

Quenching through the QCD chiral phase transition

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We present a detailed numerical and analytical study of the out-of-equilibrium dynamics of Model G, the dynamical universality class relevant to the chiral phase transition. We perform numerical 3D stochastic (Langevin) simulations of the $O(4)$ critical point for large lattices in the chiral limit. We quench the system from the high-temperature unbroken phase to the broken phase and study the nonequilibrium dynamics of pion fields. Strikingly, the nonequilibrium evolution of the two-point functions exhibits a regime of growth, a parametrically large enhancement, and a subsequent slow relaxation to equilibrium. We analyze our numerical results using dynamic critical scaling and mean-field theory. The growth of the two-point functions is determined by the nonlinear dynamics of an ideal non-Abelian superfluid, which is a limit of Model G that reflects the broken chiral symmetry. We also relate the nonequilibrium two-point functions to a long-lived parametric enhancement of soft pion yields relative to thermal equilibrium following a quench.

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I. INTRODUCTION

One of the original motivations for high-energy collisions of heavy ions was the creation of an “abnormal” state of matter with symmetry properties different from that of vacuum [1]. A prominent example is the symmetry between left- and right-handed quarks in QCD. Chiral symmetry is spontaneously broken in a vacuum, which explains the existence of light pions—the (pseudo-) Goldstone bosons.¹ After 50 years of research, there is ample evidence that a new form of chirally symmetric state of matter—the quark-gluon plasma (QGP)—is created in

high-energy nuclear collisions at the Large Hadron Collider (LHC) at CERN and the Relativistic Heavy Ion Collider (RHIC) at BNL [2]. The heavy-ion models based on the viscous hydrodynamic expansion of the QGP have shown remarkable success in describing experimental low-momentum hadron data. The sophisticated Bayesian fits have put tight constraints on the bulk properties of QGP matter, such as shear and bulk viscosities [3,4]. However, the restoration of chiral symmetry in high-temperature QGP plays a surprisingly small role in heavy-ion phenomenology [5–7].

Hydrodynamics is the universal description of the time evolution of conserved charges. However, in the vicinity of a second-order phase transition, the order parameter evolves asymptotically slowly due to critical slowing down. As such, ordinary hydrodynamics has to be supplemented by additional slowly evolving degrees of freedom to obtain a complete low-momentum description [8]. In other words, the fluid-dynamic theory has to include also the nonconserved variables that evolve on a timescale similar to the conserved charges to be consistent. The resulting effective description of the dynamics depends on the interaction between the hydrodynamical modes and the order parameter. This naturally leads to the question if presently unaccounted-for critical dynamics in heavy ion collisions might be responsible for soft pion enhancement.

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¹Chiral symmetry is also explicitly broken by small, but nonzero up and down quark masses, which leads to light, but not massless, pions.

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An intriguing indication that the current hydrodynamic models might be incomplete is the observed enhancement of low-momentum pions over theory predictions in PbPb collisions at $\sqrt{s_{\text{NN}}} = 2.76$ TeV and $\sqrt{s_{\text{NN}}} = 5.02$ TeV as measured by the ALICE experiment [9,10]. Neither complete hydrodynamic simulations [11–14], nor dedicated blast-wave type fits [9,10,15,16] are able to account for $\sim 50\%$ enhancement of pions below $p_T < 0.5$ GeV. There have been several attempts to explain the soft pion enhancement using the formation of Bose-Einstein condensation of pions [17], decays of broad resonances [18], or perturbative contributions [19]. In the early 1990s, it was proposed that a sudden quench from chirally symmetric to a broken phase could lead to the formation of domains of a disoriented chiral condensate [20–22]. A characteristic modification of a neutral to charged pion ratio was looked into in early heavy-ion data, but never confirmed [23]. Three decades after these pioneering studies, we are at the advantage of tremendous computational and technological progress to study the full extent of chiral critical dynamics in heavy-ion collisions. Thanks to the high energy available at the LHC, the number of produced hadrons can reach tens of thousands in a single lead-lead collision—an order of magnitude increase from pre-2000 experiments [6]. With the advent of high-precision data from high-luminosity LHC and the next-generation heavy-ion experiment, ALICE3 [24], it is therefore high time to study the phenomenological signatures of the chiral phase transition in high-energy nuclear collisions.

Critical dynamics in QCD are relevant in several ways. At low-energy nuclear collisions, critical fluctuations in the vicinity of the conjectured QCD critical point, which is the endpoint of the first-order phase transition line between QGP and the hadronic phase, are subject to active experimental and theoretical searches [25]. In the Hohenberg and Halperin classification scheme [8], the relevant critical theory is “Model H” [26], and has recently been studied in the QCD context in [27]; see also [28,29] for related studies. At high-energy collisions at the LHC, the high-density baryon-symmetric matter cools down and transitions from a chirally symmetric deconfined phase to a hadronic phase. While this transition is not second order *per se*, the QGP undergoes a rapid chiral crossover [30,31]. It is therefore reasonable to expect the dynamics of the order parameters to be parametrically slow. In the limiting case of chiral QCD, when $m_{\text{up}} = m_{\text{down}} = 0$, the phase transition is now known to be second order [32–35]. These computations show that even for realistic quark masses, the QCD susceptibilities are well described by universal scaling functions of the $O(4)$ static universality class. The appropriate description of the associated dynamic universality class is “Model G” [8]. Model G is an effective field theory of pions (pion EFT) and the Goldstone particles associated with the $O(4) \simeq SU(2)_L \times SU(2)_R$ spontaneous

symmetry breaking, coupled to the chiral condensate. The first attempt at resolving Model G’s dynamics in the context of QCD was presented in [36] and subsequent works have carefully studied relevant universal properties and scaling functions [37,38].

In this work we study, for the first time to our knowledge, the out-of-equilibrium dynamics of Model G. We perform 3D Langevin simulations of an $O(4)$ symmetric scalar field (which is coupled to conserved charges) quenched from the symmetric to the broken phase close to the critical point. We find that a sudden quench of the system generates a characteristic growth of the order parameter. This growth is coupled to a corresponding parametric enhancement in the (equal time) pion correlation function, which subsequently relaxes slowly back to equilibrium. While this is reminiscent of the situation in many cosmological preheating scenarios [39–41], universality and the nontrivial hydrodynamic limit of Model G’s pion EFT quantitatively control the structure of the parametric enhancement, making the dynamics novel and markedly different from these studies. In particular, the growth of the condensate is controlled by the nonlinear and nondissipative dynamics of an $SU(2)_L \times SU(2)_R$ superfluid theory, reflecting the broken chiral symmetry. A brief account of this work is given in a companion Letter [42].

In more detail, this paper is structured as follows. In Sec. II, we review Model G, which we use to investigate the nonequilibrium dynamics. In particular we discuss the relevant timescales governing the pion dynamics in the broken phase. In Sec. III, we go fully nonadiabatic and study the case of an instantaneous quench from the hot to the cold phase. We first use the scaling analysis, independent of our simulation details, to show the expected parametric behavior of the condensate and two-point correlation functions. We confirm the scaling analysis with lattice simulations of quenches. Finally, we show that the mean-field analysis, which provides a simple analytic model of the quenches, fails in several important aspects. In Sec. IV we summarize our results and present arguments of why the observed pion enhancement will persist with explicit symmetry breaking. Appendix A provides a concise summary of the leading and subleading critical behavior of the equilibrium quantities established in earlier studies. In Appendix B we provide details on mean-field analysis.

II. OVERVIEW OF MODEL G AND ITS PHASES

As already discussed in the Introduction, critical dynamics is the appropriate framework to discuss the time evolution near the chiral phase transition of QCD. The relevant dynamical critical model is Model G [43–45] and describes the universal dynamics of pions, the Goldstone particles associated to the $O(4) \simeq SU(2)_L \times SU(2)_R$ spontaneous symmetry breaking.

The static properties of QCD near the chiral transition can be modeled with an $O(4)$ order parameter $\phi_a = (\phi_0, \phi_1, \phi_2, \phi_3)$, which parametrizes the quark condensate,² $\langle \bar{q}_R q_L \rangle$. The corresponding Landau-Ginzburg free energy of the $O(4)$ model is

$$H[\phi] = \int d^d x \left[\frac{1}{2} \nabla \phi_a \cdot \nabla \phi_a + U(\phi) - H \cdot \phi \right], \quad (1)$$

where $d = 3$ is the number of spatial dimensions, and the potential $U(\phi)$ is

$$U(\phi) = \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda}{4} (\phi \cdot \phi)^2, \quad (2)$$

with m_0^2 negative. The magnetic field $H_a = (H, \vec{0})$ in the model accounts for the explicit breaking of $O(4)$ symmetry by the quark mass in QCD. In equilibrium, the order parameter is distributed according to the statistical weight:

$$P[\phi] = \frac{1}{Z} e^{-H[\phi]/T_c}. \quad (3)$$

The $O(4)$ model with $\lambda = 4$ and $H = 0$ has a second-order phase transition at $m_0^2 = m_c^2 = -4.8110(4)$ [37], where the order parameter acquires a nonzero expectation value. All models belonging to the $O(4)$ universality class, including QCD and the model in Eq. (1), will exhibit the same critical behavior as a function of relevant couplings, which are normalized by convention. In the chiral limit ($H = 0$), the model in Eq. (1) has one relevant coupling m_0^2 which can be varied around m_c^2 . We define the reduced temperature in the simulation, t_r , as the deviation of the relevant coupling from its critical value³

$$t_r \equiv \frac{m_0^2 - m_c^2}{|m_c^2|}. \quad (5)$$

Previously in [37] we matched the reduced temperature and field of Eq. (1) to a conventional parametrization of the $O(4)$ magnetic equation of state given in [46]. For QCD in the chiral limit, the only relevant coupling close for chiral

phase transition is the temperature, and the reduced temperature is $(T - T_c)/T_c$. Ultimately the relation between t_r and $(T - T_c)/T_c$ in QCD must be determined by matching lattice QCD data on the chiral condensate to the conventional parametrization discussed above [33].

The real time critical dynamics of a second-order phase transition depend on the order parameter and the number and type of conserved charges [8]. In the case of the QCD and the chiral phase transition, both the iso-vector charge, $\vec{n}_V \sim \bar{q} \gamma^0 \vec{\tau}_I q$, and iso-axial vector charge, $\vec{n}_A \sim \bar{q} \gamma^0 \gamma^5 \vec{\tau}_I q$, are conserved and couple to the order parameter. In the $O(4)$ language, $(\vec{n}_V, \vec{n}_A)$ are combined into an antisymmetric $O(4)$ tensor n_{ab} with

$$n_{0\ell} = n_A^\ell, \quad n_V^\ell = \frac{1}{2} \epsilon^{\ell \ell_1 \ell_2} n_{\ell_1 \ell_2}. \quad (6)$$

As these conserved charges are noncritical, their equilibrium distribution is Gaussian

$$H_n \equiv \int d^3 x \left[\frac{1}{4\chi_I} n_{ab} n^{ab} \right], \quad (7)$$

where χ_I is the isospin susceptibility.⁴

The interactions of the conserved charges with the order parameter arise from nontrivial Poisson brackets—see [43,45] for the expressions and explicit derivation. The effective equations of motion that describe the system in the critical region, close to equilibrium, are

$$\partial_t \phi_a + \frac{g_0}{\chi_I} n_{ab} \phi_b = \Gamma_0 \nabla^2 \phi_a - \Gamma_0 (m_0^2 + \lambda \phi^2) \phi_a + \Gamma_0 H_a + \theta_a, \quad (8a)$$

$$\partial_t n_{ab} + g_0 \nabla \cdot (\nabla \phi_{[a} \phi_{b]}) + g_0 H_{[a} \phi_{b]} = D_0 \nabla^2 n_{ab} + \partial_i \Xi_{ab}^i. \quad (8b)$$

Here $H_{[a} \phi_{b]}$ denotes the antisymmetrization, $H_a \phi_b - H_b \phi_a$. The coefficients Γ_0 and D_0 are the bare kinetic coefficients associated with the order parameter and the charges.⁵ The bare conductivity of the charges is related to the bare diffusion coefficient and the charge susceptibility, $\sigma_0 = \chi_I D_0$. The constant g_0 is a coupling of the field ϕ , and has the units of (action)⁻¹ in our conventions. Finally, θ_a and Ξ_{ab} are the appropriate noises, which are defined through their two-point correlations [8]

$$\langle \theta_a(t, x) \theta_b(t', x') \rangle = 2T_c \Gamma_0 \delta_{ab} \delta(t - t') \delta^3(x - x'), \quad (9a)$$

⁴The isospin susceptibility χ_I is unrelated to the susceptibility of the order parameter χ defined in Eq. (11).

⁵Outside of $d = 4$ (mean field), the renormalized parameters Γ and D are markedly different from the bare ones, and are defined by a pion EFT described in Sec. II B 3.

²As in our previous work $O(4)$ indices are indicated by $a, b, c = 0 \dots 3$ [37,45]. The “spatial” components describing the $O(3)$ isospin subgroup are indicated with an arrow, e.g. $\phi_a = (\phi_0, \vec{\pi})$. If necessary, an index ℓ labels the components, $\vec{\pi}_\ell = (\pi_1, \pi_2, \pi_3)$. The dot product indicates an appropriate contraction of indices when clear from context, e.g. $\phi \cdot \phi = \phi_a \phi_a$, $\vec{\pi} \cdot \vec{\pi} = \vec{\pi}_\ell \vec{\pi}_\ell$.

³In mean-field theory, $m_c^2 = 0$ and the reduced temperature is defined as

$$t_r = \frac{m_0^2}{m^2}, \quad (4)$$

where m is a dimensionful parameter, scaling as the microscopic length to the power of -1 .

TABLE I. Critical exponents of the $O(4)$ model at zero magnetic field used in this work. The high precision values for the static exponents are from Engels *et al.* [46,50]. Within the uncertainty of our simulation, these exponents are reproduced by the Langevin simulations after accounting for subleading corrections [37,38]. The dynamical critical exponent ζ is also reproduced by these simulations.

Quantity	Scaling	Relation	Exponent value
Correlation length	$\xi \propto \mathfrak{t}_r^{-\nu}$		$\nu = 0.7377(41)$
Susceptibility	$\chi \sim \xi^{2-\eta} \propto \mathfrak{t}_r^{-\gamma}$	$\gamma = \nu(2 - \eta)$	$\eta = 0.0302(16)$
Condensate	$\bar{\sigma}_{\text{eq}} \sim \xi^{-(d-2+\eta)/2} \propto (-\mathfrak{t}_r)^\beta$	$\beta = \nu(d - 2 + \eta)/2$	$\beta = 0.380(2)$
Relaxation time	$\tau_R \propto \xi^\zeta$		$\zeta = d/2$

$$\langle \Xi_{ab}^i(t, x) \Xi_{cd}^j(t', x') \rangle = 2T_c \chi_I D_0 \delta^{ij} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) \times \delta(t - t') \delta^3(x - x'). \quad (9b)$$

The dynamical equations listed above define Model G in the Halperin and Hohenberg classification scheme. The field will equilibrate to the free energy distribution, $H[\phi, n] \equiv H_\phi[\phi] + H_n[n]$ [8]. Following our previous work we have discretized the equations of motion on a lattice of size L , setting the lattice spacing a , the critical temperature T_c , and coupling g_0 to unity $g_0 = T_c = a = 1$ [37,38]. In these units isospin susceptibility and kinetic coefficients are taken to be $\chi_I = 5$, $\Gamma_0 = 1$, and $D_0 = 1/3$ [37].

Our numerical scheme is detailed in Appendix A of [37]. Briefly, the scheme uses operator splitting, first evolving the system with the ideal (Poisson-bracket) terms, using a second order position-Verlet symplectic integrator. The time step is $\Delta t = 0.24a^2/\Gamma_0$. Then, six dissipative steps (each of step size $0.04a^2/\Gamma_0$) are taken using a metropolis scheme. In this scheme, a proposal is made for the field update at a given lattice site, $\phi_a(\mathbf{x}) \rightarrow \phi_a(\mathbf{x}) + \delta\phi_a(\mathbf{x})$, with variance $\langle \delta\phi_a^2 \rangle_0 = 2T_c \Gamma_0 \Delta t / a^3$. This proposal is then accepted or rejected with probability, $\min(1, \exp(-\Delta H/T_c))$, where $\Delta \mathcal{H} = \mathcal{H}[\phi_a + \delta\phi_a] - \mathcal{H}[\phi_a]$. The accept-reject process introduces the dissipation and reproduces the fluctuation dissipation theorem (FDT). The dissipative time step, $0.04a^2/\Gamma_0$, is chosen so that the accept-reject probability is approximately one half. A similar metropolis step is also used to update the charges, reproducing the stochastic diffusion equation. Full details are given in Appendix A of [37].

As summarized below, we have previously analyzed the real-time equilibrium correlation functions using this scheme, both above and below T_c [37,38].

A. Equilibrium dynamics of the order parameter above T_c

In this section we will briefly review the dynamics of the order parameters in equilibrium and above T_c [47,48]. In this regime the correlation length ξ sets the relevant length scale, and perturbations of the order parameter relax slowly

back to equilibrium on a dynamical timescale, $\tau_R \propto \xi^\zeta$, where $\zeta = d/2$ is the dynamical critical exponent of Model G. Although we will speak informally about these scales, they can be precisely defined from specific equilibrium correlators.

The magnetization is defined from the zero mode of the order parameter

$$M_a(t) \equiv \phi_a(t, \mathbf{k})|_{\mathbf{k}=0}, \quad \phi_a(t, \mathbf{k}) \equiv \frac{1}{V} \sum_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}} \phi_a(t, \mathbf{x}), \quad (10)$$

where $V \equiv L^d$ is the three-dimensional lattice volume for $d = 3$. The magnetic susceptibility χ at $H = 0$ scales as

$$\chi \equiv \frac{V}{4} \sum_{a=1}^4 \langle M_a(t) M_a(t) \rangle_{\text{eq}} = C^+ \mathfrak{t}_r^{-\gamma} \sim \xi^{2-\eta}. \quad (11)$$

Here the average $\langle \dots \rangle_{\text{eq}}$ denotes an average over time, which is equivalent to the ensemble average. $\chi \equiv \chi(\mathfrak{t}_r)$ is a function of reduced temperature, but here and below we will drop the $\mathfrak{t}_r$ argument when confusion does not arise. The critical exponents γ and η are given in Table I. More generally at finite $\mathbf{k}$ we define the static correlation function

$$G_{\phi\phi}(k) \equiv \frac{V}{4} \sum_{a=1}^4 \langle |\phi_a(t, \mathbf{k})|^2 \rangle_{\text{eq}}, \quad (12)$$

and we will implicitly average over the three Cartesian directions for a specified $k = |\mathbf{k}|$ to increase statistics. When the correlation length must be precisely defined, for $T > T_c$ we use the standard “second moment” correlation length⁶ used in [38,49]:

⁶The motivation for the definition is the following—in continuous mean-field theory $G_{\phi\phi}(k) = \chi / ((k\xi)^2 + 1)$. Now replace k^2 in the mean-field formula with the lowest eigenvalue of the discretized Laplacian, $k^2 \rightarrow (2 \sin(ka/2)/a)^2$, with lattice spacing $a = 1$. Solving for ξ produces Eq. (13).

$$\xi \equiv \frac{1}{2 \sin(\pi/L)} \sqrt{\frac{\chi}{G_{\phi\phi}(k)|_{k=2\pi/L}} - 1} = \xi^+ t_r^{-\nu} \quad \text{for } t_r > 0. \quad (13)$$

Below T_c we will use a different definition of the correlation length based on the pion decay constant as discussed below.

References [37,38] provide a careful study of the critical behavior of this model, including the determination of relevant nonuniversal constants C^+ and ξ^+ and the first correction to scaling—we refer the reader to Table II and Appendix B of [38] for the complete parametrization of the equilibrium properties of the model obtained from Langevin dynamics of the previous section.

The order parameter relaxation time can be defined from real time response of the system. For instance, previously we analyzed the unequal time correlation function

$$G_{\phi\phi}(\Delta t, k) \equiv \frac{V}{4} \sum_a \langle \phi_a(t + \Delta t, \mathbf{k}) \phi_a(t, -\mathbf{k}) \rangle_{\text{eq}}, \quad (14)$$

and, motivated by exponential decay, defined a relaxation time based on the integral

$$\tau_R \equiv \chi^{-1} \int_0^\infty d\Delta t G_{\phi\phi}^{\text{eq}}(\Delta t, 0). \quad (15)$$

The relaxation time diverges in the vicinity of the critical point in a universal fashion

$$\tau_R = \tau^+ t_r^{-\nu\zeta} \sim \xi^\zeta, \quad (16)$$

where $\zeta = d/2$ is the dynamical critical exponent of Model G and τ^+ is a microscopic timescale. Again we previously measured the nonuniversal constant τ^+ (see Table II) and demonstrated the dynamical scaling of the model in a number of ways [37,38].

We note that all of the scaling relations above follow from the scaling hypothesis, where the correlation function takes the scaling for [47,48]

$$G_{\phi\phi}(\Delta t, k, t_r) = s^{2-\eta} G_{\phi\phi}(s^{-\zeta} \Delta t, s k, s^{1/\nu} t_r), \quad (17)$$

with s a positive number. For instance, setting $k = 0$, $\Delta t = 0$ and $s = t_r^{-\nu}$ reproduces the temperature dependence of the susceptibility $\chi \propto t_r^{-\gamma}$, after noting the relation $\gamma = \nu(2 - \eta)$ given in Table I.

B. Equilibrium dynamics of Goldstone modes below T_c

In this section, we will briefly review the equilibrium dynamics of pions below T_c following Son and Stephanov and our previous work [45,51,52]. First we provide an overview of pion propagation close to the critical point. Subsequently we review the hydrodynamic equations of

motion that lead to these waves. Finally, we connect these classical waves to a quasiparticle description.

I. Overview

Below T_c , the system spontaneously breaks the $O(4)$ symmetry, thereby developing a condensate. The typical correlation length below T_c remains of order $\xi \sim |t_r|^{-\nu}$ and the typical relaxation time for modes with $k\xi \sim 1$ remains of order $\tau_R \sim \xi^\zeta$ with $\zeta = d/2$. However, long wavelength pion modes of length $L \gg \xi$ are characterized by the longer time scales $\tau_R(L/\xi)$ and $\tau_R(L/\xi)^2$. Analyzing the statics and dynamics of long wavelength pions is the best way to precisely define the parameters ξ and τ_R below T_c .

Specifically, by analyzing the spatial correlation function of pions at large distances, one determines the spatial decay constant, f^2 , which scales as ξ^{2-d} [see [51] and the discussion around Eq. (34)]. Thus in three dimensions f^2 provides a definition of the correlation length, $\xi \propto 1/f^2$. Similarly the pion dispersion curve at small wave numbers takes the form

$$\omega(k) = vk - \frac{i}{2} D_A k^2. \quad (18)$$

Here the velocity and diffusion parameters in the simulation, v and D_A , depend critically on the reduced temperature and correlation length. The parameters can be (and were) precisely determined by analyzing pion waves close to the critical point [38]. As discussed below, the pion velocity is defined as

$$v \equiv \sqrt{\frac{f^2}{\chi_I}}, \quad (19)$$

where χ_I is the charge susceptibility of Eq. (7) and is constant near the critical point. Thus the velocity scales as

$$v \sim \xi^{1-d/2} \sim \frac{\xi}{\tau_R}, \quad (20)$$

close to the critical point [51]. See Table II and [37,38] for a parametrization including subleading corrections. The ballistic propagation of pions over length L takes a time of order, $L/v \sim \tau_R(L/\xi)$.

The damping coefficient D_A must scale as ξ^2/τ_R so that when k is of order ξ^{-1} , the frequency $\omega(k)$ is of order $\omega \sim 1/\tau_R$, in accord with the scaling hypothesis. Note that for $k\xi \sim 1$ the real and imaginary parts of ω are the same order of magnitude [51,52]. Thus, the decay time for the pion mode of size $L \gg \xi$ is of order $\tau_R(L/\xi)^2$.

To summarize, soft pions are characterized by their spatial decay constant $f^2 \propto 1/\xi$. For pions of wavelength $L \gg \xi$, the timescales associated with ballistic pion propagation and diffusion are

$$\begin{aligned} \text{ballistic time} &\equiv \frac{L}{v} \sim \tau_R \left(\frac{L}{\xi} \right), \quad \text{and} \\ \text{diffusion time} &\equiv \frac{L^2}{D_A} \sim \tau_R \left(\frac{L}{\xi} \right)^2. \end{aligned} \quad (21)$$

The pion description is valid for $k\xi \ll 1$ and times $t \gg \tau_R$. We will now elaborate on the equilibrium pion effective theory, before studying the nonequilibrium dynamics.

2. Spontaneous symmetry breaking

Below the critical temperature the system spontaneously condenses. The equilibrium value of the condensate is defined from the limiting procedure [53]

$$\bar{\sigma}_{\text{eq}} \equiv \lim_{H \rightarrow 0} \lim_{V \rightarrow \infty} \langle M_0 \rangle_{\text{eq}}, \quad (22)$$

where V is taken to infinity first, and then H is taken to zero. The mean magnetization drops towards the critical point as

$$\bar{\sigma}_{\text{eq}} = B^- (-t_r)^\beta, \quad (23)$$

where β is given in Table I and the nonuniversal constant B^- and subleading correction were determined in our previous work [37]; see Table II.

Even at zero magnetic field and finite volume the condensate can still be determined from the correlations in the system. The zero mode of the order parameter is decomposed into a magnitude $\sigma(t)$ and a direction $\hat{n}_a(t)$ that generally depends on time

$$M_a(t) \equiv \sigma(t) \hat{n}_a(t). \quad (24)$$

However, in large enough volume and below the critical point the value of σ and the direction $\hat{n}_a$ are nearly constant in time.⁷ Using $\hat{n}_a(t)$ defined from the zero mode, and an additional three orthogonal basis vectors $e_a^\ell(t)$ with $\ell = 1 \dots 3$ defined as follows,

⁷More precisely $\hat{n}_a(t)$ diffuses over the three sphere on a timescale proportional to L^3 with τ_R and ξ making up the remaining dimensions, $\sim \tau_R (L/\xi)^3$ [38]. The equilibration time of the system is proportional to $\tau_R (L/\xi)^2$ and is determined by the damping rate of Goldstone modes (see the main text). Thus, starting from a random initial condition, the mean magnetization can be defined as infinite volume limit of a time average, with time in a restricted range:

$$\bar{\sigma}_{\text{eq}} \hat{n}_a \equiv \lim_{L \rightarrow \infty} \langle M_a \rangle_{\text{eq}} \quad \text{with} \quad \tau_R \left(\frac{L}{\xi} \right)^2 \ll t \ll \tau_R \left(\frac{L}{\xi} \right)^3. \quad (25)$$

As discussed around Eq. (35), there are easier ways to determine the condensate numerically.

$$\begin{aligned} \hat{n}_a e_a^\ell &= 0, \quad e_a^{\ell_1} e_a^{\ell_2} = \delta^{\ell_1 \ell_2}, \\ (\vec{e}_a \times \vec{e}_b)^{\ell_1} &\equiv \varepsilon^{\ell_1 \ell_2 \ell_3} e_a^{\ell_2} e_b^{\ell_3}, \end{aligned} \quad (26)$$

the fields can be decomposed as

$$\phi_a(t, \mathbf{x}) \equiv \sigma(t, \mathbf{x}) \hat{n}_a(t) + \vec{\pi}(t, \mathbf{x}) \cdot \vec{e}_a(t). \quad (27)$$

Here $\vec{\pi} \cdot \vec{e}_a = \pi^\ell e_a^\ell$ and from its definition, $\pi^\ell(t, \mathbf{k} = 0) = 0$. The equal time correlation functions we will need are

$$G_{\sigma\sigma}(k) \equiv V \langle \sigma(t, \mathbf{k}) \sigma(t, -\mathbf{k}) \rangle_{\text{eq}}, \quad (28a)$$

$$G_{\pi\pi}(k) \equiv \frac{V}{3} \sum_{\ell=1}^3 \langle \pi^\ell(t, \mathbf{k}) \pi^\ell(t, -\mathbf{k}) \rangle_{\text{eq}}. \quad (28b)$$

The unequal time correlation functions such as $G_{\pi\pi}(\Delta t, k)$ are defined analogously to Eq. (14). The conserved charges are decomposed as

$$n_{ab}(t, \mathbf{x}) = \vec{n}_A(t, \mathbf{x}) \cdot (\hat{n}_a \vec{e}_b - \hat{n}_b \vec{e}_a) + \vec{n}_V(t, \mathbf{x}) \cdot (\vec{e}_a \times \vec{e}_b) \quad (29)$$

where as above, $\vec{n}_V \cdot \vec{e}_a \equiv n_V^\ell e_a^\ell$. We will not study the correlations of the charges in this work, but refer to our previous work for a detailed study [38].

3. Superfluid hydrodynamics of Goldstone modes

Next, we will review how the equilibrium parameters of the pion dispersion curve in Eq. (18) were extracted from simulations and their theoretical underpinnings. For wavelengths longer than the correlation length $\xi \ll k^{-1}$, the magnitude of the order parameter $\phi_a(t, \mathbf{x})$ is approximately constant, $\sqrt{\phi_a(t, \mathbf{x}) \phi_a(t, \mathbf{x})} \simeq \bar{\sigma}_{\text{eq}}$. Fluctuations of the order parameter are dominated by the changes in the local direction of $\phi_a(t, \mathbf{x})$ indicated with a “spin” vector $s_a(t, \mathbf{x})$, which is normalized to unity $s_a s_a = 1$ and parametrizes the Goldstone modes.

For definiteness, take $\hat{n}_a = (1, 0, 0, 0)$ as the condensate direction. Small fluctuations in the order parameter can be parametrized by three angles $\vec{\varphi}(t, \mathbf{x})$ indicating the deviation of $s_a(t, \mathbf{x})$ from $\hat{n}_a$:

$$\phi_a(t, \mathbf{x}) = (\sigma(t, \mathbf{x}), \vec{\pi}(t, \mathbf{x})) = \bar{\sigma}_{\text{eq}} s_a(t, \mathbf{x}), \quad (30)$$

with

$$s_a \simeq \left(1 - \frac{1}{2} \vec{\varphi}^2(t, \mathbf{x}), \vec{\varphi}(t, \mathbf{x}) \right). \quad (31)$$

Here we defined local angles $\vec{\pi} \simeq \bar{\sigma}_{\text{eq}} \vec{\varphi}$ from the pion field, and made the small angle approximations, $\cos \varphi \simeq 1 - \frac{1}{2} \vec{\varphi}^2$ and $\vec{\varphi} \sin(\varphi)/\varphi \simeq \vec{\varphi}$. The appropriate effective theory for

the Goldstone modes (akin to chiral perturbation theory at zero temperature) is based on parametrizing the fluctuations in spins or angles, and may be recognized as the $SU(2)_L \times SU(2)_R$ generalization of a $U(1)$ superfluid [54]. For the applications discussed in this work (e.g. Fig. 2), the small angle approximation is not enough, but it is sufficient to analyze the pion dispersion curve discussed previously.

The free energy associated with the Goldstone modes and charge fluctuations for $k \ll \xi^{-1}$ is [53]

$$\mathcal{H}[s_a, n] = \int d^d x \frac{f^2}{2} \partial_i s_a \cdot \partial^i s_a + \frac{n_{ab}^2}{4\chi_I}, \quad (32)$$

or for small fluctuations

$$\mathcal{H}[\varphi, n] \simeq \int d^d x \frac{f^2}{2} \partial_i \vec{\varphi} \cdot \partial^i \vec{\varphi} + \frac{\vec{n}_V^2}{2\chi_I} + \frac{\vec{n}_A^2}{2\chi_I}. \quad (33)$$

Here χ_I is the charge susceptibility appearing in Eq. (7) and is constant near the critical point.⁸ However, f^2 , the spatial pion decay constant, is a matching coefficient describing the pion effective theory and is determined by integrating out modes which fluctuate on length scales of order the correlation length ξ , leaving only the Goldstone modes in the effective theory. Thus, f^2 depends critically on the reduced temperature.

Specifically, near the critical point, the pion decay constant scales as $f^2 \propto \xi^{2-d}$ as we now review [51,53]. Using the Gaussian effective theory in Eq. (33), the static correlations of the order parameter at long wavelength can be straightforwardly computed⁹:

$$G_{\pi\pi}(k) \simeq \bar{\sigma}_{\text{eq}}^2 G_{\varphi\varphi}(k) = \frac{\bar{\sigma}_{\text{eq}}^2}{f^2 k^2} \sim \frac{\xi^{2-\eta}}{(k\xi)^2}. \quad (34)$$

For $k\xi \ll 1$, the static pion correlator is enhanced by a factor of $1/(k\xi)^2$ relative to the static sigma correlator, which scales as the magnetic susceptibility $\chi \sim \xi^{2-\eta}$. As $k\xi \rightarrow 1$, the equilibrium pion and sigma fluctuations must become comparable to ensure that the system ultimately realizes chiral symmetry restoration. This requires $G_{\pi\pi}$ to approach $\xi^{2-\eta}$ in this limit, thereby justifying the scaling $f^2 \propto \xi^{2-d}$ *a posteriori* [51].

The value of $\bar{\sigma}_{\text{eq}}$ can be estimated from the equilibrium expectation value, $\langle M^2 \rangle_{\text{eq}}$. The leading deviation of $\langle M^2 \rangle_{\text{eq}}$ from $\bar{\sigma}_{\text{eq}}^2$ at finite volume comes from fluctuations of long wavelength Goldstone modes and can be analyzed with the pion EFT [53]

⁸This is the charge susceptibility χ_I and not the magnetic susceptibility in Eq. (11).

⁹Note that T_c has been set to unity. If the temperature is restored, this equation is written, $G_{\pi\pi}^{\text{eq}} = T_c \bar{\sigma}_{\text{eq}}^2 / f^2 k^2$. The factor of temperature is needed when comparing the $G_{\pi\pi}$ to the pion Bose-Einstein distribution below.

$$\langle M_a M_a \rangle_{\text{eq}} = \bar{\sigma}_{\text{eq}}^2 \left[1 + \frac{0.677355}{f^2 L} + \mathcal{O}((f^2 L)^{-2}) \right]. \quad (35)$$

Here the displayed numerical coefficient and the next term in the series expansion are known analytically in terms of shape coefficients, but are of no interest here. The corrections are organized in inverse powers of $f^2 L \sim L/\xi$, showing again that $f^2 \propto \xi^{-1}$ in three dimensions. Using Eq. (35) we previously extracted the value of $\bar{\sigma}_{\text{eq}}$ and f^2 as a function of reduced temperature, including subleading corrections [38]; see Table II.

Based on this discussion we will define the correlation length below T_c as

$$\xi(t_r) \equiv \frac{\xi^-}{f^2(t_r)} \quad (t_r < 0), \quad (36)$$

where $\xi^- = 0.479(7)$ is chosen so that the correlation length $\xi(t_r) = \xi^+ |t_r|^{-\nu}$ is symmetric around $t_r = 0$, up to subleading temperature corrections shown in Table II. Furthermore, from Eq. (35) we get a practical fit to our numerical results used in the plots

$$\sqrt{\langle M_a M_a \rangle_{\text{eq}}} \approx \bar{\sigma}_{\text{eq}} \left(1 + 0.706 \frac{\xi}{L} \right). \quad (37)$$

The linearized effective hydrodynamic equations of motion for the pions and axial charge follow from the free energy in Eq. (33), the conservation laws and the Josephson constraint, and a dissipative derivative expansion. They take the form [44,51,52]

$$\partial_t \vec{\varphi} - \frac{\vec{n}_A}{\chi_I} = \frac{\Gamma}{f^2} \nabla \cdot (f^2 \nabla \vec{\varphi}), \quad (38a)$$

$$\partial_t \vec{n}_A - \nabla \cdot (f^2 \nabla \vec{\varphi}) = D \nabla^2 \vec{n}_A. \quad (38b)$$

Γ and D are defined by these equations, i.e. they are the dissipative parameters of the pion hydro EFT presented in Eq. (38). They scale with temperature as $\xi^{2-d/2} \sim \xi^2 / \tau_R$ close to the critical point and they are distinct (and of parametrically different size) from the temperature independent bare parameters, Γ_0 and D_0 in Eq. (8). However, in mean-field theory with $d = 4$, Γ and D are independent of ξ and then reduce to Γ_0 and D_0 , justifying the notation.

The linearized hydrodynamic system in Eq. (38) is easily solved and the eigen-waves have dispersion relations

$$\omega(k) = \pm vk - \frac{i}{2} D_A k^2, \quad (39)$$

where $D_A = \Gamma + D$. By systematically analyzing the unequal time correlators $G_{\pi\pi}(\Delta t, k)$ we previously extracted the pion dispersion curve numerically with high precision [38].

Finally, it is worth recording the ideal $SU(2)_L \times SU(2)_R$ superfluid equations of motion in the fully nonlinear regime,¹⁰ following Son [54]. These equations can be determined by writing down the Poisson brackets between the spins s_a and the conserved charges n_{ab} , which are fixed because the charges generate $O(4)$ rotations [43], e.g.

$$\{n_{ab}(\mathbf{x}), s_c(\mathbf{y})\} = (\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad})s_d(\mathbf{x})\delta^3(\mathbf{x} - \mathbf{y}). \quad (40)$$

Using the effective spin Hamiltonian in (32) and these Poisson brackets, the equations stemming from $\dot{s}_a = \{s_a, \mathcal{H}\}$ and $\dot{n}_{ab} = \{n_{ab}, \mathcal{H}\}$ read

$$\partial_t s_a + \frac{n_{ab}}{\chi_I} s_b = 0, \quad (41a)$$

$$\partial_t n_{ab} + \partial_i (f^2 \partial^i s_{[a} s_{b]}) = 0. \quad (41b)$$

In the linearized limit with $s_a \simeq (1, \vec{\varphi})$ the equations reproduce (38) without the dissipative terms. (The dissipative equations in the nonlinear regime are given in [44,55].) As we will see, the nonlinear ideal superfluid theory is responsible for the growth of the chiral condensate following a quench.

4. Quasiparticle description of pion waves

In the next section we will study how the medium evolves after a sudden quench. The pion dynamics post-quench will be analyzed through the time dependence of the equal-time correlation function

$$G_{\pi\pi}(t, k) \equiv \frac{V}{3} \langle \pi^\ell(t, \mathbf{k}) \pi^\ell(t, -\mathbf{k}) \rangle. \quad (42)$$

Now the brackets $\langle \dots \rangle$ notate an average over a rotationally invariant ensemble of initial conditions, determined by the quench. Here we will relate the $G_{\pi\pi}(t, k)$ to the pion yield $n_\pi(t, k)$, which evolves to equilibrium, $n_\pi^{\text{eq}}(k)$, after the quench.

After a global condensate has approximately formed, the linearized fluctuations of $\phi_a(t, \mathbf{x})$ are parametrized by the angles $\vec{\varphi}(t, \mathbf{x})$, and the hydrodynamic equations of motion are given by Eq. (38). Neglecting the dissipative dynamics, the linearized equations of motion in Eq. (38) are similar to a $U(1)$ superfluid—Eq. (38a) is the Josephson constraint relating the time derivative of the phase to the chemical potential, $\partial_t \vec{\varphi} = \vec{\mu}_A \equiv \vec{n}_A / \chi_I$ [51].

An action reproducing the (ideal) equations of motion given in (33) is [44,56]

$$S = \int d^4x \frac{1}{2} \chi_I (\partial_t \varphi)^2 - \frac{1}{2} f^2 (\nabla \varphi)^2, \quad (43)$$

which clearly describes a wave with dispersion curve $\omega(k) = vk$ and squared velocity, $v^2 = f^2 / \chi_I$. The action is appropriately normalized: the Lagrange density is identified as the superfluid contribution to the pressure [56]. Further, the canonical momentum conjugate to the phase $\vec{\varphi}(t, \mathbf{x})$ is the charge density $\vec{n}_A(t, \mathbf{x})$ and a Legendre transform of the Lagrangian yields the axial part of the free energy, $\mathcal{H}$, previously introduced in (33). Examining the action and noting that $\vec{\varphi} = \vec{\pi} / \bar{\sigma}_{\text{eq}}$, we see that the rescaled pion field $(\sqrt{\chi_I} / \bar{\sigma}_{\text{eq}}) \vec{\pi}$ is canonically normalized, and may be expanded in terms of plane wave solutions:

$$(\sqrt{\chi_I} / \bar{\sigma}_{\text{eq}}) \pi^\ell(t, \mathbf{x}) = \sum_{\mathbf{k}} a_{\mathbf{k}}^\ell \frac{e^{-i\omega(k)t + i\mathbf{k} \cdot \mathbf{x}}}{\sqrt{2\omega(k)V}} + a_{\mathbf{k}}^{\ell*} \frac{e^{i\omega(k)t - i\mathbf{k} \cdot \mathbf{x}}}{\sqrt{2\omega(k)V}}. \quad (44)$$

Substituting the mode expansion into (28b) yields a direct relation between $G_{\pi\pi}$ and the pion yield $n_\pi(t, k)$

$$\frac{\chi_I}{\bar{\sigma}_{\text{eq}}^2} G_{\pi\pi}(t, k) = \frac{n_\pi(t, k)}{\omega(k)}, \quad (45)$$

where we identified the (isospin averaged) pion yield as

$$n_\pi(t, k) = \frac{1}{3} \sum_{\ell=1}^3 \langle |a_{\mathbf{k}}^\ell|^2 \rangle. \quad (46)$$

In writing Eq. (45) we dropped rapidly oscillating terms, $\sim e^{-i2\omega(k)t}$, in a quasiparticle (or kinetic) approximation. If $G_{\pi\pi}(t, k)$ is averaged over a narrow momentum range Δk with $\Delta k \ll k$ and $v\Delta kt \gtrsim 1$ these oscillating terms average to zero. A more sophisticated treatment would match the time evolution of the two-point functions of the classical fields onto a Boltzmann description [44], but this analysis is left for future work.

When $G_{\pi\pi}$ takes the equilibrium form given in (34), $\chi_I G_{\pi\pi}^{\text{eq}} / \bar{\sigma}_{\text{eq}}^2$ evaluates to

$$\frac{\chi_I}{\bar{\sigma}_{\text{eq}}^2} G_{\pi\pi}^{\text{eq}}(t, k) = \frac{T}{\omega(k)^2} = \frac{n_\pi^{\text{eq}}(k)}{\omega(k)}. \quad (47)$$

In the last step we recognized the classical part of the Bose-Einstein distribution,

$$n_\pi^{\text{eq}}(k) = \frac{1}{e^{\omega(k)/T} - 1} \simeq \frac{T}{\omega(k)}, \quad (48)$$

and thus (47) further corroborates the identification in (45).

¹⁰The equations and steps are equivalent to [54]. However, the final equations look rather different because we are using a notation based on $O(4)$ symmetry rather than the $SU_L(2) \times SU_R(2)$ notation adopted by Son.

In what follows we will monitor the ratio $G_{\pi\pi}(t, k)/G_{\pi\pi}^{\text{eq}}(k) = n_{\pi}(t, k)/n_{\pi}^{\text{eq}}(k)$ to evaluate the time evolution of the soft-pion yield out of equilibrium.

III. OUT-OF-EQUILIBRIUM DYNAMICS: QUENCHES

A. Qualitative picture and scaling analysis

In this work we are concerned with the out-of-equilibrium dynamics of Model G, particularly the approach to equilibrium of the Goldstone modes. We will start by studying the case of instantaneous quenches from the high-temperature restored phase to the low temperature broken phase. By suddenly driving the system with a quench to the broken phase, we isolate the dynamics that lead to the development of the condensate and its impact on the conserved charges and the Goldstone modes.

Specifically, we initialize our fields in thermal equilibrium specified by reduced temperature $t_r = t_r^0 > 0$, defined in Eq. (5). Then, at time zero, the reduced temperature is instantaneously changed to $-t_r^0$ and the system relaxes to the new equilibrium. The illustration in Fig. 1 shows the transition and the equilibrium correlation length above (13) and below (36) the critical temperature [38].

The dynamics of the condensate and its excitations can be analyzed through the time evolution of equal time correlation functions such as

$$G_0(t) \equiv \frac{V}{4} \sum_{a=1}^4 \langle M_a(t) M_a(t) \rangle, \quad (49)$$

which is the nonequilibrium analog of χ . The average here and below $\langle \dots \rangle$ reflects an average over the ensemble of initial conditions, and is distinct from equilibrium average $\langle \dots \rangle_{\text{eq}}$ of the previous sections. All other equal time correlation functions such as $G_{\pi\pi}(k)$ and $G_{\sigma\sigma}(k)$ defined in Eq. (28) become functions of time, e.g.,

$$G_{\pi\pi}(k) \rightarrow G_{\pi\pi}(t, k). \quad (50)$$

We will focus on the time evolution of equal time correlation functions in this work.

With our conventions, correlation functions such as G_0 and $G_{\pi\pi}$ are independent of volume when the field is random,¹¹ i.e., the field is correlated over domains of

¹¹More explicitly, our Fourier transform $\phi(\mathbf{k}) \equiv \frac{1}{V} \sum_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{x})$ was defined with a prefactor of V^{-1} . Thus, $\phi(\mathbf{k})$ scales as a volume average of $\phi(\mathbf{x})$ for $kL \sim 1$. If $\phi(\mathbf{x})$ is a random field with domains of volume $\ell^d \ll V$, then $\phi(\mathbf{k})$ is an average formed with $N_{\text{samp}} \propto V/\ell^d$ independent samples. Statistically, this average decreases like $1/\sqrt{N_{\text{samp}}}$ or $\sqrt{\ell^d/V}$. When defining correlation functions, e.g., $G \equiv V \langle \phi(\mathbf{k}) \phi(-\mathbf{k}) \rangle$, we multiply by V to have a volume-independent correlation function for random fields.

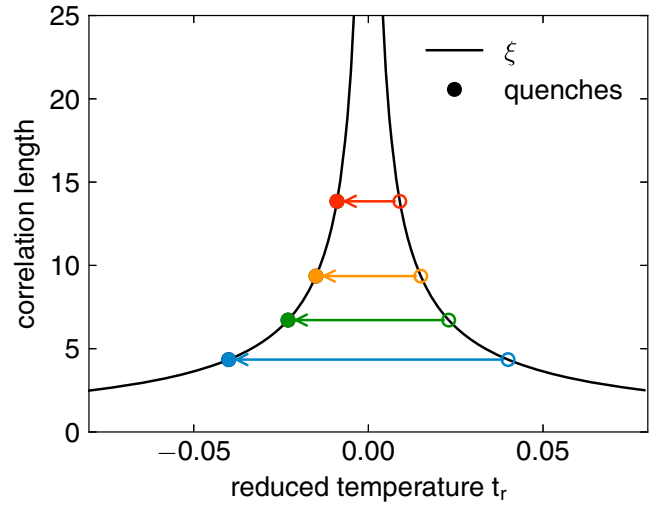


FIG. 1. Illustration of the quench protocol in terms of correlation length ξ . The quenches studied in Sec. III transition from the high temperature phase to the low temperature phase, $t_r^0 \rightarrow -t_r^0$ with $\xi(t_r^0) = \xi(-t_r^0)$ up to subleading corrections; see Table II. The colors from top to bottom correspond to $t_r^0 = 0.009, 0.015, 0.023, 0.04$ and are correlated with the lines in the upper panels of Figs. 4–6. In the lower panels of Figs. 4–7 we vary the lattice size while $t_r^0 = 0.04$.

volume ℓ^d much smaller than V . Correlation functions of “condensate like” fields, i.e., fields with domains of volume $\sim V$, scale linearly in the volume.

The quench dynamics is illustrated schematically in Fig. 2. Consider a local spatial region of length ℓ , several correlation lengths long. In the restored phase, before the quench, the potential is shown by the red dashed line, while after the quench the potential takes the familiar wine-bottle form. The order parameter averaged in this region is notated $\bar{\phi}_a$, and its value before the quench fluctuates around zero, with variance order χ/ℓ^d (see figure). After the quench, $\bar{\phi}_a$ now sits at the top of the hill and is unstable. Over a time of order $t \sim \tau_R$, fluctuations source an unstable growth of the local order parameter, which rolls down the hill in a random direction, forming a locally equilibrated domain. At this stage, the global condensate is still approximately zero, since the local condensate orientations are random, as shown in Fig. 2(b). Over a much longer timescale these the domains begin to merge to form the global condensate. As discussed further below, since the distances and times involved are much larger than the correlation length ξ and the relaxation time τ_R respectively, hydrodynamics is the right tool to describe the growth in this stage. The appropriate theory was given in Sec. II B 3 and describes a non-Abelian superfluid with broken $SU(2)_L \times SU(2)_R$ symmetry.

Scaling considerations let us go beyond this qualitative understanding and make sharp predictions on the effect of this nonequilibrium evolution on the correlators. At time $t = 0$ (the “top of the hill”), the system is in equilibrium in

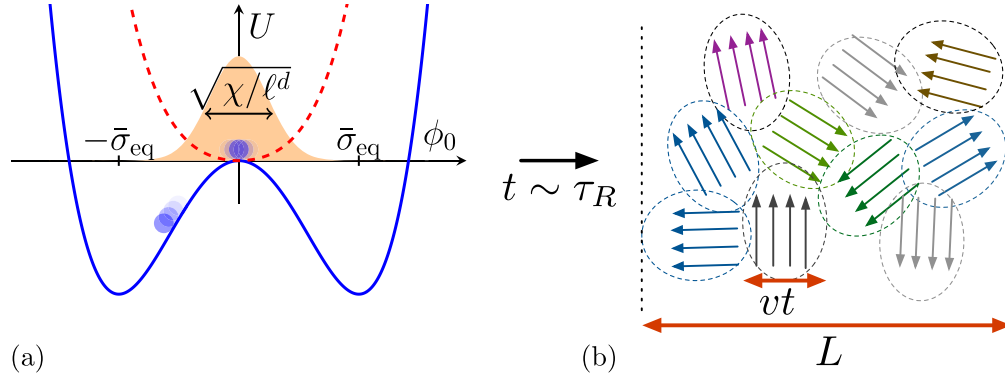


FIG. 2. (a) Schematic of a quench from the high-temperature phase to the broken phase in a spatial region of size ℓ , which is several correlation lengths long (e.g., $\ell \sim 3\xi$). The red dashed line shows the initial potential at temperature t_l^0 , and the blue line shows the post-quench potential. The ball represents a local order parameter $\bar{\phi}_a(t)$, averaged over the region. The orange Gaussian depicts the initial thermal distribution for $\bar{\phi}_a$ before the quench, with variance χ/ℓ^d . Over a time of order $t \sim \tau_R$, $\bar{\phi}_a$ evolves to the well's bottom in a random direction, forming a local equilibrated domain of size ℓ . (b) Field configuration for $t \gg \tau_R$ in a region of size $L \gg \ell \sim \xi$, showing randomly oriented domains of the chiral condensate. The domains merge over a time of order $t \sim L/v$ with dynamics governed by the ideal superfluid equations given in (41).

the restored phase and the correlator reflects the magnetic susceptibility:

$$G_0(t)|_{t=0^-} = \chi \sim \xi^{2-\eta}. \quad (51)$$

This means that at $t = 0$ the fluctuations of M_a are of order $\sim \sqrt{\chi/V}$. By contrast, at late times a single randomly oriented domain is formed with $M_a = \bar{\sigma}_{\text{eq}} \hat{n}_a$. This implies that

$$\lim_{t \rightarrow \infty} G_0(t) = \frac{V}{4} \bar{\sigma}_{\text{eq}}^2 \sim \xi^{2-\eta} \left(\frac{L}{\xi} \right)^d. \quad (52)$$

Thus, G_0 grows by a factor which is a ratio of these limiting values:

$$\text{amplification} = \frac{V \bar{\sigma}_{\text{eq}}^2}{4\chi} \sim \left(\frac{L}{\xi} \right)^d. \quad (53)$$

This factor will appear frequently below.

The scaling hypothesis quite generally dictates the dependence of G_0 on time and the correlation length

$$G_0(t, \xi, L) = \xi^{2-\eta} \mathcal{F}(t/\tau_R, \xi/L), \quad (54)$$

where here and below $\mathcal{F}$ denotes some dimensionless function, distinguished by context and arguments. For $\xi/L \ll 1$, Eq. (54) suggests a naive scaling form

$$G_0(t, \xi) = \xi^{2-\eta} \mathcal{F}(t/\tau_R) \quad \text{for } t \sim \tau_R. \quad (55)$$

The naive scaling form in (55), which is valid for $t \sim \tau_R$ and is independent of the system size, describes the formation of the local domains as discussed in Fig. 2(a). However, this

naive scaling drops secular terms which become important at late times when $t \sim \tau_R(L/\xi)$.

Indeed, as described above G_0 must be of order $(L/\xi)^d$ at late times. We anticipate that the condensate will grow in the time it takes for a pion to propagate across the box, $t \sim L/v \sim \tau_R(L/\xi)$. Recognizing these scalings, we reshuffle the parameters in Eq. (54) into a new dimensionless function

$$G_0(t, \xi, L) = \xi^{2-\eta} \left(\frac{L}{\xi} \right)^d \mathcal{F}(vt/L, \xi/L). \quad (56)$$

So far, this is a trivial rewrite of Eq. (54). Now, however, the new function is expected to be regular and order one as $L \rightarrow \infty$ with $vt/L \sim 1$ fixed, and its dependence on ξ/L can be dropped. Thus, at large volumes and at late times $vt/L \sim 1$ we expect a scaling form for the time evolution of the magnetization

$$G_0(t, \xi, L) = \xi^{2-\eta} \left(\frac{L}{\xi} \right)^d \mathcal{F}(vt/L), \quad \text{for } t \sim L/v. \quad (57)$$

The scaling form in Eq. (57) is valid at intermediate times $t \sim L/v$ when ballistic transport of pions is operative as indicated in Fig. 2(b). However, the scaling form does not capture the late-time equilibration dynamics of the Goldstone modes, which happens on the diffusive timescale, $t \sim \tau_R(L/\xi)^2 \sim (L/v)(L/\xi)$. In Eq. (57) the approach to equilibrium at late times has been discarded by ignoring possible secular terms in passing from (56) to (57).

At early times, but still with $t \gg \tau_R$, the global condensate has not formed. Then, G_0 is a correlation function of a random field, a field consisting of domains of volume $\sim (vt)^d \ll V$, i.e. Fig. 2(b) with small domains. In this regime G_0 will be independent of the system volume. As a

result, for $vt/L \ll 1$ we must have $F(vt/L) \propto (vt/L)^d$ to cancel the leading power of L^d in (57). Using the estimate $v \sim \xi/\tau_R$, we thus extract the early-time behavior of G_0 in the overlap region of (55) and (57)

$$G_0(t, \xi, L) \propto \xi^{2-\eta} \left(\frac{t}{\tau_R} \right)^d \quad \tau_R \ll t \ll \frac{L}{v}. \quad (58)$$

The timescale L/v of the scaling regime in (57) is much longer than the relaxation time τ_R . The distance scale $vt \sim L$ is much longer than the correlation length ξ . Thus, the scaling regime in (57) is a hydrodynamic regime. The appropriate hydrodynamic theory is given by an $SU(2)_L \times SU(2)_R$ superfluid described in Sec. II B 3, which accounts for the broken symmetry. As discussed above, the scaling form in (57) ignores the dissipative dynamics and thus, the relevant hydrodynamics is the nondissipative (but non-linear) superfluid dynamics given in (41). In the future, it should be possible to simulate these equations directly to describe the condensate growth.

Scaling forms for condensate growth, similar to Eq. (57), are common in analyses of coarsening in other critical systems [47,57]. However, the linear growth of the domain size, $\sim vt$, is a distinctive feature of the superfluid limit of Model G, ultimately arising from its reversible Poisson bracket structure.

The critical scaling in Eq. (57) can also be extended to small but finite momenta. For simplicity, assume that the $\mathbf{k} = \mathbf{0}$ mode, $\phi_a(\mathbf{0}) \equiv M_a$, is pointing in the $\hat{n}_a = (1, 0, 0, 0)$ direction. We then analyze order parameter fluctuations $\phi_a(\mathbf{k}) = (\sigma(\mathbf{k}), \vec{\pi}(\mathbf{k}))$ at momenta satisfying $kL \sim 1$. For $t \lesssim L/v$, at the boundary of applicability of (58), pion waves have not fully traversed the box or other distances of order $\sim 1/k$, see Fig. 2(b). As a result, the zero mode and other long wavelength modes have not had time to dramatically influence each other and form a global condensate. Both are primarily produced from random superpositions of smaller scales [see Fig. 2(b)]. Consequently, the fluctuations of $\phi_a(\mathbf{k})$ will be the same order of magnitude as the zero mode, $\phi_a(\mathbf{0})$. In addition, the decomposition of $\phi_a(\mathbf{k})$ into $(\sigma(\mathbf{k}), \vec{\pi}(\mathbf{k}))$, which is based on the direction of the zero mode $\hat{n}_a$, is not physically significant and all components of $\phi_a(\mathbf{k})$ are the same order of magnitude. Thus, the pion and sigma correlation functions will follow a scaling form similar to the zero mode, for example:

$$G_{\pi\pi}(t, \xi, k, L) = \xi^{2-\eta} \left(\frac{L}{\xi} \right)^d \mathcal{F}(vkt, kL). \quad (59)$$

Here we recognized that $\mathcal{F}(vkt, kL)$ is an order-one function when $kL \sim 1$ and $vkt \sim 1$. For later convenience, we replaced $1/L$ in the time argument of (57) with the wave number k of the relevant mode, writing $vkt = (vt/L)(kL)$.

Now in the intermediate range

$$L^{-1} \ll k \ll \xi^{-1}, \quad (60)$$

and for times $t \sim (vk)^{-1}$, which are short compared to L/v , the correlation function must again become independent of the system volume. Thus, we anticipate that for $kL \gg 1$, the function $\mathcal{F}(vkt, kL)$ simplifies, $\mathcal{F}(vkt, kL) \rightarrow \mathcal{F}(vkt)/(kL)^d$, leading to the scaling form

$$G_{\pi\pi}(t, \xi, k) = \frac{\xi^{2-\eta}}{(k\xi)^d} \mathcal{F}(vkt). \quad (61)$$

Again, this description is only valid for $t \sim (vk)^{-1}$ and does not capture the diffusive behavior at late times, $t \sim \tau_R/(k\xi)^2 \sim 1/(vk)1/(k\xi)$.

The prediction in Eq. (61), which follows from scaling and the dynamics of Model G, is quite striking. After noting that the equilibrium correlation function in Eq. (34) scales as $G_{\pi\pi}^{\text{eq}} \sim \xi^{2-\eta}(k\xi)^{-2}$, we see that the equation predicts a significant enhancement over equilibrium. In the quasiparticle regime discussed in Sec. II B 4 where $vkt \gtrsim 1$, this translates to the parametric enhancement of soft pion spectra

$$\frac{G_{\pi\pi}(t, k)}{G_{\pi\pi}^{\text{eq}}(k)} \simeq \frac{n_{\pi}(t, k)}{n_{\pi}^{\text{eq}}(k)} \sim \frac{1}{k\xi} \quad \text{with} \quad k\xi \ll 1. \quad (62)$$

This enhancement remains substantial over an extended timescale, $\tau_R/(k\xi) \lesssim t \ll \tau_R/(k\xi)^2$, demonstrating a parametrically long-lived deviation from equilibrium.

All of the scaling predictions discussed here are observed in our numerics, as presented in the next section. Before moving on, we summarize the situation with Fig. 3, which shows a schematic of a pion correlator $G_{\pi\pi}$ obtained from our numerics on a logarithmic time axis. The three relevant timescales $t \sim \tau_R$ (initial conditions), $t \sim \tau_R/(k\xi)$ (ballistic transport) and $t \sim \tau_R/(k\xi)^2$ can be clearly seen from the behavior of the correlator. The $1/k\xi$ enhancement over equilibrium during the ballistic phase is particularly striking. Again, since the timescale in the ballistic regime is long compared to τ_R , the superfluid hydrodynamic equations reflecting the broken $O(4)$ symmetry in (41) determine the dynamics of this regime.

B. Lattice simulations of quenches

In this section, we gather all our different numerical simulations of quenches and show that the above scaling analysis holds. To give a sense of the numerical effort, consider the orange quench in Fig. 1, which produces orange curves in the top two panels of Fig. 4. The relaxation time, $\tau_R \propto \xi^{d/2}$, is of order $\tau_R \sim 150(a^2/\Gamma_0)$. The box size and correlation length are of order $(L/\xi) \sim 20$ and $\xi \sim 10a$ respectively. The relaxation time is large compared to

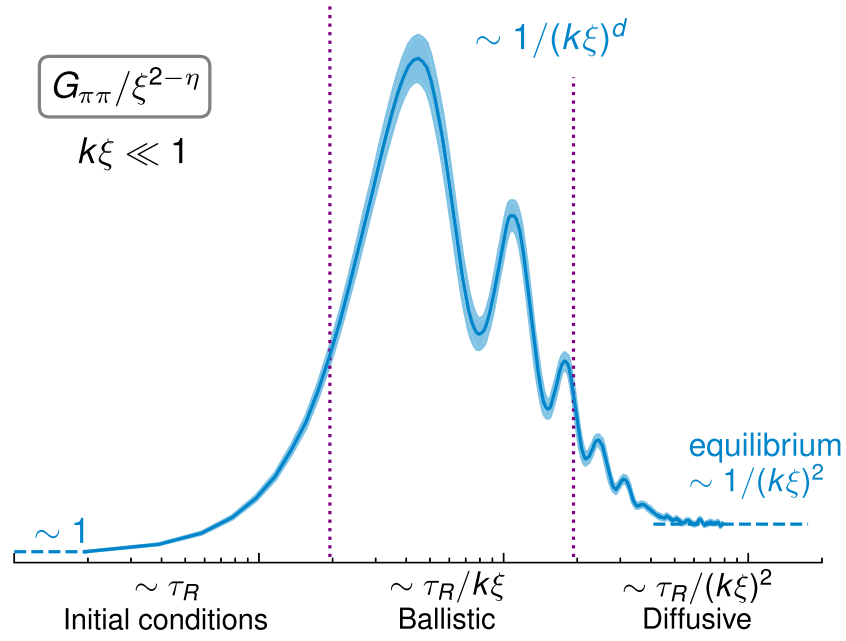


FIG. 3. A sketch of different stages in the typical evolution of the (pion) two-point function: early evolution setting the initial conditions $t \sim \tau_R$, rapid growth until the maximum, and slow decay to its equilibrium value in an oscillatory manner. See Sec. III A for detailed discussion.

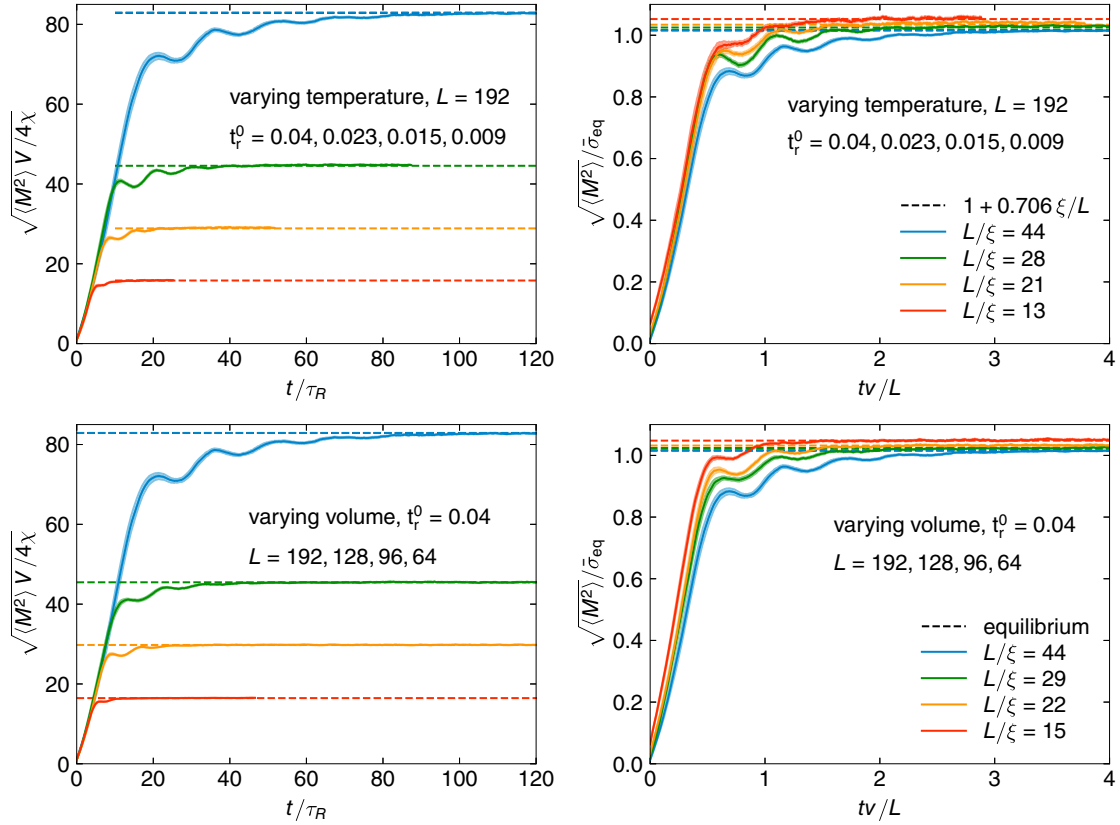


FIG. 4. Time evolution of magnetization for symmetric quenches in Fig. 1, with (top) different starting t_r^0 , but same lattice size $L = 192$ and (bottom) with $t_r^0 = 0.04$ and different lattice sizes. All curves are normalized by the initial equilibrium value, Eq. (11). Colors indicate the system size in units of correlation length L/ξ . The time is given in units of relaxation time $t/\tau_R \propto t/\xi^{d/2}$ and pion ballistic propagation time L/v . Dashed lines show the expected equilibrium value with the leading order finite-size correction Eq. (35).

a^2/Γ_0 because we are close to the critical point and the dynamics shows critical slowing down. A typical event is first thermalized by evolving for $t \sim 1200(a^2/\Gamma_0)$, with time steps of $0.24(a^2/\Gamma_0)$ as discussed above. After the quench, the system is evolved for $\sim 50\tau_R$, which is ample time for the longest wavelength Goldstone mode to relax to equilibrium for $L/\xi \sim 20$. The relaxation time for the longest Goldstones is of order $\sim \tau_R L^2/(2\pi\xi)^2$. The events are run on single NERSC CPU node consisting of 128 cores. The curves in Fig. 4 represent an average over ~ 40 events.

1. Magnetization evolution

First, in Fig. 4 we show the time evolution of the magnetization, $\sqrt{\langle M^2(t) \rangle}$. Here and below, the brackets $\langle \dots \rangle$ denote an average over events. Before the quench, when the system is in equilibrium, $\sqrt{\langle M^2 \rangle} = \sqrt{4\chi/V}$, where $\chi(t_r)$ is the magnetic susceptibility above T_c . Accordingly, in Fig. 4 (left column), the y axis is normalized by this equilibrium value so that all curves start at the same point. After the quench, the system relaxes and approaches its new equilibrium value given by $\bar{\sigma}_{\text{eq}}(t_r)$ up to finite corrections, Eq. (35). In Fig. 4 (right column), the y axis is normalized by $\bar{\sigma}_{\text{eq}}$. Here and throughout, the rescalings use the parametrizations displayed in Table II for $\chi(t_r)$, $\bar{\sigma}_{\text{eq}}(t_r)$, and other quantities.

In the top panels of Fig. 4 we keep lattice size $L = 192$ fixed and vary the initial reduced temperature t_r^0 with line colors correlated with Fig. 1. In the lower panels, we fix $t_r^0 = 0.04$, but change the lattice size. The almost identical curves illustrate the fact that the only relevant parameter is the system size in units of correlation length, i.e., L/ξ , which is the same in both panels.

The left panels focus on early-time dynamics. By rescaling the x axis by τ_R ,¹² all curves collapse at early times. This corresponds to the first regime of interest $t \sim \tau_R$ mentioned in the previous sections, Eqs. (55) and (58). The right panels focus on the ballistic regime $t \sim L/v$ captured by the scaling analysis of the previous section. Rescaling the y axis by $\bar{\sigma}_{\text{eq}}$, we expect to see all curves eventually to diffuse back to one, up to the finite volume corrections predicted by the poin EFT and given in (35). More interestingly, rescaling the time axis by v/L confirms the main hypothesis in our scaling analysis: intermediate timescales are described by a universal, regular function $\mathcal{F}(tv/L)$. This can be seen by the data collapse, which is hindered to fully happen only by the finite volume corrections of order ξ/L .¹³

¹²We assume use the relaxation time τ_R is symmetric around $t_r = 0$ and use the parametrization for $T > T_c$ in Table II.

¹³Note that we could have also rescaled the y axis by M_{eq}^2 instead of $\bar{\sigma}_{\text{eq}}$ and obtained a better collapse. We refrained from doing so as strictly speaking the scaling analysis presented in the previous section does not control the finite volume corrections responsible for the difference between these two quantities.

2. Pion and sigma correlation functions

Next, in Fig. 5 we show the evolution of pion-pion correlation function $G_{\pi\pi}(t, k)$ for the lowest momentum mode $k = 2\pi/L$, normalized by its equilibrium value. Analogously to Fig. 4, we vary the reduced temperature (upper row) and lattice size (lower row). Again, only the value of $k\xi$ is relevant to characterize the behavior of the correlator, which is the same in all panels. Normalizing by τ_R , we see again universality in the initial regime shown by the left two panels, although the scaling is not quite as good as the condensate previously shown in Fig. 4.

The ballistic transport phase is further studied in the second column, where we now multiply the correlator ratio by $k\xi$ and plot it as a function vkt . [Since the equilibrium correlator scales as $G_{\pi\pi}^{\text{eq}} \propto 1/k^2$, multiplying $G_{\pi\pi}/G_{\pi\pi}^{\text{eq}}$ by $k\xi$ removes the leading $1/(k\xi)^d$ factor from the scaling form in (61).] The data collapse is impressive, fully confirming the scaling prediction. The $1/k\xi$ enhancement over the equilibrium value is clear.

In Fig. 6 we repeat the same analysis for the sigma-sigma correlation function, $G_{\sigma\sigma}(t, k)$. In equilibrium, this correlator relaxes to nearly zero. Therefore we normalize the y axis with the pion equilibrium correlator $G_{\pi\pi}^{\text{eq}} = k^{-2}\bar{\sigma}^2/f(t_r)$, instead of $G_{\sigma\sigma}^{\text{eq}}$. As argued in the scaling section, the initial dynamics proceeds in the same way until the condensate is well formed, further motivating scaling $G_{\pi\pi}$ and $G_{\sigma\sigma}$ by the same factor. The drastic difference between the pion and sigma channel is of course that the sigma mode does not have long-lived modes, and thus quickly decays back to equilibrium. Rescaling the time with the ballistic time v/L and the amplitude with $1/(k\xi)$ leads to a reasonable data collapse.

Next we study the dependence of the pion and sigma correlations on kL at fixed $k\xi$. In Fig. 7 we show pion-pion and sigma-sigma correlation functions for fixed temperature and fixed wave number $k = n \times 2\pi/L$, but varying system size. That is for lattice sizes $L = 64, 128$, and 192 we show the $n = 1$, $n = 2$, and $n = 3$ modes correspondingly. Figure 7(a) shows the pion channel. The resulting curves are approximately independent of kL at fixed $k\xi$. However, close inspection shows that the higher Fourier modes, e.g., $k = 3(2\pi)/L$, exhibit weaker oscillations than the $n = 1$ curve. Indeed, at larger n , a specified wave number can mix with many other modes of similar frequency, leading to decoherence. In the limit where $kL \gg 1$ with fixed $k\xi$, we expect that the oscillations seen in the $n = 1$ mode of Fig. 5 will disappear entirely.

Figure 7(b) shows the sigma channel. In general the σ can mix with two pions, making it difficult to define the screening mass m_σ below T_c [58]. In real time, if the frequencies are commensurate, this mixing introduces slowly decaying contributions to the sigma channel. For $n = 2$ and $n = 3$ modes we see significant corrections compared to $n = 1$ behavior. Indeed, the sigma correlator

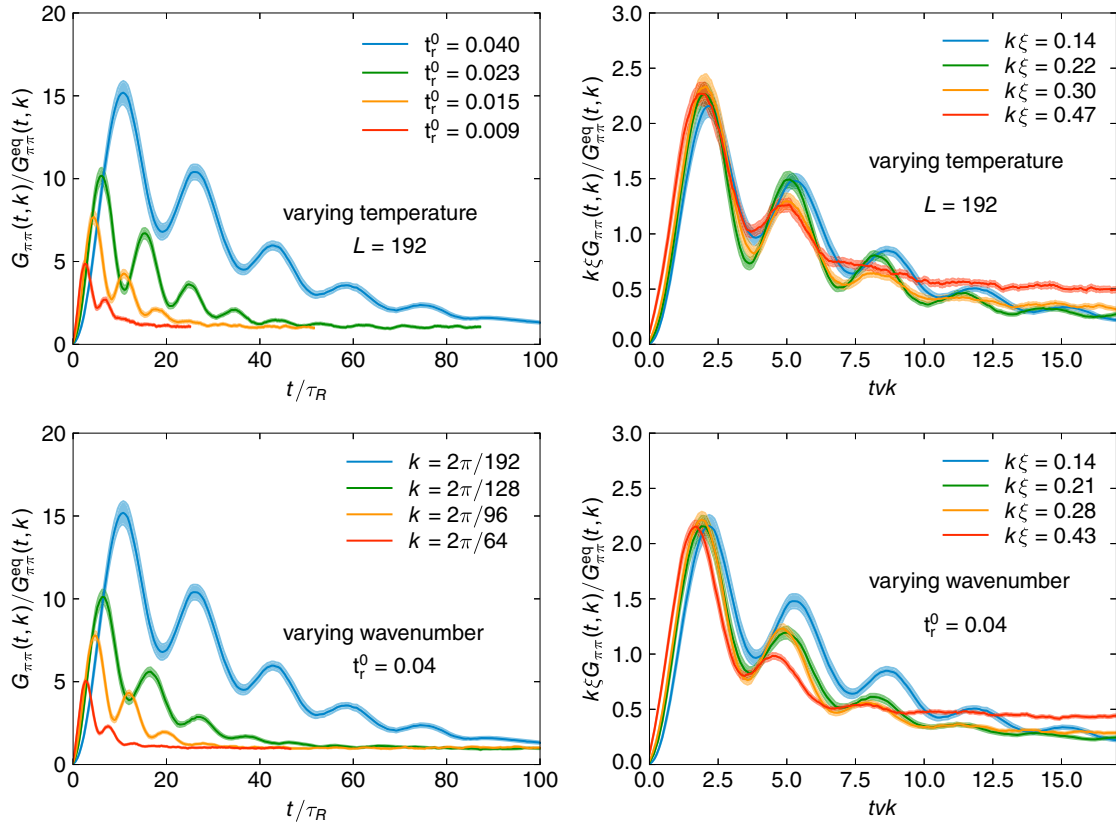


FIG. 5. Top left: enhancement of equal time pion-pion field correlator over thermal equilibrium expectation during symmetric quench for different starting t_r^0 and wave numbers $k = 2\pi/L$, where $L = 192$. The time is given in units of relaxation time $t/\tau_R \propto t/\xi^{d/2}$. Top right: the same plot multiplied with $k\xi$ on the y axis and time given in units of pion ballistic propagation time $L/v \sim \tau_R(L/\xi)$, where v is the pion velocity at corresponding $t_r = t_r^0$. Bottom left and right: enhancement of pion-pion field correlator over thermal equilibrium expectation during symmetric quench from $t_r = 0.04$ to $t_r = -0.04$ for different wave numbers $k = 2\pi/L$, where $L = 192, 128, 96, 64$.

decays on the same timescale as the pion-pion correlation function shown on the left, suggesting that this mixing is at work.

C. Mean field description of quenches

1. Overview and preliminaries

In this section, we will analyze the quench dynamics in mean-field theory, which provides a simple analytic model for the dynamics. However, because of its simplicity the mean-field description fails in several respects, both qualitatively and quantitatively.

In mean-field theory, the zero mode is separated from the remaining modes, which are then linearized around the zero mode, which is in general time dependent. The equations of motion are presented in Appendix B and [45] where the equilibrium properties of the model have been studied previously.

Briefly, to set notation, the static susceptibility above T_c follows from the mean-field free energy, which is the quadratic part of the Landau-Ginzburg free energy in Eq. (1)

$$G_{\phi\phi}(k) = \frac{1}{k^2 + m_0^2} = \frac{\chi}{(k\xi)^2 + 1}. \quad (63)$$

In the last step we identified the mean-field correlation length

$$\xi \equiv \frac{1}{|m_0|} \propto t_r^{-1/2} \quad (64)$$

and the mean-field susceptibility

$$\chi \equiv \lim_{k \rightarrow 0} G_{\phi\phi}(k) = \frac{1}{m_0^2} \equiv \xi^2. \quad (65)$$

The condensate below T_c follows by minimizing the potential $U(\phi)$ in Eq. (2),

$$\bar{\sigma}_{\text{eq}} = \frac{|m_0(t_r)|}{\sqrt{\lambda}} \propto \xi^{-1}. \quad (66)$$

From these relations we deduce familiar mean-field exponents $\beta = \nu = \frac{1}{2}$ and $\eta = 0$. These are consistent with hyperscaling in Table I only in $d = 4$.

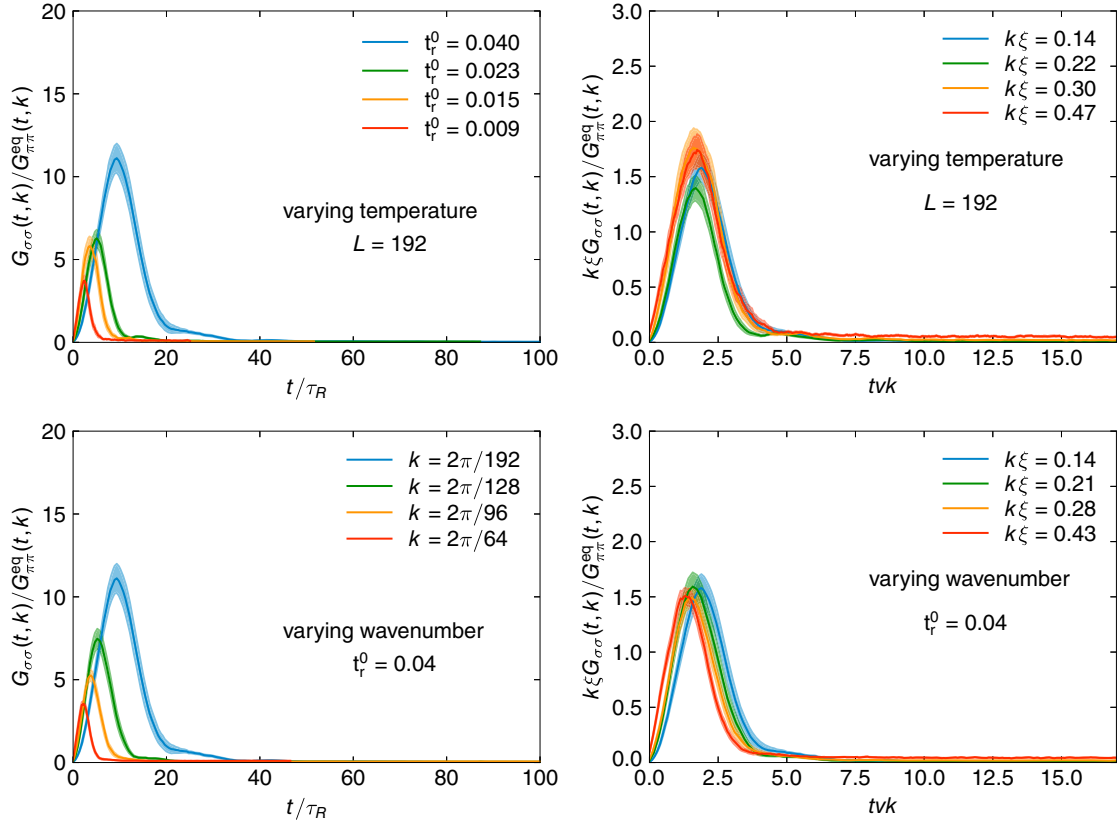


FIG. 6. Top left: enhancement of the equal time sigma-sigma field correlator over thermal equilibrium expectation of pion-pion correlator during symmetric quench for different starting t_r^0 and wave numbers $k = 2\pi/L$, where $L = 192$. The time is given in units of relaxation time $t/\tau_R \propto t/\xi^{d/2}$. Top right: the same plot multiplied with $k\xi$ on y axis and time given in units of pion ballistic propagation time $L/v \sim \tau_R(L/\xi)$, where v is the pion velocity at the corresponding $t_r = -t_r^0$. Bottom left and right: enhancement of sigma-sigma field correlator over thermal equilibrium expectation during symmetric quench from $t_r = 0.04$ to $t_r = -0.04$ for different wave numbers $k = 2\pi/L$, where $L = 192, 128, 96, 64$.

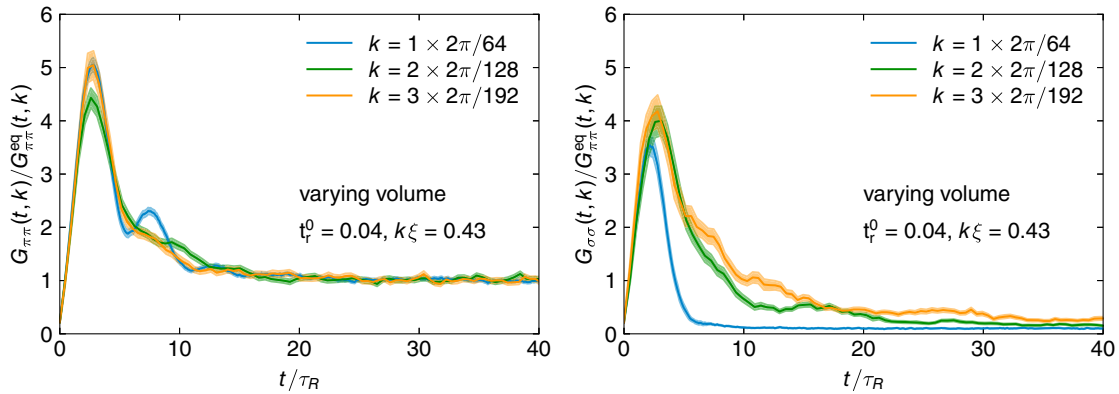


FIG. 7. Left: enhancement of the equal time pion-pion field correlator over thermal equilibrium expectation during symmetric quench from $t_r = 0.04$ to $t_r = -0.04$ for fixed wave number $k = \pi/32$, but different lattice sizes $L = 192, 128, 64$. The time is given in units of relaxation time $t/\tau_R \propto t/\xi^{d/2}$. Right: the same plot for sigma-sigma correlator.

In mean-field theory the dynamical parameter Γ_0 is constant and the typical relaxation time is

$$\tau_R \equiv \frac{1}{\Gamma_0 |m_0|^2} \propto \xi^2. \quad (67)$$

Again, this is consistent with the scaling $\tau_R \propto \xi^{d/2}$ only in $d = 4$. We will measure time in units of τ_R for our mean-field analysis. The pion dispersion curve in mean-field theory takes a familiar form of Eq. (18) [45]. The decay constant in mean field is $f^2 = \bar{\sigma}_{\text{eq}}^2 \propto \xi^{-2}$, which is consistent with the previous section in $d = 4$. With this scaling, the velocity in Eq. (19) scales as ξ/τ_R as before, with a coefficient that depends on χ_I . As a result, for modes of wavelength L the timescales τ_R , $\tau_R(L/\xi)$, and $\tau_R(L/\xi)^2$ identified in Eq. (21) valid.

As discussed previously, the qualitative picture of condensate growth following a quench is discussed in Fig. 2. Following a quench, the system is unstable at “the top of hill” and the condensate subsequently grows to its equilibrium value. We will first analyze the condensate correlator $G_0(t) = V \langle M^2 \rangle / 4$, defined in Eq. (49). As emphasized in Sec. III A, G_0 needs to grow from χ to $V \bar{\sigma}_{\text{eq}}^2 / 4$ increasing by a factor

$$\text{amplification} \equiv \frac{V \bar{\sigma}_{\text{eq}}^2}{4\chi} = \frac{1}{4\lambda} \left(\frac{L}{\xi} \right)^d. \quad (68)$$

In the last step we assumed $d = 4$, where λ is dimensionless, and the amplification factor scales as $(L/\xi)^d$. Outside of $d = 4$, Eq. (68) is simply $L^d/\lambda \xi^4$ and the mean-field results will not exhibit scaling. As we will see in the next section, in mean-field theory the growth of the condensate takes the scaling form in $d = 4$,

$$\frac{G_0}{\chi} = \left(\frac{L}{\xi} \right)^d \mathcal{F}(t/\tau_R, \xi/L), \quad (69)$$

as in the critical dynamics.

However, there are striking differences between the mean field and critical dynamics. Specifically, the condensate growth is exponential, rather than ballistic, which reflects the instability at the “top of the hill.” The e-folding time for the exponential growth is τ_R and it takes $\sim \ln(V \bar{\sigma}_{\text{eq}}^2 / \chi)$ e-foldings before G_0 can grow from its initial value of χ to its final value, $V \bar{\sigma}_{\text{eq}}^2 / 4$. Thus, since G_0 is proportional to M^2 , we will define a timescale t_L characterizing the growth of the condensate M :

$$t_L \equiv \frac{\tau_R}{2} \ln \left(\frac{V \bar{\sigma}_{\text{eq}}^2}{8\chi} \right) \sim \tau_R \ln((L/\xi)^{d/2}). \quad (70)$$

Here the factor of 8 in the logarithm (as opposed to 4) is without significance and reflects the details of the

computation given below. The timescale t_L , which depends only logarithmically on L , should be contrasted with the ballistic timescale $\tau_R(L/\xi)$ of the previous section.

2. Growth of magnetization

In a mean-field description we first look at the time evolution of the zero mode, which we find by substituting a spatially homogeneous field $\phi_a(t, \mathbf{x}) = M_a(t)$ in Eq. (8a) and averaging the equation over volume

$$\begin{aligned} \partial_t M_a &= -\Gamma_0(m_0^2 + \lambda \sigma^2) M_a + \xi, \\ \langle \xi_a(t) \xi_b(t') \rangle &= \frac{2\Gamma_0}{V} \delta_{ab} \delta(t - t'). \end{aligned} \quad (71)$$

Here and below $\sigma^2 = M_a M_a$. In the quench setup, this equation should be solved with initial conditions drawn from the equilibrium distribution at temperature t_r^0 , before evolving the system at temperature $-t_r^0$ for $t > 0$ with $m_0^2 < 0$.

The probability distribution for the magnetization M_a in the symmetric phase $T > T_c$ is Gaussian provided $L \gg \xi$:

$$P(M_a) = \mathcal{N} \exp \left[-\frac{V}{2\chi} M_a M_a \right]. \quad (72)$$

Concretely,

$$\langle M_a \rangle = 0, \quad (73)$$

$$\langle M_a M_b \rangle = \frac{\chi}{V} \delta_{ab}, \quad (74)$$

with the higher-order moments further suppressed by powers of the volume. At early times the field amplitude is small and the nonlinear terms can be dropped. The probability distribution of M_a is well characterized by the variance. On the other hand, at early times the noise plays an important role in the evolution. The equation of motion for the two-point function which follows from (71) and in this limit reads

$$\partial_t \langle M_a M_b \rangle = 2\Gamma_0 |m_0^2| \langle M_a M_b \rangle + \frac{2\Gamma_0}{V} \delta_{ab}, \quad (75)$$

which is solved by exponential growth:

$$\langle M_a M_b \rangle = \delta_{ab} \frac{\chi}{V} (2e^{2t/\tau_R} - 1) \quad (76)$$

where we recall that $\tau_R = \chi/\Gamma_0$. In terms of the schematics of Fig. 2, this stage corresponds to the initial time when the field starts rolling down the potential. The squared amplitude evolves at late times as

$$\sigma^2 = \langle M_a M_a \rangle = 2 \left(\frac{4\chi}{V} \right) e^{2t/\tau_R}. \quad (77)$$

After these initial stages, when $\sigma^2 \gg \chi/V$, the nonlinearities cannot be neglected from the mean-field equations. However, the intrinsic effect of the noise is subdominant and suppressed by volume. The appropriate simplification of Eq. (71) in this limit is

$$\partial_t \sigma^2 = -2\Gamma_0 m^2(\sigma) \sigma^2, \quad (78)$$

where $m^2(\sigma) = -|m_0^2| + \lambda\sigma^2$. The solution is given by

$$\sigma^2(t) = \frac{\bar{\sigma}_{\text{eq}}^2}{e^{-2(t-t_L)/\tau_R} + 1}, \quad (79)$$

where, at this point, e^{2t_L/τ_R} is an integration constant, which remains to be fixed.

The integration constant is chosen so that the late-time solution matches the exponential growth of the early-time behavior in a region of overlap. Comparison of (77) and (79) shows that

$$\frac{V\bar{\sigma}_{\text{eq}}^2}{8\chi} = e^{2t_L/\tau_R}, \quad (80)$$

as anticipated in the text surrounding Eq. (70). The correlation function G_0/χ takes the form

$$\frac{G_0(t, \xi, L)}{\chi} = \frac{1}{4\lambda} \left(\frac{L}{\xi} \right)^d \frac{e^{2(t-t_L)/\tau_R}}{1 + e^{2(t-t_L)/\tau_R}}. \quad (81)$$

To summarize, we have shown that in mean field the global condensate is formed on a timescale of $t_L \sim \tau_R \ln((L/\xi)^2)$. This timescale should be contrasted with the scaling results and our numerical simulations presented in Fig. 4, which shows a timescale of $\tau_R(L/\xi) \sim L/v$.

3. Equal time correlation functions

In mean-field theory, the pion and sigma correlators are obtained by linearizing the equations of motion around the time-dependent background from the previous section. For $k\xi \ll 1$ and $t \sim \tau_R$, the coupling between the pion field $\vec{\pi}$ and the conserved charge n_{ab} can be ignored. This follows from the full set of mean-field equations for the two-point functions in the time-dependent background (see Appendix B). More intuitively, due to charge conservation the charge and pion fields are coupled through a derivative interaction, which, over a limited time, has little effect at small k . Initially, the equation of motion for $\pi\pi$ matches that of $\sigma\sigma$, since before the condensate forms, there is no preferred direction in flavor space and all field components must have the same correlation functions.

At small momentum and times $t \sim \tau_R$ equations of motion for the equal time pion and sigma correlation functions read

$$\partial_t G_{\pi\pi} = -2\Gamma_0(m^2(t) + k^2)G_{\pi\pi} + 2\Gamma_0, \quad (82a)$$

$$\partial_t G_{\sigma\sigma} = -2\Gamma_0(m_\sigma^2(t) + k^2)G_{\sigma\sigma} + 2\Gamma_0, \quad (82b)$$

with the pion and sigma masses given by

$$m^2(t) = -|m_0^2| + \lambda\sigma^2(t), \quad (83a)$$

$$m_\sigma^2(t) = -|m_0^2| + 3\lambda\sigma^2(t). \quad (83b)$$

In the limit when $k\xi \ll 1$, we have $k^2 \ll |m_0^2| \sim \lambda\sigma^2(t)$ and we can drop the k^2 in (82) in this limit. Thus, in the linearized mean-field analysis the correlation functions become k independent for $k\xi \ll 1$. This sharply contrasts with the scaling analysis in Sec. III A, where the correlation functions scale as $1/(k\xi)^d$ in the same limit.

In the quench setup, the initial conditions for $G_{\pi\pi}(t, k)$ and $G_{\sigma\sigma}(t, k)$ are given by equilibrium at a reduced temperature t_l^0 [see Eq. (63)]. For $k\xi \ll 1$, both correlation functions start as $G_{\pi\pi}(0, k) = G_{\sigma\sigma}(0, k) \simeq \chi$. In this limit the equations are solved using the approximation scheme from the previous section. At early times, when $\lambda\sigma^2(t)$ is still small, the inhomogeneous term in (82) (arising from thermal noise) cannot be ignored. However, at late times, thermal noise becomes negligible while $\lambda\sigma^2(t)$ plays a dominant role. By matching the late-time solution with the initial growth, as done in the previous section, we obtain the solution for $G_{\pi\pi}$ and $G_{\sigma\sigma}$ for $\tau_R \ll t \sim t_L$:

$$\frac{G_{\pi\pi}(t)}{\chi} = \frac{1}{4\lambda} \left(\frac{L}{\xi} \right)^d \frac{e^{2(t-t_L)/\tau_R}}{1 + e^{2(t-t_L)/\tau_R}}, \quad (84)$$

$$\frac{G_{\sigma\sigma}(t)}{\chi} = \frac{1}{4\lambda} \left(\frac{L}{\xi} \right)^d \frac{e^{2(t-t_L)/\tau_R}}{(1 + e^{2(t-t_L)/\tau_R})^3}. \quad (85)$$

The time evolution of $G_{\pi\pi}(t)/\chi$ exactly follows with G_0/χ from (81). As anticipated, this implies that $G_{\pi\pi}$ is amplified by the factor $V\bar{\sigma}_{\text{eq}}^2/4\chi$ and this enhancement persists even at late times. In contrast, $G_{\sigma\sigma}/\chi$ is also amplified but reaches a peak value of $4/27$ (in units of the amplification factor) before decreasing exponentially. These analytic results hold for $k\xi \ll 1$.

In Fig. 8 we show numerical simulations for the correlation functions, G_0 , $G_{\pi\pi}$, and $G_{\sigma\sigma}$ for $L/\xi = 50$ and $k = 2\pi/L$. As expected from (81), (84), and (85), the correlation functions grow to a size of order $V\bar{\sigma}_{\text{eq}}^2/4$ over a timescale of $\sim \tau_R \log((L/\xi)^2)$. At later times, when $t \sim 1/vk$, the coupling between the pion $\vec{\pi}$ and the conserved charges n_{ab} cannot be neglected and is analyzed Appendix B. At this stage $G_{\pi\pi}$ and the axial charge correlation function G_{AA} oscillate together with an angular frequency of vk as exhibited in Eq. (B20). The amplitude of the oscillations is determined by the determinant $(G_{\pi\pi}G_{AA} - G_{A\pi}^2)/\chi_I$ of the

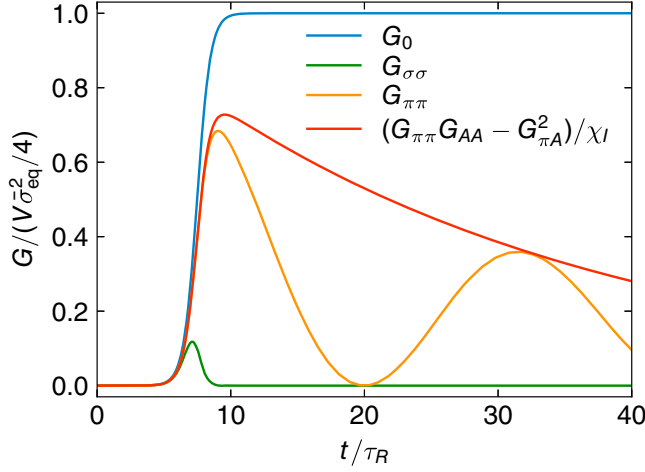


FIG. 8. Numerical results in mean-field theory ($d = 4$) for the growth of the condensate, $G_0(t)$, and related equal time correlation functions $G_{\pi\pi}(t)$ and $G_{\sigma\sigma}(t)$ following a quench at time $t = 0$ (see text). The dimensionless parameters are $L/\xi = 50$, $D_0/\Gamma_0 = 1$, $\lambda = 1/4$, and $v\tau_R/\xi = 1$. The condensate and correlators grow on a time $t_L \sim \tau_R \ln((L/\xi)^{d/2})$ [see Eq. (70) for a definition] indicated in the plot. After a time t_L , the pion waves with wave number $k = 2\pi/L$ (shown here) oscillate on a timescale of $L/2v \sim \tau_R(L/\xi)$.

coupled system [see Eq. (B21) for the analytic expression], which gradually decays on the timescale for diffusive processes $\sim 1/D_A k^2 \sim \tau_R/(k\xi)^2$.

In summary, the linearized mean-field approach accurately reproduces critical scaling and the expected parametric size $\propto (L/\xi)^d$ of the enhancement in four dimensions ($d = 4$). When a magnetic field is included as analyzed in Appendix B, the mean-field correlation functions are amplified by a factor of $1/m^4$, consistent with the general scaling prediction of $1/(m\xi)^d$ only for $d = 4$. However, in mean-field theory, the condensate grows exponentially over a timescale of $\sim \tau_R \log((L/\xi)^2)$, rather than spreading ballistically over $\sim \tau_R(L/\xi)$. Finally, the linearized mean-field approach fails spectacularly at capturing the k dependence of the enhancement. In our statistical simulations the enhanced correlator scales as $1/(k\xi)^d$ for $k\xi \ll 1$, whereas in mean-field theory the correlator is enhanced, but independent of k until $k\xi \sim 1$.

IV. DISCUSSION

Motivated by the rapid transition through the chiral crossover in high-energy heavy-ion collisions, we performed the first simulations of Model G, quenched from the high-temperature symmetric phase to the low-temperature broken phase in the chiral limit. We measured how the average chiral condensate and the equal-time correlation functions of the pion and sigma modes approach equilibrium. In the quench setup, these curves are shown in Figs. 4–6, along with a scaling analysis summarized below.

The main result is a parametric enhancement of the equal time pion-pion correlation function over equilibrium, $G_{\pi\pi}(t, k)/G_{\pi\pi}^{\text{eq}}(k)$ by a factor of order $\sim 1/(k\xi)$ (see Fig. 5). The enhancement sets in for $k\xi \ll 1$ and persists until late times, $t \sim \tau_R/(k\xi)$, before slowly decaying back to equilibrium in a time of order $\sim \tau_R/(k\xi)^2$. This is illustrated schematically in Fig. 3.

With the quasiparticle assumptions outlined in Sec. II B 4, the equal time two-point function parametrizes the pion distribution function $n_\pi(t, k)$ at small k , $\chi_I G_{\pi\pi}(t, k)/\bar{\sigma}_{\text{eq}}^2 = n_\pi(t, k)/\omega(k)$ with $\omega(k) = vk$. Our scaling results for $G_{\pi\pi}(t, k)$ can then be written¹⁴

$$n_\pi(t, k) = T_c \tau_R \left[\frac{1}{(vk\tau_R)^2} \mathcal{F}(vkt) \right]. \quad (86)$$

The scaling function $\mathcal{F}(vkt)$ approaches a constant for $vkt \gg 1$. For a given quench initial condition, v and τ_R are fixed, and the spectrum exhibits a scaling form in time and momentum, which is characteristic of a nonthermal fixed point (NTFP) [59–62].¹⁵ Alternatively, one may fix k and vary the initial conditions. In this case, the nonequilibrium spectrum still shows a critical scaling with time and the correlation length ξ , reflecting the proximity to T_c . Both scalings are shown in Fig. 5.

Noting that the equilibrium spectrum is $n_\pi^{\text{eq}} \simeq T_c/vk$ with $v \sim \xi/\tau_R$, and taking the limit $vkt \gg 1$, we can express the enhanced pion yield more precisely:

$$n_\pi(t, k) = n_\pi^{\text{eq}}(k) \left(\frac{\text{const}}{k\xi} \right) \quad \text{with} \quad \frac{\tau_R}{t} \ll k\xi \ll \left(\frac{\tau_R}{t} \right)^{1/2}. \quad (87)$$

The constant is roughly 2.0, as can be estimated from the maximum of $G_{\pi\pi}(t, k)/G_{\pi\pi}^{\text{eq}}(k)$ shown in Fig. 5. A more precise determination will require larger lattices, where the NTFP can be clearly separated from the approach to equilibrium, and where the classical evolution at high wave numbers can be matched onto a Boltzmann equation [44].

The parametric enhancement can be understood through a scaling analysis from Sec. III A, along with a physical picture based on the hydrodynamic limits of Model G, as illustrated in Fig. 2. Specifically, over a time $\tau_R \propto \xi^{d/2}$, disoriented domains of the chiral condensate form, each only a few correlation lengths long. The domains then

¹⁴This expression is simply a rewrite of $G_{\pi\pi}(t, k)$ in (61) after using $\xi \propto v\tau_R$ and restoring the temperature. It must be emphasized that the identification with the pion yield is only strictly valid for $vkt \gtrsim 1$ when the quasiparticle assumptions hold.

¹⁵At a NTFP, the spectrum takes a nonequilibrium scaling form $n(t, k) \propto t^a \mathcal{F}_S(t^b k)$ for some exponents a and b . In the present case, $a = 2$ and $b = 1$. The number of soft pions in this nonequilibrium spectrum is not conserved, and the familiar NTFP relation $a = db$ does not hold.

merge on a much longer timescale of order $t \sim L/v \sim \tau_R(L/\xi)$. Since the condensate growth timescale is much larger than the critical relaxation time τ_R , and, since the spatial scale $vt \sim L$ is much larger than ξ , the growth is determined by nonlinear dynamics of the hydrodynamic limit of Model G. In the broken phase, this hydrodynamic limit corresponds to a $O(4)$ superfluid, first written down by Son in a rather different language¹⁶ [54]. The enhancement arises when an $O(4)$ superfluid is initialized with white noise and then subsequently evolves to form a global chiral condensate. Because no direction in the $O(4)$ space is preferred, all directions try to become the global condensate, leading to a pion enhancement during the competitive growth. In the future it should be possible to simulate the superfluid limit directly without the full structure of Model G.

We also investigated quench dynamics in mean-field theory, where the pion modes are treated as linearized fluctuations on top of a time-dependent chiral condensate. Mean-field theory predicts scaling in four dimensions ($d = 4$) and produces an enhancement of the right order of magnitude. However, the growth of both the enhancement and the chiral condensate is exponential rather than ballistic, saturating in a time of order $\tau_R \ln((L/\xi)^2)$ instead of $\tau_R(L/\xi)$. Most importantly, the linearized treatment gives a qualitatively incorrect dependence on $k\xi$. It remains to be seen whether taking into account a full nonlinear treatment of mean field will give an improved prediction. We leave this discussion for future work.

An exciting phenomenological application of our result is the potential enhancement of soft-pion yields, which are currently underpredicted by hydrodynamic models that neglect critical dynamics. The difficulty in describing the experimental results with existing models is that the yield steadily rises above from the equilibrium curve only at small momenta. In current models there is no scale that would dictate this change. While it is too early to draw a firm conclusion, our analysis naturally explains why the yield of Goldstone pions should follow this trend.

Since our simulations were performed at zero magnetic field, i.e., without explicit symmetry breaking, an important question is how our results will be affected by finite pion mass m , or equivalently, a finite magnetic field H . Including the pion mass is also essential for any comparisons with experiments. While a full study is left for future work, the scaling analysis of the quenches discussed in Sec. III A can be straightforwardly extended to small but finite m . In particular, the scaling function in (61) describing $G_{\pi\pi}$ for $k\xi \ll 1$ now depends on the dimensionless ratio, k/m . In the regime $k \ll m \ll 1/\xi$, the result should be independent of k , leading to the prediction

$$G_{\pi\pi}(t, m)|_{t \sim 1/vm} \sim \xi^{2-\eta} \left(\frac{1}{m\xi} \right)^d. \quad (88)$$

This is an enhancement over equilibrium by a factor of order $\sim 1/(m\xi)$ which persists a long time, $1/vm \lesssim t \ll \tau_R/(m\xi)^2$. The result is intuitive: the $1/k\xi$ dependence is regulated by the pion mass at small momenta as is illustrated in Fig. 9. This conclusion is also supported by preliminary results of Langevin simulations at a finite magnetic field and the mean-field analysis (see Appendix B), though in the latter case, the enhancement is $1/(m\xi)^2$, reflecting the fact that the mean field is valid only in $d = 4$.

If real-world QCD lies outside the $O(4)$ scaling window [63,64], exhibiting only broken chiral symmetry and a fairly light pseudo-Goldstone mode, the qualitative reasons for the enhanced pion yield remain unchanged. Critical scaling is essential for the data collapse seen in the top two panels of Fig. 5. However, in the bottom two panels where the pion yield is also enhanced, the initial conditions are fixed and the critical scaling of the parameters is never used. Thus, outside of the scaling window we expect that the superfluid dynamics will coarsen the system up to a length scale m^{-1} , producing the enhancement as described previously. However, since the superfluid parameters, f^2 and m^2 , do not exhibit strict scaling, the data collapse with reduced temperature will not be perfect. In the language of the preceding paragraphs, outside of the $O(4)$ scaling window we expect NTFP scaling but not critical scaling.

We have considered only instantaneous quenches. In a real heavy ion collision the QGP is expanding and cooling down at a finite rate. Therefore, it will be important to study different nonequilibrium protocols. Preliminary work where the reduced temperature is changed at a finite rate (as opposed to a sudden quench) shows that the basic picture of condensate growth outlined in Fig. 2 remains

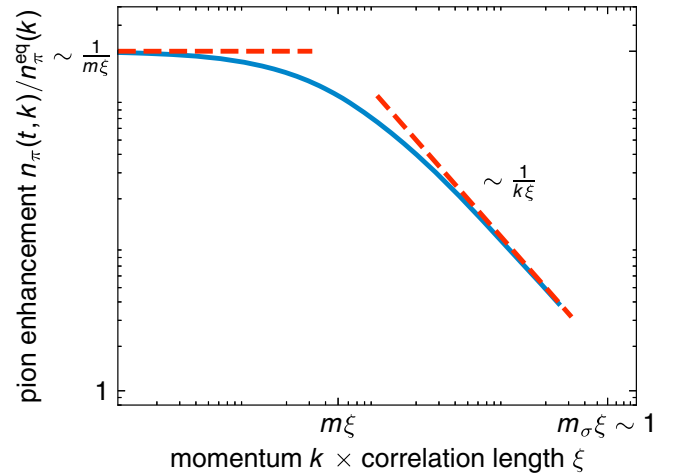


FIG. 9. Schematic illustration of the expected pion enhancement as a function of momentum in units of QCD correlation length.

¹⁶In Sec. II B 1 we rederived Son's equations with an $O(4)$ notation [see Eq. (41)] making the connection with Model G transparent.

valid [65]. Indeed, this preliminary work motivated the current study of a theoretically cleaner case of instantaneous quenches.

To summarize, we highlight the following points from our work (see also the companion Letter [42]):

- (i) We investigated the nonequilibrium dynamics of Model G, a universal model belonging to the same dynamic universality class as QCD.
- (ii) As the system passes through the phase transition, the equal-time correlation function of the Goldstone mode shows a parametric enhancement of order $1/(k\xi)$ compared to equilibrium for $k\xi \ll 1$. In the quasiparticle picture (see Sec. II B 4) this corresponds to enhancement of soft pion spectra; see Fig. 9.
- (iii) The nonequilibrium enhancement can be predicted using a general scaling argument (see Sec. III A), which identifies three distinct timescales: relaxation, ballistic transport, and diffusion, as illustrated in Fig. 3.
- (iv) The growth of the chiral condensate and the enhancement of the pion yield takes place on timescales which are long compared to the dynamical relaxation time τ_R . This is a hydrodynamic regime and the appropriate nonlinear effective theory is an ideal non-Abelian superfluid, reflecting the broken $SU(2)_L \times SU(2)_R$ symmetry below T_c [54].
- (v) The parameters of the hydrodynamic effective theory, which reflect the magnetic equation of state, can be determined from Euclidean measurements in lattice QCD [52,66]. This means that the details of the enhancement in QCD are rigorously computable using current technology.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [67].

APPENDIX A: SUBLEADING CORRECTIONS TO CRITICAL BEHAVIOR

Previous works [37,38] employed real-time simulations of Model G to study critical scaling in multiple dynamical channels. In particular, these studies extracted the nonuniversal leading and subleading amplitudes for several scaling observables. In the present work, whenever an equilibrium value is required we use the numerical values of these amplitudes together with the universal leading and subleading scaling exponents to compute the corresponding quantities. For convenience, we summarize the numerical values of the amplitudes in Table II.

Below T_c the correlation length can be defined in terms of the pion decay constant f ; see Eq. (36). By analyzing the static correlator, we previously fitted $\bar{\sigma}_{\text{eq}}^2/f^2$ with a phenomenological ansatz $C_0(-t_r)^{C_1}$ [38].¹⁷ Critical scaling predicts $\bar{\sigma}_{\text{eq}}^2/f^2 \sim (-t_r)^{\nu}$ with a small exponent $\nu \approx 0.02$ (see Table I). In practice, $\bar{\sigma}_{\text{eq}}^2/f^2 \simeq 0.9$ remains nearly constant over the temperature range relevant for our study. One could therefore use the condensate parametrization to determine the pion decay constant f and the correlation length below T_c . However, we adopt a different strategy and use the definition of the pion velocity, Eq. (19), to evaluate $f^2 = \chi_I v^2$ and the correlation length below T_c . We choose the normalization constant ξ^- such that the correlation length exhibits a symmetric leading behavior around T_c , i.e. $\xi(t_r) = \xi^+ |t_r|^{-\nu}$, where the nonuniversal amplitude ξ^+ above T_c was determined in Ref. [38]. Note that with this definition, subleading corrections to the correlation length above T_c and to the pion velocity below T_c (see Table II) lead to a slight asymmetry of ξ away from $t_r = 0$.

APPENDIX B: DETAILS REGARDING THE MEAN-FIELD THEORY

In this appendix, we write down the model in mean field. Following the initial exposition, we provide some further details on the quench from Sec. III C. As in Sec. II we will keep g_0 and T_c explicit rather than setting them to unity as in the main text.

The equations of the model are Eq.s (8a) and (8b) with the Gaussian noises given in (9a) and (9b). It is useful to decompose the field and the charge into an orthonormal basis in flavor space as

¹⁷An incorrect value of C_0 was reported in Ref. [38]; the correct fit values are $C_0 = 0.923(8)$ and $C_1 = 0.009(3)$.

TABLE II. The universal subleading scaling exponent ω and nonuniversal amplitudes of the $O(4)$ model at zero magnetic field used in this work. The first subleading exponent is taken from high-precision Monte Carlo simulations [49]. Nonuniversal leading and subleading amplitudes are taken from fits to Langevin simulations in Refs. [37,38].

Quantity	Subleading scaling	Constant values
Any scaling quantity	$Y \propto \xi^{\theta_Y} (1 + C_Y \xi^{-\omega} + \dots)$	$\omega = 0.755(5)$
Susceptibility ($T > T_c$)	$\chi_+ = C^+ \mathfrak{t}_r^{-\gamma} (1 + C_1 \mathfrak{t}_r^{\omega})$	$C^+ = 0.2077(32)$, $C_1 = 1.11(15)$
Condensate ($T < T_c$)	$\bar{\sigma}_{\text{eq}} = B^- (-\mathfrak{t}_r)^{\beta} (1 + B_1 (-\mathfrak{t}_r)^{\nu\omega})$	$B^- = 0.988(7)$, $B_1 = 0.51(7)$
Correlation length ($T > T_c$)	$\xi = \xi^+ \mathfrak{t}_r^{-\nu} (1 + \xi_1 \mathfrak{t}_r^{\omega})$	$\xi^+ = 0.4492(36)$, $\xi_1 = 0.332(85)$
Correlation length ($T < T_c$)	$\xi = \xi^- / f^2 \equiv \xi^- / (v^2 \chi_I)$	$\xi^- = \xi^+ (v^-)^2 \chi_I = 0.479(7)$
Relaxation time ($T > T_c$)	$\tau_R = \tau^+ \mathfrak{t}_r^{-\nu\zeta} (1 + \tau_1 \mathfrak{t}_r^{\omega})$	$\tau^+ = 1.570(37)$, $\tau_1 = -1.49(15)$
Pion velocity ($T < T_c$)	$v = v^- (-\mathfrak{t}_r)^{\nu/2} (1 + v_1 (-\mathfrak{t}_r)^{\nu\omega})$	$v^- = 0.462(3)$, $v_1 = 0.325(3)$

$$\phi_a(t, \mathbf{x}) \equiv \sigma(t, \mathbf{x}) \hat{n}_a + \vec{\pi}(t, \mathbf{x}) \cdot \vec{e}_a, \quad (\text{B1})$$

where $\vec{e}_a$ and $\hat{n}_a$ form an orthonormal basis in flavor space. Here, we derive the equation in the presence of the explicit breaking H . Therefore, it is more natural to have a basis fixed in time with the $\hat{n}$ component directed along the explicit breaking, $H_a = H \hat{n}_a$. The charges are in the adjoint of the group, leading to the following decomposition:

$$n_{ab}(t, \mathbf{x}) = \vec{n}_A(t, \mathbf{x}) \cdot (\hat{n}_a \vec{e}_b - \hat{n}_b \vec{e}_a) + \vec{n}_V(t, \mathbf{x}) \cdot (\vec{e}_a \times \vec{e}_b). \quad (\text{B2})$$

We can then project the equation of motions into this basis:

$$\partial_t \sigma + \frac{g_0}{\chi_I} \vec{n}_A \cdot \vec{\pi} = \Gamma_0 \nabla^2 \sigma - \Gamma_0 (m_0^2 + \lambda \sigma^2 + \lambda \vec{\pi} \cdot \vec{\pi}) \sigma + \Gamma_0 H + \theta, \quad (\text{B3a})$$

$$\partial_t \vec{n}_V + g_0 \nabla \cdot (\nabla \vec{\pi} \times \vec{\pi}) = D_0 \nabla^2 \vec{n}_V + \partial_i \vec{\Xi}_V^i, \quad (\text{B3b})$$

$$\partial_t \vec{\pi} - \frac{g_0}{\chi_I} \vec{n}_A \sigma = \Gamma_0 \nabla^2 \vec{\pi} - \Gamma_0 (m_0^2 + \lambda \sigma^2 + \lambda \vec{\pi} \cdot \vec{\pi}) \vec{\pi} + \vec{\theta}, \quad (\text{B3c})$$

$$\partial_t \vec{n}_A + g_0 \nabla \cdot (\nabla \sigma \vec{\pi} - \sigma \nabla \vec{\pi}) + g_0 H \vec{\pi} = D_0 \nabla^2 \vec{n}_A + \partial_i \vec{\Xi}_A^i. \quad (\text{B3d})$$

The nonvanishing noise variances are

$$\langle \theta(t, x) \theta(t', x') \rangle = 2T_c \Gamma_0 \delta(t - t') \delta^3(x - x'), \quad (\text{B4a})$$

$$\langle \theta^{\ell_1}(t, x) \theta^{\ell_2}(t', x') \rangle = 2T_c \Gamma_0 \delta^{\ell_1 \ell_2} \delta(t - t') \delta^3(x - x'), \quad (\text{B4b})$$

$$\langle \Xi_V^{\ell_1 i}(t, x) \Xi_{cd}^{\ell_2 j}(t', x') \rangle = 2T_c \chi_I D_0 \delta^{ij} \delta^{\ell_1 \ell_2} \delta(t - t') \delta^3(x - x'), \quad (\text{B4c})$$

$$\langle \Xi_A^{\ell_1 i}(t, x) \Xi_A^{\ell_2 j}(t', x') \rangle = 2T_c \chi_I D_0 \delta^{ij} \delta^{\ell_1 \ell_2} \delta(t - t') \delta^3(x - x'). \quad (\text{B4d})$$

Following the spirit of mean field, we linearize the equations of motion around a time-dependent and spatially uniform condensate, $\phi_a(t) = (\sigma(t), \vec{0})$. The evolution of the order condensate is nonlinear and given by

$$\partial_t \sigma = \Gamma_0 \nabla^2 \sigma - \Gamma_0 (m_0^2 + \lambda \sigma^2) \sigma + \Gamma_0 H, \quad (\text{B5})$$

which follows by substituting $\phi_a(t) = \sigma(t) n_a$ into (B3a) and averaging over volume. For simplicity, we have dropped the noise in the time evolution of the condensate since the averaged noise is suppressed by volume (see Sec. III C 2 where the noise is included). Given this equation for the condensate, we can write down the linearized equations for the spatially inhomogeneous fluctuations, $\delta\phi_a = (\delta\sigma, \vec{\pi})$,

$$\partial_t \delta\sigma = \Gamma_0 \nabla^2 \delta\sigma - \Gamma_0 m_\sigma^2 \delta\sigma + \theta, \quad (\text{B6a})$$

$$\partial_t \vec{n}_V = D_0 \nabla^2 \vec{n}_V + \partial_i \vec{\Xi}_V^i, \quad (\text{B6b})$$

$$\partial_t \vec{\pi} - \frac{g_0}{\chi_I} \vec{n}_A \sigma = \Gamma_0 \nabla^2 \vec{\pi} - \Gamma_0 m^2 \vec{\pi} + \vec{\theta}, \quad (\text{B6c})$$

$$\partial_t \vec{n}_A + g_0 \nabla \cdot (\nabla \sigma \vec{\pi} - \sigma \nabla \vec{\pi}) + g_0 H \vec{\pi} = D_0 \nabla^2 \vec{n}_A + \partial_i \vec{\Xi}_A^i. \quad (\text{B6d})$$

In writing these equations we adopted the following standard definition of the mean-field masses:

$$m_\sigma^2(\sigma) \equiv m_0^2 + 3\lambda\sigma^2, \quad m^2(\sigma) \equiv m_0^2 + \lambda\sigma^2. \quad (\text{B7})$$

We define the (connected) correlators as

$$G_{\sigma\sigma}(t, x, y) = \langle \delta\sigma(t, x) \delta\sigma(t, y) \rangle, \quad (\text{B8a})$$

$$G_{\pi\pi}^{\ell_1, \ell_2}(t, x, y) = \langle \pi^{\ell_1}(t, x) \pi^{\ell_2}(t, y) \rangle = \delta^{\ell_1 \ell_2} G_{\pi\pi}(t, x, y), \quad (\text{B8b})$$

$$G_{AA}^{\ell_1, \ell_2}(t, x, y) = \langle n_A^{\ell_1}(t, x) n_A^{\ell_2}(t, y) \rangle = \delta^{\ell_1 \ell_2} G_{AA}(t, x, y), \quad (\text{B8c})$$

$$G_{A\pi}^{\ell_1, \ell_2}(t, x, y) = \langle n_A^{\ell_1}(t, x) \pi^{\ell_2}(t, y) \rangle = \delta^{\ell_1 \ell_2} G_{A\pi}(t, x, y), \quad (\text{B8d})$$

$$G_{VV}^{\ell_1, \ell_2}(t, x, y) = \langle n_V^{\ell_1}(t, x) n_V^{\ell_2}(t, y) \rangle = \delta^{\ell_1 \ell_2} G_{VV}(t, x, y). \quad (\text{B8e})$$

Here we have neglected non-Gaussian correlations in these definitions. Therefore, the isospin dynamics are trivial, as each correlator is diagonal in flavor space.

Given this definition of equal time correlation functions, the evolution equations can be deduced from the linearized equations in (B6). At present, we will outline the essential steps only for the scalar channel to focus the discussion.

For a short time step Δt , we have

$$\delta\sigma(t + \Delta t) = \delta\sigma(t) + \Delta t(\Gamma_0 \nabla^2 \delta\sigma - \Gamma_0 m_\sigma^2 \delta\sigma) + \Delta\theta, \quad (\text{B9})$$

where the Gaussian random variable $\Delta\theta$ is

$$\Delta\theta = \int_t^{t+\Delta t} dt' \theta(t'). \quad (\text{B10})$$

With this definition, expanding in series of Δt , using the variance of the noise θ , we find

$$\begin{aligned} \partial_t G_{\sigma\sigma}(t, x, y) &= \Gamma_0 (\nabla_x^2 + \nabla_y^2) G_{\sigma\sigma}(t, x, y) \\ &\quad - 2\Gamma_0 m_\sigma^2 G_{\sigma\sigma}(t, x, y) + 2\Gamma_0 T_c \delta^3(x - y). \end{aligned} \quad (\text{B11})$$

In the same way, all the correlators can be worked out. It is more natural to express their dynamics in Fourier space, yielding

$$\partial_t G_{\sigma\sigma}(t, k) = -2\Gamma_0(k^2 + m_\sigma^2) G_{\sigma\sigma}(t, k) + 2\Gamma_0 T_c. \quad (\text{B12})$$

The vector channel correlator equation follows from (B6b)

$$\partial_t G_{VV}(t, k) = -2D_0 k^2 G_{VV} + 2D_0 k^2 T_c \chi_I. \quad (\text{B13})$$

The pion-axial correlators are coupled with the pion correlator given by

$$\begin{aligned} \partial_t G_{\pi\pi}(t, k) &= \frac{g_0 \sigma}{\chi_I} (G_{A\pi} + G_{\pi A}) - 2\Gamma_0(k^2 + m^2) \\ &\quad \times G_{\pi\pi} + 2\Gamma_0 T_c, \end{aligned} \quad (\text{B14a})$$

the axial correlator given by

$$\begin{aligned} \partial_t G_{AA}(t, k) &= -g_0(k^2 \sigma + H)(G_{A\pi} + G_{\pi A}) \\ &\quad - 2D_0 k^2 G_{AA} + 2D_0 k^2 T_c \chi_I, \end{aligned} \quad (\text{B14b})$$

and, finally, the mixed correlator given by

$$\begin{aligned} \partial_t G_{A\pi}(t, k) &= -g_0(\sigma k^2 + H) G_{\pi\pi} + \frac{g_0 \sigma}{\chi_I} G_{AA} \\ &\quad - \Gamma_0(k^2 + m^2) G_{A\pi} - D_0 k^2 G_{A\pi}. \end{aligned} \quad (\text{B14c})$$

In the following, we will set $g_0 = T_c = 1$.

1. Mean field description of a quench

Having presented the mean-field equations, we will now describe the evolution of $G_{\pi\pi}(t, k)$ and $G_{AA}(t, k)$ following a quench for $k\xi \ll 1$. This expands on the discussion given Sec. III C 3. For simplicity, the magnetic field H will be set to zero in this section.

As described in the body of the text, after the quench, when the condensate is on “top of the hill,” the initial behavior of $G_{\pi\pi}$ is characterized by exponential growth (see Fig. 3). The resulting evolution of the condensate and $G_{\pi\pi}$ are given in (79), (81), and (84).

During this short period of exponential growth, which lasts a time of order $t_L \sim \tau_R \log((L/\xi)^2)$, the axial correlators $G_{A\pi}$ and G_{AA} are constant, taking values zero and χ_I , respectively. These initial values are the equilibrium values of these correlators before the quench at temperature t_r^0 . These correlators are nearly constant because the axial charge is conserved and thus evolves slowly for $k\xi \ll 1$. For the same reason, the coupling between G_{AA} and $G_{\pi\pi}$ can be neglected for times of order t_L . However, at later times $t \sim 1/(vk) \sim \tau_R/k\xi$, the condensate has reached its equilibrium value and the coupling between the pion and axial fields cannot be ignored. Here, we will characterize the late-time oscillatory behavior of the correlators, which is seen in Fig. 8 and reflects this coupling.

At late times $t - t_L \gg \tau_R$, the condensate has reached its asymptotic value of $\sigma = \bar{\sigma}_{\text{eq}}$ and is treated as constant in what follows. Then we can write the evolution equations of the correlators in the pion-axial channel via $\vec{G} = (G_{\pi\pi}, G_{A\pi}, G_{\pi A}, G_{AA})$ as

$$\partial_t G_i = A_{ij} G_j + s_i, \quad (\text{B15})$$

where the sources are $s_i = (2\Gamma_0, 0, 0, 2D_0 \chi_I k^2)$ and the matrix is

$$A_{ij} = \begin{pmatrix} -2\Gamma_0 k^2 & \bar{\sigma}_{\text{eq}}/\chi_I & \bar{\sigma}_{\text{eq}}/\chi_I & 0 \\ -\bar{\sigma}_{\text{eq}} k^2 & -(D_0 + \Gamma_0) k^2 & 0 & \bar{\sigma}_{\text{eq}}/\chi_I \\ -\bar{\sigma}_{\text{eq}} k^2 & 0 & -(D_0 + \Gamma_0) k^2 & \bar{\sigma}_{\text{eq}}/\chi_I \\ 0 & -\bar{\sigma}_{\text{eq}} k^2 & -\bar{\sigma}_{\text{eq}} k^2 & -2D_0 k^2 \end{pmatrix}, \quad (\text{B16})$$

where we have set $H = 0$.

We can easily determine the eigenvalues of the above matrix

$$\lambda_{\pm} = -D_A k^2 \pm \sqrt{(\Gamma_0 - D_0)^2 k^4 - 4k^2 v^2}, \quad (\text{B17})$$

where we recalled the pion velocity $v^2 \equiv \sigma^2/\chi_I$, see Eq. (19), and the damping coefficient $D_A \equiv \Gamma_0 + D_0$. Sticking to small momenta, it is easy to see that the eigenfrequencies are given by

$$\omega(k) = i\lambda_{\pm} \sim \pm 2vk - iD_A k^2 + \mathcal{O}(k^3), \quad (\text{B18})$$

in line with the discussion in Sec. II B 3.

In the limit of $\Gamma_0 = D_0 = D_A/2$, these equations can be solved once the mean field reaches its constant equilibrium value at late times. The initial evolution for $t \sim t_L$ gives the initial conditions for the subsequent evolution with $t \sim 1/vk$. This initial condition can be well approximated by

$$G_{\pi\pi}(0) = \frac{V\bar{\sigma}_{\text{eq}}^2}{4}, \quad (\text{B19})$$

while the remaining components are much smaller $G_{AA} = \chi$ and $G_{A\pi} = 0$. The solutions take the form of a homogeneous solution, which depends on $G_{\pi\pi}(0)$ and is initially large, and an inhomogeneous solution, which is small and determines the late-time equilibrium behavior. The solution takes the form

$$k^2 G_{\pi\pi} = 1 + G_{\pi\pi}(0) k^2 e^{-D_A k^2 t} \cos^2(vkt), \quad (\text{B20a})$$

$$G_{AA}/\chi_I = 1 + G_{\pi\pi}(0) k^2 e^{-D_A k^2 t} \sin^2(vkt), \quad (\text{B20b})$$

$$kG_{\pi A}/\sqrt{\chi_I} = -G_{\pi\pi}(0) k^2 e^{-D_A k^2 t} \sin(vkt) \cos(vkt). \quad (\text{B20c})$$

In this solution the leading constants reflect the late-time equilibrium behavior, while the remaining terms proportional to $G_{\pi\pi}(0)$ represent the nonequilibrium homogeneous solution. In writing this solution we have made the approximation, $G_{\pi\pi}(0)k^2 - 1 \simeq G_{\pi\pi}(0)k^2$, which is uniformly valid at all times for $\xi/L \ll 1$. In the same approximation, (B20) is valid even for $\Gamma_0 \neq D_0$, and the determinant depends only on the axial diffusion constant and not on the pion velocity:

$$\frac{k^2}{\chi_I} (G_{\pi\pi} G_{AA} - G_{\pi A}^2) = 1 + G_{\pi\pi}(0) k^2 e^{-D_A k^2 t}. \quad (\text{B21})$$

For times of order $t \sim 1/vk$ the damping term $e^{-D_A k^2 t} \simeq 1$ is approximately unity and the homogeneous terms proportional to $G_{\pi\pi}(0)$ in (B20) and (B21) are dominant. These terms account for qualitative and quantitative features of the late-time behavior of the curves in Fig. 8.

2. Mean field at finite and small H

a. Overview

In Sec. III C 2 we analyzed the growth of the condensate following a quench seeded by an initial fluctuation $M_a \sim \sqrt{\chi/V}$. In this section, we will modify this analysis by considering a small magnetic field which provided a small initial value which seeds the growth of the condensate. The analysis of the subsequent growth of the condensate and the correlation functions $G_{\pi\pi}$ and $G_{\sigma\sigma}$ is quite similar to Sec. III C 2.

Consider the field configuration prequench shown in the left panel of Fig. 10. When a small magnetic field is applied, the mean value of the zero mode is nonzero even above T_c and equals

$$\langle M_0 \rangle = \sigma(0) = \chi H. \quad (\text{B22})$$

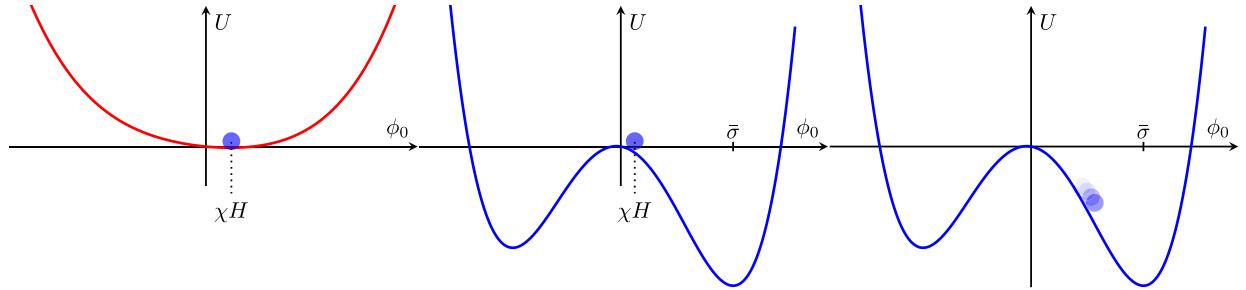


FIG. 10. Schematics of quenching from high temperature to the broken phase in the presence of explicit symmetry breaking $H \neq 0$. The order parameter is initiated in thermal equilibrium. The finite magnetic field introduces a displacement χH . The instantaneous quench brings the potential to its broken phase. The fields roll down to the bottom of one of the wells.

Here the mean-field susceptibility is given in Eq. (65). Subsequently M_0 grows exponentially from this initial size to $\bar{\sigma}_{\text{eq}}$ as the condensate evolves from the unstable configuration in the left panel of Fig. 10 to the stable one in the rightmost plot of Fig. 10. The process leads to amplification factor for M^2 of order

$$\text{amplification} = \frac{\bar{\sigma}_{\text{eq}}^2}{(\chi H)^2} = \frac{1}{(m_H \xi)^d}. \quad (\text{B23})$$

Here we recalled the definition of $\chi = \xi^2$ above T_c given in Eq. (65) and also defined the equilibrium pion screening mass

$$m_H^2 \equiv \frac{H}{\bar{\sigma}_{\text{eq}}}, \quad (\text{B24})$$

which follows from the form of the static two-point functions of the order parameter $\delta\phi_a = (\delta\sigma(k), \vec{\pi}(\mathbf{k}))$ in equilibrium below T_c and in the mean field [45]. This definition agrees with Eq. (B7) only when the condensate has reached its equilibrium value of $\bar{\sigma}_{\text{eq}}$. Finally, as in the mean-field results of Sec. III C 2, we have assume $d = 4$ when presenting the mean-field amplification in Eq. (B23). Note that we introduce a subscript H to the mass to distinguish from the rest of the discussion in the main text with the exception of the mass in (88), which is m_H .

The timescale for exponential growth is τ_R , and it takes $\bar{\sigma}_{\text{eq}}/\chi H$ e-foldings for the condensate to reach its asymptotic value. Thus, in analogy with Sec. III C 2 we define a timescale

$$t_H \equiv \tau_R \ln\left(\frac{\bar{\sigma}_{\text{eq}}}{2\chi H}\right) = \tau_R \ln\left(\frac{1}{2(m_H \xi)^{d/2}}\right), \quad (\text{B25})$$

where the additional factor of 2 which appears in the logarithm is conventional following from the computations given below.

b. Growth of the condensate and correlations

Having outlined the timescales and scaling in the previous section, we now present the details. The dynamics of condensate growth proceeds in two stages, paralleling Sec. III C 2. Initially, σ is of the order of the perturbation provided by H . Nonlinearities can be ignored, and σ grows exponentially. In the second stage, when $\sigma/\chi H \gg 1$, the perturbation H is irrelevant, but the dynamics become nonlinear.

Let us see this explicitly. The evolution of σ can be determined from Eq. (B5) by

$$\partial_t \sigma = -\Gamma_0(m^2(\sigma)\sigma - H), \quad (\text{B26})$$

where the transverse mass is given by Eq. (B7). When $\lambda\sigma^3 \ll \sigma m_0^2$, we can neglect the nonlinearities¹⁸ and obtain

¹⁸Note that this regime $\lambda\sigma^3 \ll \sigma m_0^2$ is equivalent to $\sigma \ll \sigma_{\text{eq}}$.

$$\partial_t \sigma = \Gamma_0 m_0^2 \sigma + \Gamma_0 H. \quad (\text{B27})$$

The solution in this limit reads

$$\sigma(\tau) = \sigma(0)e^{t/\tau_R} + \chi H e^{t/\tau_R} (1 - e^{-t/\tau_R}). \quad (\text{B28})$$

Upon using the initial condition Eq. (B22), we see that the condensate grows exponentially for $t \gg \tau_R$,

$$\sigma(t) = 2\chi H e^{t/\tau_R}. \quad (\text{B29})$$

Ultimately, σ cannot be considered small anymore and the nonlinearities becoming relevant.

After this initial stage, however, the magnetic field is small compared to the field $\sigma \gg \chi H$, and the magnetic field can be neglected. In the case of a vanishing magnetic field, the solution of the nonlinear equation Eq. (B26) with $H = 0$ is

$$\sigma^2(t) = \frac{\bar{\sigma}_{\text{eq}}^2}{e^{-2(t-t_H)/\tau_R} + 1}, \quad (\text{B30})$$

where at this stage the timescale t_H is an integration constant. The integration constant has to be determined by the early-time behavior in (B29) to (B30) yielding

$$e^{t_H/\tau_R} = \frac{\bar{\sigma}_{\text{eq}}}{2\chi H}, \quad (\text{B31})$$

which agrees with the previous definition of t_H given in (B25).

Summarizing, following a quench, the condensate squared (relative to its initial value) grows as

$$G_0(t) \equiv \left(\frac{\sigma(t)}{\chi H}\right)^2 = \frac{1}{(m_H \xi)^d} \frac{e^{2(t-t_H)/\tau_R}}{1 + e^{2(t-t_H)/\tau_R}}, \quad (\text{B32})$$

increasing by a factor of order $1/(m_H \xi)^d$ over a timescale of order $t_H \sim \tau_R \ln((m_H \xi)^{-d/2})$, cf. Eqs. (B23) and (B25).

c. Equal time correlation functions

The pion and sigma correlators can be determined following a similar strategy. For $k^2 \ll m_0^2$ and the approximations discussed in the body of the text (Sec. III C 2), the correlators evolve as

$$\partial_t G_{\pi\pi} = -2\Gamma_0 m^2(t) G_{\sigma\sigma} + 2\Gamma_0, \quad (\text{B33})$$

$$\partial_t G_{\sigma\sigma} = -2\Gamma_0 m_\sigma^2(t) G_{\sigma\sigma} + 2\Gamma_0, \quad (\text{B34})$$

where the time dependence of the masses stems from the time-dependent condensate, e.g. $m^2(t) = -|m_0^2| + \lambda\sigma^2(t)$. These equations are subject to the initial condition

$$G_{\sigma\sigma}(0) = G_{\pi\pi}(0) = \chi, \quad (\text{B35})$$

which reflects the equilibrium at t_H^0 just before the quench.

As previously stated, at early times, when $\lambda\sigma^2(t)$ is small compared to $-|m_0|^2$, the inhomogeneous terms (arising from thermal noise) cannot be ignored. However, at late times the inhomogeneous term becomes negligible, while $\lambda\sigma^2$ plays a dominant role. By matching the initial growth with the late-time solution with the initial growth we obtain

$$\frac{G_{\pi\pi}(t)}{\chi} = \frac{1}{2} \left(\frac{1}{m_H \xi} \right)^d \frac{e^{2(t-t_H)/\tau_R}}{1 + e^{2(t-t_H)/\tau_R}}, \quad (\text{B36})$$

$$\frac{G_{\sigma\sigma}(t)}{\chi} = \frac{1}{2} \left(\frac{1}{m_H \xi} \right)^d \frac{e^{2(t-t_H)/\tau_R}}{(1 + e^{2(t-t_H)/\tau_R})^3}. \quad (\text{B37})$$

In writing these equations we noted that the amplification factor $(\bar{\sigma}_{\text{eq}}/\chi H)^2$ can be written as $(m_H \xi)^{-d}$ via Eq. (B23). These finite H results are similar to the finite volume results; see Eqs. (84) and (85). Moreover, the correlators at finite H will look similar to those in Fig. 8, with the substitution

$$\frac{1}{4\lambda} \left(\frac{L}{\xi} \right)^d \rightarrow (m_H \xi)^{-d}. \quad (\text{B38})$$

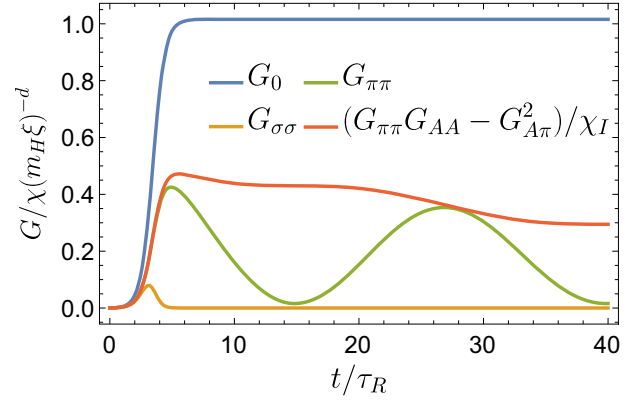


FIG. 11. Numerical results in mean-field theory ($d = 4$) for the growth of the condensate, $G_0(t)$, and related equal time correlation functions $G_{\pi\pi}(t)$ and $G_{\sigma\sigma}(t)$ following a quench at time $t = 0$ for $k = 0$ and for finite magnetic field, $m_H \xi = 2\pi/50$ (see Fig. 8 for comparison). The timescale for condensate growth is $t_H \equiv \tau_R \ln((m_H \xi)^{-d/2}/2) \simeq 4.84\tau_R$. The remaining dimensionless parameters are $D_0/\Gamma_0 = 1$, $\lambda = 1/4$, and $v\tau_R/\xi = 1$.

Omitting the details, at late times the axial charge and pion correlators will oscillate together with frequency vm_H and decay rate $\Gamma_0 m_H^2$, instead of vk and $D_A k^2$ as discussed in Eq. (B20). A numerical solution of the mean-field equations is shown in Fig. 11 and confirms this analysis.

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