

Analytic black hole perturbation theory package for gravitational scattering amplitudes

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Applications of effective field theory (EFT) and scattering amplitudes to gravitational problems have recently produced many unique results that advanced our understanding of the dynamics of compact binaries. Many of these results were made possible by comparing gravitational scattering amplitudes in EFT with exact expressions from general relativity. However, the latter expressions are not easily available as they require intricate solution techniques for the Teukolsky master equation, such as the Mano-Suzuki-Takasugi (MST) method. In this paper, we develop and present the first public package that enables computations of gravitational scattering amplitudes in black hole perturbation theory within the MST framework. Our package supports both analytic computations to a given post-Minkowskian (PM) order in the low-frequency limit and exact numerical computations for an arbitrary frequency of the perturbing field. As an application, we compute scattering phase shifts and inelasticity parameters for massless spin—0, 1, and 2 fields resulting from scattering off a rotating Kerr black hole through the third PM order and compare these results with the exact numerical solutions.

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I. INTRODUCTION

Progress in gravitational wave astronomy [1–8], culminating with the most recent release of the first part of the LIGO-Virgo-KAGRA data from the fourth observing run (O4a) [9], has provided strong motivation for novel theoretical developments in gravitational physics. In particular, the application of techniques from the high-energy physics toolbox, such as effective field theories (EFTs) and scattering amplitudes [10–32], have produced many new results that have significantly advanced our understanding of gravitational dynamics. For example, gravitational scattering amplitudes can be used to define and extract tidal responses of black holes and exotic compact objects in a manifestly gauge-invariant and coordinate-independent fashion. For instance, the conservative tidal Love numbers that capture a compact body’s responses to external gravitational fields are defined as Wilson coefficients in the worldline

EFT [10,33].¹ The universality of EFT then allows one to match Love numbers from a simple scattering amplitude calculation and then use the results of this measurement to make predictions for gravitational waveforms [33,44–47].

From the practical perspective, gravitational scattering amplitudes in EFT have to be matched to scattering amplitudes in general relativity in order to extract these Wilson coefficients. In the context of black holes, the execution of this program requires one to have analytic expressions for scattering amplitudes, which in principle can be computed from black hole perturbation theory (BHPT) [48–50] based on the Teukolsky master equation [51–53]. In particular, BHPT equations admit a semianalytic solution in terms of a series over hypergeometric functions via the Mano-Suzuki-Takasugi (MST) method [54–57].² In practice, however, the analytic scattering amplitude calculations in BHPT are quite contrived and even though they have been computed in a number of papers, e.g. [44–46,63–65], it is not easy to obtain them from scratch. This contrasts with the situation in other aspects of BHPT, where excellent numerical tools, such as the Black Hole Perturbation Toolkit

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¹See [34–43] for detailed discussions about Love numbers of black holes and their various definitions and extraction modes, as well as relations to hidden symmetries.

²See also [27,58–62] for alternative techniques.

package [66], are readily available.³ In this work, we fill this gap and provide a first public package for analytic computations of BHPT scattering amplitudes.

Our paper is structured as follows. We first critically review analytic calculations on BHPT scattering amplitudes from the Teukolsky master equation and clearly write down all the ingredients needed for such calculations in Sec. II. Then, we present a method of building the analytic solution to a given post-Minkowskian (PM) order (order in Newton's gravitational constant G) using the MST method in Sec. III. There, we also present the results of our method, which includes several novel results, such as expressions for scattering phase shifts of the gravitational Raman process with a nonzero black hole spin. Section IV draws conclusions. Appendices A, B, and C contain additional tables and plots. In this paper, we use geometrized units, that is, we set Newton's gravitational constant G and the speed of light c to unity.

II. BACKGROUND

A. Scattering in BHPT from the Teukolsky master equation

We seek a perturbative solution to scattering amplitudes in black hole perturbation theory using the Teukolsky master Equation [51], based on the Newman-Penrose formalism [67,68]. In the following discussion, M denotes the mass of the black hole, a its spin parameter, and (r, θ, ϕ) the spherical coordinates centered at the black hole singularity.

We consider three types of massless fields: the spin-0 scalar field, the spin-1 electromagnetic field, and the spin-2 perturbation of the gravitational field described by the Teukolsky equation. The field perturbation are captured by the Newman-Penrose scalars Ψ_i , which separate into radial and angular components. For example, for Ψ_4 , we may write

$$\Psi_4 \propto e^{-i\omega t} e^{im\phi} {}_sS_l^m(\theta; a\omega) {}_sR_{lm\omega}(r), \quad (1)$$

which causes the perturbative Newman-Penrose equations to separate into

$$\Delta^{-s} \frac{d}{dr} \left(\Delta^{s+1} \frac{dR}{dr} \right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda_{lm} \right) R = 0, \quad (2a)$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dS}{d\theta} \right) + \left(a^2\omega^2 \cos^2\theta - \frac{m^2}{\sin^2\theta} - \frac{2ms \cos\theta}{\sin^2\theta} - 2a\omega s \cos\theta - s^2 \cot^2\theta + s + A_{lm} \right) S = 0. \quad (2b)$$

Here, $\Delta = r^2 - 2Mr + a^2$, $K = (r^2 + a^2)\omega - am$, and $\lambda_{lm} = A_{lm} + a^2\omega^2 - 2am\omega$. In this paper, we conventionally retrieve the scattering phase shift from Equation (4) for $s \geq 0$, working with negative s only to get the necessary intermediate quantities ${}_sB_{lm\omega}^{(refl/inc)}$. We thus set $s = 0$ for scalar perturbations, $s = 1$ for electromagnetic perturbations, and $s = 2$ for gravitational perturbations. For a fixed s , the solution space has a basis enumerated by the integers l and m , for which it holds $|s| \leq l$ and $l \geq m \geq -l$, as well as by the non-negative real angular frequency ω .

To proceed, we seek a solution as an expansion in the parameter $\epsilon = 2M\omega$, as is standard in the MST approach. If no fixing of the constants G and c to unity is done, the parameter has the form $\epsilon = 2GM\omega/c^3$. Therefore, without assumptions about the other quantities in the expression for ϵ , an expansion in small ϵ is formally equivalent to an expansion in small G , corresponding to a post-Minkowskian approach. The angular Equation (2b) has well-known solutions through the spheroidal harmonics. These, as well as the value of the eigenvalue λ_{lm} have known expansions in ϵ to arbitrary order [69], and are provided through the Black Hole Perturbation Toolkit (BHPT toolkit) [70].

Now, let us discuss the radial Equation (2). To start, define the tortoise coordinate r^* to satisfy $dr^*/dr = (r^2 + a^2)/\Delta$, the horizons of the rotating black hole as $r_{\pm} = M \pm \sqrt{M^2 - a^2}$, and the quantity $\Omega_h = a/(2Mr_+)$. Then, considering extremal solutions to (2), we recover

$${}_sR_{lm\omega}(r) \sim \begin{cases} {}_sB_{lm\omega}^{(trans)} \Delta^{-s} e^{-i(\omega - m\Omega_h)r_*} & \text{for } r_* \rightarrow -\infty, \\ {}_sB_{lm\omega}^{(inc)} r^{-1} e^{-i\omega r_*} + {}_sB_{lm\omega}^{(refl)} r^{-2s-1} e^{+i\omega r_*} & \text{for } r_* \rightarrow +\infty. \end{cases} \quad (3)$$

These relations set the asymptotic scattering boundary conditions at infinity.

B. Scattering and absorption

We are particularly interested in a characteristic phase factor related to the exponentials in (3) as $r^* \rightarrow +\infty$, defined as

$$e^{2i\delta_{lm\omega}^p} = (-1)^{l+1} \frac{{}_sC_l^m(a\omega)}{(2\omega)^{2s}} \frac{{}_sB_{lm\omega}^{(refl)}}{{}_sB_{lm\omega}^{(inc)}}. \quad (4)$$

Here, ${}_sC_l^m(a\omega)$ is the Teukolsky-Starobinsky constant [71], which has values of

³bhptoolkit.org.

$$\begin{aligned}
{}_0C_l^m(a\omega) &= 1, \\
{}_1C_l^m(a\omega) &= [({}_1\lambda_{lm} + 2 + a^2\omega^2 - 2am\omega)^2 + 4am\omega - 4a^2\omega^2]^{1/2}, \\
{}_2C_l^m(a\omega) &= [(-{}_2\lambda_{lm} + 2)^2 + 4am\omega - 4a^2\omega^2] \times [(-{}_2\lambda_{lm})^2 + 36am\omega - 36a^2\omega^2] \\
&\quad + (2 - {}_2\lambda_{lm} + 3)(96a^2\omega^2 - 48am\omega)]^{1/2} + 12iPM\omega.
\end{aligned} \tag{5}$$

For $s = 2$ only, the variable $P = \pm 1$ denotes parity, and can be chosen independently of the other mode numbers.

The phase $e^{2i\delta_{lm\omega}^P}$ can be related to important quantities in scattering theory. For example, as in [63], for $s = 2$ the scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |g(\theta)|^2, \tag{6}$$

where the partial wave series take the values

$$\begin{aligned}
f(\theta) &= \frac{\pi}{i\omega} \sum_{P=\pm 1} \sum_{l=2}^{\infty} {}_{-2}S_l^2(0; a\omega) {}_{-2}S_l^2(\theta; a\omega) \\
&\quad \cdot [e^{2i\delta_{lm\omega}^P} - 1],
\end{aligned} \tag{7a}$$

$$\begin{aligned}
g(\theta) &= \frac{\pi}{i\omega} \sum_{P=\pm 1} \sum_{l=2}^{\infty} P(-1)^l {}_{-2}S_l^2(0; a\omega) {}_{-2}S_l^2(\pi - \theta; a\omega) \\
&\quad \cdot [e^{2i\delta_{lm\omega}^P} - 1].
\end{aligned} \tag{7b}$$

The absorption cross section is determined by

$$\sigma_a = \frac{4\pi^2}{\omega^2} \sum_{l=2}^{\infty} |{}_{-2}S_l^2(0; a\omega)|^2 \mathbb{T}_{l2}, \tag{8}$$

with the transmission factor $0 \leq \mathbb{T}_{lm} \leq 1$ given by

$$\mathbb{T}_{lm} = 1 - |e^{2i\delta_{lm\omega}^P}|^2. \tag{9}$$

Similarly, one can relate these cross sections to the imaginary parts of $\delta_{lm\omega}$ for $s = 0, 1$. Therefore, determining the phase factor is of critical importance for describing scattering around rotating black holes.

C. The Mano-Suzuki-Takasugi method

To proceed, the ratio ${}_sB_{lm\omega}^{(refl)} / {}_sB_{lm\omega}^{(inc)}$ needs to be determined and inserted into Eq. (4). As discussed, we seek an expansion of the solution in low $\epsilon = 2M\omega$. To that end, we invoke the MST formalism [56].

Following it, we obtain a series expansion in terms of hypergeometric functions for ${}_sR_{lm\omega}(r)$, which converges everywhere except for $r \rightarrow \infty$. Similarly, we obtain a solution in terms of Coulomb wave functions that converges everywhere outside the event horizon. Matching these two solutions at the horizon, we obtain the full solution. To tackle scattering, we are particularly interested

in the hypergeometric expansion, which converges near the singularity. The ingoing solution to Eq. (2a) can be written as

$$R_{in}^\nu = e^{i\epsilon\kappa x} (-x)^{-s-i\frac{\epsilon+\tau}{2}} (1-x)^{i\frac{\epsilon-\tau}{2}} p_{in}^\nu(x), \tag{10}$$

where we have defined $q = a/M$, $\kappa = \sqrt{1-q^2}$, $\tau = (\epsilon - mq)/\kappa$, and $x = \omega(r_+ - r)/(\epsilon\kappa)$. For $p_{in}^\nu(x)$, we have the expansion

$$p_{in}^\nu(x) = \sum_{n=-\infty}^{\infty} a_n^\nu p_{n+\nu}(x), \tag{11}$$

where $p_{n+\nu}(x)$ denotes the standard hypergeometric function

$$p_{n+\nu}(x) = F(n + \nu + 1 - i\tau, -n - \nu - i\tau; 1 - s - i\epsilon - i\tau; x). \tag{12}$$

The parameter ν in the above equation is called the renormalized angular momentum, which generalized the orbital quantum number l beyond the Newtonian limit. We discuss its determination shortly.

After inserting (10) into Eq. (2), we get that the expansion coefficients a_n^ν satisfy the three-term recurrence relation

$$\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0, \tag{13}$$

with the recurrence coefficients given by

$$\begin{aligned}
\alpha_n^\nu &= \frac{i\epsilon\kappa(n + \nu + 1 + s + i\epsilon)}{n + \nu + 1} \\
&\quad \cdot \frac{(n + \nu + 1 + s - i\epsilon)(n + \nu + 1 + i\tau)}{2n + 2\nu + 3},
\end{aligned} \tag{14a}$$

$$\begin{aligned}
\beta_n^\nu &= -\lambda_{lm} - s(s + 1) + (n + \nu)(n + \nu + 1) + \epsilon^2 \\
&\quad + \epsilon(\epsilon - mq) + \frac{\epsilon(\epsilon - mq)(s^2 + \epsilon^2)}{(n + \nu)(n + \nu + 1)},
\end{aligned} \tag{14b}$$

$$\gamma_n^\nu = -\frac{i\epsilon\kappa(n + \nu - s + i\epsilon)(n + \nu - s - i\epsilon)(n + \nu - i\tau)}{(n + \nu)(2n + 2\nu - 1)}. \tag{14c}$$

Conventionally, we take $a_0^\nu = 1$. An important point to note is that ν is a new parameter. To fix it, the solutions of the

three-point relation (13) emerging from $n \rightarrow -\infty$ match those of $n \rightarrow \infty$. This procedure is carefully examined in the next section.

To determine ν , raising and lowering continued fractions for MST coefficients are first defined as

$$R_n(\nu) = \frac{a_n^\nu}{a_{n-1}^\nu} = -\frac{\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}(\nu)}, \quad (15a)$$

$$L_n(\nu) = \frac{a_n^\nu}{a_{n+1}^\nu} = -\frac{\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n+1}(\nu)}. \quad (15b)$$

In the expansion of the MST recurrence relation in ϵ , the minimal solution for $n \rightarrow \infty$ is matched with the solution for $n \rightarrow -\infty$ by imposing connection formulas, an alternate form of Equation (13), between the raising and lowering ratios

$$\beta_u^\nu + \alpha_u^\nu R_{u+1}(\nu) + \gamma_u^\nu L_{u-1}(\nu) = 0. \quad (16)$$

Now, after expansion solutions to (2a) are obtained, we may determine ${}_s B_{lm\omega}^{(refl)} / {}_s B_{lm\omega}^{(inc)}$. First, we write

$$A_+^\nu = e^{-(\pi/2)\epsilon} e^{(\pi/2)i(\nu+1-s)} 2^{-1+s-i\epsilon} \cdot \frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1+s-i\epsilon)} \sum_{n=-\infty}^{\infty} a_n^\nu, \quad (17a)$$

$$A_-^\nu = e^{-(\pi/2)\epsilon} e^{-(\pi/2)i(\nu+1+s)} 2^{-1-s+i\epsilon} \cdot \sum_{n=-\infty}^{\infty} (-1)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} a_n^\nu. \quad (17b)$$

Finally, denoting K_ν the connection coefficient between the hypergeometric and Coulomb solution [64], we have

$${}_s B_{lm\omega}^{(inc)} = \omega^{-1} \left(K_\nu - i e^{-i\pi\nu} \frac{\sin(\pi(\nu-s+i\epsilon))}{\sin(\pi(\nu+s-i\epsilon))} \right) \cdot e^{-i(\epsilon \ln \epsilon - \frac{1-\kappa}{2}\epsilon)} A_+^\nu, \quad (18a)$$

$${}_s B_{lm\omega}^{(refl)} = \omega^{-1-2s} (K_\nu + i e^{i\pi\nu} K_{-\nu-1}) \cdot e^{i(\epsilon \ln \epsilon - \frac{1-\kappa}{2}\epsilon)} A_-^\nu. \quad (18b)$$

Relations (18a) and (18b) were obtained by considering the behavior of the solution (10) in appropriate extremal settings of the coordinate r_* .

A final point of note is that though the MST solution, when expanded in ϵ , converges well in the radial coordinate, it has a limited radius of convergence in ϵ [72] if the connection constraint (16) is solved perturbatively in ϵ . In other words, while the MST solution formally converges for any frequency in the numerical sense, the analytic solution has a finite radius of convergence in ϵ . For example, for $s = 2$, $l = 2$, $m = 0$, a discrepancy appears

in the renormalized angular momentum a little above $\epsilon = 0.7$. This is related to the development of the imaginary part for the renormalized angular momentum for ϵ greater than this value [57]. This fact will be evident later when our expansion is compared with a numerical solution obtained through the BHPT toolkit.

III. METHOD AND RESULTS

We aim to determine the analytical expansion of the scattering phase shift for low ϵ to a desired order n_0 in a robust, efficient, and repeatable way. To do this, an algorithm that automates most of the process was developed in *Mathematica*. The diagram below outlines the procedure. A notable detail is that the order that the MST coefficients are determined to in ϵ , as well as the order in ϵ of all intermediate expressions until the phase shift, might need to be bigger than the desired order in ϵ of the phase shift itself. The reason for this is that intermediate expressions might have other factors whose leading order is $O(1/\epsilon^k)$ for $k \geq 1$. To evaluate each of these expressions to a desired order, we need higher order terms in the MST coefficients and ν . Such considerations form the crux of the manual part of the procedure. Due to the high variability of such leading order terms for various starting parameters (s , l , and m), careful manual analysis is needed at every step to ensure we are obtaining every expression to the desired order.

The only systematic and user-friendly currently available public routine for determining relevant scattering parameters is numerical, through the BHPT toolkit. To recover ν , the toolkit numerically solves the transcendental connection Equation (16) via root-finding, after sufficiently expanding the continued fractions in Equations (15b) and (15a). Because it avoids analytic expansion of these fractional forms, this method does not suffer from convergence issues at large ϵ . To provide an equally convenient analytical method, we wrote and polished a new *Mathematica* notebook containing our algorithm. As we detail in the next section, our algorithm parallels the numerical approach of the BHPT toolkit. The key difference is that we solve the connection Equation (16) analytically, order by order, while the BHPT toolkit recovers $\cos(2\pi\nu)$ using monodromy eigenvalue methods [60–62]. This difference will not affect our subsequent qualitative discussion on the convergence of ν and related quantities in ϵ , as cosine is a function analytic on the whole complex plane.

In addition to presenting novel analytical expansions of the parameters and a ready-for-use analytical algorithm, we also compare the results with BHPT toolkit's numerical solutions. Specifically, for an example set of s , l , and m , the expanded ν is compared with the numerical module `RenormalizedAngularMomentum` from BHPT toolkit's `Teukolsky` package. In addition, though the BHPT toolkit offers no function to recover the scattering phase

factor directly, another new *Mathematica* notebook was written that first calculates ν numerically using the toolkit, then determines high-order MST coefficients explicitly using (15a) and (15b), and finally inserts them into (18a), (18b), and (4). This way, convergence issues are avoided as every intermediate quantity is determined numerically. In spirit, this mirrors the way the toolkit uses MST coefficients to obtain desired quantities. Notably, in addition to a module for numerically determining the phase factor, the newly written notebook also contains a module to test the numerical convergence of specified MST coefficients. The obtained numerical phase factor is then compared with our analytical expansion. In both these comparisons, we expect a discrepancy for large ϵ due to the aforementioned convergence issues of the analytical expansions.

A. An algorithm for MST coefficients

First, the expansion of the renormalized angular momentum ν and the MST coefficients a_n^ν is required to order $\tilde{n}_0 \geq n_0$ in ϵ . This process corresponds to steps one and two in Fig. 1.

The appropriate connection formula (16) also fixes the value of ν . For convenience, we choose $u = 0$. Thus, to fix ν up to order $\tilde{n}_0$ in ϵ , $R_1(\nu)$ and $L_{-1}(\nu)$ are required to order $\tilde{n}_0$ as well. To get them, the lowermost $\tilde{n}_0$ raising ratios for $n > 0$ are obtained using the continued formula (15a), with $R_{\tilde{n}_0}(\nu)$ calculated with $R_{\tilde{n}_0+1}(\nu)$ set to zero, while the uppermost $\tilde{n}_0 + \Delta n$ lowering ratios for $n < 0$ are obtained using (15b) in the same way. The ratios are $O(\epsilon)$, and thus the n th coefficient is $O(\epsilon^{|n|})$, except in certain special cases for $n < 0$ (which is why we calculate $\tilde{n}_0 + \Delta n$ lowering ratios instead of $\tilde{n}_0$, setting Δn equal up to 3. See [56] Eq. (5.6) for exact order relationships and to guide the

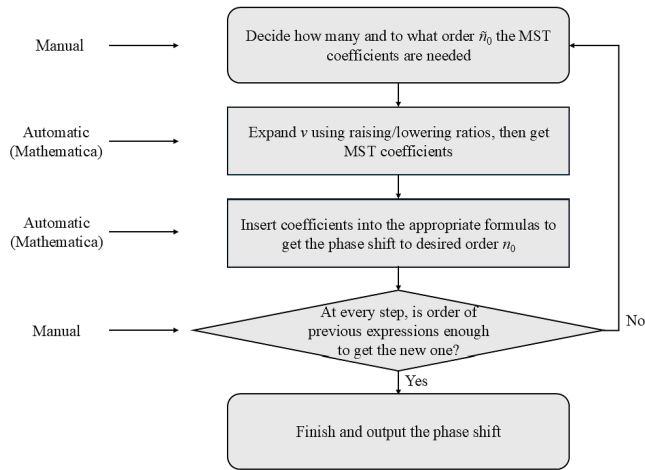


FIG. 1. Flowchart of the phase determination procedure. The manual steps are essentially choosing to what order each intermediate expression should be expanded to, while *Mathematica* carries out the automatic ones.

choice of Δn). The connection Equation (16) is solved order by order in ϵ , obtaining ν . This step is carried out using *Mathematica*'s `Eliminate` function on the connection formula.

Finally, the MST coefficients a_n^ν themselves are recovered from the definitions of the raising and lowering ratios in (15a) and (15b) and the fact that conventionally $a_0^\nu = 1$. Notably, sometimes ν is required to an order or two higher than $\tilde{n}_0$ to get the coefficients to $\tilde{n}_0$. This was the case for $s = 0$ and $l = 1$, with ν required to order 4 to get a_1^ν to order 3. Such issues are peculiarities of the order relationships from (15a) and (15b), and arise due to certain terms in the numerators or denominators of α_n^ν , β_n^ν , and γ_n^ν vanishing, changing their order in ϵ . A general way to check the order needed of ν for determination of the coefficients is not practical to implement, so each use of the algorithm requires a prior consideration of order relationships.

The obtained expansions of ν and the coefficients were found to match ones in [56] for general l , as well as for nongeneral l and $s = 0$ (which changes the order of a few ratios) in [73]. This signifies the method is working as expected. As an illustration, in Fig. 2, the real part of ν obtained using BHPT toolkit's Teukolsky package was plotted against the analytic expansion up to fourth order in ϵ for $s = 2$, $l = 2$, $m = 0$, showing an expected trend. Though it is possible to go to higher orders in ϵ , the expansion was truncated here for computational efficiency purposes in this illustrative example. As laid out in [72], the discontinuities and jumps of the numerical solution arise due to ν obtaining a non-negligible imaginary part at appreciable values of ϵ . In an eikonal approximation to the scattering problem, this imaginary part arises due to the shadow of the black hole in the impact parameter space [47]. Since the eikonal approximation pertains to large values of G and ω , the analytical expansion in low $\epsilon = 2GM\omega/c^3$ (geometrized units restored), somewhat

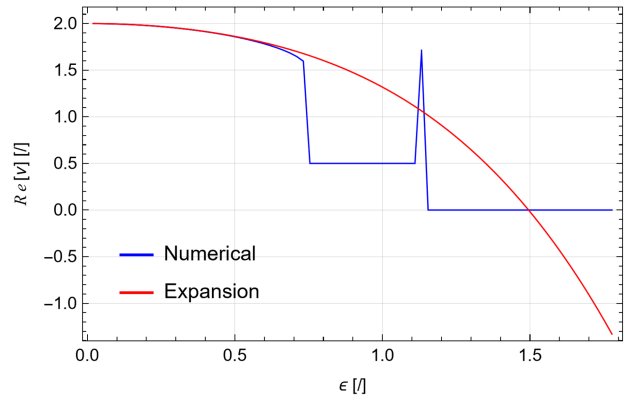


FIG. 2. Expansion of $\text{Re}[\nu]$ up to fourth order in ϵ and the numerical result of $\text{Re}[\nu]$ for $s = 2$, $l = 2$, $m = 0$. The analytic expansion and numerical result match well until around $\epsilon = 0.7$, which defines the radius of convergence of the analytic expression for the renormalized angular momentum.

expectedly, does not exhibit the jumps and has no imaginary part. Such considerations contribute to understanding the limited radius of convergence of the expansion. In the formal limit of $\epsilon \rightarrow 0$, however, convergence was observed. Though evidently not valid at large enough values of ϵ , the analytical expansion still offers an efficient and fully explicit way to recover analytical phase shifts, allowing for perturbative gravitational EFT interpretations such as the ones from the end of Subsection B.

B. The scattering phase factor

After obtaining the coefficients to order $\tilde{n}_0$, they can now be used to get the phase factor $\delta_{lm\omega}$ to order n_0 by inserting them into Equations (18a), (18b), and (10), carefully making sure the desired order is obtained after expanding at each step. This corresponds to steps 2 and 3 in Fig. 1. To save on computational resources, this step is carried out in three phases in *Mathematica*. First, $A_{\pm}^{\nu}$ are expanded using (17a) and (17b), and then, along with the exponential prefactors $e^{\pm i(\epsilon \ln \epsilon - \frac{1-\kappa}{2}\epsilon)}$, their ratio is expanded. This is known as the far zone contribution to the phase factor. Second, K_{ν} and $K_{-\nu-1}$ are expanded using their formulas from [64], Equation (A.11), by expanding first their prefactors, then the larger, more computationally intensive sum factors. They are then combined to yield the remainder of the formula for $_{-s}B_{lm\omega}^{(refl)} / _{-s}B_{lm\omega}^{(inc)}$. This is known as the near zone contribution to the phase factor. Finally, the far zone and near zone contributions are combined to give the phase factor from (4).

For $s = 0$, the obtained factor matched the expansion up to the provided order 3 in ϵ in [46]. More generally than there, the expansions for $s = 0$ with $l = 0$ and $l = 1$ were recovered in terms of $q \neq 0$. The same was done for $s = 1$ with $l = 1$ and $m = 0$, as well as $s = 2$ with $l = 2$, $m = 0$, and $P = -1$. Table I shows the newly calculated coefficients for $s = 2$, while in Appendix A, Table IV shows them for $s = 1$, and Table III shows them for $s = 0$. In addition, the expansion for general l and m was carried out for $s = 0$. Table II in Appendix A shows these results. Though technically possible to obtain too, general l expansions were not carried out for $s = 1$ and $s = 2$ due to a significant computation time needed. We note, similar to [46], that for the real part of the phase for $s = 0$ it holds

TABLE I. Expansion coefficients $_s c_{n,lm\omega}$ and $_s t_{n,lm\omega}$ of $\text{Re}[\delta_{lm\omega}]$ and $e^{-2\text{Im}[\delta_{lm\omega}]}$ for $s = 2$, $l = 2$, $m = 0$, $P = -1$.

	$2c_{n,20\omega}$	$2t_{n,20\omega}$
$n = 1$	$-\frac{5}{3} + \gamma_E$	0
$n = 2$	$\frac{107}{420}\pi$	0
$n = 3$	$\frac{107}{1260}\pi^2 - \frac{1}{3}\zeta(3) - \frac{1}{24}q^2 - \frac{29}{648}$	0

$$\begin{aligned} \text{Re}[\delta_{lm\omega}] &= \epsilon \ln(2\epsilon) + \sum_{n=1}^3 {}_0 c_{n,lm\omega} \epsilon^n \\ &+ \delta_{[l0]} \epsilon^3 \left[\frac{3}{2} - \frac{11\kappa}{12} - \gamma_E - \ln(2\kappa\epsilon) \right] \\ &+ O(\epsilon^4), \end{aligned} \quad (19)$$

where $\delta_{[l0]}$ is the Kronecker delta, while for $s = 1, 2$ we have just

$$\text{Re}[_s \delta_{lm\omega}] = \epsilon \ln(2\epsilon) + \sum_{n=1}^3 {}_s c_{n,lm\omega} \epsilon^n + O(\epsilon^4). \quad (20)$$

For the exponentiated imaginary part of the phase factor, which is necessary to determine the transmission factor in (9), for $s = 0, 1, 2$ we introduce the expansion

$$e^{-2\text{Im}[_s \delta_{lm\omega}]} = 1 + \sum_{n=1}^3 {}_s t_{n,lm\omega} \epsilon^n + O(\epsilon^4). \quad (21)$$

As a visual comparison, in Fig. 3 the obtained analytic expansion up to third order in ϵ of $\text{Re}[\delta_{lm\omega}]$ for $s = 2$, $l = 2$, $m = 0$ was compared to a precise numerical MST result obtained using the BHPT toolkit. As in the case for $\text{Re}[\nu]$, the expansion was truncated here for efficiency purposes, as well as because other state-of-the-art analytical expressions stop here [74]. For completeness, in Appendix B, the comparison process was repeated for $s = 1$ with $l = 1$ and $m = 0$, as well as for $s = 0$ with $l = 0, 1$ and $m = 0$, in Fig. 4. All the trends generally showed an expected shape here as well. While discontinuities similar to those of $\text{Re}[\nu]$ from Fig. 2 do not appear here, the expansion expectedly breaks down at a similar scale of ϵ that the one of $\text{Re}[\nu]$ does. In the limit $\epsilon \rightarrow 0$, convergence was observed for all

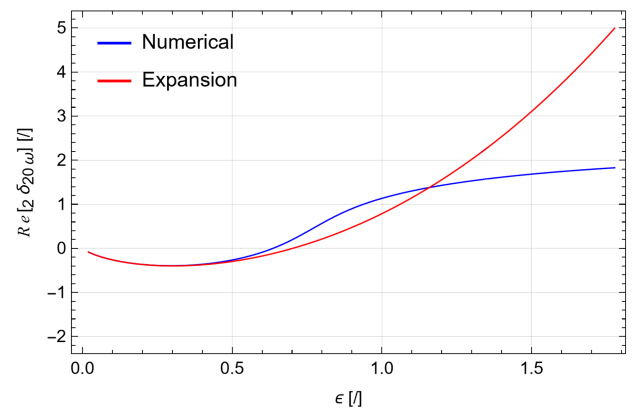


FIG. 3. Analytic expansion of $\text{Re}[_2 \delta_{20\omega}]$ for $q = 0.5$ and $P = -1$ up to third order in ϵ and the corresponding numerical result. Once again, agreement is seen until around $\epsilon = 0.75$, where the numerical result exhibits an inflection point, the same value of ϵ shown to be the convergence radius of the analytic expression for ν in this case.

calculated expansions, including the cases of $s = 0, 1$ from Appendix A. As with ν , expanding the phase factor, though not accurate at large enough ϵ , offers an explicit formula that can also be used for gravitational EFT matching calculations. Also, in Appendix B, the numerical result for the mode absorption factor, $e^{-2\text{Im}[\delta_{lmo}]}$, was compared with its expansion up to third order for $s = 0$ and $s = 1$ in Fig. 5, though for the smaller region $\epsilon \leq 0.5$ as divergence is observed earlier here. The $s = 2$ result was omitted in this case as this is a high-order effect. Finally, to see how much effect the rotation of the black hole has on the trends, $\text{Re}[\delta_{20\omega}]$ versus ϵ was plotted for various values of q . These plots are presented in Appendix C in Fig. 6.

The scattering phase shift has a clear interpretation within the framework of point-particle EFT [74], which allows one to write a gauge-invariant theory of perturbative gravity. This perturbation theory is controlled by the post-Minkowskian expansion. The low energy constants of EFT, dubbed Wilson coefficients, cannot be determined from EFT itself. These must instead be obtained by matching scattering amplitudes from post-Minkowskian perturbative GR with those of the EFT. The Wilson coefficients are directly connected to tidal Love numbers, which are critical in giving insight into a body's response to external time-independent gravitational fields. For $s = 0, 1$, the appropriate Love numbers enter into the phase shift already at 3PM order, which is part of the reason we truncate our expansion here. Our toolkit's ability to reach any desired order in ϵ for expansions of scattering parameters should be especially useful in such further investigations, as the $s = 2$ EFT tidal operator enters the scattering phase expansion at 5PM order, while the determination of the first dynamical Love number requires a 7PM order expansion.

C. Comparison with existing results and codes

While this manuscript was being prepared, Refs. [75–77] and the associated public code SFPN appeared, which can be used to carry out similar calculations. Where overlapping, our results agree to the orders in ϵ we present in this work.

Our main difference to SFPN is that our code is a documented piece of software specializing in the recovery of the scattering phase factor, among also the MST coefficients, renormalized angular momentum, and certain other Teukolsky equation solution parameters such as $A_{\pm}^{\nu}$ and K_{ν} . To determine these quantities analytically, particular care was needed in partitioning and simplifying intermediate results to achieve computational efficiency, as detailed in Fig. 1 and in our code's documentation (e.g. considering near-zone and far-zone contributions, the needed MST coefficient expansion orders, etc.). We thus offer a toolkit for immediate use within the perturbative gravitational scattering community. At the same time, our

code is well documented and adaptable to modifications at each step of its workflow, offering great robustness and adaptability to other problems in perturbative general relativity.

Having software developed independently by different groups greatly increases the robustness of the perturbative gravity results in the same way as having independent Boltzmann codes has been crucial in the establishment of precision cosmology. We hope that our code will contribute to the ambitious efforts of precision gravitational wave science, simultaneously connecting black hole perturbation theory and scattering amplitudes communities.

IV. CONCLUSIONS

Progress in the interpretation of gravitational wave data requires precision waveform models whose development benefits from the scattering amplitude computations. A particularly important problem in this context is the scattering of gravitational waves off isolated compact objects, which allows for a systematic on-shell matching of tidal effects. This problem, called the gravitational Raman scattering, is especially insightful as its amplitude can be computed exactly in general relativity in the absence of back-reaction. This a rare example of an exact highly nontrivial scattering amplitude that encodes rich gravitational physics.

In this paper, we develop and describe a module for both analytic and numerical computations of Raman scattering amplitudes in general relativity. Our module parallels the well-known black hole perturbation theory toolkit [66], but, in addition to numerical computations of scattering amplitudes, it allows one to extract analytic expressions. We reproduce some of the known results in the literature and also produce new results for the scattering phase shifts and inelasticity parameters for Kerr black holes in the post-Minkowskian limit. We also explore the issue of convergence of the analytic and numerical expansions of BHPT scattering amplitudes.

We hope that our analytic toolkit will be a useful addition to computational modules utilized by the black hole perturbation theory and scattering amplitudes communities.

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DATA AVAILABILITY

Our package is publicly available at Refs. [78,79].

APPENDIX A: EXPANSION COEFFICIENTS FOR $s=0$ AND $s=1$ TABLE II. Expansion coefficients ${}_s c_{n,l m \omega}$ of $\text{Re}[\delta_{lm\omega}]$ for $s=0$ and general l, m .

	${}_0 c_{n,l m \omega}$
$n=1$	$-\frac{1}{2} - \psi^{(0)}(1+l)$
$n=2$	$\frac{(-11+15l+15l^2)\pi}{4(-1+2l)(1+2l)(3+2l)} - \frac{mq}{2l(1+l)}$
$n=3$	$\frac{-11+15l+15l^2}{2(-1+2l)(1+2l)(3+2l)} \psi^{(1)}(1+l) + \frac{1}{6} \psi^{(2)}(1+l) + \frac{9(1+2l)(-2+3l(1+l))-6(-3+5l(1+l))mq\pi-3(1+2l)(l+l^2-3m^2)q^2}{12l(1+l)(-1+2l)(1+2l)(3+2l)}$

TABLE III. Expansion coefficients ${}_s c_{n,l m \omega}$ and $t_{n,l m \omega}$ of $\text{Re}[\delta_{lm\omega}]$ and $e^{-2\text{Im}[\delta_{lm\omega}]}$ for $s=0$ and a few specific l, m .

l	0	1	0
m	0	0	0
	${}_0 c_{n,l m \omega}$	${}_0 c_{n,l m \omega}$	${}_0 t_{n,l m \omega}$
$n=1$	$-\frac{1}{2} + \gamma_E$	$-\frac{3}{2} + \gamma_E$	0
$n=2$	$\frac{11\pi}{12}$	$\frac{19\pi}{60}$	$-\kappa - 1$
$n=3$	$\frac{11\pi^2}{36} - \frac{1}{3}\zeta(3) - \frac{1}{12} + \frac{q^2}{12}$	$\frac{19\pi^2}{180} - \frac{1}{3}\zeta(3) - \frac{1}{20}q^2$	$-\kappa\pi - \pi$

TABLE IV. Expansion coefficients ${}_s c_{n,l m \omega}$ and $t_{n,l m \omega}$ of $\text{Re}[\delta_{lm\omega}]$ and $e^{-2\text{Im}[\delta_{lm\omega}]}$ for $s=1, l=1, m=0$.

	${}_1 c_{n,10\omega}$	${}_1 t_{n,10\omega}$
$n=1$	$-\frac{5}{2} + \gamma_E$	0
$n=2$	$\frac{47}{120}\pi$	$\frac{q^8}{8}$
$n=3$	$\frac{47}{360}\pi^2 - \frac{1}{3}\zeta(3) - \frac{1}{8}q^2 + \frac{7}{96}$	0

APPENDIX B: COMPARISONS BETWEEN NUMERICAL RESULTS AND THE EXPANSIONS FOR OTHER VALUES OF s

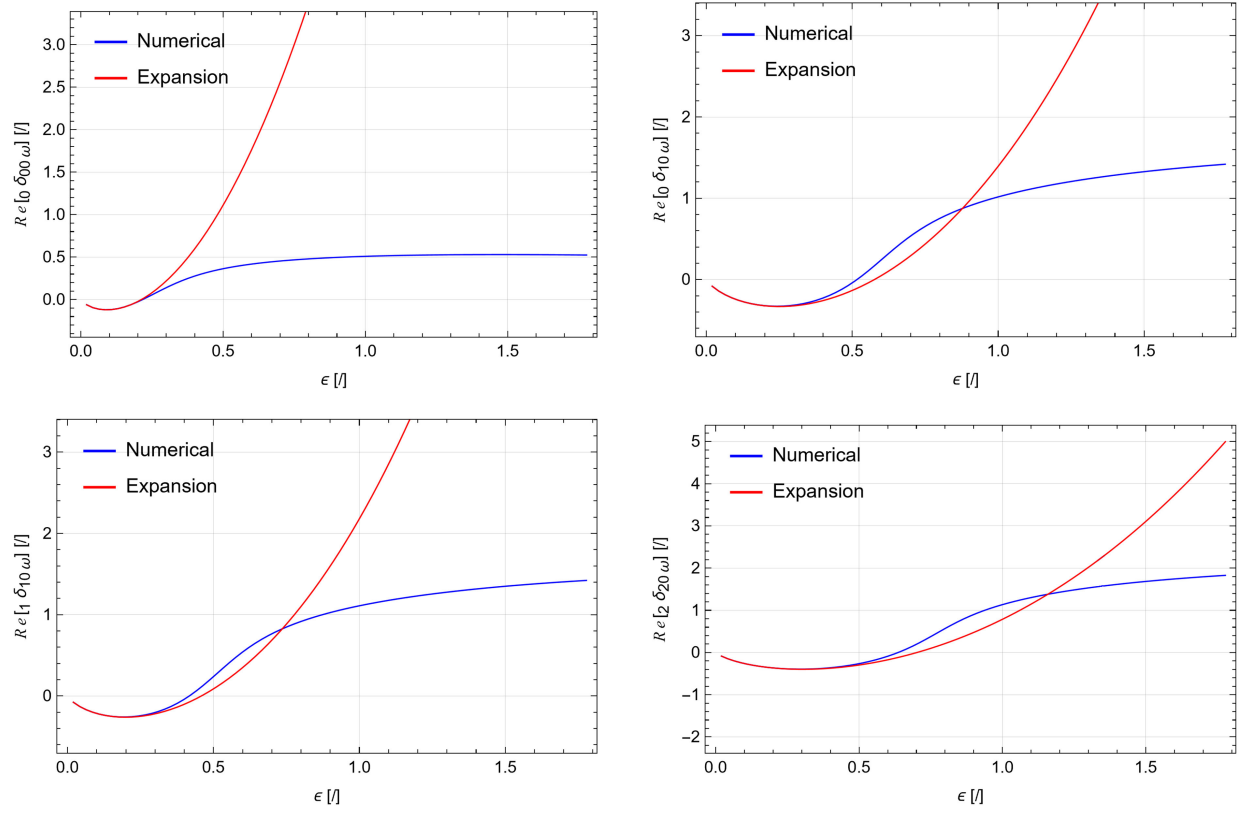


FIG. 4. Analytic expansions of $\text{Re}[s \delta_{lmo}]$ for $q = 0.5$ up to third order in ϵ and the corresponding numerical results. The plot for $s = 2$, $l = 2$, $m = 0$ from Fig. 3 was included again in the bottom right for purposes of comparison.

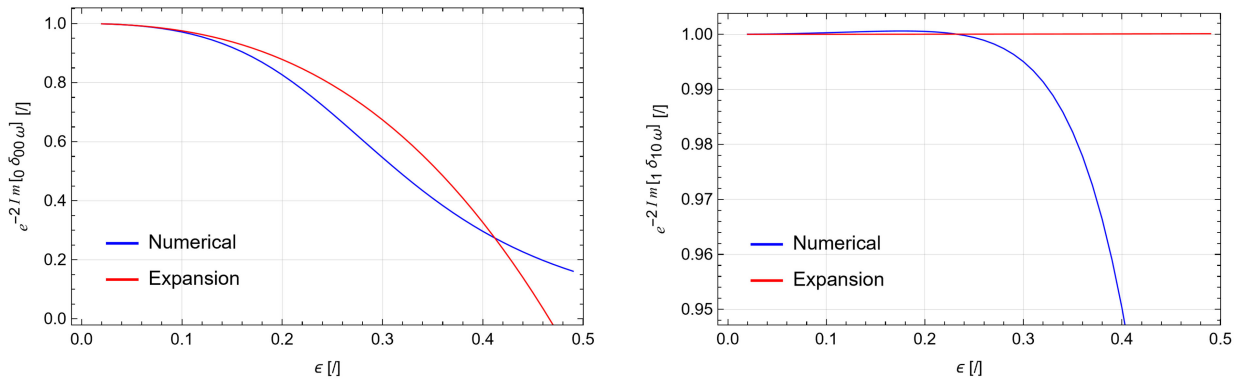


FIG. 5. Analytic expansions of $e^{-2 \text{Im}[s \delta_{lmo}]}$ for $q = 0.5$ up to third order in ϵ and the corresponding numerical results. Note that for $s = 1$, the expansion unexpectedly hovers around 1. This is due to the low expansion order in ϵ .

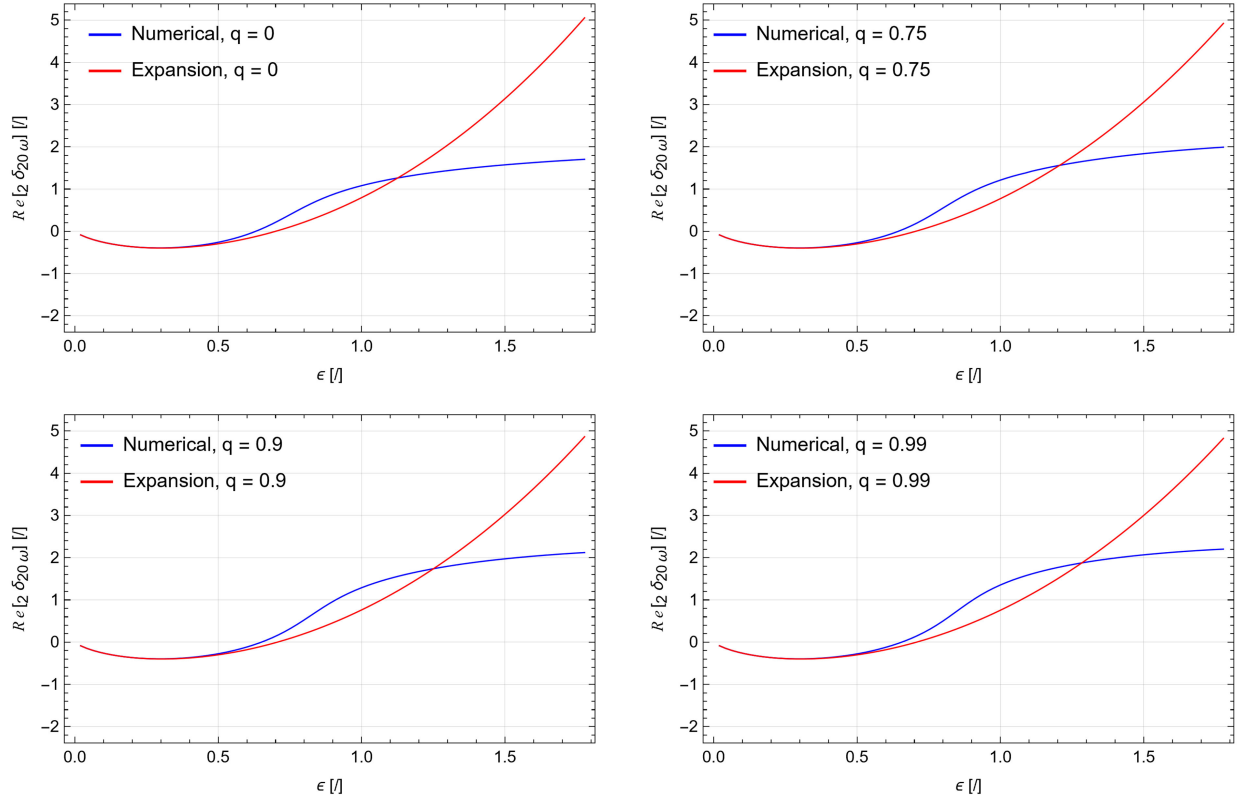
APPENDIX C: THE SCATTERING PHASE FACTOR WITH $s=2$ FOR VARIOUS VALUES OF q 

FIG. 6. Comparison between the numerical scattering phase $\text{Re}[e_{2\delta_{20\omega}}]$ and its expansion for different values of the Kerr parameter $q = a/M$. The dependence on q remains mild even close to the extremal limit.

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