

Light dilaton from top-down holographic confinement with magnetic fluxes

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A two-parameter class of higher-dimensional, strongly coupled, confining field theories in the presence of magnetic fluxes for two Abelian gauge groups admits a top-down, holographic dual description. The corresponding two-parameter family of regular background solutions of the classical equations of maximal supergravity in seven dimensions descends from maximal supergravity in 11 dimensions. We study the global and local stability properties of these solutions. We identify lines of zero-temperature first-order phase transitions, describing a polygon (a square) in the space of parameters, identified with the two fluxes. The transition separates the family of gravity solutions dual to confining theories, inside the polygon, from those outside, in which the field theory is realized in a conformal phase. In the spectrum of fluctuations of the supergravity equations, interpreted as bound states of the dual, confining field theories, we find no evidence of local instabilities (tachyons). Over a significant portion of parameter space, that extends far away from the proximity to the transition, we identify an approximate dilaton, the mass of which is one order of magnitude smaller than the scale set by confinement. Our findings complement those emerging in other holographic models discussed in the literature, in which either the dilaton mass is only mildly lower than the confinement scale (when approaching a first-order transitions), or parametrically suppressed (when reaching the proximity to a second-order one).

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I. INTRODUCTION

The advent of gauge-gravity dualities, or holography [1–4], has provided new tools to gain insight into non-perturbative properties of field theories. This is particularly effective in the strong-coupling regime of special field theories with a large number of degrees of freedom, the dual description of which consists of weakly coupled gravity theories in higher dimensions. Local operators of the field theory and their sources are in one-to-one correspondence with gravity fields and their asymptotic behavior. Nonlocal, gauge-invariant operators in the field theory correspond to geometric configurations of extended objects in the curved background of the gravity dual. The Wilson loop is a particularly clean example [5,6]—see also Refs. [7–11]—as it discriminates between conformal and confining field theories. In the holographic

treatment, confinement is realized geometrically, by the smooth shrinking of a submanifold in the gravity background [12]—see also Refs. [13–16].¹

In strongly coupled field theories that admit a gravity dual description, the free energy in the presence of external sources, and the associated response functions (condensates), can be extracted from holographic renormalization [41–43]. This is hence the ideal environment for the study of phase transitions, both at finite and zero temperature. As programmatically outlined in Ref. [44], we can use this technology to investigate the relation between the nature of phase transitions in confining gauge theories and the spectrum of their bound states, computed in proximity of the transition. This information is obtained from the study of small fluctuations of the gravity backgrounds, for which we deploy the gauge-invariant formalism developed in Refs. [45–53]. Our findings, as we shall see, fill an important, unexpected, gap in the characterization

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¹In the simplest examples, the geometry ends in correspondence to the shrinking to zero size of a circle, which generates a physical scale and mass gap in the dual theory. Other examples in which such manifold is larger exist in the context of the conifold [17], to include the celebrated cases of Refs. [18–21]—see also Refs. [22–40].

emerging from this extensive research program—see also Refs. [54–59].²

The main motivation of this research program is to provide the foundations for the physics of the dilaton, the pseudo-Nambu-Goldstone boson (PNGB) associated with scale invariance [63].³ Its possible emergence in strongly coupled field theories has distinctive implications [93], of interest both on phenomenological and theoretical grounds [94–100], especially in view of the Large Hadron Collider (LHC) program [101–107].⁴ A particular mechanism leading to the emergence of the dilaton is linked to the appearance of local, tachyonic instabilities in some region of parameter space of the theory. For example, this might happen in association with the existence of complex fixed points in the renormalization group equations [128], characterizing theories that violate the Breitenlohner-Freedman (BF) unitarity bound [129], that underlie the study of nonunitary conformal field theories [130,131].⁵

Generalizing these ideas, a dilaton might arise in strongly coupled field theories for which the holographic description displays a classical instability that is resolved by the presence of a phase transition. A first-order phase transition is characterized by the phenomena of phase coexistence, as stable and metastable solutions appear for the same choices of control parameters, but different response functions. A generic expectation is that additional, unstable solutions appear as well, for intermediate values of the response functions near the transition. The instability is expected to be signaled by at least one tachyonic state in the dual field theory. Assuming continuity it is therefore expected that the mass of this tachyonic state crosses zero somewhere in the approach to the instability, possibly along a metastable branch of solutions. The presence of the phase transition prevents the tachyonic state from being realized physically, yet does allow the possibility of a light state to arise in its proximity, and it may be associated with the dilaton. This suggestion has been tested both in the context of bottom-up holographic models [56–58]—see also precursors in Refs. [134–140]—as well as in the rigorous context of top-down constructions derived from complete supergravity theories [44,54,55,59]. Finding viable

realizations of top-down holographic models is challenging, due to the constrained nature of supergravity. Yet, fertile ground for the search of relevant supergravity background solutions has been identified in many examples. The interesting theories include Romans’s half-maximal supergravity in $D = 6$ dimensions [141–144]—see also Refs. [145–152] and Ref. [16]—the maximal supergravity in $D = 7$ space-time dimensions [153–155]—see also Refs. [156–161]—the $\mathcal{N} = 8$ maximal supergravity in $D = 5$ dimensions [162–164]—see also Refs. [165–167] and [168]—and the lower dimensional truncation of maximal supergravity in $D = 11$ dimensions discussed in Refs. [169,170]. In these explicit examples, the arising of phase transitions ensures that the tachyonic part of parameter space cannot be physically realized as vacuum of the theory.

These known examples display two possible ways in which the nature of the phase transitions is linked to the dilaton. In the presence of first-order phase transitions, the dilaton has mass just mildly lower than the confinement scale along the stable branches of vacua, while being suppressed along some metastable branches [44,54–56,58]. Conversely, in the proximity to a second-order phase transition, the dilaton displays a parametrically suppressed mass, approaching zero at the transition [57,59]. The common feature to the two classes is that it appears to be the order of the phase transition to drive the properties of the dilaton, rather than the precise nature of the instabilities associated with it. As we shall see, our present results further demonstrate this trend. While there is evidence of a first-order phase transition in the theories we analyze here, we do not find direct signals of local instabilities in the spectrum of bound states.

We focus our attention on classes of background solutions in gravity that admit a dual interpretation as confining field theories in the presence of a magnetic flux. The way they are built is a generalization of the Melvin flux-tube solutions [171], that have been studied in Refs. [172–174]. The gravity solutions are related to the literature on holographic treatment of field theories at finite temperature and chemical potential (see Ref. [175] for a pedagogical introduction). Soliton solutions with black-brane asymptotics [176,177], in which there is a finite Hawking temperature [178] for the black brane, are related by double Wick rotation to the gravity dual of confining field theories [12]. In a similar sense, soliton solutions with charged black-brane asymptotics, which have an interpretation as field theories at finite temperature and finite chemical potential [179–190], after double Wick rotation are dual to confining field theories with magnetic fluxes. Recent applications of this construction include for example Refs. [191,192], that identify a phase transition in the parameter space of the theories, and Refs. [193–199].

We work in the context of the maximal gauged supergravity in $D = 7$ space-time dimensions [153–155] (see also Refs. [156–161]), truncated to retain only the

²Related work exists using Monte Carlo numerical techniques in lattice field theory [60,61] as well as exploiting semianalytical techniques valid in special superrenormalizable lower-dimensional field theories [62].

³A composite dilaton is expected to exist in special strongly coupled field theories [64–68], although there is no clear consensus on such a scenario [69–72]. The emerging low-energy theory, in which the dilaton is coupled to ordinary PNGBs, is called the dilaton effective field theory (dEFT); it is discussed in Refs. [73–88] and has found additional novel applications, as proposed for example in Refs. [89–92].

⁴The literature on the holographic techni-dilaton is exemplified by Refs. [108–127] (besides the aforementioned Refs. [16,29,38–40,52]).

⁵Realizations of such scenarios within string theory can be found in Refs. [132,133].

Cartan subgroup of the gauged symmetry, $SO(2) \times SO(2) \subset SO(5)$, together with two real scalars coupled to gravity [183,200,201]. The dual field theory is a special deformation of the $\mathcal{N} = (2, 0)$ superconformal field theory in $D - 1 = 6$ dimensions that lives on a stack of $M5$ -branes [202–204]. Furthermore, one of the dimensions is a circle, so that under special conditions its field theory dual confines [205]. Our analysis is based on the $SO(2) \times SO(2)$ truncation presented in Ref. [200]. We find it convenient to construct the soliton solutions of interest by compactifying one dimension on a circle and reducing to $D = 6$ dimensions first. We then use this language to compute the free energy and the spectrum of bound states, using the aforementioned holographic techniques. Interesting studies of the finite temperature and finite chemical potential thermodynamics of related solutions are found also in Refs. [157,206–208].

We identify two different branches of 2-parameter classes of physically inequivalent solutions. We call them, respective, domain-wall and soliton solutions, and holographically interpret the former in terms of conformal field theories, the latter of confining theories with magnetic fluxes. The global analysis based on the (holographic) calculation of the free energy demonstrates the presence of a closed line in parameter space, describing a polygon, demarcating the phase transitions between conformal (outer) and confining (inner) portions of space. By analyzing the properties of the background solutions near such line, we conclude that along the edges of the polygon the transition is of the first order, as in Refs. [44,54–56,58].⁶ As anticipated, the spectrum of fluctuations of the soliton (confining) solutions does not exhibit pathological behavior (tachyons) anywhere in parameter space. In the spin-0 spectrum, the lightest state has suppressed mass. This suppression is a tangible effect, that can be as large as an order of magnitude in respect to the mass of the lightest spin-2 state, which we use as an estimator of the confinement scale. Furthermore, we collect evidence that this state has distinctive properties that allow us to identify it as a dilaton, and that its properties persist along a large portion of parameter space, reaching away from the transition.

The paper is organized as follows. We present the supergravity formalism in Sec. II. We start from the bosonic part of the action of maximal supergravity in $D = 7$ dimensions, recall the properties of its $SO(2) \times SO(2)$ truncation, and dimensionally reduce on a circle, in Sec. II A, to the action in $D = 6$ dimensions used in the body of the paper. In Sec. II B we present the solutions of interest, starting from the classical equations of motion and their properties, reporting in details the general ultraviolet (UV) expansions of the solutions, and present the special closed-form functional dependence of the

background functions. We compute the free energy via holographic renormalization, in Sec. III, and the spectrum of bound states in the dual field theory in Sec. IV, by applying the gauge-invariant formalism to the system of scalars coupled to gravity in $D = 6$ dimensions. We also repeat the calculation of the spectrum in the probe approximation, that we use as a diagnostic tool to identify the dilaton, in Sec. IV A. We summarize our findings and outline suggestions for future research in Sec. V. We complement the main body of the paper by an appendix, in which we report additional material that is not essential in order to understand our main result, but that can be useful to reproduce our calculations.

II. THE HOLOGRAPHIC GRAVITY THEORY

In this section we provide a brief review of the field content of maximal supergravity in $D = 7$ dimensions, along with its $SO(2) \times SO(2)$ -invariant truncation of interest,⁷ and the dimensional reduction on a circle, down to $D = 6$ dimensions. We write explicitly the bulk action used later in the paper. In the process, we fix our notation and conventions, and highlight a few technical subtleties.

Supergravity theories in $D = 7$ spacetime dimensions can be obtained from the compactification (and reduction) of maximal supergravity in $D = 11$ dimensions on an internal four-dimensional space [153–155] (see also Refs. [156–161]). While compactification on locally flat internal manifolds, (e.g., a 4-torus, T^4), leads to ungauged supergravity, replacing the internal space with a 4-sphere, S^4 , yields the maximal gauged supergravity in $D = 7$ dimensions of interest in this paper. Its gauge group, $SO(5)$, corresponds to the isometries of the internal space.

Upon Kaluza-Klein reduction on S^4 , the 128 degrees of freedom of the supergravity in $D = 11$ dimensions (44 from the graviton and 84 from the 3-form) are reorganized into the following field content in $D = 7$ dimensions: the metric, providing 14 degrees of freedom, 14 real scalars, 10 massless vector fields, each propagating 5 physical degrees of freedom, and 5 self-dual 3-forms, each providing 10 degrees of freedom. The 128 degrees of freedom match the fermionic field content, but we will only concern ourselves with the bosonic sector. Following the notation of Ref. [209], but omitting the 3-forms, as they play no part in the following, the bosonic portion of the action is the following:

$$\begin{aligned} S_7 = \int d^7x \sqrt{-\hat{g}_7} & \left[\frac{\mathcal{R}_7}{4} - \frac{1}{4} \hat{g}^{\hat{M}\hat{N}} \delta^{ik} \delta_{j\ell} P_{\hat{M}i}^j P_{\hat{N}k}^\ell \right. \\ & \left. - \frac{m^2}{8} (2\delta^i_k \delta^j_\ell - \delta^{ij} \delta_{k\ell}) T_{ij} T^{k\ell} + \frac{1}{8} (\Pi^i_\alpha \Pi^j_\beta \mathcal{F}_{\hat{M}\hat{N}}^{\alpha\beta})^2 \right], \end{aligned} \quad (1)$$

⁶At the end point along the branch of solutions the geometry has a singularity at the end of space, while all other solutions along the branch are completely smooth and regular.

⁷This is not in general a consistent truncation, but it is for solutions in which the product $F^{(1)} \wedge F^{(2)} = 0$, so that truncated fields are not excited—see discussion in Sec. 4.1 of Ref. [183].

where the matrices appearing in the action are defined as

$$P_{\hat{M}i}^j \equiv \frac{1}{2}((\Pi^{-1})_i^\alpha \mathcal{D}_{\hat{M}\alpha}^\beta \Pi_\beta^j + (i \leftrightarrow j)), \quad (2)$$

$$\mathcal{D}_{\hat{M}\alpha}^\beta \equiv \delta_\alpha^\beta \partial_{\hat{M}} + ig \mathcal{A}_{\hat{M}\alpha}^\beta, \quad (3)$$

$$T_{ij} \equiv (\Pi^{-1})_i^\alpha (\Pi^{-1})_j^\beta \delta_{\alpha\beta}, \quad (4)$$

$$\mathcal{F}_{\hat{M}\hat{N}\alpha}^\beta \equiv 2(\partial_{[\hat{M}} \mathcal{A}_{\hat{N}]}^\beta + ig \mathcal{A}_{[\hat{M}\alpha}^\gamma \mathcal{A}_{\hat{N}]\gamma}^\beta). \quad (5)$$

Space-time indexes in $D = 7$ dimensions are denoted by hatted Roman letters, $\hat{M} = 0, 1, 2, 3, 4, 6, 7$, while we use Greek indices, $\alpha = 1, \dots, 5$, for the fundamental representation of the gauge group $SO(5)$, with gauge fields $\mathcal{A}_{\hat{M}\alpha}^\beta$. The scalars are parametrized by the unit-determinant matrix, Π_α^i , with $i, j = 1, \dots, 5$, in the $SL(5, \mathbb{R})/SO(5)_c$ right coset, where $SL(5, \mathbb{R})$ is the global symmetry of the system and $SO(5)_c$ the local symmetry that identifies equivalent configurations of the scalar fields.

Following Refs. [183,200,201,208], we truncate the theory to preserve the Cartan subgroup, $SO(2) \times SO(2)$ of the rank-2 gauge group $SO(5)$. We choose the two commuting generators, T_1, T_2 , to be represented by the following matrixes:

$$T_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (6)$$

which generate rotations in the $\{1, 2\}$ and $\{3, 4\}$ planes, respectively. In the truncated theory we retain only gauge fields along the Cartan subgroup.

Out of the 14 that span the 2-index symmetric, traceless representation of $SO(5)$, we retain the two real scalars, ϕ_1 and ϕ_2 , that in the presence of generic vacuum expectation values the $SO(2) \times SO(2)$ subgroup defined by T_1 and T_2 . The specific form of scalar matrix Π_α^i is hence the exponential of a traceless matrix:

$$\Pi_\alpha^i = \text{diag} \left(e^{\frac{\phi_1}{2\sqrt{2}} + \frac{\phi_2}{2\sqrt{10}}}, e^{\frac{\phi_1}{2\sqrt{2}} + \frac{\phi_2}{2\sqrt{10}}}, e^{-\frac{\phi_1}{2\sqrt{2}} + \frac{\phi_2}{2\sqrt{10}}}, e^{-\frac{\phi_1}{2\sqrt{2}} + \frac{\phi_2}{2\sqrt{10}}}, e^{-\frac{2\phi_2}{\sqrt{10}}} \right). \quad (7)$$

This parametrization is such that for general values of $\phi_1 \neq 0$ with $\phi_2 = 0$, the $SO(5)$ is broken to its maximal $SO(4)$ subgroup, and for general $\phi_1 \neq 0$ and $\phi_2 \neq 0$ to its $SO(2) \times SO(2)$ subgroup. In the latter case, the 8 gauge bosons in the coset, $SO(5)/(SO(2) \times SO(2))$, are Higgsed and become massive, by combining with 8 of the scalars. Of the remaining $14 - 8 = 6$ scalars, while we retain the two $SO(2) \times SO(2)$ singlets, we truncate away 4 degrees of freedom, corresponding to two complex scalars, each of which is charged under one of the Abelian gauge groups.

We identify the two vector fields we retain with $\mathcal{A}_{\hat{M}}^{(1)} \equiv \mathcal{A}_{\hat{M}1}^2$ and $\mathcal{A}_{\hat{M}}^{(2)} \equiv \mathcal{A}_{\hat{M}3}^4$, respectively. The covariant derivatives reduce to standard derivatives and the action greatly simplifies [183,200,201,208],⁸

$$\begin{aligned} \mathcal{S}_7 = \frac{1}{2\pi} \int d^7x \sqrt{-\hat{g}_7} & \left\{ \frac{\mathcal{R}_7}{4} - \frac{1}{8} g^{\hat{M}\hat{N}} (\partial_{\hat{M}} \phi_1 \partial_{\hat{N}} \phi_1 + \partial_{\hat{M}} \phi_2 \partial_{\hat{N}} \phi_2) \right. \\ & \left. - \frac{1}{16} g^{\hat{M}\hat{R}} g^{\hat{N}\hat{S}} \left(e^{\sqrt{2}\phi_1 + \sqrt{5}\phi_2} \mathcal{F}_{\hat{M}\hat{N}}^{(1)} \mathcal{F}_{\hat{R}\hat{S}}^{(1)} + e^{-\sqrt{2}\phi_1 + \sqrt{5}\phi_2} \mathcal{F}_{\hat{M}\hat{N}}^{(2)} \mathcal{F}_{\hat{R}\hat{S}}^{(2)} \right) - \mathcal{V}_7(\phi) \right\}, \end{aligned} \quad (8)$$

where the Abelian field strengths, for $i = 1, 2$, are

$$\mathcal{F}_{\hat{M}\hat{N}}^{(i)} = \partial_{\hat{M}} \mathcal{A}_{\hat{N}}^{(i)} - \partial_{\hat{N}} \mathcal{A}_{\hat{M}}^{(i)}. \quad (9)$$

The potential is [200]

$$\mathcal{V}_7(\phi) = \frac{1}{8} e^{-\sqrt{5}\phi_2} \left(e^{\sqrt{10}\phi_2} - 8 e^{\sqrt{5}\phi_2} \cosh\left(\frac{\phi_1}{\sqrt{2}}\right) - 8 \right), \quad (10)$$

and it has a maximum at $\phi_1 = \phi_2 = 0$ as well as a saddle point at $\phi_1 = 0, \phi_2 = \sqrt{\frac{2}{5}} \log(2)$. The potential and the whole action are symmetric for $\phi_1 \rightarrow -\phi_1$, provided one

also exchanges $\mathcal{A}_{\hat{M}}^{(1)} \leftrightarrow \mathcal{A}_{\hat{M}}^{(2)}$. Furthermore, $\mathcal{S}_7$ is also invariant under the separate changes of sign of the two Abelian gauge fields, $\mathcal{A}_{\hat{M}}^{(1)} \rightarrow -\mathcal{A}_{\hat{M}}^{(1)}$ and $\mathcal{A}_{\hat{M}}^{(2)} \rightarrow -\mathcal{A}_{\hat{M}}^{(2)}$.

A. Dimensional reduction to six dimensions

We reduce the theory to six dimensions by assuming one of the spacelike dimensions is compactified on a circle, S^1 , that is parametrized by an angular coordinate, $\eta \in [0, 2\pi]$,

⁸This is equivalent to defining $\lambda_1 = -\frac{\phi_1}{2\sqrt{2}} - \frac{\phi_2}{2\sqrt{10}}$ and $\lambda_2 = \frac{\phi_1}{2\sqrt{2}} - \frac{\phi_2}{2\sqrt{10}}$, for $\kappa^2 = 2$ and $m^2 = 1$, in Eqs. (2.5) and (2.6) of Ref. [200].

that all background functions are independent of this angular coordinate, η , and that any η -dependent properties can be set to zero—this includes the fact that fluctuations with momentum along the circle are also ignored. Solutions for which the compact dimension smoothly shrinks to zero, at some radial position, provide a geometric realization of the notion of confinement in the dual theory. The metric ansatz takes the soliton form

$$ds_7^2 = e^{\frac{-2}{\sqrt{10}\chi}} ds_6^2 + e^{\frac{8}{\sqrt{10}\chi}} (d\eta + V_M dx^M)^2, \quad (11)$$

where unhatted Latin indexes run over six dimensions, $M = 0, 1, 2, 3, 4, 6$, with $M = 6$ the holographic direction, while the metric in $D = 6$ dimensions can be cast in domain-wall form, as

$$\begin{aligned} ds_6^2 &= e^{2A} dx_{1,4}^2 + dr^2 \\ &= e^{2A} dx_{1,4}^2 + e^{\frac{2}{\sqrt{10}\chi}} d\rho^2. \end{aligned} \quad (12)$$

The conventions in these equations are chosen so that the AdS₇ backgrounds have $\mathcal{A} = A - \chi/\sqrt{10} = \rho/2 = 4\chi/\sqrt{10}$.

Under these assumption, the action, $\mathcal{S}_7$, can be written as a lower-dimensional action, $\mathcal{S}_6$, along with an additive total derivative, by performing the integration over the circle:

$$\mathcal{S}_7 = \mathcal{S}_6 + \frac{1}{2} \int d^6 x \partial_M \left(\sqrt{-g_6} g^{MN} \partial_N \left(\frac{\chi}{\sqrt{10}} \right) \right). \quad (13)$$

The action of the six-dimensional theory is given by⁹

$$\begin{aligned} \mathcal{S}_6 = \int d^6 x \sqrt{-g_6} & \left\{ \frac{\mathcal{R}_6}{4} - \frac{1}{2} g^{MN} G_{ab} \partial_M \Phi^a \partial_N \Phi^b \right. \\ & \left. - \frac{1}{4} H_{ab} g^{MR} g^{NS} F_{MN}^a F_{RS}^b - \mathcal{V}(\Phi^a) \right\}, \end{aligned} \quad (14)$$

where the potential is $\mathcal{V}(\Phi^a) = e^{\frac{-2}{\sqrt{10}\chi}} \mathcal{V}_7(\phi)$. The sigma-model metric contains five scalars: besides ϕ_1 and ϕ_2 , it also includes χ and the two components of the Abelian fields along the circle, $\mathcal{A}_7^{(1)}$ and $\mathcal{A}_7^{(2)}$. In the basis

$\Phi^a = \{\phi_1, \phi_2, \chi, \mathcal{A}_7^{(1)}, \mathcal{A}_7^{(2)}\}$, the sigma-model metric is

$$G_{ab} = \text{diag} \left(\frac{1}{4}, \frac{1}{4}, 1, \frac{1}{4} e^{\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2 - \frac{8\chi}{\sqrt{10}}}, \frac{1}{4} e^{-\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2 - \frac{8\chi}{\sqrt{10}}} \right). \quad (15)$$

The vectors, $V_M^a = \{V_M, A_M^{(1)}, A_M^{(2)}\}$, enter the action with the following field-strength tensors,

$$F_{MN}^V = \partial_M V_N - \partial_N V_M, \quad (16)$$

$$F_{MN}^{A^{(i)}} = \partial_M \mathcal{A}_N^{(i)} - \partial_N \mathcal{A}_M^{(i)} + V_M \partial_N \mathcal{A}_7^{(i)} - V_N \partial_M \mathcal{A}_7^{(i)}, \quad (17)$$

along with their associated sigma-model metric:

$$H_{ab} = \frac{1}{4} \text{diag} \left(e^{\sqrt{10}\chi}, e^{\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2 + \frac{2}{\sqrt{10}}\chi}, e^{-\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2 + \frac{2}{\sqrt{10}}\chi} \right). \quad (18)$$

B. Background solutions

We allow the background scalars appearing in the six-dimensional action, Eq. (14), to develop a nonvanishing bulk profile, as a function of the radial coordinate, ρ , and backreact their effect on the geometry, so that the metric, which takes the form in Eq. (12), has nontrivial $A(\rho)$. The vector fields vanish in the background, hence we can use the formalism described in Appendix A. We seek background solutions that, after uplifting to the seven-dimensional space, have finite curvature everywhere, and are free of conical singularities. We also require that the uplifted solution have asymptotically AdS₇ geometry, so that all background functions approach the expansion of the backgrounds close to the solution with $\phi_1 = 0 = \phi_2$, when $\rho \rightarrow +\infty$.

1. Background equations

The background equations, derived from the action written in six dimensions, and expressed in terms of the holographic coordinate, ρ , are the following—see also Appendix A and Refs. [45–53,210]:

$$4 \frac{\partial \mathcal{V}_7}{\partial \phi_1} = \phi_1''(\rho) + \left[5A'(\rho) - \frac{1}{\sqrt{10}} \chi'(\rho) \right] \phi_1'(\rho) + \frac{e^{-4\sqrt{\frac{2}{5}}\chi}}{\sqrt{2}} \left(-e^{\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(1)'}(\rho)^2 + e^{-\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(2)'}(\rho)^2 \right), \quad (19)$$

$$4 \frac{\partial \mathcal{V}_7}{\partial \phi_2} = \phi_2''(\rho) + \left[5A'(\rho) - \frac{1}{\sqrt{10}} \chi'(\rho) \right] \phi_2'(\rho) + \frac{e^{-4\sqrt{\frac{2}{5}}\chi}}{\sqrt{10}} \left(-e^{\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(1)'}(\rho)^2 - e^{-\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(2)'}(\rho)^2 \right), \quad (20)$$

⁹Notice the different choice of normalization between Eqs. (8) and (14), that arises because of the integration over the circle, $\frac{1}{2\pi} \int d\eta = 1$.

$$-\frac{2\mathcal{V}_7}{\sqrt{10}} = \chi''(\rho) + \left[5A'(\rho) - \frac{1}{\sqrt{10}}\chi'(\rho) \right] \chi'(\rho) + \frac{e^{-4\sqrt{\frac{2}{5}}\chi}}{\sqrt{10}} \left(e^{\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(1)'}(\rho)^2 + e^{-\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(2)'}(\rho)^2 \right), \quad (21)$$

$$0 = \mathcal{A}_7^{(1)''}(\rho) + \mathcal{A}_7^{(1)'}(\rho) \left(\sqrt{2}\phi_1'(\rho) + \sqrt{\frac{2}{5}}\phi_2'(\rho) + 5A'(\rho) - \frac{9}{\sqrt{10}}\chi'(\rho) \right), \quad (22)$$

$$0 = \mathcal{A}_7^{(2)''}(\rho) + \mathcal{A}_7^{(2)'}(\rho) \left(-\sqrt{2}\phi_1'(\rho) + \sqrt{\frac{2}{5}}\phi_2'(\rho) + 5A'(\rho) - \frac{9}{\sqrt{10}}\chi'(\rho) \right), \quad (23)$$

$$\begin{aligned} -2\mathcal{V}_7 = 10A'(\rho)^2 + 4A''(\rho) - \frac{4A'(\rho)\chi'(\rho)}{\sqrt{10}} + \frac{\phi_1'(\rho)^2 + \phi_2'(\rho)^2}{4} + \chi'(\rho)^2 \\ + \frac{e^{-4\sqrt{\frac{2}{5}}\chi}}{4} \left(e^{\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(1)'}(\rho)^2 + e^{-\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(2)'}(\rho)^2 \right), \end{aligned} \quad (24)$$

$$-2\mathcal{V}_7 = 10A'(\rho)^2 - \frac{\phi_1'(\rho)^2 + \phi_2'(\rho)^2}{4} - \chi'(\rho)^2 - \frac{e^{-4\sqrt{\frac{2}{5}}\chi}}{4} \left(e^{\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(1)'}(\rho)^2 + e^{-\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} \mathcal{A}_7^{(2)'}(\rho)^2 \right), \quad (25)$$

where primed variables denote derivatives with respect to ρ , so that $f'(\rho) = \frac{\partial}{\partial \rho} f(\rho)$.

We note that Eqs. (22) and (23) may be reformulated in terms of two total derivatives:

$$\begin{aligned} \partial_\rho \left(e^{\sqrt{2}\phi_1(\rho) + \sqrt{\frac{2}{5}}\phi_2(\rho) + 5A(\rho) - \frac{9}{\sqrt{10}}\chi(\rho)} \mathcal{A}_7^{(1)'}(\rho) \right) &= 0, \\ \partial_\rho \left(e^{-\sqrt{2}\phi_1(\rho) + \sqrt{\frac{2}{5}}\phi_2(\rho) + 5A(\rho) - \frac{9}{\sqrt{10}}\chi(\rho)} \mathcal{A}_7^{(2)'}(\rho) \right) &= 0, \end{aligned} \quad (26)$$

and hence we identify two conserved quantities, $\mathcal{C}_1$ and $\mathcal{C}_2$, as

$$\begin{aligned} e^{\sqrt{2}\phi_1(\rho) + \sqrt{\frac{2}{5}}\phi_2(\rho) + 5A(\rho) - \frac{9}{\sqrt{10}}\chi(\rho)} \mathcal{A}_7^{(1)'}(\rho) &\equiv \mathcal{C}_1, \\ e^{-\sqrt{2}\phi_1(\rho) + \sqrt{\frac{2}{5}}\phi_2(\rho) + 5A(\rho) - \frac{9}{\sqrt{10}}\chi(\rho)} \mathcal{A}_7^{(2)'}(\rho) &\equiv \mathcal{C}_2. \end{aligned} \quad (27)$$

By forming the combination $-2\sqrt{10} \times (21) + (24) + (25)$, and using the definitions of the two conserved quantities, $\mathcal{C}_1$ and $\mathcal{C}_2$, identified above, we arrive at the equation:

$$20A'(\rho)^2 - 52\sqrt{\frac{2}{5}}A'(\rho)\chi'(\rho) + 2\chi'(\rho)^2 + 4A''(\rho) - 2\sqrt{10}\chi''(\rho) - 2e^{-5A(\rho) + \frac{1}{\sqrt{10}}\chi(\rho)} (\mathcal{C}_1 \mathcal{A}_7^{(1)'}(\rho) + \mathcal{C}_2 \mathcal{A}_7^{(2)'}(\rho)) = 0, \quad (28)$$

which can also be recast as a total derivative, and hence leads us to identify a third conserved quantity, $\mathcal{C}_3$, defined as

$$-2(\mathcal{C}_1 \mathcal{A}_7^{(1)}(\rho) + \mathcal{C}_2 \mathcal{A}_7^{(2)}(\rho)) + e^{5A(\rho) - \frac{\chi(\rho)}{\sqrt{10}}} (4A'(\rho) - 2\sqrt{10}\chi'(\rho)) \equiv \mathcal{C}_3. \quad (29)$$

2. UV expansions

All the solutions of interest approach the same AdS_7 background geometry, asymptotically, for $\rho \rightarrow +\infty$. The dual field theories are all deformations of the same UV fixed point. We find it convenient to replace the radial direction with $z = e^{-\frac{\rho}{2}}$, and expand in powers of small z . The asymptotic UV expansion of the background fields in the whole class is the following:

$$\begin{aligned} \mathcal{A}_7^{(1)}(z) &= \mathcal{A}_{7,U}^{(1)} + z^4 \mathcal{A}_{7,4}^{(1)} - \frac{2}{15} z^6 \left(\sqrt{2} \mathcal{A}_{7,4}^{(1)} (5\phi_{1,2} + \sqrt{5}\phi_{2,2}) \right) \\ &\quad + \frac{\mathcal{A}_{7,4}^{(1)}}{40} z^8 (\log(z) (-24\phi_{1,2}^2 - 48\sqrt{5}\phi_{2,2}\phi_{1,2} + 72\phi_{2,2}^2) - 19\phi_{1,2}^2 \\ &\quad - 2\sqrt{5}\phi_{2,2}\phi_{1,2} - 15\phi_{2,2}^2 + 20\sqrt{2}\phi_{1,4} + 4\sqrt{10}\phi_{2,4}) + O(z^{10}), \end{aligned} \quad (30)$$

$$\mathcal{A}_7^{(2)}(z) = \mathcal{A}_{7,U}^{(2)} + z^4 \mathcal{A}_{7,4}^{(2)} + \frac{2}{15} \sqrt{2} z^6 \mathcal{A}_{7,4}^{(2)} (5\phi_{1,2} - \sqrt{5}\phi_{2,2}) + \frac{\mathcal{A}_{7,4}^{(2)}}{40} z^8 (\log(z) (24\phi_{1,2}^2 - 48\sqrt{5}\phi_{2,2}\phi_{1,2} - 72\phi_{2,2}^2) + 19\phi_{1,2}^2 - 2\sqrt{5}\phi_{2,2}\phi_{1,2} + 15\phi_{2,2}^2 + 20\sqrt{2}\phi_{1,4} - 4\sqrt{10}\phi_{2,4}) + O(z^{10}), \quad (31)$$

$$\begin{aligned} \phi_1(z) = & z^2 \phi_{1,2} + \frac{1}{5} z^4 (5\phi_{1,4} - 6\sqrt{10}\phi_{1,2}\phi_{2,2} \log(z)) \\ & + \frac{1}{60} z^6 (108\phi_{1,2}^3 \log(z) - 108\phi_{2,2}^2 \phi_{1,2} \log(z) - 77\phi_{1,2}^3 + 63\phi_{2,2}^2 \phi_{1,2} - 18\sqrt{10}\phi_{2,4}\phi_{1,2} - 18\sqrt{10}\phi_{1,4}\phi_{2,2}) \\ & + \frac{1}{600} z^8 (\log^2(z) (-432\sqrt{10}\phi_{2,2}\phi_{1,2}^3 + 1296\sqrt{10}\phi_{2,2}^3 \phi_{1,2}) \\ & + \log(z) (312\sqrt{10}\phi_{2,2}\phi_{1,2}^3 + 360\phi_{1,4}\phi_{1,2}^2 - 2088\sqrt{10}\phi_{2,2}^3 \phi_{1,2} + 720\phi_{2,2}\phi_{2,4}\phi_{1,2} - 1080\phi_{1,4}\phi_{2,2}^2) \\ & - 115\sqrt{10}\phi_{2,2}\phi_{1,2}^3 + 120\phi_{1,4}\phi_{1,2}^2 + 1321\sqrt{10}\phi_{2,2}^3 \phi_{1,2} - 760\phi_{2,2}\phi_{2,4}\phi_{1,2} \\ & + 600\phi_{1,4}\phi_{2,2}^2 - 60\sqrt{10}\phi_{1,4}\phi_{2,4}) + O(z^{10}), \end{aligned} \quad (32)$$

$$\begin{aligned} \phi_2(z) = & z^2 \phi_{2,2} + \frac{1}{5} z^4 (-3\sqrt{10}\phi_{1,2}^2 \log(z) + 9\sqrt{10}\phi_{2,2}^2 \log(z) + 5\phi_{2,4}) \\ & + \frac{1}{60} z^6 (972\phi_{2,2}^3 \log(z) - 108\phi_{1,2}^2 \phi_{2,2} \log(z) - 637\phi_{2,2}^3 + 63\phi_{1,2}^2 \phi_{2,2} + 54\sqrt{10}\phi_{2,4}\phi_{2,2} - 18\sqrt{10}\phi_{1,2}\phi_{1,4}) \\ & + \frac{z^8}{2400} (1296\sqrt{10}\phi_{1,2}^4 \log^2(z) - 9504\sqrt{10}\phi_{2,2}^2 \phi_{1,2}^2 \log^2(z) + 11664\sqrt{10}\phi_{2,2}^4 \log^2(z) \\ & - \log(z) (1440\sqrt{10}\phi_{1,2}^4 + 4992\sqrt{10}\phi_{2,2}^2 \phi_{1,2}^2 - 4320\phi_{2,4}\phi_{1,2}^2 + 2880\phi_{1,4}\phi_{2,2}\phi_{1,2} + 8928\sqrt{10}\phi_{2,2}^4 + 12960\phi_{2,2}^2 \phi_{2,4}) \\ & + 759\sqrt{10}\phi_{1,2}^4 - 814\sqrt{10}\phi_{2,2}^2 \phi_{1,2}^2 + 2400\phi_{2,4}\phi_{1,2}^2 - 3040\phi_{1,4}\phi_{2,2}\phi_{1,2} - 11921\sqrt{10}\phi_{2,2}^4 - 120\sqrt{10}\phi_{1,4}^2 \\ & + 360\sqrt{10}\phi_{2,4}^2 + 4960\phi_{2,2}^2 \phi_{2,4}) + O(z^{10}), \end{aligned} \quad (33)$$

$$\begin{aligned} \chi(z) = & \chi_U - \frac{\sqrt{10}}{4} \log(z) - \frac{z^4 (\phi_{1,2}^2 + \phi_{2,2}^2)}{16\sqrt{10}} \\ & + \frac{z^6}{360} \left(\frac{6\sqrt{10}\phi_{2,2}^3 - 6\sqrt{10}\phi_{1,2}^2 \phi_{2,2} - 40\phi_{2,4}\phi_{2,2} - 40\phi_{1,2}\phi_{1,4} + 15\sqrt{10}\chi_6}{\sqrt{10}} + \chi_6 + 72 \log(z) (\phi_{1,2}^2 \phi_{2,2} - \phi_{2,2}^3) \right) \\ & + \frac{z^8}{12800} (\log^2(z) (-288\sqrt{10}\phi_{1,2}^4 + 576\sqrt{10}\phi_{2,2}^2 \phi_{1,2}^2 - 2592\sqrt{10}\phi_{2,2}^4) \\ & - \log(z) (216\sqrt{10}\phi_{1,2}^4 + 432\sqrt{10}\phi_{2,2}^2 \phi_{1,2}^2 + 960\phi_{2,4}\phi_{1,2}^2 + 1920\phi_{1,4}\phi_{2,2}\phi_{1,2} - 1944\sqrt{10}\phi_{2,2}^4 - 2880\phi_{2,2}^2 \phi_{2,4}) \\ & + 163\sqrt{10}\phi_{1,2}^4 - 270\sqrt{10}\phi_{2,2}^2 \phi_{1,2}^2 + 360\phi_{2,4}\phi_{1,2}^2 + 720\phi_{1,4}\phi_{2,2}\phi_{1,2} + 1355\sqrt{10}\phi_{2,2}^4 - 80\sqrt{10}\phi_{1,4}^2 \\ & - 80\sqrt{10}\phi_{2,4}^2 - 1080\phi_{2,2}^2 \phi_{2,4}) + O(z^{10}), \end{aligned} \quad (34)$$

$$\begin{aligned} A(z) = & A_U - \frac{5}{4} \log(z) - \frac{z^4}{32} (\phi_{1,2}^2 + \phi_{2,2}^2) \\ & + \frac{z^6}{720} (72\sqrt{10} \log(z) (\phi_{1,2}^2 \phi_{2,2} - \phi_{2,2}^3) - 40(\phi_{1,2}\phi_{1,4} + \phi_{2,2}\phi_{2,4}) - 6\sqrt{10}(\phi_{1,2}^2 \phi_{2,2} - \phi_{2,2}^3) + 15\sqrt{10}\chi_6) \\ & \times \frac{z^8}{2560} (\log(z)^2 (-288\phi_{1,2}^4 + 576\phi_{1,2}^2 \phi_{2,2}^2 - 2592\phi_{2,2}^4) \\ & + \log(z) (-216\phi_{1,2}^4 + 192\sqrt{10}\phi_{1,2}\phi_{1,4}\phi_{2,2} + 432\phi_{1,2}^2 \phi_{2,2}^2 - 1944\phi_{2,2}^4 + 96\sqrt{10}\phi_{1,2}^2 \phi_{2,4} - 288\sqrt{10}\phi_{2,2}^2 \phi_{2,4}) \\ & + 163\phi_{1,2}^4 - 80\phi_{1,4}^2 + 72\sqrt{10}\phi_{1,2}\phi_{1,4}\phi_{2,2} - 270\phi_{1,2}^2 \phi_{2,2}^2 + 1355\phi_{2,2}^4 \\ & + 36\sqrt{10}\phi_{2,2}^2 \phi_{2,4} - 108\sqrt{10}\phi_{2,2}^2 \phi_{2,4} - 80\phi_{2,4}^2) + O(z^{10}). \end{aligned} \quad (35)$$

We define χ_6 in such a way that for $\chi_6 = 0$ we recover the relation $\chi = \frac{\sqrt{10}}{5}A$, typifying the domain-wall solutions in $D = 7$ dimensions.

The UV expansions are determined by 11 parameters: they correspond to the five sources and five vacuum expectation values of the operators dual to the five sigma-model scalars, with the addition of a harmless additive constant in the warp factor, A . In these expressions, we denote the parameters as $\{\phi_{1,2}, \phi_{2,2}, \phi_{1,4}, \phi_{2,4}, \mathcal{A}_{7,U}^{(1)}, \mathcal{A}_{7,U}^{(2)}, \mathcal{A}_{7,4}^{(1)}, \mathcal{A}_{7,4}^{(2)}, \chi_U, \chi_6, A_U\}$. One recovers domain-wall solutions in $D = 7$ dimensions, with AdS_7 geometry, by setting to zero all these constants, except for $\mathcal{A}_{7,U}^{(1)}, \mathcal{A}_{7,U}^{(2)}$, and the trivial A_U .

3. Soliton (confining) solutions

The soliton solutions that are the main interest for our analysis can be written in closed form, by first rewriting the metric after a change of radial coordinate from ρ to the new q [157,200], as follows:

$$ds_7^2 = \frac{(H_1(q)H_2(q))^{\frac{1}{5}}}{f(q)} dq^2 + \frac{1}{4} (H_1(q)H_2(q))^{\frac{1}{5}} q^2 dx_{1,4}^2 + (H_1(q)H_2(q))^{-\frac{4}{5}} f(q) d\eta^2. \quad (36)$$

The new functions $H_1(q)$, $H_2(q)$, and $f(q)$ can be identified as

$$\chi(\rho) \equiv \frac{\sqrt{10}}{8} \log[(H_1 H_2)^{-\frac{4}{5}} f], \quad (37)$$

$$A(\rho) \equiv \frac{1}{2} \log \left[\frac{1}{4} (H_1 H_2)^{\frac{1}{5}} q^2 \right] + \frac{1}{\sqrt{10}} \chi(q), \quad (38)$$

$$\frac{\partial}{\partial \rho} \equiv \frac{\sqrt{f}}{(H_1 H_2)^{\frac{1}{10}}} \frac{\partial}{\partial q}. \quad (39)$$

By imposing the background equations, one arrives to explicit expressions for the background functions:

$$H_i = 1 - \frac{Q_i^2}{q^4} \quad (i = 1, 2), \quad (40)$$

$$f = -\frac{\mu}{q^4} + \frac{1}{4} q^2 H_1 H_2, \quad (41)$$

$$\phi_1 = \frac{1}{\sqrt{2}} \log \left[\frac{H_1}{H_2} \right], \quad (42)$$

$$\phi_2 = \frac{1}{\sqrt{10}} \log [H_1 H_2], \quad (43)$$

$$\mathcal{A}_7^{(i)} = \pm \left(\frac{\sqrt{\mu}(1 - H_i^{-1})}{Q_i} - \frac{\sqrt{\mu}(1 - H_i(q_0)^{-1})}{Q_i} \right). \quad (44)$$

They depend on the parameters Q_1 , Q_2 , and μ . To avoid a singularity, the conditions $q_0^2 \geq \text{Max}[|Q_i|]$ and $q_0 \geq 0$ must be satisfied, where q_0 is defined as the largest root of $f(q)$.

Notice that $\mathcal{A}_7^{(i)}$ vanishes at the end of space, when $q \rightarrow q_0$.¹⁰ The scalars ϕ_1 and ϕ_2 , that have no charge under the two Abelian gauge groups, develop profiles that are controlled by the fluxes, the magnitude of which are given by $\mathcal{A}_7^{(i)}$, and break the $SO(5)$ symmetry to its $SO(2) \times SO(2)$ subgroup.

In order to characterize these solutions in the large class identified in Sec. II B 2, we perform a UV expansion of the solutions in the parameter $z = e^{-\frac{q}{q_0}}$. We do so by first defining a new coordinate $z = \frac{1}{q}$ and writing this as a series in z :

$$z = \frac{z}{2} - \frac{1}{320} (Q_1^2 + Q_2^2) z^5 - \frac{\mu}{384} z^7 + \frac{Q_1^4 + 7Q_1^2 Q_2^2 + Q_2^4}{51200} z^9 + \frac{13(Q_1^2 + Q_2^2)}{307200} z^{11} + \mathcal{O}(z^{13}). \quad (45)$$

The soliton solutions then read as follows:

$$\mathcal{A}_7^{(1)}(z) = \frac{Q_1 \sqrt{\mu}}{Q_1^2 - q_0^4} + \frac{Q_1 \sqrt{\mu}}{16} z^4 + \frac{Q_1(3Q_1^2 - 2Q_2^2) \sqrt{\mu}}{1280} z^8 + \mathcal{O}(z^{10}), \quad (46)$$

$$\mathcal{A}_7^{(2)}(z) = \frac{Q_2 \sqrt{\mu}}{Q_2^2 - q_0^4} + \frac{Q_2 \sqrt{\mu}}{16} z^4 + \frac{2Q_2(-Q_1^2 + 3Q_2^2) \sqrt{\mu}}{1280} z^8 + \mathcal{O}(z^{10}), \quad (47)$$

$$\phi_1(z) = \frac{-Q_1^2 + Q_2^2}{16\sqrt{2}} z^4 + \frac{-Q_1^4 + Q_2^4}{2560\sqrt{2}} z^8 + \mathcal{O}(z^{10}), \quad (48)$$

$$\phi_2(z) = -\frac{Q_1^2 + Q_2^2}{16\sqrt{10}} z^4 - \frac{Q_1^4 - 8Q_1^2 Q_2^2 + Q_2^4}{2560\sqrt{10}} z^8 + \mathcal{O}(z^{10}), \quad (49)$$

$$\chi(z) = -\frac{\sqrt{10}}{4} \log(z) - \frac{\mu}{16\sqrt{10}} z^6 - \frac{3Q_1^4 - 4Q_1^2 Q_2^2 + 3Q_2^4}{20480\sqrt{10}} z^8 + \mathcal{O}(z^{10}), \quad (50)$$

¹⁰The algebraic equation $f(q_0) = 0$ admits multiple solutions, which one might be tempted to interpret in terms of multiple branches of solutions to the background equations. Unfortunately, only one such algebraic solution gives rise to a gravity background ending at q_0 . By replacing in the analytic expressions, in particular Eq. (36), solutions that violate the requirement $q_0^2 \geq \text{Max}[|Q_i|]$, results in the metric developing a singularity at $q = \text{Max}[|Q_i|] > q_0$, before the end of space. Such solutions to the algebraic relations are discarded.

$$A(z) = -\frac{5}{4}\log(z) - \frac{\mu}{768}z^6 - \frac{3Q_1^4 - 4Q_1^2Q_2^2 + 3Q_2^4}{40960}z^8 + \mathcal{O}(z^{10}). \quad (51)$$

We hence arrive at the identifications:

$$\phi_{1,2} = \phi_{2,2} = \chi_U = A_U = 0, \quad (52)$$

$$\phi_{1,4} = \frac{-Q_1^2 + Q_2^2}{16\sqrt{2}}, \quad (53)$$

$$\phi_{2,4} = \frac{-Q_1^2 - Q_2^2}{16\sqrt{10}}, \quad (54)$$

$$\mathcal{A}_{7,4}^{(i)} = \pm \frac{Q_i\sqrt{\mu}}{16}, \quad (55)$$

$$\mathcal{A}_{7,U}^{(i)} = \pm \frac{Q_i\sqrt{\mu}}{Q_i^2 - q_0^4}, \quad (56)$$

$$\chi_6 = -\frac{\mu}{16\sqrt{10}}. \quad (57)$$

The three parameters Q_1 , Q_2 , and μ can be traded for $\mathcal{A}_{7,U}^{(i)}$, $\mathcal{A}_{7,U}^{(i)}$, χ_6 . We illustrate the properties of the soliton solutions in Fig. 1, by plotting the value of the parameters as a function of q_0 , for representative choices of other parameters.

Absence of conical singularity near the end of space, at $q \rightarrow q_0$, is ensured by assessing the expansion of the induced two-dimensional metric in the plane described by polar coordinates (η, ρ) , near q_0 :

$$\begin{aligned} ds_2^2 &= d\rho^2 + e^{4\sqrt{\frac{5}{2}}\eta} d\eta^2 \\ &= \frac{(H_2 H_2)^{\frac{1}{2}}}{f} dq^2 + (H_1 H_2)^{(-\frac{4}{3})} f d\eta^2 \\ &\approx \frac{(H_1(q_0) H_2(q_0))^{\frac{1}{2}}}{f'(q_0)(q - q_0)} dq^2 \\ &\quad + (H_1(q_0) H_2(q_0))^{(-\frac{4}{3})} f'(q_0)(q - q_0) d\eta^2. \end{aligned} \quad (58)$$

Making the change of variable $\frac{(H_2 H_2)^{\frac{1}{2}}}{f'(\rho_0)(q - q_0)} dq^2 = d\rho^2$, such that $\rho = \sqrt{\frac{4(H_1(q_0) H_2(q_0))^{\frac{1}{2}}}{f'(q_0)}}(q - q_0)$, the induced metric described by the polar coordinates, (ρ, η) , reads

$$ds_2^2 = d\rho^2 + \left(\frac{f'(q_0)}{2\sqrt{H_1(q)H_2(q_0)}} \right)^2 \rho^2 d\eta^2, \quad (59)$$

which requires one to set $f'(q_0) = \pm 2\sqrt{H_1(q)H_2(q_0)}$, hence avoiding the arising of a conical singularity. This expression translates into the following additional constraint on the parameters characterizing the solution:

$$0 = Q_2^2 q_0^4 + q_0^8 + Q_1^2(q_0^4 - 3Q_2^2) \pm 4q_0^3 \sqrt{(Q_1^2 - q_0^4)(Q_2^2 - q_0^4)} + 8q_0^2 \mu. \quad (60)$$

This constraint must be satisfied at the same time as the defining equation for q_0 :

$$f(q_0) = 0 = \frac{1}{4}q_0^2 \left(1 - \frac{Q_1^2}{q_0^4} \right) \left(1 - \frac{Q_2^2}{q_0^4} \right) - \frac{\mu}{q_0^4}. \quad (61)$$

This constraint further reduces the number of free parameters to two, which can be fixed in more than one way: for example, we can fix $\mathcal{A}_{7,U}^{(1)}$ and $\mathcal{A}_{7,U}^{(2)}$, which we will generically refer to as the sources in later parts of the paper, and use the constraints to set q_0 and μ , and ultimately Q_1 and Q_2 . By inspection, we find that nonsingular, smooth soliton solutions exist only for $q_0 < \frac{4}{3}$.

By making use of Eq. (61), and of the definition of the leading-order coefficients in the UV expansions, in Eq. (55), we can rewrite

$$(Q_1^2 - q_0^4)(Q_2^2 - q_0^4) = 4\mu q_0^2, \quad (62)$$

$$Q_1^2 Q_2^2 = (4q_0^2 \mathcal{A}_{7,U}^{(1)} \mathcal{A}_{7,U}^{(2)})^2, \quad (63)$$

$$Q_1^2 + Q_2^2 = -4\frac{\mu}{q_0^2} + q_0^4 + (4\mathcal{A}_{7,U}^{(1)} \mathcal{A}_{7,U}^{(2)})^2, \quad (64)$$

and hence the constraint derived from the absence of conical singularities reads

$$(2\mu + q_0^6 - 16q_0^2 (\mathcal{A}_{7,U}^{(1)})^2 (\mathcal{A}_{7,U}^{(2)})^2)^2 = 16\mu q_0^4. \quad (65)$$

The relation to the source terms, $\mathcal{A}_{7,U}^{(i)}$, is completed by the following, obtained by combining the definition of $\mathcal{A}_{7,U}^{(i)}$ with the three expressions in Eqs. (61), (62), and (64):

$$(\mathcal{A}_{7,U}^{(1)})^2 + (\mathcal{A}_{7,U}^{(2)})^2 = \frac{256q_0^2 (\mathcal{A}_{7,U}^{(1)})^4 (\mathcal{A}_{7,U}^{(2)})^4 - 32(\mathcal{A}_{7,U}^{(1)})^2 (\mathcal{A}_{7,U}^{(2)})^2 (q_0^6 + 2\mu) + q_0^4 (q_0^6 - 4\mu)}{16\mu q_0^2}. \quad (66)$$

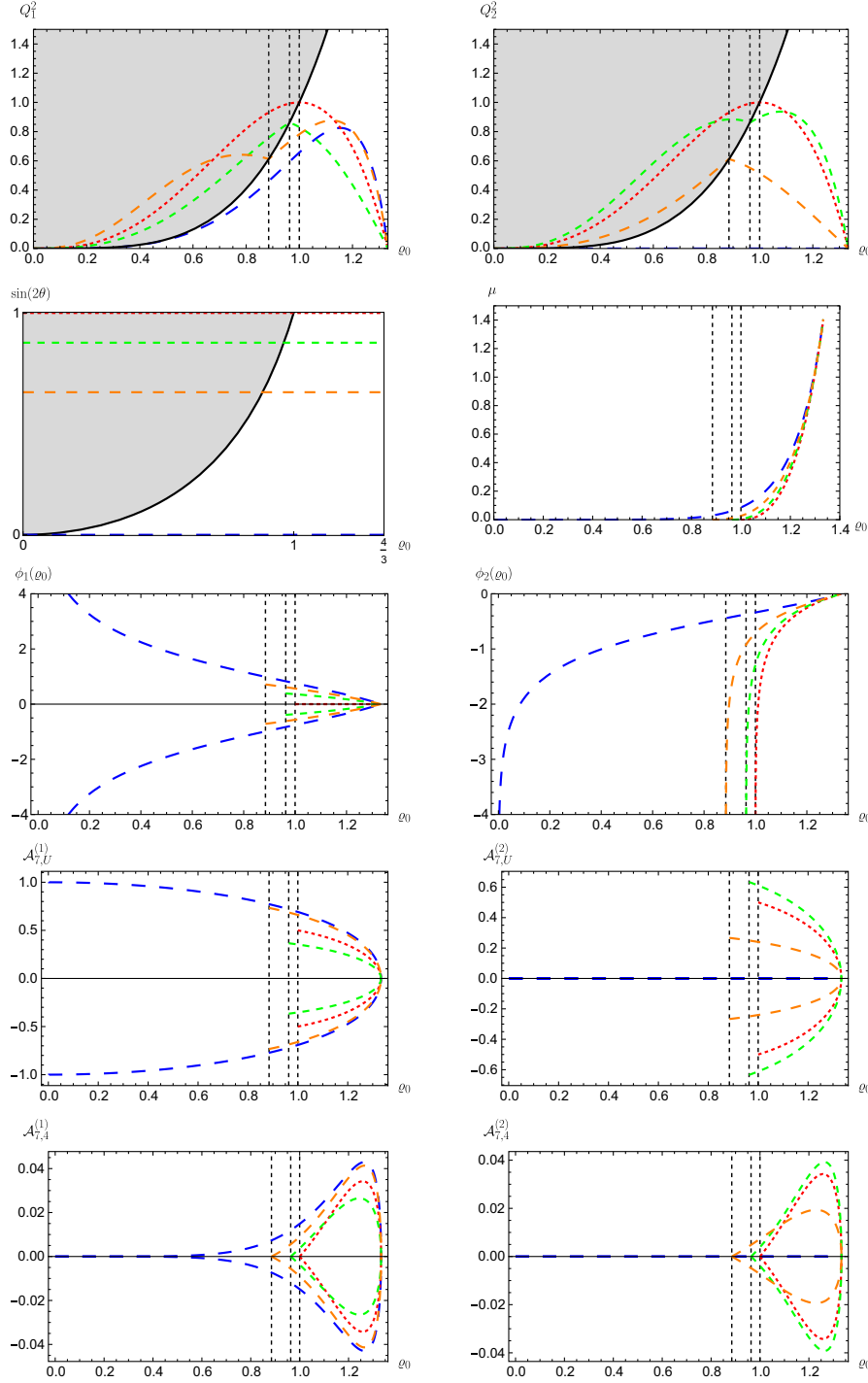


FIG. 1. Parameters and functions appearing in the soliton (confining) solutions, as a function of q_0 , that sets the end of space in the geometry. In each plot the four colors (blue, orange, green, red), with long to short dashed, correspond to branches of solutions with $\mathcal{A}_{7,U}^{(2)}/\mathcal{A}_{7,U}^{(1)} = (0, \tan(\frac{\pi}{9}), \tan(\frac{\pi}{6}), \tan(\frac{\pi}{4}))$, respectively. We shall use the same color coding also in Fig. 2(a). In the first three plots we mark a region of instability in the solutions as a dark shaded area; this is given when $q_0^2 \leq Q_i$, or equivalently $\sin(2\theta) \geq q_0^2/(2 - q_0^2)$. In these regions, at least one of the H_i functions changes sign in the region of the geometry corresponding to the deep IR, at which point the metric would become singular and would result in the background values of ϕ_1 or ϕ_2 to become imaginary. These cases are hence unphysical, and excluded from the rest of the analysis in the paper. In the fifth and sixth panels we plot the functions $\phi_1(q), \phi_2(q)$, evaluated at the end of space, $q = q_0$, where we see that ϕ_2 diverges at the boundary of parameter space.

Finally, we find it convenient to define an alternative parametrization of the magnetic fluxes by introducing polar coordinates $\mathcal{A}_{7,U}^{(1)} = \mathcal{A}_U \cos(\theta)$ and $\mathcal{A}_{7,U}^{(2)} = \mathcal{A}_U \sin(\theta)$, so that the relations between the four variables read

$$\mu = 2\varrho_0^4(2 - 2\mathcal{A}_U^2 - \varrho_0^2), \quad (67)$$

$$\sin^2(2\theta) = \pm \left(\frac{\mathcal{A}_U^4 \varrho_0^2 (-8\mathcal{A}_U^2 - 3\varrho_0^2 + 8) - 4\sqrt{2}\sqrt{-\mathcal{A}_U^8 \varrho_0^4 (2\mathcal{A}_U^2 + \varrho_0^2 - 2)}}{4\mathcal{A}_U^8} \right). \quad (68)$$

We conclude this section by noticing that we have identified two widely different classes of solutions, that can be both parametrized in terms of the leading coefficients, $\mathcal{A}_{7,U}^{(1)}$ and $\mathcal{A}_{7,U}^{(2)}$, of the expansion of the seventh component of the two Abelian gauge fields. Because of the compact nature of the seventh dimension, η , the value of such gauge fields corresponds to nontrivial magnetic fluxes in the dual field theory language. The AdS₇ solutions have geometries that are independent of $\mathcal{A}_{7,U}^{(1)}$, and $\mathcal{A}_{7,U}^{(2)}$, are interpreted in terms of conformal field theories in $D - 1 = 6$ dimensions, and exist for any choices of these two parameters.

The soliton solutions obtained by the aforementioned restrictions among the parameters μ , ρ_0 , Q_1 , and Q_2 have a more sophisticated interpretation in field theory terms. The compactification of the seventh dimension is causing it to shrink, and the dual field theory ultimately reduces to a five-dimensional one that confines. The presence of

magnetic flux is no longer harmless, as when the circle shrinks to zero size the magnetic flux must vanish as well, resulting in restrictions to the admissible values of $\mathcal{A}_{7,U}^{(1)}$ and $\mathcal{A}_{7,U}^{(2)}$. As shown explicitly in Appendix D, these solutions are smooth and regular, except for the ones with $\mu = 0$, that are singular when $\varrho \rightarrow \varrho_0$. In the next sections, we will perform a stability analysis aimed at understanding which of these two classes of solutions is physically realized, as a function of the two sources, $\mathcal{A}_{7,U}^{(1)}$ and $\mathcal{A}_{7,U}^{(2)}$.

III. GLOBAL STABILITY ANALYSIS: FREE ENERGY

The holographic renormalization prescription for the calculation of the free energy of the dual field theory starts with the evaluation of the on-shell action in the gravity dual [41–43], which in seven dimensions reads

$$S_7^{\text{on-shell}} = \frac{1}{2\pi} \int d^5x d\eta d\rho \partial_\rho \left[e^{5A(\rho) - \frac{1}{\sqrt{10}}\chi(\rho)} \left(-\frac{1}{2} \partial_\rho A(\rho) + \frac{1}{2\sqrt{10}} \partial_\rho \chi(\rho) \right) \right], \quad (69)$$

having made use of Einstein's equations, (24) and (25). To this, one adds boundary localized terms, required by holographic renormalization. The Gibbons-Hawking-York (GHY), boundary localized contribution to the action is

$$\begin{aligned} S_{\text{GHY}}^{\text{on-shell}} &= \frac{1}{2\pi} \int d^5x d\eta d\rho \sum_{i=1}^2 \delta(\rho - \rho_i) (-)^i \left[\sqrt{-\tilde{g}} \frac{\kappa}{2} \right] \\ &= \frac{1}{2\pi} \int d^5x d\eta d\rho \sum_{i=1}^2 \delta(\rho - \rho_i) (-)^i \left[e^{5A(\rho) - \frac{1}{\sqrt{10}}\chi(\rho)} \left(\frac{5}{2} \partial_\rho A(\rho) - \frac{1}{2\sqrt{10}} \partial_\rho \chi(\rho) \right) \right], \end{aligned} \quad (70)$$

where we introduce the vector orthonormal to the boundary, $N_{\hat{M}} = (0, 0, 0, 0, 0, 1, 0)$, the induced metric $\tilde{g}_{\hat{M}\hat{N}} = g_{\hat{M}\hat{N}} - N_{\hat{M}} N_{\hat{N}}$, and the extrinsic curvature, $\kappa \equiv \tilde{g}^{\hat{M}\hat{N}} \nabla_{\hat{M}} N_{\hat{N}} = -\tilde{g}^{\hat{M}\hat{N}} \Gamma_{\hat{M}\hat{N}}^{\hat{P}} N_{\hat{P}}$. Finally, there are boundary localized potentials, of the form

$$\begin{aligned} S_\lambda^{\text{on-shell}} &= \frac{1}{2\pi} \int d^5x d\eta d\rho \sum_{i=1}^2 \delta(\rho - \rho_i) (-)^i \left[\sqrt{-\tilde{g}} \lambda_i \right] \\ &= \frac{1}{2\pi} \int d^5x d\eta d\rho \sum_{i=1}^2 \delta(\rho - \rho_i) (-)^i \left[e^{5A(\rho) - \frac{1}{\sqrt{10}}\chi(\rho)} \lambda_i \right]. \end{aligned} \quad (71)$$

Putting the three contributions together yields

$$\mathcal{S} = \mathcal{S}_7^{\text{on-shell}} + \mathcal{S}_{\text{GHY}}^{\text{on-shell}} + \mathcal{S}_\lambda^{\text{on-shell}} \quad (72)$$

$$= \frac{1}{2\pi} \int d^5x d\eta d\rho \sum_{i=1}^2 \delta(\rho - \rho_i) (-)^i \left[e^{5A(\rho) - \frac{1}{\sqrt{10}}\chi(\rho)} (2\partial_\rho A(\rho) + \lambda_i) \right]. \quad (73)$$

In order to apply holographic renormalization [41–43] to the reduction onto a circle, as in Refs. [44,54–56,58], we impose the requirements $\lambda_1 = -2\partial_\rho A(\rho)$, that is necessary for the consistency of the variational problem giving the background solutions, and $\lambda_2 = \mathcal{W}_2$ (see Appendix B for the complete form of $\mathcal{W}_2$ required in general classes of solutions), that automatically cancels the divergences. We ultimately arrive to the final expression

$$\begin{aligned} \mathcal{S}^{\text{ren}} &= \mathcal{S} \\ &= \int d^5x \left[e^{5A(\rho) - \frac{1}{\sqrt{10}}\chi(\rho)} (2\partial_\rho A(\rho) + \mathcal{W}_2) \right]_{\rho=\rho_2}, \end{aligned} \quad (74)$$

where we performed the trivial integral $\int d\eta = 2\pi$. Finally, using the general relation

$$F = - \lim_{\varrho_1 \rightarrow \varrho_0} \lim_{\varrho_2 \rightarrow \infty} \mathcal{S} \equiv \int dx^6 d\eta \mathcal{F}, \quad (75)$$

we arrive at renormalized result for free energy density, valid for all the solutions in Sec. II B 2,

$$\begin{aligned} \mathcal{F} &= \frac{1}{120} e^{5A_U - \frac{z_U}{\sqrt{10}}} (-10\phi_{1,2}\phi_{1,4} + 3\sqrt{10}\phi_{2,2}^3(3\log(\phi_{2,2}) - 2) \\ &\quad + \phi_{2,2}(3\sqrt{10}\phi_{1,2}^2(2 - 3\log(\phi_{2,2})) - 10\phi_{2,4}) \\ &\quad + 15\sqrt{10}\chi_6), \end{aligned} \quad (76)$$

which in the case in which both the sources for the operators associated with the scalars ϕ_i vanish, with $\phi_{1,2} = \phi_{2,2} = 0$, simplifies to the expression

$$\mathcal{F} = \frac{\sqrt{10}}{8} e^{5A_U - \frac{z_U}{\sqrt{10}}} \chi_6. \quad (77)$$

In the case of the soliton solutions discussed in this paper, using the expansion for small z of the analytical solutions, we find that $\chi_6 = -\frac{\mu}{16\sqrt{10}}$, and hence

$$\mathcal{F} = -\frac{\mu}{128}. \quad (78)$$

From the relation in Eq. (62) one knows that $\mu \geq 0$, and hence $\mathcal{F} \leq 0$. Conversely, the domain-wall AdS₇ solutions have $\chi_6 = 0$, and hence $\mathcal{F} = 0$. Hence, the soliton solutions have lower free energy than the domain-wall solutions, as long as they exist, and phase coexistence may arise when $\mu = 0 = \chi_6 = \mathcal{F}$. We therefore find a zero-temperature

phase transition between two types of solution. The first is the soliton solution given by the metric in Eq. (36), dual to a confining gauge theory. The second is the domain-wall, AdS₇ solution with the metric in Eq. (11), which has $\chi = \frac{\sqrt{10}}{8}\rho$ and $A = \frac{5}{8}\rho$, and is dual to a conformal field theory.

The critical line is defined by the requirement $\mu = 0$, which is satisfied when $\varrho_0^2 = 2 - 2\mathcal{A}_U^2$. In this case the metric can be brought into the domain-wall form,

$$\begin{aligned} ds_7^2(\mu = 0) &= \frac{4}{\varrho^2(H_1(\varrho)H_2(\varrho))^{4/5}} d\varrho^2 \\ &\quad + \frac{1}{4} \varrho^2(H_1(\varrho)H_2(\varrho))^{1/5} (dx_{1,4}^2 + d\eta^2), \end{aligned} \quad (79)$$

and furthermore $\sin^2(2\theta) = \frac{(\mathcal{A}_U^2 - 1)^2}{\mathcal{A}_U^4}$. One then finds that $\varrho_0^4 = Q_1^2 = Q_2^2 = 4(1 - \mathcal{A}_U^2)^2$. In this case one arrives at the conclusion that the critical line in the first octant of the $(\mathcal{A}_{7,U}^{(1)}, \mathcal{A}_{7,U}^{(2)})$ plane is defined by the relation

$$\sin(\theta) = -\frac{\sqrt{2\mathcal{A}_U^2 - 1} - 1}{2\mathcal{A}_U}, \quad (80)$$

which can be recast as $\mathcal{A}_{7,U}^{(1)} + \mathcal{A}_{7,U}^{(2)} = 1$. Extending to the other quadrants by symmetry, one concludes that the condition $\mu = 0$ defines a square, with sides $\sqrt{2}$ and vertex on the axis. This is illustrated in Fig. 2, to which we will return shortly.

By exploiting the symmetries of the system, it is most insightful to look at the solutions in the two limiting cases in which $\mathcal{A}_{7,U}^{(1)} = \mathcal{A}_U$ and $\mathcal{A}_{7,U}^{(2)} = 0$ and contrast it with the case $\mathcal{A}_{7,U}^{(2)} = \mathcal{A}_{7,U}^{(1)} = \mathcal{A}_U/\sqrt{2} > 0$, as representative of the whole space of solutions. In the former case, the two constraints can be rewritten as follows:

$$(2\mu + \varrho_0^6)^2 = 16\mu\varrho_0^4, \quad (81)$$

$$(\mathcal{A}_{7,U}^{(1)})^2 = \frac{\varrho_0^4(\varrho_0^6 - 4\mu)}{16\mu\varrho_0^2}, \quad (82)$$

These admit four solutions, but, by imposing the reality constraints, $\varrho_0^2 > 0$ and $\varrho_0 > 0$, one identifies uniquely the free energy. It is a negative definite quantity for $\mathcal{A}_{7,U}^{(1)} < 1$, that vanishes when $\mathcal{A}_{7,U}^{(1)} = 1$. For larger values of $\mathcal{A}_{7,U}^{(1)}$ the globally stable solutions are domain-wall solutions with constant background field $\mathcal{A}_7^{(1)}$, vanishing ϕ_i , and vanishing $\mathcal{F}$.

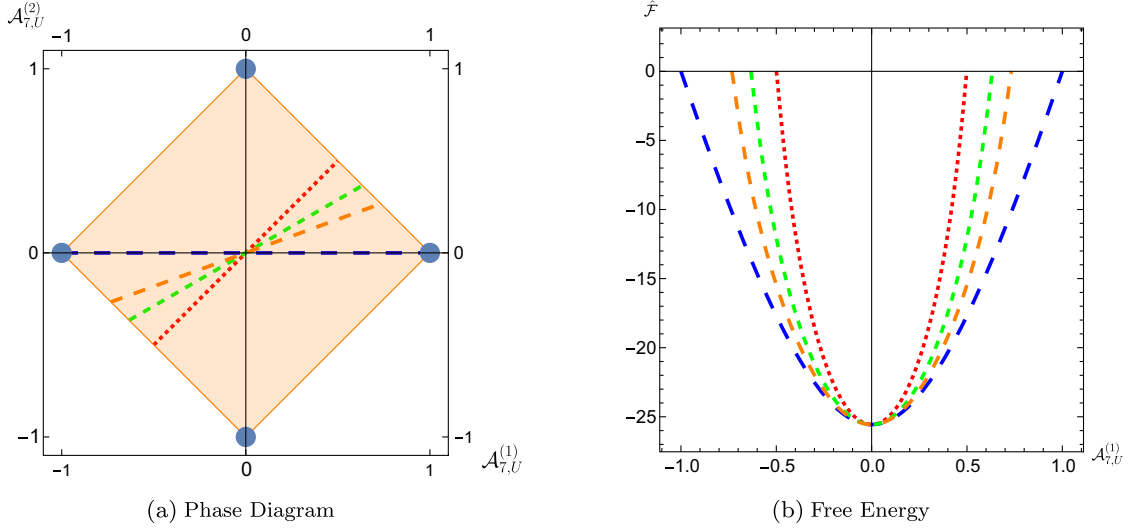


FIG. 2. Left: phase diagram of the model, in the plane defined by the sources $\mathcal{A}_{7,U}^{(1)}$ and $\mathcal{A}_{7,U}^{(2)}$. Inside the shaded area, the vacuum is given by regular soliton solutions, corresponding to confining solutions with magnetic flux in the dual theory. Outside the shaded region, the vacuum is given by domain wall solutions leading to AdS_7 geometry. Along the sides of the square one has first-order phase transitions. The dashed line correspond to lines of constant ratio $\mathcal{A}_{7,U}^{(2)}/\mathcal{A}_{7,U}^{(1)} = (0, \tan(\frac{\pi}{9}), \tan(\frac{\pi}{6}), \tan(\frac{\pi}{4}))$ (long dashed to short dashed). Right: the free energy density, $\hat{\mathcal{F}}$ of the dual, strongly coupled field theory living in six dimensions, expressed in units of the scale Λ , as a function of the parameter $\mathcal{A}_{7,U}^{(1)}$, and for values of $\mathcal{A}_{7,U}^{(2)}$ chosen along the four straight lines in the phase diagram in the left panel, with constant ratio $\mathcal{A}_{7,U}^{(2)}/\mathcal{A}_{7,U}^{(1)} = (0, \tan(\frac{\pi}{9}), \tan(\frac{\pi}{6}), \tan(\frac{\pi}{4}))$ —long dashed to short dashed lines in blue, orange, green, and red, respectively.

By contrast, setting $\sin^2(2\theta) = 1$ yields the somewhat more complicated relations

$$\left(2\mu + q_0^6 - 16q_0^2(\mathcal{A}_{7,U}^{(1)})^4\right)^2 = 16\mu q_0^4. \quad (83)$$

$$2(\mathcal{A}_{7,U}^{(1)})^2 = \frac{256q_0^2(\mathcal{A}_{7,U}^{(1)})^8 - 32(\mathcal{A}_{7,U}^{(1)})^4(q_0^6 + 2\mu) + q_0^4(q_0^6 - 4\mu)}{16\mu q_0^2}. \quad (84)$$

This system admits six solutions. Four of them have complex q_0 , which is unphysical. Of the remaining two, we must choose the one with largest q_0 , and ensure that it satisfies both $q_0 > Q_1$ and $q_0 > Q_2$. This second condition restricts the space to $\mathcal{A}_U < 1/\sqrt{2}$, as outside the square defined by $\mu = 0$ one finds that neither of the two values of q_0 is larger than both Q_1 and Q_2 . As a result the qualitative shape of the free energy is closely resembling the aforementioned $\theta = 0$ case, with $\mathcal{F} < 0$ for $\mathcal{A}_{7,U}^{(2)} = \mathcal{A}_{7,U}^{(1)} < \frac{1}{2}$, and vanishing when $\mathcal{A}_{7,U}^{(2)} = \mathcal{A}_{7,U}^{(1)} = \frac{1}{2}$. For larger values of $\mathcal{A}_{7,U}^{(2)} = \mathcal{A}_{7,U}^{(1)}$ the globally stable solutions are domain-wall solutions with constant background fields $\mathcal{A}_7^{(2)} = \mathcal{A}_7^{(1)}$, vanishing ϕ_i , and vanishing $\mathcal{F}$.

To reinstate physical units, it is convenient to define the following scale, given by the time it takes for light to reach the end of space from infinity, travelling along the holographic direction, q [211]:

$$\Lambda^{-1} \equiv \int_{\rho_0+\rho_1}^{\infty} d\rho e^{\frac{\rho}{\sqrt{10}}-A} = \int_{q_0+q_1}^{+\infty} dq \frac{2}{q\sqrt{f(q)}}, \quad (85)$$

where we require the inclusion of an IR cutoff, q_1 , in the lower limit of the integration to avoid a divergence at the end of space. We numerically integrate and take $q_1 = 10^{-6}$.

We illustrate the behavior of the free energy in Fig. 2. The left-hand panel shows the phase diagram, in the $(\mathcal{A}_{7,U}^{(1)}, \mathcal{A}_{7,U}^{(2)})$ plane, together with a selection of lines obtained by holding fixed the ratio $(\mathcal{A}_{7,U}^{(1)}/\mathcal{A}_{7,U}^{(2)})$. The phase diagram is characterized by lines of first-order phase transitions forming a polygon (a square), as expected on the basis of the symmetries of the system. The right-hand panel shows the free energy density expressed in terms of the scale Λ , as $\hat{\mathcal{F}} = \mathcal{F}\Lambda^{-6}$. The resulting plot displays a continuous curve, that is not differentiable at the transition. As expected, the endpoints of the confining branches

obtained at fixed source ratio, $\tan(\theta)$, take the form of domain-wall solutions. Unlike their AdS_7 counterparts, these solutions are singular at the end of space. This distinction results in the coexistence of two separate configurations, a hallmark of first-order phase transitions, as is the nondifferentiable character of the free energy at the transition.¹¹ The presence of a singularity indicates the incompleteness of the classical supergravity treatment, large curvature effects emerging in close proximity of the phase transitions—see Appendix D.

IV. LOCAL STABILITY ANALYSIS: FLUCTUATION SPECTRA

We perform a local stability study of the soliton solutions, restricted to the region inside the square in

$$0 = [\mathcal{D}_r^2 + 5\partial_r A \mathcal{D}_r + e^{-2A} M^2] \mathbf{a}^a - \left[V_{|c}^a - R_{bcd}^a \partial_r \Phi^b \partial_r \Phi^d + \frac{(\partial_r \Phi^a V^b + V^a \partial_r \Phi^b) G_{bc}}{\partial_r A} + \frac{V \partial_r \Phi^a \partial_r \Phi^b G_{bc}}{(\partial_r A)^2} \right] \mathbf{a}^c, \quad (86)$$

subject to the boundary condition,

$$\partial_r \Phi^a \partial_r \Phi_b \mathcal{D}_r \mathbf{a}^b|_{r_i} = \left[-2\partial_r A e^{-2A} M^2 \delta_b^a + \partial_r \Phi^a \left(\frac{V}{\partial_r A} \partial_r \Phi_b + V_b \right) \right] \mathbf{a}^b|_{r_i}, \quad (87)$$

where $\mathbf{a}^a$ are the five scalar fluctuations, and $r_i = \{r_1, r_2\}$ are the cutoffs we impose along the holographic direction, in the IR, near the end of space, r_0 , and in the far UV. It is understood that at the end of the calculation the results are extrapolated to the limits $r_1 \rightarrow r_0$ and $r_2 \rightarrow +\infty$, respectively. We denote as M^2 the mass squared of the fluctuation, in the dual five-dimensional confining theory. The full expressions for the fluctuation equations and boundary conditions written in terms of holographic variable ρ are given in Appendix E 1.

In our numerical study, we use Eq. (87) to set the IR boundary conditions for the evolution of the linearized equations. In order to improve the convergence of the numerical results we find it useful to impose the UV boundary conditions on asymptotic expansions of the scalar fluctuations, and then match the resulting, constrained

Fig. 2, in which they are regular and energetically favored. To this purpose, we compute the mass spectrum of fluctuations of the scalar and tensor fields in the gravity description, along the soliton solutions, which we interpret as bound states of the dual field theory (in Yang-Mills theories, these would be identified with glueballs). We adopt the gauge-invariant formalism developed in Refs. [45–50], for both spin-0 and spin-2 cases—see also Refs. [51–53, 210], and Appendix E, in which we fix and explain the notation.

Using the holographic direction r , the gauge-invariant combinations of the scalar fluctuations obey the following coupled equations:

expansions to the numerical solutions of the fluctuation equations at the holographic coordinate $\rho = \rho_2$. The UV expansions are presented explicitly in Appendix E 2.

The spin-2, tensor fluctuations, $\mathbf{e}^\mu_\nu$, obey the simpler differential equations

$$0 = [\partial_r^2 + (D-1)\partial_r A \partial_r + e^{-2A(r)} M^2] \mathbf{e}^\mu_v, \\ = \left[e^{2A(\rho) - \sqrt{\frac{2}{3}}\chi(\rho)} \partial_\rho^2 + 5e^{2A(\rho) - \sqrt{\frac{2}{3}}\chi(\rho)} \partial_\rho A \partial_\rho + M^2 \right] \mathbf{e}^\mu_v, \quad (88)$$

with boundary conditions given by

$$\partial_\rho \mathbf{e}^\mu_v|_{\rho=\rho_i} = 0. \quad (89)$$

We performed an extensive numerical study of spectra, focusing on four representative branches of soliton solutions, characterized by four choices of the angle controlling the ratio between the two Abelian fluxes, $\theta = 0, \pi/9, \pi/6, \pi/4$, as illustrated in Fig. 3. In all cases, we repeated the calculations for several choices of the IR and UV cutoff, looking to minimize residual numerical artifacts, within the limits of numerical precision of the calculations. We notice that the improvement makes the convergence in the UV so fast that no discernible corrections could be found by changing the UV cutoff within reasonable ranges. We did not have a similar process available for the IR, yet we found that by solving the equation rewritten in the variable ϱ , and choosing $\varrho_{\text{IR}} = 10^{-6} \varrho_0$ as the IR cutoff, the results are a

¹¹First-order phase transitions are often accompanied also by the existence of metastable and unstable branches of solutions, that are not realized at equilibrium, but can be produced under controlled conditions studied in details. This is not the case in this example: there are other solutions to the algebraic constraint defining ϱ_0 , as discussed earlier in the manuscript, but as we anticipated these additional solutions cannot be interpreted as background geometries, and hence do not have a dual field theory interpretation. This is at odds with what was found for other theories—see, e.g., Ref. [59]—in which the presence of metastable and unstable branches leads to the typical swallow-tail diagrams in sketches of the free energy as a function of the sources.

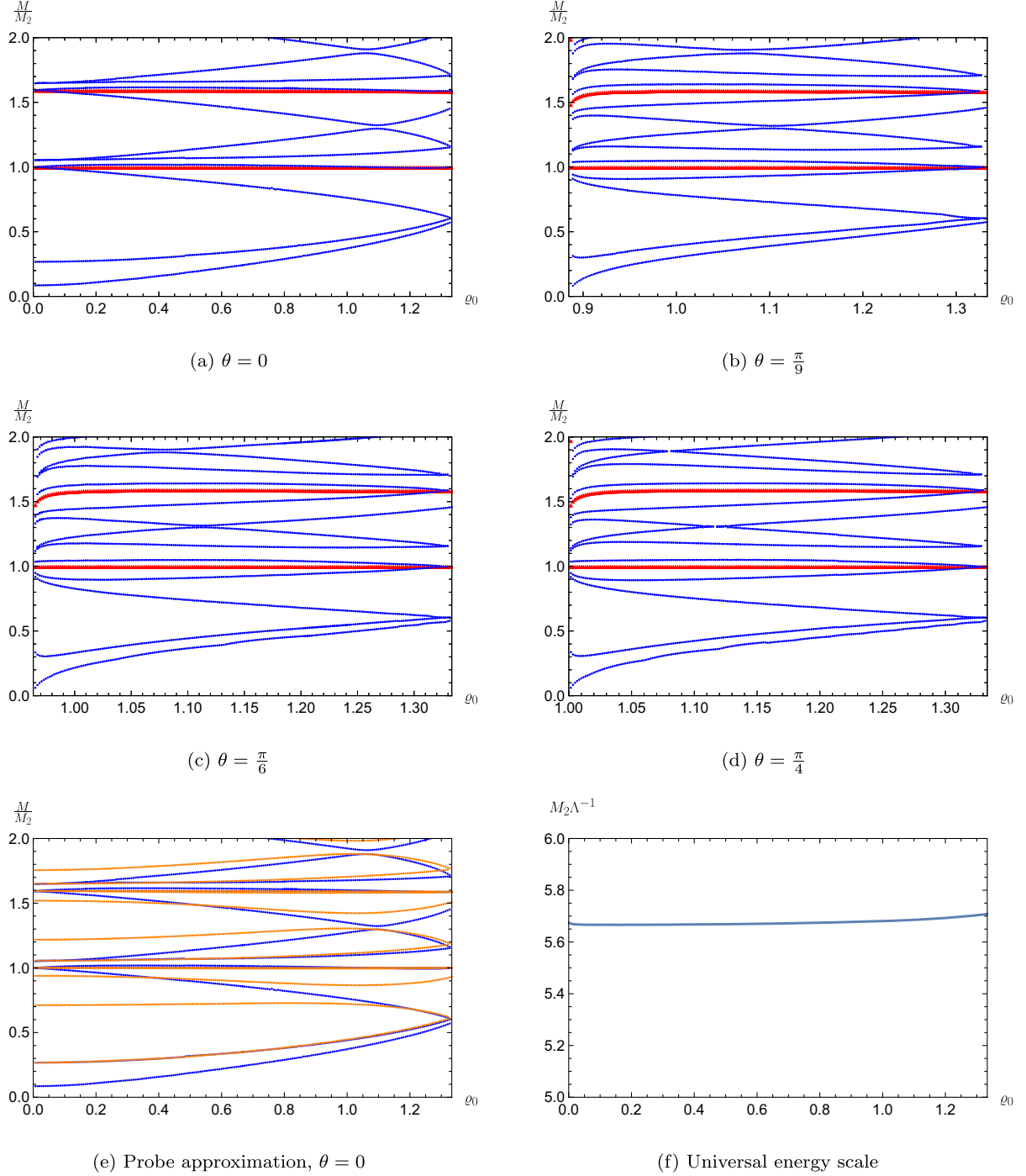


FIG. 3. Top four panels: mass spectra, normalized to the mass of the lightest spin-2 fluctuation, M_2 , as a function of q_0 , for four examples of one-parameter subclasses of soliton (confining) solutions, obtained by fixing the ratio of the two magnetic fluxes—see the free energy of the same solutions in Fig. 2(a). Spin-2 states are shown as (red) triangles and spin-0 as (blue) disks. The far left of each plot corresponds to points along the line of first order phase transitions in Fig. 2(a). Fifth panel: comparison of the spin-0 states with the result of using the probe approximation, shown with the orange rectangles, for solutions with $\theta = 0$. The probe approximation entirely misses the lightest state, suggesting this state contains a substantial contribution from the dilaton. The numerical results are obtained for $q_{\text{IR}} = 10^{-6} q_0$. Sixth panel: ratio of the mass of the lightest spin-2 states, M_2 , to the universal energy scale, Λ , for the solutions in which $\theta = 0$; the ratio is roughly constant across the parameter space, suggesting either of the two observables could be used to set the physical scale, and compare between theories of this class.

good approximation to the exact ones, for the purposes of this publication. We present in Appendix F examples demonstrating the degree of convergence of the results, as a function of the IR cutoff.

The results, shown in Fig. 3, can be summarized as follows. Firstly, we do not find evidence of local instabilities, in the form of tachyons, anywhere in the relevant portion of parameter space. This is to be contrasted with the results of Refs. [44,54,55], in which the arising of a first-order phase transition is always accompanied by the presence of a tachyonic state that emerges along the same branch of solutions, but for choices of tunable parameters past the phase transition. In the case at hand, the transition is happening at the end of the physical branches of solutions of interest, beyond which background solutions of this class do not exist.

The five scalar and one tensor gauge-invariant fluctuations all lead to discrete mass spectra, with typical spacing and scale well represented by the lightest of the spin-2 states.¹² The two lightest scalar modes are the only ones that are substantially lighter than the lightest spin-2 particle, ranging from 0.6 down to 0.05 and 0.3 for the lightest and next-to-lightest state, in units of the tensor mass, M_2 . This is happening over a sizeable portion of parameter space, not just in proximity of the transition, and deserves further discussion, after we present the results we obtained with the probe approximation, in the next section. We observe here that this behavior is also in sharp contrast to Refs. [44,54,55], that show no particular suppression of the mass of the scalar bound states along the physically realized portion of the branches of solutions, before the inset of the phase transition—in the cases in the literature, a suppression of the mass of the lightest scalar is observed only along the metastable portion of the branches of solutions, past the transition itself.

A. Probe approximation

We now turn to examining the extent to which the scalar fluctuations discussed in the previous section couple as the dilaton, with particular interest to the lightest scalars and to the regions of parameter space close to the phase transition. To investigate this point, we adopt the probe approximation, following the approach and notation of Ref. [53]. The key insight underlying this approximation, and the results presented in this section, stems from the definition of gauge-invariant scalar combinations given in Eq. (E4). These combinations consist of two additive terms: the first, φ^a , represents small fluctuations of the sigma-model scalar fields around their background values; the second is proportional to h , the trace of the five-dimensional metric

¹²As displayed by the last panel of Fig. 3, we could as well have normalized the masses with the scale Λ , with no significant qualitative change to the overall emerging picture, as the ratio M_2/Λ is effectively a constant.

fluctuations which couples to the trace of the boundary stress-energy tensor, the dilatation operator. The probe approximation refers to computing the spectrum while deliberately neglecting the contribution of h to the gauge-invariant combinations. In doing so, the resulting spectrum reflects only the dynamics of the scalar fluctuations, φ^a .

While this is an approximation, and therefore not physically complete (in particular, we are explicitly breaking gauge invariance) it is nonetheless useful for diagnostic purposes. If the probe approximation yields a spectrum that closely matches the result from the full gauge-invariant analysis, we can infer that the metric fluctuations, h , play a negligible role. This would imply that the scalar fluctuations are not significantly coupled to the dilaton operator of the dual field theory (the trace of the stress-energy tensor). Conversely, the emergence of substantial differences between the two results for the spectra would indicate that the affected scalar modes are the result of substantial mixing with the dilaton.

In the probe approximation, the equations of motion for the fluctuations simplify considerably and take the following form:

$$0 = [\mathcal{D}_r^2 + 5\partial_r A \mathcal{D}_r + e^{-2A} M^2] \mathbf{p}^a - [V_{|c}^a - R_{bcd}^a \partial_r \Phi^b \partial_r \Phi^d] \mathbf{p}^c, \quad (90)$$

while the boundary conditions reduce to Dirichlet:

$$0 = \mathbf{p}^a|_{r_i}. \quad (91)$$

We report the expression for the equations in the ρ variable in Appendix E 3. The results of our analysis are displayed in the fifth panel of Fig. 3, for the choice $\theta = 0$ in which only one of the two Abelian fields is nontrivial in the background. The comparison between the probe approximation and the result of the complete calculation with gauge-invariant variables is overall quite good: particularly for the largest values of q_0 available, and for the mass of the heaviest fluctuations, the probe approximation works well, demonstrating that the majority of states we identified have little contamination from the dilaton.

Focusing on the two lightest states shows a striking contrast. The next-to-lightest state is always well captured by the probe approximation, demonstrating that it has nothing to do with dilaton and scale invariance. The lightest state is entirely missed, demonstrating its nature as mostly an approximate dilaton. This is possibly the most striking example of the probe approximation failing completely to identify the mass of a state—see the catalog of examples in Ref. [53]—a phenomenon that persists throughout the whole parameter space, not just near the phase transition.

This observation adds to the distinctive peculiarity of this theory. We find evidence of first-order phase transitions, but not of the transition becoming weaker, or second order

(as was the case in the results discussed in Refs. [57,59]), anywhere in the region of parameter space that is open to exploration. Similarly to what emerged in the theories in Refs. [44,54,55] and the bottom-up models in Refs. [56,58], we find that there is no sense in which a parametrically light dilaton emerges near the transition—in which region the geometry exhibits large curvature, undermining the classical supergravity treatment. And yet, we find that, for $\theta = 0$, in a large portion of parameter space that extends to the whole branch of confining solutions, the lightest state is a dilaton, and its mass can be as much as one order of magnitude suppressed in respect to the typical scale set by confinement. To emphasize this finding we can compare the fifth panel of Fig. 3 to the figures in Ref. [56], particularly the summary Fig. 10. In this earlier publication, a whole class of bottom-up models were demonstrated to exhibit a first-order phase transition, accompanied by the existence of a parametrically light dilaton, along the metastable branches of solutions. Yet, along the physically realized, stable branches of solution, the dilaton is never parametrically light. Its mass is minimized in closest proximity to the transition. The ratio between mass of the dilaton and of lightest spin-2 state is found to be $M_d/M_2 \gtrsim 0.36$. In sharp contrast, in the theory discussed in this paper, this ratio is much smaller, $M_d/M_2 \sim 1/10$, over large parts of parameter space. Furthermore, this interesting behavior emerges without having to tune any of the parameters, contrary to earlier examples in the literature.

V. OUTLOOK

This research sits at the confluence of two ongoing research programs. On the one hand, it is part of the systematic exploration, by means of holography, of confining field theories for which it is possible to compute the spectrum of bound states in regions of parameter space near phase transitions, as formulated in the outlook of Ref. [44]—see also Refs. [54,55,59] in the top-down approach to holography, and Refs. [56–58]. The aim of this program is to understand under what circumstances a light dilaton emerges as a bound state, and hence to provide the foundations for dilaton effective field theory and its applications. On the other hand, the work in Ref. [191] identified a new solution-generating technique within top-down supergravity—see also Refs [192–199]—giving access to new families of nontrivial holographic descriptions of confining field theories. We are here using this solution-generating technique as part of the tools deployed for the aforementioned exploration of the relation between dilaton and phase transitions.

We have presented the results of an extensive global and local stability analysis of a two-parameter family of regular classical solutions of maximal supergravity in $D = 7$ dimensions with $SO(2) \times SO(2)$ gauge symmetry. Solutions in

this class take the soliton form, and admit an elegant interpretation in terms of a strongly coupled confining field theory obtained by circle compactification of the $\mathcal{N} = (2, 0)$ superconformal theory in $D - 1 = 6$ dimensions. The global stability analysis makes use of the holographically renormalized free energy, and confirms the presence of a first-order phase transition in the space of solutions. In a compact region of parameter space, inside a square, the soliton/confining family is energetically favored, with respect to a competing family of AdS_7 supergravity solutions with constant gauge field that take the domain-wall form and are physically realized outside of the square. The local stability analysis, performed by computing gauge-invariant fluctuations of the spin-0 and spin-2 supergravity fields, confirms that there are no further instabilities, all the accessible states having nonzero, positive mass squared.

The main element of novelty emerging from this work is that the spectrum of bound states of the dual field theory (the dilaton in particular) behaves in an unusual way, with respect to former examples found in the literature. In the two known cases in which a parametric suppression of the dilaton mass has been found [57,59], there is evidence of a nearby critical point at which a phase transition became second order, which is not the case for the theories discussed here. The dilaton we identify is not parametrically light. Yet, the suppression of its mass, with respect to the mass of the lightest spin-2 bound state, $M_d/M_2 \sim 1/10$, is strong enough that if it could be engineered to take place in a realistic new physics model with composite dynamics it would be sufficient to avoid most exclusion bounds—see the discussions of the misalignment angle in Refs. [212,213] and references therein. Furthermore, the suppression of the dilaton mass persists over a large portion of parameter space, extending far away from the first-order phase transition characterizing the system, without requiring one to fine-tune the parameters.

This finding marks important progress toward building a complete picture of the conditions upon which strongly coupled, confining gauge theories can give rise to a light dilaton in their spectrum, separated in mass from the other bound states of the theory. It also highlights how our current understanding of the underlying mechanism is still incomplete, warranting a systematic exploration of other holographic models, in other dimensions, and in theories with less symmetry. The solution generating technique we borrowed from Ref. [191] is proving a useful tool in this direction. But our study opens fundamental questions about the relation of the magnetic fluxes to the emergence of a light dilaton, and its relation to approximate scale invariance. It would be desirable to have at our disposal other such techniques to explore the broader space of relevant top-down holographic theories, and we envision starting an extensive exploration of the space of similar theories, to address the aforementioned open questions.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [214]; embargo periods may apply.

APPENDIX A: SIGMA MODEL COUPLED TO GRAVITY, GENERALITIES

We report here the general equations defining a sigma model coupled to gravity, by adopting the notation of Refs. [49,50]—see also Refs. [45–48,51–53,210]. In general D dimensions, we write the two-derivative sigma-model action, for real field Φ^a , with $a = 1, \dots, n$, as follows:

$$\mathcal{S}_D = \int d^D x \sqrt{-g_D} \left\{ \frac{\mathcal{R}}{4} - \frac{1}{2} g^{MN} G_{ab} \partial_M \Phi^a \partial_N \Phi^b - \mathcal{V}_D(\Phi^a) \right\},$$

where G_{ab} is the metric in the sigma-model space, and $\mathcal{V}_D(\Phi^a)$ is the potential. We use the following conventions for gravity. Firstly the Christoffel symbols are defined as

$$\Gamma^P_{MN} = \frac{1}{2} g^{PQ} (\partial_M g_{NQ} + \partial_N g_{MQ} - \partial_Q g_{MN}). \quad (\text{A1})$$

The Riemann tensor is

$$\mathcal{R}_{MNP}{}^Q \equiv \partial_N \Gamma^Q_{MP} - \partial_M \Gamma^Q_{NP} + \Gamma^S_{MP} \Gamma^Q_{SN} - \Gamma^S_{NP} \Gamma^Q_{SM}, \quad (\text{A2})$$

and the Ricci tensor and scalar are obtained by contracting the indices, with $\mathcal{R}_{MN} = \mathcal{R}_{MPN}{}^P$, and $\mathcal{R} = \mathcal{R}_{MN} g^{MN}$.

We assume that the background geometry is characterized by the domain wall ansatz for the metric:

$$ds_D^2 = e^{2A} dx_{1,D-2}^2 + dr^2, \quad (\text{A3})$$

where $A = A(r)$. Similarly, we allow the scalars to have a nontrivial profile depending only on the holographic direction, with $\Phi^a = \Phi^a(r)$ in the background. In the calculations, we assume the holographic direction to be bounded, as in $r_1 < r < r_2$. The two boundaries act, respectively, as an infrared (IR) and ultraviolet (UV) regulator, with it understood that the limits $r_1 \rightarrow r_0$ and

$r_2 \rightarrow +\infty$ have to be taken at the end of the calculation, to recover physical results that do not depend on the regulators. The general expression for the equations of motions, satisfied by the background scalars in general D dimensions, are the following:

$$\partial_r^2 \Phi^a + (D-1) \partial_r A \partial_r \Phi^a + \mathcal{G}^a_{bc} \partial_r \Phi^b \partial_r \Phi^c - \mathcal{V}^a = 0, \quad (\text{A4})$$

where the sigma-model derivatives are given by $\mathcal{V}^a \equiv G^{ab} \partial_b \mathcal{V}$ and $\partial_b \mathcal{V} \equiv \frac{\partial \mathcal{V}}{\partial \Phi^b}$. The Einstein equations reduce to

$$(D-1)(\partial_r A)^2 + \partial_r^2 A + \frac{4}{D-2} \mathcal{V} = 0, \quad (\text{A5})$$

$$(D-1)(D-2)(\partial_r A)^2 - 2G_{ab} \partial_r \Phi^a \partial_r \Phi^b + 4\mathcal{V} = 0. \quad (\text{A6})$$

APPENDIX B: SUPERPOTENTIAL AND SUPERSYMMETRIC SOLUTIONS

In this appendix, we report and discuss the superpotential of the theory in $D = 7$ dimensions, and its role in the process of holographic renormalization for field-theory observables computed in general, dual gravity background solutions. We start from the domain-wall ansatz for the metric, which here we write as

$$ds_D^2 = d\rho^2 + e^{2A} dx_{1,D-2}^2. \quad (\text{B1})$$

If the scalar potential, $\mathcal{V}_D$, can be written as

$$\mathcal{V}_D = \frac{1}{2} G^{ab} \frac{\partial \mathcal{W}_D}{\partial \Phi^a} \frac{\partial \mathcal{W}_D}{\partial \Phi^b} - \frac{D-1}{D-2} (\mathcal{W}_D)^2, \quad (\text{B2})$$

for a superpotential, $\mathcal{W}_D = \mathcal{W}_D(\Phi^a)$, that depends only on the scalar fields, then any solution of the following differential equations

$$\partial_\rho \mathcal{A} = -\frac{2}{D-2} \mathcal{W}_D, \quad (\text{B3})$$

$$\partial_\rho \Phi^a = G^{ab} \frac{\partial \mathcal{W}_D}{\partial \Phi^b} \quad (\text{B4})$$

is also a solution to the classical equations derived from the potential. In these equations, the background value of both $\mathcal{A}$ and the scalars, Φ^a , depend only on ρ .

In the case at hand, with $D = 7$, the potential admits the following approximation, for small fields:

$$\mathcal{V}_7 \simeq -\frac{15}{8} - \frac{1}{4} (\phi_1^2 + \phi_2^2) + \dots, \quad (\text{B5})$$

where we expanded in powers of the (small) scalar fields, $\phi_{1,2}$, starting from the AdS_7 solutions with $\phi_i = 0$ and $\mathcal{A} = \mathcal{A}_0 + \frac{1}{2}\rho$. There are four possible superpotentials that satisfy the defining relation in Eq. (B2):

$$\mathcal{W}_7 = -\frac{5}{4} - \frac{3 \pm 1}{16} \phi_1^2 - \frac{3 \pm 1}{16} \phi_2^2 + \dots. \quad (\text{B6})$$

By choosing the minus sign in both quadratic terms, so that $\mathcal{W}_7 = -\frac{5}{4} - \frac{1}{8}(\phi_1^2 + \phi_2^2) + \dots$, one finds that the first-order equations are

$$\partial_\rho \mathcal{A} = \frac{1}{2} + \dots, \quad (\text{B7})$$

$$\partial_\rho \phi_{1,2} = -\phi_{1,2} + \dots. \quad (\text{B8})$$

By setting $z = e^{-\rho/2}$, one finds that $e^{\mathcal{A}} \propto \frac{1}{z}$ and $\phi_{1,2} \propto z^2$, indicating that gravity solutions obtained as small perturbations of the AdS_7 ones have a dual description in terms of deformations of a six-dimensional CFT, in the presence of two operators of dimensions $\Delta = 4$ (so that $6 - \Delta = 2$ is the scaling dimension of their deforming coupling). When computing the free energy of a general solution, the superpotential plays the role of a counterterm in our prescription. We must require one to cancel all divergences, which in this case can be done by retaining in the superpotential terms up to cubic order in the scalars:

$$\mathcal{W}_2 = -\frac{5}{4} - \frac{\phi_1^2 + \phi_2^2}{8} + \frac{3}{4\sqrt{10}}\phi_2 \ln(\phi_2)(\phi_1^2 - \phi_2^2). \quad (\text{B9})$$

The free energy is determined by the action evaluated on shell. Its UV-divergent terms match those arising from the superpotential, and take the form $\frac{1}{z^6}[W_0 + W_2\phi_i^2 + \dots]$. The UV expansion of scalar fields takes the form $\phi_i \approx z^2\phi_{i,2} + z^4\phi_{i,4}$ where $\phi_{i,2}$ encodes the sources and $\phi_{i,4}$ is proportional to the vacuum expectation values (VEVs) of the dual operators. In the body of the paper, we restrict our attention to solutions in which the source is zero, but with nonzero VEV. Hence, because all terms in the UV expansion of the superpotential, beyond the constant term, scale as z^8 or higher, they do not contribute to the divergence in the free energy—yet, we report the general result, for completeness.

We conclude this appendix by showing an additional class of solutions, obtained using the first order equations. These domain-wall solutions are singular, have vanishing free energy, and correspond to field theories in $D = 6$ dimensions in which all deformations are trivial (though the VEVs of the two operators are not), hence they have no effect on the stability analysis discussed in the body of this paper. The superpotential conjugate to $\mathcal{W}_2$ in Eq. (B9)

admits a superpotential that can be written in closed form:

$$\mathcal{W}_{\text{susy}} = -e^{-\frac{1}{\sqrt{10}}\phi_2} \left(\frac{1}{4} e^{\sqrt{\frac{5}{2}}\phi_2} + \cosh\left(\frac{\phi_1}{\sqrt{2}}\right) \right). \quad (\text{B10})$$

This superpotential has the following small-field expansion:

$$\mathcal{W}_{\text{susy}} = -\frac{5}{4} - \frac{1}{4}(\phi_1^2 + \phi_2^2) + \dots, \quad (\text{B11})$$

hence the solutions of the first-order equations derived from it amount to turning on the subleading terms in the UV expansions of ϕ_1 and ϕ_2 , corresponding to the emergence of VEVs for field theory operators in the dual language. One then finds that for all solutions one has $\phi_{1,2} = \phi_{2,2} = \chi_6 = 0$, hence the free energy vanishes exactly, $\mathcal{F} = 0$.¹³ Due to the symmetries of the system, for any solution of the first-order equations, there exists another equivalent one with $\phi_1 \rightarrow -\phi_1$ and the other background fields unchanged.

The resulting equations of motion are the following:

$$\partial_\rho \phi_1 = -2\sqrt{2}e^{-\frac{\phi_2}{\sqrt{10}}} \sinh\left(\frac{\phi_1}{\sqrt{2}}\right), \quad (\text{B12})$$

$$\partial_\rho \phi_2 = -2\sqrt{\frac{2}{5}}e^{-\frac{\phi_2}{\sqrt{10}}} \left(e^{\sqrt{\frac{5}{2}}\phi_2} - \cosh\left(\frac{\phi_1}{\sqrt{2}}\right) \right), \quad (\text{B13})$$

$$\partial_\rho \mathcal{A} = \frac{1}{10}e^{-\frac{\phi_2}{\sqrt{10}}} \left(e^{\sqrt{\frac{5}{2}}\phi_2} + 4 \cosh\left(\frac{\phi_1}{\sqrt{2}}\right) \right). \quad (\text{B14})$$

We can solve these equations, by applying the change of variable $\partial_\rho = e^{-\frac{\phi_2}{\sqrt{10}}}\partial_\tau$, to rewrite them as follows:

$$\partial_\tau \phi_1 = -2\sqrt{2} \sinh\left(\frac{\phi_1}{\sqrt{2}}\right), \quad (\text{B15})$$

$$\partial_\tau \phi_2 = -2\sqrt{\frac{2}{5}} \left(e^{\sqrt{\frac{5}{2}}\phi_2} - \cosh\left(\frac{\phi_1}{\sqrt{2}}\right) \right), \quad (\text{B16})$$

$$\partial_\tau \mathcal{A} = \frac{1}{10} \left(e^{\sqrt{\frac{5}{2}}\phi_2} + 4 \cosh\left(\frac{\phi_1}{\sqrt{2}}\right) \right). \quad (\text{B17})$$

The general solution is

$$\phi_1^\pm(\tau) = \pm 2\sqrt{2} \text{arctanh}\left[e^{-2(\tau-\tau_o)}\right], \quad (\text{B18})$$

$$\phi_2^\pm(\tau) = \sqrt{\frac{2}{5}} \log \left[\frac{\sinh(2(\tau - \tau_o))}{\cosh(2(\tau - \tau_o)) - 1 \pm (1 - \cosh(2(\bar{\tau}_o - \tau_o)))} \right], \quad (\text{B19})$$

¹³Also $-\mathcal{W}_{\text{susy}}$ is an admissible superpotential. Its adoption would require a change of sign in the radial variable, τ .

$$\mathcal{A}(\tau) = \mathcal{A}_o - \frac{1}{2}(\tau - \tau_o) + \log \left[(e^{4(\tau - \tau_o)} - 1)^{\frac{1}{5}} (e^{4(\tau - \tau_o)} + 1 + 2(-1 \pm (1 - \cosh(2(\bar{\tau}_o - \tau_o)))e^{2(\tau - \tau_o)})^{\frac{1}{20}}) \right], \quad (\text{B20})$$

with τ_o , $\bar{\tau}_o$, and $\mathcal{A}_o$ integration constants.

APPENDIX C: LIFT TO $D=11$ DIMENSIONS

We report here the general form of the background metric, by lifting the $SO(2) \times SO(2)$ truncation back to $D=11$ dimensions. We identify the seven-dimensional action in Eq. (8) and the potential in Eq. (10) with Eqs. (4.6) and (4.7) of Ref. [157], respectively. Doing so requires setting the gauge coupling $g^2 \equiv 1$ in the latter equation. The lift is based on defining

$$X_1 \equiv e^{-\frac{1}{\sqrt{2}}\phi_1 - \frac{1}{\sqrt{10}}\phi_2}, \quad (\text{C1})$$

$$X_2 \equiv e^{+\frac{1}{\sqrt{2}}\phi_1 - \frac{1}{\sqrt{10}}\phi_2}, \quad (\text{C2})$$

$$X_0 \equiv (X_1 X_2)^{-2} = e^{\frac{4}{\sqrt{10}}\phi_2}. \quad (\text{C3})$$

We choose the ranges of the four angles describing the manifold to be the following:

$$0 \leq \tilde{\psi} \leq \frac{\pi}{2}, \quad (\text{C4})$$

$$0 \leq \tilde{\xi} \leq \pi, \quad (\text{C5})$$

$$0 \leq \tilde{\varphi}_i < 2\pi. \quad (\text{C6})$$

We solve the constraint $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$ by defining

$$\mu_0 \equiv \cos \tilde{\xi}, \quad (\text{C7})$$

$$\mu_1 \equiv \sin \tilde{\xi} \cos \tilde{\psi}, \quad (\text{C8})$$

$$\mu_2 \equiv \sin \tilde{\xi} \sin \tilde{\psi}, \quad (\text{C9})$$

and introduce the variable

$$\tilde{\Delta} \equiv \sum_{i=0}^2 X_i \mu_i^2, \quad (\text{C10})$$

which depends on the two angles, $\tilde{\varphi}_i$, along the two circles corresponding to the $SO(2) \times SO(2)$ directions. The metric in 11 dimensions takes the following form [157]:

$$ds_{11}^2 = \tilde{\Delta}^{1/3} ds_7^2 + \tilde{\Delta}^{-2/3} \left(X_0^{-1} d\mu_0^2 + \sum_{i=1,2} X_i^{-1} \left(d\mu_i^2 + \mu_i^2 (d\tilde{\varphi}_i + \mathcal{A}_M^{(i)} dx^{\hat{M}})^2 \right) \right). \quad (\text{C11})$$

As a consistency check, notice that if $\phi_1 = \phi_2 = 0 = \mathcal{A}_M^i$, then $X_1 = X_2 = X_0 = 1 = \tilde{\Delta}$, in which case,

$$\begin{aligned} ds_{11}^2(0) &= ds_7^2 + \left[d\mu_0^2 + \sum_{i=1,2} (d\mu_i^2 + \mu_i^2 d\tilde{\varphi}_i^2) \right] \\ &= ds_7^2 + \left[d\tilde{\xi}^2 + \sin^2 \tilde{\xi} (d\tilde{\psi}^2 + \cos^2 \tilde{\psi} d\tilde{\varphi}_1^2 + \sin^2 \tilde{\psi} d\tilde{\varphi}_2^2) \right]. \end{aligned} \quad (\text{C12})$$

The four-dimensional internal submanifold has indeed $\text{vol}(\mathcal{S}^4) = \frac{8\pi^2}{3}$, and furthermore the round bracket in the last form of the metric describes the 3-sphere, with angular coordinates $(\tilde{\psi}, \tilde{\varphi}_1, \tilde{\varphi}_2)$.

In principle, one could compactify one of the external dimensions on a circle, and rewrite the background solutions in $D=10$ dimensions, within type-IIA supergravity. The standard prescription to extract the field-theory behavior of the Wilson loop of the (lower dimensional) dual field theory, and the related static potential between heavy quarks [5,6], would require to compute the effect of the curved background on a probe string, allowed to develop extended configurations stretching in the space directions. We observe that in the presence of nontrivial solutions, for which the scalar fields have a nonvanishing bulk profile, $\phi_i \neq 0$, the metric of the internal, four-dimensional manifold is characterized by a warp factor, $\tilde{\Delta}^{-2/3}$, that depends on the internal angles parametrizing it, and is not a simple, smooth function. Because of the presence of the factor of $\tilde{\Delta}^{-2/3}$, performing this exercise would require one to solve nonlinear, coupled equations involving the internal angles, besides the radial direction, and doing so exceeds the scopes of this study. Furthermore, special singularities emerge in the internal part of the geometry of the solutions of interest in the body of this paper, and rendering the field-theory interpretation of the results less transparent than in Ref. [12].

APPENDIX D: GRAVITATIONAL INVARIANTS

The space-time geometry in $D = 7$ dimensions for the backgrounds we study in the body of the paper, which correspond to confining field theories in $D - 1 = 6$ dimensions compactified on a shrinking circle, is characterized by curvature invariants that can be written in the following form:

$$\mathcal{R} = -10A''(\rho) + 2\sqrt{10}A'(\rho)\chi'(\rho) - 30A'(\rho)^2 + \sqrt{\frac{2}{5}}\chi''(\rho) - \frac{11}{5}\chi'(\rho)^2 \quad (\text{D1})$$

$$= 20A'(\rho)^2 - \frac{1}{5}e^{\sqrt{\frac{2}{5}}(\phi_2(\rho)-4\chi(\rho))-\sqrt{2}\phi_1(\rho)}\left(e^{2\sqrt{2}\phi_1(\rho)}\mathcal{A}_7^{(1)'}(\rho)^2 + \mathcal{A}_7^{(2)'}(\rho)^2\right) - 2\chi'(\rho)^2 + \frac{48\mathcal{V}_7}{5} \quad (\text{D2})$$

$$= \frac{1}{10e^{32/5}((\varrho^4 - Q_1^2)(\varrho^4 - Q_2^2))^{6/5}} \left[15Q_1^4Q_2^4 + 48\mu\varrho^6(Q_1^2 + Q_2^2) - 32\mu Q_1^2Q_2^2\varrho^2 + 126\varrho^{12}(Q_1^2 + Q_2^2) \right. \\ \left. + 6Q_1^2Q_2^2\varrho^4(Q_1^2 + Q_2^2) - 3\varrho^8(3Q_1^4 + 52Q_1^2Q_2^2 + 3Q_2^4) - 105\varrho^{16} \right]. \quad (\text{D3})$$

Here, the first expression is valid as long as the vector fields all vanish in the background, the second has been obtained by making use of the equations of motion, and the third by making use of the exact soliton solutions, having changed variable in the holographic direction, but without imposing the constraint derived from the absence of conical singularities.

Upon imposing the constraint to avoid a conical singularity, the Ricci scalar calculated for solutions along the line of phase transition, where $\mu = 0$ (and $Q_1^2 = Q_2^2 = \varrho_0^4$) is given by

$$\mathcal{R}^{(\mu=0)} = -\frac{3(35\varrho^8 - 14\varrho^4\varrho_0^4 - 5\varrho_0^8)}{10\varrho^8\left(\frac{\varrho^4 - \varrho_0^4}{\varrho^8}\right)^{\frac{1}{5}}}. \quad (\text{D4})$$

This expression diverges at the end of space, when $\varrho = \varrho_0$. Conversely, in the case of the AdS_7 , domain-wall solutions, the Ricci scalar takes on the constant value $-\frac{21}{2}$.

We calculate three gravitational invariants, $\mathcal{R}$, $\mathcal{R}_{MN}\mathcal{R}^{MN}$, and $\mathcal{R}_{MNPQ}\mathcal{R}^{MNPQ}$, for the confining solutions. In Fig. 4 we display some representative examples for solutions close to (but not at) the phase transition. We also display the Ricci scalar, $\mathcal{R}$, for solutions with $\mu = 0$, which sit along the line of phase transition. We find that the confining solutions are regular, with all three invariants finite, away from the phase transition. However, the Ricci scalar diverges for solutions at the end of these branches, corresponding to the phase transition. The classical supergravity approximation in the limiting case of $\mu = 0$ is not reliable at this point, due to the large curvature.

APPENDIX E: GAUGE INVARIANT FORMALISM FOR THE FLUCTUATIONS

We summarize here the main equations entering the gauge-invariant treatment of the fluctuations of the

background, as developed in Refs. [45–53]. The sigma-model scalar fields, Φ^a , are rewritten as follows [45–49]:

$$\Phi^a(x^\mu, r) = \Phi^a(r) + \varphi^a(x^\mu, r), \quad (\text{E1})$$

where $\varphi^a(x^\mu, r)$ are small fluctuations around the background solutions, $\Phi^a(r)$. The metric fluctuations are decomposed by foliating the holographic direction as in the Arnowitt-Deser-Misner (ADM) formalism [215], as follows:

$$ds_D^2 = ((1 + \nu)^2 + \nu_\sigma \nu^\sigma)dr^2 + 2\nu_\mu dx^\mu dr \\ + e^{2A(r)}(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu, \quad (\text{E2})$$

where

$$h^\mu{}_\nu = e^\mu{}_\nu + iq^\mu \epsilon_\nu + iq_\nu \epsilon^\mu + \frac{q^\mu q_\nu}{q^2}H + \frac{1}{D-2}\delta^\mu{}_\nu h. \quad (\text{E3})$$

In these expressions, $\nu(x^\mu, r)$, $\nu^\mu(x^\mu, r)$, $e^\mu{}_\nu(x^\mu, r)$, $\epsilon^\mu(x^\mu, r)$, $H(x^\mu, r)$, and $h(x^\mu, r)$ are small fluctuations around the background metric. The gauge-invariant $\epsilon^\mu(x^\mu, r)$ and the traceless $e^\mu{}_\nu(x^\mu, r)$ are both transverse. The set of gauge-invariant (under diffeomorphism) combinations of the fluctuations can be written as follows:

$$\mathfrak{a}^a = \varphi^a - \frac{\partial_r \Phi^a}{2(D-2)\partial_r A}h, \quad (\text{E4})$$

$$\mathfrak{b} = \nu - \partial_r \left(\frac{h}{2(D-2)\partial_r A} \right), \quad (\text{E5})$$

$$\mathfrak{c} = e^{-2A}\partial_\mu \nu^\mu - \frac{e^{-2A}q^2 h}{2(D-2)\partial_r A} - \frac{1}{2}\partial_r H, \quad (\text{E6})$$

$$\mathfrak{d}^\mu = e^{-2A}P^\mu{}_\nu \nu^\nu - \partial_r \epsilon^\mu. \quad (\text{E7})$$

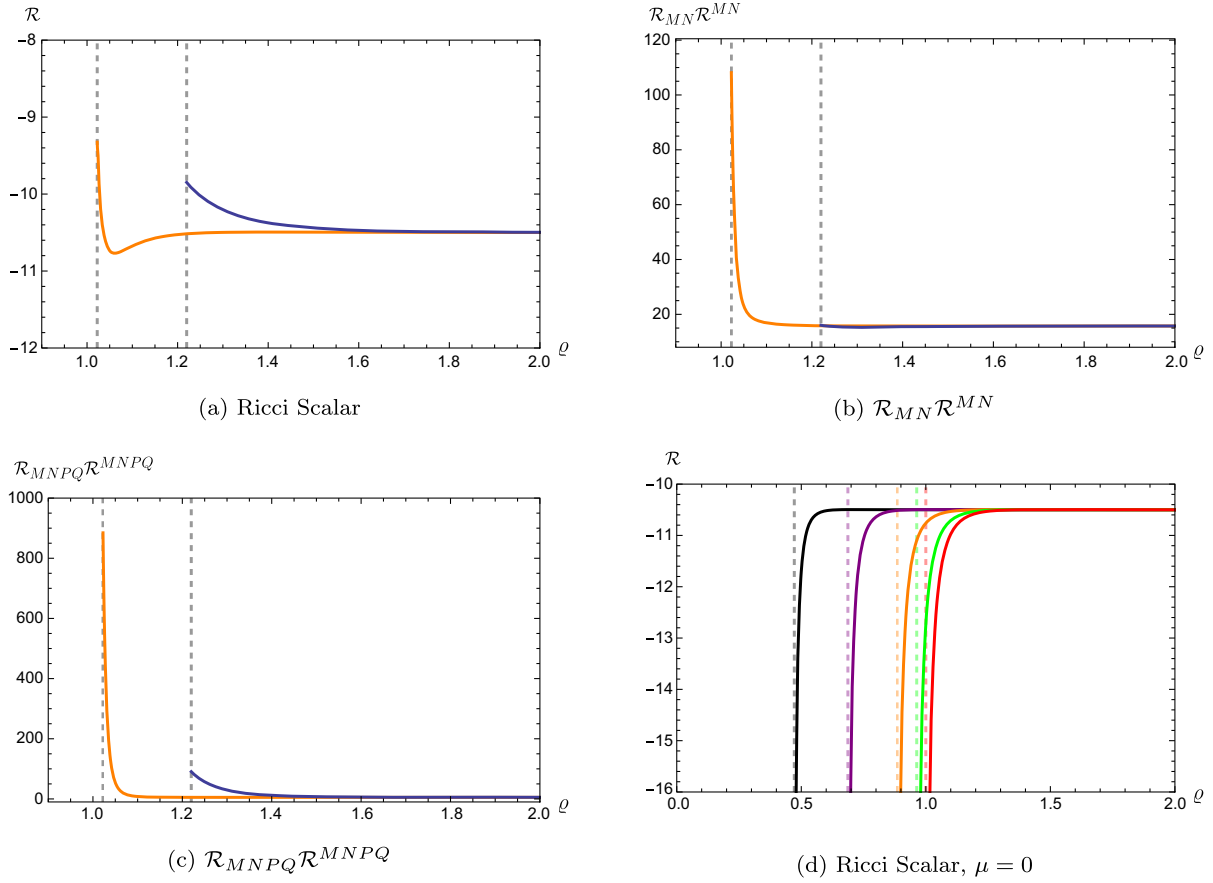


FIG. 4. Gravitational invariants for a selection of soliton (confining) solutions, as a function of the holographic direction, q . Panels (a), (b), and (c) show three gravitational invariants for two particular choices of regular confining solutions, parametrized by $\{\mathcal{A}_{7,U}^{(1)}, \mathcal{A}_{7,U}^{(2)}\} = \{0.396, 0.216\}$ and $\{0.796, 0.565\}$, in blue and orange respectively. The gray dashed vertical lines correspond to the value of q_0 , where the space ends for each solution. All invariants are found to be smooth and regular over the entire space. The bottom right panel, (d), shows the Ricci scalar calculated for a selection of solutions along the line of phase transitions, along which $\mu = 0$ and the free energy vanishes [the sides of the square in Fig. 2(a)]. The red, green, orange, purple and black lines correspond to soliton solutions for which the value of the angle $\theta = \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{9}, \frac{\pi}{20}$, and $\frac{\pi}{25}$ respectively. The dashed lines show the corresponding end of space for each solution, at which point the Ricci scalar diverges.

The equations of motion obeyed by $\mathfrak{h}$ and $\mathfrak{c}$ are algebraic, and can be decoupled in the treatment of the scalar fluctuations. The equation of motion for $\mathfrak{d}^\mu$ is also algebraic and does not lead to a spectrum of states.

The treatment of the fluctuations of the scalars is suitable to the adoption of a language that captures the curvature of the internal space, and makes use of its symmetries, in parallel to the space-time. One hence introduces the sigma-model connection, that descends from the sigma-model metric, G_{ab} , and the sigma-model derivative, $\partial_b = \frac{\partial}{\partial \Phi^b}$, to read

$$\mathcal{G}_{ab}^d \equiv \frac{1}{2} G^{dc} (\partial_a G_{cb} + \partial_b G_{ca} - \partial_c G_{ab}). \quad (\text{E8})$$

The sigma-model Riemann tensor captures the curvature of the internal sigma-model space. We use conventions in which we write

$$\mathcal{R}_{bcd}^a \equiv \partial_c \mathcal{G}_{bd}^a - \partial_d \mathcal{G}_{bc}^a + \mathcal{G}_{bd}^e \mathcal{G}_{ce}^a - \mathcal{G}_{bc}^e \mathcal{G}_{de}^a, \quad (\text{E9})$$

and hence the sigma-model covariant derivative is

$$D_b X^d{}_a \equiv \partial_b X^d{}_a + \mathcal{G}_{cb}^d X^c{}_a - \mathcal{G}_{ab}^c X^d{}_c. \quad (\text{E10})$$

Adopting this formalism, the equations of motion for $\mathfrak{a}^a$ are

$$\begin{aligned} 0 = & [\mathcal{D}_r^2 + (D-1)\partial_r A \mathcal{D}_r - e^{-2A} q^2] \mathfrak{a}^a \\ & - \left[\mathcal{V}^a{}_{|c} - \mathcal{R}_{bcd}^a \partial_r \Phi^b \partial_r \Phi^d + \frac{4(\partial_r \Phi^a \mathcal{V}^b + \mathcal{V}^a \partial_r \Phi^b) G_{bc}}{(D-2)\partial_r A} \right. \\ & \left. + \frac{16\mathcal{V} \partial_r \Phi^a \partial_r \Phi^b G_{bc}}{(D-2)^2 (\partial_r A)^2} \right] \mathfrak{a}^c, \end{aligned} \quad (\text{E11})$$

where $q^2 = \eta_{\mu\nu} q_\mu q_\nu = -M^2$, while the boundary conditions are given by

$$\frac{2e^{2A}\partial_r\Phi^a}{(D-2)q^2\partial_rA}\left[\partial_r\Phi^b\mathcal{D}_r-\frac{4\mathcal{V}\partial_r\Phi^b}{(D-2)\partial_rA}-\mathcal{V}^b\right]\mathbf{a}_b-\mathbf{a}^a\Big|_{r_i}=0. \quad (\text{E12})$$

In all these expressions, the background covariant derivative is $D_r\mathbf{a}^a \equiv \partial_r\mathbf{a}^a + \mathcal{G}^a_{bc}\partial_r\Phi^b\mathbf{a}^c$, while we use the shorthand $\mathcal{V}^a|_b \equiv \frac{\partial\mathcal{V}^a}{\partial\Phi^b} + \mathcal{G}^a_{bc}\mathcal{V}^c$.

The tensor fluctuations, $e^\mu{}_\nu$, must obey the simpler differential equations

$$[\partial_r^2 + (D-1)\partial_r A \partial_r - e^{-2A(r)}q^2]e^\mu{}_\nu = 0, \quad (\text{E13})$$

with boundary conditions given by

$$\partial_r e^\mu{}_\nu|_{r=r_i} = 0. \quad (\text{E14})$$

1. Fluctuation equations for the scalars

We report here the explicit form of the fluctuation equations used for the numerical study the results of which are reported in the body of the paper, written in terms of the variable ρ . Primed variables stand for derivatives with respect to ρ , so that $g' = \partial_\rho g(\rho)$. The five gauge-invariant scalar fluctuations, $\{\mathbf{a}^{\phi_1}, \mathbf{a}^{\phi_2}, \mathbf{a}^\chi, \mathbf{a}^{\mathcal{A}_7^{(1)}}, \mathbf{a}^{\mathcal{A}_7^{(2)}}\}$, obey the following equations:

$$\begin{aligned} 0 = & \left[160e^{2A+\sqrt{2}\phi_1+\sqrt{\frac{5}{3}}(\phi_2+4\chi)}A'^2\partial_\rho^2 + 16e^{2A+\sqrt{2}\phi_1+\sqrt{\frac{5}{3}}(\phi_2+4\chi)}A'^2(50A' - \sqrt{10}\chi')\partial_\rho \right. \\ & + 5\left(32A'^2\left(e^{2A+\frac{\phi_1}{\sqrt{2}}+\frac{5\phi_2+8\chi}{\sqrt{10}}}(1+e^{\sqrt{2}\phi_1}) - e^{2A+2\sqrt{\frac{5}{3}}\phi_2}(e^{2\sqrt{2}\phi_1}(\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) + e^{\sqrt{2}\phi_1+\sqrt{\frac{5}{3}}(\phi_2+5\chi)}M^2 \right) \right. \\ & + 16\sqrt{2}e^{2A+\frac{\phi_1}{\sqrt{2}}+\frac{5\phi_2+8\chi}{\sqrt{10}}}(-1+e^{\sqrt{2}\phi_1})A'\phi_1' + e^{2A+\frac{\phi_1}{\sqrt{2}}+4\sqrt{\frac{5}{3}}\chi}\left(8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_2}{\sqrt{2}}+\sqrt{10}\phi_2} + 4e^{\sqrt{\frac{5}{3}}\phi_2}(1+e^{\sqrt{2}\phi_1}) \right)\phi_1^2 \Big) \Big] \mathbf{a}^{\phi_1} \\ & + \left[32\sqrt{5}e^{2\sqrt{\frac{5}{3}}\phi_2}A'^2\left(3e^{\frac{\phi_1}{\sqrt{2}}+\frac{\phi_2+8\chi}{\sqrt{10}}}(-1+e^{\sqrt{2}\phi_1}) - e^{2\sqrt{2}\phi_1}(\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2 \right) \right. \\ & + 5e^{4\sqrt{\frac{5}{3}}\chi(\rho)+\frac{\phi_1(\rho)}{\sqrt{2}}}\left(4(e^{\sqrt{2}\phi_1}+1)e^{\sqrt{\frac{5}{3}}\phi_2} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} + 8e^{\frac{\phi_1}{\sqrt{2}}} \right)\phi_1'\phi_2' \\ & + 16\sqrt{2}A'e^{\sqrt{2}\phi_1+\frac{8\chi+5\phi_2}{\sqrt{10}}}\left(\sqrt{5}\phi_1'\left(3\cosh\left(\frac{\phi_1}{\sqrt{2}} \right) + \sinh\left(\sqrt{\frac{5}{2}}\phi_2 \right) - 3\cosh\left(\sqrt{\frac{5}{2}}\phi_2 \right) \right) + 5\sinh\left(\frac{\phi_1}{\sqrt{2}} \right)\phi_2' \right) \Big] e^{2A}\mathbf{a}^{\phi_2} \\ & + \left[-160\sqrt{2}e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{5}{3}}\phi_2}(A')^2\mathcal{A}_7^{(1)}\partial_\rho + e^{2A+\frac{3\phi_1}{\sqrt{2}}+\sqrt{\frac{5}{3}}\phi_2}\left(20e^{\sqrt{\frac{5}{3}}\phi_2}(2\sqrt{2}(e^{\sqrt{2}\phi_1}-1)A' + (e^{\sqrt{2}\phi_1}+1)\phi_1')\mathcal{A}_7^{(1)'} \right. \right. \\ & - 16e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{\frac{5}{3}}\phi_2}(A')^2\mathcal{A}_7^{(1)'}(25\sqrt{2}A' + \sqrt{5}(2\phi_2'-9\chi') + 10\phi_1') + 5\sqrt{2}\mathcal{A}_7^{(1)''}) + 40e^{\frac{\phi_1}{\sqrt{2}}}\phi_1'\mathcal{A}_7^{(1)'} - 5e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2}\phi_1'\mathcal{A}_7^{(1)'} \Big) \Big] \mathbf{a}^{\mathcal{A}_7^{(1)}} \\ & \times \left[160\sqrt{2}e^{2A+2\sqrt{\frac{5}{3}}\phi_2}(A')^2\mathcal{A}_7^{(2)'}\partial_\rho + e^{2A+\sqrt{\frac{5}{3}}\phi_2}\left(40e^{\sqrt{\frac{5}{3}}\phi_2}\mathcal{A}_7^{(2)'}\left(2\sqrt{2}A'\sinh\left(\frac{\phi_1}{\sqrt{2}} \right) + \phi_1'\cosh\left(\frac{\phi_1}{\sqrt{2}} \right) \right) \right. \right. \\ & + 16e^{\sqrt{\frac{5}{3}}\phi_2}(A')^2\mathcal{A}_7^{(2)'}((25\sqrt{2}A' + \sqrt{5}(2\phi_2'-9\chi') - 10\phi_1') + 5\sqrt{2}\mathcal{A}_7^{(2)''}) - 5e^{\sqrt{10}\phi_2}\phi_1'\mathcal{A}_7^{(2)'} + 40\phi_1'\mathcal{A}_7^{(2)'} \Big) \Big] \mathbf{a}^{\mathcal{A}_7^{(2)}} \\ & + \left[-16\sqrt{5}e^{2\sqrt{\frac{5}{3}}\phi_2}(A')^2\left(-2e^{2\sqrt{2}\phi_1}(\mathcal{A}_7^{(1)'})^2 + 2(\mathcal{A}_7^{(2)'})^2 + (e^{\sqrt{2}\phi_1}-1)e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right. \\ & + \sqrt{2}A'e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}}\left(40e^{\sqrt{\frac{5}{3}}\phi_2}(e^{\sqrt{2}\phi_1}-1)\chi' - \sqrt{5}\left(4e^{\sqrt{\frac{5}{3}}\phi_2}(e^{\sqrt{2}\phi_1}+1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right)\phi_1' \right) \Big) \\ & + 5\left(4e^{\sqrt{\frac{5}{3}}\phi_2}(e^{\sqrt{2}\phi_1}+1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right)\chi'e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}}\phi_1' \Big] 4e^{2A}\mathbf{a}^\chi, \end{aligned} \quad (\text{E15})$$

$$\begin{aligned}
0 = & \left[32\sqrt{5}e^2\sqrt{\frac{5}{3}}\phi_2(A')^2 \left(-e^{2\sqrt{2}\phi_1}(\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2 + 3(e^{\sqrt{2}\phi_1} - 1)e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right. \\
& + 16\sqrt{2}A'e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\sqrt{2}\phi_1} \left(5\phi_2' \sinh\left(\frac{\phi_1}{\sqrt{2}}\right) + \sqrt{5}\phi_1' \left(\sinh\left(\sqrt{\frac{5}{2}}\phi_2\right) + 3\cosh\left(\frac{\phi_1}{\sqrt{2}}\right) - 3\cosh\left(\sqrt{\frac{5}{2}}\phi_2\right) \right) \right) \\
& + 5 \left(4e^{\sqrt{\frac{5}{2}}\phi_2}(e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}}\phi_1'\phi_2' \Big] e^{2A}\mathbf{a}^{\phi_1} \\
& + \left[160(A')^2 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho^2 + 16(A')^2(50A' - \sqrt{10}\chi') e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho \right. \\
& + 32(A')^2 \left(-e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{5}{3}}\phi_2}(\mathcal{A}_7^{(1)'})^2 - e^{2A+2\sqrt{\frac{5}{3}}\phi_2}(\mathcal{A}_7^{(2)'})^2 + e^{4\sqrt{\frac{5}{3}}\chi+\sqrt{2}\phi_1} \left(5M^2 e^{\sqrt{\frac{5}{3}}(\chi+\phi_2)} - 8e^{2A}(2e^{\sqrt{10}\phi_2} - 1) \right) \right. \\
& + 9(e^{\sqrt{2}\phi_1} + 1)e^{2A+\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \Big) + 16\sqrt{10} \left(3e^{\sqrt{\frac{5}{2}}\phi_2}(e^{\sqrt{2}\phi_1} + 1) - 4e^{\frac{\phi_1}{\sqrt{2}}} - 2e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) A'\phi_2' e^{2A+4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \\
& + 5 \left(4e^{\sqrt{\frac{5}{2}}\phi_2}(e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) (\phi_2')^2 e^{2A+4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \Big] \mathbf{a}^{\phi_2} \\
& + \left[32\sqrt{10}e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{5}{3}}\phi_2}(A')^2\mathcal{A}_7^{(1)'}\partial_\rho \right. \\
& + e^{2A+\frac{3\phi_1}{\sqrt{2}}+\sqrt{\frac{5}{3}}\phi_2} \left(-8e^{\frac{\phi_1}{\sqrt{2}}}(4\sqrt{10}A' - 5\phi_2')\mathcal{A}_7^{(1)'} + 4e^{\sqrt{\frac{5}{2}}\phi_2}(e^{\sqrt{2}\phi_1} + 1)(6\sqrt{10}A' + 5\phi_2')\mathcal{A}_7^{(1)'} \right. \\
& \left. \left. - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2}(16\sqrt{10}A' + 5\phi_2')\mathcal{A}_7^{(1)'} - 16e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{\frac{5}{3}}\phi_2}(A')^2(\mathcal{A}_7^{(1)'})'(5\sqrt{10}A' - 9\chi' + 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10}\mathcal{A}_7^{(1)''}) \right) \right] \mathbf{a}^{\mathcal{A}_7^{(1)'}} \\
& + \left[-32\sqrt{10}(A')^2\mathcal{A}_7^{(2)'}\partial_\rho + 8(5\phi_2' - 4\sqrt{10}A')\mathcal{A}_7^{(2)'}(\rho) - e^{\sqrt{10}\phi_2}(16\sqrt{10}A' + 5\phi_2')\mathcal{A}_7^{(2)'} \right. \\
& + 8e^{\sqrt{\frac{5}{2}}\phi_2} \cosh\left(\frac{\phi_1}{\sqrt{2}}\right) (6\sqrt{10}A' + 5\phi_2')\mathcal{A}_7^{(2)'} - 16e^{\sqrt{\frac{5}{3}}\phi_2}(A')^2(\mathcal{A}_7^{(2)'})'(5\sqrt{10}A' - 9\chi' - 2\sqrt{5}\phi_1' + 2\phi_2') \\
& \left. + \sqrt{10}\mathcal{A}_7^{(2)''}) \right] e^{2A+2\sqrt{\frac{5}{3}}\phi_2}\mathbf{a}^{\mathcal{A}_7^{(2)'}} \\
& + \left[32e^{2\sqrt{\frac{5}{3}}\phi_2}(A')^2(e^{2\sqrt{2}\phi_1}(\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) + e^{\sqrt{\frac{5}{3}}(4\chi+5\phi_2)+\sqrt{2}\phi_1}(\sqrt{10}A'(\phi_2' - 16\chi') + 32(A')^2 - 5\chi'\phi_2') \right. \\
& - 4(e^{\sqrt{2}\phi_1} + 1)e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}}(\sqrt{10}A'(\phi_2' - 6\chi') + 12(A')^2 - 5\chi'\phi_2') \\
& \left. + 8e^{4\sqrt{\frac{5}{3}}\chi+\sqrt{2}\phi_1}(-\sqrt{10}A'(4\chi' + \phi_2') + 8(A')^2 + 5\chi'\phi_2') \right] 4e^{2A}\mathbf{a}^\chi, \tag{E16}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[16\sqrt{5}e^2\sqrt{\frac{5}{3}}\phi_2(A')^2 \left(-2e^{2\sqrt{2}\phi_1}(\mathcal{A}_7^{(1)'})^2 + 2(\mathcal{A}_7^{(2)'})^2 + (e^{\sqrt{2}\phi_1} - 1)e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right. \\
& + \sqrt{2}A'e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\sqrt{2}\phi_1} \left(\sqrt{5}\phi_1' \left(-9\sinh\left(\sqrt{\frac{5}{2}}\phi_2\right) + 8\cosh\left(\frac{\phi_1}{\sqrt{2}}\right) + 7\cosh\left(\sqrt{\frac{5}{2}}\phi_2\right) \right) - 80\chi' \sinh\left(\frac{\phi_1}{\sqrt{2}}\right) \right) \\
& - 5 \left(4e^{\sqrt{\frac{5}{2}}\phi_2}(e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) \chi' e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}}\phi_1' \Big] e^{2A}\mathbf{a}^{\phi_1} \\
& + \left[32e^{2\sqrt{\frac{5}{3}}\phi_2}(A')^2(e^{2\sqrt{2}\phi_1}(\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) + e^{\sqrt{\frac{5}{3}}(4\chi+5\phi_2)+\sqrt{2}\phi_1}(\sqrt{10}A'(\phi_2' - 16\chi') + 32(A')^2 - 5\chi'\phi_2') \right. \\
& - 4(e^{\sqrt{2}\phi_1} + 1)e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}}(\sqrt{10}A'(\phi_2' - 6\chi') + 12(A')^2 - 5\chi'\phi_2') \\
& \left. + 8e^{4\sqrt{\frac{5}{3}}\chi+\sqrt{2}\phi_1}(-\sqrt{10}A'(4\chi' + \phi_2') + 8(A')^2 + 5\chi'\phi_2') \right] e^{2A}\mathbf{a}^{\phi_2}
\end{aligned}$$

$$\begin{aligned}
& + \left[32\sqrt{10}e^{2A+2\sqrt{2}\phi_1+2\sqrt{5}\phi_2}(A')^2\mathcal{A}_7^{(1)'}\partial_\rho + e^{2A+\frac{3\phi_1}{\sqrt{2}}+\sqrt{5}\phi_2} \left(-8e^{\frac{\phi_1}{\sqrt{2}}}(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(1)'} + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2}(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(1)'} \right. \right. \\
& \left. \left. - 4e^{\sqrt{5}\phi_2}(e^{\sqrt{2}\phi_1} + 1)(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(1)'} + 16e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{5}\phi_2}(A')^2(\mathcal{A}_7^{(1)'}(5\sqrt{10}A' - 9\chi' + 2\sqrt{5}\phi'_1 + 2\phi'_2) + \sqrt{10}\mathcal{A}_7^{(1)''}) \right) \right] \mathbf{a}^{\mathcal{A}_7^{(1)}} \\
& + \left[32\sqrt{10}\phi_2(A')^2\mathcal{A}_7^{(2)'}\partial_\rho - 8(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(2)'} + e^{\sqrt{10}\phi_2}(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(2)'} - 8e^{\sqrt{5}\phi_2}\cosh\left(\frac{\phi_1}{\sqrt{2}}\right)(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(2)'} \right. \\
& \left. + 16e^{\sqrt{5}\phi_2}(A')^2(\mathcal{A}_7^{(2)'}(5\sqrt{10}A' - 9\chi' - 2\sqrt{5}\phi'_1 + 2\phi'_2) + \sqrt{10}\mathcal{A}_7^{(2)''}) \right] e^{2A+2\sqrt{5}\phi_2}\mathbf{a}^{\mathcal{A}_7^{(2)}} \\
& + \left[160(A')^2e^{2A+\sqrt{5}(4\chi+\phi_2)+\sqrt{2}\phi_1}\partial_\rho^2 + 16(A')^2(50A' - \sqrt{10}\chi')e^{2A+\sqrt{5}(4\chi+\phi_2)+\sqrt{2}\phi_1}\partial_\rho \right. \\
& + 8(A')^2 \left(-16e^{2A+2\sqrt{2}\phi_1+2\sqrt{5}\phi_2}(\mathcal{A}_7^{(1)'})^2 - 16e^{2A+2\sqrt{5}\phi_2}(\mathcal{A}_7^{(2)'})^2 + e^{4\sqrt{5}\phi_2+\sqrt{2}\phi_1} \left(20M^2e^{\sqrt{5}(4\chi+\phi_2)} - e^{2A}(e^{\sqrt{10}\phi_2} - 8) \right) \right. \\
& + 4(e^{\sqrt{2}\phi_1} + 1)e^{2A+\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} - 8\sqrt{10} \left(4e^{\sqrt{5}\phi_2}(e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) A'\chi'e^{2A+4\sqrt{5}\phi_2+\frac{\phi_1}{\sqrt{2}}} \\
& \left. + 20 \left(4e^{\sqrt{5}\phi_2}(e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) (\chi')^2 e^{2A+4\sqrt{5}\phi_2+\frac{\phi_1}{\sqrt{2}}} \right] \mathbf{a}^\chi, \tag{E17}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[160\sqrt{2}(A')^2e^{2A+\sqrt{5}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}}\mathcal{A}_7^{(1)'}\partial_\rho + 4e^{2A+4\sqrt{5}\phi_2+\sqrt{2}\phi_1} \left(-8e^{\frac{\phi_1}{\sqrt{2}}}(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(1)'} \right. \right. \\
& + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2}(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(1)'} - 4e^{\sqrt{5}\phi_2}(e^{\sqrt{2}\phi_1} + 1)(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(1)'} \\
& \left. \left. - 16e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{5}\phi_2}(A')^2(\mathcal{A}_7^{(1)'}(5\sqrt{10}A' - 9\chi' + 2\sqrt{5}\phi'_1 + 2\phi'_2) + \sqrt{10}\mathcal{A}_7^{(1)''}) \right) \right] \mathbf{a}^{\phi_1} \\
& + \left[32\sqrt{10}(A')^2e^{2A+\sqrt{5}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}}\mathcal{A}_7^{(1)'}\partial_\rho + \mathbf{a}^{\phi_2}(\rho)e^{2A+4\sqrt{5}\phi_2+\sqrt{2}\phi_1} \left(-8e^{\frac{\phi_1}{\sqrt{2}}}(4\sqrt{10}A' - 5\phi'_2)\mathcal{A}_7^{(1)'} \right. \right. \\
& + 4e^{\sqrt{5}\phi_2}(e^{\sqrt{2}\phi_1} + 1)(6\sqrt{10}A' + 5\phi'_2)\mathcal{A}_7^{(1)'} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2}(16\sqrt{10}A' + 5\phi'_2)\mathcal{A}_7^{(1)'} \\
& \left. \left. + 16e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{5}\phi_2}(A')^2(\mathcal{A}_7^{(1)'}(5\sqrt{10}A' - 9\chi' + 2\sqrt{5}\phi'_1 + 2\phi'_2) + \sqrt{10}\mathcal{A}_7^{(1)''}) \right) \right] \mathbf{a}^{\phi_2} \\
& + \left[16(A')^2e^{2A+\sqrt{5}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}}(10\partial_\rho^2 + (50A' + \sqrt{10}(2\phi'_2 - 9\chi') + 10\sqrt{2}\phi'_1)\partial_\rho) \right. \\
& + e^{\frac{3\phi_1}{\sqrt{2}}} \left(16e^{\sqrt{5}\phi_2}(A')^2(-10e^{2A+\sqrt{2}\phi_1+\sqrt{5}\phi_2}(\mathcal{A}_7^{(1)'})^2 \right. \\
& + e^{2A+4\sqrt{5}\phi_2}(-4\sqrt{10}\chi''(\rho) + 5\sqrt{2}\phi_1''(\rho) + \sqrt{10}\phi_2''(\rho) + \chi'(4\chi' - \sqrt{5}\phi'_1 - \phi'_2) \\
& - 5e^{4\sqrt{5}\phi_2} + 20e^{\frac{\phi_1}{\sqrt{2}}+\frac{3\phi_2}{\sqrt{10}}}) + 10M^2e^{\sqrt{10}\chi}) + 80\sqrt{2}(A')^3e^{2A+\sqrt{5}(4\chi+\phi_2)}(\sqrt{5}(\phi'_2 - 4\chi') + 5\phi'_1) \\
& \left. \left. + 5e^{2A+\sqrt{2}\phi_1+\frac{7\phi_2}{\sqrt{10}}}(\mathcal{A}_7^{(1)'})^2 \left(-9\sinh\left(\sqrt{\frac{5}{2}}\phi_2\right) + 8\cosh\left(\frac{\phi_1}{\sqrt{2}}\right) + 7\cosh\left(\sqrt{\frac{5}{2}}\phi_2\right) \right) \right) \right] \mathbf{a}^{\mathcal{A}_7^{(1)}} \\
& + \left[-5 \left(-4e^{\sqrt{5}\phi_2}(e^{\sqrt{2}\phi_1} + 1) - 8e^{\frac{\phi_1}{\sqrt{2}}} + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{2A+\sqrt{5}\phi_2}\mathcal{A}_7^{(1)'}\mathcal{A}_7^{(2)'} \right] \mathbf{a}^{\mathcal{A}_7^{(2)}} \\
& + \left[128\sqrt{10}(A')^2e^{2A+\sqrt{5}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}}\mathcal{A}_7^{(1)'}\partial_\rho + 4e^{2A+4\sqrt{5}\phi_2+\sqrt{2}\phi_1} \left(-8e^{\frac{\phi_1}{\sqrt{2}}}(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(1)'} \right. \right. \\
& + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2}(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(1)'} - 4e^{\sqrt{5}\phi_2}(e^{\sqrt{2}\phi_1} + 1)(\sqrt{10}A' - 5\chi')\mathcal{A}_7^{(1)'} \\
& \left. \left. - 16e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{5}\phi_2}(A')^2(\mathcal{A}_7^{(1)'}(5\sqrt{10}A' - 9\chi' + 2\sqrt{5}\phi'_1 + 2\phi'_2) + \sqrt{10}\mathcal{A}_7^{(1)''}) \right) \right] \mathbf{a}^\chi, \tag{E18}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[-160\sqrt{2}(A')^2 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \mathcal{A}_7^{(2)'} \partial_\rho + e^{2A+4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \left(-400\sqrt{2} e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{\frac{5}{3}}\phi_2} (A')^3 \mathcal{A}_7^{(2)'} \right. \right. \\
& + 40\sqrt{2} e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} - 1) A' \mathcal{A}_7^{(2)'} + 16 e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{\frac{5}{3}}\phi_2} (A')^2 (\mathcal{A}_7^{(2)'})' (\sqrt{5}(9\chi' - 2\phi_2') + 10\phi_1') - 5\sqrt{2} \mathcal{A}_7^{(2)''}) \\
& \left. + 5 \left(4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) \phi_1' \mathcal{A}_7^{(2)'} \right] \mathbf{a}^{\phi_1} \\
& + \left[32\sqrt{10}(A')^2 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \mathcal{A}_7^{(2)'} \partial_\rho + e^{2A+4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \left(-8e^{\frac{\phi_1}{\sqrt{2}}}(4\sqrt{10}A' - 5\phi_2') \mathcal{A}_7^{(2)'} \right. \right. \\
& + 4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) (6\sqrt{10}A' + 5\phi_2') \mathcal{A}_7^{(2)'} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} (16\sqrt{10}A' + 5\phi_2') \mathcal{A}_7^{(2)'} \\
& \left. + 16e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{\frac{5}{3}}\phi_2} (A')^2 (\mathcal{A}_7^{(2)'})' (5\sqrt{10}A' - 9\chi' - 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10} \mathcal{A}_7^{(2)''}) \right] \mathbf{a}^{\phi_2} \\
& + \left[5e^{2A+2\sqrt{2}\phi_1+\frac{7\phi_2}{\sqrt{10}}} \mathcal{A}_7^{(1)'} \mathcal{A}_7^{(2)'} \left(-9 \sinh\left(\sqrt{\frac{5}{2}}\phi_2\right) + 8 \cosh\left(\frac{\phi_1}{\sqrt{2}}\right) + 7 \cosh\left(\sqrt{\frac{5}{2}}\phi_2\right) \right) \right] \mathbf{a}^{\mathcal{A}_7^{(1)}} \\
& + \left[16e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} (10(A')^2 \partial_\rho^2 + (A')^2 (50A' + \sqrt{10}(2\phi_2' - 9\chi') - 10\sqrt{2}\phi_1') \partial_\rho) \right. \\
& + 16e^{\sqrt{\frac{5}{3}}\phi_2} (A')^2 \left(-10e^{\sqrt{\frac{5}{3}}\phi_2} \mathcal{A}_7^{(2)'} \right)^2 \\
& + e^{2A+4\sqrt{\frac{5}{3}}\chi+\sqrt{2}\phi_1} \left(-5\sqrt{2}\phi_1''(\rho) + \sqrt{10}(\phi_2''(\rho) - 4\chi''(\rho)) + \chi'(4\chi' + \sqrt{5}\phi_1' - \phi_2') - 5e^4\sqrt{\frac{5}{3}}\phi_2 \right) \\
& + 20e^{2A+4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}+\frac{3\phi_2}{\sqrt{10}}} + 10M^2 e^{\sqrt{2}(\sqrt{5}\chi+\phi_1)} \left. - 80\sqrt{2}(A')^3 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} (5\phi_1' - \sqrt{5}(\phi_2' - 4\chi')) \right. \\
& \left. - 5e^{2A+\sqrt{\frac{5}{3}}\phi_2} \left(e^{\sqrt{10}\phi_2} - 8e^{\sqrt{\frac{5}{3}}\phi_2} \cosh\left(\frac{\phi_1}{\sqrt{2}}\right) - 8 \right) (\mathcal{A}_7^{(2)'})^2 \right] \mathbf{a}^{\mathcal{A}_7^{(2)}} \\
& + \left[-128\sqrt{10}(A')^2 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \mathcal{A}_7^{(2)'} \partial_\rho + 4e^{2A+4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \left(-8e^{\frac{\phi_1}{\sqrt{2}}}(\sqrt{10}A' - 5\chi') \mathcal{A}_7^{(2)'} \right. \right. \\
& + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} (\sqrt{10}A' - 5\chi') \mathcal{A}_7^{(2)'} - 4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) (\sqrt{10}A' - 5\chi') \mathcal{A}_7^{(2)'} \\
& \left. - 16e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{\frac{5}{3}}\phi_2} (A')^2 (\mathcal{A}_7^{(2)'})' (5\sqrt{10}A' - 9\chi' - 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10} \mathcal{A}_7^{(2)''}) \right] \mathbf{a}^\chi. \tag{E19}
\end{aligned}$$

The boundary conditions for the five gauge-invariant, scalar fluctuations are given by the following expressions:

$$\begin{aligned}
0 = & \left[40A'(\phi'_1)^2 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho + 20\sqrt{2}e^{2A+2\sqrt{\frac{5}{3}}\phi_2} A' \phi'_1 \left(e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 - (\mathcal{A}_7^{(2)'})^2 + 2(e^{\sqrt{2}\phi_1} - 1)e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right. \\
& + 320M^2(A')^2 e^{\sqrt{\frac{5}{3}}(5\chi+\phi_2)+\sqrt{2}\phi_1} + 5 \left(4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) (\phi'_1)^2 e^{2A+4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \left. \right] \mathbf{a}^{\phi_1} \Big|_{\rho=\rho_i} \\
& + \left[40A' \phi'_1 \phi'_2 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho + e^{2A} \phi'_1 \left(4\sqrt{10}e^{2\sqrt{\frac{5}{3}}\phi_2} A' (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) - 8e^{4\sqrt{\frac{5}{3}}\chi+\sqrt{2}\phi_1} (4\sqrt{10}A' - 5\phi'_2) \right. \right. \\
& + 4(e^{\sqrt{2}\phi_1} + 1)e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} (6\sqrt{10}A' + 5\phi'_2) - e^{\sqrt{\frac{5}{3}}(4\chi+5\phi_2)+\sqrt{2}\phi_1} (16\sqrt{10}A' + 5\phi'_2) \left. \left. \right) \right] \mathbf{a}^{\phi_2} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{5}{3}}\phi_2} A' \phi'_1 \mathcal{A}_7^{(1)'} \partial_\rho \right. \\
& + 5e^{2A+2\sqrt{2}\phi_1+\frac{7\phi_2}{\sqrt{10}}} \phi'_1 \mathcal{A}_7^{(1)'} \left(-9 \sinh \left(\sqrt{\frac{5}{2}}\phi_2 \right) + 8 \cosh \left(\frac{\phi_1}{\sqrt{2}} \right) + 7 \cosh \left(\sqrt{\frac{5}{2}}\phi_2 \right) \right) \left. \right] \mathbf{a}^{\mathcal{A}_7^{(1)}} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+2\sqrt{\frac{5}{3}}\phi_2} A' \phi'_1 \mathcal{A}_7^{(2)'}(\rho) \partial_\rho - 5e^{2A+\sqrt{\frac{5}{3}}\phi_2} \phi'_1 \left(e^{\sqrt{10}\phi_2} - 8e^{\sqrt{\frac{5}{3}}\phi_2} \cosh \left(\frac{\phi_1}{\sqrt{2}} \right) - 8 \right) \mathcal{A}_7^{(2)'}(\rho) \right] \mathbf{a}^{\mathcal{A}_7^{(2)}} \Big|_{\rho=\rho_i} \\
& + \left[160A' \chi' \phi'_1 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho + 4e^{2A} \phi'_1 ((-4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) \right. \\
& - 8e^{\frac{\phi_1}{\sqrt{2}}} + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2}) e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} (\sqrt{10}A' - 5\chi') - 4\sqrt{10}e^{2\sqrt{\frac{5}{3}}\phi_2} A' (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) \left. \right] \mathbf{a}^\chi \Big|_{\rho=\rho_i}, \tag{E20}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[40A' \phi'_1 \phi'_2 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho + 5e^{2A} \phi'_2 \left(4\sqrt{2}e^{2\sqrt{\frac{5}{3}}\phi_2} A' \left(e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 - (\mathcal{A}_7^{(2)'})^2 + 2(e^{\sqrt{2}\phi_1} - 1)e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right. \right. \\
& + \left(4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \phi'_1 \left. \left. \right) \right] \mathbf{a}^{\phi_1} \Big|_{\rho=\rho_i} \\
& + \left[40A'(\phi'_2)^2 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho + 4\sqrt{10}e^{2A} A' \phi'_2 (e^{2\sqrt{\frac{5}{3}}\phi_2} (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) \right. \\
& + 2 \left(3e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) - 4e^{\frac{\phi_1}{\sqrt{2}}} - 2e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} + 320M^2(A')^2 e^{\sqrt{\frac{5}{3}}(5\chi+\phi_2)+\sqrt{2}\phi_1} \\
& + 5 \left(4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) (\phi'_2)^2 e^{2A+4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \left. \right] \mathbf{a}^{\phi_2} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{5}{3}}\phi_2} A' \phi'_2 \mathcal{A}_7^{(1)'} \partial_\rho + 5e^{2A+2\sqrt{2}\phi_1+\frac{7\phi_2}{\sqrt{10}}} \phi'_2 \mathcal{A}_7^{(1)'} \left(-9 \sinh \left(\sqrt{\frac{5}{2}}\phi_2 \right) \right. \right. \\
& + 8 \cosh \left(\frac{\phi_1}{\sqrt{2}} \right) + 7 \cosh \left(\sqrt{\frac{5}{2}}\phi_2 \right) \left. \left. \right) \right] \mathbf{a}^{\mathcal{A}_7^{(1)}} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+2\sqrt{\frac{5}{3}}\phi_2} A' \phi'_2 \mathcal{A}_7^{(2)'} \partial_\rho - 5e^{2A+\sqrt{\frac{5}{3}}\phi_2} \phi'_2 \left(e^{\sqrt{10}\phi_2} - 8e^{\sqrt{\frac{5}{3}}\phi_2} \cosh \left(\frac{\phi_1}{\sqrt{2}} \right) - 8 \right) \mathcal{A}_7^{(2)'} \right] \mathbf{a}^{\mathcal{A}_7^{(2)}} \Big|_{\rho=\rho_i} \\
& + \left[160A' \chi' \phi'_2 e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho + 4e^{2A} \phi'_2 ((-4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) - 8e^{\frac{\phi_1}{\sqrt{2}}} + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2}) e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} (\sqrt{10}A' - 5\chi') \right. \\
& - 4\sqrt{10}e^{2\sqrt{\frac{5}{3}}\phi_2} A' (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) \left. \right] \mathbf{a}^\chi \Big|_{\rho=\rho_i}, \tag{E21}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[40A'\chi'\phi_1' e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho + 5e^{2A}\chi' \left(4\sqrt{2}e^2\sqrt{\frac{2}{3}}\phi_2 A' \left(e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 - (\mathcal{A}_7^{(2)'})^2 + 2(e^{\sqrt{2}\phi_1} - 1)e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right. \right. \\
& + \left. \left(4e^{\sqrt{\frac{2}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{4\sqrt{\frac{2}{3}}\chi+\frac{\phi_1}{\sqrt{2}}}\phi_1' \right) \Big] \mathbf{a}^{\phi_1} \Big|_{\rho=\rho_i} \\
& + \left[40A'\chi'\phi_2' e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho + e^{2A}\chi' \left(4\sqrt{10}e^2\sqrt{\frac{2}{3}}\phi_2 A' (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) \right. \right. \\
& - 8e^{4\sqrt{\frac{2}{3}}\chi+\sqrt{2}\phi_1} (4\sqrt{10}A' - 5\phi_2') + 4(e^{\sqrt{2}\phi_1} + 1)e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} (6\sqrt{10}A' + 5\phi_2') \\
& \left. \left. - e^{\sqrt{\frac{2}{3}}(4\chi+5\phi_2)+\sqrt{2}\phi_1} (16\sqrt{10}A' + 5\phi_2') \right) \right] \mathbf{a}^{\phi_2} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{2}{3}}\phi_2} A'\chi'\mathcal{A}_7^{(1)'} \partial_\rho + 5e^{2A+2\sqrt{2}\phi_1+\frac{7\phi_2}{\sqrt{10}}} \chi'\mathcal{A}_7^{(1)'} \left(-9 \sinh \left(\sqrt{\frac{5}{2}}\phi_2 \right) \right. \right. \\
& + \left. \left. 8 \cosh \left(\frac{\phi_1}{\sqrt{2}} \right) + 7 \cosh \left(\sqrt{\frac{5}{2}}\phi_2 \right) \right) \right] \mathbf{a}^{\mathcal{A}_7^{(1)'}} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+2\sqrt{\frac{2}{3}}\phi_2} A'\chi'\mathcal{A}_7^{(2)'} \partial_\rho - 5e^{2A+\sqrt{\frac{2}{3}}\phi_2} \chi' \left(e^{\sqrt{10}\phi_2} - 8e^{\sqrt{\frac{2}{3}}\phi_2} \cosh \left(\frac{\phi_1}{\sqrt{2}} \right) - 8 \right) \mathcal{A}_7^{(2)'} \right] \mathbf{a}^{\mathcal{A}_7^{(2)'}} \Big|_{\rho=\rho_i} \\
& + \left[160A'(\chi')^2 e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} \partial_\rho + 4 \left(-\sqrt{10}e^{2A} A'\chi' \left(4e^2\sqrt{\frac{2}{3}}\phi_2 (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) \right. \right. \right. \\
& + \left. \left. \left(4e^{\sqrt{\frac{2}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{4\sqrt{\frac{2}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \right) + 80M^2(A')^2 e^{\sqrt{\frac{2}{3}}(5\chi+\phi_2)+\sqrt{2}\phi_1} \right. \\
& \left. \left. + 5 \left(4e^{\sqrt{\frac{2}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) (\chi')^2 e^{2A+4\sqrt{\frac{2}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \right) \right] \mathbf{a}^\chi \Big|_{\rho=\rho_i}, \tag{E22}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[40A'\phi_1' e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(1)'} \partial_\rho + 5e^{2A+\frac{\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(1)'} \left(4\sqrt{2}e^2\sqrt{\frac{2}{3}}\phi_2 A' \left(e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})'^2 - (\mathcal{A}_7^{(2)'})^2 \right. \right. \right. \\
& + \left. \left. 2(e^{\sqrt{2}\phi_1} - 1)e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) + \left(4e^{\sqrt{\frac{2}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{4\sqrt{\frac{2}{3}}\chi+\frac{\phi_1}{\sqrt{2}}}\phi_1' \right) \Big] \mathbf{a}^{\phi_1} \Big|_{\rho=\rho_i} \\
& + \left[40A'\phi_2' e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(1)'} \partial_\rho + e^{2A+\frac{\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(1)'} \left(4\sqrt{10}e^2\sqrt{\frac{2}{3}}\phi_2 A' \left(e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2 \right) \right. \right. \\
& - 8e^{4\sqrt{\frac{2}{3}}\chi+\sqrt{2}\phi_1} (4\sqrt{10}A' - 5\phi_2') + 4(e^{\sqrt{2}\phi_1} + 1)e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} (6\sqrt{10}A' + 5\phi_2') \\
& \left. \left. - e^{\sqrt{\frac{2}{3}}(4\chi+5\phi_2)+\sqrt{2}\phi_1} (16\sqrt{10}A' + 5\phi_2') \right) \right] \mathbf{a}^{\phi_2} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+\frac{5\phi_1}{\sqrt{2}}+2\sqrt{\frac{2}{3}}\phi_2} A'(\mathcal{A}_7^{(1)'})^2 \partial_\rho + 5e^{\frac{3\phi_1}{\sqrt{2}}+\sqrt{\frac{2}{3}}\phi_2} \left(64M^2 e^{\sqrt{10}\chi} (A')^2 \right. \right. \\
& + \left. \left. e^{2A} \left(8e^{\sqrt{2}\phi_1} + 4e^{\frac{\phi_1+\sqrt{5}\phi_2}{\sqrt{2}}} - e^{\sqrt{2}(\phi_1+\sqrt{5}\phi_2)} + 4e^{\frac{3\phi_1+\sqrt{5}\phi_2}{\sqrt{2}}} \right) (\mathcal{A}_7^{(1)'})^2 \right) \right] \mathbf{a}^{\mathcal{A}_7^{(1)'}} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+\frac{\phi_1}{\sqrt{2}}+2\sqrt{\frac{2}{3}}\phi_2} A'\mathcal{A}_7^{(1)'} \mathcal{A}_7^{(2)'} \partial_\rho - 5 \left(-4e^{\sqrt{\frac{2}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) - 8e^{\frac{\phi_1}{\sqrt{2}}} + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{2A+\sqrt{\frac{2}{3}}\phi_2} \mathcal{A}_7^{(1)'} \mathcal{A}_7^{(2)'} \right] \mathbf{a}^{\mathcal{A}_7^{(2)'}} \Big|_{\rho=\rho_i} \\
& + \left[160A'\chi' e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(1)'} \partial_\rho + 4e^{2A+\frac{\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(1)'} \left(\left(-4e^{\sqrt{\frac{2}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) - 8e^{\frac{\phi_1}{\sqrt{2}}} + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{4\sqrt{\frac{2}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} (\sqrt{10}A' - 5\chi') \right. \right. \\
& \left. \left. - 4\sqrt{10}e^2\sqrt{\frac{2}{3}}\phi_2 A' (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) \right) \right] \mathbf{a}^\chi \Big|_{\rho=\rho_i}, \tag{E23}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[40A'\phi_1' e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(2)'} \partial_\rho + 5e^{2A+\frac{\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(2)'} \left(4\sqrt{2}e^{2\sqrt{\frac{5}{3}}\phi_2} A' (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 - (\mathcal{A}_7^{(2)'})^2 + 2(e^{\sqrt{2}\phi_1} - 1)e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}}) \right. \right. \\
& + \left. \left(4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}}\phi_1' \right) \Big] \mathbf{a}^{\phi_1} \Big|_{\rho=\rho_i} \\
& + \left[40A'\phi_2' e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(2)'} \partial_\rho + e^{2A+\frac{\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(2)'} \left(4\sqrt{10}e^{2\sqrt{\frac{5}{3}}\phi_2} A' (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) \right. \right. \\
& - 8e^{4\sqrt{\frac{5}{3}}\chi+\sqrt{2}\phi_1} (4\sqrt{10}A' - 5\phi_2') + 4(e^{\sqrt{2}\phi_1} + 1)e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} (6\sqrt{10}A' + 5\phi_2') - e^{\sqrt{\frac{5}{3}}(4\chi+5\phi_2)+\sqrt{2}\phi_1} (16\sqrt{10}A' + 5\phi_2') \Big) \Big] \mathbf{a}^{\phi_2} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+\frac{5\phi_1}{\sqrt{2}}+2\sqrt{\frac{5}{3}}\phi_2} A' \mathcal{A}_7^{(1)'} \mathcal{A}_7^{(2)'} \partial_\rho \right. \\
& + 5e^{2A+\frac{5\phi_1}{\sqrt{2}}+\frac{7\phi_2}{\sqrt{10}}} \mathcal{A}_7^{(1)'} \mathcal{A}_7^{(2)'} \left(-9\sinh\left(\sqrt{\frac{5}{2}}\phi_2\right) + 8\cosh\left(\frac{\phi_1}{\sqrt{2}}\right) + 7\cosh\left(\sqrt{\frac{5}{2}}\phi_2\right) \right) \Big] \mathbf{a}^{\mathcal{A}_7^{(1)}} \Big|_{\rho=\rho_i} \\
& + \left[40e^{2A+\frac{\phi_1}{\sqrt{2}}+2\sqrt{\frac{5}{3}}\phi_2} A' (\mathcal{A}_7^{(2)'})^2 \partial_\rho \right. \\
& + 5e^{\sqrt{\frac{5}{3}}\phi_2} \left(64M^2 (A')^2 e^{\sqrt{10}\chi+\frac{3\phi_1}{\sqrt{2}}} + e^{2A} \left(4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) (\mathcal{A}_7^{(2)'})^2 \right) \Big] \mathbf{a}^{\mathcal{A}_7^{(2)}} \Big|_{\rho=\rho_i} \\
& + \left[160A'\chi' e^{2A+\sqrt{\frac{5}{3}}(4\chi+\phi_2)+\frac{3\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(2)'} \partial_\rho + 4e^{2A+\frac{\phi_1}{\sqrt{2}}} \mathcal{A}_7^{(2)'} \left(\left(-4e^{\sqrt{\frac{5}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) - 8e^{\frac{\phi_1}{\sqrt{2}}} + e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) e^{4\sqrt{\frac{5}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} (\sqrt{10}A' - 5\chi') \right. \right. \\
& - 4\sqrt{10}e^{2\sqrt{\frac{5}{3}}\phi_2} A' (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) \Big) \Big] \mathbf{a}^\chi \Big|_{\rho=\rho_i}. \tag{E24}
\end{aligned}$$

In the numerical routines written for the purposes of this paper, these equations are rewritten after the further change of variable to the ϱ parametrization of the holographic direction. We decided not to report such expressions, in view of the uninspiring complexity of their explicit form.

2. UV expansions

We provide the leading terms in the UV expansions of the fluctuations used to improve the numerical calculation of the spectrum, in terms of variable $\mathfrak{z} = \frac{1}{\varrho}$. In the numerical calculations of the spectra reported in the main body, we retained terms up to tenth order in these expansions. The expansions depend on ten free parameters: the dominant $\{\mathbf{a}_2^{\phi_1}, \mathbf{a}_2^{\phi_2}, \mathbf{a}_0^\chi, \mathbf{a}_0^{\mathcal{A}_7^{(1)}}, \mathbf{a}_0^{\mathcal{A}_7^{(2)}}\}$ and the subdominant $\{\mathbf{a}_4^{\phi_1}, \mathbf{a}_4^{\phi_2}, \mathbf{a}_6^\chi, \mathbf{a}_4^{\mathcal{A}_7^{(1)}}, \mathbf{a}_4^{\mathcal{A}_7^{(2)}}\}$.

$$\begin{aligned}
\mathbf{a}_{\text{UV}}^{\mathcal{A}_7^{(1)}}(\mathfrak{z}) = & \mathbf{a}_0^{\mathcal{A}_7^{(1)}} + 4\mathbf{a}_0^{\mathcal{A}_7^{(1)}} M^2 \mathfrak{z}^2 + \mathfrak{z}^4 (\mathbf{a}_4^{\mathcal{A}_7^{(1)}} - 16\mathbf{a}_0^{\mathcal{A}_7^{(1)}} M^4 \log[\mathfrak{z}]) \\
& + \mathfrak{z}^6 \left(\frac{64}{3} \mathbf{a}_0^{\mathcal{A}_7^{(1)}} M^6 \log[\mathfrak{z}] - \frac{2}{45} \sqrt{\mu} Q_1 \left(30\mathbf{a}_4^{\mathcal{A}_7^{(1)}} M^2 - 3\sqrt{2}(5\mathbf{a}_{UV,2}^{\phi_1} + \sqrt{5}(\mathbf{a}_2^{\phi_1} - 8\mathbf{a}_0^\chi M^2)) \right) \right. \\
& + 10\mathbf{a}_0^{\mathcal{A}_7^{(1)}} M^2 (32M^4 - 9Q_1^2 + 3Q_2^2) \Big) \\
& + \mathfrak{z}^8 \left(Q_1^2 \left(-16\mathbf{a}_0^{\mathcal{A}_7^{(1)}} M^4 \log(\mathfrak{z}) - 2\mathbf{a}_0^{\mathcal{A}_7^{(1)}} M^4 + \mathbf{a}_4^{\mathcal{A}_7^{(1)}} \right) \right. \\
& - \frac{2}{9} M^2 \left(\mathbf{a}_0^{\mathcal{A}_7^{(1)}} (6Q_2^2 M^2 + 9\mu - 50M^6) + 48\mathbf{a}_0^{\mathcal{A}_7^{(1)}} M^6 \log(\mathfrak{z}) - 3\mathbf{a}_4^{\mathcal{A}_7^{(1)}} M^2 \right) \\
& + \frac{Q_1 \sqrt{\mu}}{15\sqrt{2}} (M^2 (5\mathbf{a}_2^{\phi_1} + \sqrt{5}(\mathbf{a}_2^{\phi_2} - 32\mathbf{a}_0^\chi M^2)) - 24M^2 (5\mathbf{a}_2^{\phi_1} + \sqrt{5}\mathbf{a}_2^{\phi_2}) \log(\mathfrak{z}) + 15\mathbf{a}_4^{\phi_1} + 3\sqrt{5}\mathbf{a}_4^{\phi_2}) \Big)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\mathfrak{z}^{10}}{3375} \left(-24M^6 \left(5 \left(-120\mathfrak{a}_0^{A_7^{(1)}} Q_2^2 \log(\mathfrak{z}) + 22\mathfrak{a}_0^{A_7^{(1)}} Q_2^2 + 5\mathfrak{a}_4^{A_7^{(1)}} \right) \right. \right. \\
& + 15\mathfrak{a}_0^{A_7^{(1)}} Q_1^2 (139 - 240 \log(\mathfrak{z})) + 8\sqrt{10}\mathfrak{a}_0^\chi Q_1 \sqrt{\mu} (1 - 60 \log(\mathfrak{z})) \Big) \\
& - 90M^2 \left(10\mathfrak{a}_0^{A_7^{(1)}} (-15Q_1^4 + 5Q_1^2 Q_2^2 + Q_2^4) + 10\mathfrak{a}_4^{A_7^{(1)}} (6Q_1^2 + Q_2^2) + \sqrt{2}Q_1 \sqrt{\mu} (35\mathfrak{a}_4^{\phi_1} + \sqrt{5}(48\mathfrak{a}_0^\chi Q_1^2 + 7\mathfrak{a}_4^{\phi_2})) \right) \\
& + 18Q_1 \sqrt{\mu} \left(-240\sqrt{\mu}(\mathfrak{a}_0^{A_7^{(1)}} Q_1 + \mathfrak{a}_0^{A_7^{(2)}} Q_2) - 144\sqrt{10}\mathfrak{a}_0^\chi \mu \right. \\
& + \sqrt{2} \left(-60\sqrt{5}\mathfrak{a}_6^\chi + 5\mathfrak{a}_2^{\phi_1} (38Q_1^2 + 7Q_2^2) + \sqrt{5}\mathfrak{a}_2^{\phi_2} (38Q_1^2 - 37Q_2^2) \right) \Big) \\
& + 80\mathfrak{a}_0^{A_7^{(1)}} M^{10} (120 \log(\mathfrak{z}) - 157) - 12600\mathfrak{a}_0^{A_7^{(1)}} \mu M^4 + 6\sqrt{2}Q_1 \sqrt{\mu} M^4 (5\mathfrak{a}_2^{\phi_1} + \sqrt{5}\mathfrak{a}_2^{\phi_2}) (840 \log(\mathfrak{z}) - 649) \Big) \\
& + \mathcal{O}(\mathfrak{z}^{12}), \tag{E25}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{a}_{UV}^{A_7^{(2)}}(\mathfrak{z}) &= \mathfrak{a}_0^{A_7^{(2)}} + 4\mathfrak{a}_0^{A_7^{(2)}} M^2 \mathfrak{z}^2 + \mathfrak{z}^4 (\mathfrak{a}_4^{A_7^{(2)}} - 16\mathfrak{a}_0^{A_7^{(2)}} M^4 \log[\mathfrak{z}]) \\
&+ \mathfrak{z}^6 \left(\frac{64}{3} \mathfrak{a}_0^{A_7^{(2)}} M^6 \log[\mathfrak{z}] - \frac{2}{45} \sqrt{\mu} Q_1 \left(30\mathfrak{a}_4^{A_7^{(2)}} M^2 - 3\sqrt{2}(5\mathfrak{a}_{UV,2}^{\phi_1} + \sqrt{5}(\mathfrak{a}_2^{\phi_1} - 8\mathfrak{a}_0^\chi M^2)) \right) \right. \\
&+ 10\mathfrak{a}_0^{A_7^{(2)}} M^2 (32M^4 - 9Q_1^2 + 3Q_2^2) \Big) + \mathfrak{z}^8 \left(Q_2^2 \left(-16\mathfrak{a}_0^{A_7^{(2)}} M^4 \log(\mathfrak{z}) - 2\mathfrak{a}_0^{A_7^{(2)}} M^4 + \mathfrak{a}_4^{A_7^{(2)}} \right) \right. \\
&- \frac{2}{9} M^2 \left(\mathfrak{a}_0^{A_7^{(2)}} (6Q_1^2 M^2 + 9\mu - 50M^6) + 48\mathfrak{a}_0^{A_7^{(2)}} M^6 \log(\mathfrak{z}) - 3\mathfrak{a}_4^{A_7^{(2)}} M^2 \right) \\
&+ \frac{Q_2 \sqrt{\mu}}{15\sqrt{2}} (M^2 (-5\mathfrak{a}_2^{\phi_1} + \sqrt{5}(\mathfrak{a}_2^{\phi_2} - 32\mathfrak{a}_0^\chi M^2)) + 24M^2 (5\mathfrak{a}_2^{\phi_1} - \sqrt{5}\mathfrak{a}_2^{\phi_2}) \log(\mathfrak{z}) - 15\mathfrak{a}_4^{\phi_2} + 3\sqrt{5}\mathfrak{a}_4^{\phi_2}) \Big) \\
&+ \frac{\mathfrak{z}^{10}}{3375} \left(-24M^6 (5(-120\mathfrak{a}_0^{A_7^{(2)}} Q_1^2 \log(\mathfrak{z}) + 22\mathfrak{a}_0^{A_7^{(2)}} Q_1^2 + 5\mathfrak{a}_4^{A_7^{(2)}}) \right. \\
&+ 15\mathfrak{a}_0^{A_7^{(2)}} Q_2^2 (139 - 240 \log(\mathfrak{z})) + 8\sqrt{10}\mathfrak{a}_0^\chi Q_2 \sqrt{\mu} (1 - 60 \log(\mathfrak{z})) \Big) \\
&- 90M^2 \left(10\mathfrak{a}_0^{A_7^{(2)}} (-15Q_2^4 + 5Q_1^2 Q_2^2 + Q_1^4) + 10\mathfrak{a}_4^{A_7^{(2)}} (6Q_2^2 + Q_1^2) + \sqrt{2}Q_2 \sqrt{\mu} (-35\mathfrak{a}_4^{\phi_1} + \sqrt{5}(48\mathfrak{a}_0^\chi Q_2^2 + 7\mathfrak{a}_4^{\phi_2})) \right) \\
&+ 18Q_2 \sqrt{\mu} (-240\sqrt{\mu}(\mathfrak{a}_0^{A_7^{(1)}} Q_1 + \mathfrak{a}_0^{A_7^{(2)}} Q_2) - 144\sqrt{10}\mathfrak{a}_0^\chi \mu - \sqrt{2}(60\sqrt{5}\mathfrak{a}_6^\chi + 5\mathfrak{a}_2^{\phi_1} (38Q_2^2 + 7Q_1^2) \\
&+ \sqrt{5}\mathfrak{a}_2^{\phi_2} (-38Q_2^2 + 37Q_1^2))) + 80\mathfrak{a}_0^{A_7^{(2)}} M^{10} (120 \log(\mathfrak{z}) - 157) - 12600\mathfrak{a}_0^{A_7^{(2)}} \mu M^4 \\
&+ 6\sqrt{2}Q_2 \sqrt{\mu} M^4 (5\mathfrak{a}_2^{\phi_1} - \sqrt{5}\mathfrak{a}_2^{\phi_2}) (840 \log(\mathfrak{z}) - 649) + \mathcal{O}(\mathfrak{z}^{12}), \tag{E26}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{a}_{UV}^{\phi_1}(\mathfrak{z}) &= \mathfrak{a}_2^{\phi_1} \mathfrak{z}^2 + \mathfrak{z}^4 (\mathfrak{a}_4^{\phi_1} - 8\mathfrak{a}_2^{\phi_1} M^2 \log[\mathfrak{z}]) \\
&+ \mathfrak{z}^6 \left(16\mathfrak{a}_2^{\phi_1} M^4 \log[\mathfrak{z}] - 2\mathfrak{a}_4^{\phi_1} M^2 + \frac{3}{2\sqrt{5}} \mathfrak{a}_2^{\phi_2} (Q_1 - Q_2)(Q_1 + Q_2) + \frac{1}{2} \mathfrak{a}_2^{\phi_1} (Q_1^2 + Q_2^2 - 24M^4) \right) \\
&+ \frac{\mathfrak{z}^8}{90} (-12M^2 (40\sqrt{2}\sqrt{\mu}(\mathfrak{a}_0^{A_7^{(1)}} Q_1 - \mathfrak{a}_0^{A_7^{(2)}} Q_2) + 5\mathfrak{a}_2^{\phi_1} (Q_1^2 + Q_2^2) + \sqrt{5}\mathfrak{a}_2^{\phi_2} (Q_2^2 - Q_1^2)) \\
&- 24M^2 \log(\mathfrak{z}) (5\mathfrak{a}_2^{\phi_1} (3(Q_1^2 + Q_2^2) + 8M^4) + 3\sqrt{5}\mathfrak{a}_2^{\phi_2} (Q_1 - Q_2)(Q_1 + Q_2)) + 60\mathfrak{a}_2^{\phi_1} \mu + 1120\mathfrak{a}_2^{\phi_1} M^6 \\
&+ 15\mathfrak{a}_4^{\phi_1} (3(Q_1^2 + Q_2^2) + 8M^4) + 9\sqrt{5}\mathfrak{a}_4^{\phi_2} (Q_1 - Q_2)(Q_1 + Q_2) \Big)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\mathfrak{z}^{10}}{1350} \left(3\sqrt{5}\mathfrak{a}_2^{\phi_2}(\mathcal{Q}_1 - \mathcal{Q}_2)(\mathcal{Q}_1 + \mathcal{Q}_2)(57(\mathcal{Q}_1^2 + \mathcal{Q}_2^2) - 430M^4) \right. \\
& + 12(15\sqrt{5}\mathfrak{a}_4^{\phi_2}(\mathcal{Q}_2^2 - \mathcal{Q}_1^2)M^2 - 50\mathfrak{a}_4^{\phi_1}(3(\mathcal{Q}_1^2 + \mathcal{Q}_2^2)M^2 + M^6 - 3\mu) + 192\sqrt{5}\mathfrak{a}_0^\chi(\mathcal{Q}_2^2 - \mathcal{Q}_1^2)\mu \\
& - 10\sqrt{2}(15\mathfrak{a}_4^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1 - 15\mathfrak{a}_4^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2 + 6(\mathcal{Q}_1 - \mathcal{Q}_2)(\mathcal{Q}_1 + \mathcal{Q}_2)(\mathfrak{a}_0^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1 + \mathfrak{a}_0^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2) \\
& + 10(-\mathfrak{a}_0^{\mathcal{A}_7^{(2)}}\mathcal{Q}_1 + \mathfrak{a}_0^{\mathcal{A}_7^{(1)}}\mathcal{Q}_2)M^4))5\mathfrak{a}_2^{\phi_1}(81\mathcal{Q}_1^4 + 81\mathcal{Q}_2^4 - 1770\mathcal{Q}_2^2M^4 + 6\mathcal{Q}_1^2(18\mathcal{Q}_2^2 - 295M^4) - 20(70M^8 + 33M^2\mu)) \\
& + 480M^2 \log[\mathfrak{z}](3M^2(\sqrt{5}\mathfrak{a}_2^{\phi_2}(\mathcal{Q}_1 - \mathcal{Q}_2)(\mathcal{Q}_1 + \mathcal{Q}_2) + 20\sqrt{2}(\mathfrak{a}_0^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1 - \mathfrak{a}_0^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2)\sqrt{\mu}) \\
& \left. + 10\mathfrak{a}_2^{\phi_1}(2(\mathcal{Q}_1^2 + \mathcal{Q}_2^2)M^2 + M^6 - 3\mu)) + \mathcal{O}(\mathfrak{z}^{12}), \right. \tag{E27}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{a}_{\text{UV}}^{\phi_2}(\mathfrak{z}) &= \mathfrak{a}_2^{\phi_2}\mathfrak{z}^2 + \mathfrak{z}^4(\mathfrak{a}_4^{\phi_2} - 8\mathfrak{a}_2^{\phi_2}M^2 \log[\mathfrak{z}]) \\
& + \mathfrak{z}^6 \left(-12\mathfrak{a}_2^{\phi_2}M^4 - 2\mathfrak{a}_4^{\phi_2}M^2 + 16\mathfrak{a}_2^{\phi_2}M^4 \log[\mathfrak{z}] + \frac{3}{2\sqrt{5}}\mathfrak{a}_2^{\phi_2}(\mathcal{Q}_1 - \mathcal{Q}_2)(\mathcal{Q}_1 + \mathcal{Q}_2) - \frac{7}{10}\mathfrak{a}_2^{\phi_2}(\mathcal{Q}_1^2 + \mathcal{Q}_2^2 - 24M^4) \right) \\
& + \mathfrak{z}^8 \left(\frac{1}{90} \left(9\sqrt{5}\mathfrak{a}_4^{\phi_1}(\mathcal{Q}_1 - \mathcal{Q}_2)(\mathcal{Q}_1 + \mathcal{Q}_2) + 9\mathfrak{a}_4^{\phi_2}(\mathcal{Q}_1^2 + \mathcal{Q}_2^2) + 120\mathfrak{a}_4^{\phi_2}M^4 + 1120\mathfrak{a}_2^{\phi_2}M^6 \right. \right. \\
& - 12M^2(\sqrt{5}\mathfrak{a}_2^{\phi_1}(\mathcal{Q}_2^2 - \mathcal{Q}_1^2) + 9\mathfrak{a}_2^{\phi_2}(\mathcal{Q}_1^2 + \mathcal{Q}_2^2) + 8\sqrt{10}(\mathfrak{a}_0^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1 + \mathfrak{a}_0^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2)\sqrt{\mu}) + 60\mathfrak{a}_2^{\phi_2}\mu) \\
& \left. - \frac{4}{15}M^2 \log[\mathfrak{z}](3\sqrt{5}\mathfrak{a}_2^{\phi_1}(\mathcal{Q}_1 - \mathcal{Q}_2)(\mathcal{Q}_1 + \mathcal{Q}_2) + 3\mathfrak{a}_2^{\phi_2}(\mathcal{Q}_1^2 + \mathcal{Q}_2^2) + 40\mathfrak{a}_2^{\phi_2}M^4) \right) \\
& + \frac{\mathfrak{z}^{10}}{1350} \left(180\sqrt{5}\mathfrak{a}_4^{\phi_1}(\mathcal{Q}_2^2 - \mathcal{Q}_1^2)M^2 + 3\sqrt{5}\mathfrak{a}_2^{\phi_1}(\mathcal{Q}_1 - \mathcal{Q}_2)(\mathcal{Q}_1 + \mathcal{Q}_2)(57(\mathcal{Q}_1^2 + \mathcal{Q}_2^2) - 430M^4) \right. \\
& + -120\mathfrak{a}_4^{\phi_2}M^2(9(\mathcal{Q}_1^2 + \mathcal{Q}_2^2) + 5M^4) + 24(75\mathfrak{a}_4^{\phi_2}\mu - 96\mathfrak{a}_0^\chi(\mathcal{Q}_1^2 + \mathcal{Q}_2^2)\mu) \\
& - 24\sqrt{10}\sqrt{\mu}(15\mathfrak{a}_4^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1 + 15\mathfrak{a}_4^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2 + 2(\mathfrak{a}_0^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1 + \mathfrak{a}_0^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2)(3(\mathcal{Q}_1^2 + \mathcal{Q}_2^2) - 5M^4)) \\
& - \mathfrak{a}_2^{\phi_2}(279\mathcal{Q}_1^4 + 279\mathcal{Q}_2^4 + 3690\mathcal{Q}_2^2M^4 + 700M^8 + 18\mathcal{Q}_1^2(46\mathcal{Q}_2^2 + 205M^4) + 3300M^2\mu) \\
& + 480M^2 \log[\mathfrak{z}](10\mathfrak{a}_2^{\phi_2}M^6 + 3M^2(\sqrt{5}\mathfrak{a}_2^{\phi_1}(\mathcal{Q}_1 - \mathcal{Q}_2)(\mathcal{Q}_1 + \mathcal{Q}_2) + 6\mathfrak{a}_2^{\phi_2}(\mathcal{Q}_1^2 + \mathcal{Q}_2^2) \\
& \left. + 4\sqrt{10}(\mathfrak{a}_0^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1 + \mathfrak{a}_0^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2)\sqrt{\mu}) - 30\mathfrak{a}_2^{\phi_2}\mu) \right) + \mathcal{O}(\mathfrak{z}^{12}), \tag{E28}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{a}_{\text{UV}}^\chi(\mathfrak{z}) &= \mathfrak{a}_0^\chi + 2\mathfrak{a}_0^\chi M^2 \mathfrak{z}^2 + 4\mathfrak{a}_0^\chi M^4 \mathfrak{z}^4 + \mathfrak{z}^6 \left(\mathfrak{a}_6^\chi - \frac{32}{3}\mathfrak{a}_0^\chi M^6 \log[\mathfrak{z}] \right) \\
& + \mathfrak{z}^8 \left(2\mathfrak{a}_0^\chi(\mathcal{Q}_1^2 + \mathcal{Q}_2^2)M^4 - \frac{20}{3}\mathfrak{a}_0^\chi M^8 + \frac{M^2}{5}(-5\mathfrak{a}_6^\chi + 8\sqrt{10}(\mathfrak{a}_0^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1 + \mathfrak{a}_0^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2)\sqrt{\mu}10\mathfrak{a}_0^\chi\mu) + \frac{32}{3}\mathfrak{a}_0^\chi M^8 \log[\mathfrak{z}] \right) \\
& \times \frac{\mathfrak{z}^{10}}{375} \left(15(15\mathfrak{a}_6^\chi(\mathcal{Q}_1^2 + \mathcal{Q}_2^2) + 8(\sqrt{10}\mathfrak{a}_4^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1\sqrt{\mu} + \sqrt{10}\mathfrak{a}_4^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2\sqrt{\mu} + 4\mathfrak{a}_0^\chi(\mathcal{Q}_1^2 + \mathcal{Q}_2^2)\mu)) \right. \\
& - 150\mathfrak{a}_0^\chi\mathcal{Q}_1^2\mathcal{Q}_2^2M^2 + 40\mathfrak{a}_0^\chi M^{10}(39 - 40 \log[\mathfrak{z}]) - 20\mathfrak{a}_0^\chi(\mathcal{Q}_1^2 + \mathcal{Q}_2^2)M^6(23 + 120 \log[\mathfrak{z}]) \\
& \left. + 6M^4(25\mathfrak{a}_6^\chi + 350\mathfrak{a}_0^\chi\mu - 8\sqrt{10}(\mathfrak{a}_0^{\mathcal{A}_7^{(1)}}\mathcal{Q}_1 + \mathfrak{a}_0^{\mathcal{A}_7^{(2)}}\mathcal{Q}_2)\sqrt{\mu}) \right) + \mathcal{O}(\mathfrak{z}^{12}). \tag{E29}
\end{aligned}$$

For the tensor fluctuations, the expressions are simpler, and depend on two free parameters that we denote as T_0 and T_6 :

$$\begin{aligned}
T_{UV}(\mathfrak{z}) = & T_0 + 2M^2 T_0 \mathfrak{z}^2 + 4M^4 T_0 \mathfrak{z}^4 + \mathfrak{z}^6 \left(T_6 - \frac{32}{3} M^6 T_0 \log(\mathfrak{z}) \right) \\
& + \mathfrak{z}^8 \left(\frac{32}{3} M^8 T_0 \log(\mathfrak{z}) + \frac{1}{3} M^2 (-20M^6 T_0 + 6M^2(Q_1^2 + Q_2^2)T_0 + 6\mu T_0 - 3T_6) \right) \\
& + \mathfrak{z}^{10} \left(\frac{1}{75} (Q_2^2(45T_6 - 92M^6 T_0) + 6M^4(52M^6 T_0 + 70\mu T_0 + 5T_6) + Q_1^2(-92M^6 T_0 - 30M^2 Q_2^2 T_0 + 45T_6)) \right. \\
& \left. - \frac{32}{15} M^6 T_0 (2M^4 + 3(Q_1^2 + Q_2^2)) \log(\mathfrak{z}) \right) + \mathcal{O}(\mathfrak{z}^{12}). \tag{E30}
\end{aligned}$$

3. Probe approximation

We report here the equations obeyed by the scalar fluctuations treated in the probe approximation defined in the main body of the text. In this appendix, we denote the fluctuations as $\{\mathfrak{p}^{\phi_1}, \mathfrak{p}^{\phi_2}, \mathfrak{p}^\chi, \mathfrak{p}^{\mathcal{A}_7^{(1)}}, \mathfrak{p}^{\mathcal{A}_7^{(2)}}\}$, to distinguish them from the gauge-invariant variables. These fluctuations all obey Dirichlet boundary conditions, $\mathfrak{p}|_{\rho=\rho_i} = 0$.

$$\begin{aligned}
0 = & \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} (10\partial_\rho^2 + (50A' - \sqrt{10}\chi')\partial_\rho) \right. \\
& + 10 \left(-e^{2A+2\sqrt{\frac{2}{3}}\phi_2} (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) + (e^{\sqrt{2}\phi_1} + 1) e^{2A+\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} + M^2 e^{\sqrt{\frac{2}{3}}(5\chi+\phi_2)+\sqrt{2}\phi_1} \right) \Big] \mathfrak{p}^{\phi_1} \\
& + \left[2\sqrt{5} e^{2A+2\sqrt{\frac{2}{3}}\phi_2} \left(-e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2 + 3(e^{\sqrt{2}\phi_1} - 1) e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right] \mathfrak{p}^{\phi_2} \\
& + \left[\sqrt{2} e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{2}{3}}\phi_2} \left(-10\mathcal{A}_7^{(1)'} \partial_\rho - (\mathcal{A}_7^{(1)'})' (25\sqrt{2}A' + \sqrt{5}(2\phi_2' - 9\chi') + 10\phi_1') + 5\sqrt{2}\mathcal{A}_7^{(1)''} \right) \right] \mathfrak{p}^{\mathcal{A}_7^{(1)}} \\
& + \left[e^{2A+2\sqrt{\frac{2}{3}}\phi_2} \left(10\sqrt{2}\mathcal{A}_7^{(2)'} \partial_\rho + \mathcal{A}_7^{(2)'} (25\sqrt{2}A' + \sqrt{5}(2\phi_2' - 9\chi') - 10\phi_1') + 5\sqrt{2}\mathcal{A}_7^{(2)''} \right) \right] \mathfrak{p}^{\mathcal{A}_7^{(2)}} \\
& + \left[4\sqrt{5} e^{2A+2\sqrt{\frac{2}{3}}\phi_2} \left(-2e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + 2(\mathcal{A}_7^{(2)'})^2 + (e^{\sqrt{2}\phi_1} - 1) e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right] \mathfrak{p}^\chi, \tag{E31}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[2\sqrt{5} e^{2A+2\sqrt{\frac{2}{3}}\phi_2} \left(-e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2 + 3(e^{\sqrt{2}\phi_1} - 1) e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right] \mathfrak{p}^{\phi_1} \\
& + \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} (10\partial_\rho^2 + 50A' - \sqrt{10}\chi' \partial_\rho) \right. \\
& + 2e^{4\sqrt{\frac{2}{3}}\chi+\frac{\phi_1}{\sqrt{2}}} \left(e^{2A} \left(9e^{\sqrt{\frac{2}{3}}\phi_2} (e^{\sqrt{2}\phi_1} + 1) + 8e^{\frac{\phi_1}{\sqrt{2}}} - 16e^{\frac{\phi_1}{\sqrt{2}}+\sqrt{10}\phi_2} \right) + 5M^2 e^{\sqrt{\frac{2}{3}}(\chi+\phi_2)+\frac{\phi_1}{\sqrt{2}}} \right) \\
& \left. - 2e^{2A+2\sqrt{\frac{2}{3}}\phi_2} (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) \right] \mathfrak{p}^{\phi_2} \\
& + \left[e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{2}{3}}\phi_2} \left(-2\sqrt{10}\mathcal{A}_7^{(1)'} \partial_\rho - \mathcal{A}_7^{(1)'} (5\sqrt{10}A' - 9\chi' + 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10}\mathcal{A}_7^{(1)''} \right) \right] \mathfrak{p}^{\mathcal{A}_7^{(1)}} \\
& + \left[e^{2A+2\sqrt{\frac{2}{3}}\phi_2} \left(-2\sqrt{10}\mathcal{A}_7^{(2)'} \partial_\rho - \mathcal{A}_7^{(2)'} (5\sqrt{10}A' - 9\chi' - 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10}\mathcal{A}_7^{(2)''} \right) \right] \mathfrak{p}^{\mathcal{A}_7^{(2)}} \\
& + \left[4e^{2A} \left(2e^{2\sqrt{\frac{2}{3}}\phi_2} (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) - 3(e^{\sqrt{2}\phi_1} + 1) e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} + 2(e^{\sqrt{10}\phi_2} + 2) e^{4\sqrt{\frac{2}{3}}\chi+\sqrt{2}\phi_1} \right) \right] \mathfrak{p}^\chi, \tag{E32}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[-2\sqrt{5}e^{2A+2\sqrt{\frac{2}{3}}\phi_2} \left(-2e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + 2(\mathcal{A}_7^{(2)'})^2 + (e^{\sqrt{2}\phi_1} - 1)e^{\frac{8\chi+\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right) \right] \mathbf{p}^{\phi_1} \\
& + \left[2e^{2A} \left(2e^{2\sqrt{\frac{2}{3}}\phi_2} (e^{2\sqrt{2}\phi_1} (\mathcal{A}_7^{(1)'})^2 + (\mathcal{A}_7^{(2)'})^2) - 3(e^{\sqrt{2}\phi_1} + 1)e^{\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} + 2(e^{\sqrt{10}\phi_2} + 2)e^{4\sqrt{\frac{2}{3}}\chi+\sqrt{2}\phi_1} \right) \right] \mathbf{p}^{\phi_2} \\
& + \left[e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{2}{3}}\phi_2} \left(4\sqrt{10}\mathcal{A}_7^{(1)'}\partial_\rho + 2(\mathcal{A}_7^{(1)'})'(5\sqrt{10}A' - 9\chi' + 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10}\mathcal{A}_7^{(1)''} \right) \right] \mathbf{p}^{\mathcal{A}_7^{(1)}} \\
& + \left[e^{2A+2\sqrt{\frac{2}{3}}\phi_2} \left(4\sqrt{10}\mathcal{A}_7^{(2)'}\partial_\rho + 2(\mathcal{A}_7^{(2)'})'(5\sqrt{10}A' - 9\chi' - 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10}\mathcal{A}_7^{(2)''} \right) \right] \mathbf{p}^{\mathcal{A}_7^{(2)}} \\
& + \left[-16e^{2A+2\sqrt{2}\phi_1+2\sqrt{\frac{2}{3}}\phi_2} (\mathcal{A}_7^{(1)'})^2 - 16e^{2A+2\sqrt{\frac{2}{3}}\phi_2} (\mathcal{A}_7^{(2)'})^2 + e^{4\sqrt{\frac{2}{3}}\chi+\sqrt{2}\phi_1} \left(20M^2e^{\sqrt{\frac{2}{3}}(\chi+\phi_2)} - e^{2A}(e^{\sqrt{10}\phi_2} - 8) \right) \right. \\
& \left. + 4(e^{\sqrt{2}\phi_1} + 1)e^{2A+\frac{8\chi+5\phi_2}{\sqrt{10}}+\frac{\phi_1}{\sqrt{2}}} \right] \mathbf{p}^\chi, \tag{E33}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)} (10\sqrt{2}\mathcal{A}_7^{(1)'}\partial_\rho + \mathcal{A}_7^{(1)'}(25\sqrt{2}A' + \sqrt{5}(2\phi_2' - 9\chi') + 10\phi_1') + 5\sqrt{2}\mathcal{A}_7^{(1)''}) \right] \mathbf{p}^{\phi_1} \\
& + \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)} (2\sqrt{10}\mathcal{A}_7^{(1)'}\partial_\rho + \mathcal{A}_7^{(1)'}(5\sqrt{10}A' - 9\chi' + 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10}\mathcal{A}_7^{(1)''}) \right] \mathbf{p}^{\phi_2} \\
& + \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)} \left(10\partial_\rho^2 + (50A' + \sqrt{10}(2\phi_2' - 9\chi') + 10\sqrt{2}\phi_1')\partial_\rho \right. \right. \\
& + e^{2A}(\phi_1'(25\sqrt{2}A' - \sqrt{5}\chi') - (\chi' - 5\sqrt{10}A')(\phi_2' - 4\chi') - 4\sqrt{10}\chi''(\rho) + 5\sqrt{2}\phi_1''(\rho) + \sqrt{10}\phi_2''(\rho) \\
& \left. \left. - 5e^{4\sqrt{\frac{2}{3}}\phi_2} + 20e^{\frac{\phi_1}{\sqrt{2}}+\frac{3\phi_2}{\sqrt{10}}}) + 10M^2e^{\sqrt{\frac{2}{3}}\chi} \right) - 10e^{2A+\sqrt{2}\phi_1+2\sqrt{\frac{2}{3}}\phi_2} (\mathcal{A}_7^{(1)'})^2 \right] \mathbf{p}^{\mathcal{A}_7^{(1)}} \\
& + \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)} (-8\sqrt{10}\mathcal{A}_7^{(1)'}\partial_\rho - 4(\mathcal{A}_7^{(1)'})'(\rho)(5\sqrt{10}A' - 9\chi' + 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10}\mathcal{A}_7^{(1)''}) \right] \mathbf{p}^\chi, \tag{E34}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} (-10\sqrt{2}\mathcal{A}_7^{(2)'}(\rho)\partial_\rho + \mathcal{A}_7^{(2)'}(-25\sqrt{2}A' + \sqrt{5}(9\chi' - 2\phi_2') + 10\phi_1') - 5\sqrt{2}\mathcal{A}_7^{(2)''}) \right] \mathbf{p}^{\phi_1} \\
& + \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} (2\sqrt{10}\mathcal{A}_7^{(1)'}\partial_\rho + \mathcal{A}_7^{(2)'}(5\sqrt{10}A' - 9\chi' - 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10}\mathcal{A}_7^{(2)''}) \right] \mathbf{p}^{\phi_2} \\
& + \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} (10\partial_\rho^2 + (50A' + \sqrt{10}(2\phi_2' - 9\chi') - 10\sqrt{2}\phi_1')\partial_\rho) \right. \\
& + e^{\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\frac{\phi_1}{\sqrt{2}}} \left(e^{\frac{\phi_1}{\sqrt{2}}} (e^{2A}(5\sqrt{2}A'(\sqrt{5}(\phi_2' - 4\chi') - 5\phi_1') - 5\sqrt{2}\phi_1''(\rho) + \sqrt{10}(\phi_2''(\rho) - 4\chi''(\rho)) + \chi'(4\chi' + \sqrt{5}\phi_1' - \phi_2')) \right. \\
& \left. + 10M^2e^{\sqrt{\frac{2}{3}}\chi} - 5e^{2A+\frac{\phi_1}{\sqrt{2}}+4\sqrt{\frac{2}{3}}\phi_2} + 20e^{2A+\frac{3\phi_2}{\sqrt{10}}}) - 10e^{2A+2\sqrt{\frac{2}{3}}\phi_2} (\mathcal{A}_7^{(2)'})^2 \right] \mathbf{p}^{\mathcal{A}_7^{(2)}} \\
& + \left[e^{2A+\sqrt{\frac{2}{3}}(4\chi+\phi_2)+\sqrt{2}\phi_1} (-8\sqrt{10}\mathcal{A}_7^{(2)'}\partial_\rho - 4(\mathcal{A}_7^{(2)'})'(5\sqrt{10}A' - 9\chi' - 2\sqrt{5}\phi_1' + 2\phi_2') + \sqrt{10}\mathcal{A}_7^{(2)''}) \right] \mathbf{p}^\chi, \tag{E35}
\end{aligned}$$

APPENDIX F: INFRARED CUTOFF ARTIFACTS

In Fig. 5, we show representative examples of the IR-cutoff dependence of our calculations of the spectrum of bound states. For four representative backgrounds, we repeated the calculation of the spectrum by changing the IR cutoff, denoted here as q_{IR} . The results show that $q_{\text{IR}} = 10^{-6}q_0$ is sufficiently small as to yield a good approximation of the physical spectrum. At smaller values of the IR cutoff, we start to see the effect of the finite working precision of our numerical routines. Although we notice the persistence of a small yet visible IR dependence in the second to lightest scalar state, these results are sufficiently accurate to the purposes of this study.

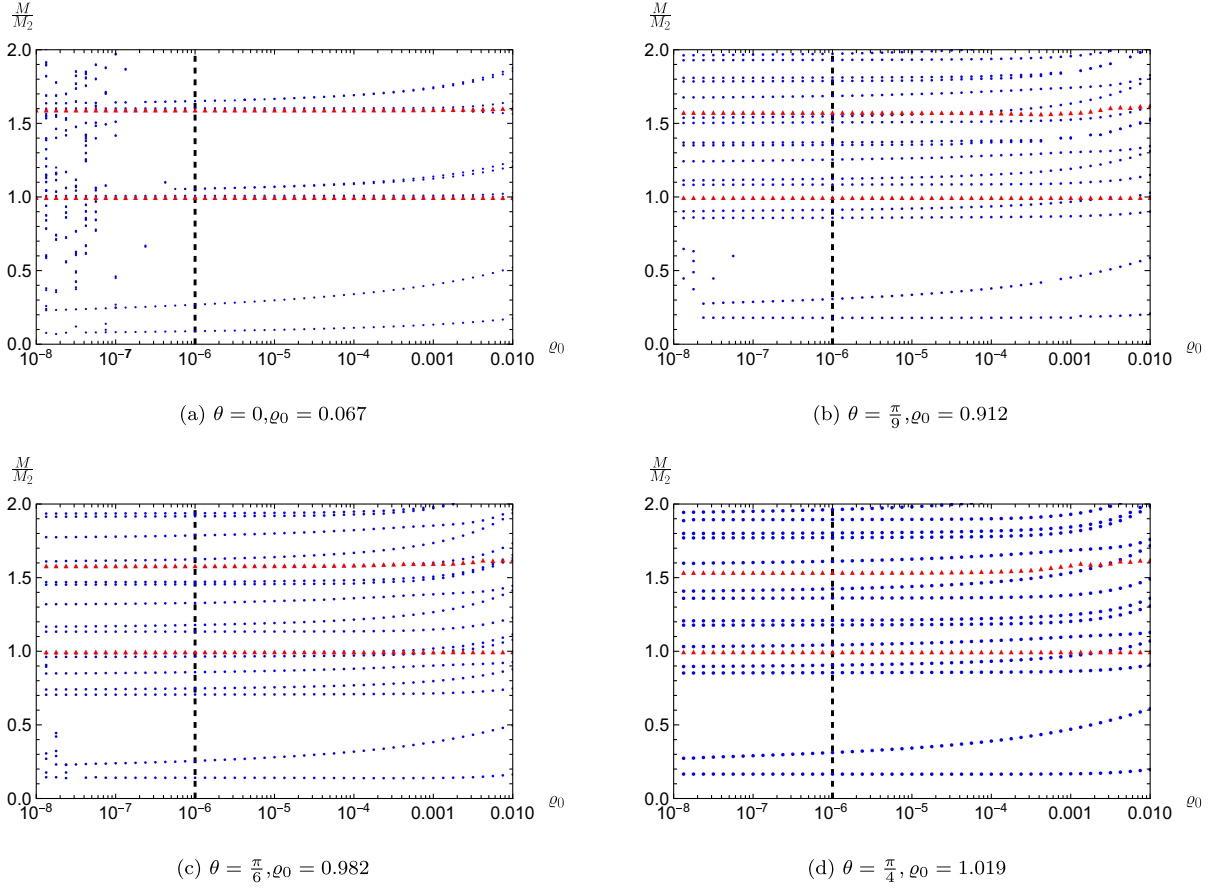


FIG. 5. Examples of mass spectra, normalized to the mass of the lightest spin-2 fluctuation, M_2 , as a function of q_0 , for four examples of one-parameter subclasses of soliton (confining) solutions, obtained by fixing the ratio of the two magnetic fluxes—see the free energy of the same solutions in Fig. 2(a)—and choosing a representative value of q_0 . Spin-2 states shown as (red) triangles and spin-0 as (blue) disks. To demonstrate the suppressed IR cutoff dependence of the spectra, we evaluated them at values of q_0 close to, but not at, the phase transition. The black dashed vertical line shows the value of the cutoff used in the calculations presented in the main body of the paper, with $q_{\text{IR}} = 10^{-6}q_0$.

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