

Quantum simulation of QC_2D on a two-dimensional small lattice

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We study the Hamiltonian formulation of $SU(2)$ Yang-Mills theory with staggered fermions in a $(2+1)$ -dimensional small lattice system. We construct a gauge-invariant and finite-dimensional Hilbert space for the theory by applying the loop-string-hadron formulation and specifically map the model to a spin system. We classically emulate digital quantum simulation and observe the real-time evolution of the single-site entanglement entropy, the fermion entanglement entropy, and the fermion pair production.

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I. INTRODUCTION

Gauge theory plays a crucial role in constructing the framework of modern particle physics. In particular, the $SU(3)$ gauge theory provides the foundation for quantum chromodynamics (QCD) that describes the strong interaction among quarks and gluons. One important feature of QCD is asymptotic freedom, which implies that perturbation theory becomes invalid at low energy scales. Therefore, researchers have developed several nonperturbative approaches to investigate the low-energy physics of QCD. One of the most successful approaches is the first-principles calculation based on the path integral formulation for discrete variables, known as lattice QCD [1].

By numerically evaluating the path integral and taking the continuum limit, one can, in principle, investigate thermal expectation values of observables even in the nonperturbative region.

Conventionally, this integration is performed using the Markov-Chain Monte Carlo method, which does not work

when the integrand is nonreal positive and highly oscillating. This kind of difficulty in numerical integration is known as the sign problem [2–5], which occurs in a wide range of fields in physics. In lattice QCD calculations, one encounters the sign problem in situations such as those involving topological effects induced by the theta term, the finite baryon chemical potential region, and real-time evolution.

Even though there have been various approaches to challenge the sign problem based on the Lagrangian formalism, such as the complex Langevin method [6–8], Lefschetz thimble method [9–11] and some related methods [12–15], their success has yet to lead to practical applications in QCD.

The Hamiltonian formulation on a lattice has recently gained attention as an alternative approach to studying gauge theories [16,17]. This method avoids the sign problem by directly dealing with quantum states in their Hilbert space and calculating observables, rather than relying on Monte Carlo integration. However, the Hamiltonian formulation introduces a different practical challenge: the vastness of the Hilbert space to be handled. The Hilbert space of gauge fields is infinitely large because they are bosonic theories. Therefore, in practice, it is necessary to truncate the theories to make their Hilbert space finite. Several truncation methods have been proposed, including the loop-string-hadron (LSH) approach [18–20], q deformation [21–23], the local multiplet basis [24,25], and the quantum link model [26]. Each formulation has its own distinct features. None of the approaches

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are currently applicable to $3 + 1$ dimensional QCD. Researchers typically start with simpler cases than QCD, such as lower spatial dimensions, fewer flavors and colors, and by neglecting fermionic degrees of freedom.

Nevertheless, even after truncation, the Hilbert space often remains significantly large. To address this issue, quantum computation emerges as a promising tool. A quantum computer with N qubits has a memory capacity of $O(2^N)$, whereas a classical computer with N classical bits has a memory capacity of only $O(N)$. Due to the development of quantum computing, it is convincing that a quantum computer will be able to deal with this huge Hilbert space in the future as its computing resource increases exponentially with the increase in the number of qubits.

With the development in the formulation, significant progress has also been made in numerical simulations of gauge theories. Although numerous efforts have been made to investigate the quantum simulation of lattice gauge theories, there are still many steps to bridge the gap to QCD. For example, many studies focus on $1 + 1$ dimensional physics, such as the Schwinger model [27–35]. In the higher spatial dimensions such as $2 + 1$ D and $3 + 1$ D, there are studies investigating the dynamics of non-Abelian gauge theory without fermions [21–25, 36–46].

However, simulations of non-Abelian gauge theory with fermions in more than two spatial dimensions are still lacking, making it an open and attractive question.

One of the most interesting applications of the model is its real-time quantum evolution to investigate the thermalization of SU(2) gauge theory with fermions. This problem is closely related to the understanding of the thermalization process in the early stages of relativistic heavy-ion collisions [47, 48]. The theoretical picture of relativistic heavy-ion collisions based on perturbative QCD suggests that the initial state of the bulk matter produced is a far-from-equilibrium system of gluons [49–52]. This gluonic matter thermalizes due to the intrinsic dynamics of QCD, while fermions are produced, eventually evolving into a quark-gluon plasma. However, our understanding of the non-equilibrium processes remains incomplete due to the numerical difficulty of simulating the real-time nonequilibrium evolution of QCD.

In recent years, significant progress has been made in understanding quantum thermalization, particularly in the field of condensed matter physics. It has been suggested that quantum thermalization can be explained by the eigenstate thermalization hypothesis, which is also known to be closely related to entanglement and quantum chaos [53–55]. Motivated by these developments, increasing attention has been given to investigating thermalization mechanisms in quantum gauge theories using Hamiltonian-based simulations, even in finite, low-dimensional SU(2) systems [38, 40, 42, 44, 56]. Building upon this research

direction, investigating the thermalization of SU(2) gauge theory by incorporating fermions is a worthwhile endeavor. By including fermions, we can further explore fermion production and fermion entropy.

In this paper, we study the real-time quantum evolution of the SU(2) Yang-Mills theory with fermions in a $2 + 1$ dimensional small isolated system by using the LSH formulation. We consider the Kogut-Susskind Hamiltonian formulation [16, 17] on a single square lattice and introduce staggered fermions [17, 57] on the vertices to circumvent the fermion doubling [58, 59]. Using the Schwinger boson representation [18, 60] of the SU(2) gauge field, we can construct the physical states on the lattice by solving the Gauss law and figure out the actions of the Hamiltonian [18, 60]. Our simulation is based on a quantum computer emulator [18, 61]. To perform the quantum simulation, we map the model to a spin system.

This paper is organized as follows. In Sec. II, we first review the Kogut-Susskind Hamiltonian and show the necessary formulations. Then, we introduce the construction of the physical states by applying gauge invariant constraints and specifically calculate actions of the Hamiltonian. In Sec. III, we show how to map the model to a spin system and discuss our numerical results, which mainly focus on entanglement and pair production. In Sec. IV, we briefly discuss how to generalize our method to a larger lattice. Section V is summary and outlook. In Appendix A, we show the actions of all necessary gauge invariant operators. In Appendix B, we show the exact identity between the lattice system and the spin system. Besides, we also show all terms of the Hamiltonian matrix.

II. HAMILTONIAN FORMULATION

A. Formulation

We review the Hamiltonian formulation of the SU(2) lattice Yang-Mills theory, which is the so-called Kogut-Susskind Hamiltonian [16, 17], on a two-dimensional spatial lattice with the staggered lattice fermion. The Hamiltonian is given as

$$H = H_E + H_B + H_F, \quad (1)$$

where

$$H_E = K \sum_{\mathbf{x}, \mu} E^2(\mathbf{x}, \mu), \quad (2)$$

$$H_B = -\frac{1}{2K} \sum_P \text{tr}[U_\mu(\mathbf{x}) U_\nu(\mathbf{x} + \hat{e}_\mu) U_\mu^\dagger(\mathbf{x} + \hat{e}_\nu) U_\nu^\dagger(\mathbf{x})] + (\text{H.c.}), \quad (3)$$

$$H_F = \sum_{\mu, \mathbf{x}} [\eta_\mu(\mathbf{x}) \{\psi^\dagger(\mathbf{x}) U_\mu(\mathbf{x}) \psi(\mathbf{x} + \hat{e}_\mu) + (\text{H.c.})\} - (-1)^{x_1+x_2} m \psi^\dagger(\mathbf{x}) \psi(\mathbf{x})], \quad (4)$$

here $\eta_1(\mathbf{x}) = (-1)^{x_1+x_2}$ and $\eta_2(\mathbf{x}) = (-1)^{x_2}$. In this Hamiltonian, $E(\mathbf{x}, \mu)$, defined on the ends of links, is the chromo-electric field and $U_\mu(\mathbf{x})$ is the link variable, which is defined on the link. Besides, $\psi(x)$ is the staggered fermion field, which is defined on the sites. Under this definition, H_E , H_B , and H_F are the electric part, magnetic part, and fermionic part of the Hamiltonian, respectively. In this formula, $\mathbf{x}$ and $\mu(=x^1, x^2)$, respectively, represent the site position of two-dimensional spatial lattice and directions with $\hat{e}_\mu$ being the unit vector along the direction μ . Besides, P denotes the plaquette. Note that we redefine the Hamiltonian, shown in Eq. (1), by multiplying a factor $2a$ with a being the lattice constant. According to this, the parameter K is equal to g^2 with g being the coupling constant and the mass parameter, m , is equal to $2m_0a$ with m_0 being the intrinsic mass of fermion field. However, throughout this paper, unless otherwise specified, other physical quantities are normalized by the lattice spacing a .

There are two types of representations of the chromo-electric field $E(\mathbf{x}, \mu)$ on the lattice. One is the left electric field $E_L^a(\mathbf{x}, \mu)$, located at position $\mathbf{x}$ on the edge of the link connecting $\mathbf{x}$ to $\mathbf{x} + \hat{\mu}$. The other is the right electric field $E_R^a(\mathbf{x}, \mu)$, located at position $\mathbf{x}$ on the edge of the link connecting $\mathbf{x} - \hat{\mu}$ to $\mathbf{x}$. For the lattice SU(2) Yang-Mills theory, the nonvanishing commutation relation among the link variable and electric fields are given as

$$[E_L^a(\mathbf{x}, \mu), U_\nu(\mathbf{y})] = -\frac{\sigma^a}{2} U_\mu(\mathbf{x}) \delta_{\mu\nu} \delta_{\mathbf{x}\mathbf{y}}, \quad (5)$$

$$[E_R^a(\mathbf{x}, \mu), U_\nu(\mathbf{y})] = U_\mu(\mathbf{x}) \frac{\sigma^a}{2} \delta_{\mu\nu} \delta_{\mathbf{x}\mathbf{y}}, \quad (6)$$

$$[E_L^a(\mathbf{x}, \mu), E_L^b(\mathbf{y}, \nu)] = i\epsilon^{abc} E_L^c(\mathbf{x}, \mu) \delta_{\mu\nu} \delta_{\mathbf{x}\mathbf{y}}, \quad (7)$$

$$[E_R^a(\mathbf{x}, \mu), E_R^b(\mathbf{y}, \nu)] = i\epsilon^{abc} E_R^c(\mathbf{x}, \mu) \delta_{\mu\nu} \delta_{\mathbf{x}\mathbf{y}}, \quad (8)$$

where $\sigma^{a=1,2,3}$ are the Pauli matrices, $\delta_{\mu\nu}$ is the Kronecker delta, and ϵ^{abc} is the Levi-Civita symbol with $\epsilon^{123} = 1$. These relations mean that the left and right electric fields serve as generators for the link variable in the SU(2) gauge transformation. Note that we have the equality condition between the two types of electric fields as

$$\sum_a E_L^a(\mathbf{x}, \mu) E_L^a(\mathbf{x}, \mu) = \sum_a E_R^a(\mathbf{x}, \mu) E_R^a(\mathbf{x}, \mu), \quad (9)$$

indicating that the electric part of the Hamiltonian, given in Eq. (1), does not depend on the representation used. The physical states $|\Psi\rangle$ defined on the sites should satisfy the Gauss law constraint,

$$\left(\sum_\mu [E_L^a(\mathbf{x}, \mu) + E_R^a(\mathbf{x} - \hat{e}_\mu, \mu)] + Q^a(\mathbf{x}) \right) |\Psi\rangle = 0, \quad (10)$$

where the charge operator Q^a is defined as $Q^a(\mathbf{x}) = \psi^\dagger(\mathbf{x}) \frac{\sigma^a}{2} \psi(\mathbf{x})$. This charge operator also satisfies the SU(2) algebra as

$$[Q^a(\mathbf{x}), Q^b(\mathbf{y})] = i\epsilon^{abc} Q^c(\mathbf{x}) \delta_{\mathbf{x}\mathbf{y}}. \quad (11)$$

To construct a gauge-invariant basis for the Hilbert space and to calculate the matrix elements of observables in this basis, we use the Schwinger boson representation of the gauge field operator [18,60]. The electric field can be rewritten as

$$E_L^a = \alpha_i^\dagger \frac{\sigma_{ij}^a}{2} \alpha_j, \quad (12)$$

$$E_R^a = \beta_i^\dagger \frac{\sigma_{ij}^a}{2} \beta_j, \quad (13)$$

where $\alpha_{i=1,2}$ and $\beta_{i=1,2}$ are both boson annihilation operators whose specific meaning will be given in the next subsection. For magnetic field, we define the link variable as $U = U_L(\alpha) U_R(\beta)$ with

$$U_L(\alpha) = \frac{1}{\sqrt{N_L + 1}} \begin{pmatrix} \alpha_2^\dagger & \alpha_1 \\ -\alpha_1^\dagger & \alpha_2 \end{pmatrix}, \quad (14)$$

$$U_R(\beta) = \begin{pmatrix} \beta_1^\dagger & \beta_2^\dagger \\ -\beta_2 & \beta_1 \end{pmatrix} \frac{1}{\sqrt{N_R + 1}}, \quad (15)$$

which is the well-known form, reproducing the commutation relation of gauge field operators.

As the Fock space for Schwinger bosons, we can define the Hilbert space. Note that the defined Hilbert space is larger than the Hilbert space for the original theory even after limiting it to the subspace that satisfies the Gauss law. The equality condition (9) indeed imposes an additional constraint on the physical state, in addition to the Gauss law, as

$$N_L(\mathbf{x}, \mu) |\Psi_L\rangle = N_R(\mathbf{x}, \mu) |\Psi_R\rangle, \quad (16)$$

where $N_L = \sum_i \alpha_i^\dagger \alpha_i$ and $N_R = \sum_i \beta_i^\dagger \beta_i$ are the number operators of Schwinger bosons. Here $|\Psi_L\rangle, |\Psi_R\rangle$ are the physical states located at the left and right sites of the link $(\mathbf{x}, \mu)$. This implies that the number of Schwinger bosons on both sides of one link must be the same. This constraint is called the Abelian Gauss law [18].

Based on the above formulas, we can calculate the real-time evolution of the two-dimensional square lattice system according to the Schrödinger equation. But before this, we need to construct the physical state $|\Psi\rangle$ to satisfy the Gauss law, shown in Eq. (10), and the equality condition, shown in Eq. (16).

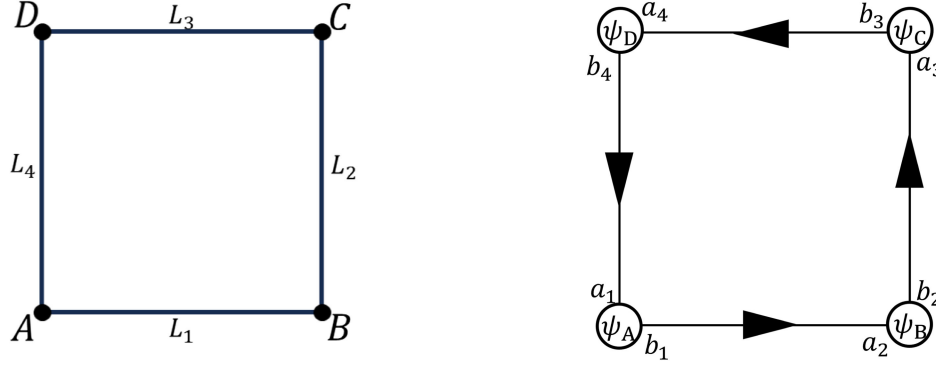


FIG. 1. Left: labels of links and sites on the $(2 + 1)$ dimension small lattice model. The links are labeled as L_i with $i = 1, 2, 3$, and 4 . The sites are labeled as A, B, C, and D, each representing the site at $(x^1, x^2) = (0, 0), (1, 0), (1, 1)$, and $(0, 1)$, respectively. Right: labels of the boson and fermion operators on sites on the $(2 + 1)$ dimension small lattice model. Along a path moving counterclockwise on the square lattice, we define the bosons associated with a given lattice point in the order, $b \rightarrow a$.

B. Construction of physical state

In this section, we intend to construct the physical state on the $(2 + 1)$ -dimensional little lattice. Basically, we have two constraints: the Gauss law constraint (10) for a local state at a single site and the equality condition (16) for connecting two adjacent sites. The labels for links and sites are given in the left of Fig. 1. The labels for the annihilation operators of bosons and fermions at the sites are shown on the right in Fig. 1. The labels for the creation operators are obtained by adding a dagger symbol to each annihilation operator, such as $a_i^\dagger$, $b_i^\dagger$, and $\psi_A^\dagger$. At this point, we do not distinguish whether the boson operator represents the left or right electric field, or whether the fermion operator represents a fermion or antifermion. However, later, we see that the fermions at sites B and D represent fermions, while those at sites A and C represent antifermions.

First, let us focus on the local states at single sites that preserve Gauss law. Such a state can be expressed by gauge-invariant operators as follows [18]:

$$|s, r_1, r_2, v\rangle = \frac{1}{\sqrt{C_n}} (\mathbf{b}^\dagger \cdot \tilde{\mathbf{a}}^\dagger)^s (\mathbf{b}^\dagger \cdot \tilde{\psi}^\dagger)^{r_1} (\mathbf{a}^\dagger \cdot \tilde{\psi}^\dagger)^{r_2} (\psi^\dagger \cdot \tilde{\psi}^\dagger)^v |0\rangle, \quad (17)$$

where s, r_1, r_2, v denote the quantum number of the state, $(\mathbf{a}^\dagger, \mathbf{b}^\dagger)$ and $\psi^\dagger$ denote the creation operators of Schwinger bosons and the staggered fermion, respectively, $|0\rangle$ is the ground state of the Fock space defined by these operators, and $C_n = 4^v s! (s + r_1 + r_2 + 1)$ is the normalization factor. Besides, the dot means inner product in color space with $\tilde{\mathbf{a}} = i\sigma^2 \mathbf{a}$. The parameter s can be any natural number, while r_1, r_2, v can only be 0 or 1 because of the anti-commutation relation of the fermion operator. We have two useful relations

$$(\mathbf{b}^\dagger \cdot \tilde{\psi}^\dagger)(\mathbf{a}^\dagger \cdot \tilde{\psi}^\dagger) = \frac{1}{2} (\mathbf{b}^\dagger \cdot \tilde{\mathbf{a}}^\dagger)(\psi^\dagger \cdot \tilde{\psi}^\dagger), \quad (18)$$

$$(\mathbf{a}^\dagger \cdot \tilde{\psi}^\dagger)(\psi^\dagger \cdot \tilde{\psi}^\dagger) = (\mathbf{b}^\dagger \cdot \tilde{\psi}^\dagger)(\psi^\dagger \cdot \tilde{\psi}^\dagger) = 0, \quad (19)$$

which indicate that states with $r_1 = r_2 = 1, v = 0$ are equivalent to states with $r_1 = r_2 = 0, v = 1$ and thus any two parameters of r_1, r_2, v can not be 1 at the same time. The expression shown in Eq. (17) is symmetric with respect to the exchange of a and b . The local state, given in Eq. (17), satisfies the following Gauss condition,

$$(E_a + E_b + Q_\psi)|\Psi\rangle = 0, \quad (20)$$

with $E_a = \mathbf{a}^\dagger \frac{\sigma}{2} \mathbf{a}$, $E_b = \mathbf{b}^\dagger \frac{\sigma}{2} \mathbf{b}$, and $Q_\psi = \psi^\dagger \frac{\sigma}{2} \psi$. This method to construct the local gauge-invariant state is known as the LSH formulation [18].

Let us consider the local state given in Eq. (17) in terms of another representation to discuss. Using the fact that the electric field and charge operators all satisfy the $SU(2)$ algebra, the Gauss law shown in Eq. (20) can be interpreted as the singlet condition of the physical state, namely requiring the “total angular momentum,” $E_a + E_b + Q_\psi$, to vanish. Such a singlet state can be expressed in terms of angular momentum quantum numbers as $|j_b, j_a, j_\psi\rangle$, where j_b, j_a , and j_ψ denote the angular momentum quantum numbers associated with E_b, E_a , and Q_ψ , respectively. However, the theory has a conserved quantity: the net fermion number. As a result, the fermion number is required as an additional label to distinguish states with different fermion numbers and uniquely specify the physical state. This is reflected in the relationship between the angular momentum and quantum numbers for the states in the LSH formulation,

$$j_b = \frac{1}{2}(s + r_1), \quad j_a = \frac{1}{2}(s + r_2), \quad j_\psi = \frac{1}{2}(r_1 + r_2), \quad (21)$$

which shows that different values of the label “ v ” can result in the same angular momentum.

Since the Hilbert space of a bosonic theory is infinite, we need to introduce some truncation to do the quantum simulation practically. Here, we only treat local states where all the quantum numbers of the angular momentum are lower than $j_{\max} = \frac{1}{2}$. The interpretation of the truncation implemented as a cutoff on the angular momentum quantum numbers becomes clearer from the viewpoint of the Schwinger boson representation, as discussed in Ref. [38]. While the electric term of the Hamiltonian does not change the number of Schwinger bosons, the magnetic term does. Thus, these can be regarded as the kinetic and interaction terms, respectively, in the Schwinger boson representation. When the coupling constant K is large, the magnetic term becomes negligible, and the Hamiltonian becomes nearly diagonal. In this regime, the effects of truncation on the Hamiltonian dynamics are more controlled. In contrast, when K is small, the magnetic term induces significant fluctuations, and the system enters a strongly correlated regime. In this case, the effect of truncation on physical observables becomes larger. Therefore, to perform reliable analyses in the small- K regime, one needs to carefully assess the truncation effects and choose a sufficiently high truncation level to suppress their influence. Under such a setup, we have six local states based on Eq. (17) as

$$\begin{aligned} &|0, 0, 0, 0\rangle, |1, 0, 0, 0\rangle, \\ &|0, 0, 1, 0\rangle, |0, 1, 0, 0\rangle, \\ &|0, 0, 0, 1\rangle, |1, 0, 0, 1\rangle. \end{aligned} \quad (22)$$

Here states at the first, second, and third lines have different eigenvalues for the number operator, $N_F \equiv \psi^\dagger \psi$, each having $N_F = 0, 1$ and 2 , respectively. To simplify the notation, we relabel these states as

$$\begin{aligned} |0\rangle &= |0, 0, 0, 0\rangle, \\ |2\rangle &= |0, 0, 0, 1\rangle, \\ |1\rangle &= |0, 0, 1, 0\rangle, \\ |\hat{1}\rangle &= |0, 1, 0, 0\rangle, \\ |\hat{0}\rangle &= |1, 0, 0, 0\rangle, \\ |\hat{2}\rangle &= |1, 0, 0, 1\rangle, \end{aligned} \quad (23)$$

where the left-hand side is our new notation with the number being N_f , and the hat means that the local state has a finite number of b bosons, $\langle b^\dagger b \rangle \neq 0$. It should be noted that the number operator N_f is not the fermion number operator, as there are no fermions but rather antifermions at sites A and C. Therefore, at sites A and C, $N_f - 2$ serves as the fermion number operator. This implies that the total fermion number operator across all sites, $N_f^{\text{tot}} \equiv N_{f,A} + N_{f,B} + N_{f,C} + N_{f,D}$, only differs from the net fermion number by the constant 4, and thus it is conserved.

For global state, the constraint (16) implies

$$(s + r_1)_i = (s + r_2)_f, \quad (24)$$

where the i, f mean the start and end of the link along the direction of counterclockwise around the square lattice. Thus, we can get all possible two point neighbor states as

$$\begin{aligned} &|00\rangle, |02\rangle, |20\rangle, |22\rangle, |\hat{0}\hat{0}\rangle, |\hat{0}\hat{2}\rangle, |\hat{2}\hat{0}\rangle, |\hat{2}\hat{2}\rangle, |0\hat{1}\rangle, \\ &|2\hat{1}\rangle, |\hat{0}\hat{1}\rangle, |\hat{2}\hat{1}\rangle, |10\rangle, |12\rangle, |\hat{1}\hat{0}\rangle, |\hat{1}\hat{2}\rangle, |1\hat{1}\rangle, |1\hat{1}\rangle, \end{aligned} \quad (25)$$

the left and right labels in the ket states correspond to the states at the neighboring sites, respectively. Based on the two point states above, we can spell out all global states like $|2020\rangle$, where the labels in the state are ordered as (A, B, C, D) from left to right.

To show the global states, we divide them into different groups according to the different eigenvalues of the total number operator N_f^{tot} . This is a benefit for both calculation and understanding the physics because of the fermion number conservation. First, we have the states with $N_f^{\text{tot}} = 0$ as

$$|0000\rangle, |\hat{0}\hat{0}\hat{0}\hat{0}\rangle. \quad (26)$$

There is no state that has an odd fermion number. Thus, next we show the $N_f^{\text{tot}} = 2$ states as

$$|2000\rangle, |\hat{2}\hat{0}\hat{0}\hat{0}\rangle, |\hat{1}\hat{1}00\rangle, |1\hat{1}\hat{0}\hat{0}\rangle, |\hat{1}\hat{0}10\rangle. \quad (27)$$

Here we only list representative states because cyclic permutations of labels in any global state automatically give us other global states. Thus, the number of states with $N_f^{\text{tot}} = 2$ should be 20. We list states with $N_f^{\text{tot}} = 4$ in the same manner as

$$\begin{aligned} &|2200\rangle, |2020\rangle, |\hat{2}\hat{2}\hat{0}\hat{0}\rangle, |\hat{2}\hat{0}\hat{2}\hat{0}\rangle, |\hat{1}\hat{1}\hat{1}\hat{1}\rangle, |1\hat{1}\hat{0}\hat{2}\rangle, \\ &|\hat{1}\hat{1}20\rangle, |1\hat{1}\hat{2}\hat{0}\rangle, |\hat{1}\hat{2}\hat{1}0\rangle, |12\hat{1}\hat{0}\rangle, |\hat{1}\hat{1}02\rangle, \end{aligned} \quad (28)$$

which gives that the number of states with $N_f^{\text{tot}} = 4$ is 38 after cyclic permutation of labels. To get the states with $N_f^{\text{tot}} = 6$ and 8, we only need to exchange $|0\rangle, |\hat{0}\rangle$ and $|2\rangle, |\hat{2}\rangle$ in states with $N_f^{\text{tot}} = 2$ and 0, respectively. The states with $N_f^{\text{tot}} = 6$ are given by

$$|0222\rangle, |\hat{0}\hat{2}\hat{2}\hat{2}\rangle, |\hat{1}\hat{1}22\rangle, |1\hat{1}\hat{2}\hat{2}\rangle, |\hat{1}\hat{2}12\rangle, \quad (29)$$

and cyclic permuted states, whose number is 20. Finally, for states with $N_f = 8$, we have the two states,

$$|2222\rangle, |\hat{2}\hat{2}\hat{2}\hat{2}\rangle. \quad (30)$$

Thus, the number of all possible global states is 82. Even though we have 82 global states totally, we can deal with

them piece by piece because of the fermion number conservation. Later, we focus on the set of states with the vanishing net fermion number, which is characterized by $N_f^{\text{tot}} = 4$.

C. Actions of the Hamiltonian

In this section, we introduce the action of the Hamiltonian on the set of states with the vanishing net fermion number, given in Eq. (28), which means to calculate their matrix elements in the basis vectors. The general action of a gauge-invariant operator on states is derived for the LSH basis in Ref. [61] and for the angular momentum basis of pure gauge fields in Ref. [38]. However, it should be noted that there are slight differences in the definitions between these references. We summarize the action of a gauge-invariant operator on states in our notation in Appendix A.

The electric part Hamiltonian in our model is given as

$$\begin{aligned} H_E &= K \sum_{\mu, x} E^2(x, \mu) \\ &= K [E_{L_1}^2 + E_{L_2}^2 + E_{L_3}^2 + E_{L_4}^2]. \end{aligned} \quad (31)$$

It should be noted that L_i does not specify the left electric field here; it simply denotes the label of the link, as shown in the left of Fig. 1. Since $E_{L_i}^2$ is a Casimir operator, it acts on the local state as

$$E_{L_i}^2 |s, r_1, r_2, v\rangle = j_i(j_i + 1) |s, r_1, r_2, v\rangle, \quad (32)$$

where j_i can be either j_a or j_b on that link. It is obvious that the global states given in the previous section are the eigenstates of H_E , the sum of the Casimir operators. The action of H_E on global states is determined by the number of hatted labels, such as

$$|\hat{1}120\rangle \rightarrow \frac{3}{4} |\hat{1}120\rangle, \quad (33)$$

$$|\hat{1}\hat{2}10\rangle \rightarrow \frac{3}{2} |\hat{1}\hat{2}10\rangle, \quad (34)$$

$$|\hat{1}\hat{2}\hat{0}1\rangle \rightarrow \frac{9}{4} |\hat{1}\hat{2}\hat{0}1\rangle, \quad (35)$$

$$|\hat{2}\hat{2}\hat{0}\hat{0}\rangle \rightarrow 3 |\hat{2}\hat{2}\hat{0}\hat{0}\rangle, \quad (36)$$

where we omit the coupling constant K for convenient.

For the magnetic part of the Hamiltonian, we have a specific Hamiltonian as

$$\begin{aligned} H_B &= -\frac{1}{2K} \sum_{i,j,k,l=\pm} \mathcal{L}^{i,j}(a_1, b_1) \mathcal{L}^{j,k}(a_2, b_2) \mathcal{L}^{k,l}(a_3, b_3) \\ &\quad \times \mathcal{L}^{l,i}(a_4, b_4) + (\text{H.c.}), \end{aligned} \quad (37)$$

where the operator $\mathcal{L}$ is the gauge invariant operator [18]. The pure boson gauge invariant operators are given as

$$\mathcal{L}^{++}(a, b) = \frac{1}{\sqrt{N_b + 1}} (a^\dagger \cdot \tilde{b}^\dagger) \frac{1}{\sqrt{N_a + 1}}, \quad (38)$$

$$\mathcal{L}^{-+}(a, b) = -\frac{1}{\sqrt{N_b + 1}} (a \cdot b^\dagger) \frac{1}{\sqrt{N_a + 1}}, \quad (39)$$

$$\mathcal{L}^{+-}(a, b) = \frac{1}{\sqrt{N_b + 1}} (a^\dagger \cdot b) \frac{1}{\sqrt{N_a + 1}}, \quad (40)$$

$$\mathcal{L}^{--}(a, b) = \frac{1}{\sqrt{N_b + 1}} (a \cdot \tilde{b}) \frac{1}{\sqrt{N_a + 1}}, \quad (41)$$

where the N_a and N_b denote the number operator of a and b bosons, respectively. The actions of these operators are given in Appendix A. Note that there are only pure boson operators in Eq. (37). Therefore, the actions of H_B do not change the numbers of fermion at each sites. Besides, the summation in Eq. (37) indicates that the actions of H_B add or subtract one boson at each link. Therefore, in our notation, the actions of H_B always add a hat on the states without a hat and subtract the hat on the states with a hat. Therefore, under the current truncation, $j_{\text{max}} = \frac{1}{2}$, indeed, we only have one nonvanishing action of H_B on each global state as

$$|0220\rangle \rightarrow 2|\hat{0}\hat{2}\hat{2}\hat{0}\rangle, \quad (42)$$

$$|02\hat{1}1\rangle \rightarrow -|\hat{0}\hat{2}\hat{1}\hat{1}\rangle, \quad (43)$$

$$|\hat{1}120\rangle \rightarrow -|\hat{1}\hat{2}\hat{0}\rangle, \quad (44)$$

$$|\hat{1}\hat{2}10\rangle \rightarrow -|\hat{1}\hat{2}\hat{0}\rangle, \quad (45)$$

$$|\hat{1}\hat{1}\hat{1}\hat{1}\rangle \rightarrow \frac{1}{2} |\hat{1}\hat{1}\hat{1}\hat{1}\rangle, \quad (46)$$

$$|12\hat{1}\hat{0}\rangle \rightarrow -|\hat{1}\hat{2}10\rangle, \quad (47)$$

$$|1\hat{1}\hat{2}\hat{0}\rangle \rightarrow -|\hat{1}120\rangle, \quad (48)$$

$$|\hat{0}\hat{2}\hat{1}\hat{1}\rangle \rightarrow -|02\hat{1}1\rangle, \quad (49)$$

$$|\hat{0}\hat{2}\hat{2}\hat{0}\rangle \rightarrow 2|0220\rangle, \quad (50)$$

where we have also ignored the factor $-\frac{1}{2K}$; see also Appendix A for some details. Here, we present only representative examples, as performing an n -fold cyclic permutation of the labels on both sides of the states mentioned above also results in a nonvanishing action of H_B . It is observed that the magnetic part of the Hamiltonian changes the quantum number of bosons on every site.

For the fermionic part, we first divide the Hamiltonian shown in Eq. (4) into two parts as $H_F = H_{FK} + H_{FM}$ where H_{FK} and H_{FM} denote the kinetic term and the mass term, respectively. Therefore, in our model, the mass term is given as

$$H_{FM} = -m\psi_A^\dagger\psi_A + m\psi_B^\dagger\psi_B - m\psi_C^\dagger\psi_C + m\psi_D^\dagger\psi_D. \quad (51)$$

The minus sign reflects the fact that there is an antiparticle at the sites, A and C. It is not difficult to write down the nonvanishing action of H_{FM} as

$$|2020\rangle \rightarrow -4|2020\rangle, \quad (52)$$

$$|0202\rangle \rightarrow 4|0202\rangle, \quad (53)$$

$$|\hat{1}120\rangle \rightarrow -2|\hat{1}120\rangle, \quad (54)$$

$$|\hat{0}1\hat{1}\hat{2}\rangle \rightarrow 2|\hat{0}1\hat{1}\hat{2}\rangle, \quad (55)$$

$$|102\hat{1}\rangle \rightarrow -2|102\hat{1}\rangle, \quad (56)$$

$$|\hat{1}102\rangle \rightarrow 2|\hat{1}102\rangle, \quad (57)$$

where we omit the mass factor m . As with listing the nonvanishing actions of H_B , we show only representative examples here, as performing a $2n$ -fold cyclic permutation on the labels on both sides of the states listed above also results in a nonvanishing action of H_{FM} .

Finally, for the kinetic term in our model, we can rewrite it using gauge invariant operators as

$$\begin{aligned} H_{FK} &= [\mathcal{S}^{++}(\psi_A, b_1)\mathcal{S}^{+-}(a_2, \psi_B) + \mathcal{S}^{+-}(b_1, \psi_A)\mathcal{S}^{++}(\psi_B, a_2)] \\ &+ [\mathcal{S}^{++}(\psi_B, b_2)\mathcal{S}^{+-}(a_3, \psi_C) + \mathcal{S}^{+-}(b_2, \psi_B)\mathcal{S}^{++}(\psi_C, a_3)] \\ &+ [\mathcal{S}^{++}(\psi_C, b_3)\mathcal{S}^{+-}(a_4, \psi_D) + \mathcal{S}^{+-}(b_3, \psi_C)\mathcal{S}^{++}(\psi_D, a_4)] \\ &+ [\mathcal{S}^{+-}(a_1, \psi_A)\mathcal{S}^{++}(\psi_D, b_4) + \mathcal{S}^{++}(\psi_A, a_1)\mathcal{S}^{+-}(b_4, \psi_D)] \\ &+ (\text{H.c.}), \end{aligned} \quad (58)$$

which is a summation of interaction terms acting on two neighboring sites; see Ref. [18] for details. The gauge-invariant operator $\mathcal{S}$ containing one boson and one fermion operator is given as follows:

$$\mathcal{S}^{++}(\psi, b) = \frac{1}{\sqrt{N_b + 1}}(\psi^\dagger \cdot \tilde{b}^\dagger), \quad (59)$$

$$\mathcal{S}^{++}(\psi, a) = \frac{1}{\sqrt{N_a + 1}}(\psi^\dagger \cdot \tilde{a}^\dagger), \quad (60)$$

$$\mathcal{S}^{+-}(b, \psi) = (b^\dagger \cdot \psi) \frac{1}{\sqrt{N_b + 1}}, \quad (61)$$

$$\mathcal{S}^{+-}(a, \psi) = (a^\dagger \cdot \psi) \frac{1}{\sqrt{N_a + 1}}, \quad (62)$$

whose actions are also shown in Appendix A. In Eq. (58), the first through fourth brackets act in the same way on the sites at both ends of links 1 through 4, respectively. The Hermitian conjugate terms correspond to actions that reverse the processes applied by these four brackets. Thus, we only need to determine all possible actions on both ends of link 1 to fully understand the action of H_{FK} . To do this, we first consider how the gauge-invariant operators in the first bracket act on the states located at sites A and B. Then, all the nonvanishing actions of the single gauge-invariant operator included in the first bracket are given as follows:

$$\mathcal{S}^{++}(\psi_A, b_1)|0n_B\rangle = -|\hat{1}n_B\rangle, \quad (63)$$

$$\mathcal{S}^{++}(\psi_A, b_1)|1n_B\rangle = -\frac{\sqrt{2}}{2}|\hat{2}n_B\rangle, \quad (64)$$

$$\mathcal{S}^{++}(\psi_B, a_2)|n_A 0\rangle = -(-1)^{n_A}|n_A 1\rangle, \quad (65)$$

$$\mathcal{S}^{++}(\psi_B, a_2)|n_A \hat{1}\rangle = (-1)^{n_A}\frac{\sqrt{2}}{2}|n_A \hat{2}\rangle, \quad (66)$$

$$\mathcal{S}^{+-}(b_1, \psi_A)|2n_B\rangle = \sqrt{2}|\hat{1}n_B\rangle, \quad (67)$$

$$\mathcal{S}^{+-}(b_1, \psi_A)|1n_B\rangle = -|\hat{0}n_B\rangle, \quad (68)$$

$$\mathcal{S}^{+-}(a_2, \psi_B)|n_A 2\rangle = (-1)^{n_A}\sqrt{2}|n_A 1\rangle, \quad (69)$$

$$\mathcal{S}^{+-}(a_2, \psi_B)|n_A \hat{1}\rangle = (-1)^{n_A}|n_A \hat{0}\rangle, \quad (70)$$

where n_A and n_B are the labels of the local states at the sites A and B, respectively. By combining these actions, we can obtain all the nonvanishing actions of first bracket terms as

$$|20\rangle \rightarrow -\sqrt{2}|\hat{1}1\rangle, \quad (71)$$

$$|02\rangle \rightarrow -\sqrt{2}|\hat{1}1\rangle, \quad (72)$$

$$|1\hat{1}\rangle \rightarrow \frac{\sqrt{2}}{2}(|\hat{2}\hat{0}\rangle + |\hat{0}\hat{2}\rangle), \quad (73)$$

$$|0\hat{1}\rangle \rightarrow -|\hat{1}\hat{0}\rangle, \quad (74)$$

$$|10\rangle \rightarrow -|\hat{0}1\rangle, \quad (75)$$

$$|2\hat{1}\rangle \rightarrow |\hat{1}\hat{2}\rangle, \quad (76)$$

$$|12\rangle \rightarrow |\hat{2}1\rangle. \quad (77)$$

To fully describe the actions of H_{FK} , we only need to apply the same transformation rule to every pair of neighboring sites. Therefore, we can obtain them as

$$|0220\rangle \rightarrow -\sqrt{2}|\hat{1}120\rangle - \sqrt{2}|02\hat{1}1\rangle, \quad (78)$$

$$|2020\rangle \rightarrow -\sqrt{2}|\hat{1}120\rangle - \sqrt{2}|20\hat{1}1\rangle - \sqrt{2}|2\hat{1}10\rangle - \sqrt{2}|102\hat{1}\rangle, \quad (79)$$

$$|\hat{1}120\rangle \rightarrow |\hat{1}\hat{2}10\rangle - \sqrt{2}|\hat{1}\hat{1}\hat{1}1\rangle - |\hat{0}12\hat{1}\rangle, \quad (80)$$

$$|02\hat{1}1\rangle \rightarrow |0\hat{1}\hat{2}1\rangle - \sqrt{2}|\hat{1}\hat{1}\hat{1}1\rangle - |12\hat{1}\hat{0}\rangle, \quad (81)$$

$$|\hat{1}\hat{2}10\rangle \rightarrow -|\hat{1}\hat{2}\hat{0}1\rangle - |\hat{0}\hat{2}\hat{1}1\rangle, \quad (82)$$

$$|\hat{0}12\hat{1}\rangle \rightarrow -|\hat{0}1\hat{1}\hat{2}\rangle + |\hat{0}\hat{2}\hat{1}\hat{1}\rangle, \quad (83)$$

$$|\hat{1}\hat{1}\hat{1}1\rangle \rightarrow \frac{\sqrt{2}}{2}(|\hat{1}\hat{2}\hat{0}1\rangle + |\hat{1}\hat{0}\hat{2}1\rangle + |\hat{0}1\hat{1}\hat{2}\rangle + |\hat{2}\hat{1}\hat{1}\hat{0}\rangle), \quad (84)$$

$$|\hat{1}\hat{2}\hat{0}1\rangle \rightarrow \frac{\sqrt{2}}{2}|\hat{2}\hat{2}\hat{0}\hat{0}\rangle + \frac{\sqrt{2}}{2}|\hat{0}\hat{2}\hat{0}\hat{2}\rangle, \quad (85)$$

$$|\hat{0}\hat{2}\hat{1}\hat{1}\rangle \rightarrow \frac{\sqrt{2}}{2}|\hat{0}\hat{2}\hat{2}\hat{0}\rangle + \frac{\sqrt{2}}{2}|\hat{0}\hat{2}\hat{0}\hat{2}\rangle, \quad (86)$$

and their inverse process is described by the Hermitian conjugate terms. Here we show only representative examples, as performing an n -times cyclic permutation on the labels on both sides of the states listed above also results in a nonvanishing action of H_{FK} . Note that the actions always add or remove a hat on the total states.

Now, all actions of the total Hamiltonian are provided as matrix elements in the specific basis. For quantum computation purposes, we need to translate these matrix elements into the spin basis.

III. QUANTUM SIMULATION

A. Mapping to spin system

To perform quantum simulations, we need to map our model into a spin system. Then, we can write down the Hamiltonian matrix using the identity and Pauli matrices, which together can represent any general 2×2 Hermitian matrix. Since the number of the global states we consider is 38, we need six qubits at least to represent them. Note that there are specific rules for the number of the hats of the global states that are changed when the Hamiltonian acts on them, as discussed in Sec. II C. This suggests that dividing the global states into different groups is convenient. Therefore, we classify the global states into five groups based on the number of hats. We use the first three qubits to represent the groups as

$$\begin{aligned} \text{group A:} & \quad |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle, \\ \text{group B:} & \quad |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle, \\ \text{group C}_1: & \quad |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle, \\ \text{group C}_2: & \quad |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle, \\ \text{group D:} & \quad |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle, \\ \text{group E:} & \quad |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle, \end{aligned} \quad (87)$$

where the groups A, B, C, D, and E correspond to the global states with 0, 1, 2, 3, and 4 hats, respectively. The group C has ten patterns, so we divide it into groups C_1 and C_2 : states where hats are adjacent are classified as group C_1 , while all other states are classified as group C_2 . Then we use the remaining three qubits to specify the state in each group, as shown in Appendix B 1. Based on this mapping, regardless of which term in the Hamiltonian acts, all global states within one group are transformed into states within a single specific group, which may be the same as or different from the original group, which is convenient for figuring out the Hamiltonian matrix.

Now, we can write down the Hamiltonian matrix with Pauli matrices $X = \sigma^1$, $Y = \sigma^2$, $Z = \sigma^3$ and identity matrix I . Since there are numerous terms, we just take an example to show the steps and give the full results in Appendix B 2. Let us focus on the action, $|\hat{1}120\rangle \rightarrow -\sqrt{2}|\hat{1}\hat{1}\hat{1}1\rangle$, which is the action from the group B to the group C_2 . Thus, we can directly find the matrix for the first three qubits based on the mapping rule for the groups (87) as

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (88)$$

Then, according to the mapping rule (Appendix B 1), we can write the matrix for the last three qubits as

$$-\sqrt{2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (89)$$

Using Pauli matrices, we can represent the term and its transpose as

$$\begin{aligned} & -\frac{\sqrt{2}}{16} (I + Z) \{ X(I - Z) [(I + Z + X)(I + Z + X) - YY] \\ & \quad + Y(I - Z) [(I + Z + X)Y + Y(I + Z + X)] \} I, \end{aligned} \quad (90)$$

where we ignore the symbol $\otimes$ for simplicity. We show all results in Appendix B 2. Based on these Hamiltonian matrices, we can perform quantum simulations, which are shown in the next section.

B. Real-time simulation

Here we introduce our quantum simulation algorithm. The real-time evolution of a quantum system is described by the Schrödinger equation as

$$i\partial_t|\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \quad (91)$$

Since the Hamiltonian in our model is time independent, the evolution of the quantum state can be described as

$$|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle, \quad (92)$$

where $e^{-i\hat{H}t}$ denotes the real-time evolution operator. Note that, in quantum simulation, the quantum gate for a real-time evolution operator can be very difficult to construct. Therefore, we need to construct the quantum gate approximately using the so-called the Suzuki-Trotter decomposition formula [62,63], which is

$$e^{i(H_1+H_2)\delta t} = e^{iH_1\delta t}e^{iH_2\delta t} + O(\delta t^2), \quad (93)$$

where H_1 and H_2 are two noncommuting operators. Then, for a long time quantum simulation, we can divide the time evolution operator into many small steps as

$$\lim_{n \rightarrow \infty} (e^{iH_1 \frac{t}{n}} e^{iH_2 \frac{t}{n}})^n = e^{i(H_1+H_2)t}, \quad (94)$$

which can be generalized to the situation with more noncommuting operators. Since we have represented the total Hamiltonian with Pauli matrices, the quantum gate $e^{iH_j t}$ can be exactly constructed where H_j can be any single term in the total Hamiltonian.

Up to this point, we provide the mapping of the lattice gauge theory with the staggered fermion to the spin system. Through such a mapping, the ultimate goal is to perform quantum computations of the gauge theory. However, the current mapping involves numerous nonlocal operations, which are significantly affected by noise on existing quantum computers, presenting challenges for practical implementation. For this reason, we performed the calculations using the simulator Qiskit, provided by IBM [64]. Even though calculations on quantum computers are currently difficult, clarifying the correspondence between the specific gauge theory and a spin system is an advancement both theoretically and computationally. Furthermore, the proposed mapping is expected to serve as an important building block for developing more efficient mappings and algorithms in the future.

Later in this section, in order to investigate the thermalization of non-Abelian gauge theory with fermions and the process of fermion production, we calculate the evolution of the entanglement and the pair production.

1. Entanglement of the system

To measure the thermalization of non-Abelian gauge theory with fermions, we first introduce two kinds of

entanglement entropy. The entanglement entropy quantifies the mixedness of a subsystem of interest, arising from the entanglement between the subsystem and the rest of the system. The density matrix of the whole system is defined as

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|. \quad (95)$$

Focusing on the entanglement between the local state at the single site A and states on the other sites, we define the reduced density matrix by tracing out the Hilbert space at sites B, C, and D as

$$\hat{\rho}_A(t) = \sum_{i \in \text{all basis states}} \langle \psi_{BCD}^i | \hat{\rho}(t) | \psi_{BCD}^i \rangle, \quad (96)$$

where A, B, C, and D denote the labels of sites shown in Fig. 1 and $|\psi_{BCD}^i\rangle$ denotes a basis vector for the Hilbert space at the sites B, C, and D: for example $|0_B 2_C 0_D\rangle$. Note that the local state on point A can be uniquely determined by the states on sites B, C, and D because of the constraint (24). Therefore, the reduced density matrix $\hat{\rho}_A(t)$ has only diagonal elements, which denote the probabilities of being in the corresponding local states. Then, the entanglement entropy is defined as

$$S_A(t) = -\text{Tr}[\hat{\rho}_A(t) \ln \hat{\rho}_A(t)]. \quad (97)$$

Later we refer to $S_A(t)$ as the single-site entanglement entropy.

In addition to the entanglement entropy for a single lattice site, we also consider the entanglement entropy defined from the density matrix obtained by tracing out the boson Hilbert space. This entropy corresponds to the entropy carried by fermions and is more directly related to the entropy estimated from, for example, the hadron multiplicities observed in relativistic heavy-ion collision experiments. The reduced density matrix is defined as

$$\hat{\rho}_F(t) = \sum_{i \in \text{all basis states}} \langle \psi_{\text{boson}}^i | \hat{\rho}(t) | \psi_{\text{boson}}^i \rangle, \quad (98)$$

where $|\psi_{\text{boson}}^i\rangle$ denotes the state for Schwinger bosons. Note that $|\psi_{\text{boson}}^i\rangle$ can be identified as the total state without fermion number on the sites. Finally, we have another entangle entropy as

$$S_F(t) = -\text{Tr}[\hat{\rho}_F(t) \ln \hat{\rho}_F(t)]. \quad (99)$$

Later we refer to $S_F(t)$ as the fermion entropy.

We show the time evolution of the single-site entanglement entropy with four different initial conditions, $|1\hat{1}1\hat{1}\rangle$, $|\hat{1}\hat{2}10\rangle$, $|\hat{1}\hat{1}20\rangle$, $|\hat{1}\hat{2}\hat{0}1\rangle$ at $K=1$ and $m=1$, in the left of Fig. 2. As expected, the entropies are found to be initially 0, increase rapidly at first, and then fluctuate around some constant. This finding indicates that, among

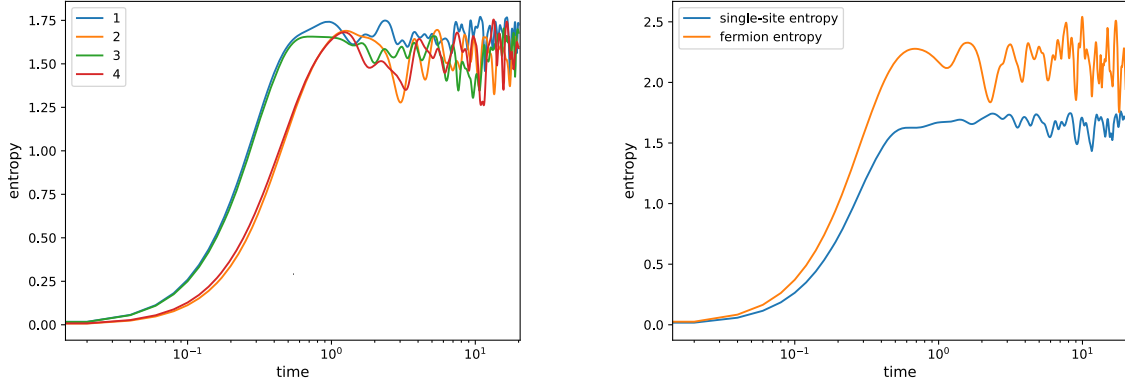


FIG. 2. Left: time evolution of the single-site entropy with different initial conditions. Labels “1, 2, 3, 4” denote the time evolution of $|\hat{1}\hat{1}\hat{1}\hat{1}\rangle$, $|\hat{1}\hat{2}\hat{1}0\rangle$, $|\hat{1}\hat{1}20\rangle$, $|\hat{1}\hat{2}\hat{0}1\rangle$, respectively. Right: time evolution of both two kinds of entropy with initial state $|\hat{1}201\rangle$. In both figures, we set the parameters as $K = 1$, $m = 1$ which make the eigenvalue of H_E, H_B, H_{FK}, H_{FM} close. This ensures the balance among the four parts of the Hamiltonian.

the initial conditions we test, the system always reaches some kind of “thermalized” state. The observation that thermal equilibration is implied, even though the system size is significantly small, is consistent with the results in Ref. [38], which reports numerical simulations of the pure SU(2) gauge theory on a small lattice. We should note that the fluctuations are visually large. This may reflect the fact that the Hilbert space we are considering is small. In addition, the emergence of two distinct patterns in the evolution of entropies is not easily explained and remains unresolved. We consider this an intriguing issue and intend to explore it further in future work.

We also show the real-time evolution of both two kinds of entanglement entropy on the right of Fig. 2. We can clearly find that these two kinds of entropy have the same behavior as the time evolves. Since the dimension of Hilbert space for a single site is 6, the single-site entropy mathematically has a maximum value that is $\ln 6 (\approx 1.79)$. For the same reason, fermion entropy also has a maximum value, which is $\ln 16 (\approx 2.77)$. Therefore, these two entropies reach high values, exceeding 80% of their respective maximum values. This indicates that, in the current setup, the reduced density matrices for the single-site and fermion subsystems spread widely within their respective Hilbert spaces over a long time.

Note that in our Hamiltonian, we have two parameters: the fermion mass m and the coupling constant K . Regarding the fermion mass m , we present the result in the next section that, as m increases, fermion excitation is suppressed, and accordingly, the fermion entropy is also suppressed. Regarding the coupling constant K , throughout this paper, we fix $K = 1$ for practical reasons. Since the electric part of the Hamiltonian is proportional to the coupling constant, $H_E \propto K$, and the magnetic part is proportional to the inverse of the coupling constant, $H_B \propto K^{-1}$, both smaller and larger values of K lead to an enhancement of the Hamiltonian. As a result, numerical simulations become more challenging: a

smaller δt is required to maintain the validity of the Suzuki-Trotter decomposition (93). The investigation of K dependence in this model is beyond the scope of this work. Moreover, since the physical system and the Hilbert space are quite small, it is reasonable to expect that the numerical results contain finite-size effects. Thus, in relating our results to the physical behavior of a real many-body system, it is important to distinguish between real physics and artifacts. This can be achieved by enlarging the size of the lattice, which is left as an important future work.

2. Pair production

In our simulation, we choose the global states that have vanishing fermion number. Because of the fermion number conservation, fermions and antifermions are always excited in pairs. Thus, we refer to the fermion number, associated with the sites at point B and D, as the pair production,

$$N_{\text{pair}} \equiv 4 - N_{f,A} - N_{f,C} = N_{f,B} + N_{f,D}. \quad (100)$$

We show the time evolution of the pair productions in Fig. 3 at 1 and $m = 0.2, 1$ and 5 with initial conditions, $|2020\rangle$ and $|\hat{2}\hat{0}\hat{2}\hat{0}\rangle$, which are the only states with vanishing pair production, $N_{\text{pair}} = 0$.

In both figures, as expected, when the mass is not too large, which means the evolutions for $m = 1$ and $m = 0.2$, the pair productions increase rapidly at first and then fluctuate around some constants. Besides, when the mass is large enough, it should be too difficult to produce fermion pairs. Therefore, in both figures, the pair productions denoted as $m = 5$ finally fluctuate at some low levels. We can also clearly see the mass dependence of the pair productions. Both the maximum value and the average value of the pair production decrease as the mass become larger. Note that because of the anticommutation relation of the fermion, the pair production is limited by the size of the lattice. Therefore, when both evolutions have small enough

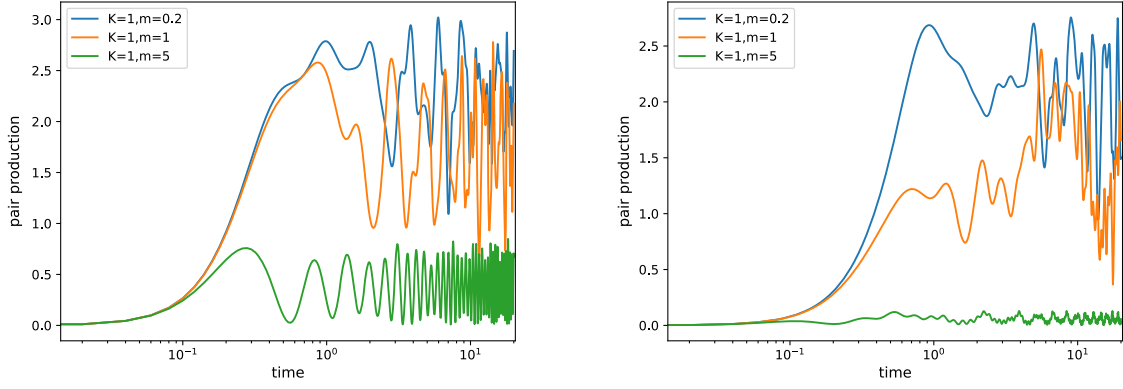


FIG. 3. Left: time evolution of the pair productions with initial state $|2020\rangle$ and different fermion mass m . Right: time evolution of the pair productions with initial state $|\hat{2}\hat{0}\hat{2}\hat{0}\rangle$ and different fermion mass m . In both figures, we set the coupling constant as $K = 1$.

masses, the distinctions may be not be clear even though the masses are significantly different, which is shown in the left of Fig. 3. The fluctuations are also visually large, which may have the same reason as the evolutions of entropies. After all, these results still satisfy the physical intuition.

To study the relation between pair production and fermion entropy, we also show the comparison of their time evolutions in Fig. 4. We have already known that the fermion mass can have a great influence on the pair production. Therefore, we show the time evolutions with small mass and large mass on the left and right of Fig. 4, respectively. These two results approximately represent the situations where fermions are produced and where fermions are not produced. We can clearly see that the fermion entropy is increasing when the fermions are being produced and fluctuating when the pair production is fluctuating. Besides, if there is hardly any pair production, the fermion entropy will also be at a low level. The coincidence between the behavior of the fermion number and the entropy in the nonequilibrium evolution is nontrivial while, in a thermal equilibrium state, both the entropy and the particle number are considered extensive quantities and proportional to each other. This coincidence suggests that,

even in nonequilibrium, there may be cases where the entropy can be estimated from the fermion number. In addition, it is reasonable to expect that the present results may contain artifacts arising from the small lattice size and the minimal truncation employed in the simulation, as may also be the case for entanglement entropies. Thus, it will be important to extend our simulation to larger lattice, which will be a future project.

IV. DISCUSSION

While the present construction is limited to a small lattice with minimal truncation, we briefly comment on possible directions for extending the framework to larger systems. One challenge in such extensions is ensuring the uniqueness of the Gauss-law singlet at each site, as degeneracy can arise when more than four fields meet at a single site. The point-splitting method [65], which treats multiple fields at a site as if they were located at virtually distinguishable points, is useful for avoiding such degeneracy.

As the system size increases, the Hilbert space grows rapidly, and constructing the mapping by hand becomes difficult. A computational strategy may therefore be

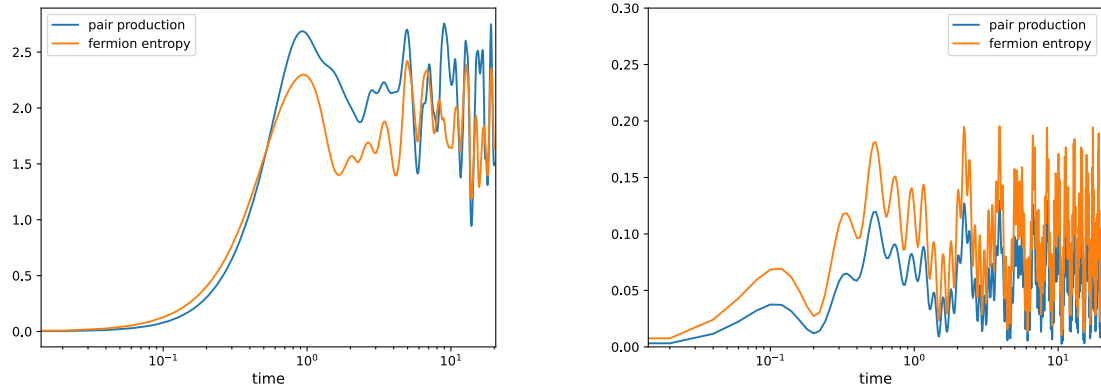


FIG. 4. Left: time evolution of pair production in comparison to the fermion entropy with $K = 1, m = 0.2$. Right: time evolution of pair production in comparison to the fermion entropy with $K = 1, m = 5$. In both figures, the initial states are $|\hat{2}\hat{0}\hat{2}\hat{0}\rangle$.

considered. In this approach, the Hermitian Hamiltonian is embedded into a Hilbert space of dimension 2^n and interpreted as acting on a tensor product of n two-level systems. The Hamiltonian can then be expanded in terms of Pauli string operators as

$$\mathcal{M}(H_{2^n}) = \sum_{\mu} c_{\mu} \hat{\sigma}_{\mu}, \quad \text{with} \quad \hat{\sigma}_{\mu} = \prod_{l=1}^n \sigma_{\mu_l}^{(l)}, \quad (101)$$

where each $\sigma_{\mu_l}^{(l)}$ is chosen from the identity and Pauli matrices and the coefficients $c_{\mu} \in \mathbb{R}$ encode the weight of each basis operator in the mapping. It should be emphasized, however, that this discussion merely outlines a formal and mechanical mapping procedure. In practice, efficient quantum implementation demands problem-specific strategies, including reducing circuit depth and maintaining locality, meaning that interactions or operators should act only on nearby qubits. These remain separate and nontrivial challenges. The insights obtained in the present work may serve as useful references when addressing such problems.

In addition to the strategy described above, we also present a brief discussion of a more explicit mapping. Although this construction is restricted to the minimal truncation case with $j_{\max} = \frac{1}{2}$, it remains applicable to lattices of arbitrary size. After point splitting, the basis states at each site are characterized by $|j_a, j_b, j_c\rangle$ or $|j_a, j_b, N_F\rangle$, where the bosonic angular momentum quantum numbers j_a, j_b , and j_c are associated with the attached links, and N_F denotes the fermion number at the site. Each bosonic angular momentum takes only two values, 0 or $\frac{1}{2}$, so the states labeled by $|j_a, j_b, j_c\rangle$ can be represented using spin degrees of freedom. For the states labeled by $|j_a, j_b, N_F\rangle$, the number of allowed values for N_F depends on the combination of j_a and j_b , and is either one or two. This constraint arises from the triangle inequalities, $|j_a - j_b| \leq j_F = \frac{1}{2}N_F \leq j_a + j_b$. The remaining arbitrariness can also be encoded using spin variables. In total, all basis states in the minimally truncated Hilbert space can be expressed in terms of spin degrees of freedom that assign values to the bosonic angular momenta and distinguish among the possible fermionic states. Each spin is associated with either a link or a site of the original lattice. Therefore, if qubits are arranged with the same geometry as the original lattice, the spatial structure of the system can be approximately preserved in the layout of the quantum hardware. This offers the advantage that nonlocal interactions, which are a major source of error in digital quantum computation, do not arise from the mapping.

V. SUMMARY AND OUTLOOK

We have simulated the real-time evolution of SU(2) gauge theory with fermions on a small two-dimensional lattice using the LSH formulation. The LSH formulation

provides a gauge-invariant Hilbert space for gauge theories with fermions and, moreover, allows us to truncate the Hilbert space while preserving gauge invariance. We have mapped the formulated model to a spin system, laying the groundwork for future quantum simulations on actual quantum computers. However, we note that the mapping we have introduced involves many nonlocal interactions, which necessitate further algorithmic improvements for practical quantum computations. Then, using a quantum computer emulator, we have tracked the real-time quantum evolution of the system starting from nonequilibrium states and have studied the entanglement entropy and fermion production.

We have investigated the thermalization of the system through the entanglement entropy of a single lattice site. The results show that entropy initially grows significantly and then oscillates around a certain value. This behavior indicates that the entanglement of a single site with other lattice sites increases as it interacts with them, suggesting the thermalization of the system. Even in a small lattice, the system exhibits thermal equilibriumlike behavior, which is consistent with previous studies on pure SU(2) Yang-Mills theory [38]. Physically, this suggests that the Hilbert space is already dense enough for quantum chaos to emerge.

Furthermore, to explore the relationship between the entropy of fermions and the fermion particle number, we have calculated the entanglement entropy of the density matrix obtained by performing a partial trace over the bosonic Hilbert space, as well as the evolution of the fermion number from an initial nonequilibrium state with a vanishing fermion number. Our results show that, within the range of initial states we have simulated, both quantities exhibit similar behavior. While entropy is known to be proportional to the particle number in equilibrium, the relationship between them in nonequilibrium states remains largely unexplored. This finding suggests that even in far-from-equilibrium systems, there exists a close relationship between the entropy and the particle number production of fermions.

To achieve our objective of simulating real QCD, we need to improve our model step by step. In particular, to investigate more aspects of SU(2) Yang-Mills theory, we need to increase the lattice size and consider higher-energy states, which may provide a testing ground for thermalization. Furthermore, we would like to consider a more realistic scenario, namely, SU(3) Yang-Mills theory with fermions. Once we achieve this objective, we hope to gain deeper insights into the physics of QCD.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: ACTION OF GAUGE INVARIANT OPERATORS

In this section, we summarize the actions of all necessary gauge invariant operators. For pure boson type operators that appear in the magnetic part of the Hamiltonian, we have

$$\mathcal{L}^{++}(a, b) = \frac{1}{\sqrt{N_b + 1}} (\mathbf{a}^\dagger \cdot \tilde{\mathbf{b}}^\dagger) \frac{1}{\sqrt{N_a + 1}}, \quad (\text{A1})$$

$$\mathcal{L}^{-+}(a, b) = -\frac{1}{\sqrt{N_b + 1}} (\mathbf{a} \cdot \mathbf{b}^\dagger) \frac{1}{\sqrt{N_a + 1}}, \quad (\text{A2})$$

$$\mathcal{L}^{+-}(a, b) = \frac{1}{\sqrt{N_b + 1}} (\mathbf{a}^\dagger \cdot \mathbf{b}) \frac{1}{\sqrt{N_a + 1}}, \quad (\text{A3})$$

$$\mathcal{L}^{--}(a, b) = \frac{1}{\sqrt{N_b + 1}} (\mathbf{a} \cdot \tilde{\mathbf{b}}) \frac{1}{\sqrt{N_a + 1}}, \quad (\text{A4})$$

which have the same form as the formula in Ref. [38]. Besides, in the fermionic part of the Hamiltonian, we have the operators including fermions as

$$S^{++}(\psi, b) = \frac{1}{\sqrt{N_b + 1}} (\boldsymbol{\psi}^\dagger \cdot \tilde{\mathbf{b}}^\dagger),$$

$$S^{++}(\psi, a) = \frac{1}{\sqrt{N_a + 1}} (\boldsymbol{\psi}^\dagger \cdot \tilde{\mathbf{a}}^\dagger),$$

$$S^{+-}(b, \psi) = (\mathbf{b}^\dagger \cdot \boldsymbol{\psi}) \frac{1}{\sqrt{N_b + 1}},$$

$$S^{+-}(a, \psi) = (\mathbf{a}^\dagger \cdot \boldsymbol{\psi}) \frac{1}{\sqrt{N_a + 1}}. \quad (\text{A5})$$

Note that, because of the anticommutation relation of fermions and the order of gauge invariant operators when defining physical states, it is necessary to distinguish bosons a, b in Eq. (A5). Besides, all N_a, N_b should be treated as operators. Before we introduce the details for calculating the actions, we show the relation between the LSH basis and angular momentum basis again as

$$j_b = \frac{1}{2}(s + r_1), \quad j_a = \frac{1}{2}(s + r_2), \quad j_\psi = \frac{1}{2}(r_1 + r_2). \quad (\text{A6})$$

Besides, the particle number can be easily gotten from the LSH basis as

$$N_b = (s + r_1), \quad N_a = (s + r_2), \quad N_\psi = (2v + r_1 + r_2). \quad (\text{A7})$$

We first calculate the action of the pure boson operator $\mathcal{L}^{++}(a, b)$ as

$$\begin{aligned} \mathcal{L}^{++}(a, b)|j_b, j_a, j_\psi\rangle &= \frac{1}{\sqrt{C_n}} \frac{1}{\sqrt{2j_b + 2}} (\mathbf{a}^\dagger \cdot \tilde{\mathbf{b}}^\dagger) \frac{1}{\sqrt{2j_a + 1}} (\mathbf{b}^\dagger \cdot \tilde{\mathbf{a}}^\dagger)^s (\mathbf{b}^\dagger \cdot \tilde{\boldsymbol{\psi}}^\dagger)^{r_1} (\mathbf{a}^\dagger \cdot \tilde{\boldsymbol{\psi}}^\dagger)^{r_2} (\boldsymbol{\psi}^\dagger \cdot \tilde{\boldsymbol{\psi}}^\dagger)^v |0\rangle \\ &= -\sqrt{\frac{(j_b + j_a - j_\psi + 1)(j_b + j_a + j_\psi + 2)}{(2j_b + 2)(2j_a + 1)}} \left| j_b + \frac{1}{2}, j_a + \frac{1}{2}, j_\psi \right\rangle, \end{aligned}$$

where we have used the relation $\widehat{N}_a|j_b, j_a, j_\psi\rangle = 2j_a|j_b, j_a, j_\psi\rangle, \widehat{N}_b|j_b, j_a, j_\psi\rangle = 2j_b|j_b, j_a, j_\psi\rangle$. Then, for operator $\mathcal{L}^{+-}(a, b)$, we have

$$\begin{aligned} \mathcal{L}^{+-}(a, b)|j_b, j_a, j_\psi\rangle &= \frac{1}{\sqrt{C_n}} \frac{1}{\sqrt{2j_b}} (\mathbf{a}^\dagger \cdot \mathbf{b}) \frac{1}{\sqrt{2j_a + 1}} (\mathbf{b}^\dagger \cdot \tilde{\mathbf{a}}^\dagger)^s (\mathbf{b}^\dagger \cdot \tilde{\boldsymbol{\psi}}^\dagger)^{r_1} (\mathbf{a}^\dagger \cdot \tilde{\boldsymbol{\psi}}^\dagger)^{r_2} (\boldsymbol{\psi}^\dagger \cdot \tilde{\boldsymbol{\psi}}^\dagger)^v |0\rangle \\ &= \begin{cases} \frac{1}{\sqrt{(2j_a + 1)2j_b}} \left| j_b - \frac{1}{2}, j_a + \frac{1}{2}, j_\psi \right\rangle, & r_1 = 1 \\ 0, & r_1 = 0 \end{cases} \end{aligned}$$

where we have used the commutation relations $[(\mathbf{b}^\dagger \cdot \mathbf{a}), (\mathbf{b}^\dagger \cdot \tilde{\mathbf{a}}^\dagger)] = 0$ and $[(\mathbf{b}^\dagger \cdot \mathbf{a}), (\mathbf{a}^\dagger \cdot \tilde{\boldsymbol{\psi}}^\dagger)] = (\mathbf{b}^\dagger \cdot \tilde{\boldsymbol{\psi}}^\dagger)$. Similarly, for operator $\mathcal{L}^{-+}(a, b)$, we have

$$\begin{aligned}
& \mathcal{L}^{+-}(a, b)|j_b, j_a, j_\psi\rangle \\
&= -\frac{1}{\sqrt{C_n}} \frac{1}{\sqrt{2j_b+2}} (a \cdot b^\dagger) \frac{1}{\sqrt{2j_a+1}} \\
&\quad \times (b^\dagger \cdot \tilde{a}^\dagger)^s (b^\dagger \cdot \tilde{\psi}^\dagger)^{r_1} (a^\dagger \cdot \tilde{\psi}^\dagger)^{r_2} (\psi^\dagger \cdot \tilde{\psi}^\dagger)^v |0\rangle \\
&= \begin{cases} -\frac{1}{\sqrt{(2j_b+2)(2j_a+1)}} |j_b + \frac{1}{2}, j_a - \frac{1}{2}, j_\psi\rangle, & r_2 = 1 \\ 0, & r_2 = 0 \end{cases}.
\end{aligned}$$

Finally, for operator $\mathcal{L}^{--}(a, b)$, we have

$$\begin{aligned}
& \mathcal{L}^{--}(a, b)|j_b, j_a, j_\psi\rangle \\
&= \frac{1}{\sqrt{C_n}} \frac{1}{\sqrt{2j_b}} (a \cdot \tilde{b}) \frac{1}{\sqrt{2j_a+1}} \\
&\quad \times (b^\dagger \cdot \tilde{a}^\dagger)^s (b^\dagger \cdot \tilde{\psi}^\dagger)^{r_1} (a^\dagger \cdot \tilde{\psi}^\dagger)^{r_2} (\psi^\dagger \cdot \tilde{\psi}^\dagger)^v |0\rangle \\
&= -\sqrt{\frac{(j_b + j_a - j_\psi)(j_b + j_a + j_\psi + 1)}{(2j_a + 1)(2j_b)}} |j_b - \frac{1}{2}, j_a - \frac{1}{2}, j_\psi\rangle,
\end{aligned}$$

where we have used the formulas $[(b \cdot \tilde{a}), (b^\dagger \cdot \tilde{a}^\dagger)] = 2 + \widehat{N}_a + \widehat{N}_b$ and $s = j_a + j_b - j_\psi$. Note that, considering the constraint of j_ψ , these results are the same as that in Ref. [38].

Now, let us consider the actions of operators with a fermion. For operator $\mathcal{S}^{++}(\psi, b)$, we have

$$\begin{aligned}
& \mathcal{S}^{++}(\psi, b)|j_b, j_a, j_\psi\rangle \\
&= \frac{1}{\sqrt{C_n}} \frac{1}{\sqrt{2j_b+2}} (\psi^\dagger \cdot \tilde{b}^\dagger) \\
&\quad \times (b^\dagger \cdot \tilde{a}^\dagger)^s (b^\dagger \cdot \tilde{\psi}^\dagger)^{r_1} (a^\dagger \cdot \tilde{\psi}^\dagger)^{r_2} (\psi^\dagger \cdot \tilde{\psi}^\dagger)^v |0\rangle \\
&= \begin{cases} -\sqrt{\frac{2j_b+1}{2j_b+2}} |j_b + \frac{1}{2}, j_a, j_\psi - \frac{1}{2}\rangle, & r_2 = 1 \\ -|j_b + \frac{1}{2}, j_a, j_\psi + \frac{1}{2}\rangle, & r_1 = r_2 = v = 0 \\ 0, & \text{other cases} \end{cases}
\end{aligned}$$

where we have used the Eq. (18). Similarly, for operator $\mathcal{S}^{++}(\psi, a)$, we have

$$\begin{aligned}
& \mathcal{S}^{++}(\psi, a)|j_b, j_a, j_\psi\rangle \\
&= \frac{1}{\sqrt{C_n}} \frac{1}{\sqrt{2j_a+2}} (\psi^\dagger \cdot \tilde{a}^\dagger) \\
&\quad \times (b^\dagger \cdot \tilde{a}^\dagger)^s (b^\dagger \cdot \tilde{\psi}^\dagger)^{r_1} (a^\dagger \cdot \tilde{\psi}^\dagger)^{r_2} (\psi^\dagger \cdot \tilde{\psi}^\dagger)^v |0\rangle \\
&= \begin{cases} \sqrt{\frac{2j_a+1}{2j_a+2}} |j_b, j_a + \frac{1}{2}, j_\psi - \frac{1}{2}\rangle, & r_1 = 1 \\ -|j_b, j_a + \frac{1}{2}, j_\psi + \frac{1}{2}\rangle, & r_1 = r_2 = v = 0 \\ 0, & \text{other cases} \end{cases}.
\end{aligned}$$

Then, for operator $\mathcal{S}^{+-}(b, \psi)$, we have

$$\begin{aligned}
& \mathcal{S}^{+-}(b, \psi)|j_b, j_a, j_\psi\rangle \\
&= \frac{1}{\sqrt{C_n}} \frac{1}{\sqrt{(2j_b+1)}} (b^\dagger \cdot \psi) \\
&\quad \times (b^\dagger \cdot \tilde{a}^\dagger)^s (b^\dagger \cdot \tilde{\psi}^\dagger)^{r_1} (a^\dagger \cdot \tilde{\psi}^\dagger)^{r_2} (\psi^\dagger \cdot \tilde{\psi}^\dagger)^v |0\rangle \\
&= \begin{cases} -|j_b + \frac{1}{2}, j_a, j_\psi - \frac{1}{2}\rangle, & r_2 = 1 \\ \sqrt{\frac{2j_b+2}{2j_b+1}} |j_b + \frac{1}{2}, j_a, j_\psi + \frac{1}{2}\rangle, & v = 1 \\ 0, & \text{other cases} \end{cases},
\end{aligned}$$

where we have used the commutation relation $[(b^\dagger \cdot \psi), (a^\dagger \cdot \tilde{\psi}^\dagger)] = -(b^\dagger \cdot \tilde{a}^\dagger)$. Similarly, for operator $\mathcal{S}^{+-}(a, \psi)$, we have

$$\begin{aligned}
& \mathcal{S}^{+-}(a, \psi)|j_b, j_a, j_\psi\rangle \\
&= \frac{1}{\sqrt{C_n}} \frac{1}{\sqrt{(2j_b+1)}} (a^\dagger \cdot \psi) \\
&\quad \times (b^\dagger \cdot \tilde{a}^\dagger)^s (b^\dagger \cdot \tilde{\psi}^\dagger)^{r_1} (a^\dagger \cdot \tilde{\psi}^\dagger)^{r_2} (\psi^\dagger \cdot \tilde{\psi}^\dagger)^v |0\rangle \\
&= \begin{cases} |j_b, j_a + \frac{1}{2}, j_\psi - \frac{1}{2}\rangle, & r_1 = 1 \\ \sqrt{\frac{2j_a+2}{2j_a+1}} |j_b, j_a + \frac{1}{2}, j_\psi + \frac{1}{2}\rangle, & v = 1 \\ 0, & \text{other cases} \end{cases}.
\end{aligned}$$

These actions of gauge invariant operators are consistent with the results of Ref. [61].

APPENDIX B: HAMILTONIAN MATRIX UNDER SPIN BASIS

1. Exact mapping to spin system

Here, we list all the correspondences between the basis states of the LSH representation and the physical states encoded in six qubits. In Sec. III, we use the first three qubits to represent groups that the physical states belong to, namely

$$\begin{aligned}
& \text{group A: } |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
& \text{group B: } |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle, \\
& \text{group C}_1: |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle, \\
& \text{group C}_2: |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle, \\
& \text{group D: } |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
& \text{group E: } |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle,
\end{aligned} \tag{B1}$$

where the groups A, B, C, D, and E correspond to the global states with 0, 1, 2, 3, and 4 hats, respectively. For all physical states mentioned in Sec. II B, we show the exact mapping from the physical states to spin basis with three other qubits. For the group A, we have

$$\begin{aligned}
|0220\rangle &= |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|2200\rangle &= |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle, \\
|2002\rangle &= |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle, \\
|0022\rangle &= |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle, \\
|2020\rangle &= |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|0202\rangle &= |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle,
\end{aligned} \tag{B2}$$

where we omit the first three qubits, $|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle$, that label the group A. Later, we always omit the first three qubits that label the group. For the group B, we have

$$\begin{aligned}
|\hat{1}\hat{2}\hat{0}1\rangle &= |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|\hat{2}\hat{0}1\hat{1}\rangle &= |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle, \\
|\hat{0}1\hat{1}\hat{2}\rangle &= |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle, \\
|\hat{1}\hat{1}\hat{2}\hat{0}\rangle &= |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle, \\
|\hat{0}\hat{2}1\hat{1}\rangle &= |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|\hat{2}1\hat{1}\hat{0}\rangle &= |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle, \\
|\hat{1}\hat{1}\hat{0}\hat{2}\rangle &= |\downarrow\rangle|\downarrow\rangle|\uparrow\rangle, \\
|\hat{1}\hat{0}\hat{2}1\rangle &= |\downarrow\rangle|\downarrow\rangle|\downarrow\rangle.
\end{aligned} \tag{B6}$$

For the group E, we have

$$\begin{aligned}
|\hat{1}120\rangle &= |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|120\hat{1}\rangle &= |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle, \\
|20\hat{1}1\rangle &= |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle, \\
|0\hat{1}12\rangle &= |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle, \\
|02\hat{1}1\rangle &= |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|2\hat{1}10\rangle &= |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle, \\
|\hat{1}102\rangle &= |\downarrow\rangle|\downarrow\rangle|\uparrow\rangle, \\
|102\hat{1}\rangle &= |\downarrow\rangle|\downarrow\rangle|\downarrow\rangle.
\end{aligned} \tag{B3}$$

$$\begin{aligned}
|\hat{2}\hat{2}\hat{0}\hat{0}\rangle &= |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|\hat{2}\hat{0}\hat{0}\hat{2}\rangle &= |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle, \\
|\hat{0}\hat{0}\hat{2}\hat{2}\rangle &= |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle, \\
|\hat{0}\hat{2}\hat{2}\hat{0}\rangle &= |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle, \\
|\hat{2}\hat{0}\hat{2}\hat{0}\rangle &= |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|\hat{0}\hat{2}\hat{0}\hat{2}\rangle &= |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle.
\end{aligned} \tag{B7}$$

For the group C₁, we have

$$\begin{aligned}
|\hat{1}\hat{2}10\rangle &= |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|\hat{2}10\hat{1}\rangle &= |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle, \\
|10\hat{1}\hat{2}\rangle &= |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle, \\
|0\hat{1}\hat{2}1\rangle &= |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle, \\
|\hat{0}12\hat{1}\rangle &= |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|12\hat{1}\hat{0}\rangle &= |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle, \\
|2\hat{1}\hat{0}1\rangle &= |\downarrow\rangle|\downarrow\rangle|\uparrow\rangle, \\
|\hat{1}\hat{0}12\rangle &= |\downarrow\rangle|\downarrow\rangle|\downarrow\rangle.
\end{aligned} \tag{B4}$$

For the group C₂, we have

$$\begin{aligned}
|\hat{1}1\hat{1}1\rangle &= |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle, \\
|1\hat{1}1\hat{1}\rangle &= |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle.
\end{aligned} \tag{B5}$$

For the group D, we have

2. All terms of the Hamiltonian matrix

In Sec. III, we have showed the approach to get the Hamiltonian matrix through an example. Here, according to the actions shown in Sec. II C, we give all terms of the Hamiltonian matrix using Pauli matrices. For the electric part, we have

$$\begin{aligned}
H_E &= \frac{3}{32}(I+Z)(I+Z)(I-Z)III \\
&\quad + \frac{3}{16}(I+Z)(I-Z)(I+Z)III \\
&\quad + \frac{3}{64}(I+Z)(I-Z)(I-Z)(I+Z)(I+Z)I \\
&\quad + \frac{9}{32}(I-Z)(I+Z)(I+Z)III \\
&\quad + \frac{3}{16}(I-Z)(I+Z)(I-Z)(I+Z)II \\
&\quad + \frac{3}{32}(I-Z)(I+Z)(I-Z)(I-Z)(I+Z)I,
\end{aligned} \tag{B8}$$

where we omit the sign $\otimes$ for simplicity. Later, we always do this. For the magnetic part, we have

$$\begin{aligned}
H_B = & \frac{1}{8} [X(I+Z)X - Y(I+Z)Y](I-Z)(I+Z)X - \frac{1}{8} [X(I+Z)X + Y(I+Z)Y](I-Z)II \\
& - \frac{1}{8} Y(I+Z)Y(I+Z)XX - \frac{1}{8} X(I+Z)Y(I+Z)XY - \frac{1}{8} Y(I+Z)Y(I+Z)IX + \frac{1}{8} X(I+Z)Y(I+Z)IY \\
& + \frac{1}{16} [X(I+Z)X - Y(I+Z)Y](I+Z)(XX+IX) + \frac{1}{16} [Y(I+Z)X + X(I+Z)Y](I+Z)(IY-XY) \\
& - \frac{1}{16} (I+Z)(I-Z)(I+Z)(XIX - YIY) - \frac{1}{16} (I+Z)(I-Z)(I+Z)(XXX + YXY) \\
& + \frac{1}{64} (I+Z)(I-Z)(I-Z)(I+Z)(I+Z)X.
\end{aligned} \tag{B9}$$

For the fermion mass part, we have

Finally, for the fermion kinetic part, we first divide it into different parts as

$$\begin{aligned}
H_{FM} = & \frac{m}{8} (I-Z)(I+Z)(I-Z)(I-Z)(I+Z)Z \\
& - \frac{m}{8} (I+Z)(I+Z)(I+Z)(I-Z)(I+Z)Z \\
& - \frac{m}{4} (I+Z)(I+Z)(I-Z)ZIZ \\
& + \frac{m}{4} (I-Z)(I+Z)(I+Z)IIZ.
\end{aligned} \tag{B10}$$

$$H_{FK} = H_{AB} + H_{BC} + H_{CD} + H_{DE} \tag{B11}$$

where H_{AB} represents the Hamiltonian matrix based on the actions that transport the group A to group B and the others are similar. For H_{AB} , we have

$$\begin{aligned}
H_{AB} = & -\frac{\sqrt{2}}{16} (I+Z)(I+Z)X(I-Z)(I+Z+X)X + \frac{\sqrt{2}}{16} (I+Z)(I+Z)Y(I-Z)YX \\
& - \frac{\sqrt{2}}{16} (I+Z)(I+Z)(XX+YY)(I+Z+X)I - \frac{\sqrt{2}}{16} (I+Z)(XY-YX)YI \\
& - \frac{\sqrt{2}}{8} (I+Z)(I+Z)(XI+XZ+XX-YY)II.
\end{aligned} \tag{B12}$$

For H_{BC} , we have

$$\begin{aligned}
H_{BC} = & \frac{1}{8} (I+Z)(XX+YY)(I+Z-X)II + \frac{1}{8} (I+Z)(YX-XY)YII \\
& + \frac{1}{16} (I+Z)(XX+YY)(XXX+YXY) + \frac{1}{16} (I+Z)(YX-XY)(YXX-XXY) \\
& + \frac{1}{16} (I+Z)(XX+YY)(XIX-YIY) + \frac{1}{16} (I+Z)(YX-XY)(YIX+XIY) \\
& - \frac{1}{16} (I+Z)(XX+YY)(I-Z)IX + \frac{1}{16} (I+Z)(XY-YX)(I-Z)IY \\
& - \frac{1}{16} (I+Z)(XX+YY)(I-Z)XX - \frac{1}{16} (I+Z)(XY-YX)(I-Z)XY \\
& - \frac{\sqrt{2}}{16} (I+Z)X(I-Z)(I+Z+X)(I+Z+X)I - \frac{\sqrt{2}}{16} (I+Z)X(I-Z)YYI \\
& + \frac{\sqrt{2}}{16} (I+Z)Y(I-Z)(I+Z+X)YI + \frac{\sqrt{2}}{16} (I+Z)Y(I-Z)Y(I+Z+X)I.
\end{aligned} \tag{B13}$$

For H_{CD} , we have

$$\begin{aligned}
H_{\text{CD}} = & \frac{1}{8}(XX + YY)(I + Z)(I - Z)II - \frac{1}{8}(XX + YY)(I + Z)(I + Z + X)II \\
& - \frac{1}{8}(XY - YX)(I + Z)YII + \frac{1}{8}(XX + YY)(I + Z)XXI + \frac{1}{8}(YX - XY)(I + Z)YXI \\
& + \frac{\sqrt{2}}{32}(XX + YY)(XX + YY)(I + Z + X)X + \frac{\sqrt{2}}{32}(YX - XY)(YX - XY)(I + Z + X)X \\
& + \frac{\sqrt{2}}{32}(XX + YY)(YX - XY)YX - \frac{\sqrt{2}}{32}(YX - XY)(XX + YY)YX \\
& + \frac{\sqrt{2}}{32}(XX + YY)X(I + Z)(I + Z + X)I + \frac{\sqrt{2}}{32}(XX + YY)Y(I + Z)YI \\
& + \frac{\sqrt{2}}{32}(YX - XY)Y(I + Z)(I + Z + X)I - \frac{\sqrt{2}}{32}(YX - XY)X(I + Z)YI.
\end{aligned} \tag{B14}$$

For H_{DE} , we have

$$\begin{aligned}
H_{\text{DE}} = & \frac{\sqrt{2}}{16}(I - Z)(I + Z)X(I + Z)II + \frac{\sqrt{2}}{32}(I - Z)(I + Z)X(I - Z + X)(X + I + Z)I \\
& - \frac{\sqrt{2}}{32}(I - Z)(I + Z)YY(X + I + Z)I + \frac{\sqrt{2}}{32}(I - Z)(I + Z)XYXI \\
& + \frac{\sqrt{2}}{32}(I - Z)(I + Z)Y(I + X - Z)YI + \frac{\sqrt{2}}{32}(I - Z)(I + Z)(XX + YY)XX \\
& + \frac{\sqrt{2}}{32}(I - Z)(I + Z)(XY - YX)XY + \frac{\sqrt{2}}{32}(I - Z)(I + Z)(XX + YY)IX \\
& - \frac{\sqrt{2}}{32}(I - Z)(I + Z)(XY - YX)IY.
\end{aligned} \tag{B15}$$

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