{h/q}^{\text{FEC}}(z, \chi) = \sum_{i=1}^{N_f} D_{h/i}^{\text{FEC}}(z, \chi) + D_{h/\bar{i}}^{\text{FEC}}(z, \chi), \\ D_{h/i}^{\text{FEC,NS}}(z, \chi) = D_{h/i}^{\text{FEC}}(z, \chi) + D_{h/\bar{i}}^{\text{FEC}}(z, \chi) \\ - \frac{1}{N_f} D_{h/q}^{\text{FEC}}(z, \chi). \quad (31)$$

The associated energy flow operator $\hat{\mathcal{E}}(\chi)$ acts only on the collinear sector, thus, if $\hat{\mathcal{E}}(\chi)$ were set to the identity, i.e., in the absence of detectors in the jet fragmentation region (JFR), FECs reduce to standard FFs and the resulting factorization formula coincides with that of SIA, up to a factor of 1/2 factor accounting for final-state double counting. From this perspective, the Wilson coefficients are identical to the SIA coefficient functions [8–10], and the UV renormalization of FECs matches that of the conventional FFs, for instance,

$$\begin{bmatrix} D_{h/q}^{\text{FEC}}\left(\ln \frac{E_h \sin \chi}{\mu}, \mu\right) \\ D_{h/g}^{\text{FEC}}\left(\ln \frac{E_h \sin \chi}{\mu}, \mu\right) \end{bmatrix} = \begin{bmatrix} D_{h/q}^{\text{FEC0}}\left(\ln \frac{E_h \sin \chi}{\mu}, \mu\right) \\ D_{h/g}^{\text{FEC0}}\left(\ln \frac{E_h \sin \chi}{\mu}, \mu\right) \end{bmatrix} \otimes \hat{Z}_S^{\text{FF}}, \quad (32)$$

where the FECs are and must be parametrized by hadronic logarithm $\ln E_h \sin \chi / \mu$, E_h being the energy of the examined hadron h , in electron-positron annihilation $E_h = xQ/2$. The multiplicative renormalization factor $\hat{Z}_S^{\text{FF}}$ is the standard FF UV renormalization factor for the singlets, the associated Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [115–117] evolution kernel reads

$$\hat{P}_S(x, \alpha_s) = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix}. \quad (33)$$

Alternatively, one could consider the Bjorken- x weighted ECs in Mellin space, defined in Eq. (10). To this end, we first introduce the Mellin moment of FECs,

$$\mathcal{D}_{h/i}^{\text{FEC}}\left(N, \ln \frac{E_a \sin \chi}{\mu}\right) = \int_0^1 dz z^{N-1} \mathcal{D}_{h/i}^{\text{FEC}}\left(z, \ln \frac{E_a \sin \chi}{\mu}\right), \quad (34)$$

where the reparametrization is defined by

$$\mathcal{D}_{h/i}^{\text{FEC}}\left(z, \ln \frac{E_a \sin \chi}{\mu}, \mu\right) = D_{h/i}^{\text{FEC}}\left(z, \ln \frac{E_h \sin \chi}{\mu}, \mu\right), \quad (35)$$

with $E_a = E_h/z$ as z denotes the momentum fraction of the observed hadron. In electron-positron annihilation, the energy of the active parton is $E_a = \hat{x}Q/2$, where $\hat{x} = x/z$. The FECs, when parametrized in terms of a partonic logarithm rather than hadronic variables, obey a modified DGLAP evolution equation,

$$\frac{d\mathcal{D}_{h/i}^{\text{FEC}}}{d \ln \mu^2}\left(N, \ln \frac{E_a \sin \chi}{\mu}, \mu\right) \\ = \sum_j \int_0^1 d\xi \xi^{N-1} \mathcal{D}_{h/j}^{\text{FEC}}\left(N, \ln \frac{\xi E_a \sin \chi}{\mu}, \mu\right) P_{ji}(\xi, \mu). \quad (36)$$

In Fig. 4, we present the numerical solution to the modified DGLAP RG equation at NNLL accuracy. The NNLL FECs incorporate both the NNLO DGLAP splitting functions [118–120] and the NNLO twist-2 matching coefficients. The latter will be derived in the next section, while in this section we solve the RG equation within perturbation theory, which consists of the standard DGLAP evolution and a nonlinear residual term,

$$\mathcal{D}_{h/i}^{\text{FEC}}\left(N, \ln \frac{E_a \sin \chi}{\mu}, \mu\right) \\ = \mathcal{R}_i^N(\mu_0, \mu) + \sum_j \mathcal{D}_{j/N}^{\text{FEC}}\left(N, \ln \frac{E_a \sin \chi}{\mu_0}, \mu_0\right) \\ \times \mathcal{D}_{ji}^N(\mu_0, \mu), \quad (37)$$

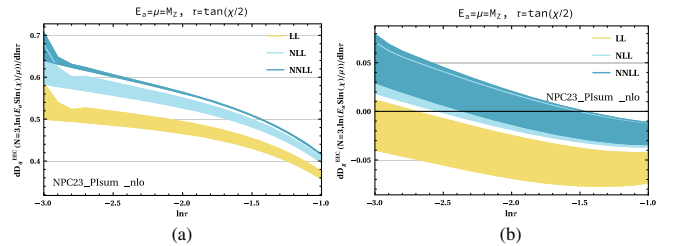


FIG. 4. Mellin moments of FEC singlets renormalized at fragmented parton energy $\mu = E_a = m_Z$. Panel (a) shows the shape of the renormalized quark fragmentation energy correlator, while panel (b) shows the corresponding gluon correlator.

where $\mathcal{D}_{ij}^N(\mu_0, \mu)$ is the standard DGLAP evolution,

$$\begin{aligned}\mathcal{D}^N(\mu_0, \mu) &= \exp \left[\int_{\mu_0}^{\mu} d \ln \bar{\mu}^2 P(N, \bar{\mu}) \right] \\ &= \exp [-2A_{P(N)}(\mu, \mu_0)].\end{aligned}\quad (38)$$

The nonlinear modification term is given by

$$\begin{aligned}\mathcal{R}_i^N(\mu_0, \mu) &= \sum_{m=1}^{\infty} \int_{\mu_0}^{\mu} d \ln \bar{\mu}^2 \mathcal{D}^N(\mu_0, \bar{\mu}) P^{(m)}(N, \bar{\mu}) \mathcal{D}^N(\bar{\mu}, \mu) \\ &\times \mathcal{D}_{h/i}^{\text{FEC}(m)} \left(N, \ln \frac{E_a \sin \chi}{\mu_0}, \mu_0 \right),\end{aligned}\quad (39)$$

where the boundary value of FECs at renormalization scale μ_0 are reorganized into a power expansion in $\ln \xi$

$$\begin{aligned}\mathcal{D}_{h/i}^{\text{FEC}} \left(N, \ln \frac{\xi E_a \sin \chi}{\mu_0}, \mu_0 \right) \\ = \mathcal{D}_{h/i}^{\text{FEC}} \left(N, \ln \frac{E_a \sin \chi}{\mu_0}, \mu_0 \right) \\ + \sum_{m=1}^{\infty} \ln^m \xi \times \mathcal{D}_{h/i}^{\text{FEC}(m)} \left(N, \ln \frac{E_a \sin \chi}{\mu_0}, \mu_0 \right),\end{aligned}\quad (40)$$

and the modified splitting functions are defined as

$$P^{(m)}(N, \mu) = \int_0^1 dx x^{N-1} \ln^m x P(x, \mu). \quad (41)$$

In summary, the Mellin space factorization formula is

$$\begin{aligned}\sigma_h^{\text{EC}}(N, Q, \chi) &\simeq \frac{1}{2} \frac{4\pi\alpha_e^2}{3Q^2} N_c \int_0^1 d\hat{x} \hat{x}^{N-1} \left\{ \sum_{i=1}^{N_f} Q_i^2 \mathcal{D}_{h/i}^{\text{FEC,NS}} \left(N, \ln \frac{\hat{x}Q/2 \sin \chi}{\mu}, \mu \right) \mathcal{C}_i^{\text{NS}}(\hat{x}, Q, \mu) + \left(\sum_{i=1}^{N_f} Q_i^2 \right) \right. \\ &\times \left[\frac{1}{N_f} \mathcal{D}_{h/q}^{\text{FEC}} \left(N, \ln \frac{\hat{x}Q/2 \sin \chi}{\mu}, \mu \right) \mathcal{C}_q^{\text{S}}(\hat{x}, Q, \mu) + \mathcal{D}_{h/g}^{\text{FEC}} \left(N, \ln \frac{\hat{x}Q/2 \sin \chi}{\mu}, \mu \right) 2\mathcal{C}_g^{\text{S}}(\hat{x}, Q, \mu) \right] \Big\}. \quad (42)\end{aligned}$$

The NNLL prediction at a center-of-mass energy corresponding to the Z boson threshold is shown in Fig. 5, where we vary the initial boundary and factorization scales around $(\mu_0, \mu_f) \sim (Q/2 \sin \chi, Q)$. Unlike the back-to-back jets, where Sudakov suppression arises due to the recoil of soft gluons, soft radiation is suppressed by the energy weight in the jet fragmentation region. As a result, the distribution at small angles is not suppressed.

V. MATCHING WITH COLLINEAR FACTORIZATION

The factorization formulas in Eqs. (20), (30), and (42) do not require a strict scale hierarchy between $q_T \sim Q/2 \sin \chi$

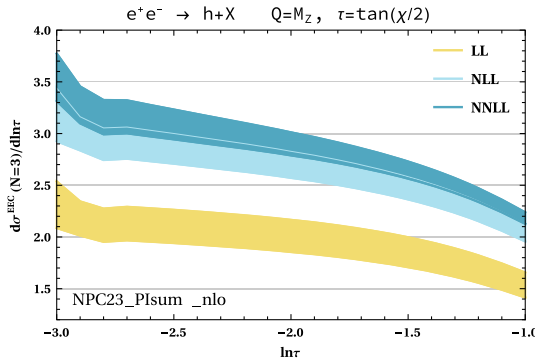


FIG. 5. NNLL prediction for semi-inclusive energy correlations in the collinear limit.

and Λ_{QCD} . Nevertheless, in the perturbative regime $Q \gg q_T \gg \Lambda_{\text{QCD}}$, consistency between the TMD/JFR factorization and the standard collinear factorization implies that the TMDs and FECs admit matching onto conventional FFs. The perturbative twist-2 part of the TMD fragmentation function are obtained in [121,122]. On the other hand, analogous to the observation in Refs. [12,123], the FECs allow an operator product expansion onto energy-weighted collinear FFs as follows:

$$\begin{aligned}\mathcal{D}_{h/i}^{\text{FEC}} \left(z, \ln \frac{E_a \sin \chi}{\mu}, \mu \right) \\ = -d_i(z, \mu) + \int_z^1 \frac{d\xi}{\xi} \frac{1}{\xi} d_j(\xi, \mu) \mathcal{C}_{ji} \left(\frac{z}{\xi}, \ln \frac{E_a \sin \chi}{\mu}, \alpha_s(\mu) \right) \\ + \mathcal{O} \left(\frac{\Lambda_{\text{QCD}}}{q_T} \right).\end{aligned}\quad (43)$$

The matching coefficients $\mathcal{C}_{ij}$ is insensitive to the hadronic state h , thus can be computed from partonic FECs, and renormalized according to Eq. (32). After UV renormalization, the FECs are reparametrized by the partonic logarithm through replacement $\ln \frac{E_a}{\mu} \rightarrow \ln \frac{E_a}{\mu} + \ln z$, and matched onto energy densities $1/\xi d_j(\xi, \mu)$. The renormalization group equation governing the coefficient function follows directly from Eqs. (36) and (43),

$$\frac{dC_{ij}}{d \ln \mu^2} \left(z, \ln \frac{E_a \sin \chi}{\mu} \right) = \int_z^1 \frac{d\xi}{\xi} \left[C_{ik} \left(\frac{z}{\xi}, \ln \frac{\xi E_a \sin \theta}{\mu} \right) P_{kj}(\xi) - \frac{P_{ik}(\xi)}{\xi} C_{kj} \left(\frac{z}{\xi}, \ln \frac{E_a \sin \chi}{\mu} \right) \right]. \quad (44)$$

The first term in Eq. (44) originates from RG flow with respect to the UV cutoff from above, while the second term arises from RG flow toward the IR, the solution up to NNLO reads

$$\begin{aligned} C_{ij} \left(z, \ln \frac{E_a \sin \chi}{\mu} \right) &= \delta_{ij} \delta(1-z) + a_s(\mu) (C_{ij}^{(1)}(z) + L_q(p_{ij}^{(0)}(z) - P_{ij}^{(0)}(z))) + a_s^2(\mu) \left(C_{ij}^{(2)}(z) + L_q^2 \right. \\ &\quad \times \left[\frac{p_{ik}^{(0)} \otimes p_{kj}^{(0)}(z)}{2} + \frac{P_{ik}^{(0)} \otimes P_{kj}^{(0)}(z)}{2} - p_{ik}^{(0)} \otimes P_{kj}^{(0)}(z) - \frac{1}{2} \beta_0 (p_{ij}^{(0)}(z) - P_{ij}^{(0)}(z)) \right] \\ &\quad + L_q [p_{ik}^{(0)} \otimes C_{kj}^{(1)}(z) - C_{ik}^{(1)} \otimes P_{kj}^{(0)}(z) - \beta_0 C_{ij}^{(1)}(z) + p_{ij}^{(1)}(z) - P_{ij}^{(1)}(z) + 2(p_{ik}^{(0)} - P_{ik}^{(0)}) \otimes \tilde{P}_{kj}^{(0)}(z)] \\ &\quad \left. + \mathcal{O}(a_s^3(\mu)) \right), \end{aligned} \quad (45)$$

where $a_s(\mu) = \alpha_s(\mu)/(4\pi)$ and $L_q = 2 \ln \frac{E_a \sin \chi}{\mu}$. The modified splitting functions are given by

$$p_{ij}^{(m)}(z) = 1/z P_{ij}^{(m)}(z), \quad \tilde{P}_{ij}^{(0)}(z) = \ln z P_{ij}^{(0)}(z). \quad (46)$$

The results above serve as boundary conditions for the NNLL resummation for the semi-inclusive energy correlations. They also directly yield the collinear limit of inclusive EEC, as the latter is given by

$$\sum_{\{A,B\}} \frac{E_A}{Q/2} \frac{E_B}{Q/2} = \sum_A \left(\frac{E_A}{Q/2} \right)^2 \frac{1}{2} \sum_B \frac{E_B}{E_A}. \quad (47)$$

As a result, the collinear limit of inclusive EEC is obtained by taking the Mellin moment $N=3$ in Eq. (42) and summing over all species of the identified hadron $\sigma^{\text{EEC}}(Q, \chi) = \sum_h \sigma_h^{\text{EEC}}(N=3, Q, \chi)$,

$$\begin{aligned} \sigma^{\text{EEC}}(Q, \chi) &\simeq \sigma_0^{\text{tot}} \sum_{i=q,g} \mathcal{J}_i \left(\ln \frac{Q/2 \sin \chi}{\mu} + \cdot, \mu \right) \odot \mathcal{C}_i^S(\cdot, Q, \mu) \\ &\quad + \mathcal{O} \left(\frac{\Lambda_{\text{QCD}}}{q_T} \right). \end{aligned} \quad (48)$$

where $\sigma_0^{\text{tot}} = \sum_{f=1}^{N_f} \sigma_{f\bar{f}}^{(0)}$, $\sigma_{f\bar{f}}^{(0)} = \frac{4\pi\alpha_s^2}{3Q^2} N_c Q_f^2$. Note that the FEC nonsinglets do not contribute to inclusive EEC in the collinear limit. Furthermore, the convolution is defined by

$$\begin{aligned} \mathcal{J}_i \left(\ln \frac{Q/2 \sin \chi}{\mu} + \cdot, \mu \right) \odot \mathcal{C}_i^S(\cdot, Q, \mu) \\ = \int_0^1 d\hat{x} \hat{x}^2 \mathcal{J}_i \left(\ln \frac{\hat{x} Q/2 \sin \chi}{\mu}, \mu \right) \mathcal{C}_i^S(\hat{x}, Q, \mu). \end{aligned} \quad (49)$$

Owing to the sum rule of collinear FFs, the collinear EEC jet function is given by the sum of third Mellin moment of

FEC twist-2 matching coefficients, reproducing the result of Refs. [47,95],

$$\mathcal{J}_i \left(\ln \frac{E_a \sin \chi}{\mu}, \mu \right) = \sum_j C_{ji}^S \left(N=3, \ln \frac{E_a \sin \chi}{\mu}, \mu \right). \quad (50)$$

Our analysis is consistent with dihadron framework [14–16] where the contact term of the dihadron fragmentation function contributes to the ECs Eq. (6) by

$$\begin{aligned} \frac{1}{2} \sigma_0^{\text{tot}} \sum_{h'} \int_0^{1-x} \frac{x' dx'}{x} \int_{x+x'}^1 \frac{dz}{z^2} D_{i \rightarrow hh'} \left(\frac{x}{z}, \frac{x'}{z} \right) \mathcal{C}_i(z, Q) \\ = \frac{1}{2} \sigma_0^{\text{tot}} \int_x^1 \frac{dz}{z} \left(\frac{1}{z} - 1 \right) d_i(z, \mu) \mathcal{C}_i \left(\frac{x}{z}, Q, \mu \right), \end{aligned} \quad (51)$$

where x' denotes the momentum fraction of an another hadron [45,124–127], and for convenience we slightly adjust the normalization of the SIA coefficient functions in Eq. (30) so that the convention matches those used in Refs. [8,14]. To the EEC observable, the first term $d_i(z, \mu)/z$ in Eq. (51) gives an IR finite contribution while the second term matches the first term of the collinear expansion in Eq. (43), and yields an IR-divergent contribution,

$$-\frac{1}{2} \sigma_0^{\text{tot}} \sum_h \sum_i d_i^h(N=3, \mu) \mathcal{C}_i^S(N=3, Q, \mu), \quad (52)$$

which exactly cancels with the contact term contribution in the EEC, thereby ensuring the IR safety of the EEC observable. It is worth emphasizing that the factor of $1/2$ plays a crucial role in guaranteeing this cancellation, which has the same origin as in Eq. (2.18) of Ref. [8] for the inclusive cross section. After isolating the contact term, the remaining contributions from the dihadron fragmentation functions arise entirely from the hard-collinear region [123].

As a concluding remark in this section, we point out that the holomorphic part of bare partonic FECs are obtained from crossings of nucleon energy correlators (NECs) [12,114]. The crossing relations have an overall minus sign compared to those for the TMDs due to the energy weight [120]. For the quark FECs/NECs, the crossing relation is

$$D_{i/\bar{q}}^{\text{FEC}}(x, E_P, \chi) = (-1)^{1+i_F} x^{1-2\epsilon} F_{q/i}^{\text{NEC}}\left(-\frac{e^{i\pi}}{x}, -E_P, \chi\right), \quad (53)$$

where $i_F = 1$ if i is a fermion, and 0 if boson. The minus sign originates from crossing a fermion from initial state to final state. For the gluon FECs/NECs, the crossing relation is

$$D_{i/g}^{\text{FEC}}(x, E_P, \chi) = (-1)^{i_F} x^{1-2\epsilon} F_{g/i}^{\text{NEC}}\left(-\frac{e^{i\pi}}{x}, -E_P, \chi\right). \quad (54)$$

The exact meaning of the inversion $x \rightarrow 1/x$ is illustrated in Fig. 6. In the first step, a representative path from $x \rightarrow 1/x$ is drawn in the base space: it first approaches the neighborhood of threshold branch point $\bar{x} = 1 - x = 0$, encloses it counterclockwise via the map $I = [0, 1] \xrightarrow{\omega} \mathbb{S}^1$ where $\omega(x) = e^{2\pi i x}$ and then proceeds to destination $1/x$ without enclosing any further branch points. In the second step, the path is lifted to its covering space, taking the spacelike NECs $F_{j/i}^{\text{NEC}}(x, \dots)$ as the initial value of the lift, which is unambiguously given since it is defined in the Euclidean region. Owing to the unique lifting property, the time-like FECs is obtained by directly computing the monodromy of individual functions. The complete analytic expressions for the coefficient functions are provided in the Supplemental Material [128], along with

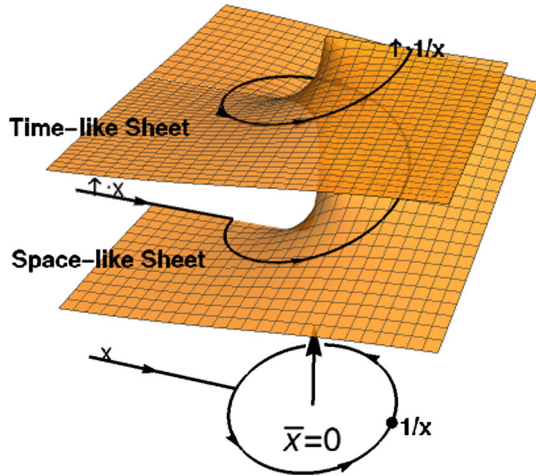


FIG. 6. The inversion relation between Bjorken variable x_B and Feynman variable $z_F = 1/x_B$.

numerical routines implemented in PolyLogsTools [129]. The crossing relation between dynamics of TFR and JFR even persists beyond the paradigm of amplitudes, in Refs. [120,130,131], a reciprocity relation was proposed between spacelike and timelike anomalous dimensions of twist-2 operators. However, a rigorous proof in QCD is still lacking and remains an open problem for future investigations.

VI. SUMMARY AND OUTLOOK

In this work, we investigate energy correlations for single-inclusive hadron production in electron-positron annihilation via photon decay. Within SCET, we demonstrate factorization into universal nonperturbative correlation functions in both small- and wide-angle limits. We compute the twist-2 matching of FECs at NNLO and achieve joint N³LL/NNLL renormalization-group-improved predictions in the Sudakov and jet fragmentation regions; results consistent with previous findings [23,47,95]. These high-precision predictions enable a precise extraction of TMD fragmentation functions in the Sudakov region and improve theoretical control over hadronization dynamics in the jet fragmentation region.

Looking ahead, several further improvements are worth pursuing. First, the Z-boson channel presents a unique opportunity to study hadronization dynamics jointly with parton-spin transport, as its chirality-violating couplings generate spin asymmetries in the final-state hadrons. This distinctive sensitivity provides solid theoretical motivation for future high-energy e^+e^- colliders such as CEPC [132], FCC-ee [133]. Second, the Mellin-space factorization fails at Mellin moment $N = 2$, requiring an all-order resummation of the associated logarithms to access the small- x regime. Third, extending the current framework to chiral-odd operators, thereby enabling probes of quark transversity [134], represents an interesting direction for ongoing work.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [128].

APPENDIX: SMALL- x LOGARITHMS

In this appendix, we collect the small- x logarithms of the twist-2 matching coefficients for the fragmentation energy correlators. The complete analytic expressions of the coefficient functions are provided in the Supplemental Material [128],

$$\begin{aligned} z^2 \mathcal{C}_{qq}^{(1)}(z) &\simeq 0, & z^2 \mathcal{C}_{gq}^{(1)}(z) &\simeq 8C_F \ln z, \\ z^2 \mathcal{C}_{qg}^{(1)}(z) &\simeq 0, & z^2 \mathcal{C}_{gg}^{(1)}(z) &\simeq 8C_A \ln z, \end{aligned} \quad (\text{A1})$$

$$z^2 \mathcal{C}_{qq}^{(2)}(z) \simeq \frac{32}{3} C_F N_f \ln^2 z + \frac{16}{3} C_F N_f \ln z + C_F N_f \left(-\frac{296}{27} + \frac{16}{3} \zeta_2 \right), \quad (\text{A2})$$

$$\begin{aligned} z^2 \mathcal{C}_{gq}^{(2)}(z) &\simeq -\frac{80}{3} C_A C_F \ln^3 z - \frac{212}{3} C_A C_F \ln^2 z + \left(-\frac{736}{7} C_F^2 + C_A C_F \left(\frac{1204}{9} - 32\zeta_2 \right) \right) \ln z \\ &+ C_A C_F \left(\frac{4424}{27} - \frac{160}{3} \zeta_2 - 120\zeta_3 \right), \end{aligned} \quad (\text{A3})$$

$$z^2 \mathcal{C}_{qg}^{(2)}(z) \simeq \frac{32}{3} C_A N_f \ln^2 z + \frac{16}{3} C_A N_f \ln z + C_A N_f \left(-\frac{296}{27} + \frac{16}{3} \zeta_2 \right), \quad (\text{A4})$$

$$\begin{aligned} z^2 \mathcal{C}_{gg}^{(2)}(z) &\simeq -\frac{80}{3} C_A^2 \ln^3 z + \left(-\frac{220}{3} C_A^2 - \frac{8}{3} C_A N_f + \frac{16}{3} C_F N_f \right) \ln^2 z + \left(-\frac{92}{9} C_A N_f + C_A^2 \left(\frac{536}{9} - 32\zeta_2 \right) \right) \ln z \\ &+ C_A N_f \left(\frac{562}{27} - \frac{16}{3} \zeta_2 \right) + C_F N_f \left(-\frac{1124}{27} + \frac{32}{3} \zeta_2 \right) + C_A^2 \left(\frac{1616}{9} - \frac{176}{3} \zeta_2 - 120\zeta_3 \right). \end{aligned} \quad (\text{A5})$$

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