

Four-dimensional spectrum generating algebra for superstrings

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We present the spectrum generating algebra for the hybrid string with manifest $\mathcal{N} = 1$ super Poincaré symmetry in $\mathbb{R}^{3,1} \times \mathbb{R}^6$. Our Del-Giudice–Di-Vecchia–Fubini operators establish a one-to-one correspondence with the conventional superstring spectrum, while making its four-dimensional structure manifest. We also discuss part of the antifield spectrum, and introduce a simple realization of the supersymmetry charges involving the $\mathbb{R}^6$ directions. As an application, we algebraically compute the helicity partition function. These results provide new tools for analyzing four-dimensional on-shell superspaces and may extend naturally to phenomenologically relevant compactifications.

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Introduction. String theory stands out as a self-consistent framework incorporating gravity at the quantum level. Despite this achievement, a string vacuum that precisely matches the matter content and parameters of our Universe has not yet been identified. Among other notable features, string theory requires additional spacetime dimensions; these are typically argued to be Planck-scale small or part of large extra dimension scenarios (see [1,2]). Their precise geometry and size determine the four-dimensional physics, contributing to the vast string landscape of possible vacua (see [3] for a recent review). Experimental constraints are stringent but have not yet ruled out their existence, in particular because their nature is highly model dependent [4]. From a phenomenological perspective, it is then natural to try to formulate string theory in a way that makes four-dimensional physics manifest. This can be achieved to different degrees of convenience through compactification in the more traditional formalisms (e.g., [5]), though not without technical challenges.

In [6,7], the critical spinning string was recast in terms of a model in which super Poincaré symmetry is manifest in four spacetime dimensions, with minimal supersymmetry. This hybrid string has the desirable features of the Ramond-Neveu-Schwarz and Green-Schwarz formalisms, while sidestepping certain complicating ingredients. It has a unique phenomenological appeal, enabling covariant

quantization in $d = 4$ without the need for spectrum projections and sum over spin structures. The hybrid formalism has been studied and applied in different contexts, in both its $d = 4$ and $d = 6$ [8] formulations. Earlier results showed that it is well suited to describe string theory in Ramond-Ramond backgrounds [9–11]. More recently, it has been used to study the $\text{AdS}_3/\text{CFT}_2$ correspondence [12,13], and second-quantized heterotic strings [14,15], showcasing its versatility.

Regarding physical states, the covariant construction of vertex operators in the hybrid string becomes impractical at higher mass levels. In $d = 4$, beyond the massless spectrum only the first massive level of the open string has been explored—either in flat space or in a constant electromagnetic background [16,17]. The $d = 6$ formulation is in this sense more developed (e.g., [18–20]). Spectrum generating algebras have been successfully obtained in AdS_3 backgrounds [21–23], extending the Del-Giudice–Di-Vecchia–Fubini (DDF) construction [24]. Analogous progress in the $d = 4$ version has not yet been reported.

Such a gap becomes particularly substantial given the recent applications found for DDF operators. They have been used, for example, in the analysis of the decay and scattering of highly excited strings, and their chaotic behavior [25,26] (see also [27]). The latter motivated the introduction of a new measure for chaos in scattering amplitudes [28,29]. In these results, the focus was on DDF operators of the bosonic string, with the tachyon as ground state. In the spinning string [30], the creation-annihilation algebra of Ramond vertices leads to operators of arbitrarily high picture number (see, e.g., [31]). This hinders a more systematic study of scattering amplitudes. In the pure spinor superstring this obstacle is naturally absent

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[32–34]. However, amplitude computations are more convoluted because of the pure spinor prescription, in particular when brought down to $d = 4$.

In this work, we present a spectrum generating algebra for the hybrid string in $\mathbb{R}^{3,1} \times \mathbb{R}^6$, introducing a direct vertex operator map to more traditional superstring formalisms. In addition to making manifest the $d = 4$ structure of the vertex operators, our construction bypasses current limitations of DDF operators in superstring theories. We also derive some simple partition functions of the spectrum, including the distribution of helicities through different mass levels. This data is helpful in understanding interactions and selection rules sensitive to helicity, and the counting of physical states relevant for helicity amplitudes. More generally, it helps to expose the structure of four-dimensional on-shell superspaces and spinor-helicity descriptions in string theory. Furthermore, our results establish a framework for the algebraic description of the physical spectrum and the moduli space of superstrings on backgrounds of phenomenological interest, such as the T^6/Z_3 heterotic orbifold [35,36].

The hybrid formalism. We will split the target space coordinates of the string, X^μ , with $\mu = 0, \dots, 9$, into x^m , with $m = 0, \dots, 3$, and $\{\bar{y}^I, y_I\}$, with $I = 1, 2, 3$. Their world-sheet action is given by

$$S_X = \frac{1}{2\pi\alpha'} \int d^2z \{ \partial x^m \bar{\partial} x^n \eta_{mn} + \partial \bar{y}^I \bar{\partial} y_I \}, \quad (1)$$

where η_{mn} is the flat metric. The remaining world-sheet fields are described by the action

$$S_f = \frac{1}{2\pi} \int d^2z \{ p_\alpha \bar{\partial} \theta^\alpha + \bar{p}_{\dot{\alpha}} \bar{\partial} \bar{\theta}^{\dot{\alpha}} + \bar{\psi}^I \bar{\partial} \psi_I + \frac{1}{2} \partial \rho \bar{\partial} \rho + \dots \}, \quad (2)$$

where $\alpha, \dot{\alpha} = \pm$, with θ^α and $\bar{\theta}^{\dot{\alpha}}$ denoting the spacetime superpartners of x^m , and $\bar{\psi}^I$ and ψ_I the twisted world-sheet superpartners of $\bar{y}^I$ and y_I . In addition, we have a chiral scalar ρ , satisfying $\bar{\partial} \rho = 0$, and the ellipsis denotes an antiholomorphic sector. The latter could be the usual mirrored fields of the open string or heterotic/type II completions of the closed string. For simplicity, we will focus on the holomorphic sector, and specific results for open or closed strings follow the standard steps.

The four-dimensional spacetime supersymmetry charges are realized through

$$q_\alpha = \oint \left(p_\alpha - \frac{1}{\alpha'} \sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \partial x_m - \frac{1}{4\alpha'} \partial \theta_\alpha \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} \right), \quad (3a)$$

$$\bar{q}_{\dot{\alpha}} = \oint \left(\bar{p}_{\dot{\alpha}} - \frac{1}{\alpha'} \sigma_{\alpha\dot{\alpha}}^m \theta^\alpha \partial x_m - \frac{1}{4\alpha'} \partial \bar{\theta}_{\dot{\alpha}} \theta^\alpha \theta_\alpha \right), \quad (3b)$$

which satisfy $\{q_\alpha, q_\beta\} = \{\bar{q}_{\dot{\alpha}}, \bar{q}_{\dot{\beta}}\} = 0$ and

$$\{q_\alpha, \bar{q}_{\dot{\alpha}}\} = -\frac{2}{\alpha'} \sigma_{\alpha\dot{\alpha}}^m \oint \partial x_m. \quad (4)$$

The Pauli matrices are denoted by $\sigma_{\alpha\dot{\alpha}}^m$, and spinor indices are raised (lowered) with the antisymmetric tensor $\epsilon^{\alpha\beta}$ ($\epsilon_{\alpha\beta}$), e.g., $\theta_\alpha = \epsilon_{\alpha\beta} \theta^\beta$, with $\epsilon^{\alpha\gamma} \epsilon_{\gamma\beta} = \delta_\beta^\alpha$. There is an analogous construction for the dotted indices. Unlike in the spinning string, the $d = 4$ supersymmetry algebra in the hybrid formalism closes without picture changing.

In order to have manifest supersymmetry, the usual invariants are introduced,

$$d_\alpha = p_\alpha + \frac{1}{2\alpha'} \bar{\theta}^{\dot{\alpha}} (2\sigma_{\alpha\dot{\alpha}}^m \partial x_m + \bar{\theta}_{\dot{\alpha}} \partial \theta_\alpha - \partial \bar{\theta}_{\dot{\alpha}} \theta_\alpha), \quad (5)$$

$$\bar{d}_{\dot{\alpha}} = \bar{p}_{\dot{\alpha}} + \frac{1}{2\alpha'} \theta^\alpha (2\sigma_{\alpha\dot{\alpha}}^m \partial x_m + \theta_\alpha \partial \bar{\theta}_{\dot{\alpha}} - \partial \theta_\alpha \bar{\theta}_{\dot{\alpha}}), \quad (6)$$

$$\Pi^m = \partial x^m + \frac{1}{2} \sigma_{\alpha\dot{\alpha}}^m (\partial \theta^\alpha \bar{\theta}^{\dot{\alpha}} + \partial \bar{\theta}^{\dot{\alpha}} \theta^\alpha). \quad (7)$$

The hybrid string Becchi-Rouet-Stora-Tyutin (BRST) current is given by

$$J_{\text{BRST}} = e^\rho \Delta + \frac{4}{\alpha'} \partial \bar{y}^I \psi_I, \quad (8)$$

while the b ghost, due to various field redefinitions, is recast as

$$b = -\frac{\alpha'}{16} e^{-\rho} \bar{\Delta} - \frac{1}{4} \partial y_I \bar{\psi}^I, \quad (9)$$

where

$$\Delta = \epsilon^{\alpha\beta} (d_\alpha, d_\beta), \quad (10)$$

$$\bar{\Delta} = \epsilon^{\dot{\alpha}\dot{\beta}} (\bar{d}_{\dot{\alpha}}, \bar{d}_{\dot{\beta}}). \quad (11)$$

The ordering operation is defined by

$$(A, B)(z) = \oint \frac{dw}{(w-z)} A(w) B(z). \quad (12)$$

The holomorphic component of the total energy-momentum tensor, given by

$$T = -\frac{1}{\alpha'} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha - \bar{p}_{\dot{\alpha}} \partial \bar{\theta}^{\dot{\alpha}} - \frac{1}{\alpha'} \partial \bar{y}^I \partial y_I - \bar{\psi}^I \partial \psi_I - \frac{1}{2} \partial \rho \partial \rho - \frac{1}{2} \partial^2 \rho, \quad (13)$$

is BRST exact. This follows from the Operator Product Expansion (OPE)

$$J_{\text{BRST}}(z)b(y) \sim \frac{2}{(z-y)^3} + \frac{J}{(z-y)^2} + \frac{T}{(z-y)}.$$

Here, J is a $U(1)$ current that is identified as the ghost number, and expressed as

$$J = \psi_I \bar{\psi}^I - \partial \rho. \quad (14)$$

Lastly, we introduce

$$\eta = -2\alpha' e^{-2\rho} \Psi \bar{\Delta} + 4e^{IJK} e^{-\rho} \partial y_I \psi_J \psi_K, \quad (15)$$

with ϵ^{IJK} denoting the totally antisymmetric tensor with $\epsilon^{123} = +1$, and $\Psi = (\epsilon^{IJK} \psi_I \psi_J \psi_K)/6$. The operator η comes from the fermionization of the superconformal ghosts of the spinning string, again after field redefinitions and similarity transformations.

The massless spectrum. Physical states U in the hybrid string belong to the ghost number one cohomology of the BRST charge

$$Q = \oint J_{\text{BRST}}. \quad (16)$$

Furthermore, the physical spectrum should be in the so-called small Hilbert space. This condition is inherited from the spinning string, and can be expressed as $\eta_0 \cdot U = 0$, i.e., physical states are also annihilated by the zero mode of η .

Since our goal is to have manifest Lorentz symmetry in four dimensions, we will boost all momentum eigenstates to the x^m directions. In this case, the massless spectrum can be divided into two sets. The first corresponds to states that are singlets with respect to boosts in the $\{\bar{y}^I, y_I\}$ directions,

$$\begin{aligned} U_{4d} &= \eta_0 \cdot \mathcal{V}(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}), \\ &= -2\alpha' [\partial(\Psi e^{-2\rho}) \bar{D}^2 \mathcal{V} + 2\Psi e^{-2\rho} (\bar{d}_{\dot{\alpha}}, \bar{D}^{\dot{\alpha}} \mathcal{V})]. \end{aligned} \quad (17)$$

Here $\mathcal{V}$ is a real superfield and $\bar{D}_{\dot{\alpha}}$ (D_α) denotes the superderivative, with $\bar{D}^2 = \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}$. BRST closedness requires that

$$D_\alpha \bar{D}^2 D^\alpha \mathcal{V} = \bar{D}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} \mathcal{V} = 0, \quad (18)$$

which is the equation of motion for Maxwell's superfield. The gauge invariance $\delta \mathcal{V} = \Lambda + \bar{\Lambda}$, where Λ ($\bar{\Lambda}$) is a chiral (antichiral) superfield, follows directly from η_0 -exact (BRST-exact) states. On the mass shell, $\mathcal{V}$ encodes two bosonic and two fermionic degrees of freedom, respectively the two circular polarizations of the photon and superpartners. In the case of closed strings, they would describe the chiral half of the massless multiplet.

The remaining physical states at the massless level can be mapped to the vertex operator

$$U_{6d} = \psi_I \bar{\Phi}^I + e^{-\rho} \epsilon^{IJK} \psi_J \psi_K \Phi_I, \quad (19)$$

satisfying the massless superfield equations

$$D^2 \Phi_I = \bar{D}^2 \bar{\Phi}^I = 0, \quad (20)$$

together with the chirality conditions $D_\alpha \bar{\Phi}^I = \bar{D}_{\dot{\alpha}} \Phi_I = 0$. The massless equation of motion for Φ_I follows from BRST closedness, while annihilation by η_0 implies that $\bar{D}_{\dot{\alpha}} \Phi_I = 0$. For $\bar{\Phi}^I$, it is the other way around, with Q and η_0 playing a dual role. In fact, $\bar{\Phi}^I$ and Φ_I are conjugate to each other in the hybrid formalism.¹ All together, they describe six bosonic and six fermionic degrees of freedom. In [6] there is a detailed analysis of the massless cohomology in the heterotic case in a Calabi-Yau background.

It is interesting to point out that the hybrid string also inherits the picture degeneracy of the spinning string. Therefore, a given physical state admits multiple (equivalent) descriptions with vertex operators at different pictures. For example, we can define $\tilde{U}_{6d} = \frac{1}{2} \eta_0 \cdot (e^{-\rho} \psi_I \Omega^I)$, such that

$$\begin{aligned} \tilde{U}_{6d} &= 4e^{-2\rho} \Psi \partial y_I \Omega^I - \alpha' e^{-3\rho} \Psi \partial \psi_I (\bar{\Delta}, \Omega^I) \\ &\quad + \alpha' e^{-3\rho} (\partial^2 \Psi - 4\partial \Psi \partial \rho) \psi_I (\bar{d}_{\dot{\alpha}}, \bar{D}^{\dot{\alpha}} \Omega^I) \\ &\quad + \alpha' [\partial \Psi \partial^2 \rho - 2\partial \Psi (\partial \rho)^2] e^{-3\rho} \psi_I \bar{D}^2 \Omega^I \\ &\quad + \alpha' \left(\partial^2 \Psi \partial \rho - \frac{1}{3!} \partial^3 \Psi \right) e^{-3\rho} \psi_I \bar{D}^2 \Omega^I. \end{aligned} \quad (21)$$

It is then straightforward to evaluate the action of the BRST charge,

$$Q \cdot \tilde{U}_{6d} = -\alpha' e^{-2\rho} \Psi \partial \psi_I \bar{D}^2 D^2 \Omega^I. \quad (22)$$

In this case, $D^2 \Omega^I$ defines the antichiral superfield $\bar{\Phi}^I$, and BRST closedness leads to $\bar{D}^2 \bar{\Phi}^I = 0$.

The vertices (17) and (19) also have the usual integrated version. The respective integrands are

$$\begin{aligned} V_{4d} &= i\partial \theta^\alpha D_\alpha \mathcal{V} - i\partial \bar{\theta}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} \mathcal{V} - \frac{i\alpha'}{4} (d_\alpha, \bar{D}^2 D^\alpha \mathcal{V}) \\ &\quad + \frac{i\alpha'}{4} (\bar{d}_{\dot{\alpha}}, D^2 \bar{D}^{\dot{\alpha}} \mathcal{V}) + \frac{i}{2} \sigma_{\alpha\dot{\alpha}}^m (\Pi_m, [\bar{D}^{\dot{\alpha}}, D^\alpha] \mathcal{V}), \end{aligned} \quad (23)$$

in which $\delta \mathcal{V} = \Lambda + \bar{\Lambda}$ leads to $\delta V_{4d} = i\partial(\Lambda - \bar{\Lambda})$, and

¹Hermiticity properties have a nontrivial realization in the hybrid description [37].

$$V_{6d} = \partial \bar{y}^I \Phi_I + \frac{\alpha'}{2} e^\rho \bar{\psi}^I (d_\alpha, D^\alpha \Phi_I) + \partial y_I \bar{\Phi}^I + \frac{\alpha'}{2} e^{-\rho} \psi_I (\bar{d}_{\dot{\alpha}}, \bar{D}^{\dot{\alpha}} \bar{\Phi}^I). \quad (24)$$

The latter should be equivalent to the large radius limit of marginal deformations of the Calabi-Yau manifold, which were described in [6]. Both V_{4d} and V_{6d} are annihilated by the BRST charge and by η_0 up to total derivatives. For example, we can show that

$$Q \cdot V_{4d} = \partial[\partial e^\rho D^2 \mathcal{V} + 2e^\rho (d_\alpha, D^\alpha \mathcal{V})], \quad (25)$$

$$\eta_0 \cdot V_{6d} = -4\alpha' \partial[e^{-\rho} \epsilon^{IJK} \psi_J \psi_K \Phi_I], \quad (26)$$

using the relevant superfield equations of motion.

Note that from (24), we obtain the missing supersymmetry charges of the model. They are given by

$$q_\alpha^I = \oint \left(e^\rho \bar{\psi}^I d_\alpha - \frac{2}{\alpha'} \partial \bar{y}^I \theta_\alpha \right), \quad (27a)$$

$$\bar{q}_{I\dot{\alpha}} = \oint \left(e^{-\rho} \psi_I \bar{d}_{\dot{\alpha}} - \frac{2}{\alpha'} \partial y_I \bar{\theta}_{\dot{\alpha}} \right), \quad (27b)$$

and satisfy

$$\{q_\alpha^I, \bar{q}_{J\dot{\alpha}}\} = -\frac{2}{\alpha'} \delta_J^I \sigma_{\alpha\dot{\alpha}}^m \oint \partial x_m. \quad (28)$$

The remaining brackets, including with q_α and $\bar{q}_{\dot{\alpha}}$, are either BRST exact or η_0 exact. This is compatible with the fact that the $\mathcal{N} = 1$ ten-dimensional supersymmetry algebra only closes on shell. It is straightforward to show that (17) and (19) are mapped to each other using (27).

As for the massless antifield spectrum, i.e., the ghost number two cohomology, it is easy to find the analog of (17), for instance. A generic η_0 -exact vertex operator can be cast as

$$U_{4d}^* = e^{-\rho} \Psi \Phi^*(x, \theta, \bar{\theta}), \quad (29)$$

with $\Phi^* \equiv \bar{D}_{\dot{\alpha}} \bar{A}^{\dot{\alpha}}$ for a given superfield $\bar{A}^{\dot{\alpha}}$. BRST closedness implies the equation of motion $D^2 \Phi^* = 0$. BRST-exact states translate to the gauge invariance $\delta \bar{A}^{\dot{\alpha}} = \bar{\Lambda}^{\dot{\alpha}}$, with $\bar{\Lambda}^{\dot{\alpha}}$ denoting an antichiral superfield parameter. In order to manifest the field content of an antifield multiplet, we can fix some algebraic redundancy, such that

$$\Phi^* = \theta^\alpha \chi_\alpha^* + \theta^\alpha \theta_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \xi_\alpha^* + \theta^\alpha \theta_{\dot{\alpha}} F^* + \theta^\alpha \bar{\theta}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^m a_m^* - \frac{1}{4} \theta^\alpha \theta_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_{\dot{\alpha}} \partial^m a_m^*. \quad (30)$$

$D^2 \Phi^* = 0$ then leads to

$$F^* = 0, \quad (31a)$$

$$\partial^m a_m^* = 0, \quad (31b)$$

$$\xi_\alpha^* = \frac{1}{4} \epsilon^{\alpha\beta} \sigma_{\alpha\dot{\alpha}}^m \partial_m \chi_\beta^*, \quad (31c)$$

matching the expected antifield dynamics of the massless vector multiplet.

Spectrum generating algebra. While the analysis of the massless spectrum is relatively simple, the computation of massive string vertices quickly becomes impractical. Results for the first massive level can be found in [16,17]. For higher mass levels, there is a systematic procedure to build vertex operators of arbitrary mass level using only the massless ones, the well-known DDF operators [24]. Simply put, they correspond to massless vertices rewritten in a specific light-cone frame.

We will denote the four-dimensional directions x^m by $x^\pm = (x^0 \pm x^3)/\sqrt{2}$, $x = (x^1 + ix^2)/\sqrt{2}$, and $\bar{x} = (x^1 - ix^2)/\sqrt{2}$, such that the nonzero components of the flat-space metric are $\eta^{+-} = \eta^{-+} = -1$ and $\eta^{x\bar{x}} = \eta^{\bar{x}x} = 1$. In this case, the Pauli matrices have a simple form, with $\sigma_{\alpha\dot{\alpha}}^\pm = \sqrt{2} \delta_\alpha^\pm \delta_{\dot{\alpha}}^\pm$, $\sigma_{\alpha\dot{\alpha}}^x = \sqrt{2} \delta_\alpha^+ \delta_{\dot{\alpha}}^+$, and $\sigma_{\alpha\dot{\alpha}}^{\bar{x}} = \sqrt{2} \delta_\alpha^+ \delta_{\dot{\alpha}}^-$.

Our first step is to introduce the ground states of the DDF representation. They will be defined through the state-operator map of the gauge-fixed massless unintegrated vertex operators of (17) and (19) with momentum $\sqrt{2}k$ in the light-cone direction x^- . Up to normalizations, the bosonic states are identified by

$$|a, k\rangle \cong \Psi e^{-2\rho} \bar{d}_+ \theta^+ e^{ik(\sqrt{2}x^- - \theta^- \bar{\theta}^-)}, \quad (32a)$$

$$|\bar{a}, k\rangle \cong \Psi e^{-2\rho} \bar{d}_- \theta^- e^{ik\sqrt{2}x^-}, \quad (32b)$$

$$|b, I, k\rangle \cong e^{-\rho} \epsilon^{IJK} \psi_J \psi_K e^{ik(\sqrt{2}x^- - \theta^- \bar{\theta}^-)}, \quad (32c)$$

$$|\bar{b}, I, k\rangle \cong \psi_I e^{ik(\sqrt{2}x^- + \theta^- \bar{\theta}^-)}, \quad (32d)$$

and the fermionic states by

$$|\chi, k\rangle \cong -2i\Psi e^{-2\rho} \bar{d}_+ \theta^+ e^{ik\sqrt{2}x^-}, \quad (32e)$$

$$|\bar{\chi}, k\rangle \cong \frac{1}{k} \Psi e^{-2\rho} \bar{d}_- \theta^- e^{ik(\sqrt{2}x^- + \theta^- \bar{\theta}^-)}, \quad (32f)$$

$$|\xi, I, k\rangle \cong e^{-\rho} \epsilon^{IJK} \psi_J \psi_K \theta^- e^{ik\sqrt{2}x^-}, \quad (32g)$$

$$|\bar{\xi}, I, k\rangle \cong \psi_I \bar{\theta}^- e^{ik\sqrt{2}x^-}. \quad (32h)$$

For convenience, these states will be collectively denoted by $|0, k\rangle$, indicating they are massless. It is straightforward to compute their supersymmetry transformations using (3),

e.g., $q_-|\bar{a}, k\rangle = k|\bar{\chi}, k\rangle$ and $\bar{q}_-|\bar{\xi}, I\rangle = -|\bar{b}, I\rangle$. In particular, it is possible to show that both $q_+|a, k\rangle$ and $q_+|\chi, k\rangle$ are mapped to pure gauge operators.

For the DDF algebra, we will work with integrated vertex operators derived from (23) and (24), with momentum $\sqrt{2}k$ in the light-cone direction x^+ . They are given by

$$A(k) = \oint \left[\sqrt{2}\Pi^x\Phi - \left(\frac{\alpha'}{2}\bar{d}_- + \frac{i}{k}\partial\theta^- \right) D_+\Phi \right], \quad (33a)$$

$$\bar{A}(k) = \oint \left[\sqrt{2}\Pi^x\bar{\Phi} - \left(\frac{\alpha'}{2}d_- + \frac{i}{k}\partial\bar{\theta}^- \right) \bar{D}_+\bar{\Phi} \right], \quad (33b)$$

$$B(k) = \oint \left[\partial\bar{y}^I\Phi_I + \frac{\alpha'}{2}e^\rho\bar{\psi}^I(d_-, D_+\Phi_I) \right], \quad (33c)$$

$$\bar{B}(k) = \oint \left[\partial y_I\bar{\Phi}^I + \frac{\alpha'}{2}e^{-\rho}\psi_I(\bar{d}_-, \bar{D}_+\bar{\Phi}^I) \right], \quad (33d)$$

with

$$\Phi = (\phi + \theta^+\chi)e^{ik(\sqrt{2}X^+ - \theta^+\bar{\theta}^+)}, \quad (34a)$$

$$\bar{\Phi} = (\bar{\phi} + \bar{\theta}^+\bar{\chi})e^{ik(\sqrt{2}X^+ + \theta^+\bar{\theta}^+)}, \quad (34b)$$

$$\Phi_I = (\phi_I + \theta^+\chi_I)e^{ik(\sqrt{2}X^+ - \theta^+\bar{\theta}^+)}, \quad (34c)$$

$$\bar{\Phi}^I = (\bar{\phi}^I + \bar{\theta}^+\bar{\chi}^I)e^{ik(\sqrt{2}X^+ + \theta^+\bar{\theta}^+)}, \quad (34d)$$

in which $\{\phi, \bar{\phi}, \phi_I, \bar{\phi}^I\}$ and $\{\chi, \bar{\chi}, \chi_I, \bar{\chi}^I\}$, respectively, denote the set of bosonic and fermionic polarizations. These operators satisfy

$$[\bar{A}(k), A(\tilde{k})] = -ik\sqrt{2}\alpha'\delta_{k+\tilde{k}}\left(\phi\bar{\phi} + \frac{i}{2k}\chi\bar{\chi}\right)\oint dx^+, \quad (35a)$$

$$[\bar{B}(k), B(\tilde{k})] = -ik\sqrt{2}\alpha'\delta_{k+\tilde{k}}\left(\phi_I\bar{\phi}^I + \frac{i}{2k}\chi_I\bar{\chi}^I\right)\oint dx^+, \quad (35b)$$

and the remaining commutators vanish.

Equation (35) can be interpreted as a creation-annihilation algebra for ground states (32) with specific momentum. Schematically, any mass level N state in the physical spectrum of the hybrid string can be expressed as

$$|N, \{p, \bar{p}, q, \bar{q}\}\rangle = \prod_{n=1}^L (A(nk))^{p_n} (\bar{A}(nk))^{\bar{p}_n} \times (B(nk))^{q_n} (\bar{B}(nk))^{\bar{q}_n} \left| 0, \frac{1}{\alpha'k} \right\rangle, \quad (36)$$

with mass $m^2 = 4N/\alpha'$. Here, $p_n, \bar{p}_n, q_n$, and $\bar{q}_n$ are non-negative integers, such that

$$\sum_{n=1}^L n(p_n + \bar{p}_n + q_n + \bar{q}_n) = N. \quad (37)$$

Annihilation operators come with negative n , e.g.,

$$A(-k) \left| 0, \frac{1}{\alpha'k} \right\rangle = 0. \quad (38)$$

Since we have 16 DDF ground states, cf. (32), and 16 creation operators for each n , half bosonic and half fermionic, it is trivial to establish a one-to-one map between the light-cone states of the hybrid string and other superstring formalisms.

Some partition functions. From the DDF algebra, we can easily compute some of the partition functions of the hybrid string. They are defined as the weighted sum over the states in the physical Hilbert space, $\mathcal{H}_{\text{phys}}$, which is spanned by (36). While these are connected to the one-loop vacuum amplitude of the theory (see, e.g., [38]), we are mostly interested in keeping track of some quantum numbers, in particular because the DDF states have fixed momentum. We will use the open string as a case study, but the extension to the closed string is mostly obvious.

The mass distribution of the physical spectrum can be read from

$$Z(q) = \text{Tr}_{\mathcal{H}_{\text{phys}}}(q^{L_0}), \quad (39)$$

$$= 16 \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^8, \quad (40)$$

which is a well-known result, where L_0 is the mass-level operator. If we insert the fermion number operator, $(-1)^F$, i.e., the supertrace in (39), the result vanishes because supersymmetry is not broken.

Since we are working with the little group $SO(2) \cong U(1)$, the helicity structure of the DDF states is also very transparent. In (39), the polarizations $\{\phi, \bar{\phi}, \chi, \bar{\chi}\}$ can be assigned helicity $\left\{+1, -1, +\frac{1}{2}, -\frac{1}{2}\right\}$, while the R -charged polarizations $\{\phi_I, \bar{\phi}^I, \chi_I, \bar{\chi}^I\}$ have helicity $\left\{0, 0, -\frac{1}{2}, +\frac{1}{2}\right\}$. The helicity partition function is defined as

$$H(q, y) = \text{Tr}_{\mathcal{H}_{\text{phys}}}(q^{L_0} y^{2h}), \quad (41)$$

where h is the helicity operator. It is straightforward to show that

$$H(q, y) = \frac{(1+y)^4}{y^2} \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^6 \frac{(1+q^n y)^4}{1-q^n y^2} \frac{(1+q^n/y)^4}{1-q^n/y^2}, \quad (42)$$

which explicitly enumerates supermultiplet content in terms of helicities. The supertrace version of (41) is given by $H(q, -y)$.

There are other possible refinements, such as the inclusion of the R charge, but we will not discuss them here. They become more interesting with additional spacetime structures (such as orbifolds) or compactifications that break supersymmetry. A detailed account of refined partition functions can be found in [39], where the open superstring is analyzed in even-dimensional compactifications.

Final remarks. The DDF framework in string theory exposes a clear map between vertex operators and light-cone states. Here we have established this construction for the hybrid string in $\mathbb{R}^{3,1} \times \mathbb{R}^6$. The advantage of the hybrid model over other formalisms is the embedded $d = 4$ superspace realization of the vertex operators. Our construction is well placed to extend recent developments on bosonic DDF operators [25,26,28,29] to the superstring.

More intriguingly, our results may help to bridge four-dimensional methods for field-theory scattering amplitudes with string scattering. The DDF operators defined in (33) admit a semicovariant formulation using a reference vector, analogous to the spinor-helicity formalism. In particular, massive vertex operators naturally involve two such vectors—one associated with the ground state and another with the creation operators—mirroring the structure of massive spinor helicity [40].

Regarding nontrivial backgrounds, our results extend naturally to $\mathbb{R}^{3,1} \times T^6$, with the exception of compactification modes. Such states have nonzero (discrete) momentum in the T^6 directions, as well as winding number. A relevant future investigation is to determine how the states in the Narain lattice fit the framework of DDF operators. To our knowledge, this topic has received only partial

treatment, without a unified spectrum-generating algebra in most of the literature. Perhaps more interestingly, we would like to analyze a similar construction in the heterotic orbifold T^6/Z_3 . In this background, the string spectrum resembles the minimal supersymmetric standard model, with chiral fermions and a realistic gauge group, while remaining fully accessible through the powerful methods of free conformal field theory. The explicit construction of the vertex operators requires the use of twist fields [35,36], and the hybrid formalism may offer a natural framework for this, particularly the massless sector.

Beyond the spectrum generating algebra, it would be interesting to explore the hybrid formalism in the context of chiral strings [41,42]. Its tensionless limit should be connected to some variant of the ambitwistor string [43], with four-dimensional covariance. Since the Virasoro constraints are not manifest in the hybrid string, we expect additional ingredients to play a role. A possible connection to twistor strings [44,45] might become more evident in this case. Furthermore, this investigation might help to introduce alternative Cachazo-He-Yuan formulas [46] for the scattering of gluons and gravitons, now in superspace.

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