

CMB constraints on quantized spatial curvature Ω_K in globally *CPT*-symmetric universes

Wei-Ning Deng^{*} and Will Handley[†]

*Astrophysics Group, Cavendish Laboratory, J. J. Thomson Avenue,
Cambridge, CB3 0HE, United Kingdom*

and Kavli Institute for Cosmology, Madingley Road, Cambridge, CB3 0HA, United Kingdom



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The periodic solution of the Friedmann equation in conformal time implies that only cosmological perturbations exhibiting corresponding symmetries are physically permissible, leading to a discrete spectrum of allowed wave vectors. Furthermore, in a spatially closed universe, these wave vectors are independently constrained to be integers. Matching these two distinct quantization conditions provides a novel theoretical constraint on the possible values of spatial curvature. In this work, we numerically solve the cosmological perturbation equations, incorporating radiation anisotropy and higher-order Boltzmann terms, to calculate these discrete wave vectors with improved precision. Subsequently, we generate cosmic microwave background power spectra for different characteristic spacings of these quantized wave vectors. Finally, we apply the constraint to Planck 2018 observational data to determine the cosmological parameters. This analysis yields a discrete set of allowed values for the spatial curvature, $\Omega_{K,0}$, including $[-0.076, -0.039, -0.024, -0.016, -0.012, \dots]$.

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I. INTRODUCTION

The spatial curvature of our Universe, denoted by Ω_K , has been a subject of wide interest and ongoing investigation. Data from Planck 2018 indicate a preference for a closed universe, with $-0.095 < \Omega_{K,0} < -0.007$ at 99% confidence level [1]. However, the inclusion of other observational datasets, such as those from baryon acoustic oscillations (BAOs) and Planck's lensing, tend to favor a spatially flat universe [1].

Nevertheless, this conclusion has been challenged in recent literature. Studies by Di Valentino, Melchiorri, and Silk [2] and Handley [3] demonstrate Bayesian evidence supporting a closed universe. This tension is often attributed to a disagreement between the lensing amplitude (A_L) and the implicit assumption of a fiducial Λ CDM cosmology within BAO analyses. The preference for a flat universe can be significantly weakened when these underlying assumptions are relaxed [4].

If the Universe is indeed closed, its finite spatial volume imposes a quantization condition, restricting the allowed

comoving wave vectors to integer multiples of a fundamental mode related to the curvature radius. However, this is not the sole potential constraint on wave vectors. Solving the Friedmann equation in conformal time, rather than physical time, reveals a periodic solution for the scale factor [5,6]. Considering this periodicity shows important insight to our Universe. For example, it led Boyle and Turok to further postulate that the Universe is charge, parity, and time (*CPT*) symmetric at the big bang [6–9]. Such a *CPT*-symmetric framework offers novel explanations for several outstanding problems in cosmology and particle physics without necessarily introducing additional model complexity. On the other hand, Lasenby *et al.* also investigated the evolution of cosmological perturbations within such a periodic framework and found that physical consistency requires these perturbations to also be periodic, thereby constraining their wave vectors to a discrete set of values [10].

In [11], we pointed out that the constraints on wave vectors arising from a spatially closed geometry and those from the periodicity of the Universe must be consistent. This consistency requirement imposes a strong theoretical constraint on the value of spatial curvature, implying that only specific, discrete values of $\Omega_{K,0}$ are permissible. Building upon this foundation, the present work incorporates more realistic cosmological components, namely, matter (Ω_m) and radiation anisotropy. To achieve this, we adapt the numerical solutions in [12] and modify it to include spatial curvature. By subsequently imposing the

^{*}Contact author: wnd22@cam.ac.uk

[†]Contact author: wh260@cam.ac.uk

quantization condition from our previous work, we determine the permissible curvature values within this more realistic model. The final step is to compare these discrete values with observational data to identify which are empirically viable.

We begin by reviewing the underlying theory in Sec. II. We then numerically solve for the discrete wave vectors within a realistic cosmological model in Sec. III. Subsequently, in Sec. IV, we generate the corresponding cosmic microwave background (CMB) power spectra and compare them with Planck 2018 data. Our final results and discussion are presented in Sec. V.

II. THEORY

Let us first review the formulation of the theory. Throughout this paper, we adopt natural units where $8\pi G = c = \hbar = k_B = 1$.

A. Perturbation equations and solution

The perturbed Friedmann-Lemaître-Robertson-Walker metric in the Newtonian gauge can be written as

$$ds^2 = a^2(\eta) \left[-(1 + 2\Psi)d\eta^2 + (1 - 2\Phi) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right) \right], \quad (1)$$

where $a(\eta)$ is the scale factor as a function of conformal time η , Φ and Ψ are the scalar metric perturbations, and κ is the spatial curvature parameter.

The zeroth-order Friedmann equations in conformal time can be expressed in dimensionless form as

$$\begin{aligned} 3\dot{a}^2 &= \lambda(a^4 - 3\tilde{\kappa}a^2 + \tilde{m}a + \tilde{r}), \\ 6\ddot{a} &= \lambda(4a^3 - 6\tilde{\kappa}a + \tilde{m}), \end{aligned} \quad (2)$$

where $\dot{a} \equiv da/d\eta$ and $\tilde{\kappa} = \kappa/\lambda$, $\tilde{m} = m/\lambda$, and $\tilde{r} = r/\lambda$ are dimensionless parameters representing curvature, matter, and radiation energy densities, respectively, scaled by a parameter λ related to the cosmological constant.

Lasenby solved the first-order perturbation equations within the periodic framework and found that, for physical consistency (avoiding divergences or unphysical behavior in the “anti-universe” phase), the perturbation solutions must be periodic [10]. These solutions can be either symmetric or antisymmetric at $\eta = \eta_\infty$ (the “end” of the Universe in one cycle), leading to the quantization condition $\tilde{k}\eta_\infty/\sqrt{3} = n\pi/2$, for $n \in \mathbb{N}$, where $\tilde{k} = k_{\text{com}}/\sqrt{\lambda}$ is the dimensionless comoving wave number. However, Lasenby also noted a complication: The matter density $\rho_m \propto 1/a^3$ would lead to negative value in the anti-universe phase (where $a < 0$ in the periodic mathematical extension). Such negative energy density is problematic and

would also break the symmetry of the background evolution. To address this, Boyle proposed that the matter density should scale as $\rho_m \propto 1/|a|^3$. This modification arises because the classical fluid approximation $\rho_m \propto a^{-3}$ is expected to break down near the big bang singularity and the end of the Universe, where a quantum field theoretical description becomes necessary. For a scalar field, it is easy to check that the solution (of Klein-Gordon) maintains symmetry in both situations. However, for fermionic matter, which constitutes standard matter, is more complex due to spin connection terms in the Dirac equation in curved spacetime; a full treatment is deferred to future work.

If one accepts the $\rho_m \propto 1/|a|^3$ scaling, the evolution of the Universe and its anti-universe becomes symmetric. In this scenario, only perturbation modes that are symmetric with respect to the bang and η_∞ are considered physically admissible. This restricts the quantization condition to $\tilde{k}\eta_\infty/\sqrt{3} = n\pi$, for $n \in \mathbb{N}$ [13]. The discrete dimensionless wave vectors then have a characteristic spacing.

B. Basic idea

In our previous work [11], we extended Lasenby’s analysis [10] to spatially curved universes. Spatial curvature affects the conformal age of the Universe, η_∞ , and consequently, the spacing of the discrete dimensionless wave vectors, $\Delta\tilde{k}$, becomes dependent on this curvature. (Recall that $\tilde{k} \equiv k_{\text{com}}/\sqrt{\lambda}$.)

For a spatially closed universe ($\kappa > 0$), the finite spatial volume independently quantizes the allowed comoving wave numbers. Specifically, $k \equiv k_{\text{com}}/\sqrt{\kappa}$ must effectively be an integer, $k \in \mathbb{N}$. This implies that the dimensionless wave vectors $\tilde{k} = k_{\text{com}}/\sqrt{\lambda}$ are quantized as $\tilde{k}_j = j\sqrt{\tilde{\kappa}}$ for some integer j , leading to a characteristic spacing related to $\sqrt{\tilde{\kappa}}$. For consistency between the periodic-derived quantization and the geometric quantization, the periodic-derived spacing $\Delta\tilde{k} = N\sqrt{\tilde{\kappa}}$, where $N \in \mathbb{N}$. Requiring such consistency restricts the possible values of spatial curvature, $\tilde{\kappa}$ (and, thus, Ω_K), to a discrete set, as demonstrated in [11]. In this work, we extend this framework to a realistic cosmological model and aim to identify the observationally favored discrete curvature values by comparing these theoretical predictions with CMB data.

III. REALISTIC COSMOLOGICAL MODEL

To apply this model to the observable Universe, we first establish the transformation between the dimensionless parameters of the theory ($\tilde{m}, \tilde{\kappa}, \tilde{r}$) and standard cosmological density parameters ($\Omega_m, \Omega_K, \Omega_r, \Omega_\Lambda$), omitting the subscript 0 for simplicity. Subsequently, we calculate the periodic-derived wave vector spacing $\Delta\tilde{k}$ by incorporating the effects of radiation anisotropy and higher-order terms in the Boltzmann hierarchy, adapting the methods of [12,13].

A. Parameter transformation

We begin by relating the dimensionless parameters $\tilde{r}$, $\tilde{m}$, and $\tilde{\kappa}$ to the present-day cosmological density parameters Ω_Λ , Ω_r , Ω_m , and Ω_K . The transformations are given by

$$\Omega_m = \tilde{m}\Omega_\Lambda/a_0^3, \quad \Omega_K = -3\tilde{\kappa}\Omega_\Lambda/a_0^2, \quad \Omega_r = \tilde{r}\Omega_\Lambda/a_0^4, \quad (3)$$

where a_0 is the present-day scale factor. Using the Friedmann constraint equation at the present epoch, $\Omega_\Lambda + \Omega_K + \Omega_m + \Omega_r = 1$, and substituting the definitions from Eq. (3), we obtain (after dividing by Ω_Λ)

$$1 - 3\frac{\tilde{\kappa}}{a_0^2} + \frac{\tilde{m}}{a_0^3} + \frac{\tilde{r}}{a_0^4} = \frac{1}{\Omega_\Lambda}. \quad (4)$$

Without loss of generality, we set $\tilde{r} = 1$. This choice normalizes the scale factor such that $a = 1$ at the epoch where the energy density contribution from λ equals that from radiation (i.e., $r/a^4 = \lambda$, so $a^4 = r/\lambda = \tilde{r}$). Consequently, from the definition of Ω_r , the present-day scale factor a_0 is given by $a_0 = (\Omega_\Lambda/\Omega_r)^{1/4}$ [since $\Omega_r = (1 \cdot \Omega_\Lambda)/a_0^4$ with $\tilde{r} = 1$]. Equation (4) then becomes a quartic equation for a_0 :

$$a_0^4 - 3\tilde{\kappa}a_0^2 + \tilde{m}a_0 + \left(1 - \frac{1}{\Omega_\Lambda}\right) = 0.$$

Solving this quartic equation for a_0 yields four roots; the physically relevant solution is the unique positive real root. We choose to fix the present-day radiation density parameter Ω_r , as its value is relatively well determined. We assume a photon density parameter $\Omega_r h^2 = 2.47 \times 10^{-5}$ and an effective number of relativistic species $N_{\text{eff}} = 3.046$, so that

$$\Omega_r = (1 + N_{\text{eff}}(7/8)(4/11)^{4/3})\Omega_\gamma. \quad (5)$$

With Ω_r fixed and a_0 determined, we calculate $\Omega_\Lambda = a_0^4 \Omega_r$. Subsequently, Ω_K and Ω_m are found using Eq. (3). In summary, given values for $\tilde{\kappa}$, $\tilde{m}$ (with $\tilde{r} = 1$), and the observational value of Ω_r (which depends on h), we can determine a_0 and subsequently the full set of standard density parameters Ω_Λ , Ω_K , and Ω_m .

B. Calculating $\Delta\tilde{k}$ considering radiation anisotropy and higher-order terms

To accurately calculate the periodic-derived wave vector spacing $\Delta\tilde{k}$, we must solve the cosmological perturbation equations, identify the symmetric solutions, and determine the spacing between their allowed wave vectors. Following the methodology of [13], we evolve the perturbations using a perfect fluid approximation up to the epoch of recombination. After recombination, we incorporate radiation anisotropy by solving the Boltzmann hierarchy for photons.

This involves tracking the momentum-averaged Legendre moments of the photon distribution function $F_{r\ell}$ and its polarization components $G_{r\ell}$.

We begin by modifying the perturbation equations [Eqs. (7)–(15) in [13]] to include terms for nonzero spatial curvature, based on the formalism presented in, e.g., [14]:

$$\begin{aligned} (k_{\text{com}}^2 - 3\kappa)\Phi &= -\frac{1}{2}a^2 \sum_i \left(\delta\rho_i - 3\frac{\dot{a}}{a}(\bar{\rho}_i + \bar{P}_i)v_i \right), \\ \Phi &= -\frac{\dot{a}}{a}\Psi - \frac{1}{2}a^2 \sum_i (1 + w_i)\rho_i v_i, \\ k_{\text{com}}^2(\Phi - \Psi) &= \frac{3}{2}a^2 \sum_i (\bar{\rho}_i + \bar{P}_i)\sigma_i, \\ \dot{\delta}_i &= (1 + w_i)(3\dot{\Phi} + v_i k_{\text{com}}^2), \end{aligned} \quad (6)$$

$$\begin{aligned} \dot{v}_i &= 3\frac{\dot{a}}{a} \left(w_i - \frac{1}{3} \right) v_i - \frac{w_i}{1 + w_i} \left(\delta_i - \frac{2}{3} \left(1 - \frac{3\kappa}{k_{\text{com}}^2} \right) \Pi_i \right) \\ &\quad - \Psi - \frac{an_e\sigma_T}{k_{\text{com}}^2}(\theta_b - \theta_r). \end{aligned} \quad (7)$$

Here, $\theta_i = -k_{\text{com}}^2 v_i$ (for scalar velocity v_i), σ_i is the anisotropic stress, Π_i is related to σ_i (e.g., $\sigma_i = \frac{1}{6}\Pi_i$ or similar for curved space), n_e is the proper mean density of free electrons, and σ_T is the Thomson scattering cross section.

After recombination, we assume photons are free-streaming. In this regime, the Thomson scattering term proportional to $n_e\sigma_T$ vanishes. With this simplification, the photon polarization moments $G_{r\ell}$ decouple from the intensity moments $F_{r\ell}$ and other perturbations and are, thus, neglected for calculating $\Delta\tilde{k}$. The evolution equations for $F_{r\ell}$ are

$$\dot{F}_{r\ell} = \frac{k_{\text{com}}}{2\ell + 1} [\ell F_{r(\ell-1)} - (\ell + 1)F_{r(\ell+1)}] - n_e\sigma_T |a| F_{r\ell}. \quad (8)$$

Note the relations $F_{r0} = \delta_r$ (density perturbation), $F_{r1} \propto k_{\text{com}} v_r$ (dipole or velocity), and $F_{r2} \propto \Pi_r$ (quadrupole or anisotropic stress), which show how the macroscopic radiation fluid perturbations are coupled to the moments $F_{r\ell}$.

We then follow the procedure in [12] to calculate the linear evolution mapping for perturbations from recombination to the end of the Universe (η_∞). This mapping, combined with the symmetry conditions imposed at η_∞ and the perfect fluid solutions at recombination, allows determination of the allowed perturbation modes. Calculating this mapping involves integrating the perturbation equations backward in time from near η_∞ to recombination. To mitigate numerical instabilities near η_∞ , the backward integration is initiated from $\eta' = \eta_\infty - \Delta\eta$. The solutions at η' are then extrapolated to η_∞ using a Taylor expansion, whose coefficients are derived from the coupled

perturbation equations (6)–(8). Please refer to the Appendix for the coefficients.

It is noteworthy that [12] performed calculations using fixed Planck 2018 best-fit cosmological parameters [1]. In contrast, this analysis treats these parameters as free, enabling direct comparison with observational data. In dimensionless models, $\Delta\tilde{k}$ depends only on $\tilde{m}$ and $\tilde{\kappa}$. Standard cosmological parameters become relevant when converting $\tilde{k}$ to physical wave vectors $\Delta k_{\text{phys}} = \Delta\tilde{k} \sqrt{\lambda}/a_0$, where

$$\lambda = \frac{3H_0^2\Omega_\Lambda}{8\pi G}, \quad a_0 = (\Omega_\Lambda/\Omega_r)^{1/4}.$$

However, in our more realistic treatment (perfect fluid prerecombination, Boltzmann hierarchy postrecombination), $\Delta\tilde{k}$ itself acquires a dependence on parameters affecting recombination, such as recombination redshift z_{rec} , which is a function of $\Omega_b h^2$ and $H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$, in addition to $\tilde{m}$ and $\tilde{\kappa}$. Note that, while z_{rec} is also dependent on $\Omega_r h^2$, this parameter is held fixed in this work (see Sec. III A). Consequently, our modified computational framework yields $\Delta\tilde{k}(\tilde{m}, \tilde{\kappa}, \Omega_b, h)$.

IV. CMB POWER SPECTRUM AND COMPARISON WITH PLANCK DATA

Having established the theoretical framework and the method for calculating the discrete wave vector spacing $\Delta\tilde{k}$, we now turn to comparing the model with observational data. Our procedure involves two main steps: first, generating theoretical CMB power spectra that incorporate the required wave vector discretization and, second, comparing these spectra with Planck 2018 data to identify the cosmological parameters that satisfy our model's consistency condition.

A. Modification of the Boltzmann angular power spectrum code

To compute the CMB power spectra for our model, we modified the Boltzmann solver CLASS [15]. The standard code integrates over a continuous spectrum of comoving wave numbers, k_{com} . For a closed universe within our theoretical framework, however, only a discrete set of modes is physically permissible. The consistency between the geometric quantization (due to finite volume) and the periodicity-induced quantization (from the *CPT*-symmetric solution) requires the dimensionless wave number $k \equiv k_{\text{com}}/\sqrt{\kappa}$ to be restricted to integer multiples of a fundamental integer N . That is, the allowed modes are given by $k_{j'} = j'N$ for $j' \in \mathbb{N}$. Our modification to CLASS, therefore, restricts the computation of the CMB power spectrum to this discrete basis of comoving wave numbers, $k_{\text{com},j'} = j'N\sqrt{\kappa}$.

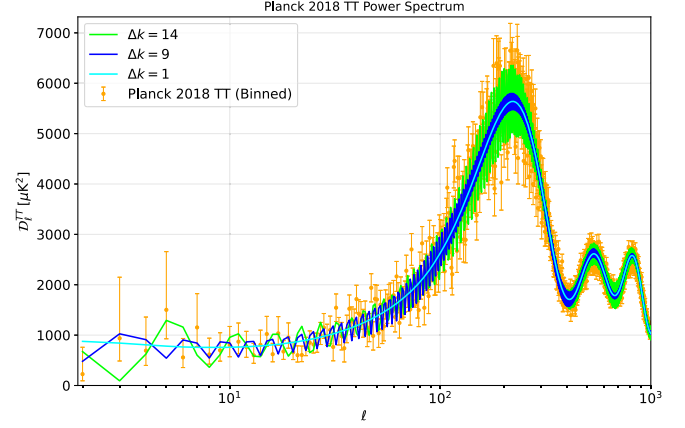


FIG. 1. The CMB temperature power spectrum calculated for different effective fundamental mode spacings, characterized by $\Delta k = N$. Here, $N = 1$ corresponds to the standard closed Λ CDM model. For this illustrative plot, we fix Ω_K and other cosmological parameters to be Planck best-fit values and isolate the effect of the mode spacing. Larger values of N (a sparser sampling in k space) lead to more fluctuations on large angular scales.

The value of κ itself is not a free parameter for a given N but is determined by solving the consistency condition $\Delta\tilde{k}(\tilde{m}, \tilde{\kappa}, \Omega_b, h) = N\sqrt{\kappa}$. The spacing of the physical modes, and, thus, the potential for observable discreteness features in the CMB power spectrum, depends on the value of the fundamental mode $N\sqrt{\kappa}$. As illustrated in Fig. 1, a sparser sampling of modes can induce more fluctuating features in the CMB power spectrum.

However, a key aspect of our model mitigates this effect. The value of $\Delta\tilde{k}$ is primarily determined by the conformal age of the Universe, η_∞ , which is not expected to vary dramatically across the observationally viable parameter space. The constraint $\Delta\tilde{k} = N\sqrt{\kappa}$, therefore, implies an inverse relationship: Larger integer values of N necessitate smaller values of $\tilde{\kappa}$ (and, consequently, smaller magnitudes of spatial curvature $|\Omega_K|$). This ensures that the effective fundamental mode in dimensionless units, $N\sqrt{\kappa} = \Delta\tilde{k}$, remains relatively constant. As a result, the physical fundamental mode $k_{\text{phys}} = (N\sqrt{\kappa})/a_0 = \Delta\tilde{k} \sqrt{\lambda}/a_0$ does not become excessively large, and the fluctuation in the CMB spectrum remain subtle for the parameter space favored by observations.

B. Postprocessing planck 2018 MCMC samples

Given that the discreteness features are not expected to be prominent, we can employ an efficient method to constrain our model's parameters without performing a full, computationally expensive Markov chain Monte Carlo (MCMC) analysis. Instead of fitting our modified CLASS code directly to the data, we postprocess the publicly available Planck 2018 MCMC chains from the standard Λ CDM + Ω_K analysis (base_omegak_plikHM_TTTEEE_lowl_lowE).

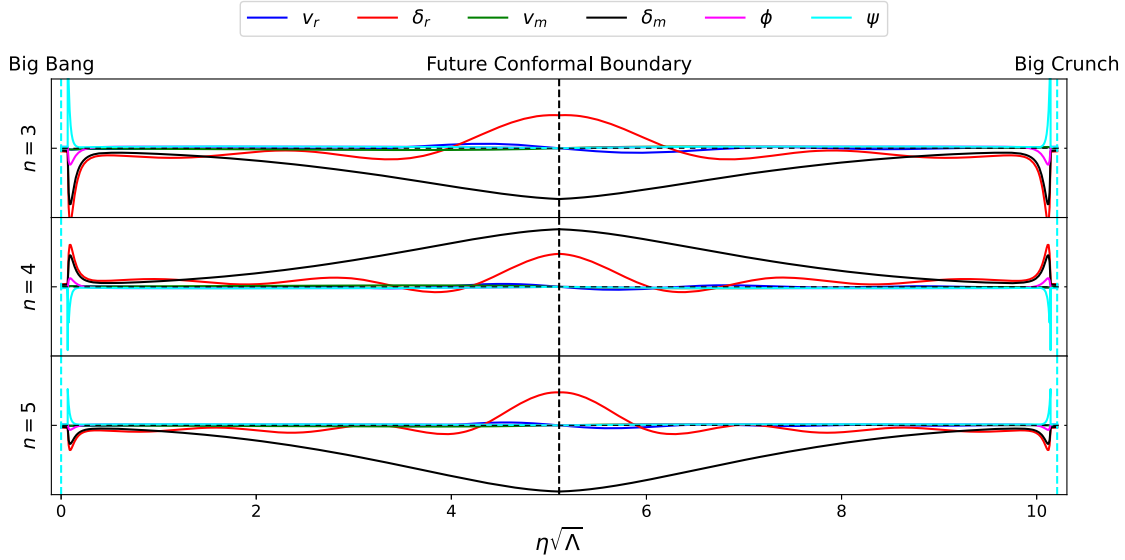


FIG. 2. Numerical solutions for several allowed perturbation modes (modes 3–5 are shown to clarify the oscillatory behavior). As expected, the solutions are totally symmetric or antisymmetric at the future conformal boundary. The plot uses best-fit parameters corresponding to the $\Delta k = 4$ case: $\Omega_m = 0.459$, $\Omega_K = -0.039$, $h = 0.547$, and $\Omega_b = 0.075$.

This approach is valid because the likelihood surface of the standard model serves as a good approximation for our model.

Our procedure is as follows: For each sample point in the Planck MCMC chain, we use its cosmological parameters ($\Omega_m, \Omega_K, \Omega_b, h$, etc.) to calculate $\Delta k \equiv \Delta \tilde{k}(\tilde{m}, \tilde{\kappa}, \Omega_b, h) / \sqrt{\tilde{\kappa}}$. We then filter the chains, retaining only those samples for which this calculated value is approximately equal ($\pm 2\%$) to an integer N . This filtering process isolates the regions of the standard model’s posterior distribution that are consistent with our theoretical constraint. These filtered subsamples are then used to construct the marginalized posterior distributions for each allowed integer N , which we visualize using ANESTHETIC [16].

V. RESULTS AND DISCUSSION

Figure 2 displays the numerical solutions for several allowed perturbation modes. As theoretically expected, these solutions exhibit total symmetry or antisymmetry at the future conformal boundary. The solutions are generated using the best-fit cosmological parameters for the $\Delta k = 4$ case, a choice we justify later as being favored by Planck 2018 data (Fig. 4).

These perturbation solutions are a crucial input for our analysis. Specifically, they are used to compute the function $\Delta \tilde{k}(\tilde{m}, \tilde{\kappa}, \Omega_b, h)$, which, in turn, allows us to filter the Planck 2018 MCMC data samples by selecting only those that satisfy the theoretical constraint $\Delta \tilde{k} / \sqrt{\tilde{\kappa}} = N$. The outcome of this filtering process, which constitutes the primary result of our analysis, is shown in Fig. 3. This figure presents the marginalized posterior distributions derived from the constrained MCMC samples. The full posterior for the standard Λ CDM + Ω_K

model is shown in orange, while the colored contours highlight the subsets of samples that satisfy our consistency condition, $\Delta k \approx N$, for different integer values of N . The plot clearly demonstrates that our theoretical framework restricts the allowed parameter space to a series of discrete “islands,” most prominently visible in the posterior for the spatial curvature parameter, Ω_K . This corresponds to a discrete set of allowed values for the spatial curvature, with the most favored values

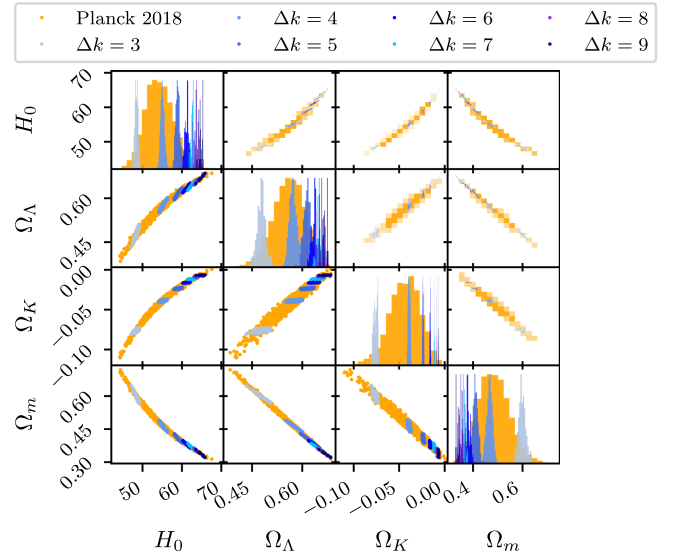


FIG. 3. Marginalized posterior distributions from Planck 2018 MCMC samples for the `base_omegak_plikHM_TTTEEE_lowl_lowE` likelihood combination. The orange contours represent the full posterior for the standard Λ CDM + Ω_K model. The colored regions highlight the subsets of samples that satisfy our consistency constraint $\Delta k \approx N$ for various integers N .

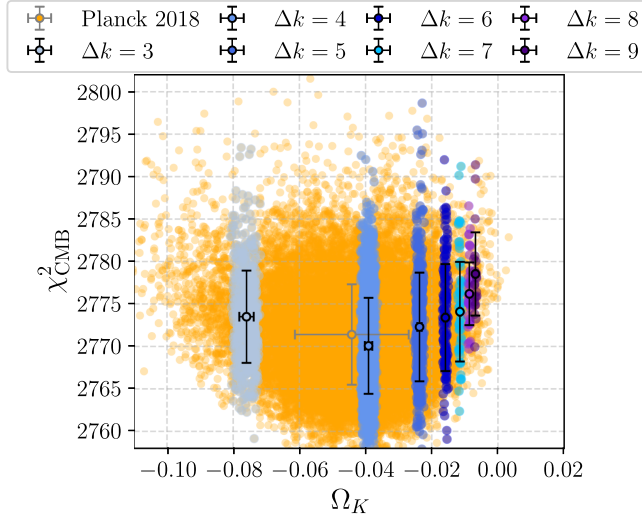


FIG. 4. A scatter plot of χ^2 versus Ω_K for the filtered MCMC samples. The points show the mean value and 1σ error bars for each integer solution $\Delta k = N$. The minimum χ^2 occurs for $N = 4$, indicating it is the solution most favored by the Planck 2018 data.

being $\Omega_K \in [-0.076, -0.039, -0.024, -0.016, -0.012, \dots]$ for $N = 3, 4, 5, 6, 7, \dots$, respectively.

As predicted by our theoretical considerations in Sec. IV A, the results show a distinct inverse relationship between the integer N and the magnitude of the spatial curvature $|\Omega_K|$. Larger values of N correspond to solutions that are closer to a spatially flat universe. This is a direct consequence of the constraint $\Delta\tilde{k} = N\sqrt{\tilde{k}}$ coupled with the relative stability of the calculated $\Delta\tilde{k}$ across the parameter space.

To determine which of these discrete parameters is most favored by the data, we examine the χ^2 value for each family of parameters. Figure 4 plots χ^2 against Ω_K for the filtered samples. The analysis reveals a clear minimum at $N = 4$, indicating that this is the most probable solution according to the Planck 2018 data alone.

It is important to discuss the cosmological parameters associated with the best-fit solution ($N = 4$). The preferred values, such as $\Omega_m = 0.467^{+0.012}_{-0.013}$ and $H_0 = 55.045^{+0.518}_{-0.485} \text{ km s}^{-1} \text{ Mpc}^{-1}$, are different from the values derived from a flat Λ CDM cosmology and other cosmological probes. This discrepancy is a well-known consequence of the strong degeneracy within the CMB data: A wide range of parameter combinations, particularly involving Ω_K , Ω_m , and H_0 , can produce nearly identical CMB power spectra.

To break this degeneracy and robustly test our model, it is necessary to perform a joint analysis with other cosmological datasets, such as BAOs and type Ia supernovae. However, such an analysis requires careful consideration of underlying assumptions. Many BAO analyses, for instance, assume a fiducial flat cosmology, which can inadvertently

bias the results against a curved universe. A rigorous test demands that spatial curvature be treated as a free parameter consistently across all datasets. Indeed, analyses that do so, such as the work by Glanville, Howlett, and Davis [4] on BAO data, have also shown a preference for a closed universe. A comprehensive joint analysis, which we defer to future work, will therefore be crucial for discriminating between the allowed integer solutions and determining the true quantized value of spatial curvature.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [17,18].

APPENDIX: TAYLOR EXPANSION COEFFICIENTS

To determine the cosmological perturbation solutions at the end of the Universe, denoted by $\mathbf{X}^\infty$, we employ a Taylor expansion. For this purpose, it is advantageous to reexpress the governing perturbation equations in terms of the reciprocal scale factor, $s \equiv 1/a$. This is because the end of the Universe (where the scale factor $a \rightarrow \infty$) corresponds to $s \rightarrow 0$, a limit around which the coefficients of the differential equations behave well, allowing for a systematic expansion. The solutions at $s = 0$ (conformal time η_∞) are then extrapolated from solutions at a slightly earlier conformal time $\eta' = \eta_\infty - \Delta\eta$, where $\Delta\eta$ is a small, positive interval and serves as the expansion parameter. The relationship between the solutions at η' and $\mathbf{X}^\infty$ is given by

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{y}'_{2:4} \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \mathbf{X}^\infty. \quad (\text{A1})$$

Here, $\mathbf{X}^\infty$ represents the vector of perturbation modes at η_∞ (note that $\phi^\infty = \psi^\infty = 0$)

$$\mathbf{X}^\infty = (\delta_r^\infty, \delta_m^\infty, v_r^\infty, v_m^\infty)^T. \quad (\text{A2})$$

The vectors $\mathbf{x}'$ and $\mathbf{y}'_{2:4}$ contain the perturbation variables at η' :

$$\begin{aligned}\mathbf{x}' &= (\phi', \psi', \delta'_r, \delta'_m, v'_r, v'_m)^T, \\ \mathbf{y}'_{2:4} &= (F'_{r2}, F'_{r3})^T.\end{aligned}\quad (\text{A3})$$

Note that periodic solutions satisfy $F_{r\ell}' = 0$ for $\ell > 3$, as they must exhibit (anti-)symmetry at the cosmic boundary [12].

The matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ contain the Taylor expansion coefficients.

The coefficients within these matrices depend on various cosmological parameters and the wave number k . We define the following dimensionless combinations:

$$c_1 = \frac{H_\infty^3 \Omega_M}{s_0^3 \Omega_\Lambda}, \quad c_2 = \frac{H_\infty^4 \Omega_R}{s_0^4 \Omega_\Lambda}, \quad c_3 = \frac{H_\infty^2 \Omega_K}{s_0^2 \Omega_\Lambda} \quad (\text{A4})$$

and a recurring denominator term:

$$\mathcal{K}_c = k^2 + 3c_3. \quad (\text{A5})$$

For brevity in presenting $\mathbf{X}_1$, we define the following intermediate expressions:

$$A = \frac{6k^6 + 26k^4 c_3 + 288c_2 c_3 + 12k^2(2c_3^2 - 7c_2)}{45\mathcal{K}_c}, \quad (\text{A6})$$

$$B = \frac{c_1}{\mathcal{K}_c} \left(-\frac{9}{2} \Delta\eta + \frac{2k^2 - 3c_3}{4} (\Delta\eta)^3 \right), \quad (\text{A7})$$

$$C = \frac{c_1}{\mathcal{K}_c} \left(-\frac{27}{2} \Delta\eta - \frac{3(k^2 + 12c_3)}{4} (\Delta\eta)^3 \right), \quad (\text{A8})$$

$$D = \frac{3k^2 + 4c_3}{120}. \quad (\text{A9})$$

The matrix $\mathbf{X}_1$ is then given by

$$\mathbf{X}_1 = \begin{pmatrix} 0 & \frac{c_1}{\mathcal{K}_c} \left(-\frac{3}{2} \Delta\eta - \frac{c_3}{4} (\Delta\eta)^3 \right) & \frac{c_2}{\mathcal{K}_c} \left(6\Delta\eta + \frac{k^2 + 8c_3}{5} (\Delta\eta)^3 \right) & \frac{c_1}{\mathcal{K}_c} \left(-\frac{9}{2} \Delta\eta - \frac{3(k^2 + 4c_3)}{4} (\Delta\eta)^3 \right) \\ 0 & \frac{c_1}{\mathcal{K}_c} \left(-\frac{3}{2} \Delta\eta - \frac{c_3}{4} (\Delta\eta)^3 \right) & \frac{c_2}{\mathcal{K}_c} \left(6\Delta\eta + \frac{7k^2 + 16c_3}{5} (\Delta\eta)^3 \right) & \frac{c_1}{\mathcal{K}_c} \left(-\frac{9}{2} \Delta\eta - \frac{3(k^2 + 4c_3)}{4} (\Delta\eta)^3 \right) \\ 1 - \frac{1}{6} k^2 (\Delta\eta)^2 & \frac{c_1}{\mathcal{K}_c} \left(-6\Delta\eta - \frac{3c_3 - 2k^2}{3} (\Delta\eta)^3 \right) & -\frac{4k^4 + 12k^2 c_3 - 72c_2}{3\mathcal{K}_c} \Delta\eta + A(\Delta\eta)^3 & \frac{c_1}{\mathcal{K}_c} (-18\Delta\eta - (k^2 + 12c_3)(\Delta\eta)^3) \\ 0 & 1 + B & \frac{c_2}{\mathcal{K}_c} \left(18\Delta\eta + \frac{24c_3 - 7k^2}{5} (\Delta\eta)^3 \right) & \frac{1}{2} k^2 (\Delta\eta)^2 + C \\ \frac{1}{4} \Delta\eta - D(\Delta\eta)^3 & -\frac{3c_1}{2\mathcal{K}_c} (\Delta\eta)^2 & 1 - \frac{3k^4 + 13k^2 c_3 + 12(c_3^2 - 5c_2)}{10\mathcal{K}_c} (\Delta\eta)^2 & -\frac{9c_1}{2\mathcal{K}_c} (\Delta\eta)^2 \\ 0 & -\frac{3c_1}{2\mathcal{K}_c} (\Delta\eta)^2 & \frac{6c_2}{\mathcal{K}_c} (\Delta\eta)^2 & -\Delta\eta - \frac{9c_1}{2\mathcal{K}_c} (\Delta\eta)^2 - \frac{c_3}{6} (\Delta\eta)^3 \end{pmatrix}. \quad (\text{A10})$$

The matrix $\mathbf{X}_2$ is given by

$$\mathbf{X}_2 = \begin{pmatrix} \frac{1}{15} k^2 (\Delta\eta)^2 - \frac{k^2(15k^2 + 14c_3)}{3150} (\Delta\eta)^4 & -\frac{4k^2 c_1}{15\mathcal{K}_c} (\Delta\eta)^3 & \frac{8}{15} k^2 \Delta\eta - \frac{4k^2(30k^4 + 118k^2 c_3 + 84(c_3^2 - 5c_2))}{1575\mathcal{K}_c} (\Delta\eta)^3 & -\frac{4c_1 k^2}{5\mathcal{K}_c} (\Delta\eta)^3 \\ -\frac{1}{105} k^3 (\Delta\eta)^3 & \frac{k^3 c_1}{35\mathcal{K}_c} (\Delta\eta)^4 & -\frac{4}{35} k^3 (\Delta\eta)^2 + \frac{k^3(30k^4 + 118k^2 c_3 + 84(c_3^2 - 5c_2))}{3675\mathcal{K}_c} (\Delta\eta)^4 & \frac{3k^3 c_1}{35\mathcal{K}_c} (\Delta\eta)^4 \end{pmatrix}. \quad (\text{A11})$$

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