

Chiral anomaly of Kogut-Susskind fermions in the (3 + 1)-dimensional Hamiltonian formalism

Shoto Aoki*

*Interdisciplinary Theoretical and Mathematical Sciences Program (iTHEMS),
RIKEN, Wako 351-0198, Japan*

Yoshio Kikukawa[†] and Toshinari Takemoto[‡]

Graduate School of Arts and Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8902, Japan



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We consider Kogut-Susskind fermions (also known as staggered fermions) in a (3 + 1)-dimensional Hamiltonian formalism and examine a chiral transformation and its associated chiral anomaly. The Hamiltonian of the massless Kogut-Susskind fermion has symmetry under the shift transformations in each space direction S_k ($k = 1, 2, 3$), and the product of the three shift transformations in particular (the odd shifts in general) may be regarded as a unitary discrete chiral transformation, modulo two-site translations. The Hermitian part of the transformation kernel $\Gamma = \sqrt{-1}S_1S_2S_3$ can define an axial charge as $Q_A = (1/2) \sum_x \chi^\dagger(x)(\Gamma + \Gamma^\dagger)\chi(x)$, which is non-on site, nonquantized, and commutative with the vector charge, analogous to $\tilde{Q}_A = (1/2) \sum_n (\chi_n^\dagger \chi_{n+1} + \chi_{n+1}^\dagger \chi_n)$ for the (1 + 1)-dimensional Kogut-Susskind fermion. However, our Q_A cannot be expressed in terms of any quantized charges in a generalized Onsager algebra. Although Q_A does not commute with the fermion Hamiltonian in general when coupled to background link gauge fields, we show that they become commutative for a class of $U(1)$ link configurations carrying nontrivial magnetic and electric fields. We then verify numerically that the vacuum expectation value of Q_A satisfies the anomalous conservation law of axial charge in the continuum two-flavor theory under an adiabatic evolution of the link gauge field.

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I. INTRODUCTION

Recently, remarkable progress has been made in understanding chiral symmetry and the associated chiral anomaly of lattice fermions in the Hamiltonian formalism, particularly for Kogut-Susskind (KS) fermions [1–11] in (1 + 1) dimensions [12–15] and also in (3 + 1) dimensions [16–18]. In these formalisms, discrete shift symmetries under shift transformations in each spatial direction, S_k ($k = 1, \dots, D$), play an essential role in the chiral symmetry. Among these, the odd shifts—and in particular the diagonal shift, defined as the product of D individual shifts—can be regarded as unitary discrete axial transformations, modulo two-site translations [16].

In (1 + 1) dimensions, Dempsey *et al.* [12] derived a mass counterterm to improve the Hamiltonian of the massless Schwinger model, such that the shift transformation precisely induces a variation of the θ parameter by π . On the other hand, Shao *et al.* [14] constructed a non-on site axial charge Q_A by applying the shift transformation to a single Majorana (imaginary) component. This operator is conserved and quantized, does not commute with the onsite vector charge Q_V , and together with Q_V generates an Onsager algebra [19]. They also introduced another non-on site axial charge, $\tilde{Q}_A$, which is conserved but not quantized, commutes with the vector charge, and still reproduces the correct axial anomaly when coupled to a $U(1)$ link field in a gauge-covariant manner.

For (3 + 1) dimensions, Catterall *et al.* [16] extended the above analyses and introduced a non-on site, conserved, and quantized charge—analogue to Q_A in (1 + 1) dimensions—by applying the diagonal shift transformation $S_1S_2S_3$ only to a single Majorana (imaginary) component. Furthermore, Onogi and Yamaoka [17] identified non-on site, conserved, and quantized charges Q_x , ($x = \hat{1}, \hat{2}, \hat{3}$), defined by acting the shift transformations S_k ($k = 1, 2, 3$) on a single Majorana component. These charges were shown

*Contact author: shoto.aoki@riken.jp

[†]Contact author: kikukawa@hep1.c.u-tokyo.ac.jp

[‡]Contact author: takemoto@hep1.c.u-tokyo.ac.jp

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to generate the $U(1)_F$ subgroup of the continuum symmetry group $SU(2)_L \times SU(2)_R \times U(1)_A$.¹

In this paper, we further study KS fermions in the $(3+1)$ -dimensional Hamiltonian formalism and examine another non-on site axial charge and its associated chiral anomaly.² The Hermitian part of the unitary kernel, $\Gamma = \sqrt{-1}S_1S_2S_3$, of the discrete chiral transformation defines a non-on site axial charge $Q_A = (1/2) \sum_x \chi^\dagger (\Gamma + \Gamma^\dagger) \chi$, which is conserved, nonquantized, and commutes with the vector charge Q_V . It is analogous to $\tilde{Q}_A = (1/2) \sum_x (\chi_x^\dagger \chi_{x+1} + \chi_{x+1}^\dagger \chi_x)$ for the $(1+1)$ -dimensional KS fermion [14,16], but, as we will see below, it is not related to any quantized charges of the generalized Onsager algebra in $(3+1)$ dimensions [23], in contrast to the $(1+1)$ -dimensional case where $\tilde{Q}_A = (1/2)(Q_1 + Q_{-1})$.³ We note that $\Gamma = \sqrt{-1}S_1S_2S_3$ is rather related to the chiral operator Γ_5 in the Euclidean formulation [5–10].

Although Q_A does not generally commute with the fermion Hamiltonian when coupled to a background link gauge field, we show that they become commutative for a particular class of $U(1)$ link configurations carrying non-trivial magnetic and electric fields. We then verify numerically, with controlled accuracy, that the vacuum expectation value of Q_A satisfies the anomalous axial charge conservation law of the continuum two-flavor theory [24–27] under adiabatic evolution of the link gauge field.⁴

This paper is organized as follows. In Sec. II, we review the symmetries of the KS fermion system and define the diagonal shift operator corresponding to the chiral symmetry. We also construct the generalized Onsager algebra [23] associated with all shift operators. In Sec. III, we study the vacuum expectation value of the chiral charge and demonstrate that it satisfies the chiral anomaly equation. Finally, we summarize our results and present concluding remarks in Sec. IV.

II. FREE KS FERMION HAMILTONIAN AND SYMMETRIES IN $(3+1)$ -DIMENSIONAL SPACETIME

We consider KS fermion systems in $(3+1)$ -dimensional spacetime. We are interested in the case where the only time is continuous, but the space is discrete. In this section, we review the Hamiltonian of the KS fermion system and symmetries. We also define the chiral transformation in the

lattice [16]. The analysis on the taste basis is discussed in Appendix A.

We denote the site on the three-dimensional lattice space by

$$x = (x^1, x^2, x^3), \quad x^i = 0, \dots, N-1, \quad (1)$$

where N is an even integer so that lattice translations do not violate periodic boundary conditions $x^i + N \sim x^i$. The Hamiltonian of the KS fermion is defined by [1]

$$H = \sum_x \chi(x)^\dagger h \chi(x), \quad h = \sum_{i=1}^3 \frac{T_i - T_i^\dagger}{2\sqrt{-1}} + m_{\text{lat}} \epsilon(x), \quad (2)$$

where $\eta_i(x) = (-1)^{x^1 + \dots + x^{i-1}}$, $\epsilon(x) = (-1)^{x^1 + x^2 + x^3}$, and $T_i = \eta_i(x) \delta_{x+\hat{i}, y}$ is a shift operator in the i direction,

$$T_i \chi(x) = \sum_y (T_i)_{xy} \chi(y) = \eta_i(x) \chi(x + \hat{i}), \quad (3)$$

where $\hat{i}$ represents a unit vector in the i th direction. The commutation relations are given by

$$T_i T_j + T_j T_i = 2\delta_{ij} T_j^2. \quad (4)$$

A. Parity, time reversal, and charge conjugation

The free massive KS fermion system has discrete symmetries, such as parity, time reversal, and charge conjugation. We define the parity operator as [5,11]

$$P^{-1} \chi(x) P = \epsilon(x) \chi(-x). \quad (5)$$

It converges to $(\beta \otimes 1) \psi(-r)$ on the taste basis in the continuum limit (see Appendix A).

Time reversal is defined as⁵

$$T^{-1} \chi(x) T = (-1)^{x^2} T_1 T_3 \chi(x) = -(-1)^{x^1} \chi(x + \hat{1} + \hat{3}), \\ T^{-1} \sqrt{-1} T = -\sqrt{-1}, \quad (8)$$

and commutes with the Hamiltonian. In the continuum limit, $\psi(r)$ changes to $(\alpha_1 \alpha_3 \otimes 1) \psi(r)$ on the taste basis.

¹For discussions of other 't Hooft anomalies in KS fermion systems, see [20].

²The chiral anomaly on the lattice space can be interpreted as the Lieb-Schultz-Mattis anomalies [21,22].

³It differs from any of the two axial charges introduced by Catterall *et al.*, but it anticommutes with the time reversal transformation [16].

⁴For related studies using Wilson and overlap fermions in the Hamiltonian formalism, see [28] and [29–33].

⁵There is another time reversal symmetry as

$$T'^{-1} \chi(x) T' = \epsilon(x) \chi(x), \quad T'^{-1} \sqrt{-1} T' = -\sqrt{-1}, \quad (6)$$

which acts on ψ as

$$T'^{-1} \psi(r) T' = T_2 S_2^{-1} \psi(r). \quad (7)$$

The continuum limit is given by $-\alpha_1 \alpha_3 \otimes \tau_2$.

We also find charge conjugation,

$$C^{-1}\chi(x)C = (-1)^{x^2}\epsilon T_2\chi^*(x) = (-1)^{x^3}\chi^*(x + \hat{2}). \quad (9)$$

This transformation corresponds to $\psi(r) \rightarrow (\beta\alpha_2 \otimes 1)\psi^*(r)$.

B. Shift symmetries as axial flavor and axial $U(1)$ symmetries

The massless KS fermion system has shift symmetries. We define other shift operators by a single site along the x^i direction by

$$S_i\chi(x) = \xi_i(x)\chi(x + \hat{i}), \quad (10)$$

where $\xi_i(x) = (-1)^{x^{i+1}+\dots+x^3}$ [5,16]. The commutation relations are given by

$$\begin{aligned} S_i S_j + S_j S_i &= 2\delta_{ij} S_i^2, & S_i T_j - T_j S_i &= 0, \\ S_i \epsilon + \epsilon S_i &= 0. \end{aligned} \quad (11)$$

Taking the continuum limit, this operator converges to $\gamma^5 \otimes \sigma_i$ on the taste basis, which is the generator of the axial $SU(2)$ rotation with $\frac{\pi}{2}$ (see Appendix A). Thus, the shift operators can be taken to be equivalent to axial $SU(2)$ rotations in the lattice space.

We also define the diagonal shift operator as⁶

$$\begin{aligned} \Gamma\chi(x) &= \sqrt{-1}S_1S_2S_3\chi(x) \\ &= -\sqrt{-1}T_1T_2T_3\chi(x) \\ &= \sqrt{-1}(-1)^{x^2}\chi(x + T), \end{aligned} \quad (12)$$

where $T = \hat{1} + \hat{2} + \hat{3}$. Γ commutes with the massless Hamiltonian and satisfies the lattice continuity Eq. (B8). In the continuum limit, this operator corresponds to $\gamma_5 \otimes 1$ on the taste basis. Thus, we regard Γ as the chiral operator in the lattice space. Note that Γ is a unitary rather than a Hermitian. The (lattice) regularized chiral charge and its density are defined as

$$\begin{aligned} Q_{\text{reg}} &= \sum_x j_{\text{reg}}^0(x) = \sum_x \chi^\dagger(x) \Gamma \chi(x), \\ j_{\text{reg}}^0(x) &= \chi^\dagger(x) \Gamma \chi(x). \end{aligned} \quad (13)$$

We denote its Hermitian and anti-Hermitian parts by Q_A and $\tilde{Q}$ as

$$Q_A = \sum_x j_A^0(x) = \sum_x \chi^\dagger(x) \frac{\Gamma + \Gamma^\dagger}{2} \chi(x), \quad (14)$$

$$j_A^0(x) = \frac{1}{2} \sqrt{-1} (-1)^{x^2} (\chi^\dagger(x) \chi(x + T) - \chi^\dagger(x + T) \chi(x)), \quad (15)$$

$$\tilde{Q} = \sum_x \chi^\dagger(x) \frac{\Gamma - \Gamma^\dagger}{2\sqrt{-1}} \chi(x) = \sum_x \tilde{j}^0(x), \quad (16)$$

$$\tilde{j}^0(x) = \frac{1}{2} (-1)^{x^2} (\chi^\dagger(x) \chi(x + T) + \chi^\dagger(x + T) \chi(x)). \quad (17)$$

As we will see below, the axial anomaly arises from the Hermitian part Q_A , rather than from the anti-Hermitian part $\tilde{Q}$, and Q_A is essentially different from the nonquantized charge discussed in [14].

C. Onsager algebra

We consider the massless Hamiltonian and investigate algebraic relations between conserved charges with integer eigenvalues associated with shift operators (10). We show that these charges generate the generalized Onsager algebra [19,23].

In the massless case, there is an on-site charge conjugation,

$$\chi(x) \mapsto \chi^*(x). \quad (18)$$

This transformation leads to

$$\psi(r) \mapsto (\beta \otimes 1) T_2 S_2^{-1} \psi(r) \quad (19)$$

on the taste basis. In the method proposed in the previous works [14,16,17], the staggered fermion χ is decomposed into two Majorana fermions under the on-site charge conjugation as

$$\chi(x) = \frac{1}{2} (a(x) + \sqrt{-1}b(x)), \quad (20)$$

where

$$\begin{aligned} \{a(x), a(x')\} &= \{b(x), b(x')\} = 2\delta_{xx'}, \\ \{a(x), b(x')\} &= 0. \end{aligned} \quad (21)$$

Then, the massless Hamiltonian is given by

$$H = \frac{\sqrt{-1}}{4} \sum_x \eta_i(r) (a(x)a(x + \hat{i}) + b(x)b(x + \hat{i})). \quad (22)$$

As we saw before, this Hamiltonian has shift symmetries under S_i . These operators generate a symmetry group

$$G = \langle -I, S_1, S_2, S_3 | S_i S_j = -S_j S_i (i \neq j) \rangle, \quad (23)$$

where I is an identity element. Note that G is a central extension of $\mathbb{Z}^3$ by $\mathbb{Z}_2$. The translational operator $\hat{S}_i^{(b)}$ that

⁶This operator is odd under Catterall's T transformation [16].

acts on only $b(x)$ in the i th direction is given by

$$\begin{aligned}\hat{S}_i^{(b)} a(x) (\hat{S}_i^{(b)})^{-1} &= a(x), \\ \hat{S}_i^{(b)} b(x) (\hat{S}_i^{(b)})^{-1} &= S_i b(x) = \xi_i(x) b(x + \hat{i}).\end{aligned}\quad (24)$$

We extend this representation to any group element in G . If $g \in G$ is written as $g = S_{g_1} S_{g_2} \cdots S_{g_n}$ ($g_i = 1, 2, 3$), we can define

$$g(x) = x + \sum_{i=1}^n \hat{g}_i, \quad (25)$$

$$\xi_g(x) = \xi_{g_1} \left(x + \sum_{i=2}^n \hat{g}_i \right) \xi_{g_2} \left(x + \sum_{i=3}^n \hat{g}_i \right) \cdots \xi_{g_n}(x), \quad (26)$$

and the translational operator $\hat{S}_g^{(b)}$ as

$$\begin{aligned}\hat{S}_g^{(b)} b(x) (\hat{S}_g^{(b)})^{-1} &= \hat{S}_{g_1}^{(b)} \cdots \hat{S}_{g_n}^{(b)} b(x) (\hat{S}_{g_n}^{(b)})^{-1} \cdots (\hat{S}_{g_1}^{(b)})^{-1} \\ &= S_{g_n} \cdots S_{g_1} b(x) = b(g(x)) \xi_g(x).\end{aligned}\quad (27)$$

We assume that $g \mapsto (\hat{S}_g^{(b)})$ is a group homomorphism. Then, ξ should satisfy

$$\xi_{\pm I}(x) = \pm 1, \quad \xi_{gh}(x) = \xi_g(h(x)) \xi_h(x) \quad (28)$$

for $g, h \in G$, and

$$\xi_{g^{-1}}(x) = \xi_g(g^{-1}(x)) \quad (29)$$

for the inverse element of g . Here, we use $\xi_g(x) = \pm 1$.

We define the quantized and conserved charges as

$$Q_I = \sum_x \left[\chi^\dagger(x) \chi(x) - \frac{1}{2} \right] = \frac{\sqrt{-1}}{2} \sum_x a(x) b(x), \quad (30)$$

$$Q_g = \hat{S}_g^{(b)} Q_I (\hat{S}_g^{(b)})^{-1} = \frac{\sqrt{-1}}{2} \sum_x \xi_g(x) a(x) b(g(x)), \quad (31)$$

and the auxiliary generator as

$$G_{g,h} = \frac{\sqrt{-1}}{2} \sum_x [\xi_g(x) a(x) a(g(x)) - \xi_h(x) b(x) b(h(x))]. \quad (32)$$

Q_I is a vector $U(1)$ charge whose eigenvalue expresses the fermion number.⁷ These operators commute with the massless Hamiltonian and satisfy the G -Onsager algebra relations [23],

⁷ Q_I is equivalent to Q_V or Q_0 in [14,16,17,23].

$$[Q_g, Q_h] = \sqrt{-1} G_{g^{-1}h, hg^{-1}}, \quad (33)$$

$$[Q_g, G_{h,k}] = \sqrt{-1} (Q_{gh^{-1}} + Q_{k^{-1}g} - Q_{gh} - Q_{kg}), \quad (34)$$

$$\begin{aligned}[G_{g_1, h_1}, G_{g_2, h_2}] &= \sqrt{-1} (G_{g_2 g_1, h_1 h_2} - G_{g_2^{-1} g_1, h_1 h_2^{-1}} \\ &\quad - G_{g_1 g_2, h_2 h_1} + G_{g_1 g_2^{-1}, h_2^{-1} h_1}).\end{aligned}\quad (35)$$

We denote the infinite-dimensional Lie algebra $\{Q_g, G_{h,k}\}$ by $\mathbf{Ons}_3$.⁸

The regularized chiral charge Q_{reg} belongs to $\mathbf{Ons}_3$. Setting $S_T = S_3 S_2 S_1$, we can rewrite Q_{reg} as

$$\begin{aligned}Q_{\text{reg}} &= Q_A + \sqrt{-1} \tilde{Q} \\ &= \frac{\sqrt{-1}}{4} \sum_x (-1)^{x^2} (a(x) a(x+T) + b(x) b(x+T) \\ &\quad + \sqrt{-1} (a(x) b(x+T) - a(x) b(x-T))) \\ &= \frac{1}{2} G_{S_T, -S_T} + \sqrt{-1} \frac{Q_{S_T} + Q_{S_T^{-1}}}{2}.\end{aligned}\quad (36)$$

Since S_T is a central element of G , $Q_A = \frac{1}{2} G_{S_T, -S_T}$ commutes with all elements of $\mathbf{Ons}_3$. That is, the Hermitian chiral operator Q_A ⁹ can be interpreted as the central charge of the KS fermion.

On the other hand, Q_{S_T} acts on two Majorana fields as

$$[Q_{S_T}, a(x)] = -\sqrt{-1} (-1)^{x^2} b(x+T), \quad (37)$$

$$[Q_{S_T}, b(x)] = -\sqrt{-1} (-1)^{x^2} a(x-T). \quad (38)$$

These equations lead to

$$[Q_{S_T}, \psi(r)] \rightarrow -\sqrt{-1} (\beta \alpha_2 \otimes \tau_2) \psi^*(r) \quad (39)$$

in the continuum limit [17]. This transformation looks like the charge conjugation (9) with a flavor rotation rather than the chiral transformation. The derivation is implemented in Appendix A. Our Q_{S_T} is constructed from the translation of the Majorana b field, in a manner similar to that in [14]. However, the diagonal translation operator S_T cannot be interpreted as C^R , which corresponds to the charge conjugation of right-moving particles in $(1+1)$ dimensions. In fact, while the commutation relation of ψ resembles that of charge conjugation, the quantized charge in $(1+1)$ dimensions is associated with the axial transformation.

⁸ $\mathbf{Ons}_3$ contains Onsager subalgebras, such as those generated by Q_I and Q_{S_T} discussed in [16], and those generated by Q_I and Q_{S_3} discussed in [17].

⁹Our Q_A is not quantized and is consistent with the Nielsen-Ninomiya theorem [34]. However, this does not mean that there is no well-defined definition on the lattice.

Therefore, although the construction is formally similar to that in [14], its physical interpretation is different.

III. INTERACTION WITH ELECTRO-MAGNETIC FIELD

In this section, we consider the chiral anomaly in the presence of a $U(1)$ gauge field. We begin by reviewing the continuum theory and summarizing the results relevant to chiral symmetry in Sec. III A, while the detailed calculations are presented in Appendix C. In Secs. III B and III C, we investigate the chiral anomaly in the KS Hamiltonian with $U(1)$ link variables.

A. Continuum theory

We briefly review the single Dirac fermion system with a background $U(1)$ gauge field on the continuum theory [26],

$$H = \int_{\mathbb{R}^3} d^3x \psi^\dagger h \psi(x, t),$$

$$(h = \alpha^i (-\sqrt{-1} \partial_i + A_i(x, t))), \quad (40)$$

where $\alpha^i = \sigma_3 \otimes \sigma_i$. We assume that the electric field $E_i = \partial_0 A_i$ and magnetic field $B_i = \epsilon^{ijk} \partial_j A_k$ are constant for x and t .

This system has a chiral symmetry generated by the chiral operator,

$$\gamma_5 = -\sqrt{-1} \alpha^1 \alpha^2 \alpha^3 = \sigma_3 \otimes 1. \quad (41)$$

Its charge density and current are given by

$$j_{A,c}^0(x, t) = \lim_{y \rightarrow x} \psi^\dagger(x, t) \gamma^5 e^{\sqrt{-1} \int_y^x A_i(z, t) dz^i} \psi(y, t), \quad (42)$$

$$j_{A,c}^i(x, t) = \lim_{y \rightarrow x} \psi^\dagger(x, t) \alpha^i \gamma^5 e^{\sqrt{-1} \int_y^x A_i(z, t) dz^i} \psi(y, t), \quad (43)$$

where the subscript c means “continuum.”

We solve the eigenvalue problem of h at fixed time t and find the positive and negative energy states as

$$hu(\Omega, x, t) = E(\Omega, t)u(\Omega, x, t), \quad (44)$$

$$hv(\Omega, x, t) = -E(\Omega, t)v(\Omega, x, t), \quad (45)$$

for energy $E(\Omega, t) > 0$.¹⁰ Ω is a set of all parameters that characterize the wave functions u and v . The normalization of them is determined by

$$\int d^3x u^\dagger(\Omega, x, t) u(\Omega', x, t) = \delta_{\Omega\Omega'}, \quad (46)$$

¹⁰There is an ambiguity as to whether the zero modes should be regarded as positive or negative energy states.

$$\int d^3x v^\dagger(\Omega, x, t) v(\Omega', x, t) = \delta_{\Omega\Omega'}, \quad (47)$$

$$\int d^3x u^\dagger(\Omega, x, t) v(\Omega', x, t) = 0, \quad (48)$$

$$\int d\Omega [u(\Omega, x, t) u^\dagger(\Omega, x', t) + v(\Omega, x, t) v^\dagger(\Omega, x', t)]$$

$$= \delta^{(3)}(x - x'). \quad (49)$$

Let us expand the Dirac fermion field by the creation and annihilation operators of the wave functions,

$$\psi(x, t) = \int d\Omega [u(\Omega, x, t) b(\Omega, t) + v(\Omega, x, t) d^\dagger(\Omega, t)], \quad (50)$$

where the annihilation operators $b(\Omega)$ and $d(\Omega)$ satisfy the standard anticommutation relations,

$$\{b(\Omega, t), b^\dagger(\Omega', t)\} = \{d(\Omega, t), d^\dagger(\Omega', t)\} = \delta_{\Omega\Omega'}, \quad (51)$$

and the others are zero. The vacuum state at time t is defined as

$$b(\Omega, t)|0, t\rangle = d(\Omega, t)|0, t\rangle = 0. \quad (52)$$

According to the argument in Appendix C, the vacuum expectation value of j_A^0 is given by

$$\langle j_{A,c}^0(x, t) \rangle := \langle 0, t | j_{A,c}^0(x, t) | 0, t \rangle$$

$$= \lim_{y \rightarrow x} \int d\Omega v^\dagger(\Omega, x) \gamma^5 v(\Omega, y)$$

$$= \lim_{y \rightarrow x} \sqrt{-1} \frac{B_i}{2\pi^2} \frac{(y - x)^i}{||y - x||^2}. \quad (53)$$

As a result of the point splitting, the value becomes a pure imaginary number. Furthermore, the chiral current satisfies the chiral anomaly equation,

$$\langle \partial_\mu j_{A,c}^\mu(x, t) \rangle = -\frac{E_i B_i}{2\pi^2}, \quad (54)$$

and

$$\frac{d}{dt} \langle Q_{A,c}(t) \rangle = -\sum_x \frac{E_i B_i}{2\pi^2}. \quad (55)$$

B. Lattice theory

In the presence of link variables, the chiral operator (12) does not commute with the massless Hamiltonian.¹¹ However, there is a certain magnetic field configuration

¹¹If we consider the dynamical link variables, there is a shift symmetry under $\chi(x) \rightarrow \sqrt{-1}(-1)^{x^2} \chi(x + T)$ and $U_i(x) \rightarrow U_i(x + T)$ [11].

in which Γ commutes. Imposing the electric field adiabatically, we solve the eigenvalue problem of the Hamiltonian and determine the vacuum state. Then, we show that the expectation value of the chiral charge density (17) satisfies the anomaly equation in the kinetic normal ordering [26].

We fix link variables as

$$U_1(x, t) = \begin{cases} e^{\sqrt{-1}Bx^3} & (x^1 \neq N-1) \\ e^{\sqrt{-1}Bx^3} e^{-\sqrt{-1}BNx^2} & (x^1 = N-1) \end{cases}, \quad (56)$$

$$U_2(x, t) = e^{\sqrt{-1}B(x^1-x^3)}, \quad (57)$$

$$U_3(x, t) = \begin{cases} e^{\sqrt{-1}\frac{2\pi}{N}t} & (x^3 \neq N-1) \\ e^{\sqrt{-1}BN(x^1-x^2)} e^{\sqrt{-1}\frac{2\pi}{N}t} & (x^3 = N-1) \end{cases}, \quad (58)$$

where B is a magnetic flux through one plaquette and takes a discrete value

$$B = \frac{2\pi n}{N^2} (n \in \mathbb{Z}) \quad (59)$$

from the periodic boundary condition for the KS fermion. This configuration generates a uniform magnetic field B and electric field E ,

$$B_1 = B_2 = B_3 = B, \quad E_1 = E_2 = 0, \quad E_3 = \frac{2\pi}{N}. \quad (60)$$

In the presence of the link variables, the shift operators are modified as

$$T_i^U \chi(x, t) = \eta_i(x) U_i(x, t) \chi(x + \hat{i}, t). \quad (61)$$

Since $T_i^U T_j^U + T_j^U T_i^U \neq 2\delta_{ij}(T_j^U)^2$ in general, the existence of the chiral symmetry is nontrivial. However,

$$[T_i^U, T_1^U T_2^U T_3^U] = 0 \quad (62)$$

holds under our link variables. Then, we can define the chiral transformation commuting with the massless Hamiltonian in the presence of the magnetic field as

$$\Gamma^U \chi(x, t) = (-\sqrt{-1}) e^{-\sqrt{-1}B/2} T_1^U T_2^U T_3^U \chi(x, t). \quad (63)$$

Here, $e^{-\sqrt{-1}B/2}$ is the normalization factor so that

$$(\Gamma^U)^N = (-1)^n e^{\sqrt{-1}2\pi n}. \quad (64)$$

Note that the KS fermion effectively becomes an antiperiodic function when n is odd. We also define the chiral charge Q_A and its density $j_A^0(x, t)$ as well as Eq. (14).

C. Numerical results

We solve the eigenvalue problem of the one-particle Hamiltonian h with $m = 0$ on each time slice. Unlike the case of $(1+1)$ -dimensional, it is hard to solve the eigenvalue problem of the Hamiltonian analytically, so we only calculate it numerically. Fixing $N = 8$, $m = 0$, and $t = 0$, the energy is plotted against the eigenvalue of Γ^U in Fig. 1 with $n = -1$ and $n = 2$. When $n = -1$, we can see the gap generated by the antiperiodicity $(\Gamma^U)^N = -1$. On the other hand, when $n = 2$, there are zero modes at $\Gamma^U = +1$ and $\Gamma^U = e^{\pm\sqrt{-1}\pi} = -1$.

We also investigate the time evolution of the energy, shown in Fig. 2, with $N = 8$ and $m = 0$. The color gradation represents the eigenvalue of Γ^U : a positive imaginary part is indicated in red, a negative one in blue, and the real part is encoded in the brightness. We observe that the $\Gamma^U = \pm 1$ modes cross $E = 0$ at $t = 0.5$ for $n = -1$, and at $t = 0$ and $t = 1$ for $n = 2$. Our numerical results confirm that zero modes appear at $t \in \mathbb{Z} + \frac{n}{2}$.¹²

We compute the expectation values of j_A^0 and $\tilde{j}^0$ in the same manner as in Sec. III A. Here, the zero modes with $\Gamma^U = 1$ are assigned to the negative energy states to define the vacuum. The spatially averaged values are plotted as functions of time in the left and right panels of Fig. 3, where the parameters are fixed in the same way as in Fig. 1. The averages are defined by

$$q_A(t) = \frac{1}{N^3} \sum_x \langle j_A^0(x, t) \rangle = \frac{1}{N^3} \langle Q_A(t) \rangle, \quad (65)$$

$$\tilde{q}(t) = \frac{1}{N^3} \sum_x \langle \tilde{j}^0(x, t) \rangle = \frac{1}{N^3} \langle \tilde{Q}(t) \rangle, \quad (66)$$

and error bars are given by

$$\sigma_A(t) = \sqrt{\frac{1}{N^3} \sum_x [\langle j_A^0(x, t) \rangle - q_A(t)]^2}, \quad (67)$$

$$\tilde{\sigma}(t) = \sqrt{\frac{1}{N^3} \sum_x [\langle \tilde{j}^0(x, t) \rangle - \tilde{q}(t)]^2}. \quad (68)$$

The solid lines represent the fitting functions, defined as

$$q_{A,\text{fitting}}(t) = -\frac{4|n|}{N^3} \left(\left\{ \frac{n}{|n|} t - \frac{n}{2} \right\} - \frac{1}{2} \right), \quad (69)$$

$$\tilde{q}_{\text{fitting}}(t) = \frac{2n}{\pi N^2}, \quad (70)$$

where the curly bracket is the sawtooth function, which takes the fractional part of the given real number. We find

¹²We do not provide an analytic proof of this statement.

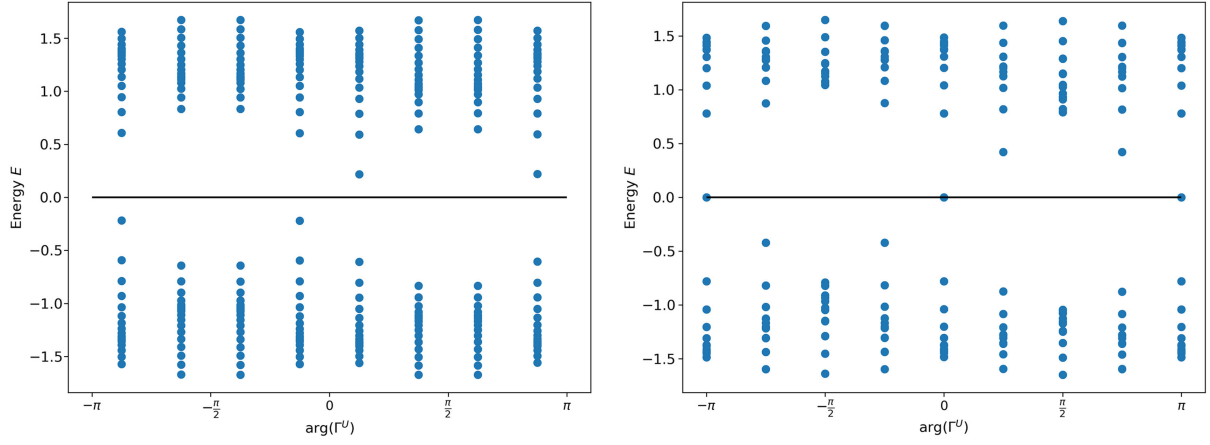


FIG. 1. Energy spectrum represented by the argument (phase) of Γ^U at $N = 8$, $m = 0$, and $t = 0$. The left and right panels correspond to $n = -1$ and $n = 2$, respectively.

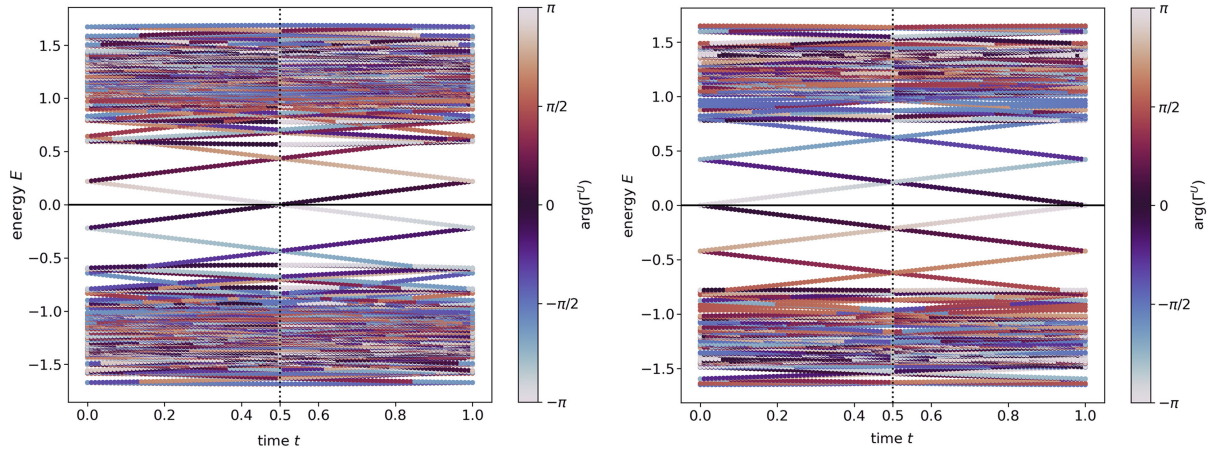


FIG. 2. Time evolution of the energy spectrum at $N = 8$ and $m = 0$. The left and right panels correspond to $n = -1$ and $n = 2$, respectively. The color gradation represents the eigenvalues of Γ^U : the positive and negative imaginary parts are shown in red and blue, respectively, while the real part is encoded in the brightness.

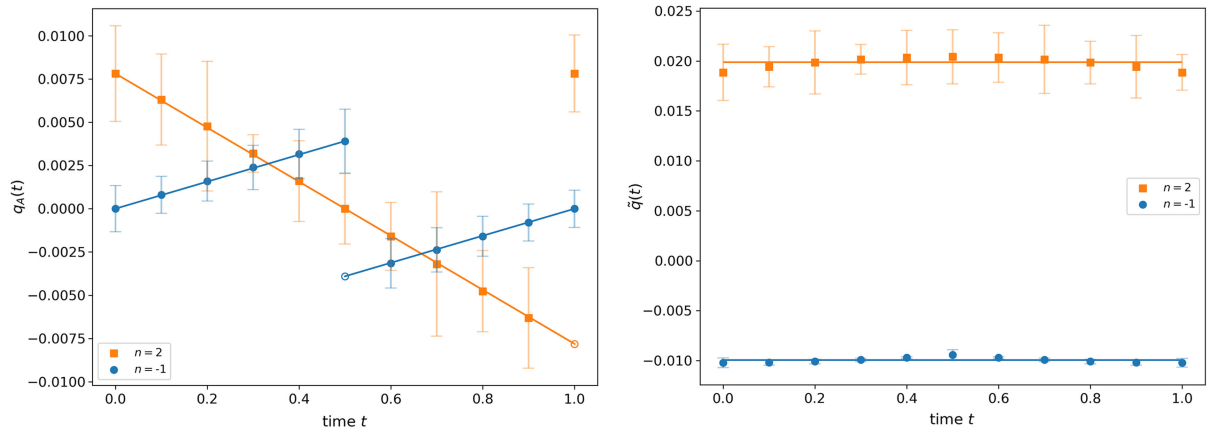
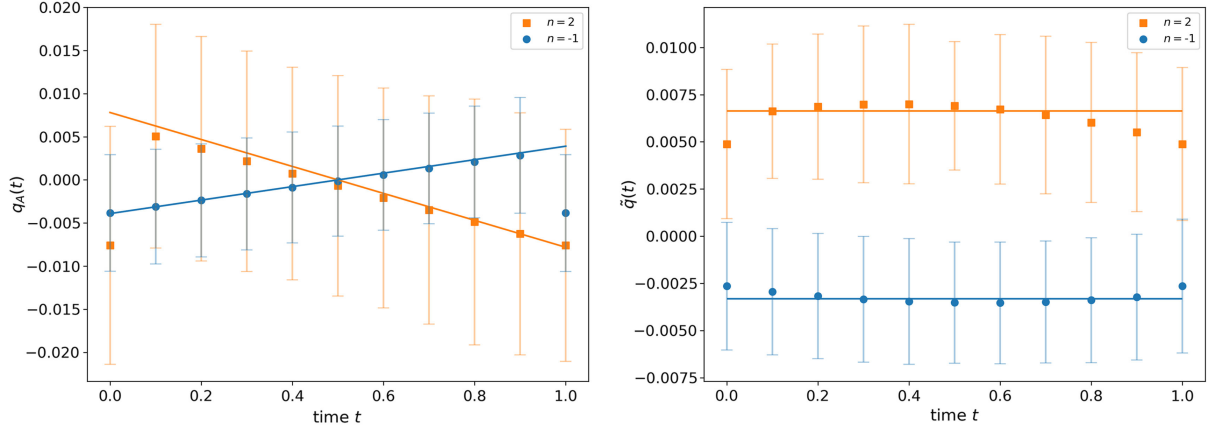


FIG. 3. Left panel: time evolution of the expectation value of the chiral charge density j_A^0 under a magnetic field in the diagonal direction and an electric field applied along the x^3 -axis. Filled symbols indicate spatially averaged values, with error bars showing spatial variations. Solid lines represent the fitting functions defined in Eq. (69). The white circle marks the discontinuity. Right panel: the same plot for $\tilde{j}_A^0$, where the fitting function is defined in Eq. (70). We fix $N = 8$ and $m = 0$, and choose $n = -1$ and $n = 2$.

FIG. 4. Same plots as in Fig. 3, but with both the magnetic and electric fields aligned along the x^3 -axis.

that our lattice data are in good agreement with Eq. (69). The $q_A(t)$ exhibit discontinuities at $t \in \mathbb{Z} + \frac{n}{2}$, where zero modes appear. This phenomenon is consistent with the interpretation in [28,35–37]. The electric field accelerates particles, leading to the emergence of positive energy excitations from the Dirac sea, while others are driven into negative energy states and sink back into the sea. As a result, the chirality of the Dirac sea decreases discontinuously. Note that the value of $q_A(t)$ at $t \in \mathbb{Z} + \frac{n}{2}$ depends on the choice of the definition of positive and negative energy states.

Our numerical results are in good agreement with the continuum predictions. The time derivative of $\langle Q_A(t) \rangle$ in the continuum region is estimated as

$$\frac{d}{dt} \langle Q_A(t) \rangle \simeq -\sum_x \frac{4n}{N^3} = -2 \sum_x \frac{B_i E_i}{2\pi^2}. \quad (71)$$

Compared with the single Dirac fermion case (55), this equation is equivalent to the chiral anomaly equation of a two-flavor Dirac fermion.

On the other hand, what does $\tilde{Q}$ or $\tilde{j}^0$ mean? The expectation value is estimated as

$$\langle \tilde{j}^0(x, t) \rangle \simeq \tilde{q}_{\text{fitting}} = 2 \frac{B}{2\pi^2}, \quad (72)$$

which is equivalent to the imaginary part of Eq. (53) when $y = x + T$ and $B_i = B$. The factor 2 means the number of Dirac fermions in the continuum limit. Thus, we conclude that

$$\langle j_{\text{reg}}^0(x, t) \rangle = \langle j_A^0(x, t) \rangle + \sqrt{-1} \langle \tilde{j}^0(x, t) \rangle \rightarrow 2 \langle j_{A,c}^0(x, t) \rangle, \quad (73)$$

$$\begin{aligned} \frac{d}{dt} \langle Q_{\text{reg}}(t) \rangle &= \frac{d}{dt} (\langle Q_A(t) \rangle + \sqrt{-1} \langle \tilde{Q}(t) \rangle) \\ &\rightarrow 2 \frac{d}{dt} \langle Q_{A,c}(t) \rangle, \end{aligned} \quad (74)$$

and the diagonal shift operator Γ^U converges to the chiral operator $\gamma_5 \otimes 1$ in the continuum limit.

One may be wondering whether our definition can capture the chiral anomaly under link variables that break the commutativity with H and Q_A . We try the same computation under

$$\begin{aligned} U_1(x, t) &= 1, & U_2(x, t) &= e^{\sqrt{-1} B x^1}, \\ U_3(x, t) &= e^{\sqrt{-1} \frac{2\pi}{N} t}, \end{aligned} \quad (75)$$

which generate a uniform magnetic and electric fields in the x^3 direction. We can define the chiral operator (63), but it no longer commutes with the Hamiltonian.

Anyway, we plot the expectation value of the chiral charge density at $N = 8$, $n = -1$, and $n = 2$ in Fig. 4. The deviations are bigger than the previous examples. This reflects the fact that the chiral operator is not an exact symmetry of the Hamiltonian. Nevertheless, the averages agree with the continuum predictions indicated by solid lines,

$$q_{A,\text{fitting}}(t) = -2 \frac{B_3 E_3}{2\pi^2} t (0 < t < 1), \quad (76)$$

$$\tilde{q}_{\text{fitting}}(t) = 2 \frac{B_3}{2\pi^2} \frac{1}{3}. \quad (77)$$

Thus, these results agree with the continuum prediction.

IV. CONCLUSION

We studied the $(3+1)$ -dimensional KS fermion system in Hamiltonian formalism. We confirmed that the diagonal shift operator can be interpreted as the chiral operator on the lattice.

In the presence of link variables, the diagonal shift symmetry is violated because of their spatial dependence. Nevertheless, we found a specific configuration that preserves the diagonal shift symmetry in Sec. III. In this case, the magnetic field aligns with the diagonal direction,

consistent with the point-splitting construction of the chiral charge on the continuum space. Note that the diagonal shift operator is unitary rather than Hermitian. We defined two charges Q_A and $\tilde{Q}$ corresponding to the Hermitian and anti-Hermitian parts of the diagonal shift operator. We numerically calculated the expectation values of Q_A and $\tilde{Q}$ and confirmed two things. One is that the time evolution of $\langle Q_A \rangle$ satisfies the chiral anomaly equation for a two-flavor Dirac fermion. The other is that the expectation value of $\tilde{Q}$ is consistent with the continuum prediction. Thus, we concluded that the diagonal shift operator can be interpreted as the lattice regularized chiral charge. This implies that Q_A is equivalent to the Chern-Simons term [38,39] cohomologically.

We also investigated the relation with the Onsager algebra. In Sec. II C, we constructed the Onsager algebra associated with all shift operators in the three-dimensional space and found that Q_A and $\tilde{Q}$ belong to the Onsager algebra as $\tilde{Q} = (Q_{S_T} + Q_{S_T^{-1}})/2$ and $Q_A = G_{S_T, -S_T}/2$. Since Q_A commutes with all elements in the algebra, it can be interpreted as a central charge. This result supports the statement in [17] that Q_{S_T} is not identified as the chiral charge. In the first place, the chiral anomaly is calculated from a triangle diagram with two vector and one axial currents. We need to take into account the current algebra [26,40,41].

Our results suggest that the construction of a discrete chiral operator [12] performed in the Schwinger model is also possible in higher dimensions. As in the $(1+1)$ dimension, a shift operator in a specific direction induces a chiral charge in the $(3+1)$ dimension. Therefore, if we clarify the relationship between this chiral charge and the discrete chiral operator, we can expect to construct a quantum electrodynamics Hamiltonian that explicitly possesses chiral symmetry. This will significantly improve the convergence of numerical calculations to the continuum limit, which is likely to contribute to the simulation of physical phenomena involving theta terms.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this

research project. The data are available from the authors upon reasonable request.

APPENDIX A: KOGUT-SUSSKIND HAMILTONIAN ON THE TASTE BASIS

The KS fermion field $\chi(x)$ is a single-component fermion. However, KS fermions on $2^3 = 8$ cube sites are equivalent to $2^{\frac{3+1}{2}} = 2$ -flavor Dirac fermions, which is the so-called “taste basis” [4,16]. The Dirac fermions live on the blocked lattice whose sites are labeled by r , and internal cubic lattice sites are identified by A whose component takes the value 0 or 1. Then the original lattice sites are $x = 2r + A$. Now, we set $U_i(x) = 1$ and introduce the Dirac fermion fields as

$$u_{\alpha,f}(r) = \frac{1}{2} \sum_A \chi(2r + A) \sigma_{\alpha f}^{(A)}, \quad (\text{A1})$$

$$d_{\alpha,f}(r) = \frac{1}{2} \sum_A \chi(2r + A) \epsilon(A) \sigma_{\alpha f}^{(A)}, \quad (\text{A2})$$

where α and f denote the spinor and flavor indices, respectively. u and d correspond to the upper and lower components of the $2^{\frac{3+1}{2}} = 4$ -component Dirac spinor.¹³ $\sigma^{(A)}$ is a Pauli matrix determined by

$$\sigma^{(A)} = \sigma_1^{A_1} \sigma_2^{A_2} \sigma_3^{A_3}. \quad (\text{A3})$$

Setting $T = \hat{1} + \hat{2} + \hat{3}$ and $\bar{B} = T - B$, we have

$$\sigma^{(\bar{A})} = \sigma^{(T-A)} = \sqrt{-1}(-1)^{A_2} \sigma^{(A)} \quad (\text{A4})$$

and

$$\text{tr}(\sigma^{(A)\dagger} \sigma^{(B)}) = 2(\delta_{AB} + \sqrt{-1}(-1)^{A_2} \delta_{A\bar{B}}). \quad (\text{A5})$$

Then, χ is written as

$$\chi(2r + A) = \frac{1}{2} \text{tr}(\sigma^{(A)\dagger} u(r) + \epsilon(A) \sigma^{(A)\dagger} d(r)) \quad (\text{A6})$$

in terms of u and d . Using the completeness relations

$$\sum_A \sigma_{\alpha f}^{(A)} (\sigma^{(A)})_{g\beta}^\dagger = 4\delta_{\alpha\beta} \delta_{fg}, \quad \sum_A \sigma_{\alpha f}^{(A)} \epsilon(A) (\sigma^{(A)})_{g\beta}^\dagger = 0, \quad (\text{A7})$$

the mass term becomes

$$\sum_x \chi(x)^\dagger \epsilon(x) \chi(x) = \sum_r \text{tr}(d^\dagger(r) u(r) + u^\dagger(r) d(r)). \quad (\text{A8})$$

¹³When the spatial dimension is D , a $2^{\frac{D+1}{2}}$ -flavor Dirac fermion with $2^{\frac{D+1}{2}}$ components emerges [16].

On the other hand, the kinetic term is

$$\begin{aligned} \sum_x \eta_i(x) \chi^\dagger(x) (\chi(x + \hat{i}) - \chi(x - \hat{i})) &= \sum_{r, A_i=0} \eta_i(A) \chi^\dagger(2r + A) (\chi(2r + A + \hat{i}) - \chi(2(r - \hat{i}) + A + \hat{i})) \\ &+ \sum_{r, A_i=1} \eta_i(A) \chi^\dagger(2r + A) (\chi(2(r + \hat{i}) + A - \hat{i}) - \chi(2r + A - \hat{i})). \end{aligned} \quad (\text{A9})$$

The first is written by

$$\begin{aligned} &\sum_{A_i=0} \eta_i(A) \chi^\dagger(2r + A) \chi(2r + A + \hat{i}) \\ &= \sum_{A_i=0} (u(r) + \epsilon(A) d(r))_{f\alpha}^\dagger \sigma_{\alpha f}^{(A)} \eta(A) (\sigma^{(A+\hat{i})})_{g\beta}^\dagger (u(r) + \epsilon(A + \hat{i}) d(r))_{\beta g} \\ &= \sum_{A_i=0} (u(r) + \epsilon(A) d(r))_{f\alpha}^\dagger \sigma_{\alpha f}^{(A)} \eta(A) (\sigma^{(A+\hat{i})})_{g\beta}^\dagger (u(r) - \epsilon(A) d(r))_{\beta g} \frac{1 + (-1)^{A_i}}{2} \\ &= \frac{1}{2} (u^\dagger(r) (\sigma^i \otimes 1) u(r) - d^\dagger(r) (\sigma^i \otimes 1) d(r) - u^\dagger(r) (1 \otimes {}^t\sigma^i) d(r) + d^\dagger(r) (1 \otimes {}^t\sigma^i) u(r)). \end{aligned} \quad (\text{A10})$$

The first term of the tensor product is a spin matrix, and the second one is a taste matrix.

We calculate the others, and we have

$$\begin{aligned} H &= \frac{1}{2} \sum_r \left[u(r)^\dagger \sigma_i \otimes 1 \frac{\nabla_i - \nabla_i^\dagger}{2\sqrt{-1}} u(r) - d(r)^\dagger \sigma_i \otimes 1 \frac{\nabla_i - \nabla_i^\dagger}{2\sqrt{-1}} d(r) \right. \\ &\quad \left. + u(r)^\dagger 1 \otimes {}^t\sigma_i \frac{\nabla_i + \nabla_i^\dagger}{2\sqrt{-1}} d(r) - d(r)^\dagger 1 \otimes {}^t\sigma_i \frac{\nabla_i + \nabla_i^\dagger}{2\sqrt{-1}} u(r) + 2m(u^\dagger(r) d(r) + d^\dagger(r) u(r)) \right], \end{aligned} \quad (\text{A11})$$

where $\nabla_i u(x) = u(x + \hat{i}) - u(x)$ and $\nabla_i^\dagger u(r) = u(r - \hat{i}) - u(r)$.

Let us define the $(d + 1)$ -dimensional spinor by

$$\psi_f = \begin{pmatrix} u_f \\ d_f \end{pmatrix}. \quad (\text{A12})$$

This allows us to translate the Hamiltonian into the taste basis,

$$H = \frac{1}{2} \sum_r \psi^\dagger \left[(\alpha^i \otimes 1) \frac{\nabla_i - \nabla_i^\dagger}{2\sqrt{-1}} \right. \quad (\text{A13})$$

$$\left. - (\beta \gamma_5 \otimes {}^t\sigma^i) \frac{\nabla_i + \nabla_i^\dagger}{2\sqrt{-1}} + 2m(\beta \otimes 1) \right] \psi, \quad (\text{A14})$$

where

$$\begin{aligned} \alpha^i &= \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \gamma_5 &= -\sqrt{-1} \alpha^1 \alpha^2 \alpha^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (\text{A15})$$

The second term is regarded as a Wilson term, but mixes the flavors. In the continuum limit, this term gives doublers

heavy mass, and the KS fermion system converges to the $N_f = 2$ -flavor Dirac fermion system.

Note that the shift operator T_i acts on u , d , and ψ as

$$\begin{aligned} T_i u(r) &= (\sigma_i \otimes 1) u(r) + \frac{1}{2} (\sigma_i \otimes 1) \nabla_i u(r) \\ &\quad + \frac{1}{2} (1 \otimes {}^t\sigma_i) \nabla_i d(r), \end{aligned} \quad (\text{A16})$$

$$\begin{aligned} T_i d(r) &= -(\sigma_i \otimes 1) d(r) - \frac{1}{2} (\sigma_i \otimes 1) \nabla_i d(r) \\ &\quad - \frac{1}{2} (1 \otimes {}^t\sigma_i) \nabla_i u(r), \end{aligned} \quad (\text{A17})$$

$$\begin{aligned} T_i \psi(r) &= (\alpha^i \otimes 1) \psi(r) \\ &\quad + \frac{1}{2} (\alpha^i \otimes 1 - \beta \gamma_5 \otimes {}^t\sigma^i) \nabla_i \psi(r). \end{aligned} \quad (\text{A18})$$

Thus, it is equivalent to $\alpha^i \otimes 1$ in the continuum limit.

1. Parity, time reversal, and charge conjugation

Parity (5), time reversal (6), and charge conjugation (9) can be rewritten as

$$P^{-1}\psi(r)P = (\beta\gamma_5 \otimes 1)\Gamma^{-1}\psi(-r) \rightarrow (\beta \otimes 1)\psi(-r), \quad (A19)$$

$$T^{-1}\psi(r)T = T_1T_3\psi(r) \rightarrow (\alpha_1\alpha_3 \otimes 1)\psi(r), \quad (A20)$$

$$C^{-1}\psi(r)C = (\beta \otimes 1)T_2\psi^*(r) \rightarrow (\beta\alpha_2 \otimes 1)\psi^*(r), \quad (A21)$$

on the taste basis. We prove these equations in this section.

Parity changes u to

$$\begin{aligned} P^{-1}u(r)P &= P^{-1}\frac{1}{2}\sum_A \chi(2r+A)\sigma^{(A)}P \\ &= \frac{1}{2}\sum_A \chi(-2r-A)\epsilon(A)\sigma^{(A)}. \end{aligned} \quad (A22)$$

Replacing A with $T-A$, we have

$$\begin{aligned} P^{-1}u(r)P &= \frac{1}{2}\sum_A \chi(-2r+A-T)\epsilon(T-A)\sigma^{(T-A)} \\ &= -\frac{1}{2}\sum_A \sqrt{-1}(-1)^{A_2}\chi(-2r+A-T)\epsilon(A)\sigma^{(A)} \\ &= -\Gamma^{-1}d(-r). \end{aligned} \quad (A23)$$

Similarly, d turns into

$$\begin{aligned} P^{-1}d(r)P &= \frac{1}{2}\sum_A \chi(-2r-A)\sigma^{(A)} \\ &= \frac{1}{2}\sum_A \chi(-2r+A-T)\sigma^{(T-A)} \\ &= \Gamma^{-1}u(-r). \end{aligned} \quad (A24)$$

Then, parity acts on ψ as

$$\begin{aligned} P^{-1}\psi(r)P &= \begin{pmatrix} -\Gamma^{-1}d(-r) \\ \Gamma^{-1}u(-r) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Gamma^{-1}u(-r) \\ -\Gamma^{-1}d(-r) \end{pmatrix} \\ &= (\beta\gamma_5 \otimes 1)\Gamma^{-1}\psi(-r). \end{aligned} \quad (A25)$$

Next, time reversal transforms u as

$$\begin{aligned} Tu(r)T^{-1} &= \frac{1}{2}\sum_A T\chi(2r+A)T^{-1}(\sigma^{(A)})^* \\ &= \frac{1}{2}\sum_A T_1T_3\chi(2r+A)\sigma^{(A)} \\ &= T_1T_3u(r). \end{aligned} \quad (A26)$$

This leads to

$$T^{-1}\psi(r)T = T_1T_3\psi(r). \quad (A27)$$

Finally, charge conjugation for u is given by

$$\begin{aligned} C^{-1}u(r)C &= \frac{1}{2}\sum_A (-1)^{A_2}\epsilon(A)T_2\chi^*(2r+A)\sigma^{(A)} \\ &= \frac{1}{2}\sum_A \epsilon(A)T_2\chi^*(2r+A)(\sigma^{(A)})^* \\ &= T_2d^*(r), \end{aligned} \quad (A28)$$

and we have

$$C^{-1}\psi(r)C = (\beta \otimes 1)T_2\psi^*(r). \quad (A29)$$

2. Axial flavor symmetry

On the taste basis, the axial flavor transformation S_i can be translated as [11]

$$\begin{aligned} S_i\psi(r) &= (\gamma^5 \otimes {}^t\sigma_i)\psi(r) + \frac{1}{2}(\gamma^5 \otimes {}^t\sigma_i - \beta\alpha_i \otimes 1)\nabla_i\psi(r) \\ &\rightarrow (\gamma^5 \otimes {}^t\sigma_i)\psi(r). \end{aligned} \quad (A30)$$

3. Transformation by Q_{S_T} on the taste basis

We derive the action of Q_{S_T} on the taste basis. At first, Q_{S_T} acts on u as

$$\begin{aligned} [Q_{S_T}, u(r)] &= \sum_A (-1)^{A_2}(a(2r+A-T) - \sqrt{-1}b(2r+A+T))\sigma^{(A)} \\ &= \sum_A (-1)^{A_2}\chi^*(2r+A+T)\sigma^{(A)} + (\text{difference term}). \end{aligned} \quad (A31)$$

The first term becomes

$$\begin{aligned} &\sum_A (-1)^{A_2}\chi^*(2r+A+T)\sigma^{(A)} \\ &= \sum_A \chi^*(2r+A+T)(\sigma^{(A)})^* \\ &= \sum_A \epsilon(A)T_1S_2T_3\chi(2r+A)^*(\sigma^{(A)})^* \\ &= T_1S_2T_3d^*(r). \end{aligned} \quad (A32)$$

We perform the same calculation for d , and we have

$$\begin{aligned} [Q_{S_T}, \psi(r)] &= (\beta \otimes 1)T_1S_2T_3\psi^*(r) + (\text{difference term}) \\ &\rightarrow -\sqrt{-1}\beta\alpha_2 \otimes {}^t\sigma_2\psi^*(r). \end{aligned} \quad (A33)$$

Note that the difference term can be ignored in the continuum limit.

APPENDIX B: VECTOR AND AXIAL CURRENT ON THE LATTICE

One may ask whether the conservation laws of the vector and axial currents are preserved on the lattice. In what follows, we derive the conservation laws associated with the vector $U(1)$ and axial $U(1)$ symmetries in lattice space, under the assumption that the field χ obeys the Schrödinger equation,

$$\frac{\partial}{\partial t}\chi(x, t) = -\sqrt{-1}h\chi(x, t). \quad (\text{B1})$$

The vector $U(1)$ charge and current densities are defined as

$$j_V^0(x, t) = \chi^\dagger(x, t)\chi(x, t), \quad (\text{B2})$$

$$j_V^i(x, t) = \frac{1}{2}(\chi^\dagger(x, t)S_i\chi(x, t) + (S_i\chi(x, t))^\dagger\chi(x, t)). \quad (\text{B3})$$

These satisfy the lattice continuity equation,

$$\frac{\partial}{\partial t}j_V^0(x, t) - \nabla_i^\dagger j_V^i(x, t) = 0, \quad (\text{B4})$$

where the difference operator $\nabla_i^\dagger$ is given by

$$\nabla_i^\dagger j_V^i(x, t) = j_V^i(x - \hat{i}, t) - j_V^i(x, t). \quad (\text{B5})$$

We define the axial $U(1)$ charge and current densities as [11]

$$j_A^0(x, t) = \frac{1}{2}(\chi^\dagger(x, t)\Gamma\chi(x, t) + \text{H.c.}), \quad (\text{B6})$$

$$j_A^i(x, t) = \frac{1}{4}(\chi^\dagger(x, t)S_i\Gamma\chi(x, t) + (S_i\chi(x, t))^\dagger\Gamma\chi(x, t) + \text{H.c.}). \quad (\text{B7})$$

They satisfy the lattice continuity equation with an additional source term,

$$\begin{aligned} \frac{\partial}{\partial t}j_A^0(x, t) - \nabla_i^\dagger j_A^i(x, t) \\ = \frac{1}{2}(\sqrt{-1}\chi^\dagger(x, t)[h, \Gamma]\chi(x, t) + \text{H.c.}). \end{aligned} \quad (\text{B8})$$

In particular, if $[h, \Gamma] = 0$, the axial current is strictly conserved.

APPENDIX C: ANOMALY EQUATION IN CONTINUUM SPACETIME

In this section, we review Ref. [26] and derive the chiral anomaly equation on the $(3+1)$ -dimensional Dirac fermion system. The Hamiltonian is given by

$$H = \int_{\mathbb{R}^3} d^3x \psi^\dagger h \psi(x), \quad \left(h = \alpha^i(-\sqrt{-1}\partial_i + A_i)\right), \quad (\text{C1})$$

where $\alpha^i = \sigma_3 \otimes \sigma^i$. We apply the homogeneous magnetic field in the x^3 direction,

$$A_1 = A_3 = 0, \quad A_2 = Bx^1, \quad (\text{C2})$$

for the positive constant $B > 0$. The chiral operator is defined by

$$\gamma_5 = -\sqrt{-1}\alpha^1\alpha^2\alpha^3 = \sigma^3 \otimes 1. \quad (\text{C3})$$

We solve the eigenvalue problem of h . Since there are translational symmetries in x^2 and x^3 directions, we can take

$$\psi(x) = \frac{e^{\sqrt{-1}(k_2x^2 + k_3x^3)}}{2\pi} \xi_s \otimes \eta(x^1). \quad (\text{C4})$$

Then, the Hamiltonian becomes

$$h\psi(x) = \frac{e^{\sqrt{-1}(k_2x^2 + k_3x^3)}}{2\pi} \xi_s \otimes s \begin{pmatrix} k_3 & a(k_2) \\ a^\dagger(k_2) & -k_3 \end{pmatrix} \eta(x^1), \quad (\text{C5})$$

where

$$a(k_2) = -\sqrt{-1}(\partial_1 + k_2 + Bx^3), \quad (\text{C6})$$

$$a^\dagger(k_2) = -\sqrt{-1}(\partial_1 - k_2 - Bx^3), \quad (\text{C7})$$

and the commutation relation is given by

$$[a(k_2), a^\dagger(k_2)] = 2B. \quad (\text{C8})$$

We find a normalizable mode $f(k_2 + Bx^1)$ annihilated by $a(k_2)$,

$$f(k_2 + Bx^1) = \exp\left(-\frac{1}{2}(k_2 + Bx^1)^2\right). \quad (\text{C9})$$

The normalization factor for $(a^\dagger(k_2))^n f$ is determined by

$$\begin{aligned} C_n^2 &= \int dx^1 |(a^\dagger)^n f|^2 \\ &= (2B)^n n! \int dx^1 e^{-(k_2 + Bx^1)^2} \\ &= (2B)^n n! \frac{\sqrt{\pi}}{B}. \end{aligned} \quad (\text{C10})$$

Setting

$$\begin{aligned} \eta_+(n, k_2, k_3, x) &= \frac{1}{\sqrt{(k_3 + E)^2 + 2Bn}} \left(\frac{(k_3 + E) \frac{(a^\dagger)^{n-1}}{C_{n-1}} f}{\sqrt{2Bn} \frac{(a^\dagger)^n}{C_n} f} \right), \quad (C11) \end{aligned}$$

$$\begin{aligned} \eta_-(n, k_2, k_3, x) &= \frac{1}{\sqrt{(k_3 + E)^2 + 2Bn}} \left(\frac{\sqrt{2Bn} \frac{(a^\dagger)^{n-1}}{C_{n-1}} f}{-(k_3 + E) \frac{(a^\dagger)^n}{C_n} f} \right) \quad (C12) \end{aligned}$$

for $n = 1, 2, \dots$, and

$$\eta(n = 0, k_2, k_3, x) = \begin{pmatrix} 0 \\ \frac{1}{C_0} f \end{pmatrix} \quad (C13)$$

for $n = 0$, we find the positive and negative energy states

$$u(s, n, k_2, k_3, x) = \frac{e^{\sqrt{-1}(k_2 x^2 + k_3 x^3)}}{2\pi} \xi_s \otimes \eta_s(n, k_2, k_3, x^1), \quad (C14)$$

$$v(s, n, k_2, k_3, x) = \frac{e^{\sqrt{-1}(k_2 x^2 + k_3 x^3)}}{2\pi} \xi_s \otimes \eta_{-s}(n, k_2, k_3, x^1) \quad (C15)$$

for $n = 1, 2, \dots$, and

$$\begin{aligned} u(n = 0, k_2, k_3, x) &= \frac{e^{\sqrt{-1}(k_2 x^2 + k_3 x^3)}}{2\pi} \xi_{-\text{sign}(k_3)} \otimes \eta(n = 0, k_2, k_3, x^1), \quad (C16) \end{aligned}$$

$$\begin{aligned} v(n = 0, k_2, k_3, x) &= \frac{e^{\sqrt{-1}(k_2 x^2 + k_3 x^3)}}{2\pi} \xi_{\text{sign}(k_3)} \otimes \eta(n = 0, k_2, k_3, x^1) \quad (C17) \end{aligned}$$

for $n = 0$. Let $\Omega = \{(s, n, k_2, k_3)\}$ be a set of parameters characterizing the energy states. The normalizations are given by

$$\int d^3 x u^\dagger(\Omega, x, t) u(\Omega', x, t) = \delta_{\Omega\Omega'}, \quad (C18)$$

$$\int d^3 x v^\dagger(\Omega, x, t) v(\Omega', x, t) = \delta_{\Omega\Omega'}, \quad (C19)$$

$$\int d^3 x u^\dagger(\Omega, x, t) v(\Omega', x, t) = 0, \quad (C20)$$

$$\begin{aligned} \int d\Omega [u(\Omega, x, t) u^\dagger(\Omega, x', t) + v(\Omega, x, t) v^\dagger(\Omega, x', t)] \\ = \delta^{(3)}(x - x'), \quad (C21) \end{aligned}$$

where

$$\int d\Omega = \int dk_2 dk_3 \sum_{n=1}^{\infty} \sum_{s=\pm} + \int dk_2 dk_3 \delta_{n,0}. \quad (C22)$$

Then, the Dirac spinor ψ can be expanded as

$$\psi(x) = \int_{-\infty}^{\infty} dk [b(k) u(k, x) + d^\dagger(k) v(k, x)], \quad (C23)$$

where $b(k)$ and $d(k)$ are annihilation operators with momentum k . The commutator relations are given by

$$\{b(k), b^\dagger(k')\} = \{d(k), d^\dagger(k')\} \delta_{kk'} \quad (C24)$$

and all others are zero. We define the vacuum state such that it is annihilated by all $b(k)$ and $d(k)$,

$$b(k)|0\rangle = d(k)|0\rangle. \quad (C25)$$

The massless Hamiltonian commutes with γ_5 , which is called chiral symmetry. The chiral current is defined as

$$j_{A,c}^0(x) = \lim_{y \rightarrow x} \psi^\dagger(x) \gamma^5 e^{-\sqrt{-1} A_i(y-x)^i} \psi(y), \quad (C26)$$

$$j_{A,c}^i(x) = \lim_{y \rightarrow x} \psi^\dagger(x, t) \alpha^i \gamma^5 e^{-\sqrt{-1} A_i(y-x)^i} \psi(y, t), \quad (C27)$$

with point splitting. Then, we have the vacuum expectation value,¹⁴

$$\begin{aligned} \langle j_{A,c}^0(x) \rangle &= \lim_{y \rightarrow x} \int d\Omega v^\dagger(\Omega, x) \gamma^5 v(\Omega, y) \\ &= \lim_{y \rightarrow x} \int_{-\infty}^{\infty} dk^2 dk^3 \frac{e^{\sqrt{-1}(k_2(y^2-x^2) + k_3(y^3-x^3))}}{(2\pi)^2} \left[\text{sign}(k_3) \eta(0, k_2, k_3, x^1)^2 + \sum_{n=1}^{\infty} (\eta_-(n, k_2, k_3, x^1)^2 - \eta_+(n, k_2, k_3, x^1)^2) \right] \\ &= \lim_{y \rightarrow x} \int_{-\infty}^{\infty} dk^2 dk^3 \frac{e^{\sqrt{-1} k_3(y^3-x^3)}}{(2\pi)^2} \text{sign}(k_3) \frac{1}{C_0^2} e^{-(k_2+Bx^1)^2} \\ &= \lim_{y \rightarrow x} \frac{1}{(2\pi)^2} B 2\sqrt{-1} \frac{1}{y^3-x^3} = \lim_{y \rightarrow x} \sqrt{-1} \frac{B}{2\pi^2} \frac{1}{y^3-x^3}. \quad (C28) \end{aligned}$$

Using the rotational transformation, the general form is obtained by

¹⁴We regularize by inserting $e^{-\varepsilon|k_3|}$ and then take the limit $\varepsilon \rightarrow 0^+$.

$$\langle j_{A,c}^0(x) \rangle = \lim_{y \rightarrow x} \sqrt{-1} \frac{B_i}{2\pi^2} \frac{(y-x)^i}{||y-x||^2}. \quad (\text{C29})$$

Finally, we derive the anomaly equation by adding an adiabatic electric field. We carry out the same calculation as the previous section, and we have

$$\langle \partial_\mu j_{A,c}^\mu(x, t) \rangle = \lim_{y \rightarrow x} \sqrt{-1} \dot{A}_i (y-x)^i \sqrt{-1} \frac{B_j}{2\pi^2} \frac{(y-x)^j}{||y-x||^2}. \quad (\text{C30})$$

Unlike the case of $(1+1)$ -dimensional, there are many paths from y to x , and the limit depends on the choice of

them. According to [26], we should take the limit along the magnetic axis. That is, we set $y = \epsilon B + x$ and take the limit of $\epsilon \rightarrow 0$ as $\lim_{y \rightarrow x}$. Then, we find the anomalous divergence

$$\langle \partial_\mu j_{A,c}^\mu(x, t) \rangle = -\frac{E_i B_i}{2\pi^2} \quad (\text{C31})$$

and

$$\frac{d}{dt} \langle \mathcal{Q}_{A,c}(t) \rangle = \frac{d}{dt} \int d^3x j_{A,c}^0(x, t) = - \int d^3x \frac{E_i B_i}{2\pi^2}. \quad (\text{C32})$$

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