Homotopy kinematic algebras at null infinity

Felipe Díaz-Jaramillo,[1,*] Silvia Nagy,[2,†] and Giorgio Pizzolo[2,‡]

[1]*Institute for Physics, Humboldt University Berlin, Zum Großen Windkanal 6, D-12489 Berlin, Germany*
[2]*Department of Mathematical Sciences, Durham University, Durham, DH1 3LE, United Kingdom*

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We present the first formulation of a homotopy algebra adapted to a $1/r$ expansion near future null infinity ($\mathcal{I}^+$). Focusing on self-dual Yang-Mills theory in Bondi coordinates, we demonstrate that imposing the homotopy algebra relations naturally yields the physically consistent fall-off behavior of the fields near $\mathcal{I}^+$. Furthermore, we employ this framework to systematically construct kinematic algebras, uncovering novel infinite families of such algebras that satisfy the Jacobi identity on slices near $\mathcal{I}^+$.

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I. INTRODUCTION

Holography, initially in the context of the anti–de Sitter/conformal field theory (AdS/CFT) correspondence [1], has garnered significant attention for its profound insights on both sides of the duality. More recently, celestial holography, which represents its flat-space counterpart, has been gaining a lot of traction.[1] In this framework, an important step is understanding the behavior of theories close to the relevant boundary of flat space-time, which is null infinity. This in turn controls the so-called asymptotic symmetries, which govern the behaviors or scattering amplitudes, and have natural counterparts in the study of CFT correlators.

Yet another connection between gravity and quantum field theory emerges through the double copy program, which posits that gravity can be obtained by "squaring" Yang-Mills theory [4,5]. In the context of scattering amplitudes, this process relies on the so-called color-kinematics duality which requires that the color and kinematic information of Yang-Mills[2] amplitudes obey the same algebraic relations. The color information is encoded in the structure constants (or traces of generators) of a Lie algebra, while the kinematic information is contained in the polarization vectors and momenta of the external particles participating in the scattering event. Thus, color-kinematics duality suggests the existence of an algebraic structure describing the kinematics of Yang-Mills theory dubbed the *kinematic algebra*. Due to color being encoded in a Lie algebra, the search for the kinematic algebra was initially also focused on Lie algebras.

Early success was found in self-dual Yang-Mills theory (SDYM) [6] where a convenient scalar description of the theory was available [7–15]. This has led to an elegant double copy to self-dual gravity and numerous extensions [6,16–39]. Kinematic algebras have also been revealed in Chern-Simons theory [40–42], the nonlinear sigma model [43], a certain formulation of YM following from heavy-mass effective field theory [25,44,45], via pure spinor actions [46,47], and even in Abelian theories [48].

In recent years, the algebraic structures underlying perturbative field theories, called homotopy algebras, have been used to formalise the construction of kinematic algebras and systematize the implementation of the double copy [41,49–56]. Homotopy algebras, as we shall explain in Sec. III, are generalizations of more commonly known algebras in the physics literature, such as *strict* Lie algebras,[3] that encode the symmetries and dynamics of perturbative field theories. Formulating physical theories with color degrees of freedom in this framework allows one to algebraically factorize the theories in a well-defined manner into color and kinematics, construct a kinematic algebra, and finally take the tensor product of two copies of the kinematic algebra which yields a new homotopy algebra that encodes a gravitational theory.

Notably, in general, the kinematic algebra that arises from the homotopy-algebraic framework is not a strict Lie algebra. Rather, the general structure obtained by these means is a so-called $\mathrm{BV}^{\square}_{\infty}$ algebra, first introduced by

*Contact author: felipe.diaz-jaramillo@hu-berlin.de
†Contact author: silvia.nagy@durham.ac.uk
‡Contact author: giorgio.pizzolo@durham.ac.uk

[1]See reviews [2,3] and references within.

[2]This also holds for other theories with color degrees of freedom.

[3]These are generally just referred to as Lie algebras in the physics literature, however, we use the term strict to differentiate from homotopy algebras which will feature in the article.

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TABLE I. Slices manifesting strict kinematic algebras. The slices are defined in Eq. (1.2). Note that while solutions 1-3 can be obtained via gauge choices, the families of solutions 4-7 require further elimination of unphysical d.o.f. and/or a restriction to a subsector of the theory. In the limit $k \to -\infty$ and upon imposing the Lorenz gauge condition, slices 6 and 7 reproduce two versions of the area-preserving diffeomorphisms in [6]. The families of solutions 6–9 in the shaded boxes only become strict upon imposing additional constrains coming from certain components of the e.o.m.. See Sec. VI for more details.

	$k=-1$				$k \leq -2$						
	t^{-1}_{μ}	$f^{-1}_{1,\mu}$	$f^{-1}_{2,\mu}$	$f^{-1}_{3,\mu}$	t^{k}_{μ}	$f^{k}_{4,\mu}$	$f^{k}_{5,\mu}$	$f^{k}_{6,\mu}$	$f^{k}_{7,\mu}$	$f^{k}_{8,\mu}$	$f^{k}_{9,\mu}$
r	-3	-2	-2	-3	$k-2$	-2	$k-2$	$k-1$	-2	-2	$k-1$
u	-2	-1	0	0	$k-1$	$k+1$	0	0	k	0	k
z	-1	0	0	-1	k	0	k	0	$k+1$	$k+1$	0
$\bar{z}$	-2	0	-1	0	$k-1$	$k+1$	0	k	0	k	0

Reiterer in [49] to understand color-kinematics duality. To this end, however, one can work with simpler, strict algebras at the cost of locality [52,54,57] or by making particular gauge choices. Moreover, in the context of holography, homotopy algebras have been used to transfer information from the bulk to the boundary [58,59].

Thus homotopy algebras hold great promise in both the study of holography, and that of the double copy. In this paper, we make this idea more tangible by focusing on self-dual Yang-Mills theory at null infinity. We demonstrate that the physical fall-off behavior of fields, gauge parameters, and equations of motion can be derived from a homotopy algebraic perspective. Furthermore, we identify a strict kinematic Lie algebra that arises in carefully selected slices of the radial expansion near the boundary, providing a novel approach to its formulation.

Our strategy is as follows: we begin by recasting the homotopy algebra formulation of SDYM in flat Bondi

TABLE II. Homotopy algebras defined in Sec. III.

Algebra	Definition	Physics
L_∞	Cochain complex $(\mathcal{X}, B_1)$ equipped with a set of graded symmetric (3.4) multilinear maps (n-brackets) $\{B_n\colon \mathcal{X}^{\otimes n} \to \mathcal{X}\}_{n\geq 1}$ obeying generalized Jacobi relations, the first of which are (3.5) and (3.8). Homotopy generalization of *Lie algebras*. If B_n vanishes for all $n \geq 3$ it is a differential graded Lie algebra (dgLa).	Encodes perturbative field theories: the differential B_1 and the higher maps B_n describe the free dynamics and interactions, respectively, whereas the generalized Jacobi relations ensure the consistency of the theory (e.g. gauge covariance and closure of the gauge algebra). See for instance Eq. (3.22).
C_∞	Cochain complex $(\mathcal{K}, m_1)$ equipped with a set of graded symmetric (3.24) multilinear maps (n-products) $\{m_n\colon \mathcal{K}^{\otimes n} \to \mathcal{K}\}_{n\geq 1}$ obeying homotopy associativity relations, the first of which are (3.25)–(3.27). Homotopy generalization of *associative commutative algebras*. If m_n vanishes for all $n \geq 3$ it is a differential graded commutative algebra (dgca).	Encodes the color-stripped version of theories with color degrees of freedom within the framework of homotopy algebras. For instance, given the L_∞ algebra describing self-dual Yang-Mills, the corresponding C_∞ algebra comes from the decomposition (3.29), by equipping the graded vector space $\mathcal{K}^{\mathrm{SDYM}}$ with the two-product m_2 arising from (3.31).
$\mathrm{BV}^{\Box}_{\infty}$	Collection $(\mathcal{K}, m_1, b, \Box, m_2, b_2, \theta_3, \ldots)$ where (i) $(\mathcal{K}, m_1, m_2, \ldots)$ is a C_∞ algebra, (ii) $b\colon \mathcal{K} \to \mathcal{K}$ is a nilpotent map of degree -1 satisfying the defining relation (3.36) with the operator $\Box$, (iii) b_2 is the kinematic two-bracket (3.38), satisfying (3.40), (iv) θ_3 is the trilinear map appearing in (3.40). Homotopy generalization of *Poisson algebras*.	*Kinematic algebra* encoding the relations that the kinematic numerators in a theory with color degrees of freedom have to obey, in the context of the color-kinematics duality. For instance, for self-dual Yang-Mills it is constructed from the dgca $(\mathcal{K}^{\mathrm{SDYM}}, m_1, m_2)$ by introducing the adjoint de Rham operator b (3.37) and the d'Alembert operator $\Box$.

coordinates and then perform an expansion in powers of 1/r. Without making any physical assumptions, we consider slices corresponding to approximations of varying precision near null infinity. By requiring that these slices preserve all the defining conditions of the homotopy algebras, we derive constraints on the fall-off behavior of the fields that precisely match the physical fall-off conditions described in [60,61] and reviewed in Sec. II B

$$A_r = \frac{A_r^{(-2)}}{r^2} + \cdots, \qquad A_u = A_u^{(0)} + \frac{A_u^{(-1)}}{r} + \cdots,$$
$$A_z = A_z^{(0)} + \cdots, \qquad A_{\bar{z}} = A_{\bar{z}}^{(0)} + \cdots \tag{1.1}$$

as detailed in Sec. V.

Armed with a well-defined homotopy algebra near null infinity, we then construct the kinematic algebra, and specifically look for strict Lie algebra versions of these. We manage to find a number of novel ways to partially gauge fix and truncate the fields in order to reveal new strict kinematic algebras at null infinity. The solutions are of the form

$$A_\mu = \sum_{n=t_\mu^k}^{f_{i,\mu}^k} r^n A_\mu^{(n)} \tag{1.2}$$

where t_μ^k and $f_{i,\mu}^k$, with $t_\mu^k \leq f_{i,\mu}^k \leq 0$, define "slices" in the expansion in $1/r$. We refer to t_μ^k as the truncation, as it encodes the precision to which we work in $1/r$, whereas $f_{i,\mu}^k$ is determined by the fall-off (1.1), together with possible additional constraints. We find infinite families of solutions labeled by k, where $|k|$ encodes the "thickness" of the slice, summarized in the table below

The article is organized as follows: in Sec. II, we start with a brief overview of the two main topics to be studied. The first is the physical fall-off of fields, equations of motion and gauge parameters in YM in a radial expansion near null infinity, and the second is the kinematic algebra arising as a consequence of color-kinematics duality in the double copy programme. In Sec. III we review our main mathematical tool, i.e. the homotopy algebras. Please see Table II for a concise summary. In the brief Sec. IV we present the homotopy algebra formulation of SDYM in flat Bondi coordinates, which are best adapted for studying asymptotic phenomena in this theory. In Sec. V, we formulate and solve the constraints arising from the requirement that the projection to slices near null infinity preserves the defining conditions of homotopy algebras, giving an alternative derivation of the physical fall-off (1.1). In Sec. VI, we employ the homotopy algebra machinery to find new infinite families of kinematic algebras, as summarized in Table I. Then in Sec. VII we introduce some further homotopy algebra machinery, namely cochain maps and quasi-isomorphisms, and use these to give an alternative proof for the fall-off. We conclude in Sec. VIII. Appendix A contains the analytic proof of the solution presented in Sec. V, while Appendix B details the construction of the strict kinematic algebras. Finally, Appendix C gives details of the homotopy transfer procedure.

II. PHYSICAL MOTIVATION

In this section we review the two areas where we will be applying the homotopy algebra techniques. In Sec. II A, we give a very brief overview of colo-kinematics duality, and the emergence of the kinematic algebra, focusing on SDYM. Then, in Sec. II B, we give a quick description of gauge fields near null infinity.

A. Color-kinematics duality and the kinematic algebra

Color-kinematics duality is a revolutionary observation about scattering amplitudes for non-Abelian gauge theory [4,5], which has led to immense progress in relating gravity to gauge theory via the double copy. To state it, we first formulate n-point amplitudes as sums over cubic diagrams

$$\mathcal{A}_n = g^{n-2} \sum_i \frac{c_i n_i}{D_i}, \tag{2.1}$$

where c_i are the color factors (consisting of products of the structure constants of the gauge group), n_i are kinematic factors (depending on momenta and polarization vectors), and D_i are products of Feynman propagators. The observation is that there exists a gauge freedom at the level of the expression above, such that we can choose kinematic factors which exactly mimic the behavior of the color factors. The two relevant ingredients are antisymmetry under exchange of particle labels

$$c_i \to -c_i \Rightarrow n_i \to -n_i \tag{2.2}$$

and the Jacobi identity

$$c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0. \tag{2.3}$$

The left-hand side (lhs) of the above is of course just a consequence of the fact that the gauge algebra is a *strict* Lie algebra.[4] This suggests the existence of a strict Lie algebra governing the kinematics of the theory, which has been dubbed the kinematic algebra.

A very clean manifestation of this algebra arises in the self-dual sector of Yang-Mills theory, given by the following constraint in Lorentzian signature:

$$F_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \tag{2.4}$$

[4]Strict Lie algebras are usually just referred to as Lie algebras in the physics literature, but we need to be more precise in order to differentiate from the structures introduced later in the paper.

The above encapsulates both the e.o.m. and Bianchi identities. Working in light-cone coordinates

$$x^- = \frac{1}{\sqrt{2}}(t-z), \qquad x^+ = \frac{1}{\sqrt{2}}(t+z),$$
$$w = \frac{1}{\sqrt{2}}(x+iy), \qquad \bar{w} = \frac{1}{\sqrt{2}}(x-iy), \tag{2.5}$$

with line element

$$ds^2 = -2dx^- dx^+ + 2dwd\bar{w}. \tag{2.6}$$

In order to reveal the kinematic algebra, we first impose the light-cone gauge

$$A_+ = 0. \tag{2.7}$$

Then additionally solving the constraint equations coming from (2.4), we find

$$A_w = 0. \tag{2.8}$$

Working with the remaining nonvanishing components $A^m = (A^+, A^w) = (A_-, A_{\bar{w}})$, we find a kinematic bracket in the form of a Schouten-Nijenhuis bracket ([62])

$$[\mathcal{A}_1, \mathcal{A}_2]^m_{\mathrm{SN}} = \mathcal{A}_1^n \partial_n \mathcal{A}_2^m - \mathcal{A}_2^n \partial_n \mathcal{A}_1^m. \tag{2.9}$$

In the above, $\mathcal{A}^m$ is the so-called color-stripped version on A^m (see Sec. III for details), i.e. we are treating it as a standard 2 dimensional vector. The bracket (3.45) corresponds to a diffeomorphism algebra, which is additionally area-preserving due to the constraint $\partial_m \mathcal{A}^m = 0$, arising from (2.4). This, of course, satisfies Jacobi, and is the relevant bracket underpinning the kinematic factors. Upon further expressing the gauge fields in terms of scalars, we can recover the famed original expression in [6] (see [62] for details).

Let us pause to take stock of how the kinematic algebra arose. Note that we had to

(i) find a convenient gauge choice,
(ii) eliminate a nonpropagating degree of freedom,

in order to reveal the presence of a strict kinematic algebra.

Can we systematize the search for setups manifesting strict kinematic algebras? A promising tool are the so-called homotopy algebras, reviewed in detail in Sec. III. Within this context, there exists an algorithmic way of constructing the kinematic algebra, which eliminates the guesswork [49]. The trade-off is that the outcome is not usually a strict algebra, i.e. the Jacobi Identity (2.3) is relaxed. Fortunately, the failure to satisfy (2.3) is precisely encoded in an object called the Jacobiator, also constructed in an algorithmic way from the data in the theory. This greatly simplifies the task of determining under which circumstances the kinematic algebra will be a strict one, by looking at how to set the Jacobiator to 0.

Even though the double copy procedure can be performed even in the absence of a strict kinematic algebra [42,50,51,55,63–65], its presence provides dramatic simplifications of the Yang-Mills theory, and consequently of the map to the resulting gravity theory. This has exploited for example in pushing the remit of the double copy to curved backgrounds [27,33].

Later in the article we will show that there is a novel way of revealing strict kinematic algebras, which will be related to taking appropriate slices in the fall-of of fields close to null infinity.

B. Review of physical fall-off at null infinity

We will work in the so-called flat Bondi coordinates $x^\mu = (r, u, z, \bar{z})$ in which the leading order part of the metric is

$$ds^2 = -2drdu + 2r^2 dzd\bar{z}. \tag{2.10}$$

In these coordinates, the celestial sphere becomes a complex plane spanned by z and $\bar{z}$. These are particularly suited for working in the self-dual sector, and they have a particularly simple map to the light-cone coordinates (2.5)

$$x^- = rz\bar{z} + u, \qquad x^+ = r, \qquad w = rz, \qquad \bar{w} = r\bar{z}. \tag{2.11}$$

The fall-off will be completely analogous to the more standard Bondi coordinates.

Here we will give a brief review of the physical motivation for the fall-offs that are generally assumed for the gauge potential at $\mathcal{I}$ [60,61]

$$A_r = \frac{A_r^{(-2)}}{r^2} + \cdots, \qquad A_u = A_u^{(0)} + \frac{A_u^{(-1)}}{r} + \cdots,$$
$$A_z = A_z^{(0)} + \cdots, \qquad A_{\bar{z}} = A_{\bar{z}}^{(0)} + \cdots \tag{2.12}$$

In the above, we assume that the coefficients in the r-expansion are functions of $y^\alpha = (u, z, \bar{z})$. The fall-off above and the motivation below apply to both YM and Maxwell theory. By extension, it will also apply to the self-dual sector of these theories.[5] We are seeking the most general fall-off that yields physically reasonable solutions. To start off, we recall that the electric field should fall off as

$$E_r = \frac{E_r^{(-2)}}{r^2} + \cdots \tag{2.13}$$

The corresponding field strength component in Bondi coordinates is

[5]Recall that (2.4) encodes the general YM equations; this can be seen by contracting with the gauge-covariant derivative.

$$F_{ur} = \frac{F_{ur}^{(-2)}}{r^2} + \cdots \tag{2.14}$$

This already tells us that A_r starts at $\mathcal{O}(\frac{1}{r^2})$, so we can further conclude that

$$F_{rz} = \frac{F_{rz}^{(-2)}}{r^2} + \cdots \tag{2.15}$$

We then consider the energy flux, encoded in the T_{uu} component of the energy momentum tensor

$$T_{\mu\nu} = \mathrm{Tr}\Big(F_{\mu\rho}F^{\rho}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}\Big). \tag{2.16}$$

Requiring that T_{uu} falls of like

$$T_{uu} = \frac{T_{uu}^{(-2)}}{r^2} + \cdots \tag{2.17}$$

which is needed in order to get a finite integrated flux, and using (2.14) and (2.15) we get

$$T_{uu} \approx \frac{F_{uz}F_{u\bar{z}}}{r^2} + \cdots \tag{2.18}$$

where the ... refer to terms with a sharper fall-off. We deduce that

$$F_{uz} = F_{uz}^{(0)} + \cdots \tag{2.19}$$

Finally, it is straightforward to see that the fall-offs in (2.12) follow from (2.14), (2.15), and (2.19) above. We remark that it is very common to set

$$A_u^{(0)} = 0, \tag{2.20}$$

however this does not strictly follow from the physical considerations above. Generally, (2.20) can be thought of as a partial gauge fixing; indeed it is often imposed in conjunction with Lorenz gauge $\nabla^\mu A_\mu = 0$, or radial gauge $A_r = 0$. In both these cases, the residual symmetry in these gauges is sufficient to impose (2.20). A useful consequence of (2.20) is that the soft gluon mode N_z can be written straightforwardly as a difference between the free data $A_z^{(0)}$ at future timelike infinity and spatial infinity. Specifically, we have

$$\begin{aligned} N_z &= \lim_{\omega\to 0}\int_{-\infty}^{\infty} du F_{uz}^{(0)} e^{i\omega u} = \int_{-\infty}^{\infty} du F_{uz}^{(0)} \\ &= A_z^{(0)}|_{\mathcal{I}_+^+} - A_z^{(0)}|_{\mathcal{I}_-^+} \end{aligned} \tag{2.21}$$

where to get to the last line we used (2.20).

We pause here to comment on an implicit assumption in the fall-offs above: we have assumed the absence of log terms. More generally, the gauge field could be expanded as a polyhomogeneous function

$$A_\mu = \sum_{n,k\geq 0} A_\mu^{(-n;k)}(y^\alpha)\frac{\log^k r}{r^n}, \tag{2.22}$$

The assumption that the log terms are absent (i.e. $A_\mu^{(-n;k)}(y^\alpha) = 0$ for $k \neq 0$) is a well motivated one at tree-level, and is supported by the polynomial tree level fall-offs in the u variable which relate to subleading effects.[6] In certain specific gauges, such as harmonic gauge [67], or when including loop effects, it may be that one needs to relax the fall-off assumption to something of the form in (2.22). We leave the study of this more general scenario for future work.

The fall-off in the gauge parameter following from the considerations above is

$$\Lambda = \Lambda^{(0)} + \cdots \tag{2.23}$$

where $\Lambda^{(0)}$ is a function of $(z, \bar{z})$. The celebrated Weinberg soft theorem in gauge theory [68] can be derived, at leading order, as a Ward identity [60,69] for the large gauge symmetry with parameter $\Lambda^{(0)}$, where the term large refers to the fact that $\Lambda^{(0)}$ does not vanish in the limit $r \to \infty$, unlike a standard gauge symmetry.

III. HOMOTOPY ALGEBRAS

An algebra is a vector space equipped with a bilinear map. We classify algebras according to the types of relations that the map obeys, which here we call *defining relations*. For example, an associative algebra is a vector space equipped with a bilinear map that is associative, while a Lie algebra is a vector space equipped with an antisymmetric bilinear map that obeys the Jacobi identity. We call these algebras *strict* because their defining relations are obeyed strictly. Homotopy algebras are generalizations of strict algebras in that the defining relations are relaxed: they are not obeyed strictly, but *up to homotopy*. Mathematically, this means that there exist higher multiplicity maps (sometimes called higher brackets or higher products) that control the failure of the defining relations to hold strictly.

In this section, we present the definition of the homotopy generalization of Lie algebras, called L_∞ algebras, as well as the homotopy generalization of associative commutative algebras, called C_∞ algebras, and finally, more exotic algebras called $\mathrm{BV}_\infty^\square$ algebras. These algebras have been observed to be relevant in the mathematical description of the double copy program. In the following, we present

[6] See e.g. [66] for more details.

these mathematical concepts in the context of physical theories, in particular, in the context of self-dual Yang-Mills theory.

A. L_∞ algebras

To define an L_∞ algebra, it will be useful to first define a linear structure called a *cochain complex*. A cochain complex $(\mathcal{X}, B_1)$ is a graded vector space $\mathcal{X} = \bigoplus_i X_i$ equipped with a nilpotent linear map $B_1 \colon \mathcal{X} \to \mathcal{X}$ called a *differential*. The integer label $i \in \mathbb{Z}$ is called the *degree* of the vector space, and it is inherited by the elements of each space, i.e., for an element $x \in X_i$ we say that x has degree i. In our conventions the differential has degree $+1$, namely $B_1 \colon X_i \to X_{i+1}$, and nilpotency means $B_1(B_1(x)) = 0$. Diagrammatically, we can represent cochain complexes as follows:

$$\ldots \xrightarrow{B_1} X_{-1} \xrightarrow{B_1} X_0 \xrightarrow{B_1} X_1 \xrightarrow{B_1} \ldots \ . \tag{3.1}$$

Let us remark that the differential can act differently on each space, as long as it is nilpotent.

The cohomology of the differential in degree i is defined as the equivalence class

$$H^i = \frac{\mathrm{Ker} B_1^{(i)}}{\mathrm{Im}\, B_1^{(i-1)}}, \tag{3.2}$$

where the superscripts in the differential denote the space where it is acting (we omit the superscripts in the following) and the kernel and image of the differential are subspaces of the cochain complex defined as

$$\begin{aligned} \mathrm{Ker} B_1^{(i)} &= \{x \in X_i | B_1(x) = 0\}, \\ \mathrm{Im}\, B_1^{(i-1)} &= \{x \in X_i | x = B_1(y) \text{ with } y \in X_{i-1}\}. \end{aligned} \tag{3.3}$$

We say that an element of the kernel of the differential is closed, while an element of the image is exact. The cohomology is then the space of elements of $\mathcal{X}$ that are closed but not exact. A prominent example of a cochain complex is the de Rahm complex in differential geometry, where the graded vector space is the sum of the spaces of forms of all degrees and the differential is the de Rahm differential. In this case, the cohomology is related to the Euler number of the manifold where the differential forms are defined. In the L_∞ formulation of field theories, the cohomology of the differential H^i encodes the physics of the theory, as we shall see below when we formulate self-dual Yang-Mills theory in this framework.

An L_∞ algebra is a cochain complex $(\mathcal{X}, B_1)$ equipped with a set of multilinear maps (or n-brackets) $B_n \colon \mathcal{X}^{\otimes n} \to \mathcal{X}$ with $n \geq 1$ that obey a set of relations called generalized Jacobi relations. The maps B_n are graded symmetric, which means that we can exchange the order of adjacent input elements of the maps at the cost of a sign determined by the degree of the elements, namely

$$B_n(\ldots, x_l, x_{l+1}, \ldots) = (-1)^{x_l x_{l+1}} B_n(\ldots, x_{l+1}, x_l, \ldots), \tag{3.4}$$

where the x's in the exponent denote the degree of the elements x_l and x_{l+1}. The number of nontrivial generalized Jacobi relations depends on the number of nontrivial maps B_n which, in principle, may be infinite. The first few generalized Jacobi relations are the nilpotency of the differential

$$B_1(B_1(x)) = 0, \tag{3.5}$$

the Leibniz rule of the differential with respect to the two-bracket B_2

$$\begin{aligned} &B_1(B_2(x_1, x_2)) + B_2(B_1(x_1), x_2) \\ &\quad + (-1)^{x_1} B_2(x_1, B_1(x_2)) = 0, \end{aligned} \tag{3.6}$$

and the Jacobi identity *up to homotopy*

$$\begin{aligned} &B_2(B_2(x_1, x_2), x_3) + (-1)^{x_1(x_2+x_3)} B_2(B_2(x_2, x_3), x_1) \\ &\quad + (-1)^{x_3(x_1+x_2)} B_2(B_2(x_3, x_1), x_2) \\ &\quad + B_1(B_3(x_1, x_2, x_3)) + B_3(B_1(x_1), x_2, x_3) \\ &\quad + (-1)^{x_1} B_3(x_1, B_1(x_2), x_3) \\ &\quad + (-1)^{x_1+x_2} B_3(x_1, x_2, B_1(x_3)) = 0. \end{aligned} \tag{3.7}$$

The first line of this relation is the combination of terms that would vanish if B_2 obeyed the (graded) Jacobi identity (also called the Jacobiator). The relation states that in an L_∞ algebra we allow the strict Jacobi identity to fail by terms involving the differential B_1 and a higher map B_3 or, in more technical terms, that the Jacobi identity is obeyed in the cohomology of the differential. For notational convenience and compactness, in the following, we write the Leibniz rule and the Jacobi identity up to homotopy as

$$\begin{aligned} \mathrm{Leib}(B_1, B_2) &\equiv [B_1, B_2] = 0, \\ \mathrm{Jac}(B_2) + [B_1, B_3] &= 0, \end{aligned} \tag{3.8}$$

respectively, with the Jacobiator $\mathrm{Jac}(B_2)$ defined as

$$\begin{aligned} \mathrm{Jac}(B_2)(x_1, x_2, x_3) = {} & B_2(B_2(x_1, x_2), x_3) \\ & + (-1)^{x_1(x_2+x_3)} B_2(B_2(x_2, x_3), x_1) \\ & + (-1)^{x_3(x_1+x_2)} B_2(B_2(x_3, x_1), x_2). \end{aligned} \tag{3.9}$$

We also introduced the graded commutator of multilinear maps

$$[O_1, O_2] = O_1 O_2 - (-1)^{O_1 O_2} O_2 O_1, \tag{3.10}$$

which in the case of the homotopy Jacobi relation is given by

$$\begin{aligned}[B_1,B_3](x_1,x_2,x_3) &= B_1(B_3(x_1,x_2,x_3)) \\ &+ B_3(B_1(x_1),x_2,x_3) \\ &+ (-1)^{x_1} B_3(x_1,B_1(x_2),x_3) \\ &+ (-1)^{x_1+x_2} B_3(x_1,x_2,B_1(x_3)). \end{aligned} \tag{3.11}$$

Let us emphasize that, in general, there are higher relations between the maps B_2, B_3, and B_4, and so on. These higher relations, however, will play no role in this paper, so we omit them.

L_∞ algebras are relevant in physics because one can encode perturbative field theories as data of L_∞ algebras. Generally speaking, the differential B_1 and the higher maps B_n encode the free dynamics and interactions of the theory, respectively, whereas the generalized Jacobi relations encode the consistency of the theory, including gauge covariance, closure of the gauge algebra, and so on. In the following, we will describe the L_∞ formulation of self-dual Yang-Mills theory. Let us recall that self-dual Yang-Mills theory consists of a gauge field (or one-form) $A = A^a_\mu dx^\mu \otimes t_a$ valued in a gauge algebra $\mathfrak{g}$ with generators t_a. In the following, for the sake of compactness, we shall use the formalism of differential forms for which we use the following conventions:

$$\begin{aligned}\omega &= \frac{1}{r!}\omega_{\mu_1\ldots\mu_r} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}, \\ \mathrm{d}\omega &= \frac{1}{r!}\partial_\nu \omega_{\mu_1\ldots\mu_r} \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}, \\ *\omega &= \frac{\sqrt{|g|}}{r!(4-r)!}\omega_{\mu_1\ldots\mu_r}\epsilon^{\mu_1\ldots\mu_r}{}_{\nu_{r+1}\ldots\nu_4} \mathrm{d}x^{\nu_{r+1}} \wedge \ldots \wedge \mathrm{d}x^{\nu_4}. \end{aligned}$$

The dynamics of the gauge field are determined by the first-order equation

$$2P_- dA + P_-[A,A] = 0, \tag{3.12}$$

where the projector onto antiself-dual two-forms is defined as

$$P_- := \frac{1}{2}(1-*), \tag{3.13}$$

while the bracket $[\cdot,\cdot]$ is a combination of the wedge product of forms and the structure constants of the Lie algebra of the color Lie group, namely, for two color-valued forms ω and η, the bracket is defined as

$$[\omega,\eta] = \omega^a \wedge \eta^b f_{ab}{}^c t_c. \tag{3.14}$$

The self-dual Yang-Mills field equations (3.12) are covariant with respect to the non-Abelian gauge transformation

$$\delta A = d\Lambda + [A,\Lambda], \tag{3.15}$$

generated by the $\mathfrak{g}$-valued zero-form $\Lambda = \Lambda^a t_a$.

In practice, we start formulating field theories in the framework of homotopy algebras by organizing the elements of the theory (gauge parameters, fields, and field equations) in a cochain complex. In the particular case at hand, we organize the elements of the theory in a cochain complex $(\mathcal{X}^{\mathrm{SDYM}}, B_1)$ with $\mathcal{X}^{\mathrm{SDYM}} = \bigoplus_{i=-1}^{1} X_i$ as

$$\begin{array}{ccccc} X_{-1} & \xrightarrow{B_1} & X_0 & \xrightarrow{B_1} & X_1 \\ \Lambda & & A & & E \end{array}, \tag{3.16}$$

where the space X_{-1} is the space of gauge parameters ($\mathfrak{g}$-valued zero-forms) $\Lambda = \Lambda^a t_a$, X_0 is the space of gauge fields ($\mathfrak{g}$-valued one-forms) $A = A^a_\mu dx^\mu \otimes t_a$, and X_1 is the space of field equations ($\mathfrak{g}$-valued antiself-dual two-forms) $E = \frac{1}{2}E^a_{\mu\nu} dx^\mu \wedge dx^\nu \otimes t_a$. Next, to read off the action of the differential and the higher maps, we assume that the field equations can be written perturbatively as

$$B_1(A) + \frac{1}{2}B_2(A,A) = 0, \tag{3.17}$$

while the gauge transformations are assumed to be

$$\delta A = B_1(\Lambda) + B_2(A,\Lambda). \tag{3.18}$$

Comparing the field equations (3.12) and gauge transformation (3.15) with (3.17) and (3.18), we conclude that the differential and the two-bracket act as

$$\begin{aligned} &B_1(A) = 2P_- dA \in X_1, \quad B_1(\Lambda) = d\Lambda \in X_0 \\ &B_2(A^1,A^2) = 2P_-[A^1,A^2] \in X_1, \quad B_2(A,\Lambda) = [A,\Lambda] \in X_0. \end{aligned} \tag{3.19}$$

As we advertised previously, the differential B_1 encodes the free theory, that is, the linearized field equations and the linearized gauge transformation, while the bracket B_2 encodes the nonlinearities in the field equations and gauge transformations. In addition to the above maps, there also exist the following maps that arise from the algebra of gauge transformations and gauge covariance of the field equations

$$\begin{aligned} B_2(\Lambda^1,\Lambda^2) &= -[\Lambda^1,\Lambda^2] \in X_{-1}, \\ B_2(\Lambda,E) &= -[\Lambda,E] \in X_1. \end{aligned} \tag{3.20}$$

Notice that in this case there are no higher maps than the two-bracket B_2.

Finally, one checks that the maps obey the generalized Jacobi relations mentioned above. Due to the absence of higher brackets, in self-dual Yang-Mills theory we only

need to check the nilpotency of the differential, the Leibniz rule of the differential with respect to the two-bracket, and the strict graded Jacobi identity. These relations encode the consistency of the theory. As an example, let us comment on the nilpotency of the differential. Let us act twice with the differential on a gauge parameter

$$\begin{aligned} B_1(B_1(\Lambda)) &= B_1(d\Lambda) \\ &= 2P_- d(d\Lambda) \equiv 0, \end{aligned} \tag{3.21}$$

where in the last step we used $d^2 = 0$. Notice that the outermost B_1 acts on $B_1(\Lambda) \in X_0$ which is a field, which justifies the presence of the projectors P_-. The above simple computation is the statement of gauge invariance of the linearized theory. Consequently, it encodes information about the physical propagating degrees of freedom of the theory which are elements of the cohomology of the differential B_1. Indeed, the cohomology of B_1 in degree zero is

$$H^0 = \{A \in X_0 | B_1(A) = 0 \text{ and } A \neq B_1(\Lambda)\}, \tag{3.22}$$

which are field configurations that satisfy the linear field equations modulo linearized gauge transformations, which correspond to the theory's physical or propagating degrees of freedom. In that sense, the generalized Jacobi relations encode the consistency of the theory. Proving the Leibniz rule and the graded Jacobi identity of B_2 amounts to checking that the theory is gauge covariant, that gauge transformations close, and so on. We will not explicitly show this in this paper.

Let us close this section by noting that since self-dual Yang-Mills theory has only up to bilinear terms in the field equations and gauge transformations, we only have up to bilinear brackets B_2, and no higher brackets. This algebraic structure is called a *strict* L_∞ algebra, or a *differential graded Lie algebra* (dgLa). In the full theory of Yang-Mills, however, the field equations have up to trilinear terms (coming from the quartic interactions in the action), and as a consequence, the theory also has a map B_3. In contrast, in the perturbative expansion of general relativity, the field equations have infinite coupling terms. Thus the algebra that describes general relativity has an infinite number of higher maps.

B. C_∞ algebras

The double copy program in scattering amplitudes relies on *color decomposition* or *color stripping*: the fact that Yang-Mills scattering amplitudes[7] factorize into color and kinematic information. Similarly, when formulating Yang-Mills theory in the framework of homotopy algebras, a color decomposition also occurs. More precisely, the L_∞ algebra $\mathcal{X}^{\mathrm{YM}}$ that describes Yang-Mills theory factorizes as the tensor product[8] $\mathcal{X}^{\mathrm{YM}} = \mathcal{K}^{\mathrm{YM}} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra that encodes color, while $\mathcal{K}^{\mathrm{YM}}$ is a homotopy algebra called C_∞ algebra that generalizes associative commutative algebras. In this section, we define C_∞ algebras and discuss the algebraic color decomposition and C_∞ algebra of self-dual Yang-Mills theory.

C_∞ algebras are homotopy generalizations of associative commutative algebras, where the associativity relation is relaxed to hold up to homotopy. More precisely, a C_∞ algebra is a cochain complex $(\mathcal{K}, m_1)$

$$\ldots \xrightarrow{m_1} K_0 \xrightarrow{m_1} K_1 \xrightarrow{m_1} K_2 \xrightarrow{m_1} \ldots \ , \tag{3.23}$$

equipped with a set of multilinear products $m_n \colon \mathcal{K}^{\otimes n} \to \mathcal{K}$ that obey a set of homotopy associativity relations. In our conventions, the products m_n have degree $2-n$ and have the symmetry property of *vanishing on shuffles* which for the two-product means that it is graded symmetric, namely

$$m_2(u_1, u_2) = (-1)^{u_1 u_2} m_2(u_2, u_1), \tag{3.24}$$

while for the higher products, there are more complicated relations that will not be relevant in this paper. The first few homotopy associativity relations are the nilpotency of the differential

$$m_1(m_1(u)) = 0, \tag{3.25}$$

the Leibniz rule of the differential with respect to the two-product m_2

$$\begin{aligned} &m_1(m_2(u_1, u_2)) - m_2(m_1(u_1), u_2) \\ &\quad - (-1)^{u_1} m_2(u_1, m_1(u_2)) = 0, \end{aligned} \tag{3.26}$$

and associativity up to homotopy

$$\begin{aligned} &m_2(m_2(u_1, u_2), u_3) - m_2(u_1, m_2(u_2, u_3)) \\ &= m_1(m_3(u_1, u_2, u_3)) + m_3(m_1(u_1), u_2, u_3) \\ &\quad + (-1)^{u_1} m_3(u_1, m_1(u_2), u_3) \\ &\quad + (-1)^{u_1 + u_2} m_3(u_1, u_2, m_1(u_3)). \end{aligned} \tag{3.27}$$

Similarly to the Jacobi identity up to homotopy in L_∞ algebras, the homotopy associativity relation above tells us

[7] Or amplitudes of any other theory with color degrees of freedom.

[8] This factorization also holds for other theories with color degrees of freedom such as Chern-Simons theory and, as we show more concretely at the end of the section, self-dual Yang-Mills theory.

that the failure of the product m_2 to be associative is governed by a higher product m_3 and the differential m_1.

Let us now turn to the color decomposition of the dgLa underlying self-dual Yang-Mills theory. Recall that the elements of the vector space $\mathcal{X}^{\mathrm{SDYM}}$ are $\mathfrak{g}$-valued differential forms. Hence, a generic element of $\mathcal{X}^{\mathrm{SDYM}}$ can be written as

$$x = u^a \otimes t_a, \tag{3.28}$$

where u^a are either zero-, one-, or antiself-dual two-forms, suggesting, analogously to Yang-Mills theory, the following factorization of the vector space

$$\mathcal{X}^{\mathrm{SDYM}} = \mathcal{K}^{\mathrm{SDYM}} \otimes \mathfrak{g}, \tag{3.29}$$

where $\mathcal{K}^{\mathrm{SDYM}}$ is an infinite-dimensional graded vector space encoding kinematics and spacetime dependence, and $\mathfrak{g}$ is a finite-dimensional vector space encoding color. Moreover, the maps that act on $\mathcal{X}^{\mathrm{SDYM}}$ exhibit such factorization. Indeed, given that the differential B_1 acts as a first-order differential operator, its action factorizes as

$$B_1(x) = m_1(u^a) \otimes t_a. \tag{3.30}$$

Similarly, the two-bracket B_2 factorizes as

$$B_2(x_1, x_2) = (-1)^{x_1} m_2(u_1^b, u_2^c) \otimes f_{bc}^a t_a, \tag{3.31}$$

where the sign is conventional and f_{bc}^a are the structure constants of $\mathfrak{g}$.

The vector space $\mathcal{K}^{\mathrm{SDYM}} = \bigoplus_{i=0}^{2} K_i$ equipped with the differential m_1 makes up a cochain complex

$$\begin{array}{ccccc} K_0 & \xrightarrow{m_1} & K_1 & \xrightarrow{m_1} & K_2 \\ \lambda & & \mathcal{A} & & \mathcal{E} \end{array}, \tag{3.32}$$

where we have dropped the color indices because they are irrelevant in this kinematic setting, and we use a different font for the elements of $\mathcal{K}$ relative to (3.16) to avoid confusion between colorful and color-less objects. Equipping this cochain complex with the two-product m_2 yields a strict C_∞ algebra also called a differential graded commutative algebra, or dgca, which can be easily shown by checking the homotopy associativity relations mentioned above using the explicit action of the maps on the elements of $\mathcal{K}^{\mathrm{SDYM}}$, which reads

$$\begin{aligned} m_1(\lambda) &= d\lambda \in K_1, \qquad m_1(\mathcal{A}) = 2P_- d\mathcal{A} \in K_2 \\ m_2(\lambda_1, \lambda_2) &= \lambda_1 \wedge \lambda_2 \in K_0, \qquad m_2(\lambda, \mathcal{A}) = \lambda \wedge \mathcal{A} \in K_1 \\ m_2(\mathcal{A}_1, \mathcal{A}_2) &= 2P_-(\mathcal{A}_1 \wedge \mathcal{A}_2) \in K_2, \\ m_2(\lambda, \mathcal{E}) &= \lambda \wedge \mathcal{E} \in K_2. \end{aligned} \tag{3.33}$$

C. Kinematic algebras and $\mathrm{BV}_\infty^\square$ algebras

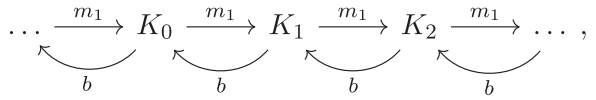
As discussed in Sec. II A, a necessary condition for the double copy of scattering amplitudes is that color-kinematics duality be fulfilled. In that section, we discussed the duality in terms of algebraic relations that the kinematic numerators of Yang-Mills amplitudes have to obey. We further showed that, for the particular case of self-dual Yang-Mills theory in light-cone gauge, there is a *kinematic algebra* underlying the relations between the numerators, namely the Lie algebra of area-preserving diffeomorphisms. Recently, in [62], using the framework of homotopy algebras, a more general, gauge-independent algebra for self-dual Yang-Mills theory was constructed which, upon imposing light-cone gauge and eliminating an unphysical degree of freedom, led to the Lie algebra of area-preserving diffeomorphisms. The gauge-independent algebraic structure constructed in [62] is a so-called kinematic $\mathrm{BV}_\infty^\square$ algebra. $\mathrm{BV}_\infty^\square$ algebras were first introduced by Reiterer in [49] to construct a kinematic algebra for pure Yang-Mills theory and later used in [42,50,51,55,63–65] to construct gravity (and double field theory) using off-shell algebraic methods. The main goal of this section is to introduce $\mathrm{BV}_\infty^\square$ algebras and explain how they can be constructed in the context of color-kinematics duality and the double-copy, especially in self-dual Yang-Mills theory.

Before formally introducing $\mathrm{BV}_\infty^\square$ algebras, it will be illustrative to first recall the definition of a Poisson algebra. A Poisson algebra is a vector space V equipped with a commutative associative two-product $\circ : V \times V \to V$ and a Lie bracket $\{\cdot,\cdot\} : V \times V \to V$ that obey the following compatibility relation:

$$\{f \circ g, h\} - \{g, h\} \circ f - \{f, h\} \circ g = 0, \qquad f, g, h \in V. \tag{3.34}$$

A common Poisson algebra in physics is the Poisson algebra of classical mechanics, where V is the phase space, the associative commutative product $\circ$ is the product of functions of phase space, and the Lie bracket is the Poisson bracket of classical mechanics. In words, Poisson algebras consist of an associative, commutative subsector given by $\circ$, and a Lie subsector given by $\{\cdot,\cdot\}$. Both subsectors are united by the compatibility relation (3.34). $\mathrm{BV}_\infty^\square$ algebras are homotopy generalizations of Poisson algebras, as we show in the following.

To construct a kinematic $\mathrm{BV}_\infty^\square$ algebra for self-dual Yang-Mills, we start with the dgca $(\mathcal{K}^{\mathrm{SDYM}}, m_1, m_2)$ that we found after color stripping in the previous section, and introduce a nilpotent map b: $\mathcal{K}^{\mathrm{SDYM}} \to \mathcal{K}^{\mathrm{SDYM}}$ of degree -1. Diagrammatically, b acts as



(3.35)

and (graded) commutes with the differential m_1 to the d'Alembert operator in Minkowski space,[9] namely

$$[b, m_1](u) \equiv m_1(b(u)) + b(m_1(u)) = \Box u, u \in \mathcal{K}^{\mathrm{SDYM}}. \tag{3.36}$$

The operator b that satisfies the condition above is the adjoint de Rham operator

$$b = d^\dagger := -\star d \star. \tag{3.37}$$

This operator, as one can explicitly check, does not obey the Leibniz rule with respect to m_2. We parametrize this failure by defining a *kinematic two-bracket*

$$\begin{aligned} b_2(u_1, u_2) &:= [b, m_2](u_1, u_2) \equiv b m_2(u_1, u_2) \\ &\quad - m_2(b u_1, u_2) - (-1)^{u_1} m_2(u_1, b u_2), \end{aligned} \tag{3.38}$$

which acts explicitly as

$$\begin{aligned} b_2(\lambda, \mathcal{A}) &= d^\dagger(\mathcal{A} \wedge \lambda) - d^\dagger \mathcal{A} \wedge \lambda \in K_0, \\ b_2(\mathcal{A}_1, \mathcal{A}_2) &= 2 d^\dagger P_-(\mathcal{A}_1 \wedge \mathcal{A}_2) - 2 d^\dagger \mathcal{A}_{[1} \wedge \mathcal{A}_{2]} \in K_1, \\ b_2(\lambda, \mathcal{E}) &= \star(\mathcal{E} \wedge d\lambda) \in K_1, \\ b_2(\mathcal{A}, \mathcal{E}) &= 2 P_-(\mathcal{A} \wedge d^\dagger \mathcal{E}) - d^\dagger \mathcal{A} \wedge \mathcal{E} \in K_2, \end{aligned} \tag{3.39}$$

A $\mathrm{BV}_\infty^\Box$ algebra is a collection $(\mathcal{K}, m_1, b, \Box, m_2, b_2, \theta_3, \ldots)$ where the two-product m_2 and the two-bracket b_2 obey a generalization of the Poisson compatibility relation (3.34). More precisely, the maps obey the Poisson compatibility relation up to terms involving a trilinear map θ_3, i.e.,

$$\begin{aligned} &b_2(m_2(u_1, u_2), u_3) - (-1)^{u_1(u_2+u_3)} m_2(b_2(u_2, u_3), u_1) \\ &\quad - (-1)^{u_3(u_1+u_2)} m_2(b_2(u_3, u_1), u_2) = [m_1, \theta_3](u_1, u_2, u_3), \end{aligned} \tag{3.40}$$

with

$$\begin{aligned} [m_1, \theta_3](u_1, u_2, u_3) &:= m_1 \theta_3(u_1, u_2, u_3) - \theta_3(m_1 u_1, u_2, u_3) \\ &\quad - (-1)^{u_1} \theta_3(u_1, m_1 u_2, u_3) \\ &\quad - (-1)^{u_1+u_2} \theta_3(u_1, u_2, m_1 u_3), \end{aligned} \tag{3.41}$$

[9]The explicit form of $\Box$ naturally depends on our choice of coordinates.

Notice that, upon identifying b_2 with $\{\cdot, \cdot\}$ and m_2 with $\circ$, the left-hand side of Eq. (3.40) has the same combination of terms (albeit graded) as the left-hand side of equation (3.34). In the former, however, the right-hand side is nonvanishing and the trilinear map θ_3 is a homotopy map that controls the failure of m_2 and b_2 to be Poisson compatible. Indeed, θ_3 plays the same role as B_3 in L_∞ algebras or m_3 in C_∞ algebras. The homotopy map θ_3 has the following two nonvanishing components:

$$\begin{aligned} \theta_3(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) &= -\star(\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \mathcal{A}_3) \in K_1, \\ \theta_3(\mathcal{E}, \mathcal{A}_1, \mathcal{A}_2) &= 2 P_-\{\star(\mathcal{E} \wedge \mathcal{A}_{[1}) \wedge \mathcal{A}_{2]}\} \in K_2. \end{aligned} \tag{3.42}$$

which were determined by explicit computation in [62]. To higher order in inputs, a $\mathrm{BV}_\infty^\Box$ algebra has further compatibility relations that the nonlinear maps obey, which means that the algebra may have higher homotopies, e.g., a quadrilinear map θ_4, and so on. In this paper, however, we only consider this algebra up to trilinear maps because we are interested in finding a strict algebra with a vanishing θ_3, i.e., we wish to find a regime where m_2 and b_2 are Poisson compatible and there are no higher homotopies θ_n.

Let us remark that the general definition of a $\mathrm{BV}_\infty^\Box$ algebra allows one to have associativity up to homotopy, as is the case in pure Yang-Mills theory, where after color stripping one obtains a C_∞ algebra with a nontrivial m_3. For a more detailed description of the construction of $\mathrm{BV}_\infty^\Box$ algebras, refer to [49,50].

The failure of m_2 and b_2 to be Poisson compatible, and the fact that b_2 is derived from m_2, implies that the kinematic bracket b_2 does not obey the strict Jacobi identity. Instead, it obeys the Jacobi identity up to homotopy and *modulo box*, i.e.,

$$\begin{aligned} &\mathrm{Jac}(b_2)(u_1, u_2, u_3) + [m_1, b_3](u_1, u_2, u_3) \\ &\quad + [\Box, \theta_3](u_1, u_2, u_3) = 0, \end{aligned} \tag{3.43}$$

where the kinematic three-bracket b_3 is derived as $b_3 = -[b, \theta_3]$ and the graded commutators that parametrize the failure of the Jacobi identity are given by

$$\begin{aligned} [m_1, b_3](u_1, u_2, u_3) &:= m_1 b_3(u_1, u_2, u_3) + b_3(m_1 u_1, u_2, u_3) \\ &\quad + (-1)^{u_1} b_3(u_1, m_1 u_2, u_3) \\ &\quad + (-1)^{u_1+u_2} b_3(u_1, u_2, m_1 u_3), \\ [\Box, \theta_3](u_1, u_2, u_3) &:= \Box \theta_3(u_1, u_2, u_3) - \theta_3(\Box u_1, u_2, u_3) \\ &\quad - \theta_3(u_1, \Box u_2, u_3) - \theta_3(u_1, u_2, \Box u_3), \\ [b, \theta_3](u_1, u_2, u_3) &:= b \theta_3(u_1, u_2, u_3) - \theta_3(b u_1, u_2, u_3) \\ &\quad - (-1)^{u_1} \theta_3(u_1, b u_2, u_3) \\ &\quad - (-1)^{u_1+u_2} \theta_3(u_1, u_2, b u_3). \end{aligned} \tag{3.44}$$

Hence, the gauge-invariant kinematic $\mathrm{BV}_\infty^\square$ we constructed here does not have a strict Lie algebra subsector. Notice, however, that for a vanishing θ_3, which in turn implies a vanishing b_3, b_2 would obey the strict Jacobi identity. Indeed, as shown in [62], if one imposes light-cone gauge and integrates out an unphysical degree of freedom, then the kinematic bracket b_2 reduces, in general, to the Schouten-Nijenhuis bracket of polyvectors, which for the particular case of two gauge fields is the Lie bracket of area-preserving diffeomorphisms

$$b_2(\mathcal{A}_1,\mathcal{A}_2)^m=[\mathcal{A}_1,\mathcal{A}_2]^m_{\mathrm{SN}}=\mathcal{A}_1^n\partial_n\mathcal{A}_2^m-\mathcal{A}_2^n\partial_n\mathcal{A}_1^m. \tag{3.45}$$

This tells us that, to significantly simplify the kinematic algebra and obtain a strict kinematic Lie algebra, one needs to find either particular gauge choices or regimes of the self-dual Yang-Mills.

D. Summary and notation

IV. HOMOTOPY ALGEBRA FORMULATION OF SDYM IN BONDI COORDINATES

We now wish to translate the results of the previous section in a language appropriate for taking the limit to null infinity. We choose to work with the flat Bondi coordinates

$$x^\mu=(r,u,z,\bar z)\equiv(r,y^\alpha). \tag{4.1}$$

due to the simplicity of the map from the light-cone coordinates, see (2.11). In the flat Bondi coordinates, the celestial sphere is replaced by the complex plane via a stereographic projection. The flat Bondi metric, given in (2.10), is reproduced below for convenience

$$ds^2=-2drdu+2r^2dzd\bar z. \tag{4.2}$$

A. L_∞ algebra

Let $\mathcal{M}$ be a manifold with global coordinates $x^\mu=(r,y^\alpha)$. We define $C^\infty_r(\mathcal{M})$ as the space of all C^∞ functions on the manifold $\mathcal{M}$ that admit an expansion in powers of r

$$C^\infty_r(\mathcal{M}):=\left\{f\in C^\infty(\mathcal{M})|f(r,y^\alpha)=\sum_{n\in\mathbb{Z}}r^nf^{(n)}(y^\alpha)\right\}. \tag{4.3}$$

As in Sec. III A, we define the following graded vector space

$$\mathcal{X}:=\textstyle\bigoplus_{i=-1}^1X_i. \tag{4.4}$$

where X_{-1}, X_0 and X_1 are, respectively, the spaces of scalars λ, covariant vectors A_μ and skew-symmetric antiself-dual (0,2)-tensors $E_{\mu\nu}$ over the manifold M. Moreover, $\forall\ \psi\in\chi$, ψ is $\mathfrak{g}$-valued and has the expansion

$$\psi(r,u,z,\bar z)=\sum_{n=-\infty}^{+\infty}r^n\psi^{(n)}(u,z,\bar z). \tag{4.5}$$

Note that at this point we have not yet truncated, i.e. we consider all powers in r. Using the skew-symmetry and the antiself-duality of elements in X_1, we find that a suitable basis is given by $\mathcal{B}_{X_1}=\{\underline{e}^i_{\mu\nu}\}_{i=1,2,3}$, with

$$\underline{e}^1_{\mu\nu}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&r^2\\0&0&-r^2&0\end{pmatrix},\quad \underline{e}^2_{\mu\nu}=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{pmatrix},$$
$$\underline{e}^3_{\mu\nu}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}. \tag{4.6}$$

Therefore, a generic element $E_{\mu\nu}\in X_1$ is of the form

$$E_{\mu\nu}=\begin{pmatrix}0&E_{ru}&0&E_{r\bar z}\\-E_{ru}&0&E_{uz}&0\\0&-E_{uz}&0&r^2E_{ru}\\-E_{r\bar z}&0&-r^2E_{ru}&0\end{pmatrix}. \tag{4.7}$$

In other words, in order to determine completely a tensor in X_1 it is sufficient to specify three of its components, namely E_{ru}, $E_{r\bar z}$, and E_{uz}. The B_1 operator ($B_1:\mathcal{X}\to\mathcal{X}$) defined in (3.19) becomes, in Bondi coordinates,

$$\begin{aligned}B_1(\Lambda)_\mu&=\partial_\mu\Lambda\in X_0\\B_1(A)_{\mu\nu}&=2(\partial_{[r}A_{u]}+r^{-2}\partial_{[z}A_{\bar z]})\underline{e}^1_{\mu\nu}\\&\quad+4\partial_{[r}A_{\bar z]}\underline{e}^2_{\mu\nu}+4\partial_{[u}A_{z]}\underline{e}^3_{\mu\nu}\in X_1.\end{aligned} \tag{4.8}$$

Similarly $B_2:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ can be written as follows:

$$\begin{aligned}B_2(\Lambda_1,\Lambda_2)&=-[\Lambda_1,\Lambda_2]\in X_{-1}\\B_2(\Lambda,A)_\mu&=-[\Lambda,A_\mu]\in X_0\\B_2(\Lambda,E)_{\mu\nu}&=-[\Lambda,E_{\mu\nu}]\in X_1\\B_2(A_1,A_2)_{\mu\nu}&=2([A_{1[r},A_{2u]}]+r^{-2}[A_{1[z},A_{2\bar z]}])\underline{e}^1_{\mu\nu}\\&\quad+4[A_{1[r},A_{2\bar z]}]\underline{e}^2_{\mu\nu}+4[A_{1[u},A_{2z]}]\underline{e}^3_{\mu\nu}\in X_1.\end{aligned} \tag{4.9}$$

Then $(\mathcal{X},B_1,B_2)$ is an L_∞ algebra, with B_1 and B_2 satisfying the generalized Jacobi relations (3.5), (3.6), and (3.7) with $B_3=0$.

B. Color stripping and kinematic algebra

We color-strip as in Sec. III B, to construct the differential graded commutative algebra $(\mathcal{K}, m_1, m_2)$,[10] with

$$\begin{aligned} m_1(\lambda)_\mu &= \partial_\mu\lambda \in K_1 \\ m_1(\mathcal{A})_{\mu\nu} &= 2(\partial_{[r}\mathcal{A}_{u]} + r^{-2}\partial_{[z}\mathcal{A}_{\bar{z}]})\underline{e}^1_{\mu\nu} \\ &\quad + 4\partial_{[r}\mathcal{A}_{\bar{z}]}\underline{e}^2_{\mu\nu} + 4\partial_{[u}\mathcal{A}_{z]}\underline{e}^3_{\mu\nu} \in K_2 \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} m_2(\lambda_1,\lambda_2) &= \lambda_1\lambda_2 \in K_0 \\ m_2(\lambda,\mathcal{A})_\mu &= \lambda\mathcal{A}_\mu \in K_1 \\ m_2(\lambda,\mathcal{E})_{\mu\nu} &= \lambda\mathcal{E}_{\mu\nu} \in K_2 \\ m_2(\mathcal{A}_1,\mathcal{A}_2)_{\mu\nu} &= 2(\mathcal{A}_{[1r}\mathcal{A}_{2]u} + r^{-2}\mathcal{A}_{[1z}\mathcal{A}_{2]\bar{z}})\underline{e}^1_{\mu\nu} \\ &\quad + 4\mathcal{A}_{[1r}\mathcal{A}_{2]\bar{z}}\underline{e}^2_{\mu\nu} + 4\mathcal{A}_{[1u}\mathcal{A}_{2]z}\underline{e}^3_{\mu\nu} \in K_2. \end{aligned} \tag{4.11}$$

The construction of the kinematic algebra

$$\mathfrak{Kin} = (\mathcal{K}, m_1, b, \Box, m_2, b_2, \theta_3) \tag{4.12}$$

proceeds as in Sec. III C. The main difference is that for the coordinate system (4.2), not all Christoffel symbols vanish, so, for example, the action of b is

$$\begin{aligned} b(\mathcal{A}) &= -\nabla^\mu\mathcal{A}_\mu = 2(\partial_{(r}\mathcal{A}_{u)} - r^{-2}\partial_{(z}\mathcal{A}_{\bar{z})} + r^{-1}\mathcal{A}_u) \in K_0 \\ b(\mathcal{E})_\mu &= \nabla^\nu\mathcal{E}_{\mu\nu} \in K_1, \end{aligned} \tag{4.13}$$

while

$$\Box := dd^\dagger + d^\dagger d = -\nabla^2 = -\partial^2 + 2r^{-1}\mathfrak{c}, \tag{4.14}$$

where we define

$$\nabla^2 := \nabla_\mu\nabla^\mu, \quad \partial^2 := \partial_\mu\partial^\mu = -2\partial_r\partial_u + 2r^{-2}\partial_z\partial_{\bar{z}}, \tag{4.15}$$

and the operator $\mathfrak{c}$ (that comes from the nonvanishing Christoffel symbols) acts as

$$\begin{aligned} \mathfrak{c}(\Lambda) &= \partial_u\Lambda \\ \mathfrak{c}(\mathcal{A})_\mu &= \partial_\mu\mathcal{A}_u - \delta_{\mu r}(2\partial_{[r}\mathcal{A}_{u]} - 2r^{-2}\partial_{(z}\mathcal{A}_{\bar{z})} + r^{-1}\mathcal{A}_u) \\ \mathfrak{c}(\mathcal{E})_{\mu\nu} &= (\partial_u\mathcal{E}_{ru} - r^{-2}\partial_{\bar{z}}\mathcal{E}_{uz})\underline{e}^1_{\mu\nu} + 2\partial_{\bar{z}}\mathcal{E}_{ru}\underline{e}^2_{\mu\nu}. \end{aligned} \tag{4.16}$$

Finally, we will need the explicit form of the kinematic bracket b_2 (between two gauge fields) and of the trilinear map θ_3 in these coordinates. From (3.38) and (3.42) we obtain, respectively,

$$\begin{aligned} \frac{1}{2}b_2(\mathcal{A}_1,\mathcal{A}_2)_\mu &= \begin{pmatrix} \mathcal{A}_{1(r}\partial_{u)}\mathcal{A}_{2r} \\ \mathcal{A}_{1(r}\partial_{u)}\mathcal{A}_{2u} \\ \mathcal{A}_{1u}\partial_r\mathcal{A}_{2z} + \mathcal{A}_{1[r}\partial_{z]}\mathcal{A}_{2u} - \mathcal{A}_{1[u}\partial_{z]}\mathcal{A}_{2r} \\ \mathcal{A}_{1r}\partial_u\mathcal{A}_{2\bar{z}} - \mathcal{A}_{1[r}\partial_{\bar{z}]}\mathcal{A}_{2u} + \mathcal{A}_{1[u}\partial_{\bar{z}]}\mathcal{A}_{2r} \end{pmatrix} - \frac{1}{r}\begin{pmatrix} 0 \\ 0 \\ \mathcal{A}_{1u}\mathcal{A}_{2z} \\ \mathcal{A}_{1u}\mathcal{A}_{2\bar{z}} \end{pmatrix} \\ &\quad - \frac{1}{r^2}\begin{pmatrix} \mathcal{A}_{1[r}\partial_{\bar{z}]}\mathcal{A}_{2z} - \mathcal{A}_{1[r}\partial_{z]}\mathcal{A}_{2\bar{z}} + \mathcal{A}_{1\bar{z}}\partial_z\mathcal{A}_{2r} \\ -\mathcal{A}_{1[u}\partial_{\bar{z}]}\mathcal{A}_{2z} + \mathcal{A}_{1[u}\partial_{z]}\mathcal{A}_{2\bar{z}} + \mathcal{A}_{1z}\partial_{\bar{z}}\mathcal{A}_{2u} \\ \mathcal{A}_{1(z}\partial_{\bar{z})}\mathcal{A}_{2z} \\ \mathcal{A}_{1(z}\partial_{\bar{z})}\mathcal{A}_{2\bar{z}} \end{pmatrix} - (1 \leftrightarrow 2) \end{aligned} \tag{4.17}$$

and

$$\theta_3(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)_\mu = 6\sqrt{\det g}\,\varepsilon_{\mu\nu\rho\sigma}g^{\nu\tau}g^{\rho\delta}g^{\sigma\epsilon}\mathcal{A}_{[1\tau}\mathcal{A}_{2\delta}\mathcal{A}_{3]\epsilon} = 6\begin{pmatrix} r^{-2}\mathcal{A}_{[1r}\mathcal{A}_{2z}\mathcal{A}_{3]\bar{z}} \\ -r^{-2}\mathcal{A}_{[1u}\mathcal{A}_{2z}\mathcal{A}_{3]\bar{z}} \\ -\mathcal{A}_{[1r}\mathcal{A}_{2u}\mathcal{A}_{3]z} \\ \mathcal{A}_{[1r}\mathcal{A}_{2u}\mathcal{A}_{3]\bar{z}} \end{pmatrix} \in K_1, \tag{4.18}$$

$$\begin{aligned} \theta_3(\mathcal{E},\mathcal{A}_1,\mathcal{A}_2)_{\mu\nu} &= 2r^{-2}(\mathcal{E}_{uz}\mathcal{A}_{[1r}\mathcal{A}_{2]\bar{z}} - \mathcal{E}_{r\bar{z}}\mathcal{A}_{[1u}\mathcal{A}_{2]z})\underline{e}^1_{\mu\nu} \\ &\quad - [4\mathcal{E}_{ru}\mathcal{A}_{[1r}\mathcal{A}_{2]\bar{z}} - 2\mathcal{E}_{r\bar{z}}(\mathcal{A}_{[1r}\mathcal{A}_{2]u} + r^{-2}\mathcal{A}_{[1z}\mathcal{A}_{2]\bar{z}})]\underline{e}^2_{\mu\nu} \\ &\quad + [4\mathcal{E}_{ru}\mathcal{A}_{[1u}\mathcal{A}_{2]z} - 2\mathcal{E}_{uz}(\mathcal{A}_{[1r}\mathcal{A}_{2]u} + r^{-2}\mathcal{A}_{[1z}\mathcal{A}_{2]\bar{z}})]\underline{e}^3_{\mu\nu} \in K_2, \end{aligned} \tag{4.19}$$

[10]Recall that we use a different font relative to the previous section to denote the color-stripped scalars, vectors, and antiself-dual 2-forms.

for $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \in K_1$ and $\mathcal{E} \in K_2$. One of our main goals in this article is to look for strict kinematic algebras close to null infinity, in other words study the conditions under which θ_3 vanishes.

To connect with previous work, if one chooses the gauge $A_u = 0$, which is equivalent to light-cone gauge in Bondi coordinates, and one partially solves the field equations which impose $A_z = 0$ and $\partial_u \mathcal{A}_r = r^{-2}\partial_z \mathcal{A}_{\bar{z}}$, one finds the components of the strict kinematic bracket

$$\begin{aligned} b_2(\mathcal{A}_1,\mathcal{A}_2)_r &= 2\mathcal{A}_{[1r}\partial_u \mathcal{A}_{2]r} - r^{-2} \\ &\quad \times(-2\mathcal{A}_{[1r}\partial_z \mathcal{A}_{2]\bar{z}} + 4\mathcal{A}_{[1\bar{z}}\partial_z \mathcal{A}_{2]r}) \\ &= 4\mathcal{A}_{[1r}\partial_u \mathcal{A}_{2]r} - 4r^{-2}\mathcal{A}_{[1\bar{z}}\partial_z \mathcal{A}_{2]r} = -2[\mathcal{A}_1,\mathcal{A}_2]^{\mathrm{SN}}_r \\ b_2(\mathcal{A}_1,\mathcal{A}_2)_u &= b_2(\mathcal{A}_1,\mathcal{A}_2)_z = 0 \\ b_2(\mathcal{A}_1,\mathcal{A}_2)_{\bar{z}} &= 4\mathcal{A}_{[1r}\partial_u \mathcal{A}_{2]\bar{z}} - 2\mathcal{A}_{[1\bar{z}}\partial_u \mathcal{A}_{2]r} - 2r^{-2}\mathcal{A}_{[1\bar{z}}\partial_z \mathcal{A}_{2]\bar{z}} \\ &= 4\mathcal{A}_{[1r}\partial_u \mathcal{A}_{2]\bar{z}} - 4r^{-2}\mathcal{A}_{[1\bar{z}}\partial_z \mathcal{A}_{2]\bar{z}} \\ &= -2[\mathcal{A}_1,\mathcal{A}_2]^{\mathrm{SN}}_{\bar{z}}, \end{aligned} \tag{4.20}$$

which correspond to the components of the Schouten-Nijenhuis bracket (3.45) in Bondi coordinates.

V. PHYSICAL FALL-OFFS FROM L_∞ ALGEBRAS

In this section and the following one, we formulate self-dual Yang-Mills theory when approaching $\mathcal{I}$, i.e. when $r \to \infty$, using the homotopy algebra formalism again. That is to say, starting from the algebra $(\mathcal{X}, B_1, B_2)$ presented in Sec. IV A, we build a new L_∞ algebra

$$(\mathcal{X}^{\mathcal{I}}, B_1^{\mathcal{I}}, B_2^{\mathcal{I}}) \tag{5.1}$$

that encodes self-dual Yang-Mills in the regime close to null infinity.

In practice, when doing computations close to null infinity, one expands gauge fields and parameters in powers of the radial coordinate r and then truncates the expansion at a certain order. The tools that we introduce here will allow us to implement such truncations algebraically to construct L_∞ algebras that describe the physics of Yang-Mills theory in certain regimes or slices close to null infinity, and consequently, their respective kinematic algebras.

A. Requirements on the L_∞ algebra near $\mathcal{I}$

Before moving on to the explicit construction of the algebra near $\mathcal{I}$, let us focus on the properties that the space and the brackets in (5.1) shall satisfy (in addition to the L_∞ relations) as a consequence of the restriction of the initial theory close to the null infinity regime.

(1) The graded vector space $\mathcal{X}^{\mathcal{I}}$ contains the gauge parameters, gauge fields and field equations near $\mathcal{I}$, where they admit an expansion in (non-negative) powers of $1/r$. Thus, its elements have the same form as (4.4), except now the sum (4.5) does not include the terms with $n > 0$, that are divergent at null infinity (where $r \to \infty$):

$$\mathcal{X}^{\mathcal{I}} \subseteq \mathcal{X}^{n\leq 0} \subset \mathcal{X} \tag{5.2}$$

where we defined $\mathcal{X}^{n\leq 0} := \{\psi \in \mathcal{X} | \psi^{(n)} = 0 \text{ for } n > 0\}$. Note that we allow $\mathcal{X}^{\mathcal{I}}$ to be a subspace of $\mathcal{X}^{n\leq 0}$, to account for the possibility that the following requirements in this list impose further restrictions on $\mathcal{X}^{\mathcal{I}}$ (as is indeed the case). Moreover, for reasons that will be clarified shortly, it is useful to consider the projection

$$\Pi : \mathcal{X} \to \mathcal{X}^{\mathcal{I}} \tag{5.3}$$

from the initial space to the one encoding the physics at null infinity; this map is well-defined by virtue of (5.2). The terms in the expansion (4.5) that diverge at null infinity are all contained in the projection to the subspace complementary to $\mathcal{X}^{\mathcal{I}}$,

$$\Pi^{\complement} := 1_{\mathcal{X}} - \Pi. \tag{5.4}$$

Notice that $\Pi^{\complement}\Pi = \Pi\Pi^{\complement} = 0$, or equivalently $\mathrm{Im}\,\Pi^{\complement} = \mathrm{Ker}\Pi$. In other words, $\mathcal{X} = \mathrm{Im}\,\Pi \oplus \mathrm{Im}\,\Pi^{\complement}$, i.e. for all $\psi \in \mathcal{X}$ it is always possible to write a unique decomposition of the form $\psi = \Pi(\psi) + \Pi^{\complement}(\psi)$.

(2) Since our goal is to transfer the structure of the algebra $(\mathcal{X}, B_1, B_2)$ in the bulk to the new L_∞ algebra close to null infinity, the brackets of the latter shall be related to the ones of the former. Thus, the most natural way to define $B_1^{\mathcal{I}}$ and $B_2^{\mathcal{I}}$ is to restrict the domain of B_1 and B_2 to $\mathcal{X}^{\mathcal{I}}$ and $\mathcal{X}^{\mathcal{I}} \otimes \mathcal{X}^{\mathcal{I}}$, respectively, and then project their image via the map Π, namely

$$B_i^{\mathcal{I}} := \Pi B_i|_{(\mathcal{X}^{\mathcal{I}})^{\otimes i}}, \tag{5.5}$$

for $i = 1, 2$; in the above and in the rest of the paper we use the notation

$$B_i|_{(\mathcal{X}^{\mathcal{I}})^{\otimes i}} : (\mathcal{X}^{\mathcal{I}})^{\otimes i} \to \mathcal{X} \tag{5.6}$$

to denote the restriction of the domain of B_i to the subspace $(\mathcal{X}^{\mathcal{I}})^{\otimes i} \subset \mathcal{X}^{\otimes i}$. In this way, both the domain and the codomain of the brackets are the correct ones.

(3) Recall that the physics of the theory in the bulk is encoded in the cohomology (3.2) of the differential B_1. To transfer this information to the algebra close to null infinity, we promote the projection Π to be a

cochain map, namely a morphism $\Pi:(\mathcal{X},B_1)\to(\mathcal{X}^{\mathcal{I}},B_1^{\mathcal{I}})$ of cochain complexes[11] such that

$$\Pi B_1 = B_1^{\mathcal{I}}\Pi \tag{5.7}$$

or, equivalently, such that the following diagram commutes:

$$\begin{array}{ccc} X_i & \xrightarrow{B_1} & X_{i+1} \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi} \\ X_i^{\mathcal{I}} & \xrightarrow{B_1^{\mathcal{I}}} & X_{i+1}^{\mathcal{I}} \end{array}, \tag{5.8}$$

i.e., if, starting from the space X_i, we can follow either of the two possible routes to $X_{i+1}^{\mathcal{I}}$ and obtain the same result. As we will explain in details in Sec. VII A, a consequence of the above requirement is that Π induces morphisms $H^i(\mathcal{X})\to H^i(\mathcal{X}^{\mathcal{I}})$ on the cohomology groups defined in (3.2).

A different perspective that further explains the necessity of imposing (5.7) arises from the following situation. As explained above, $\forall\ \psi\in\mathcal{X}$, when going close to null infinity, only $\Pi(\psi)$ survives while $\Pi^{\complement}(\psi)$ is disregarded because it encloses the divergent part. Now, consider the action of the bracket B_1 on ψ and then its projection close to $\mathcal{I}$

$$\Pi B_1(\psi) = \Pi B_1(\Pi(\psi)+\Pi^{\complement}(\psi)). \tag{5.9}$$

In general, $\Pi B_1\Pi^{\complement}(\psi)\neq 0$, meaning that the divergent terms in ψ can become finite after applying the differential in the bulk, thus contributing to the projected $B_1(\psi)$ close to null infinity. Promoting Π to a cochain map is equivalent to asking that this can never happen. In other words, we want the second term in the right-hand side (rhs) of (5.9) to vanish for all $\psi\in\mathcal{X}$, namely

$$\mathrm{Im}\,\Pi^{\complement}\subseteq\mathrm{Ker}(\Pi B_1). \tag{5.10}$$

By virtue of (5.5) and the definition of $\Pi^{\complement}$, this condition is indeed equivalent to (5.7).

(4) Finally, in order for (5.1) to be a homotopy Lie algebra, we impose the L_∞ relations on its brackets. Note that our definition of the map $B_1^{\mathcal{I}}$ together with condition (5.7) and Eq. (3.5) already imply the nilpotency of the differential at null infinity,

$$(B_1^{\mathcal{I}})^2 = B_1^{\mathcal{I}}\Pi B_1|_{\mathcal{X}^{\mathcal{I}}} = \Pi B_1 B_1|_{\mathcal{X}^{\mathcal{I}}} = 0. \tag{5.11}$$

In principle, one should allow for the appearance of higher (than two-) brackets and higher homotopy relations in the homotopy algebra that describes the theory close to null infinity. Indeed, it is not obvious that truncating the theory by selecting powers of r should preserve its original algebraic structure. However, the constraints that will be imposed on the projector Π will lead to a dgLa close to null infinity, as detailed at the end of Appendix A 2. So, for simplicity and brevity, let us for now assume that the theory is encoded by a dgLa in this regime, and only impose the Leibniz rule of $B_1^{\mathcal{I}}$ with respect to the two-bracket and the (strict) Jacobi identity for $B_2^{\mathcal{I}}$,

$$\mathrm{Leib}(B_1^{\mathcal{I}},B_2^{\mathcal{I}})=0 \quad\text{and}\quad \mathrm{Jac}(B_2^{\mathcal{I}})=0. \tag{5.12}$$

A triplet $(\mathcal{X}^{\mathcal{I}},B_1^{\mathcal{I}},B_2^{\mathcal{I}})$ satisfying all the above requirements successfully encodes self-dual Yang-Mills theory in the null infinity region. To find such an algebra (or such algebras, as for now it is not clear whether the solution of the constraints listed here is unique), note that the graded vector space close to null infinity can be implicitly defined as the image of the projection introduced in (5.3),

$$\mathcal{X}^{\mathcal{I}} \coloneqq \mathrm{Im}\,\Pi. \tag{5.13}$$

Recalling Eq. (5.5), we point out that both the graded vector space and the brackets in $(\mathcal{X}^{\mathcal{I}},B_1^{\mathcal{I}},B_2^{\mathcal{I}})$ can be written in terms of the initial L_∞ algebra $(\mathcal{X},B_1,B_2)$ and the map Π. Hence, the problem of determining the homotopy Lie algebra close to null infinity can be reformulated as the task of finding a projection Π satisfying the system

$$\mathrm{Im}\,\Pi\subseteq\mathcal{X}^{n\leq 0} \tag{5.14a}$$

$$\mathrm{Ker}\Pi\subseteq\mathrm{Ker}(\Pi B_1) \tag{5.14b}$$

$$\mathrm{Leib}(\Pi B_1|_{\mathrm{Im}\,\Pi},\Pi B_2|_{\mathrm{Im}\,\Pi\otimes\mathrm{Im}\,\Pi})=0 \tag{5.14c}$$

$$\mathrm{Jac}(\Pi B_2|_{\mathrm{Im}\,\Pi\otimes\mathrm{Im}\,\Pi})=0, \tag{5.14d}$$

that is just a reformulation of conditions (5.2), (5.10)[12] and (5.12) using the definition of $\mathcal{X}^{\mathcal{I}}$ and $B_i^{\mathcal{I}}$ in terms of Π. (5.14) plays a central role in our analysis and will be referenced frequently throughout the paper; thus, for convenience, we shall refer to it as the *master system*. In what follows, we prove the existence of an infinite family of solutions $\{\Pi_k\}_{k\in\mathbb{Z}^{\leq 0}}$ for the system above, by explicit construction.

[11] We remark that, *a priori*, $(\mathcal{X}^{\mathcal{I}},B_1^{\mathcal{I}})$ is not guaranteed to be a cochain complex, since we have not imposed the nilpotency of the differential, yet. However, $(B_1^{\mathcal{I}})^2$ vanishes automatically from (5.5) and (5.7), as we prove in (5.11), meaning that the map Π satisfying condition (5.7) is indeed a morphism of cochain complexes.

[12] Recall that $\mathrm{Im}\,\Pi^{\complement}=\mathrm{Ker}\Pi$.

B. Solutions of the master system

Equation (5.14a) states that the action of the projector Π on an element $\psi \in \mathcal{X}$ must set to zero all the terms with positive powers of r in its expansion—see the definition of the space $\mathcal{X}^{n\leq 0}$ below Eq. (5.2). Based on this observation, one can conjecture that the solution of the first equation of the master system is a map π_{trial} that acts on $\psi \in \mathcal{X}$ as

$$\pi_{\text{trial}}(\psi) \coloneqq \sum_{n\leq 0} r^n \psi^{(n)}(y^a). \tag{5.15}$$

However, this truncation of the sum (4.5) does not preserve, in general, the antiself-duality of the elements in X_1 (recall that X_1 is the space of field equations, whose elements are antiself-dual two-forms), namely

$$(1+*)\pi_{\text{trial}}(E)_{\mu\nu} \neq 0 \tag{5.16}$$

for $E_{\mu\nu} \in X_1$. The underlying reason is the presence of different powers of r in different components of the basis element $\underset{\mu\nu}{e^1}$ in (4.6). Thus, truncating the r-expansion of $E_{\mu\nu}$ uniformly in all its components produces a tensor that does not have the general form (4.7). Formally stated, π_{trial} cannot be a solution for Π in (5.14a) because $\text{Im}\,\pi \not\subseteq \mathcal{X}^{n\leq 0}$.[13] This consideration implies that the map solving the master system should act nonuniformly on the components of the vectors and tensors of $\mathcal{X}$. To find it, we must first construct a suitable truncation function that operates on individual components, which we do in the following.

1. Projection $\mathcal{P}_N$

We now construct a map that allows us to select specific terms in the r-expansion of each component of the objects in the bulk. Note that the components of the vectors/tensors in $\mathcal{X}$ are elements of the space $C^\infty_r(\mathcal{M})$ defined in (4.3), where $\mathcal{M}$ is the spacetime with Bondi coordinates $x^\mu = (r, y^\alpha) = (r, u, z, \bar{z})$. On this space, given a set[14] $N \subseteq \mathbb{Z}$, we define an endomorphism $\mathcal{P}_N$ that acts on functions $f \in C^\infty_r(\mathcal{M})$ as

$$\mathcal{P}_N f(r, y^\alpha) \coloneqq \sum_{n\in N} r^n f^{(n)}(y^\alpha) \tag{5.17}$$

To give a concrete example, for $N = \{-20, -2, 0, 3\}$ we have

$$\mathcal{P}_N f = r^3 f^{(3)} + f^{(0)} + \frac{f^{(-2)}}{r^2} + \frac{f^{(-20)}}{r^{20}}. \tag{5.18}$$

In the above equation and in what follows, we drop the explicit coordinate dependence of f and $f^{(n)}$ for the sake of readability.

The map (5.17) satisfies the properties (A2) listed in Appendix A; in particular, the first of these relations implies that $\mathcal{P}_N$ is a projection $\forall\ N$, making it an ideal candidate to serve as a building block for the solution of the master system.

2. Projection $\pi_{\mathcal{N}}$

Now that we have developed the machinery necessary to manipulate the r-expansion of the single components of the elements in $\mathcal{X}$, we can construct a function that truncates them nonuniformly, employing the projection defined above.

To do so, we need to specify how $\mathcal{P}$ acts on each component of a generic element of the space $\mathcal{X}$. In other words, we shall define a collection of sets

$$\mathcal{N} \coloneqq \{N_\Lambda, N_{A_\mu}, N_{E_{\mu\nu}} \subseteq \mathbb{Z}\} \tag{5.19}$$

that contains all the information about how to select the powers in the r-expansion of the components of ψ. For this reason, we refer to the sets in $\mathcal{N}$ as the *power selection sets*. For example, the u component of all elements in X_0 is truncated using $\mathcal{P}_{N_{A_u}}$, where $N_{A_u} \in \mathcal{N}$ is a set of integer numbers. This means that we truncate all the gauge parameters in X_{-1} using $\mathcal{P}_{N_\Lambda}$,[15] we truncate the r component of all the gauge fields in X_0 using $\mathcal{P}_{N_{A_r}}$, and so on.[16]

The next step is to introduce a function that simultaneously applies $\mathcal{P}$ to all the components of a given $\psi \in \mathcal{X}$, using the power selection sets in (5.19). We observe that it is not guaranteed that after truncating the components of an element $E_{\mu\nu} \in X_1$, we obtain an antiself-dual tensor, as explained below Eq. (5.16). In other words, the result of the projection on an arbitrary equation of motion in X_1 does not belong, in general, to (a subspace of) X_1. For this reason, we start by introducing the following graded vector space

$$\begin{aligned}\mathcal{C} &\coloneqq \textstyle\bigoplus_{i=-1}^{1} C_i \\ &= \overbrace{C^\infty_r(\mathcal{M})}^{C_{-1}} \oplus \overbrace{C^\infty_r(\mathcal{M})^4}^{C_0} \oplus \overbrace{(C^\infty_r(\mathcal{M})^4 \otimes C^\infty_r(\mathcal{M})^4)}^{C_1},\end{aligned} \tag{5.20}$$

[13] Remember that, by definition, $\mathcal{X}^{n\leq 0}$ is a subspace of $\mathcal{X}$. In particular, the tensors in $X_1^{n\leq 0} \subset X_1$ are antiself-dual.

[14] Note that we do not require N to be a discrete interval in $\mathbb{Z}$ (i.e. a set of consecutive integers). In fact, most of the proofs in Appendix A hold for arbitrary sets. However, we ultimately reduce to the case where all the sets consist of consecutive integers without gaps in order to extract the physical solution.

[15] That is to say, given two gauge parameters Λ_1, Λ_2, we truncate them in the same way: $\mathcal{P}_{N_\Lambda}\Lambda_1$ and $\mathcal{P}_{N_\Lambda}\Lambda_2$, respectively.

[16] The cardinality of (5.19) is $|\mathcal{N}| = 1 + 4 + 16 = 21$.

and we define a morphism[17] $\pi_{\mathcal{N}}:\mathcal{X}\to\mathcal{C}$ given by

$$\pi_{\mathcal{N}}(\psi) \coloneqq \begin{cases} \mathcal{P}_{N_\Lambda}(\Lambda) & \text{if } \psi\in X_{-1} \\ \mathcal{P}_{N_{A_\mu}}(A_\mu) & \text{if } \psi\in X_0 \\ \mathcal{P}_{N_{E_{\mu\nu}}}(E_{\mu\nu}) & \text{if } \psi\in X_1 \end{cases}. \tag{5.21}$$

We remark that $\pi_{\mathcal{N}}$ is well defined since it applies the projection $\mathcal{P}_N$ to each component of the objects in $\mathcal{X}$, and those components are elements of $C_r^\infty(\mathcal{M})$. As an example, consider the following choice for the power selection sets for the gauge fields:

$$N_{A_r}=\{-3,-2\}, \qquad N_{A_u}=N_{A_{\bar z}}=\{-2,-1,0\},$$
$$N_{A_z}=\{-1,0\}, \tag{5.22}$$

then the image of an arbitrary gauge field $A_\mu\in X_0$ through $\pi_{\mathcal{N}}$ is

$$(\pi_{\mathcal{N}}A)_\mu = \begin{pmatrix} \mathcal{P}_{N_{A_r}}A_r \\ \mathcal{P}_{N_{A_u}}A_u \\ \mathcal{P}_{N_{A_z}}A_z \\ \mathcal{P}_{N_{A_{\bar z}}}A_{\bar z} \end{pmatrix} = \begin{pmatrix} \frac{A_r^{(-2)}}{r^2}+\frac{A_r^{(-3)}}{r^3} \\ A_u^{(0)}+\frac{A_u^{(-1)}}{r}+\frac{A_u^{(-2)}}{r^2} \\ A_z^{(0)}+\frac{A_z^{(-1)}}{r} \\ A_{\bar z}^{(0)}+\frac{A_{\bar z}^{(-1)}}{r}+\frac{A_{\bar z}^{(-2)}}{r^2} \end{pmatrix} \in C_0. \tag{5.23}$$

Observe that $\pi_{\mathcal{N}}$ is not a projection (as $\mathcal{C}\not\subseteq\mathcal{X}$, since $C_1\not\subseteq X_1$), despite being defined in terms of the projection $\mathcal{P}$. Notice that, in contrast to pure Yang-Mills, the z and $\bar z$ components have different truncations. This asymmetry arises from the self-duality condition.

At this point, it is finally possible to solve the first equation of the master system: $\Pi=\pi_{\mathcal{N}}$ is a solution for (5.14a) if the power selection sets in $\mathcal{N}$ satisfy

$$N_I\subseteq\mathbb{Z}^{\le 0} \quad \forall\ I\in\{\Lambda,A_\mu,E_{\mu\nu}\} \tag{5.24a}$$

$$N_{E_{\mu\nu}}=N_{E_{\nu\mu}} \tag{5.24b}$$

$$N_{E_{z\bar z}}=\mathcal{T}_2 N_{E_{ru}} \tag{5.24c}$$

where $\mathbb{Z}^{\le 0}$ is the set of nonpositive integers and $\mathcal{T}$ is the translation of sets, i.e. $\forall\ s\in\mathbb{Z}$ and $\forall\ U\subseteq\mathbb{Z}$

$$\mathcal{T}_s U \coloneqq \{u+s, u\in U\}\subseteq\mathbb{Z}. \tag{5.25}$$

The first condition of the system above ensures that no positive power of r appears in the expansion of an element in $\operatorname{Im}\pi_{\mathcal{N}}$. Moreover, having in mind the general form (4.7) of the equations of motion, it is easy to see that the second condition above ensures that $\pi_{\mathcal{N}}(E)_{\mu\nu}$ are skew-symmetric tensors $\forall\ E_{\mu\nu}\in X_1$, while the third guarantees their anti-self-duality. Therefore, the morphism $\pi_{\mathcal{N}}$ with $\mathcal{N}$ satisfying (5.24) is a projection[18] and its image is a subspace of $\mathcal{X}^{n\le 0}$, so it is a solution (in fact, the most general one) of the master system's first equation.

The above considerations, combined with the fact that E_{rz} always vanishes, indicate that it is sufficient to specify the sets $N_{E_{ru}}$, $N_{E_{r\bar z}}$ and $N_{E_{uz}}$ to completely define the action of $\pi_{\mathcal{N}}$ on an arbitrary equation of motion in X_1, which can thus be written as a linear combination of (4.6), as required.

3. An infinite family of L_∞ algebras

We are now ready to present the solution of the master system (5.14), which relies on the formalism built above. The complete proof is detailed in Appendix A, where we show that the system is indeterminate and its physically relevant solution is a family of projections $\{\Pi_k\}_{k\in\mathbb{Z}^{\le 0}}$ of the form

$$\Pi_k=\pi_{\mathcal{N}_k}, \qquad \mathcal{N}_k=\{N_\Lambda^k,N_{A_\mu}^k,N_{E_{\mu\nu}}^k\subseteq\mathbb{Z}^{\le 0}\} \tag{5.26}$$

with

$$N_I^k=\begin{cases}\{k,\dots,0\} & \text{if } I=A_z,E_{uz}\\ \{k-1,\dots,0\} & \text{if } I=\Lambda,A_u,A_{\bar z}\\ \{k-2,\dots,-2\} & \text{if } I=A_r,E_{ru},E_{r\bar z}\end{cases}, \quad \text{for } k\le 0. \tag{5.27}$$

Using these projections, we build an infinite family of L_∞ algebras $\{(\mathcal{X}^{(k)},B_1^{(k)},B_2^{(k)})\}_{k\in\mathbb{Z}^{\le 0}}$ where

$$\mathcal{X}^{(k)}\coloneqq\operatorname{Im}\Pi_k, \qquad B_i^{(k)}\coloneqq\Pi_k\circ B_i|_{(\operatorname{Im}\Pi_k)^{\otimes i}}. \tag{5.28}$$

This will determine a series of natural "slices" $\mathfrak{S}_k$, displayed in Fig. 1, on which we can define our theory and construct the kinematic algebra. Remarkably, the upper boundary, which gives the so-called fall-off of the fields, is identical for all slices and independent of k, and it coincides exactly with the physical fall-off conditions presented in [60,61] and reviewed in Sec. II B. Such fall-offs therefore arise from general properties of the master system (see Fig. 2 for an explanation of how each condition of the system prohibits a specific power in the fall-off). The lower boundary of the slice is simply the choice of precision in the expansion in $1/r$. In particular, we are interested in the fall-offs of the gauge fields. Below, we define two objects that encode the approximation (truncation from below) and the

[17]A morphism between two graded vector spaces $\mathcal{V}=\bigoplus_{i\in I}V_i$ and $\mathcal{W}=\bigoplus_{i\in I}W_i$ is a linear map $f:\mathcal{V}\to\mathcal{W}$ of degree zero, i.e. $f(V_i)\subseteq W_i\ \ \forall\ i\in I$.

[18]This justifies the title of the current subsection.

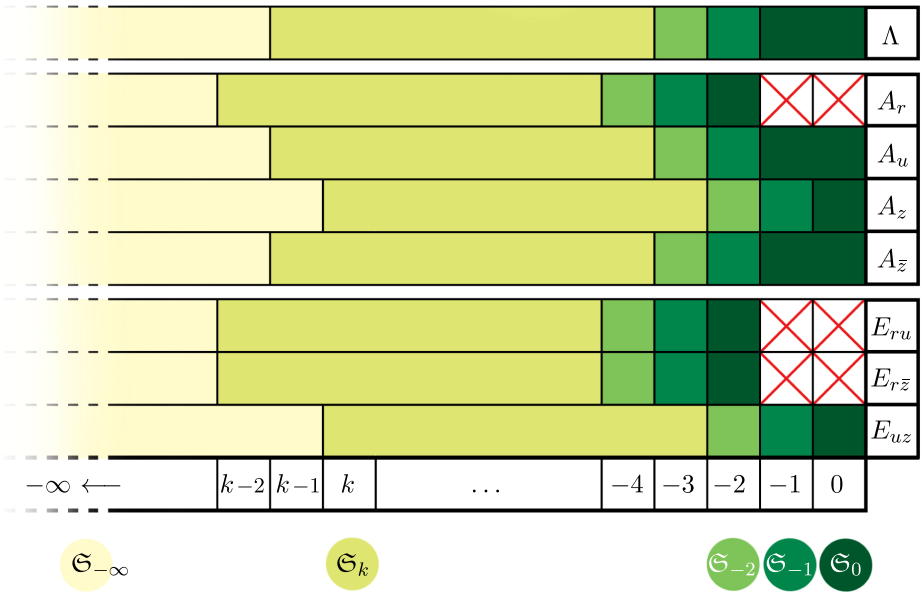


FIG. 1. The drawing illustrates the slices $\mathfrak{S}_k$ (where $k \in \mathbb{Z}^{\leq 0}$) obtained by projecting the initial graded vector space $\mathcal{X}$ via the maps Π_k in (5.26), that are the solutions of the master system (5.14). Each slice is represented in a different shade of green. The lighter slices always include the darker ones, in other words $\mathfrak{S}_{k-1} \subset \mathfrak{S}_k$ for all k. Moreover, all slices share the same maxima and therefore the same fall-offs.

fall-offs (truncation from above) of each slice $\mathfrak{S}_k$

$$t^k_\mu := \min N^{(k)}_{A_\mu} = (k-2, k-1, k, k-1)$$
$$\text{(truncation from below)}$$
$$f_\mu := \max N^{(k)}_{A_\mu} = (-2, 0, 0, 0) \quad \text{(fall-off)}. \tag{5.29}$$

Finally, we remark that the homotopy algebra construction automatically ensures that the truncated gauge fields entering our slices are exactly the ones needed to write the truncated equations of motion. This emerges directly when solving the master system (5.14) as shown in Sec. A, i.e., we did not need to impose it as an additional condition.

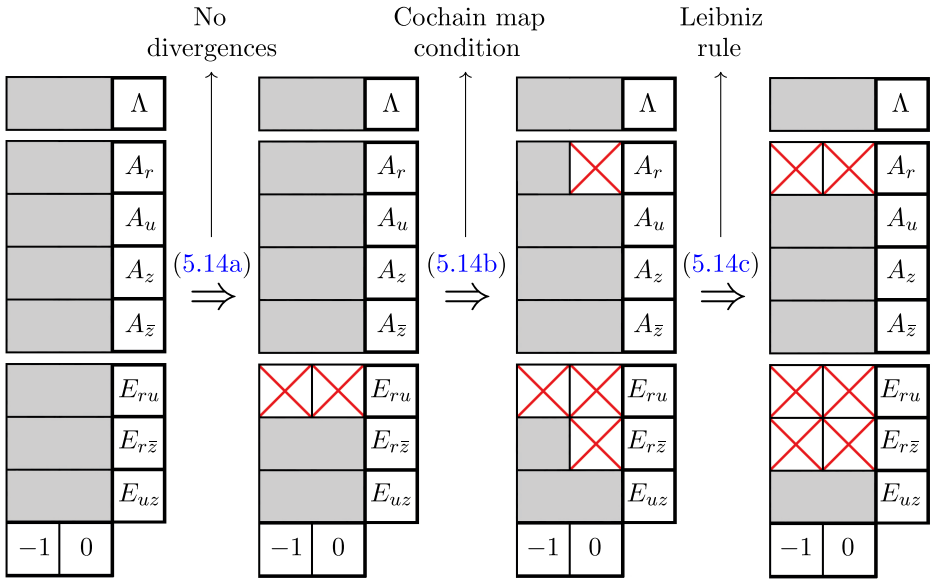


FIG. 2. The fall-offs follow directly from the general properties of the master system: each condition in the system restricts specific powers in the fall-off, and the figure illustrates how these conditions determine the excluded powers, which are represented by red crosses.

To wrap up this paragraph, we provide, as an illustrative example, the explicit form of the power selection sets that define the slice $\mathfrak{S}_0$

$$N_I^0 = \begin{cases} \{0\} & \text{if } I = A_z, E_{uz} \\ \{-1,0\} & \text{if } I = \Lambda, A_u, A_{\bar{z}} \\ \{-2\} & \text{if } I = A_r, E_{ru}, E_{r\bar{z}} \end{cases}. \tag{5.30}$$

Thus, the gauge parameters, gauge fields and e.o.m. in $\mathcal{X}^{(0)} \coloneqq \operatorname{Im}\Pi_0$ are, respectively,

$$\Lambda = \Lambda^{(0)} + \frac{\Lambda^{(-1)}}{r}, \qquad A_\mu = \begin{pmatrix} \frac{A_r^{(-2)}}{r^2} \\ A_u^{(0)} + \frac{A_u^{(-1)}}{r} \\ A_z^{(0)} \\ A_{\bar{z}}^{(0)} + \frac{A_{\bar{z}}^{(-1)}}{r} \end{pmatrix},$$

$$E_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_{ru}^{(-2)}}{r^2} & 0 & \frac{E_{r\bar{z}}^{(-2)}}{r^2} \\ -\frac{E_{ru}^{(-2)}}{r^2} & 0 & E_{uz}^{(0)} & 0 \\ 0 & -E_{uz}^{(0)} & 0 & E_{ru}^{(-2)} \\ -\frac{E_{r\bar{z}}^{(-2)}}{r^2} & 0 & -E_{ru}^{(-2)} & 0 \end{pmatrix}. \tag{5.31}$$

VI. STRICT KINEMATIC ALGEBRAS

Let us take stock of the situation. Thanks to the method explained in Sec. V, we have found a way to define an infinite family of strict L_∞ algebras (in fact, differential graded Lie algebras) that describe self-dual Yang Mills near null infinity. Each algebra $(\mathcal{X}^{(k)}, B_1^{(k)}, B_2^{(k)})$ is labeled by a nonpositive integer k, that corresponds to the approximation we adopt when truncating "from below" the r-expansion of the elements in $\mathcal{X}$, the graded vector space of the initial L_∞ algebra encoding SDYM in the bulk.

A. Kinematic algebras on slices

At this point, we may proceed with the construction of the kinematic algebra on each slice $\mathfrak{S}_k$. We follow the procedure outlined in Secs. III B, III C and employed in Sec. IV B, with the sole difference that we now use the projection Π_k defined above to ensure that the kinematic algebra closes on the slice. First, by color-stripping, we obtain a differential graded commutative algebra $(\mathcal{K}^{(k)}, m_1^{(k)}, m_2^{(k)})$, where $m_i^{(k)} = \Pi_k m_i$ for $i = 1, 2$.[19] Next, we equip the graded vector space $\mathcal{K}^{(k)}$ with the map $b^{(k)} \coloneqq \Pi_k b$, which is the projection on the slice k of the differential b introduced in (4.13). As we show in the Appendix, $(\mathcal{K}^{(k)}, b^{(k)})$ is a chain complex (see remark B.1). Finally, we define the box operator and the kinematic bracket, respectively, as

$$\Box^{(k)} \coloneqq [b^{(k)}, m_1^{(k)}], \qquad b_2^{(k)} \coloneqq [b^{(k)}, m_2^{(k)}], \tag{6.1}$$

where the commutator in the second equation acts as in (3.38). As expected, the two-product $m_2^{(k)}$ and the kinematic bracket satisfy a generalized version of the Poisson compatibility relation. The failure to satisfy this relation strictly is governed by a trilinear map $\theta_3^{(k)}$. Putting all these ingredients together, we obtain a $\mathrm{BV}_\infty^{\Box^{(k)}}$ algebra

$$\mathfrak{Kin}^{(k)} \coloneqq (\mathcal{K}^{(k)}, m_1^{(k)}, b^{(k)}, \Box^{(k)}, m_2^{(k)}, b_2^{(k)}, \theta_3^{(k)}), \tag{6.2}$$

which is the kinematic algebra on the slice $\mathfrak{S}_k$. In Appendix B we show that the box operator of this algebra is simply the projection on the slice of the d'Alembert operator, $\Box^{(k)} = \Pi_k \Box$ (see remark B.2). In particular, this implies that $\mathfrak{Kin}^{(k)}$ is a $\mathrm{BV}_\infty^{\Pi_k \Box}$ algebra. Similarly, one can show that $b_2^{(k)} = \Pi_k b_2$ and $\theta_3^{(k)} = \Pi_k \theta_3$. In light of these observations, a simple relation between the kinematic algebra on the slice and the one in the bulk can be written

$$\mathfrak{Kin}^{(k)} = \Pi_k \circ \mathfrak{Kin}, \tag{6.3}$$

cf. Eqs. (4.12) and (6.2). By this formal expression, we mean that every element (whether a space or a map) that defines $\mathfrak{Kin}^{(k)}$ is obtained by applying the projection Π_k to the corresponding element of $\mathfrak{Kin}$.

As explained at the end of Sec. III C, the kinematic algebra contains a strict Lie algebra as a subsector iff it is itself strict. In other words, the kinematic bracket $b_2^{(k)}$ satisfies the strict Jacobi identity iff the ternary bracket $\theta_3^{(k)}$ vanishes. In order to study when this occurs, it is useful find the explicit expression for $\theta_3^{(k)}$. This is achieved by applying the projection Π_k to Eq. (3.42), and by employing the identities derived in the Appendices.[20] The components of the trilinear map applied to three gauge fields $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \in K_1^{(k)}$ are given by

[19] Recall that $\Pi_k : \mathcal{X} \to \mathcal{X}^{(k)}$, where $\mathcal{X} = \mathcal{K} \otimes \mathfrak{g}$ and $\mathcal{X}^{(k)} = \mathcal{K}^{(k)} \otimes \mathfrak{g}$. Let $\Pi_k^{\mathrm{kin}} : \mathcal{K} \to \mathcal{K}^{(k)}$ be the projection that acts directly on the color-stripped graded vector space. Since the color-stripping procedure does not affect the r-expansion, we can identify Π_k^{kin} with Π_k. For this reason, by an abuse of notation, we suppress the superscript "kin" whenever the domain of the projection is clear from the context.

[20] In particular, see lemma A.1 and Eq. (B22).

$$\theta_3^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)_r = \sum_{\substack{m,n,l\in\{k-2,\ldots,-2\}\\ p\in\{k-1,\ldots,0\}}} 6\delta_{m,n+l+p} r^m \mathcal{A}^{(n)}_{[1r}\mathcal{A}^{(l+2)}_{2z}\mathcal{A}^{(p)}_{3]\bar{z}}$$

$$\theta_3^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)_u = -\sum_{\substack{m,n_1,n_3\in\{k-1,\ldots,0\}\\ n_2\in\{k-2,\ldots,-2\}}} 6\delta_{m,n_1+n_2+n_3} r^m \mathcal{A}^{(n_1)}_{[1u}\mathcal{A}^{(n_2+2)}_{2z}\mathcal{A}^{(n_3)}_{3]\bar{z}}$$

$$\theta_3^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)_z = -\sum_{\substack{m,n_3\in\{k,\ldots,0\}\\ n_1\in\{k-2,\ldots,-2\}\\ n_2\in\{k-1,\ldots,0\}}} 6\delta_{m,n_1+n_2+n_3} r^m \mathcal{A}^{(n_1)}_{[1r}\mathcal{A}^{(n_2)}_{2u}\mathcal{A}^{(n_3)}_{3]z}$$

$$\theta_3^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)_{\bar{z}} = \sum_{\substack{m,n_2,n_3\in\{k-1,\ldots,0\}\\ n_1\in\{k-2,\ldots,-2\}}} 6\delta_{m,n_1+n_2+n_3} r^m \mathcal{A}^{(n_1)}_{[1r}\mathcal{A}^{(n_2)}_{2u}\mathcal{A}^{(n_3)}_{3]\bar{z}}, \tag{6.4}$$

while the components of $\theta_3^{(k)}$ on an equation of motion $\mathcal{E}\in K_2^{(k)}$ and two gauge fields are

$$\theta_3^{(k)}(\mathcal{E},\mathcal{A}_1,\mathcal{A}_2)_{ru} = \sum_{\substack{m,n_1,n_2\in\{k-2,\ldots,-2\}\\ n_3\in\{k-1,\ldots,0\}}} 2r^m\delta_{m,n_1+n_2+n_3}\left(\mathcal{E}^{(n_1+2)}_{uz}\mathcal{A}^{(n_2)}_{[1r}\mathcal{A}^{(n_3)}_{2]\bar{z}} - \mathcal{E}^{(n_1)}_{r\bar{z}}\mathcal{A}^{(n_3)}_{[1u}\mathcal{A}^{(n_2+2)}_{2]z}\right)$$

$$\theta_3^{(k)}(\mathcal{E},\mathcal{A}_1,\mathcal{A}_2)_{r\bar{z}} = \sum_{\substack{m,n_1,n_2\in\{k-2,\ldots,-2\}\\ n_3\in\{k-1,\ldots,0\}}} 2r^m\delta_{m,n_1+n_2+n_3}\left[\mathcal{E}^{(n_1)}_{r\bar{z}}\left(\mathcal{A}^{(n_2)}_{[1r}\mathcal{A}^{(n_3)}_{2]u} + \mathcal{A}^{(n_2+2)}_{[1z}\mathcal{A}^{(n_3)}_{2]\bar{z}}\right) - 2\mathcal{E}^{(n_1)}_{ru}\mathcal{A}^{(n_2)}_{[1r}\mathcal{A}^{(n_3)}_{2]\bar{z}}\right]$$

$$\theta_3^{(k)}(\mathcal{E},\mathcal{A}_1,\mathcal{A}_2)_{uz} = -\sum_{\substack{m,n_1\in\{k,\ldots,0\}\\ n_2\in\{k-2,\ldots,-2\}\\ n_3\in\{k-1,\ldots,0\}}} 2r^m\delta_{m,n_1+n_2+n_3}\left[\mathcal{E}^{(n_1)}_{uz}\left(\mathcal{A}^{(n_2)}_{[1r}\mathcal{A}^{(n_3)}_{2]u} + \mathcal{A}^{(n_2+2)}_{[1z}\mathcal{A}^{(n_3)}_{2]\bar{z}}\right) - 2\mathcal{E}^{(n_2)}_{ru}\mathcal{A}^{(n_3)}_{[1u}\mathcal{A}^{(n_1)}_{2]z}\right] \tag{6.5}$$

Inspection of the above equations reveals that $\theta_3^{(k)}$ vanishes for all $\mathcal{A}$, $\mathcal{E}$ if $k=0$, meaning that the kinematic algebra on the thinnest slice, i.e. $\mathfrak{Kin}^{(0)}$, is strict. However, this case is trivial and uninteresting, as the kinematic bracket between two fields vanishes upon imposing $A_u^{(0)}=0$, a standard gauge choice.[21]

In general, the equations above show that the kinematic algebra $\mathfrak{Kin}^{(k)}$ on a slice $\mathfrak{S}_k$ with k nonvanishing is not strict. Since our goal is to construct kinematic brackets that strictly satisfies the Jacobi identity, we must seek alternative means of strictifying the $\mathrm{BV}_\infty^{\Pi_k\square}$ algebra on $\mathfrak{S}_{k\neq 0}$. Note that one could attempt to carefully impose relations among the components of the gauge fields for each $\mathfrak{S}_k$, so as to force $\theta_3^{(k)}$ to vanish. For example, if one were to require that all the components of A_μ be identical, the trilinear map in (6.4) would vanish by symmetry. We leave this approach for future work.

In contrast, our strategy is to *refine* $\mathfrak{S}_k$ by allowing the fall-offs (i.e., the maxima of the power selection sets) to vary, in such a way that the kinematic algebra defined on these refined slices becomes strict. This method enables us to construct the desired brackets without introducing additional relations among the components of the field, but merely by carefully selecting terms of their r-expansion.[22] In what follows, we develop this idea and show that for each slice, at least two distinct refinements[23] can be defined on which the kinematic algebra is strictified.

B. Strict kinematic algebras on refined slices

In what follows, as anticipated above, we allow the fall-off of the various components of the gauge fields to vary, by further projecting to zero some of the leading order coefficients of $A_\mu\in X_0^{(k)}$. This adjustment enables the construction of *refined slices* $\mathfrak{S}_{k,f}\subseteq\mathfrak{S}_k$ on which a strict kinematic algebra can be defined (we add the subscript f to indicate that these new slices depend on a modified fall-off). As a result, we obtain a kinematic bracket that satisfies the Jacobi identity, thereby forming a strict Lie algebra.

Before proceeding, a brief yet essential clarification: one might think that modifying the fall-offs (by projecting to zero particular coefficients) is not possible since, as shown

[21]The kinematic bracket between two gauge fields on the slice $k=0$ is $b_2^{(0)}(\mathcal{A}_1,\mathcal{A}_2)=\Pi_0 b_2(\mathcal{A}_1,\mathcal{A}_2)$, where $\mathcal{A}_1,\mathcal{A}_2\in K_1^{(0)}$ are the color-stripped version of the fields given in (5.31). By explicit computation, one can check that the only nonzero component is $b_2^{(0)}(\mathcal{A}_1,\mathcal{A}_2)_{\bar{z}}=-4r^{-1}\mathcal{A}^{(0)}_{[1u}\mathcal{A}^{(0)}_{2]\bar{z}}$, which vanishes upon setting $A_u^{(0)}=0$.

[22]Together with a small precaution that we will explain below.

[23]We wish to stress that, in the present context, the term *refinement* is employed in a sense that diverges from its conventional meaning in topology.

in Sec. V, the projections $\Pi_k = \pi_{\mathcal{N}_k}$ (that define the slices $\mathfrak{S}_k$) are the only physical solutions to the master system, and the power selection sets in $\mathcal{N}_k$ have fixed maxima, i.e. fall-off. Any modification of these maxima would violate the L_∞ relations, resulting in the loss of the L_∞ algebras on $\mathfrak{S}_k$. This issue can be resolved by constructing the refined slices through an additional projection that not only further truncates r-expansion but also modifies the functional dependence of the gauge parameters, thus ensuring that the L_∞ relations are obeyed in $\mathfrak{S}_{k,f}$, preserving the L_∞ structure. Since this detail is not essential to the following discussion, we defer its analysis to Appendix B 2.

The crucial point is that for each slice $\mathfrak{S}_k$, refined slices $\mathfrak{S}_{k,f}$ can be defined, each depending on a specific choice of the fall-off of the gauge fields. More precisely, $\mathfrak{S}_{k,f}$ are constructed following the same procedure described in Sec. V B 3 but using a modified version of Π_k. This new projection, which we call $\Pi_{k,f}$ and whose explicit definition is given in Appendix B 2, truncates the r-expansion of the gauge fields through power selection sets $N^{(k,f)}_{A_\mu} \subseteq N^{(k)}_{A_\mu}$ defined as

$$N^{(k,f)}_{A_\mu} := \begin{cases} \{k-2,\dots,f^k_r\} & \mu = r \\ \{k-1,\dots,f^k_u\} & \mu = u \\ \{k,\dots,f^k_z\} & \mu = z \\ \{k-1,\dots,f^k_{\bar{z}}\} & \mu = \bar{z} \end{cases} \tag{6.6}$$

where

$$f^k_\mu := \max N^{(k,f)}_{A_\mu} \in \mathbb{Z} \text{ s.th. } \begin{cases} k-2 \le f^k_r \le -2 \\ k-1 \le f^k_u, f^k_{\bar{z}} \le 0 \\ k \le f^k_z \le 0 \end{cases} \tag{6.7}$$

are the modified fall-off. The inequalities above are necessary to ensure that the sets in (6.6) are well defined, and that they are subsets of $N^{(k)}_{A_\mu}$. We point out that the refined slices are obtained by modifying the fall-off but not the truncation of the r-expansion from "below," which is not affected by the refinement procedure. In other words, $\min N^{(k,f)}_{A_\mu} \equiv \min N^{(k)}_{A_\mu} = t^k_\mu$ for all k and for all f, cf. Eqs. (6.6) and (5.29). Indeed, if we choose the fall-offs to be the same as those in (5.29), then the resulting refined slices coincide with the initial ones, as expected

$$\mathfrak{S}_{k,f} \equiv \mathfrak{S}_k \Leftrightarrow f^k_\mu \equiv f_\mu = (-2,0,0,0). \tag{6.8}$$

In a completely analogous way to what was done for the slices $\mathfrak{S}_k$, we can construct a kinematic algebra on each refined slice $\mathfrak{S}_{k,f}$, which we denote by $\mathfrak{Kin}^{(k,f)}$. The steps remain the same as before: we use the projection $\Pi_{k,f}$ to define a subspace $\mathcal{X}^{(k,f)} \subseteq \mathcal{X}$, on which we build an L_∞ algebra by projecting the brackets B_1, B_2, then apply color-stripping, and finally obtain the algebra $\mathfrak{Kin}^{(k,f)} = \Pi_{k,f} \circ \mathfrak{Kin}$. Such an algebra, for a generic choice of fall-off f^k_μ, is a homotopy algebra (to be precise, it is a $\mathrm{BV}^{\square^{(k,f)}}_\infty$ algebra) and, as such, is not strict. However, crucially, there exist specific choices of fall-offs that strictify the kinematic algebra on the corresponding refined slice $\mathfrak{S}_{k,f}$. Specifically

$$\mathfrak{Kin}^{(k,f)} \text{ is strict} \Leftrightarrow f^k_\mu \text{ satisfy} \begin{cases} f^k_r + f^k_u + f^k_z < k \\ f^k_r + f^k_u + f^k_{\bar{z}} < k-1 \\ f^k_r + f^k_z + f^k_{\bar{z}} < k \\ f^k_u + f^k_z + f^k_{\bar{z}} < k+1 \\ f^k_r + f^k_u < k \\ f^k_r + f^k_{\bar{z}} < k \\ f^k_u + f^k_z < k+2 \\ f^k_z + f^k_{\bar{z}} < k+2. \end{cases} \tag{6.9}$$

This statement, which we prove in the appendix (see result B.3), provides a general criterion for finding (refined) slices on which a kinematic Lie bracket can be constructed. The first four inequalities correspond to the requirement that $\theta^{(k,f)}_3(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$ vanishes for any three fields $\mathcal{A}_i$, while the last four ensure that $\theta^{(k,f)}_3(\mathcal{E},\mathcal{A}_1,\mathcal{A}_2) = 0$.

First of all, we observe that the fall-off f_μ in (5.29) is a solution of the system above if and only if k vanishes. This fact together with Eq. (6.8) imply that the kinematic algebra $\mathfrak{Kin}^{(k)}$ on the (unrefined) slice $\mathfrak{S}_k$ is strict only when $k = 0$, in full agreement with what we found in Sec. VI A.

In general, given a slice $\mathfrak{S}_k$ with $k \le -1$, there exist multiple possible refined slices $\mathfrak{S}_{k,f}$ with f^k_μ satisfying the system above. We seek to minimize the number of constraints imposed on the fields, so as to find the most general refined slices manifesting a strict kinematic algebra. Of course any subslice of a refined slice (together with appropriate restrictions on the gauge parameters) will also be a solution.

The expansion for the gauge field corresponding to these solutions is

$$A_\mu = \sum_{n=t^k_\mu}^{f^k_{i,\mu}} r^n A^{(n)}_\mu \tag{6.10}$$

Below, we present the solutions for the minimal nontrivial slice $k = -1$, and the general slices $k \le -2$, respectively.

1. Strict kinematic algebra for $q\mathfrak{S}_{-1}$

For $k = -1$, we obtain 3 different slices, summarized in Table III below.

TABLE III. The three refined slices $\mathfrak{S}_{-1,f_i}$ that strictify $\mathfrak{Kin}^{(-1)}$.

	$k=-1$			
	t_μ^{-1}	$f_{1,\mu}^{-1}$	$f_{2,\mu}^{-1}$	$f_{3,\mu}^{-1}$
r	-3	-2	-2	-3
u	-2	-1	0	0
z	-1	0	0	-1
$\bar{z}$	-2	0	-1	0

Each of the solution above arises as a simple (partial) gauge choice on the slice.[24] We collect the explicit kinematic brackets below.

(i) Slice $\mathfrak{S}_{(-1,f_1)}$

$$f_{1,\mu}^{-1}=(-2,-1,0,0)\Rightarrow A_\mu = \begin{pmatrix} \frac{A_r^{(-2)}}{r^2}+\frac{A_r^{(-3)}}{r^3} \\ \frac{A_u^{(-1)}}{r}+\frac{A_u^{(-2)}}{r^2} \\ A_z^{(0)}+\frac{A_z^{(-1)}}{r} \\ A_{\bar{z}}^{(0)}+\frac{A_{\bar{z}}^{(-1)}}{r}+\frac{A_{\bar{z}}^{(-2)}}{r^2} \end{pmatrix} \in X_0^{(-1,f_1)}, \quad (6.11)$$

with the nonvanishing components of the kinematic bracket given by

$$\begin{aligned} b_2(\mathcal{A}_1,\mathcal{A}_2)_u &= 2r^{-2}\partial_u(\mathcal{A}^{(0)}_{[1z}\mathcal{A}^{(0)}_{2]\bar{z}}) \\ b_2(\mathcal{A}_1,\mathcal{A}_2)_{\bar{z}} &= 2r^{-2}\Big(\partial_u\mathcal{A}^{(-2)}_{[1r}\mathcal{A}^{(0)}_{2]\bar{z}} \\ &\quad + 2\mathcal{A}^{(-2)}_{[1r}\partial_u\mathcal{A}^{(0)}_{2]\bar{z}} - \mathcal{A}^{(-1)}_{[1u}\mathcal{A}^{(0)}_{2]\bar{z}} \\ &\quad - \mathcal{A}^{(0)}_{[1\bar{z}}\partial_z\mathcal{A}^{(0)}_{2]\bar{z}} - \mathcal{A}^{(0)}_{[1z}\partial_{\bar{z}}\mathcal{A}^{(0)}_{2]\bar{z}}\Big). \end{aligned} \quad (6.12)$$

(ii) Slice $\mathfrak{S}_{(-1,f_2)}$

$$f_{2,\mu}^{-1}=(-2,0,0,-1)\Rightarrow A_\mu = \begin{pmatrix} \frac{A_r^{(-2)}}{r^2}+\frac{A_r^{(-3)}}{r^3} \\ A_u^{(0)}+\frac{A_u^{(-1)}}{r}+\frac{A_u^{(-2)}}{r^2} \\ A_z^{(0)}+\frac{A_z^{(-1)}}{r} \\ \frac{A_{\bar{z}}^{(-1)}}{r}+\frac{A_{\bar{z}}^{(-2)}}{r^2} \end{pmatrix} \in X_0^{(-1,f_2)}, \quad (6.13)$$

with kinematic bracket given by

$$\begin{aligned} b_2(\mathcal{A}_1,\mathcal{A}_2)_r &= 4r^{-3}\mathcal{A}^{(-2)}_{[1r}\mathcal{A}^{(0)}_{2]u} \\ b_2(\mathcal{A}_1,\mathcal{A}_2)_u &= 2r^{-2}\Big(\mathcal{A}^{(-2)}_{[1r}\partial_u\mathcal{A}^{(0)}_{2]u} - \mathcal{A}^{(0)}_{[1u}\mathcal{A}^{(-1)}_{2]u} \\ &\quad + 2\partial_z\mathcal{A}^{(0)}_{[1u}\mathcal{A}^{(0)}_{2]z} + \mathcal{A}^{(0)}_{[1u}\partial_z\mathcal{A}^{(0)}_{2]z}\Big) \\ b_2(\mathcal{A}_1,\mathcal{A}_2)_z &= -4r^{-1}\mathcal{A}^{(0)}_{[1u}\mathcal{A}^{(0)}_{2]z} \\ b_2(\mathcal{A}_1,\mathcal{A}_2)_{\bar{z}} &= -2r^{-2}\Big[2\mathcal{A}^{(0)}_{[1u}\mathcal{A}^{(-1)}_{2]\bar{z}} + \partial_{\bar{z}}(\mathcal{A}^{(-2)}_{[1r}\mathcal{A}^{(0)}_{2]u})\Big]. \end{aligned} \quad (6.14)$$

Note that if we also set $\mathcal{A}_u^{(0)}=0$, then $b_2(\mathcal{A}_1,\mathcal{A}_2)=0$.

(iii) Slice $\mathfrak{S}_{(-1,f_3)}$

$$f_{3,\mu}^{-1}=(-3,0,-1,0)\Rightarrow A_\mu = \begin{pmatrix} \frac{A_r^{(-3)}}{r^3} \\ A_u^{(0)}+\frac{A_u^{(-1)}}{r}+\frac{A_u^{(-2)}}{r^2} \\ \frac{A_z^{(-1)}}{r} \\ A_{\bar{z}}^{(0)}+\frac{A_{\bar{z}}^{(-1)}}{r}+\frac{A_{\bar{z}}^{(-2)}}{r^2} \end{pmatrix} \in X_0^{(-1,f_3)}, \quad (6.15)$$

with the nonvanishing components of the kinematic bracket given by

$$b_2(\mathcal{A}_1,\mathcal{A}_2)_\mu = 2r^{-2}\mathcal{A}^{(0)}_{[1\mu}\Big(\mathcal{A}^{(-1)}_{2]u} - \partial_z\mathcal{A}^{(0)}_{2]\bar{z}}\Big), \quad \mu=u,\bar{z} \quad (6.16)$$

Note that if we also set $\mathcal{A}_u^{(0)}=0$, then the only nonvanishing component is $\bar{z}$.

2. Strict kinematic algebra for $\mathfrak{S}_{k\le -2}$

Refined slices $\mathfrak{S}_{k,f_i}$ with $k\le -2$ are collected in the table below:

We have obtained six infinite families of solutions, valid for any $k\le -2$. In contrast with the previous section, these slices require, in addition to some gauge choice, the elimination of certain unphysical degrees of freedom, or the restriction of the solution space.[25] Additionally, in each of the slices with $k\le -2$, we found that restrictions were needed on two of the components, resulting in $\binom{4}{2}=6$ families of solutions.

The kinematic algebra brackets can be obtained by plugging in the expansion for the gauge field given in (6.10), for the explicit values of t_μ^k and $f_{i,\mu}$, with $i\in\{4,5,6,7,8,9\}$ given in the table, into the expression in (3.39), together with the projection in (B17).

[24] It is hence possible to reach these via projections which satisfy the cochain map condition (5.7). We will not present these here explicitly, as our projected fields can be obtained by a simple field redefinition of the ones arising from imposing the cochain map condition.

[25] And hence they might not necessarily by obtainable by a projection satisfying the cochain map condition.

Solutions $f^k_{4,\mu}$ and $f^k_{5,\mu}$ automatically strictify the kinematic algebra, as they satisfy all eight conditions in (6.9). For example, the nonvanishing components of the kinematic bracket between two gauge fields for the refined slice $\mathfrak{S}_{k,f_5}$ are

$$b_2^{(k,f_5)}(\mathcal{A}_1,\mathcal{A}_2)_\mu = 2r^{k-1}\mathcal{A}^{(0)}_{[1\mu}\Big[(k+2)A^{(k)}_{2]u} - \partial_z \mathcal{A}^{(k+1)}_{2]\bar{z}}\Big], \qquad \mu = u,\bar{z}. \tag{6.17}$$

Notice that the above expression with $k=-1$ reduces to (6.16), the kinematic bracket of the slice $\mathfrak{S}_{-1,f_3}$.

The families of solutions in the shaded columns of the table, i.e. $\mathfrak{S}_{k,f_6}$ to $\mathfrak{S}_{k,f_9}$, only satisfy the first four conditions in (6.9). We have checked that they strictify the kinematic algebra upon imposing further constrains arising from the nondynamical components of the equations of motion. Specifically, they turn out to be a subset of the components for which the quadratic part of the e.o.m. vanishes.

Interestingly, for slice $\mathfrak{S}_{k,f_7}$ we find that these conditions are exactly required by the cochain map condition, which amounts to imposing the following constraints on the projected gauge fields:

$$\begin{cases} \partial_u A_z^{(k+1)} = 0 \\ \partial_z A_{\bar{z}}^{(n+2)} = \partial_u A_r^{(n)} \quad n\in\{k,\ldots,-2\} \end{cases}. \tag{6.18}$$

Let us now recall the Schouten-Nijenhuis bracket previously obtained in [62], and which reduces to the Poisson bracket of Monteiro and O'Connell [6] upon writing the gauge field in terms of a scalar. This was reviewed in Sec. III C and presented in Bondi coordinates in Sec. IV B. A natural question is whether this can arise as some kind of limit of one of the solutions in Table IV. Indeed this is the case. By taking the $k\to-\infty$ limit of slice 7 we have

$$\lim_{k\to-\infty}\mathfrak{S}_{k,f_7} \Rightarrow A_u = A_z = 0. \tag{6.19}$$

In this limit, the first condition in (6.18) is trivially satisfied, while the second one reduces to the Lorenz gauge condition[26] discussed in [62], and hence the kinematic bracket reduces to the Schouten-Nijenhuis bracket in the 2-dimensional space spanned by r and $\bar{z}$, see (4.20).

It turns out that the limit $k\to-\infty$ of slice 6 gives a very similar result

$$\lim_{k\to-\infty}\mathfrak{S}_{k,f_6} \Rightarrow A_r = A_{\bar{z}} = 0, \tag{6.20}$$

for which the kinematic bracket is just the Schouten-Nijenhuis in the 2-dimensional space spanned by u and z. Indeed, this just results from an alternative gauge choice of the solution in [62], which leads to the elimination of a different unphysical degree of freedom.

It would be very interesting to study the other families of solutions in more detail, and see whether they correspond to any useful subspaces of the theory, perhaps for the study of subleading infrared phenomena. We leave this for future work.

TABLE IV. Refined slices with $k\le-2$. Slices $\mathfrak{S}_{k,f_4}$ and $\mathfrak{S}_{k,f_5}$ fully strictify $\mathfrak{Kin}^{(k)}$, by setting all θ_3's in (4.18) and (4.19) to 0. Slices $\mathfrak{S}_{k,f_6}$ to $\mathfrak{S}_{k,f_9}$ are obtained from the weaker requirement that only $\theta_3(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$ in (4.18) vanishes.

	$\forall\ k\le-2$						
	t^k_μ	$f^k_{4,\mu}$	$f^k_{5,\mu}$	$f^k_{6,\mu}$	$f^k_{7,\mu}$	$f^k_{8,\mu}$	$f^k_{9,\mu}$
r	$k-2$	-2	$k-2$	$k-1$	-2	-2	$k-1$
u	$k-1$	$k+1$	0	0	k	0	k
z	k	0	k	0	$k+1$	$k+1$	0
$\bar{z}$	$k-1$	$k+1$	0	k	0	k	0

VII. FALL-OFFS AND QUASI-ISOMORPHISM

In the previous sections, we have shown how to construct L_∞ that encode SDYM in different slices with different powers of the radial coordinate r. More precisely, we have defined projection maps that truncate the r expansion of all the elements of the L_∞ algebra that describes SDYM. Naturally, performing such truncations or projections can lead to a loss of physical information. Indeed, as we mentioned earlier and will explain in detail in the following subsection, such projections induce maps from the cohomology of the original theory to the cohomology of a truncated version of the theory. In general, these induced maps are not invertible (isomorphic), which implies that the cohomologies of the original and truncated theories are not the same, indicating the loss of physical information.

To understand this from a mathematical perspective and to implement truncations that preserve all the physical information of the theory, we begin this section by introducing the notion *quasi-isomorphisms*, which are morphisms between chain complexes that induce isomorphisms between their cohomologies. Following that initial formal discussion, we introduce maps we use to construct L_∞ algebras with isomorphic cohomologies.

A. Cochain maps and quasi-isomorphisms

Intuitively, we want to identify subspaces of $\mathcal{X}^{\mathrm{SDYM}}$ that can describe physics close to null infinity. Therefore, we need to understand how to relate the physical information of the whole space $\mathcal{X}^{\mathrm{SDYM}}$ with that of the smaller subspaces, which we shall denote by $\bar{\mathcal{X}}$, that we wish to consider. In the L_∞ formulation of field theories, the physical information of the theory is encoded in the

[26]Notice that the Lorenz condition in Bondi coordinates is given by $b(A)=0$, where b is the operator in (4.13).

cohomology of the differential B_1. Indeed, recalling the cochain complex of the theory

$$\begin{array}{ccccc} X_{-1} & \xrightarrow{B_1} & X_0 & \xrightarrow{B_1} & X_1 \\ \Lambda & & A & & E \end{array}, \tag{7.1}$$

we see that the cohomology of B_1 corresponds to field configurations that satisfy the linear field equations modulo linearized gauge transformations, which correspond to the theory's physical or propagating degrees of freedom. Consequently, to understand the relation between the physics of the full theory and that of the subspaces, we need to understand the relation between the cohomologies of the cochain complexes $(B_1, \mathcal{X}^{\mathrm{SDYM}})$ and $(\bar{B}_1, \bar{\mathcal{X}})$ with $\bar{\mathcal{X}} \subset \mathcal{X}^{\mathrm{SDYM}}$, where $\bar{B}_1$ is the differential that describes the theory in the desired regime, i.e., at the desired power of the radial coordinate r. In the subsequent general discussion, we follow [70].

The first step in establishing the relation between the physics of the full theory and that of the subspaces is to introduce a projection map $\pi\colon \mathcal{X}^{\mathrm{SDYM}} \to \bar{\mathcal{X}}$ which acts diagrammatically as

$$\begin{array}{ccccc} X_{-1} & \xrightarrow{B_1} & X_0 & \xrightarrow{B_1} & X_1 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \bar{X}_{-1} & \xrightarrow{\bar{B}_1} & \bar{X}_0 & \xrightarrow{\bar{B}_1} & \bar{X}_1 \end{array}. \tag{7.2}$$

If the projection map π obeys the following relation:

$$\pi \circ B_1 = \bar{B}_1 \circ \pi, \tag{7.3}$$

then we say that the π is a *cochain map*. Diagrammatically, we say that π is a cochain map if the following diagram commutes

$$\begin{array}{ccc} X_i & \xrightarrow{B_1} & X_{i+i} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \bar{X}_i & \xrightarrow{\bar{B}_1} & \bar{X}_{i+1} \end{array}, \tag{7.4}$$

for all i. In words, the diagram commutes if, starting from the space X_i, we can follow either of the two possible routes to $\bar{X}_{i+1}$ and obtain the same result. Naturally, cochain maps induce maps on the cohomologies H^i and $\bar{H}^i$ of the differentials B_1 and $\bar{B}_1$, respectively, as we shall prove in the following. Consider an element $x \in H^i$, that is, $x \sim x + B_1(y)$, where $\sim$ indicates that they are equivalent in H^i. We denote the whole equivalence class as $[x]$. The projection map acts on an arbitrary representative of H^i $[x + B_1(y)]$ as

$$\begin{aligned} \bar{H}^i \ni [\pi(x + B_1(y))] &= [\pi(x) + \pi(B_1(y))] \\ &= [\pi(x) + \bar{B}_1(\pi(y))] \\ &= [\pi(x)], \end{aligned} \tag{7.5}$$

where in the second line we used the cochain map condition (7.3) and to get to the last line we used the fact that, in the cohomology $\bar{H}^i$, $\pi(x) \sim \pi(x) + \bar{B}_1(\pi(y))$. The above computation proves that, indeed, π induces a well-defined, representative independent map on cohomologies, which, in general, is noninvertible. Physically, π takes an arbitrary element of the full theory in $\mathcal{X}^{\mathrm{SDYM}}$ and projects it to the regime we wish to consider. As we shall define in detail in the following sections, for our purposes, π truncates powers of the radial coordinate r to obtain expansions in this coordinate of the form (2.12).

In this paper, we also consider a map $\iota\colon \bar{\mathcal{X}} \to \mathcal{X}^{\mathrm{SDYM}}$ called the *inclusion map* that acts diagrammatically as

$$\begin{array}{ccccc} X_{-1} & \xrightarrow{B_1} & X_0 & \xrightarrow{B_1} & X_1 \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} \\ \bar{X}_{-1} & \xrightarrow{\bar{B}_1} & \bar{X}_0 & \xrightarrow{\bar{B}_1} & \bar{X}_1 \end{array}, \tag{7.6}$$

which tells us what the elements of $\bar{\mathcal{X}}$ look like in the original, bigger space $\mathcal{X}^{\mathrm{SDYM}}$. To induce a map in cohomologies, analogously to π, ι shall obey the cochain map condition

$$B_1 \circ \iota = \iota \circ \bar{B}_1. \tag{7.7}$$

Thus, the inclusion map ι induces a map inverse to that induced by the projection map π in cohomologies, i.e.,

$$\begin{aligned} H^i \ni [\iota(\bar{x} + \bar{B}_1(\bar{y}))] &= [\iota(\bar{x}) + \iota(\bar{B}_1(\bar{y}))] \\ &= [\iota(\bar{x}) + B_1(\iota(\bar{y}))] \\ &= [\iota(\bar{x})], \end{aligned} \tag{7.8}$$

with $\bar{x}, \bar{y} \in \bar{\mathcal{X}}$. In other words, there exists an isomorphism on cohomologies and we say that there exists a *quasi-isomorphism* between the two chain complexes $(B_1, \mathcal{X}^{\mathrm{SDYM}})$ and $(\bar{B}_1, \bar{\mathcal{X}})$. Physically, this indicates that both chain complexes contain the same physical information.

B. Cochain map conditions

The goal is to construct a new L_∞-algebra $(\bar{\mathcal{X}}, \bar{B}_1, \bar{B}_2)$ that encodes the physics of the initial theory near $\mathcal{I}$. To this end, we shall define two maps $\pi\colon \mathcal{X} \to \bar{\mathcal{X}}$ and $\iota\colon \bar{\mathcal{X}} \to \mathcal{X}$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \xrightarrow{B_1} & X_{-1} & \xrightarrow{B_1} & X_0 & \xrightarrow{B_1} & X_1 & \xrightarrow{B_1} & 0 \\ & & \iota\uparrow\downarrow\pi & & \iota\uparrow\downarrow\pi & & \iota\uparrow\downarrow\pi & & \\ 0 & \xrightarrow{\bar{B}_1} & \bar{X}_{-1} & \xrightarrow{\bar{B}_1} & \bar{X}_0 & \xrightarrow{\bar{B}_1} & \bar{X}_1 & \xrightarrow{\bar{B}_1} & 0 \end{array} \tag{7.9}$$

commutes. In other words, we require π and ι to be cochain maps. In general, the first two brackets on $\bar{\mathcal{X}}$ are defined in terms of the brackets on $\mathcal{X}$ via the cochain maps as

$$\bar{B}_i \coloneqq \pi B_i \iota, \quad i = 1, 2. \tag{7.10}$$

We observe that

$$\begin{aligned} (7.9)\,\text{commutes} &\Leftrightarrow \begin{cases} \bar{B}_1\pi = \pi B_1 \\ B_1\iota = \iota\bar{B}_1 \end{cases} \\ &\Leftrightarrow \begin{cases} \mathrm{Im}(1_{\mathcal{X}} - \iota\pi) \subseteq \mathrm{Ker}(\pi B_1) \\ \mathrm{Im}(B_1\iota) \subseteq \mathrm{Ker}(1_{\mathcal{X}} - \iota\pi) \end{cases}, \end{aligned} \tag{7.11}$$

where in the second equivalence we used the definition of $\bar{B}_1$. We want $\bar{\mathcal{X}}$ to satisfy an equation similar to (5.2), namely

$$\bar{\mathcal{X}} \subseteq \mathcal{X}^{n\leq 0} \subset \mathcal{X}. \tag{7.12}$$

In particular, we require $\bar{\mathcal{X}}$ to be a subspace of $\mathcal{X}$. Hence, the maps π, ι shall be, respectively, the projection to $\bar{\mathcal{X}}$ and the canonical inclusion

$$\begin{aligned} \pi &: \mathcal{X} \to \bar{\mathcal{X}} \qquad \iota : \bar{\mathcal{X}} \hookrightarrow \mathcal{X} \\ \psi &\mapsto \pi(\psi), \qquad \bar{\psi} \mapsto \bar{\psi}. \end{aligned} \tag{7.13}$$

Notice that for an arbitrary $\psi \in \mathcal{X}$, we can write $\iota\pi(\psi) = \pi(\psi)$, where both sides of this equality are viewed as elements of $\mathcal{X}$ due to (7.12). In particular,

$$1_{\mathcal{X}} - \iota\pi = 1_{\mathcal{X}} - \pi =: \pi^{\complement}, \tag{7.14}$$

where $\pi^{\complement}$ is the projection to the subspace complementary to $\bar{\mathcal{X}}$ (see discussion in Sec. V A). Moreover, $\mathcal{X} = \mathrm{Im}\,\pi \oplus \mathrm{Im}\,\pi^{\complement}$, where $\mathrm{Im}\,\pi = \bar{\mathcal{X}}$. Of course, $\mathrm{Im}\,\pi^{\complement} = \mathrm{Ker}\pi$ (and this identity is symmetric under $\pi \leftrightarrow \pi^{\complement}$). This implies that (7.9) commutes iff

$$\mathrm{Im}\,\pi^{\complement} \subseteq \mathrm{Ker}(\pi B_1) \tag{7.15a}$$

$$\mathrm{Im}(B_1\iota) \subseteq \mathrm{Ker}\pi^{\complement} = \mathrm{Im}\,\pi = \bar{\mathcal{X}}. \tag{7.15b}$$

In particular, the second inclusion above implies that the subspace $\bar{\mathcal{X}}$ is closed under the action of the differential B_1. Thus, $\bar{B}_1 = B_1|_{\bar{\mathcal{X}}}$.

In Sec. V we showed that (7.12) is solved by

$$\bar{\mathcal{X}} \coloneqq \{\psi \in \mathrm{Im}\,\pi_{\mathcal{N}} | \mathcal{N}\ \text{satisfies (5.24)}\}, \tag{7.16}$$

where $\mathcal{N}$ is the collection of power selection sets introduced in (5.19) and $\pi_{\mathcal{N}}$ is the morphism defined in (5.21). In other words, $\pi = \pi_{\mathcal{N}}$. In what follows, we will prove that there is only one choice of sets in $\mathcal{N}$ such that (7.15) with $\pi = \pi_{\mathcal{N}}$ is satisfied. (For this reason, $\bar{\mathcal{X}}$ in (7.16) does not carry the label $\mathcal{N}$.)

C. Fall-off from cochain map conditions

Result 7.1. Let D be the diagram in (7.9) with $\bar{\mathcal{X}}$ as in (7.16), projection $\pi = \pi_{\mathcal{N}}$ and canonical inclusion ι. Then D commutes iff the power selection sets in $\mathcal{N}$ are

$$\begin{aligned} N_\Lambda = N_{A_\alpha} = N_{E_{uz}} &= \mathbb{Z}^{\leq 0} \\ N_{A_r} = N_{E_{ru}} = N_{E_{r\bar{z}}} &= \mathbb{Z}^{<0}\backslash\{-1\}. \end{aligned} \tag{7.17}$$

Proof. From Eq. (7.15) it follows that

$$\begin{aligned} D\,\text{commutes} &\Leftrightarrow \begin{cases} \mathrm{Im}\,\pi^{\complement}_{\mathcal{N}} \subseteq \mathrm{Ker}(\pi_{\mathcal{N}} B_1) \\ \mathrm{Im}(B_1\iota) \subseteq \mathrm{Ker}\pi^{\complement}_{\mathcal{N}} \end{cases} \\ &\Leftrightarrow \begin{cases} \Psi_{\mathcal{N}}(\psi) \coloneqq \pi_{\mathcal{N}} B_1 \pi^{\complement}_{\mathcal{N}}(\psi) = 0 & \forall\ \psi \in \mathcal{X} \\ \Phi_{\mathcal{N}}(\bar{\psi}) \coloneqq \pi^{\complement}_{\mathcal{N}} B_1 \pi_{\mathcal{N}}(\bar{\psi}) = 0 & \forall\ \bar{\psi} \in \bar{\mathcal{X}}, \end{cases} \end{aligned} \tag{7.18}$$

where $\pi^{\complement}_{\mathcal{N}} \coloneqq 1_{\mathcal{X}} - \pi_{\mathcal{N}}$ is the projection to the complement of $\bar{\mathcal{X}}$ and we used $\iota(\bar{\psi}) = \bar{\psi} = \pi_{\mathcal{N}}(\bar{\psi})$. We want to find the conditions that the power selection sets in $\mathcal{N}$ must obey so that the second system above holds. For the first equation of this system, the answer is provided by lemma A.2 of Appendix A. There, we prove that $\Psi_{\mathcal{N}}(\psi)$ vanishes for all $\psi \in \mathcal{X}$ iff the sets in $\mathcal{N}$ satisfy the inclusions listed in (A8). To solve the second equation in (7.18), we observe that $\Phi_{\mathcal{N}}$ can be obtained from $\Psi_{\mathcal{N}}$ by switching $\pi_{\mathcal{N}} \leftrightarrow \pi^{\complement}_{\mathcal{N}}$. Moreover, from (5.21) and (A10) it follows that $\pi^{\complement}_{\mathcal{N}} = \pi_{\mathcal{N}^{\complement}}$, where $\mathcal{N}^{\complement}$ is the collection of sets containing the complements in $\mathbb{Z}$ of the sets in $\mathcal{N}$. Therefore, we deduce that $\Phi_{\mathcal{N}} = \Psi_{\mathcal{N}^{\complement}}$. So, the system in (7.18) is solved by a projector $\pi_{\mathcal{N}}$ with $\mathcal{N}$ such that both $\mathcal{N}$ and $\mathcal{N}^{\complement}$ satisfy the inclusions in (A8). Such a collection $\mathcal{N}$ consists of sets satisfying the system

$$N_{A_r}\backslash\{-1\} = \mathcal{T}_{-1} N_\Lambda \backslash\{-1\} \tag{7.19a}$$

$$N_{A_\alpha} = N_\Lambda \tag{7.19b}$$

$$N_{E_{ru}} = N_{A_r} = \mathcal{T}_{-1} N_{A_u} = \mathcal{T}_{-2} N_{A_z} = \mathcal{T}_{-2} N_{A_{\bar{z}}} \tag{7.19c}$$

$$N_{E_{r\bar{z}}} = N_{A_r} \tag{7.19d}$$

$$N_{E_{r\bar{z}}} \backslash \{-1\} = \mathcal{T}_{-1} N_{A_{\bar{z}}} \backslash \{-1\} \tag{7.19e}$$

$$N_{E_{uz}} = N_{A_u} = N_{A_z}, \tag{7.19f}$$

where $\mathcal{T}$ is the translation defined in (5.25). To find the above equations, we used the following fact: given $M, N, L \subseteq \mathbb{Z}$ and $m \in \mathbb{Z}$, one can prove that

(i) if $M\backslash\{m\} \subseteq N$ and $M^{\complement}\backslash\{m\} \subseteq N^{\complement}$, then $M\backslash\{m\} = N\backslash\{m\}$;
(ii) if $L \subseteq M \cap N$ and $L^{\complement} \subseteq M^{\complement} \cap N^{\complement}$, then $L = M = N$.

At this point, it is easy to see that (7.17) is the only possible solution for both (5.24) (which is our hypothesis) and the system (7.19). ∎

Equation (7.17) provides a proof of the fall-offs given in (2.12); thus, we are effectively proving the same result already established in Sec. V, but in a more concise and elegant way. Moreover, by comparing the power selection sets found solving the master system in Sec. V and the ones found here, we observe that

$$\bar{\mathcal{X}} = \lim_{k \to -\infty} \mathcal{X}^{(k)}, \tag{7.20}$$

cf. Eqs. (7.17) and (5.27).

D. $(\bar{\mathcal{X}}, \bar{B}_1, \bar{B}_2)$ is an L_∞ subalgebra

Lemma 7.2. $\bar{B}_2 = B_2|_{\bar{\mathcal{X}} \otimes \bar{\mathcal{X}}}$

Proof. From the definition of $\bar{B}_2$, see (7.10), it follows that the result we want to prove is equivalent to the statement that $\bar{\mathcal{X}}$ is closed under the action of B_2,[27] i.e.

$$\mathrm{Im}[B_2(\iota \otimes \iota)] \subseteq \bar{\mathcal{X}} = \mathrm{Im}\,\pi_{\mathcal{N}} = \mathrm{Ker}\pi^{\complement}_{\mathcal{N}}. \tag{7.21}$$

In other words, it suffices to show that $\Xi \coloneqq \pi^{\complement}_{\mathcal{N}} B_2(\iota \otimes \iota) = \pi^{\complement}_{\mathcal{N}} B_2(\pi_{\mathcal{N}} \otimes \pi_{\mathcal{N}})$ vanishes on $\bar{\mathcal{X}} \otimes \bar{\mathcal{X}}$. This is always true, as one can show using the explicit form of B_2 and the properties of the projectors.[28] For instance, for arbitrary $\bar{\Lambda}_1, \bar{\Lambda}_2 \in \bar{X}_{-1}$,

$$\Xi(\bar{\Lambda}_1, \bar{\Lambda}_2) = -\mathcal{P}_{\mathbb{Z}^{>0}}[\mathcal{P}_{\mathbb{Z}^{\leq 0}} \bar{\Lambda}_1, \mathcal{P}_{\mathbb{Z}^{\leq 0}} \bar{\Lambda}_2] = -\mathcal{P}_{\mathbb{Z}^{>0} \cap \mathbb{Z}^{\leq 0}}[\bar{\Lambda}_1, \bar{\Lambda}_2] = 0. \tag{7.22}$$

Analogously, one can prove that $\Xi(\psi_1, \psi_2)$ vanishes for all other choices of $\psi_1, \psi_2 \in \bar{\mathcal{X}}$. ∎

This implies that $(\bar{\mathcal{X}}, \bar{B}_1, \bar{B}_2)$ is an L_∞ subalgebra of $(\mathcal{X}, B_1, B_2)$. In Appendix C, we provide a proof of this result based on the concept of *homotopy transfer* which consists of the transfer of algebraic information between different chain complexes. Moreover, we show how the kinematic algebra for these choices of fall-offs can be constructed.

VIII. CONCLUSIONS

In this article, we have employed the power of homotopy algebras to construct robust truncations of self-dual Yang-Mills in a $\frac{1}{r}$ expansion near null infinity. We have found that the physical fall-off of gauge fields can be *derived* from the requirement that slices preserve the L_∞ relations of the original theory. We then proceeded to construct novel strict kinematic algebras on various slices near null infinity.

In our search for a strict kinematic algebra, we restricted ourselves to axial-type (partial) gauge constrains in the current article. We anticipate that there are a number of other options for revealing the strict kinematic algebra, via the vanishing of the θ terms in (6.4) and (6.5), which exactly parametrize the failure of the algebra to be strict. We leave these for future work.

The natural next step is to extend our results to self-dual gravity. In this framework, it would be intriguing to explore whether the fall-off behavior of the metric components, stemming from asymptotic flatness, can similarly be derived from a homotopy algebraic perspective. Additionally, incorporating the BMS symmetry group into this analysis presents an exciting avenue for future work. The presence of the strict kinematic algebras identified in this article also suggests that the double copy will proceed straightforwardly.[29]

So far, our analysis has centred on the self-dual subsectors of field theories, where the cochain complex exhibits a simpler structure, and the L_∞ algebra is limited to brackets no higher than B_2. Extending this framework to encompass full Yang-Mills and gravity will require working with a more intricate structure, including the presence of higher-order brackets. Nevertheless, we anticipate that the master equation in (5.14) will admit a natural generalization of the following form:

$$\mathrm{Im}\,\Pi \subseteq \mathcal{X}^{n \leq 0} \tag{8.1a}$$

$$\mathrm{Ker}\Pi \subseteq \mathrm{Ker}(\Pi B_1) \tag{8.1b}$$

$$\mathrm{Leib}(\Pi B_1|_{\mathrm{Im}\,\Pi}, \Pi B_2|_{\mathrm{Im}\,\Pi \otimes \mathrm{Im}\,\Pi}) = 0 \tag{8.1c}$$

$$\mathrm{Jac}(\Pi B_2|_{\mathrm{Im}\,\Pi \otimes \mathrm{Im}\,\Pi}) + [\Pi B_1|_{\mathrm{Im}\,\Pi}, \Pi B_3|_{\mathrm{Im}\,\Pi \otimes \mathrm{Im}\,\Pi \otimes \mathrm{Im}\,\Pi}] = 0 \tag{8.1d}$$

[27] We observe that this is equivalent to saying that B_2 commutes with the inclusion ι, since $\mathrm{Im}[B_2(\iota \otimes \iota)] \subseteq \mathrm{Ker}\pi^{\complement}_{\mathcal{N}} = \mathrm{Ker}(1_{\mathcal{X}} - \iota\pi_{\mathcal{N}})$ iff $B_2(\iota \otimes \iota) = \iota \bar{B}_2$.

[28] In particular, use (A2g).

[29] For some previous work on the double copy at null infinity, see [24,29,71–74].

$$\Pi B_2|_{\mathrm{Im}\,\Pi\otimes\mathrm{Im}\,\Pi}\Pi B_3|_{\mathrm{Im}\,\Pi\otimes\mathrm{Im}\,\Pi\otimes\mathrm{Im}\,\Pi} + \Pi B_3|_{\mathrm{Im}\,\Pi\otimes\mathrm{Im}\,\Pi\otimes\mathrm{Im}\,\Pi}\Pi B_2|_{\mathrm{Im}\,\Pi\otimes\mathrm{Im}\,\Pi} = 0, \tag{8.1e}$$

where in the second to last relation we included a term containing the three-bracket B_3 that captures the quartic interactions of Yang-Mills, and the last relation schematically represents a higher homotopy compatibility relation between the two-bracket B_2 and the higher bracket B_3.

The study of infrared phenomena and their relations to asymptotic symmetries has more recently been extended to include subleading effects [29,67,75–93]. This has unveiled infinite symmetry algebras,[27,30,33,89,90,94–113], which appear in the OPE's of the CFT on the boundary in celestial holography. On the YM side, the relevant algebra is the so-called S-algebra (see [114] and [115] for its phase space realization). It would be interesting to include the overleading large gauge parameters (which are known to lead to subleading effects, see e.g. [24] and [29] for SDYM) in our analysis to see if the homotopy algebra gives rise to some known deformations of the S-algebra. On the gravity side, the corresponding infinite algebra is the $w_{1+\infty}$ algebra; here a potentially useful starting point is [30], where the $w_{1+\infty}$ algebra was derived from the kinematic algebra in its original presentation as an area-preserving diffeomorphism in the self-dual sector [6]. Finally, another long-term goal to explore how to use the homotopy-transfer-based framework developed in the context of AdS/CFT in [58] in this setting.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: CONSTRUCTION OF THE MASTER SYSTEM'S SOLUTIONS

In this Appendix, we provide a detailed proof of the solutions (5.26) for the master system (5.14), using the formalism introduced in Sec. V B. We start by listing some useful properties of the projection $\mathcal{P}_N$, then we formulate the conditions that the power selection sets in (5.19) must satisfy in order for $\pi_{\mathcal{N}}$ to solve the system. Finally, we construct the infinite family of physically relevant solutions $\{\mathcal{N}_k\}_{k\in\mathbb{Z}^{\leq 0}}$ for these conditions, which in turn allows us to derive the L_∞ algebras encoding self-dual YM theory near null infinity.

1. Properties of $\mathcal{P}_N$

In this subsection, we provide some useful properties of the projection $\mathcal{P}_N$ defined in (5.17), which will be frequently referenced in the remainder of this Appendix.

We begin by introducing some notation. Let $M, N \subseteq \mathbb{Z}$. We define the complement of M in $\mathbb{Z}$ and the addition of the sets M and N, respectively, as

$$M^{\complement} := \mathbb{Z}\backslash M, \qquad M+N := \{m+n | m\in M, n\in N\} \subseteq \mathbb{Z}. \tag{A1}$$

Moreover, we adopt the following notation for the projector $\mathcal{P}_N$: if N is a singleton $\{n\}$, we simply write $\mathcal{P}_n := \mathcal{P}_{\{n\}}$. With this in mind, and recalling that we work with Bondi coordinates $x^\mu = (r, y^\alpha) = (r, u, z, \bar{z})$, we can prove the properties below.

Lemma A.1 (Properties of projection). Let $\mathcal{P}_N$ be the map

$$\mathcal{P}_N f(r, y^\alpha) := \sum_{n\in N} r^n f^{(n)}(y^\alpha) \tag{5.17}$$

and $\mathcal{T}$ the translation in (5.25). Then $\forall\ s\in\mathbb{Z}$, $\forall\ L, M, N \subseteq \mathbb{Z}$ and $\forall\ f, g \in C^\infty_r(\mathcal{M})$ the following relations hold:

$$\mathcal{P}_M\mathcal{P}_N = \mathcal{P}_{M\cap N} \tag{A2a}$$

$$\mathcal{P}_M - \mathcal{P}_N = \mathcal{P}_{M\backslash N} \quad \text{if } N\subseteq M \tag{A2b}$$

$$\mathcal{P}_M(r^s f) = r^s \mathcal{P}_{\mathcal{T}_{-s}M} f \tag{A2c}$$

$$\mathcal{P}_M\partial_r = \mathcal{P}_{M\backslash\{-1\}}\partial_r = \partial_r\mathcal{P}_{\mathcal{T}_1 M\backslash\{0\}} = \partial_r\mathcal{P}_{\mathcal{T}_1 M} \tag{A2d}$$

$$\mathcal{P}_M\partial_\alpha = \partial_\alpha\mathcal{P}_M \tag{A2e}$$

$$\mathcal{P}_M(fg) = \sum_{m\in M}\sum_{n\in\mathbb{Z}} \mathcal{P}_{m-n}(f)\mathcal{P}_n(g) \tag{A2f}$$

$$\mathcal{P}_L(\mathcal{P}_M(f)\mathcal{P}_N(g)) = \begin{cases} \mathcal{P}_M(f)\mathcal{P}_N(g) & \text{if } L\supseteq M+N \\ 0 & \text{if } L\subseteq (M+N)^{\complement} \end{cases} \tag{A2g}$$

In particular, $\mathcal{P}_N$ is a projector and the commutator $[\mathcal{P}_M, \mathcal{P}_N]$ vanishes for all M, N.

Proof. $\mathcal{P}_M$ can be rewritten as

$$\mathcal{P}_M = \sum_{m\in M}\mathcal{P}_m, \tag{A3}$$

where $\mathcal{P}_m$ acts as

$$\mathcal{P}_m(f) := \sum_{n\in\mathbb{Z}} \delta_{mn} r^n f^{(n)} = r^m f^{(m)}. \tag{A4}$$

By virtue of this trivial identity, we can prove the above lemma in the simple case when L, M and N are singletons; the general result can then be recovered using (A3). $\forall\ l,m,n,s\in\mathbb{Z}$ and $\forall\ f,g\in C^\infty_r(\mathcal{M})$ the following identities hold:

$$\mathcal{P}_m\mathcal{P}_n f=\mathcal{P}_m(r^n f^{(n)})=\delta_{mn}r^n f^{(n)}=\delta_{mn}\mathcal{P}_n f \tag{A5a}$$

$$\mathcal{P}_m(r^s f)=\sum_{n\in\mathbb{Z}}\delta_{m,n+s}r^{n+s}f^{(n)}=r^m f^{(m-s)}=r^s\mathcal{P}_{m-s}f \tag{A5b}$$

$$\mathcal{P}_m\partial_r f=\sum_{n\in\mathbb{Z}}n\delta_{m,n-1}r^{n-1}f^{(n)}=(m+1)r^m f^{(m+1)} \tag{A5c}$$

$$=\partial_r(r^{m+1}f^{(m+1)})=\partial_r\mathcal{P}_{m+1}f$$

$$\mathcal{P}_m(fg)=\sum_{l,n\in\mathbb{Z}}\delta_{m,l+n}r^{l+n}f^{(l)}g^{(n)}=r^m\sum_{n\in\mathbb{Z}}f^{(m-n)}g^{(n)} \tag{A5d}$$

$$=\sum_{n\in\mathbb{Z}}\mathcal{P}_{m-n}(f)\mathcal{P}_n(g)$$

$$\mathcal{P}_l[\mathcal{P}_m(f)\mathcal{P}_n(g)]=\delta_{l,m+n}r^{m+n}f^{(m)}g^{(n)}=\delta_{l,m+n}\mathcal{P}_m(f)\mathcal{P}_n(g). \tag{A5e}$$

The first equation above implies that $\mathcal{P}_m$ is a projector (since $\mathcal{P}_m^2=\mathcal{P}_m$), while from the third it is clear that

$$\mathcal{P}_{-1}(\partial_r f)=0\quad\forall\ f\in C^\infty_r(\mathcal{M}). \tag{A6}$$

Note that this equation holds because we do not include logarithms in the definition of $C^\infty_r(\mathcal{M})$, see the expansion (4.3). It is easy to see that $\mathcal{P}_m$ commutes with ∂_α. Equations (A2a)–(A2f) follow straightforwardly. For instance

$$\mathcal{P}_M\mathcal{P}_N=\sum_{m\in M}\sum_{n\in N}\mathcal{P}_m\mathcal{P}_n=\sum_{m\in M}\sum_{n\in N}\mathcal{P}_m\delta_{mn}=\sum_{m\in M\cap N}\mathcal{P}_m=\mathcal{P}_{M\cap N}, \tag{A7}$$

thus $\mathcal{P}_M$ is a projector. ∎

2. Conditions on the power selection sets

As discussed in Sec. V B 2, the morphism $\pi_\mathcal{N}$ with power selection sets satisfying (5.24) is the most general solution for Π for the first equation of the master system. Below, we find the additional conditions that the sets in $\mathcal{N}$ must satisfy in order for $\pi_\mathcal{N}$ to solve all the other equations of the master system, too. In this way, we end up with a system of equations for the power selection sets. A projection $\pi_\mathcal{N}$ with $\mathcal{N}$ solving such a system will be the desired solution for Π in (5.14).

a. Cochain map condition

We start by looking at the second equation of the master system (5.14), the one encoding the request that Π is a cochain map.

Lemma A.2 (Cochain map condition). Let $\pi_\mathcal{N}$ be the morphism defined in (5.21) with the sets in $\mathcal{N}$ obeying condition (5.24). Then $\pi_\mathcal{N}$ is a solution for Π in Eq. (5.14b) iff the corresponding power selection sets satisfy all the inclusions[30]

$$N_{A_r}\backslash\{-1\}\subseteq\mathcal{T}_{-1}N_\Lambda \tag{A8a}$$

$$N_{A_\alpha}\subseteq N_\Lambda \tag{A8b}$$

$$N_{E_{ru}}\subseteq N_{A_r}\cap\mathcal{T}_{-1}N_{A_u}\cap\mathcal{T}_{-2}(N_{A_z}\cap N_{A_{\bar z}}) \tag{A8c}$$

$$N_{E_{r\bar z}}\subseteq N_{A_r} \tag{A8d}$$

$$N_{E_{r\bar z}}\backslash\{-1\}\subseteq\mathcal{T}_{-1}N_{A_{\bar z}} \tag{A8e}$$

$$N_{E_{uz}}\subseteq N_{A_u}\cap N_{A_z}. \tag{A8f}$$

Proof. We want to find the constraints that the sets in $\mathcal{N}$ shall satisfy in order to have $\mathrm{Ker}\pi_\mathcal{N}\subseteq\mathrm{Ker}(\pi_\mathcal{N}B_1)$. Following the same approach discussed in Sec. V A,[31] it is convenient to rewrite this expression as $\mathrm{Im}\,\pi^\complement_\mathcal{N}\subseteq\mathrm{Ker}(\pi_\mathcal{N}B_1)$, where $\pi^\complement_\mathcal{N}:=1_\mathcal{X}-\pi_\mathcal{N}$ is the projection to the complement of $\mathrm{Im}\,\pi_\mathcal{N}$ in $\mathcal{X}$.[32] This condition is equivalent to asking that

$$\Psi_\mathcal{N}(\psi):=\pi_\mathcal{N}B_1\pi^\complement_\mathcal{N}(\psi)=0\quad\forall\ \psi\in\mathcal{X}, \tag{A9}$$

where we introduce the notation $\Psi_\mathcal{N}$ to increase readability in what follows. From Eqs. (5.21) and (A2b) we find that, $\forall\ \psi\in\mathcal{X}$,

$$\pi^\complement_\mathcal{N}(\psi)=\psi-\pi_\mathcal{N}(\psi)=\begin{cases}(\mathcal{P}_\mathbb{Z}-\mathcal{P}_{N_\Lambda})\Lambda=\mathcal{P}_{N^\complement_\Lambda}\Lambda & \text{if } \psi\in X_{-1}\\ (\mathcal{P}_\mathbb{Z}-\mathcal{P}_{N_{A_\mu}})A_\mu=\mathcal{P}_{N^\complement_{A_\mu}}A_\mu & \text{if } \psi\in X_0\\ (\mathcal{P}_\mathbb{Z}-\mathcal{P}_{N_{E_{\mu\nu}}})E_{\mu\nu}=\mathcal{P}_{N^\complement_{E_{\mu\nu}}}E_{\mu\nu} & \text{if } \psi\in X_1\end{cases}. \tag{A10}$$

where $N^\complement_I:=\mathbb{Z}\backslash N_I$ is the complement of the set N_I, with $I\in\{\Lambda,A_\mu,E_{\mu\nu}\}$.

[30] $\mathcal{T}$ is the translation defined in (5.25).

[31] Cf. Eqs. (5.4) and (5.10).

[32] Recall that, by hypothesis, $\mathrm{Im}\,\pi_\mathcal{N}\subseteq\mathcal{X}^{n\leq0}\subset\mathcal{X}$, see Eqs. (5.2) and (5.14a).

This observation allows us to rewrite $\Psi_{\mathcal{N}}(\psi)$ in terms of the projection $\mathcal{P}$, so that we can manipulate it using the properties listed in (A2). If $\psi = \Lambda \in X_{-1}$, then $\Psi_{\mathcal{N}}(\Lambda)$ is an element of X_0 ($\Psi_{\mathcal{N}}$ has degree $+1$, since it contains the differential) and its components are[33]

$$\begin{aligned}\Psi_{\mathcal{N}}(\Lambda)_r &= \mathcal{P}_{N_{A_r}} \partial_r \mathcal{P}_{N_\Lambda^{\mathbb{C}}} \Lambda = \mathcal{P}_{N_{A_r}\setminus\{-1\}\cap \mathcal{T}_{-1} N_\Lambda^{\mathbb{C}}} \partial_r \Lambda \\ \Psi_{\mathcal{N}}(\Lambda)_\alpha &= \mathcal{P}_{N_{A_\alpha}} \partial_\alpha \mathcal{P}_{N_\Lambda^{\mathbb{C}}} \Lambda = \mathcal{P}_{N_{A_\alpha}\cap N_\Lambda^{\mathbb{C}}} \partial_\alpha \Lambda. \end{aligned} \tag{A11}$$

To study when these expressions reduce to zero, note that given any subsets $U, V \subseteq \mathbb{Z}$, the intersection $U \cap V^{\mathbb{C}}$ vanishes iff $U \subseteq V$. Moreover, the translation $\mathcal{T}_s U$ of the set U by an arbitrary integer s satisfies the relation $(\mathcal{T}_s U)^{\mathbb{C}} = \mathcal{T}_s(U^{\mathbb{C}})$. Thus, the components in (A11) vanish $\forall\ \Lambda \in X_{-1}$ iff the inclusions (A8a) and (A8b) are satisfied. A similar argument applies when $\psi = A_\mu \in X_0$. In this case, $\Psi_{\mathcal{N}}(A)$ is an equation of motion in X_1 and its independent components are given by

$$\begin{aligned}\Psi_{\mathcal{N}}(A)_{ru} &= 2\mathcal{P}_{N_{E_{ru}}}[\partial_{[r}(\mathcal{P}_{N_A^{\mathbb{C}}} A)_{u]} + r^{-2}\partial_{[z}(\mathcal{P}_{N_A^{\mathbb{C}}} A)_{\bar{z}]}] \\ &= \mathcal{P}_{N_{E_{ru}}}[\mathcal{P}_{\mathcal{T}_{-1}N_{A_u}^{\mathbb{C}}} \partial_r A_u - \mathcal{P}_{N_{A_r}^{\mathbb{C}}} \partial_u A_r \\ &\quad + \mathcal{P}_{\mathcal{T}_{-2}N_{A_{\bar{z}}}^{\mathbb{C}}}(r^{-2}\partial_z A_{\bar{z}}) - \mathcal{P}_{\mathcal{T}_{-2}N_{A_z}^{\mathbb{C}}}(r^{-2}\partial_{\bar{z}} A_z)] \\ \Psi_{\mathcal{N}}(A)_{r\bar{z}} &= 4\mathcal{P}_{N_{E_{r\bar{z}}}} \partial_{[r}(\mathcal{P}_{N_A^{\mathbb{C}}} A)_{\bar{z}]} \\ &= 2\mathcal{P}_{N_{E_{r\bar{z}}}}(\mathcal{P}_{\mathcal{T}_{-1}N_{A_{\bar{z}}}^{\mathbb{C}}\setminus\{-1\}} \partial_r A_{\bar{z}} - \mathcal{P}_{N_{A_r}^{\mathbb{C}}} \partial_{\bar{z}} A_r) \\ \Psi_{\mathcal{N}}(A)_{uz} &= 4\mathcal{P}_{N_{E_{uz}}} \partial_{[u}(\mathcal{P}_{N_A^{\mathbb{C}}} A)_{z]} \\ &= 2\mathcal{P}_{N_{E_{uz}}}(\mathcal{P}_{N_{A_z}^{\mathbb{C}}} \partial_u A_z - \mathcal{P}_{N_{A_u}^{\mathbb{C}}} \partial_z A_u). \end{aligned} \tag{A12}$$

Since we want these expressions to vanish for arbitrary A_μ, we require that each term above vanishes independently (because different terms in a given component of $\Psi_{\mathcal{N}}(A)_{\mu\nu}$ depend on different components of A_μ). For instance,

$$\begin{aligned}\Psi_{\mathcal{N}}(A)_{uz} = 0 \quad \forall\ A_\mu &\Leftrightarrow \begin{cases} \mathcal{P}_{N_{E_{uz}}\cap N_{A_z}^{\mathbb{C}}} \partial_u A_z = 0\ \forall\ A_z \\ \mathcal{P}_{N_{E_{uz}}\cap N_{A_u}^{\mathbb{C}}} \partial_z A_u = 0\ \forall\ A_u \end{cases} \\ &\Leftrightarrow N_{E_{uz}} \subseteq N_{A_u} \cap N_{A_z}, \end{aligned} \tag{A13}$$

which gives Eq. (A8f). Similarly, one can prove that $\Psi_{\mathcal{N}}(A)_{ru}$ and $\Psi_{\mathcal{N}}(A)_{r\bar{z}}$ vanish for all A_μ iff Eqs. (A8c), (A8d) and (A8e) hold. Finally, by degree, $\Psi_{\mathcal{N}}(E) = 0$ for all $E \in X_1$, so no additional information on $\mathcal{N}$ can be extracted from this condition. ∎

b. Leibniz rule condition

The next step is to determine the additional conditions that the power selection sets in $\mathcal{N}$ must satisfy to ensure that $\pi_{\mathcal{N}}$ is a solution of the Leibniz rule in (5.14c).

Lemma A.3 (Leibniz rule condition). Let $\pi_{\mathcal{N}}$ be the morphism defined in (5.21) with the sets in $\mathcal{N}$ obeying conditions (5.24) and (A8). Then $\pi_{\mathcal{N}}$ is a solution for Π in Eq. (5.14c) iff the corresponding power selection sets satisfy all the following equations:[34]

$$N_{A_r}\setminus\{-1\} \cap [\mathcal{T}_{-1}N_\Lambda \setminus (N_{A_r} \cup \{-1\}) + N_\Lambda] = \varnothing \tag{A14a}$$

$$N_{A_\alpha} \cap (N_\Lambda \setminus N_{A_\alpha} + N_\Lambda) = \varnothing \tag{A14b}$$

$$N_{E_{ru}} \cap [(N_{A_r} \cup \mathcal{T}_{-1}(N_{A_u}\setminus\{0\}) \cup \mathcal{T}_{-2}(N_{A_z} \cup N_{A_{\bar{z}}}))\setminus N_{E_{ru}} + N_\Lambda] = \varnothing \tag{A14c}$$

$$N_{E_{r\bar{z}}}\setminus\{-1\} \cap [(N_{A_r} \cup \mathcal{T}_{-1}(N_{A_{\bar{z}}}\setminus\{0\}))\setminus N_{E_{r\bar{z}}} + N_\Lambda] = \varnothing \tag{A14d}$$

$$N_{E_{uz}} \cap [(N_{A_u} \cup N_{A_z})\setminus N_{E_{uz}} + N_\Lambda] = \varnothing, \tag{A14e}$$

together with the conditions

$$N_{A_r} \ni -1 \Rightarrow N_\Lambda \ni 0 \tag{A15a}$$

$$N_{E_{r\bar{z}}} \ni -1 \Rightarrow N_{A_{\bar{z}}} \ni 0. \tag{A15b}$$

Proof. The goal is to determine the conditions on the sets in the collection $\mathcal{N}$ so that

$$\mathrm{Leib}_{\mathcal{N}}(\psi_1, \psi_2) = 0 \quad \forall\ \psi_1, \psi_2 \in \mathrm{Im}\,\pi_{\mathcal{N}}, \tag{A16}$$

where, for brevity, we defined $\mathrm{Leib}_{\mathcal{N}} \coloneqq \mathrm{Leib}(\pi_{\mathcal{N}} B_1|_{\mathrm{Im}\,\pi_{\mathcal{N}}}, \pi_{\mathcal{N}} B_2|_{\mathrm{Im}\,\pi_{\mathcal{N}} \otimes \mathrm{Im}\,\pi_{\mathcal{N}}})$. By degree, the above equation is a trivial identity when the sum of the degree of ψ_1 and ψ_2 is non-negative. Moreover, the Leibniz rule is graded symmetric.[35] Therefore, the problem reduces to the

[33] Recall Eqs. (5.21), (4.8) and (A10) for $\pi_{\mathcal{N}}$, B_1 and $\pi_{\mathcal{N}}^{\mathbb{C}}$, respectively. Moreover, use properties (A2a), (A2d) and (A2e).

[34] As a reminder, we reference the notation defined in (A1).

[35] From Eqs. (3.4) and (3.6) it follows that $\mathrm{Leib}_{\mathcal{N}}(\psi_1, \psi_2) = (-1)^{\psi_1\psi_2}\mathrm{Leib}_{\mathcal{N}}(\psi_2, \psi_1)$.

study of the two cases with $\psi_1 \in X_{-1}$ and ψ_2 belonging to either X_{-1} or X_0.

To begin, we apply $\mathrm{Leib}_{\mathcal{N}}$ to two gauge parameters $\Lambda_1, \Lambda_2 \in \mathrm{Im}\,\pi_{\mathcal{N}} \cap X_{-1}$, obtaining the following gauge field:

$$\begin{aligned}\mathrm{Leib}_{\mathcal{N}}(\Lambda_1,\Lambda_2)_\mu &= \pi_{\mathcal{N}} B_1 \pi_{\mathcal{N}} B_2(\Lambda_1,\Lambda_2)_\mu + \pi_{\mathcal{N}} B_2(\pi_{\mathcal{N}} B_1(\Lambda_1)_\mu,\Lambda_2) - \pi_{\mathcal{N}} B_2(\Lambda_1,\pi_{\mathcal{N}} B_1(\Lambda_2)_\mu)\\ &= \mathcal{P}_{N_{A_\mu}}(\partial_\mu \pi_{\mathcal{N}} B_2(\Lambda_1,\Lambda_2) + [\pi_{\mathcal{N}} B_1(\Lambda_1)_\mu,\Lambda_2] + [\Lambda_1,\pi_{\mathcal{N}} B_1(\Lambda_2)_\mu])\\ &= \mathcal{P}_{N_{A_\mu}}(-\partial_\mu \mathcal{P}_{N_\Lambda}[\Lambda_1,\Lambda_2] + 2[\mathcal{P}_{N_{A_\mu}} \partial_\mu \Lambda_{[1},\Lambda_{2]}]),\end{aligned} \tag{A17}$$

where we used the expressions (4.8) and (4.9) for B_1 and B_2, respectively, and the definition of $\pi_{\mathcal{N}}$. At this point, it is worth noting the following fact: since Λ_i (with $i = 1, 2$) is an element of $\mathrm{Im}\,\pi_{\mathcal{N}}$ and $\pi_{\mathcal{N}}$ is idempotent, it is always possible to write

$$\Lambda_i = \mathcal{P}_{N_\Lambda} \Lambda_i. \tag{A18}$$

By applying this observation, together with the properties in (A2), the r-component of the expression above takes the form

$$\begin{aligned}\mathrm{Leib}_{\mathcal{N}}(\Lambda_1,\Lambda_2)_r &= -\mathcal{P}_{N_{A_r}\backslash\{-1\}} \partial_r \mathcal{P}_{N_\Lambda}[\Lambda_1,\Lambda_2] + 2\mathcal{P}_{N_{A_r}}[\mathcal{P}_{N_{A_r}\backslash\{-1\}} \partial_r \Lambda_{[1},\Lambda_{2]}]\\ &= -2\mathcal{P}_{N_{A_r}\backslash\{-1\} \cap \mathcal{T}_{-1} N_\Lambda}[\mathcal{P}_{\mathcal{T}_{-1} N_\Lambda \backslash\{-1\}} \partial_r \Lambda_{[1},\Lambda_{2]}] + 2\mathcal{P}_{N_{A_r}}[\mathcal{P}_{N_{A_r}\backslash\{-1\}} \partial_r \Lambda_{[1},\Lambda_{2]}].\end{aligned} \tag{A19}$$

The projector $\mathcal{P}_{N_{A_r}}$ in the last line can be written as $\mathcal{P}_{N_{A_r}\backslash\{-1\}} + \delta_{A_r,-1}\mathcal{P}_{-1}$, where

$$\delta_{A_r,-1} := \begin{cases} 1 & \text{if } N_{A_r} \ni -1 \\ 0 & \text{otherwise} \end{cases} \tag{A20}$$

Moreover, Eq. (A8a) (which holds by hypothesis) implies $N_{A_r}\backslash\{-1\} \cap \mathcal{T}_{-1} N_\Lambda = N_{A_r}\backslash\{-1\}$. Thus the above expression becomes

$$\begin{aligned}\mathrm{Leib}_{\mathcal{N}}(\Lambda_1,\Lambda_2)_r &= -2\mathcal{P}_{N_{A_r}\backslash\{-1\}}[(\mathcal{P}_{\mathcal{T}_{-1} N_\Lambda \backslash\{-1\}} - \mathcal{P}_{N_{A_r}\backslash\{-1\}}) \partial_r \Lambda_{[1},\Lambda_{2]}] + 2\delta_{A_r,-1}\mathcal{P}_{-1}[\mathcal{P}_{N_{A_r}\backslash\{-1\}} \partial_r \Lambda_{[1},\Lambda_{2]}]\\ &= -2\mathcal{P}_{N_{A_r}\backslash\{-1\}}[\mathcal{P}_{\mathcal{T}_{-1} N_\Lambda \backslash (N_{A_r} \cup \{-1\})} \partial_r \Lambda_{[1},\mathcal{P}_{N_\Lambda}\Lambda_{2]}] + 2\delta_{A_r,-1}\mathcal{P}_{-1}[\mathcal{P}_{N_{A_r}\backslash\{-1\}} \partial_r \Lambda_{[1},\mathcal{P}_{N_\Lambda}\Lambda_{2]}],\end{aligned} \tag{A21}$$

where in the second equality we used (A2b). The last line vanishes for all Λ_1, Λ_2. We prove this in two steps. First, observe that the hypotheses of this lemma imply that

$$N_{A_r} \not\ni 0. \tag{A22}$$

To show why this is true, we argue by contradiction. Suppose $N_{A_r} \ni 0$, then from (A8a) we conclude that $1 \in N_\Lambda$. However, this contradicts (5.24a), since all the sets we are considering here must not contain positive integers. Thus, $N_{A_r} \not\ni 0$. The second step is to note that $\max N_\Lambda \leq 0$ and $\max N_{A_r} < 0$, thanks to (5.24a) and (A22), hence[36]

$$\max(N_{A_r}\backslash\{-1\} + N_\Lambda) < -1 \Rightarrow -1 \notin N_{A_r}\backslash\{-1\} + N_\Lambda, \tag{A23}$$

which implies that the second term in the rhs of (A21) is zero by virtue of (A2g). Employing this property once more, we observe that the other term vanishes for all Λ_1, Λ_2 iff Eq. (A14a) is satisfied. Now, consider the other components of (A17), namely $\mu = \alpha$. By following the same line of reasoning as above, we find[37]

[36]Here we use the fact that for all sets $U, V \subseteq \mathbb{Z}$, the maximum of the sum of the two sets is the sum of their maxima, i.e. $\max(U + V) = \max U + \max V$.

[37]Use properties (A2a), (A2b), (A2e), (A2g), (A8b) and Eq. (A18).

$$\begin{aligned}\mathrm{Leib}_{\mathcal{N}}(\Lambda_1,\Lambda_2)_\alpha &= \mathcal{P}_{N_{A_\alpha}}(-\partial_\alpha\mathcal{P}_{N_\Lambda}[\Lambda_1,\Lambda_2]+2[\mathcal{P}_{N_{A_\alpha}}\partial_\alpha\Lambda_{[1},\Lambda_{2]}])\\ &= -2\mathcal{P}_{N_{A_\alpha}\cap N_\Lambda}[\mathcal{P}_{N_\Lambda}\partial_\alpha\Lambda_{[1},\Lambda_{2]}]+2\mathcal{P}_{N_{A_\alpha}}[\mathcal{P}_{N_{A_\alpha}}\partial_\alpha\Lambda_{[1},\Lambda_{2]}]\\ &= -2\mathcal{P}_{N_{A_\alpha}}[\mathcal{P}_{N_\Lambda\backslash N_{A_\alpha}}\partial_\alpha\Lambda_{[1},\mathcal{P}_{N_\Lambda}\Lambda_{2]}],\end{aligned} \tag{A24}$$

that is zero for all $\Lambda_1,\Lambda_2\in\mathrm{Im}\,\pi_{\mathcal{N}}\cap X_{-1}$ iff N_Λ and N_{A_α} satisfy Eq. (A14b).

We now turn to the case where we apply the Leibniz rule to a gauge parameter $\Lambda\in\mathrm{Im}\,\pi_{\mathcal{N}}\cap X_{-1}$ and a gauge field $A_\mu\in\mathrm{Im}\,\pi_{\mathcal{N}}\cap X_0$:

$$\mathrm{Leib}_{\mathcal{N}}(\Lambda,A)_{\mu\nu}=\pi_{\mathcal{N}}B_1\pi_{\mathcal{N}}B_2(\Lambda,A)_{\mu\nu}+\pi_{\mathcal{N}}B_2(\pi_{\mathcal{N}}B_1(\Lambda),A)_{\mu\nu}-\pi_{\mathcal{N}}B_2(\Lambda,\pi_{\mathcal{N}}B_1(A))_{\mu\nu}. \tag{A25}$$

The line of reasoning is the same as before, although the expressions are more involved. Let us analyze the three independent components of (A25) one by one, starting from the ru-component

$$\begin{aligned}\mathrm{Leib}_{\mathcal{N}}(\Lambda,A)_{ru} &= 2\mathcal{P}_{N_{E_{ru}}}(\partial_{[r}\pi_{\mathcal{N}}B_2(\Lambda,A)_{u]}+r^{-2}\partial_{[z}\pi_{\mathcal{N}}B_2(\Lambda,A)_{\bar z]}+[\pi_{\mathcal{N}}B_1(\Lambda)_{[r},A_{u]}]\\ &\quad+r^{-2}[\pi_{\mathcal{N}}B_1(\Lambda)_{[z},A_{\bar z]}]+\frac{1}{2}[\Lambda,\pi_{\mathcal{N}}B_1(A)_{ru}])\\ &= 2\mathcal{P}_{N_{E_{ru}}}(-\partial_{[r}\pi_{\mathcal{N}}([\Lambda,A])_{u]}-r^{-2}\partial_{[z}\pi_{\mathcal{N}}([\Lambda,A])_{\bar z]}+[\pi_{\mathcal{N}}(\partial\Lambda)_{[r},A_{u]}]\\ &\quad+r^{-2}[\pi_{\mathcal{N}}(\partial\Lambda)_{[z},A_{\bar z]}]+[\Lambda,\mathcal{P}_{N_{E_{ru}}}(\partial_{[r}A_{u]}+r^{-2}\partial_{[z}A_{\bar z]})]).\end{aligned} \tag{A26}$$

Let $T_{1,ru}$ be the sum of the terms above that contain A_r and A_u. At this point, we apply the same trick as before: we take advantage of the identities $\Lambda=\mathcal{P}_{N_\Lambda}\Lambda$ and $A_\mu=\mathcal{P}_{N_{A_\mu}}A_\mu$, together with the equations in (A2), to find

$$\begin{aligned}T_{1,ru} &\coloneqq 2\mathcal{P}_{N_{E_{ru}}}(-\partial_{[r}\pi_N([\Lambda,A])_{u]}+[\pi_{\mathcal{N}}(\partial\Lambda)_{[r},A_{u]}]+[\Lambda,\mathcal{P}_{N_{E_{ru}}}\partial_{[r}A_{u]}])\\ &= \mathcal{P}_{N_{E_{ru}}}\{\mathcal{P}_{N_{A_r}}([\mathcal{P}_{N_\Lambda}\partial_u\Lambda,A_r]+[\Lambda,\mathcal{P}_{N_{A_r}}\partial_uA_r])\\ &\quad-\mathcal{P}_{\mathcal{T}_{-1}N_{A_u}\backslash\{-1\}}([\mathcal{P}_{\mathcal{T}_{-1}N_\Lambda\backslash\{-1\}}\partial_r\Lambda,A_u]+[\Lambda,\mathcal{P}_{\mathcal{T}_{-1}N_{A_u}\backslash\{-1\}}\partial_rA_u])\\ &\quad+[\mathcal{P}_{N(A_r)\backslash\{-1\}}\partial_r\Lambda,A_u]-[\mathcal{P}_{N_{A_u}}\partial_u\Lambda,A_r]+2[\Lambda,\mathcal{P}_{N_{E_{ru}}}\partial_{[r}A_{u]}]\}\\ &= \mathcal{P}_{N_{E_{ru}}}([\mathcal{P}_{N_\Lambda}\partial_u\Lambda,A_r]+[\Lambda,\mathcal{P}_{N_{A_r}}\partial_uA_r]-[\mathcal{P}_{\mathcal{T}_{-1}N_\Lambda\backslash\{-1\}}\partial_r\Lambda,A_u]\\ &\quad-[\Lambda,\mathcal{P}_{\mathcal{T}_{-1}N_{A_u}\backslash\{-1\}}\partial_rA_u]+[\mathcal{P}_{N(A_r)\backslash\{-1\}}\partial_r\Lambda,A_u]-[\mathcal{P}_{N_{A_u}}\partial_u\Lambda,A_r]\\ &\quad+2[\Lambda,\mathcal{P}_{N_{E_{ru}}}\partial_{[r}A_{u]}]),\end{aligned} \tag{A27}$$

where in the last equality we make use of (A8a)–(A8c) to simplify the projectors.[38] Employing property (A2b), we obtain

$$\begin{aligned}T_{1,ru} &= \mathcal{P}_{N_{E_{ru}}}([\mathcal{P}_{N_\Lambda\backslash N_{A_u}}\partial_u\Lambda,A_r]-[\mathcal{P}_{\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r}\cup\{-1\})}\partial_r\Lambda,A_u]\\ &\quad+[\Lambda,\mathcal{P}_{N_{A_r}\backslash N_{E_{ru}}}\partial_uA_r]-[\Lambda,\mathcal{P}_{\mathcal{T}_{-1}N_{A_u}\backslash(N_{E_{ru}}\cup\{-1\})}\partial_rA_u]).\end{aligned} \tag{A28}$$

Let us focus on the first term on the rhs. We want to analyze it using the properties of the projector $\mathcal{P}$. To this end, define the set

$$S\coloneqq N_{E_{ru}}\cap(N_\Lambda\backslash N_{A_u}+N_{A_r})\subseteq\mathbb{Z}^{\leq 0}. \tag{A29}$$

By virtue of the second line in Eq. (A2g), we know that the first term in the rhs of (A28) vanishes for all Λ,A_μ when the set S above is empty. To determine when $S=\varnothing$, consider the following key observations.

(i) From Eqs. (A8c) and (A14b) we deduce that

$$N_{E_{ru}}\subseteq\mathcal{T}_{-1}N_{A_u}\subseteq\mathcal{T}_{-1}(N_\Lambda\backslash N_{A_u}+N_\Lambda)^{\complement}. \tag{A30}$$

[38] In particular, Eq. (A8c) implies that $N_{E_{ru}}\subseteq\mathcal{T}_{-1}N_{A_u}\backslash\{-1\}$, since from (5.24a) together with (5.24c) it follows that $N_{E_{ru}}\ni-1$.

(ii) Equation (A8a) implies

$$N_{A_r} = \begin{cases} N_{A_r}\backslash\{-1\} \cup \{-1\} \subseteq \mathcal{T}_{-1}N_\Lambda \cup \{-1\} & \text{if } N_{A_r} \ni -1 \\ N_{A_r}\backslash\{-1\} \subseteq \mathcal{T}_{-1}N_\Lambda & \text{if } N_{A_r} \not\ni -1. \end{cases} \tag{A31}$$

(iii) Let $U, V \subseteq \mathbb{Z}$ and $s \in \mathbb{Z}$. It is always true that

$$\mathcal{T}_s(U+V) = \mathcal{T}_s U + V = U + \mathcal{T}_s V. \tag{A32}$$

We emphasize that this identity means that the addition of sets of integers defined in (A1) and the usual notion of union of sets behave differently with the translation (5.25).[39] Moreover, the identity element for the addition of sets is the singleton containing the identity element of the addition in $\mathbb{Z}$, that is to say $U + \{0\} = \{0\} + U = U$ for all U. Together, these two facts suggest a useful relation

$$\begin{aligned} U + (\mathcal{T}_{-1}V \cup \{-1\}) &= \mathcal{T}_{-1}(U + (V \cup \{0\})) \\ &= \mathcal{T}_{-1}[(U+V) \cup U]. \end{aligned} \tag{A33}$$

We combine Eq. (A29) with (A30)–(A33) and we distinguish two different cases:

(1) if $N_{A_r} \ni -1$, then

$$\begin{aligned} S &\subseteq \mathcal{T}_{-1}(N_\Lambda\backslash N_{A_u} + N_\Lambda)^{\complement} \\ &\quad \cap \mathcal{T}_{-1}[(N_\Lambda\backslash N_{A_u} + N_\Lambda) \cup N_\Lambda\backslash N_{A_u}] \\ &= \mathcal{T}_{-1}\{[(N_\Lambda\backslash N_{A_u} + N_\Lambda)^{\complement} \cap (N_\Lambda\backslash N_{A_u} + N_\Lambda)] \\ &\quad \cup [(N_\Lambda\backslash N_{A_u} + N_\Lambda)^{\complement} \cap N_\Lambda\backslash N_{A_u}]\} = \varnothing \Leftrightarrow N_\Lambda \ni 0, \end{aligned} \tag{A34}$$

where we used the fact that for $U, V \subseteq \mathbb{Z}^{\leq 0}$ we have

$$(U+V)^{\complement} \cap U = \varnothing \Leftrightarrow U \subseteq U + V \Leftrightarrow V \ni 0. \tag{A35}$$

(2) If $N_{A_r} \not\ni -1$, then

$$S \subseteq \mathcal{T}_{-1}(N_\Lambda\backslash N_{A_u} + N_\Lambda)^{\complement} \cap \mathcal{T}_{-1}(N_\Lambda\backslash N_{A_u} + N_\Lambda) = \varnothing. \tag{A36}$$

Hence, we find that the first term in (A28) vanishes $\forall\ \Lambda, A_\mu$ if we impose condition (A15a), which comes from the first case above. The second term in (A28) is zero as well, as one can see by employing a similar argument, which results in the following chain of inclusions:

$$\begin{aligned} & N_{E_{ru}} \cap [\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r} \cup \{-1\}) + N_{A_u}] \\ \subseteq\ & N_{A_r}\backslash\{-1\} \cap [\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r} \cup \{-1\}) + N_\Lambda] \\ \subseteq\ & [\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r} \cup \{-1\}) + N_\Lambda]^{\complement} \\ & \cap [\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r} \cup \{-1\}) + N_\Lambda] = \varnothing. \end{aligned} \tag{A37}$$

Conversely, the two terms in the second line of (A28) do not vanish for arbitrary Λ, A_μ unless

$$\begin{cases} N_{E_{ru}} \cap (N_{A_r}\backslash N_{E_{ru}} + N_\Lambda) = \varnothing \\ N_{E_{ru}} \cap [\mathcal{T}_{-1}N_{A_u}\backslash(N_{E_{ru}} \cup \{-1\}) + N_\Lambda] = \varnothing. \end{cases} \tag{A38}$$

At this point, we focus on the remaining terms in (A26), denoting their sum as $T_{2,ru}$. After some algebra [completely analogous to what we do above for $T_{1,ru}$, with the only difference that now there is an additional coefficient r^{-2} that needs to be treated carefully, via the identity (A2c)], we rearrange the terms in $T_{2,ru}$ as

$$\begin{aligned} T_{2,ru} &:= 2\mathcal{P}_{N_{E_{ru}}}(-r^{-2}\partial_{[z}\pi_{\mathcal{N}}([\Lambda, A])_{\bar{z}]} + r^{-2}[\pi_{\mathcal{N}}(\partial\Lambda)_{[z}, A_{\bar{z}]}] \\ &\quad + [\Lambda, \mathcal{P}_{N_{E_{ru}}}(r^{-2}\partial_{[z}A_{\bar{z}]})]) \\ &= -r^{-2}\mathcal{P}_{\mathcal{T}_2 N_{E_{ru}}}([\mathcal{P}_{N_\Lambda\backslash N_{A_z}}\partial_z\Lambda, A_{\bar{z}}] \\ &\quad + [\Lambda, \mathcal{P}_{N_{A_{\bar{z}}}\backslash\mathcal{T}_2 N_{E_{ru}}}\partial_z A_{\bar{z}}]) - (z \leftrightarrow \bar{z}). \end{aligned} \tag{A39}$$

One can prove that the first term is zero by means of the following sequence of inclusions

$$\begin{aligned} \mathcal{T}_2 N_{E_{ru}} \cap (N_\Lambda\backslash N_{A_z} + N_{A_{\bar{z}}}) &\subseteq N_{A_z} \cap (N_\Lambda\backslash N_{A_z} + N_\Lambda) \\ &\subseteq (N_\Lambda\backslash N_{A_z} + N_\Lambda)^{\complement} \cap (N_\Lambda\backslash N_{A_z} + N_\Lambda) = \varnothing, \end{aligned} \tag{A40}$$

whereas the second term vanishes iff

$$\mathcal{T}_2 N_{E_{ru}} \cap (N_{A_{\bar{z}}}\backslash\mathcal{T}_2 N_{E_{ru}} + N_\Lambda) = \varnothing. \tag{A41}$$

Analogous results hold for the last two terms in $T_{2,ru}$ (the ones with $z \leftrightarrow \bar{z}$). By putting (A38) and (A41) together, and using (A32), one obtains equation (A14c).

The proof of the last two equations of this lemma is conceptually analogous to this last one. The $r\bar{z}$-component of the Leibniz rule (A25) is

[39] Recall that $\mathcal{T}_s(U \cup V) = \mathcal{T}_s U \cup \mathcal{T}_s V$.

$$\begin{aligned}\mathrm{Leib}_{\mathcal{N}}(\Lambda,A)_{r\bar{z}} = 4\mathcal{P}_{N_{E_{r\bar{z}}}}(&-\partial_{[r}\pi_{\mathcal{N}}([\Lambda,A])_{\bar{z}]}\\ &+[\pi_{\mathcal{N}}(\partial\Lambda)_{[r},A_{\bar{z}}]]+[\Lambda,\mathcal{P}_{N_{E_{r\bar{z}}}}\partial_{[r}A_{\bar{z}}]]).\end{aligned} \tag{A42}$$

This expression can be expanded using (5.21), obtaining a sum of many terms. We find it useful to group them as follows:

$$\begin{aligned}T_{1,r\bar{z}} \coloneqq\ & 2\mathcal{P}_{N_{E_{r\bar{z}}}\backslash\{-1\}}([\mathcal{P}_{N_\Lambda\backslash N_{A_{\bar{z}}}}\partial_{\bar{z}}\Lambda,A_r]\\ &-[\mathcal{P}_{\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r}\cup\{-1\})}\partial_r\Lambda,A_{\bar{z}}])\\ &+2\delta_{E_{r\bar{z}},-1}\mathcal{P}_{-1}([\Lambda,\mathcal{P}_{N_{A_r}\backslash N_{E_{r\bar{z}}}}\partial_{\bar{z}}A_r]\\ &+[\mathcal{P}_{N_{A_r}\backslash\{-1\}}\partial_r\Lambda,A_{\bar{z}}]\\ &+[\Lambda,\mathcal{P}_{N_{E_{r\bar{z}}}\backslash\{-1\}}\partial_rA_{\bar{z}}])\end{aligned} \tag{A43a}$$

$$\begin{aligned}T_{2,r\bar{z}} \coloneqq\ & 2\mathcal{P}_{N_{E_{r\bar{z}}}\backslash\{-1\}}([\Lambda,\mathcal{P}_{N_{A_r}\backslash N_{E_{r\bar{z}}}}\partial_{\bar{z}}A_r]\\ &-[\Lambda,\mathcal{P}_{\mathcal{T}_{-1}N_{A_{\bar{z}}}\backslash(N_{E_{r\bar{z}}}\cup\{-1\})}\partial_rA_{\bar{z}}])\\ &+2\delta_{E_{r\bar{z}},-1}\mathcal{P}_{-1}[\mathcal{P}_{N_\Lambda\backslash N_{A_{\bar{z}}}}\partial_{\bar{z}}\Lambda,A_r],\end{aligned} \tag{A43b}$$

where $\delta_{E_{r\bar{z}},-1} \coloneqq 1$ if $N_{E_{r\bar{z}}} \ni -1$ and it vanishes otherwise. Once again, we use a sequence of inclusions and Eq. (A2g) to prove that $T_{1,r\bar{z}} = 0$ for all Λ, A_μ. Let us start by focusing on the first term of (A43a). We distinguish two cases:

(1) if $N_{A_r} \ni -1$, then[40]

$$\begin{aligned}&N_{E_{r\bar{z}}}\backslash\{-1\}\cap(N_\Lambda\backslash N_{A_{\bar{z}}}+N_{A_r})\\ &\quad\subseteq\mathcal{T}_{-1}N_{A_{\bar{z}}}\cap[N_\Lambda\backslash N_{A_{\bar{z}}}+(\mathcal{T}_{-1}N_\Lambda\cup\{-1\})]\\ &=\mathcal{T}_{-1}N_{A_{\bar{z}}}\cap\mathcal{T}_{-1}[(N_\Lambda\backslash N_{A_{\bar{z}}}+N_\Lambda)\cup N_\Lambda\backslash N_{A_{\bar{z}}}]\\ &\quad\subseteq\mathcal{T}_{-1}\{(N_\Lambda\backslash N_{A_{\bar{z}}}+N_\Lambda)^{\complement}\\ &\quad\cap[(N_\Lambda\backslash N_{A_{\bar{z}}}+N_\Lambda)\cup N_\Lambda\backslash N_{A_{\bar{z}}}]\}=\varnothing.\end{aligned} \tag{A44}$$

The last line above is equal to the empty set since we require condition (A15a) to hold [the steps to prove this closely mirror those in (A34)].

(2) If $N_{A_r} \not\ni -1$, then

$$\begin{aligned}&N_{E_{r\bar{z}}}\backslash\{-1\}\cap(N_\Lambda\backslash N_{A_{\bar{z}}}+N_{A_r})\\ &\quad\subseteq\mathcal{T}_{-1}(N_\Lambda\backslash N_{A_{\bar{z}}}+N_\Lambda)^{\complement}\\ &\quad\cap\mathcal{T}_{-1}(N_\Lambda\backslash N_{A_{\bar{z}}}+N_\Lambda)=\varnothing.\end{aligned} \tag{A45}$$

In both cases, the first term of $T_{1,r\bar{z}}$ vanishes. The second term in (A43a) is zero as well, thanks to the following chain of inclusions:

$$\begin{aligned}&N_{E_{r\bar{z}}}\backslash\{-1\}\cap[\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r}\cup\{-1\})+N_{A_{\bar{z}}}]\\ &\quad\subseteq N_{A_r}\backslash\{-1\}\cap[\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r}\cup\{-1\})+N_\Lambda]\\ &\quad\subseteq[\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r}\cup\{-1\})+N_\Lambda]^{\complement}\\ &\quad\cap[\mathcal{T}_{-1}N_\Lambda\backslash(N_{A_r}\cup\{-1\})+N_\Lambda]=\varnothing.\end{aligned} \tag{A46}$$

We observe that the other terms in $T_{1,r\bar{z}}$ vanish when $N_{E_{r\bar{z}}} \not\ni -1$, because of the overall factor $\delta_{E_{r\bar{z}},-1}$. Hence, suppose $N_{E_{r\bar{z}}} \ni -1$. Then, from Eqs. (A8d) and (A15a) we deduce that $N_{A_r} \ni -1$ and $N_\Lambda \ni 0$. Moreover, $N_{E_{r\bar{z}}}$ cannot contain zero.[41] Thus,

$$\max N_\Lambda = 0, \qquad \max N_{A_r} = \max N_{E_{r\bar{z}}} = -1, \tag{A47}$$

which implies that

$$\begin{aligned}&\max(N_{A_r}\backslash N_{E_{r\bar{z}}}+N_\Lambda)<-1\Rightarrow-1\notin N_{A_r}\backslash N_{E_{r\bar{z}}}+N_\Lambda\\ &\max(N_{A_r}\backslash\{-1\}+N_{A_{\bar{z}}})<-1\Rightarrow-1\notin N_{A_r}\backslash\{-1\}+N_{A_{\bar{z}}}\\ &\max(N_{E_{r\bar{z}}}\backslash\{-1\}+N_\Lambda)<-1\Rightarrow-1\notin N_{E_{r\bar{z}}}\backslash\{-1\}+N_\Lambda.\end{aligned} \tag{A48}$$

This means that each term in the second and third line of (A43a) vanish.[42] Therefore, we conclude that $T_{1,r\bar{z}} = 0$ for all Λ, A_μ, and it does not provide any additional conditions on the power selection sets. Conversely, $T_{2,r\bar{z}} = 0$ for all Λ, A_μ iff

$$\begin{cases}N_{E_{r\bar{z}}}\backslash\{-1\}\cap(N_{A_r}\backslash N_{E_{r\bar{z}}}+N_\Lambda)=\varnothing\\ N_{E_{r\bar{z}}}\backslash\{-1\}\cap[\mathcal{T}_{-1}N_{A_{\bar{z}}}\backslash(N_{E_{r\bar{z}}}\cup\{-1\})+N_\Lambda]=\varnothing\\ \max(N_\Lambda\backslash N_{A_{\bar{z}}}+N_{A_r})<-1.\end{cases} \tag{A49}$$

The first two conditions are equivalent to (A14d). Given (A47), we deduce that the third equation in the system above is true only when $N_{A_{\bar{z}}} \ni 0$, so that the maximum of the set $N_\Lambda\backslash N_{A_{\bar{z}}}$ is strictly less than zero. In other words, the third equation in (A49) corresponds to condition (A15b). This concludes the study of the $r\bar{z}$-component of the Leibniz rule.

Finally, the uz-component of (A25) is

$$\begin{aligned}\mathrm{Leib}_{\mathcal{N}}(\Lambda,A)_{uz} &= 4\mathcal{P}_{N_{E_{uz}}}(-\partial_{[u}\pi_{\mathcal{N}}([\Lambda,A])_{z]}\\ &\quad+[\pi_{\mathcal{N}}(\partial\Lambda)_{[u},A_{z]}]+[\Lambda,\mathcal{P}_{N_{E_{uz}}}\partial_{[u}A_{z]}])\\ &=-2\mathcal{P}_{N_{E_{uz}}}([\mathcal{P}_{N_\Lambda\backslash N_{A_u}}\partial_u\Lambda,A_z]\\ &\quad+[\Lambda,\mathcal{P}_{N_{A_z}\backslash N_{E_{uz}}}\partial_uA_z])-(u\leftrightarrow z).\end{aligned} \tag{A50}$$

[40] Use Eqs. (A8e), (A8a), (A33), (A14b), (A35) and (A15a), in this order.

[41] $N_{E_{r\bar{z}}}$ is contained in N_{A_r} by hypothesis, see (A8d), so (A22) implies that $N_{E_{r\bar{z}}} \not\ni 0$.

[42] By virtue of (A2g).

By applying one last time the procedure described multiple times in this proof, one can show that the first term above is always zero thanks to the inclusion

$$N_{E_{uz}} \cap (N_\Lambda \backslash N_{A_u} + N_{A_z}) \subseteq N_{A_u} \cap (N_\Lambda \backslash N_{A_u} + N_\Lambda) = \varnothing, \tag{A51}$$

while the necessary and sufficient condition for the second term in (A50) to vanish is $N_{E_{uz}} \subseteq (N_{A_u} \backslash N_{E_{uz}} + N_\Lambda)^{\complement}$. Similar considerations hold for the other two terms in (A50), with $u \leftrightarrow z$. Thus, with the hypotheses of this lemma, $\mathrm{Leib}_{\mathcal{N}}(\Lambda, A)_{uz}$ vanishes for all Λ, A_μ iff the power selection sets satisfy equation (A14e). ∎

c. Physically relevant solutions

Let us highlight that the two lemmas above hold for arbitrary sets $N_I \in \mathcal{N}$, with $I \in \{\Lambda, A_\mu, E_{\mu\nu}\}$, since in none of the steps of their proofs did we rely on the explicit form of these sets. In particular, N_I may contain gaps, or they may only consist of negative numbers far from zero. In such cases, however, the power selection sets describe a physically noninteresting situation, since the presence of gaps in the powers of the $1/r$ expansion and/or the absence of the leading powers (the ones closer to zero) mean that the hierarchy of subleading terms is not respected. To eliminate these scenarios and extract the physical solution from the master system, we shall introduce two additional (and quite natural) assumptions on the power selection sets:

(i) N_I are discrete intervals (i.e. each set contains consecutive integers without gaps). For convenience, we introduce here the following notation: for any two integers $a \leq b$, we define the discrete interval

$$[\![a, b]\!] := [a, b] \cap \mathbb{Z} = \{a, \ldots, b\}, \tag{A52}$$

where $[a, b] \subseteq \mathbb{R}$ is an interval in the usual sense. With this notation, the assumption we are making can be expressed as follows: the power selection sets in $\mathcal{N}$ are of the form $N_I = [\![a_I, b_I]\!]$ for some integers $a_I \leq b_I$ and for all I.

(ii) We assume that each power selection set N_I contains the maximum integer allowed, where by "allowed" we mean compatible with (5.24) and with the constraints given by lemmas A.2 and A.3. A more formal way to state this assumption is the following: for each I, define

$$\mathfrak{N}_I^{\mathrm{ms}} := \{N_I \subseteq \mathbb{Z} \text{ satisfying (5.24), (A.8), (A.14) and (A.15)}\}. \tag{A53}$$

In other words, $\mathfrak{N}_I^{\mathrm{ms}}$ is the space of solutions of the first three equations of the master system for the power selection set N_I. Then, we shall restrict to the subspace

$$\tilde{\mathfrak{N}}_I^{\mathrm{ms}} := \{\tilde{N}_I \in \mathfrak{N}_I^{\mathrm{ms}} | \max \tilde{N}_I = \max_{N_I \in \mathfrak{N}_I} (\max N_I)\}. \tag{A54}$$

To sum up, building on the two assumptions above, we state that the physically relevant solutions of the first three equations of the master system are given by projectors $\pi_{\mathcal{N}}$ with power selection sets $N_I \in \mathcal{N}$ of the form

$$N_I = [\![a_I, b_I]\!] \in \tilde{\mathfrak{N}}_I^{\mathrm{ms}}, \tag{A55}$$

for some nonpositive integers $a_I \leq b_I$ and for all $I \in \{\Lambda, A_\mu, E_{\mu\nu}\}$. This means that each power selection set in the collection $\mathcal{N}$ is a discrete interval whose maximum is as close to zero as possible, consistent with (5.24) and the constraints given by lemmas A.2 and A.3.

d. Master system for $\mathcal{N}$

One last condition in the master system needs to be examined: the Jacobiator of the two-bracket, Eq. (5.14d). It turns out that a morphism $\pi_{\mathcal{N}}$ with power selection sets of the form (A55)[43] solves the Jacobiator equation for Π automatically, without the need to impose further constraints on $\mathcal{N}$. This can be proven by following a procedure that is fully analogous to the proofs of lemmas A.2 and A.3, although considerably longer and more intricate. A different strategy to show this result is to check that the projections $\pi_{\mathcal{N}_k}$ in (5.26) (which, as we will see shortly, are the solutions of the first three equations of the master system) solve the Jacobiator, too. One can do that explicitly for arbitrary values of $k \in \mathbb{Z}^{\leq 0}$ by employing the *Mathematica* package developed by one of the authors, to be presented in [116].

To conclude, in this subsection we proved that the most general solution for the first three equations of the master system (5.14) is a projection of the form $\Pi = \pi_{\mathcal{N}}$ with the sets in the collection $\mathcal{N}$ satisfying all the relations contained in (5.24), (A8), (A14) and (A15). For brevity, we call this set of 16 equations the *master system for* $\mathcal{N}$. We reproduce it here for convenience

[43]We emphasize that (A55) implies, by definition, that N_I satisfies (5.24), (A8), (A14) and (A15) $\forall\, I$.

$$
\begin{aligned}
&N_I \subseteq \mathbb{Z}^{\leq 0}\ \forall\, I \in \{\Lambda, A_\mu, E_{\mu\nu}\} && \text{(5.24a)}\\
&N_{E_{\mu\nu}} = N_{E_{\nu\mu}} && \text{(5.24b)}\\
&N_{E_{z\bar{z}}} = \mathcal{T}_2 N_{E_{ru}} && \text{(5.24c)}\\
&N_{A_\mu} \subseteq \mathcal{T}_{-\delta_{r\mu}} N_\Lambda && \text{(A:8a, A:8b, A:15a)}\\
&N_{E_{ru}} \subseteq N_{A_r} \cap \mathcal{T}_{-1} N_{A_u} \cap \mathcal{T}_{-2}(N_{A_z} \cap N_{A_{\bar{z}}}) && \text{(A.8c)}\\
&N_{E_{r\bar{z}}} \subseteq N_{A_r} \cap \mathcal{T}_{-1} N_{A_{\bar{z}}} && \text{(A:8d, A:8e, A:15b)}\\
&N_{E_{uz}} \subseteq N_{A_u} \cap N_{A_z} && \text{(A.8f)}\\
&N_{A_r} \backslash \{-1\} \cap [\mathcal{T}_{-1} N_\Lambda \backslash (N_{A_r} \cup \{-1\}) + N_\Lambda] = \varnothing && \text{(A.14a)}\\
&N_{A_\alpha} \cap [N_\Lambda \backslash N_{A_\alpha} + N_\Lambda] = \varnothing && \text{(A.14b)}\\
&N_{E_{ru}} \cap [(N_{A_r} \cup \mathcal{T}_{-1}(N_{A_u} \backslash \{0\}) \cup \mathcal{T}_{-2}(N_{A_z} \cup N_{A_{\bar{z}}})) \backslash N_{E_{ru}} + N_\Lambda] = \varnothing && \text{(A.14c)}\\
&N_{E_{r\bar{z}}} \backslash \{-1\} \cap [(N_{A_r} \cup \mathcal{T}_{-1}(N_{A_{\bar{z}}} \backslash \{0\})) \backslash N_{E_{r\bar{z}}} + N_\Lambda] = \varnothing && \text{(A.14d)}\\
&N_{E_{uz}} \cap [(N_{A_u} \cup N_{A_z}) \backslash N_{E_{uz}} + N_\Lambda] = \varnothing. && \text{(A.14e)}
\end{aligned}
\tag{A56}
$$

Moreover, we showed how to impose additional natural conditions on the power selection sets to ensure that the corresponding solutions of the master system for $\mathcal{N}$ are physically relevant. In the next paragraph, we show how to construct a family $\{\mathcal{N}_k\}_{k\in\mathbb{Z}^{\leq 0}}$ of such solutions.

3. Solutions of the master system for $\mathcal{N}$

In this subsection, thanks to the observations above, we are finally able to provide the proof of the main result of Sec. V, that is to say equations (5.26) and (5.27).

Let us recap the situation up to this point. The master system (5.14) is solved by a projector $\pi_{\mathcal{N}}$ with power selection sets $N_I \in \mathcal{N}$ satisfying the master system for $\mathcal{N}$ above. Moreover, we know that the physically interesting solutions of this system are sets N_I as in Eq. (A55). In particular, N_I are discrete intervals, so in order to determine them it is sufficient to find their extrema. Equation (A53) tells us that the maximum of each interval N_I must be the greatest possible number compatible with the system above.

The following result not only allows us to find $\max N_I$ for every I, but also provides a prescription to construct the solution of the master system for $\mathcal{N}$.

Lemma A.4 (Fall-off). Let $\mathcal{N} = \{N_\Lambda, N_{A_\mu}, N_{E_{\mu\nu}}\}$ be a collection of power selection sets obeying condition (A55). Then

$$
N_{A_r} \cup N_{E_{ru}} \cup N_{E_{r\bar{z}}} \not\ni -1, 0. \tag{A57}
$$

Proof. By hypothesis,[44] the sets $N_I \in \mathcal{N}$ are discrete intervals satisfying the master system for $\mathcal{N}$.[45] In particular, as discussed in the proof of lemma A.3, we know that

$$
\begin{aligned}
&(5.24a), (5.24c) \Rightarrow N_{E_{ru}} \not\ni -1, 0\\
&(5.24a), (A.8a), (A.8d) \Rightarrow N_{A_r} \cup N_{E_{r\bar{z}}} \not\ni 0,
\end{aligned}
\tag{A58}
$$

see, for example, the line of reasoning below Eqs. (A22). Thus, what is left to prove is $N_{A_r} \cup N_{E_{r\bar{z}}} \not\ni -1$. Notice that if N_{A_r} does not contain -1, $N_{E_{r\bar{z}}}$ does not contain it either, by virtue of (A8d).

Demonstrating that $N_{A_r} \not\ni -1$ requires additional steps. We argue by contradiction: suppose N_{A_r} does contain -1. We then show that the following collection of power selection sets

$$
\begin{aligned}
\mathcal{N}^{\text{test}} := \{N_\Lambda, N_{A_\mu}, N_{E_{\mu\nu}} | N_I \text{ are physical solutions of } &(5.24),\\
&(A.8), (A.15)\}
\end{aligned}
\tag{A59}
$$

fails to satisfy the Leibniz rule condition (A14). Consequently, if $N_{A_r} \ni -1$ there is no solution for the master system for $\mathcal{N}$.

The first step of this proof by contradiction is to understand the structure of the sets $N_I \in \mathcal{N}^{\text{test}}$. As above, the term "physical" in the definition of $\mathcal{N}^{\text{test}}$ means that

[44] See Eqs. (A53)–(A55).

[45] Recall that the master system for $\mathcal{N}$ consists of Eqs. (5.24), (A8), (A14), and (A15).

$$N_I = [\![\min N_I, \max N_I]\!] \tag{A60}$$

are discrete intervals with maxima as close to zero as possible. Moreover, by definition, $N_I \in \mathcal{N}^{\text{test}}$ satisfy the no divergences condition (5.24) and the cochain map condition (A8). In particular, they cannot contain positive integers and Eq. (A58) holds. This allows us to determine the maxima of all the power selection sets in $\mathcal{N}^{\text{test}}$

$$\max N_I = \begin{cases} 0 & \text{if } I = \Lambda, A_\alpha, E_{uz} \\ -1 & \text{if } I = A_r, E_{r\bar{z}} \\ -2 & \text{if } I = E_{ru}. \end{cases} \tag{A61}$$

Now, observe that Eqs. (A8c)–(A8f) and (A15b) can be rearranged as follows:

$$\begin{aligned} N_{A_r} &\supseteq N_{E_{ru}} \cup N_{E_{r\bar{z}}} \\ N_{A_u} &\supseteq \mathcal{T}_1 N_{E_{ru}} \cup N_{E_{uz}} \\ N_{A_z} &\supseteq \mathcal{T}_2 N_{E_{ru}} \cup N_{E_{uz}} \\ N_{A_{\bar{z}}} &\supseteq \mathcal{T}_2 N_{E_{ru}} \cup \mathcal{T}_1 N_{E_{r\bar{z}}}, \end{aligned} \tag{A62}$$

while Eqs. (A8a), (A8b) and (A15a) can be rewritten as

$$N_\Lambda \supseteq \mathcal{T}_1 N_{A_r} \cup N_{A_u} \cup N_{A_z} \cup N_{A_{\bar{z}}}. \tag{A63}$$

In the list of inclusions above, the "innermost" sets are the intervals $N_{E_{\mu\nu}}$. This means that these intervals are not constrained by (A8) and (A15), but only by the system (5.24), which fixes their maxima. Therefore, the sets $N_{E_{\mu\nu}} \in \mathcal{N}^{\text{test}}$ are of the form

$$N_{E_{ru}} = [\![m, -2]\!], \quad N_{E_{r\bar{z}}} = [\![n, -1]\!], \quad N_{E_{uz}} = [\![l, 0]\!] \tag{A64}$$

for some integers $m \leq -2$, $n \leq -1$ and $l \leq 0$. This is the starting point to find the other intervals in $\mathcal{N}^{\text{test}}$. Indeed, using (A62) we find

$$\begin{aligned} N_{A_r} &\supseteq [\![\min(m, n), -1]\!] \\ N_{A_u} &\supseteq [\![\min(m+1, l), 0]\!] \\ N_{A_z} &\supseteq [\![\min(m+2, l), 0]\!] \\ N_{A_{\bar{z}}} &\supseteq [\![\min(m+2, n+1), 0]\!]. \end{aligned} \tag{A65}$$

On the one hand, notice that the maxima of the intervals in the rhs of the inclusions above coincide with the values found in (A61). On the other hand, we do not have any further constraints on the minima. For example, the set $N_{A_r} \in \mathcal{N}^{\text{test}}$ is of the form

$$N_{A_r} = [\![\min N_{A_r}, -1]\!], \qquad \min N_{A_r} \leq \min(m, n). \tag{A66}$$

Thus, from (A63) we obtain

$$N_\Lambda \supseteq [\![\min_\mu(\min N_{A_\mu} + \delta_{\mu r}), 0]\!]. \tag{A67}$$

Nonetheless, for simplicity, let us restrict to the case where all the inclusions in (A65) and (A67) reduce to equalities

$$\begin{aligned} N_{A_r} &= [\![\min(m, n), -1]\!] \\ N_{A_u} &= [\![\min(m+1, l), 0]\!] \\ N_{A_z} &= [\![\min(m+2, l), 0]\!] \\ N_{A_{\bar{z}}} &= [\![\min(m+2, n+1), 0]\!] \\ N_\Lambda &= [\![\min(m+1, n+1, l), 0]\!]. \end{aligned} \tag{A68}$$

One can prove that all the steps that follow in this proof remain valid even in the general case. Therefore, the restriction we are assuming here does not affect the generality or validity of our proof.

To sum up, we found that the sets $N_I \in \mathcal{N}^{\text{test}}$ are given by (A64) and (A68). A visual representation of such intervals is provided in Fig. 3. One can verify that they satisfy both equations (A14a) and (A14b). However, crucially, they violate condition (A14c), as we shall prove below. Let $\mathfrak{lhs}$ be the left-hand side of Eq. (A14c). Using the explicit expressions for N_I found above, we obtain

$$\begin{aligned} \mathfrak{lhs} \coloneqq N_{E_{ru}} \cap (J + N_\Lambda) = [\![m, -2]\!] \\ \cap (J + [\![\min(m+1, n+1, l), 0]\!]), \end{aligned} \tag{A69}$$

where we defined

$$\begin{aligned} J &\coloneqq (N_{A_r} \cup \mathcal{T}_{-1}(N_{A_u} \backslash \{0\}) \cup \mathcal{T}_{-2}(N_{A_z} \cup N_{A_{\bar{z}}})) \backslash N_{E_{ru}} \\ &= ([\![\min(m, n), -1]\!] \cup [\![\min(m, l-1), -2]\!] \cup [\![\min(m, l-2), -2]\!] \cup [\![\min(m, n-1), -2]\!]) \backslash [\![m, -2]\!] \\ &= [\![\min(m, n-1, l-2), -1]\!] \backslash [\![m, -2]\!] \\ &= \{-1\} \cup \begin{cases} \varnothing & \text{if } \min(m, n-1, l-2) = m \\ [\![\min(n-1, l-2), m-1]\!] & \text{otherwise.} \end{cases} \end{aligned} \tag{A70}$$

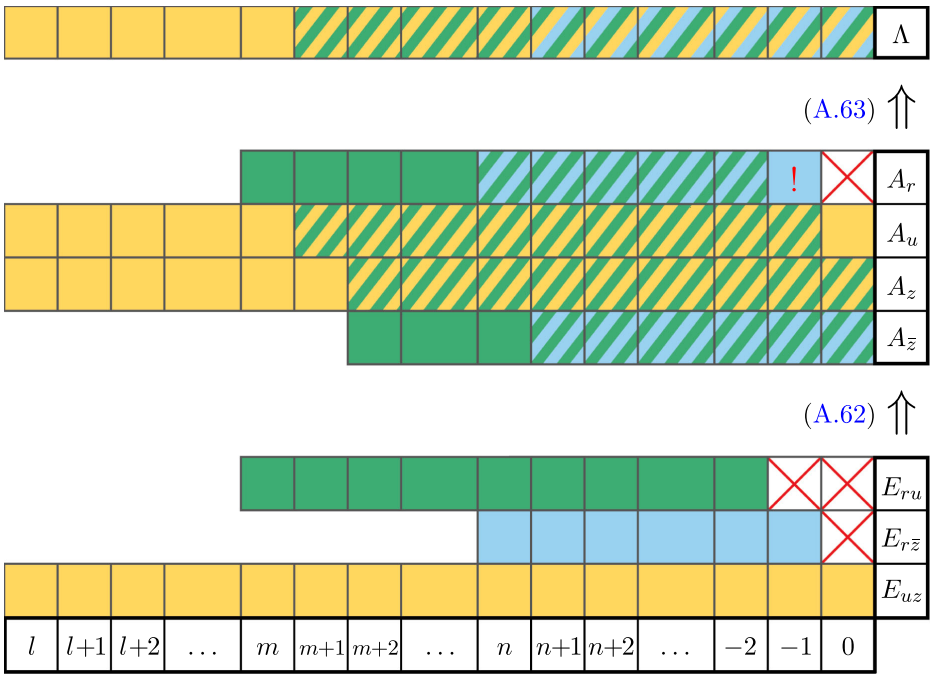


FIG. 3. Visual representation of the discrete intervals $N_I \in \mathcal{N}^{\text{test}}$, see (A64) and (A68). Each square represents an integer. The red crosses indicate the numbers forbidden by (A58). The square with the exclamation mark corresponds to the element $-1 \in N_{A_r}$, that is the hypothesis of the proof by contradiction. The use of colors help illustrate how the intervals N_{A_μ} and N_Λ depend on $N_{E_{\mu\nu}}$ through Eqs. (A62) and (A63), respectively.

Notice that, when $\min(m, n-1, l-2) \neq m$, one can prove that

$$\min(n-1, l-2) + \min(m+1, n+1, l) = 2\ \min(n, l-1). \tag{A71}$$

Using this identity, we rewrite (A69) as

$$\begin{aligned} \text{Ɬɦȿ} &= [\![m, -2]\!] \cap \left([\![\min(m, n, l-1), -1]\!] \cup \begin{Bmatrix} \varnothing \\ [\![2\ \min(n, l-1), m-1]\!] \end{Bmatrix} \right) \\ &= [\![m, -2]\!] \cap [\![\min(m, n, l-1), -1]\!] = [\![m, -2]\!], \end{aligned} \tag{A72}$$

where the two cases in the first line correspond to the same conditions of the cases in (A70). We have established that $\text{Ɬɦȿ} \neq \varnothing$, which implies that Eq. (A14c) is violated. Thus, we have reached the anticipated contradiction. Consequently, we conclude that our initial hypothesis is false: $N_{A_r} \not\ni -1$.

As a consistency check, one can show that the sets in the collection

$$\mathcal{N}^{\text{sol}} := \{N_I \in \mathcal{N}^{\text{test}} | N_{A_r} \not\ni -1\} \tag{A73}$$

do satisfy the Leibniz rule condition (A14), thus providing the most general physically relevant solution of the master system for $\mathcal{N}$. By repeating the construction outlined above, we find that $N_I \in \mathcal{N}^{\text{sol}}$ are of the form

$$\begin{aligned} N_\Lambda &\supseteq [\![\min_\mu(\min N_{A_\mu} + \delta_{\mu r}), 0]\!] \\ N_{A_r} &\supseteq [\![\min(m, n), -2]\!] \\ N_{A_u} &\supseteq [\![\min(m+1, l), 0]\!] \\ N_{A_z} &\supseteq [\![\min(m+2, l), 0]\!] \\ N_{A_{\bar{z}}} &\supseteq [\![\min(m+2, n+1), 0]\!] \\ N_{E_{ru}} &= [\![m, -2]\!] \\ N_{E_{r\bar{z}}} &= [\![n, -2]\!] \\ N_{E_{uz}} &= [\![l, 0]\!] \end{aligned} \tag{A74}$$

for some integers $m, n \leq -2$ and $l \leq 0$. They differ from the sets in $\mathcal{N}^{\text{test}}$ only in that $\max N_{A_r} = \max N_{E_{r\bar{z}}} = -2$ [recall

that the second equality comes from (A8d)]. Using the same reasoning as before, one can verify that the intervals in (A74) solve all the inclusions in (A14). In particular, the left-hand side of (A14c)[46] reduces to

$$\mathfrak{lhs} = [\![m,-2]\!] \cap \begin{cases} \varnothing & \text{if } \min(m,n-1,l-2) = m \\ [\![2\ \min(n,l-1),m-1]\!] & \text{otherwise} \end{cases} = \varnothing. \tag{A75}$$

This completes the proof of (A57).

With lemma A.4, we can finally conclude this Appendix. Indeed, the result above not only provides a proof of the fall-offs given in Eq. (2.12), but also allows us to explicitly construct the power selection sets that solve the master system for $\mathcal{N}$. In fact, starting from (A74) and taking the minimal sets[47] for N_Λ and N_{A_μ} and sets of equal cardinality for each component of the equations of motion (i.e. setting $m = n = l-2$), we obtain result (5.27), which in turn gives the projection (5.26) solving the master system (5.14).

APPENDIX B: OTHER PROOFS

1. $\mathfrak{Kin}^{(k)}$

Remark B.1 (Projected chain complex). Let Π_k with $k \leq 0$ be the projection in (5.26), and let $\mathcal{K}^{(k)}$ be the graded vector space resulting from the color-stripping of $\mathcal{X}^{(k)}$ in (5.28). Moreover, define $b^{(k)} := \Pi_k b$, where b is the differential in (4.13)[48]. Then $(\mathcal{K}^{(k)}, b^{(k)})$ is a chain complex.

Proof. This is equivalent to show that $b^{(k)}$ is a nilpotent map of degree -1. Since the projection Π_k does not affect the degree of b (that is -1) we just need to prove that $b^{(k)}$ is nilpotent. To begin with, we write the explicit action of b on gauge fields $\mathcal{A}$ and equations of motions $\mathcal{E}$ in the initial space $\mathcal{K}$, using (4.13)

$$b(\mathcal{A}) = 2(\partial_{(r}\mathcal{A}_{u)} - r^{-2}\partial_{(z}\mathcal{A}_{\bar{z})} + r^{-1}\mathcal{A}_u) \in K_0 \tag{B1}$$

$$b(\mathcal{E})_\mu = \begin{pmatrix} r^{-2}\partial_z\mathcal{E}_{r\bar{z}} - (\partial_r + 2r^{-1})\mathcal{E}_{ru} \\ r^{-2}\partial_{\bar{z}}\mathcal{E}_{uz} + \partial_u\mathcal{E}_{ru} \\ \partial_r\mathcal{E}_{uz} + \partial_z\mathcal{E}_{ru} \\ \partial_u\mathcal{E}_{r\bar{z}} - \partial_{\bar{z}}\mathcal{E}_{ru} \end{pmatrix} \in K_1. \tag{B2}$$

From the definition of $b^{(k)}$, employing lemma A.1 and the idempotence of Π_k, one can prove that

$$b^{(k)}(\mathcal{A}) = \mathcal{P}_{[\![k-1,-2]\!]}\, b(\mathcal{A}) + 2\mathcal{P}_{-1}(r^{-1}\mathcal{A}_u) \in K_0^{(k)} \tag{B3}$$

$$b^{(k)}(\mathcal{E})_\mu = \begin{pmatrix} \mathcal{P}_{[\![k-2,-3]\!]}\, b(\mathcal{E})_r \\ \mathcal{P}_{[\![k-1,-2]\!]}\, b(\mathcal{E})_u \\ \mathcal{P}_{[\![k,-2]\!]}\, b(\mathcal{E})_z \\ \mathcal{P}_{[\![k-1,-2]\!]}\, b(\mathcal{E})_{\bar{z}} \end{pmatrix} \in K_1^{(k)}, \tag{B4}$$

where now $\mathcal{A} \in K_1^{(k)}$ and $\mathcal{E} \in K_2^{(k)}$. For instance, the r-component of $b^{(k)}(\mathcal{E})$ results from

$$\begin{aligned} b^{(k)}(\mathcal{E})_r &= (\Pi_k b(\mathcal{E}))_r = (\Pi_k b(\Pi_k \mathcal{E}))_r \\ &= \mathcal{P}_{N^k_{A_r}}(r^{-2}\partial_z \mathcal{P}_{N^k_{E_{r\bar{z}}}}\mathcal{E}_{r\bar{z}} - (\partial_r + 2r^{-1})\mathcal{P}_{N^k_{E_{ru}}}\mathcal{E}_{ru}) \\ &= \mathcal{P}_{N^k_{A_r}\cap\mathcal{T}_{-2}N^k_{E_{r\bar{z}}}}(r^{-2}\partial_z\mathcal{E}_{r\bar{z}}) - \mathcal{P}_{N^k_{A_r}\cap\mathcal{T}_{-1}N^k_{E_{ru}}}[(\partial_r + 2r^{-1})\mathcal{E}_{ru}] \\ &= \mathcal{P}_{[\![k-2,-4]\!]}(r^{-2}\partial_z\mathcal{E}_{r\bar{z}}) - \mathcal{P}_{[\![k-2,-3]\!]}[(\partial_r + 2r^{-1})\mathcal{E}_{ru}] \\ &= \mathcal{P}_{[\![k-2,-3]\!]}\, b(\mathcal{E})_r - r^{-2}\partial_z\mathcal{P}_{-1}\mathcal{E}_{r\bar{z}}, \end{aligned} \tag{B5}$$

[46] In the simplified case where we take all the inclusions in (A74) to be equalities. The proof for the general case is analogous.

[47] That is to say, the sets with minimum cardinality. This choice is made to avoid including terms in the $1/r$-expansion of the gauge fields (parameters) that would not contribute to the corresponding equations of motion (gauge fields).

[48] See footnote 19 for a clarification regarding Π_k acting on the color-stripped space.

where the second term in the last line vanishes.[49] Clearly, $b^{(k)}(b^{(k)}(u))$ vanishes for all $u \in K_0^{(k)} \cup K_1^{(k)}$, by degree. Using (B3)–(B4) we find

$$\begin{aligned} b^{(k)}(b^{(k)}(\mathcal{E})) &= \mathcal{P}_{[\![k-1,-2]\!]} b(b^{(k)}(\mathcal{E})) + 2r^{-1}\mathcal{P}_0 b^{(k)}(\mathcal{E})_u \\ &= 2\mathcal{P}_{[\![k-1,-2]\!]}(\partial_{(r} b^{(k)}(\mathcal{E})_{u)} - r^{-2}\partial_{(z} b^{(k)}(\mathcal{E})_{\bar{z})} + r^{-1} b^{(k)}(\mathcal{E})_u) \\ &= \mathcal{P}_{[\![k-1,-2]\!]}(\mathcal{P}_{[\![k-2,-3]\!]}\partial_r b(\mathcal{E})_u + \mathcal{P}_{[\![k-2,-3]\!]}\partial_u b(\mathcal{E})_r - \mathcal{P}_{[\![k-3,-4]\!]}(r^{-2}\partial_z b(\mathcal{E})_{\bar{z}}) \\ &\quad - \mathcal{P}_{[\![k-2,-4]\!]}(r^{-2}\partial_{\bar{z}} b(\mathcal{E})_z) + 2\mathcal{P}_{[\![k-2,-3]\!]}(r^{-1} b(\mathcal{E})_u)) \\ &= 2\mathcal{P}_{[\![k-1,-3]\!]}(\partial_{(r} b(\mathcal{E})_{u)} - r^{-2}\partial_{(z} b(\mathcal{E})_{\bar{z})} + r^{-1} b(\mathcal{E})_u) + 2r^{-2}\mathcal{P}_{-1}\partial_{(z} b(\mathcal{E})_{\bar{z})} \end{aligned} \tag{B6}$$

where the canceled terms vanish because of (B4).[50] Therefore,

$$b^{(k)}(b^{(k)}(\mathcal{E})) = \mathcal{P}_{[\![k-1,-3]\!]} b(b(\mathcal{E})) = 0, \tag{B7}$$

which follows from the nilpotency of b.

Remark B.2 (Projected box operator). Let Π_k with $k \leq 0$ be the projection in (5.26), and let $\Box^{(k)}$ be the operator defined in (6.1). Then $\Box^{(k)} = \Pi_k \Box$.

Proof. We observe that

$$\begin{aligned} \Box^{(k)} - \Pi_k \Box &= [b^k,\, m_1^k] - \Pi_k [b,\, m_1] \\ &= \Pi_k (b\Pi_k m_1 + \not{m}_1 \Pi_k b - b m_1 - \not{m}_1 b) \\ &= \Pi_k b \Pi_k^{\complement} m_1, \end{aligned} \tag{B8}$$

where we used the definition of $m_1^{(k)}$, $b^{(k)}$ and the equation $\Pi_k m_1 \Pi_k = \Pi_k m_1$ (the color-stripped differential $m_1^{(k)}$ inherits such identity from the commuting diagram (5.8) of the differential $B_1^{(k)}$). Thus, it suffices to prove that $\Pi_k b \Pi_k^{\complement} m_1(u)$ vanishes for all[51] $u \in \mathcal{K}^{(k)}$. This can be done using the solution (5.26) for Π_k and the properties of the projection $\mathcal{P}$ listed in Appendix A. For instance, consider the case where $u = \Lambda \in K_0^{(k)}$:

$$\begin{aligned} \Pi_k b \Pi_k^{\complement} m_1(\Lambda) &= \pi_{\mathcal{N}^k} \nabla^\mu \pi_{(\mathcal{N}^k)^{\complement}} \partial_\mu \Lambda \\ &= -\mathcal{P}_{N_\Lambda^k}\{(\partial_u \mathcal{P}_{(N_{A_r}^k)^{\complement}} \partial_r \Lambda + r \leftrightarrow u) \\ &\quad - r^{-2}(\partial_{\bar{z}} \mathcal{P}_{(N_{A_z}^k)^{\complement}} \partial_z \Lambda + z \leftrightarrow \bar{z}) \\ &\quad + 2r^{-1}\mathcal{P}_{(N_{A_u}^k)^{\complement}} \partial_u \Lambda\} \\ &= -2\partial_r \partial_u \mathcal{P}_1 \Lambda + 2r^{-2}\partial_z \partial_{\bar{z}} \mathcal{P}_{\{1,2\}} \Lambda \\ &\quad + 2r^{-1}\partial_u \mathcal{P}_1 \Lambda = 0, \end{aligned} \tag{B9}$$

where we used the fact that $\mathcal{P}_n \Lambda = 0$ for all positive n.

[49] Indeed, $\mathcal{P}_{-1}\mathcal{E}_{r\bar{z}} = \mathcal{P}_{\{-1\}\cap[\![k-2,-2]\!]}\mathcal{E}_{r\bar{z}} = 0$.

[50] For instance, $\mathcal{P}_0 b^{(k)}(\mathcal{E})_u = \mathcal{P}_{\{0\}\cap[\![k-1,-2]\!]} b(\mathcal{E})_u = 0$.

[51] Notice that $\Pi_k b \Pi_k^{\complement} m_1(u) = 0$ for all $u \in K_2^{(k)}$, by degree, so there is no need to consider this case.

2. $\mathfrak{Kin}^{(k,f)}$

In Sec. V, our goal was to define an L_∞ algebra to encode SDYM near $\mathcal{I}$ by suitably truncating the r-expansion of the elements (4.5) in the initial graded vector space $\mathcal{X}$. We showed that the only way to achieve this is by defining subspaces $\mathcal{X}^{(k)} \subseteq \mathcal{X}$ through the projection Π_k and the power selection sets in $\mathcal{N}_k$, on which the required L_∞ algebras can be constructed—see Eqs. (5.26)–(5.28). In particular, the maxima of the power selection sets (which correspond to the fall-off) are fixed and common to all the slices $\mathfrak{S}_k$, as we emphasize in Fig. 1. Our goal here is to construct strict kinematic algebras by further projecting components of the r-expansion to zero.

To this end, we construct the refined slices $\mathfrak{S}_{k,f}$ which are defined by projecting components of the r-expansion of $\psi \in \mathcal{X}$ to zero, and introducing a projection that modifies the functional dependence of the gauge parameters $\Lambda \in X_{-1}$. Indeed, if one makes a gauge choice for the fields, the gauge choice imposes constraints on the gauge parameters. To clarify this point, let us consider an explicit example. As we show in Sec. VI B 1, one way to refine $\mathfrak{S}_{-1}$ in order to strictify its kinematic algebra is to modify the fall-off of A_u from 0 to -1, i.e. to replace the power selection set of A_u as follows:

$$N_{A_u}^{(-1)} = \{-2,-1,0\} \to N_{A_u}^{(-1,f)} \coloneqq \{-2,-1\}. \tag{B10}$$

Let $\Pi_{-1,f}$ be the projection defined as in (5.26), but with the substitution above for the set $N_{A_u}^{(-1)}$. As expected, the triple $(\mathcal{X}^{(-1,f)}, B_1^{(-1,f)}, B_2^{(-1,f)})$ constructed accordingly[52] does not form an L_∞ algebra. Nevertheless, one can prove that the brackets $B_i^{(-1,f)}$ satisfy the L_∞ relations up to terms proportional to $\partial_u \Lambda^{(0)}$. This means that enforcing $\Lambda^{(0)}$ to be independent of u guarantees the restoration of an L_∞ algebra, thus resolving the problem. Therefore, we require that the projection $\Pi_{-1,f}$ acts on $\Lambda \in X_{-1}$ as

[52] That is, using the definition (5.28) with Π_{-1} replaced by $\Pi_{-1,f}$.

$$\Lambda(r,u,z,\bar{z}) = \sum_{n=-\infty}^{\infty} r^n \Lambda^{(n)}(u,z,\bar{z}) \xrightarrow{\Pi_{-1,f}} \Lambda^{(0)}(z,\bar{z}) + \sum_{n=-2}^{-1} r^n \Lambda^{(n)}(u,z,\bar{z}) \tag{B11}$$

Physically, this procedure has a clear interpretation: the replacement (B10) simply amounts to setting $A_u^{(0)} = 0$, and the consistency of the gauge transformation then implies that $\partial_u \Lambda^{(0)}$ vanishes.

Building on this reasoning and generalizing it, we now provide the explicit definition of the projector $\Pi_{k,f}$ introduced in Sec. VI B, which is used to construct the refined slices $\mathfrak{S}_{k,f}$. The first ingredient that we need is the following collection of power selection sets:

$$\mathcal{N}_{k,f} \coloneqq \{N_\Lambda^{(k,f)}, N_{A_\mu}^{(k,f)}, N_{E_{\mu\nu}}^{(k)} \subseteq \mathbb{Z}^{\leq 0}\}, \tag{B12}$$

where the sets $N_I^{(k,f)} \subseteq N_I^{(k)}$, $I = \Lambda, A_\mu$, are defined as

$$N_\Lambda^{(k,f)} \coloneqq \{k-1,\ldots,f_r^k+1,0\}$$
$$N_{A_\mu}^{(k,f)} \coloneqq \begin{cases} \{k-2,\ldots,f_r^k\} & \mu = r \\ \{k-1,\ldots,f_u^k\} & \mu = u \\ \{k,\ldots,f_z^k\} & \mu = z \\ \{k-1,\ldots,f_{\bar{z}}^k\} & \mu = \bar{z} \end{cases}, \tag{6.6}$$

and the new variable fall-off are given by

$$f_\mu^k \coloneqq \max N_{A_\mu}^{(k,f)} \in \mathbb{Z} \quad \text{s.th.} \begin{cases} k-2 \leq f_r^k \leq -2 \\ k-1 \leq f_u^k, f_{\bar{z}}^k \leq 0\,. \\ k \leq f_z^k \leq 0 \end{cases} \tag{6.7}$$

The second ingredient is an operator $\mathfrak{D}_{k,f}$ that generalizes (B11), by appropriately modifying the functional dependence of the gauge parameter. To construct such operator, which of course depends on the modified fall-offs f_μ^k, we need one more definition. Let $m \in \mathbb{Z}$ and $\alpha \in \{u,z,\bar{z}\}$. We introduce a map $\mathfrak{d}_{\alpha|m} \in \mathrm{End}(\mathcal{C}_r^\infty(\mathcal{M}))$ that acts on $g \in \mathcal{C}_r^\infty(\mathcal{M})$ as

$$\mathfrak{d}_{\alpha|m}(g) \coloneqq \sum_{n=-\infty}^{m} \mathcal{P}_n g + \sum_{n=m+1}^{\infty} \mathcal{P}_n g \Big|_{\partial_\alpha \mathcal{P}_n g = 0}. \tag{B14}$$

We list below two examples to clarify how this map works

$$\mathfrak{d}_{u|4}(g) = \sum_{n=-\infty}^{4} r^n g^{(n)}(u,z,\bar{z}) + \sum_{n=5}^{\infty} r^n g^{(n)}(z,\bar{z})$$
$$\mathfrak{d}_{z|-1}(\mathfrak{d}_{u|4}(g)) = \sum_{n=-\infty}^{-1} r^n g^{(n)}(u,z,\bar{z}) + \sum_{n=0}^{4} r^n g^{(n)}(u,\bar{z}) + \sum_{n=5}^{\infty} r^n g^{(n)}(\bar{z}). \tag{B15}$$

The operator $\mathfrak{D}_{k,f}$ is then defined as

$$\mathfrak{D}_{k,f} \coloneqq \prod_\alpha \mathfrak{d}_{\alpha|f_\alpha^k} \in \mathrm{End}(\mathcal{C}_r^\infty(\mathcal{M})). \tag{B16}$$

At this point, we can finally state the definition of the desired projection from $\mathcal{X}$ to the refined slice $\mathfrak{S}_{k,f}$ (with $k \leq 0$ and fall-off f_μ^k):

$$\Pi_{k,f}(\psi) \coloneqq \begin{cases} \mathfrak{D}_{k,f} \circ \pi_{\mathcal{N}_{k,f}}(\psi) & \psi \in X_{-1} \\ \pi_{\mathcal{N}_{k,f}}(\psi) & \text{otherwise} \end{cases} \tag{B17}$$

To provide a couple of examples, we now explicitly write how $\Pi_{k,f}$ projects the gauge fields when selecting the first two fall-offs from Table IV

$$\mathfrak{D}_{k,f_4}(\Lambda) = \sum_{n=k+2}^{0} r^n \Lambda^{(n)}(z) + \sum_{n=k-1}^{k+1} r^n \Lambda^{(n)}(u,z,\bar{z}) \in X_{-1}^{(k,f_4)}$$
$$\mathfrak{D}_{k,f_5}(\Lambda) = \Lambda^0(u,\bar{z}) + r^{k-1}\Lambda^{(k-1)}(u,z,\bar{z}) \in X_{-1}^{(k,f_5)}. \tag{B18}$$

Thanks to the projection above, for any non positive integer k and choice of fall-off f_μ^k satisfying (6.7), it is possible to construct a kinematic algebra $\mathfrak{Kin}^{(k,f)}$ on the refined slice $\mathfrak{S}_{k,f}$ following a procedure analogous to that presented in Sec. VI A. Hence, we can prove the following result.

Result B.3 (Strict kinematic algebra). Let the $\mathrm{BV}_\infty^{\square(k,f)}$ algebra $\mathfrak{Kin}^{(k,f)}$ be the kinematic algebra of the refined slice $\mathfrak{S}_{k,f}$, for some $k \in \mathbb{Z}^{\leq 0}$ and f_μ^k satisfying (6.7). Then

$$\mathfrak{Kin}^{(k,f)} \text{is strict} \Leftrightarrow f_\mu^k \text{satisfy} \begin{cases} f_r^k + f_u^k + f_z^k < k \\ f_r^k + f_u^k + f_{\bar{z}}^k < k-1 \\ f_r^k + f_z^k + f_{\bar{z}}^k < k \\ f_u^k + f_z^k + f_{\bar{z}}^k < k+1 \\ f_r^k + f_u^k < k \\ f_r^k + f_{\bar{z}}^k < k \\ f_u^k + f_z^k < k+2 \\ f_z^k + f_{\bar{z}}^k < k+2. \end{cases}$$

Note that the system above is symmetric under $f_r^k \leftrightarrow f_z^k - 2$ and $f_u^k \leftrightarrow f_{\bar{z}}^k$.

Proof. The kinematic algebra $\mathfrak{Kin}^{(k,f)}$ of a refined slice $\mathfrak{S}_{k,f}$ with fall-off f^k_μ is strict if the corresponding trilinear bracket $\theta_3^{(k,f)}$ vanishes (see discussion at the end of Sec. III C). Thus, proving the above result is equivalent to finding the conditions such that

$$\theta_3^{(k,f)}(u_1,u_2,u_3)=0 \quad \forall\ u_1,u_2,u_3\in\mathcal{K}^{(k,f)} \tag{B19}$$

To study when this happens, recall that

$$\begin{aligned}\theta_3^{(k,f)}(u_1,u_2,u_3)&=\Pi_{k,f}\theta_3(u_1,u_2,u_3)\\&=\Pi_{k,f}\theta_3(\Pi_{k,f}u_1,\Pi_{k,f}u_2,\Pi_{k,f}u_3).\end{aligned} \tag{B20}$$

We start by considering $\theta_3^{(k,f)}$ acting on three gauge fields. The r-component is

$$\begin{aligned}\theta_3^{(k,f)}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)_r=\mathcal{P}_{N^{(k,f)}_{A_r}}[&6r^{-2}(\mathcal{P}_{N^{(k,f)}_{A_r}}\mathcal{A}_{[1r)}\\&\times(\mathcal{P}_{N^{(k,f)}_{A_z}}\mathcal{A}_{2z})(\mathcal{P}_{N^{(k,f)}_{A_{\bar z}}}\mathcal{A}_{3]\bar z})]\end{aligned} \tag{B21}$$

To determine when this expression vanishes, we employ a generalization of identity (A2g) of Appendix A. Let $i\in\mathbb{N}$ and $L,M_i\subseteq\mathbb{Z}$, one can show that[53]

$$\mathcal{P}_L\left(\prod_i\mathcal{P}_{M_i}g_i\right)=0 \quad \forall\ g_i\in C^\infty_r(\mathcal{M})\Leftrightarrow L\cap\sum_i M_i=\varnothing. \tag{B22}$$

Thus, we conclude that (B21) vanishes for all $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\in K_1^{(k,f)}$ iff

$$\begin{aligned}\varnothing&=\mathcal{T}_2N^{(k,f)}_{A_r}\cap(N^{(k,f)}_{A_r}+N^{(k,f)}_{A_z}+N^{(k,f)}_{A_{\bar z}})\\&=[\![k,f^k_r+2]\!]\cap([\![k-2,f^k_r]\!]+[\![k,f^k_z]\!]+[\![k-1,f^k_{\bar z}]\!])\\&=[\![k,f_r+2]\!]\cap[\![3k-3,f^k_r+f^k_z+f^k_{\bar z}]\!].\end{aligned} \tag{B23}$$

We note that $f^k_r+2\geq 3k-3$, since by hypothesis[54] $f^k_r\geq k-2$ and $k\leq 0$. Thus, the above equation is true iff

$$f^k_r+f^k_z+f^k_{\bar z}<k. \tag{B24}$$

Repeating the same procedure for the other components, we find that

$$\theta_3^{(k,f)}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)=0 \quad \forall\ \mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\in K_1^{(k,f)}\Leftrightarrow\begin{cases}\sum_{\mu\neq r}f^k_\mu<k+1\\ \sum_{\mu\neq u}f^k_\mu<k\\ \sum_{\mu\neq z}f^k_\mu<k-1\\ \sum_{\mu\neq\bar z}f^k_\mu<k\end{cases}, \tag{B25}$$

which prove the first four inequalities of the system in (6.9). Now, let us turn to the study of $\theta_3^{(k,f)}(\mathcal{E},\mathcal{A}_1,\mathcal{A}_2)$, where $\mathcal{E}\in K_2^{(k,f)}$. Its ru-component is

$$\begin{aligned}\theta_3^{(k,f)}(\mathcal{E},\mathcal{A}_1,\mathcal{A}_2)_{ru}=\mathcal{P}_{N^{(k)}_{E_{ru}}}\{&2r^{-2}[(\mathcal{P}_{N^{(k)}_{E_{uz}}}\mathcal{E}_{uz})\\&\times(\mathcal{P}_{N^{(k,f)}_{A_r}}\mathcal{A}_{[1r)}(\mathcal{P}_{N^{(k,f)}_{A_{\bar z}}}\mathcal{A}_{2]\bar z})\\&-(\mathcal{P}_{N^{(k)}_{E_{r\bar z}}}\mathcal{E}_{r\bar z})(\mathcal{P}_{N^{(k,f)}_{A_u}}\mathcal{A}_{[1u)}(\mathcal{P}_{N^{(k,f)}_{A_z}}\mathcal{A}_{2]z})]\}\end{aligned} \tag{B26}$$

The above expression vanishes $\forall\ \mathcal{A}_1,\mathcal{A}_2,\in K_1^{(k,f)}$ and $\forall\ \mathcal{E}\in K_2^{(k,f)}$ iff

$$\begin{aligned}\varnothing&=\mathcal{T}_2N^{(k)}_{E_{ru}}\cap(N^{(k)}_{E_{uz}}+N^{(k,f)}_{A_r}+N^{(k,f)}_{A_{\bar z}})\\&=[\![k,0]\!]\cap([\![k,0]\!]+[\![k-2,f^k_r]\!]+[\![k-1,f^k_{\bar z}]\!])\\&=[\![k,0]\!]\cap[\![3k-3,f^k_r+f^k_{\bar z}]\!]\end{aligned} \tag{B27}$$

and

$$\begin{aligned}\varnothing&=\mathcal{T}_2N^{(k)}_{E_{ru}}\cap(N^{(k)}_{E_{r\bar z}}+N^{(k,f)}_{A_u}+N^{(k,f)}_{A_z})\\&=[\![k,0]\!]\cap([\![k-2,-2]\!]+[\![k-1,f^k_u]\!]+[\![k,f^k_z]\!])\\&=[\![k,0]\!]\cap[\![3k-3,f^k_u+f^k_z-2]\!]\end{aligned} \tag{B28}$$

Since $k\leq 0$, the above equation hold, respectively, iff

$$f^k_r+f^k_{\bar z}<k \quad\text{and}\quad f^k_u+f^k_z\leq k+2. \tag{B29}$$

Applying the same procedure to the other two independent components, we obtain that

[53]The procedure is analogous to the proof of (A2g).
[54]See Eq. (6.7).

$$\theta_3^{(k,f)}(\mathcal{E},\mathcal{A}_1,\mathcal{A}_2)=0 \quad \forall\ \mathcal{A}_1,\mathcal{A}_2\in K_1^{(k,f)},$$
$$\mathcal{E}\in K_2^{(k,f)} \Leftrightarrow \begin{cases} f_r^k+f_u^k<k \\ f_r^k+f_{\bar z}^k<k \\ f_u^k+f_z^k<k+2 \\ f_u^k+f_z^k<k+2 \end{cases}, \tag{B30}$$

thus completing the proof of the system in (6.9).

APPENDIX C: HOMOTOPY TRANSFER

In this Appendix, we introduce the notion of homotopy transfer, a mathematical tool used to transfer algebraic (nonlinear) structure from one cochain complex to another. In the following, we illustrate how homotopy transfer works for L_∞ algebras. However, one can extend this concept to other algebras such as C_∞ or BV_∞ algebras.

Given an L_∞ algebra $(\mathcal{X},B_1,B_n)$ defined on a graded vector space $\mathcal{X}$ with differential B_1 and multilinear maps B_n with $n>1$, we wish to find an L_∞ algebra structure on a subspace $\bar{\mathcal{X}}\subseteq\mathcal{X}$ equipped with a differential $\bar B_1$. To that end, following our discussion on quasi-isomorphisms in VII A, we introduce two cochain maps: a projection map $\pi:\mathcal{X}\to\bar{\mathcal{X}}$ and an inclusion map $\imath:\bar{\mathcal{X}}\to\mathcal{X}$, i.e.,

$$\pi\circ B_1=\bar B_1\circ\pi, \qquad \imath\circ\bar B_1=B_1\circ\imath, \tag{C1}$$

which trivially obey

$$\pi\circ\imath=1_{\bar{\mathcal{X}}}, \tag{C2}$$

where $1_{\bar{\mathcal{X}}}$ is the identity map on $\bar{\mathcal{X}}$. From now on we drop the $\circ$ symbol to denote composition to simplify notation. Homotopy transfer further requires that these maps obey

$$\imath\pi-1_{\mathcal{X}}=[B_1,h], \tag{C3}$$

which tells us that the inclusion map $\imath$ is the inverse of the projection map π *up to homotopy*, namely that the failure of $\imath$ to be the inverse of π is governed by the differential B_1 and a linear *homotopy map* h of degree -1 that obeys the so-called side conditions

$$\pi h=h\imath=h^2=0. \tag{C4}$$

Even though all the above relations only involve linear maps, these relations, together with the multilinear brackets of the B_n of the algebra defined on $\mathcal{X}$, lead to the definition of the multilinear maps $\bar B_n:\bar{\mathcal{X}}^{\otimes n}\to\bar{\mathcal{X}}$ that define a new L_∞ algebra on the subspace $\bar{\mathcal{X}}$ which we present here up to trilinear order

$$\begin{aligned}\bar B_2(\bar x_1,\bar x_2)&=\pi B_2(\imath\bar x_1,\imath\bar x_2),\\ \bar B_3(\bar x_1,\bar x_2,\bar x_3)&=\pi B_3(\imath\bar x_1,\imath\bar x_2,\imath\bar x_3)+\pi B_2(hB_2(\imath\bar x_1,\imath\bar x_2),\imath\bar x_3)\\ &\quad+(-1)^{\bar x_1(\bar x_2+\bar x_2)}\pi B_2(hB_2(\imath\bar x_2,\imath\bar x_3),\imath\bar x_1)\\ &\quad+(-1)^{\bar x_3(\bar x_1+\bar x_2)}\pi B_2(hB_2(\imath\bar x_3,\imath\bar x_1),\imath\bar x_2).\end{aligned} \tag{C5}$$

For a general expression for all maps $\bar B_n$ and a more detailed explanation of homotopy transfer refer to [70]. Notice that, even without a map B_3 in our original algebra defined on $\mathcal{X}$, in principle, a nontrivial three-bracket $\bar B_3$ may exist. This implies that if one starts with a strict algebra on $\mathcal{X}$, the transferred algebraic structure on a subspace $\bar{\mathcal{X}}$ is not generally strict. In the following, we perform homotopy transfer of self-dual Yang-Mills theory to the slices that we considered in the main body of the paper.

1. Homotopy map

Result C.1. The homotopy map $h:\mathcal{X}\to\mathcal{X}$ is given by

$$\begin{aligned}h(A)&=-\mathcal{P}_{\mathbb{N}^+}\frac{1}{\partial_r}A_r\\ h(E)_r&=\frac{1}{2\partial_{\bar z}}\mathcal{P}_{-1}E_{r\bar z}\\ h(E)_u&=\mathcal{P}_{\mathbb{N}^+}\frac{1}{\partial^2}\left[2\partial_u E_{ru}-\frac{1}{r^2}\left(\frac{1}{\partial_r}\partial_u\partial_z E_{r\bar z}-\partial_{\bar z}E_{uz}\right)\right]\\ h(E)_z&=\frac{1}{\partial_u}\left(\partial_z h(E)_u-\frac{1}{2}\mathcal{P}_{\mathbb{N}^+}E_{uz}\right)\\ h(E)_{\bar z}&=-\mathcal{P}_{\mathbb{N}^+}\frac{1}{2\partial_r}E_{r\bar z},\end{aligned} \tag{C6}$$

where $\mathbb{N}^+:=\mathbb{N}\backslash\{0\}$ and ∂^2 is defined in (4.15).

Proof. We want to prove that the map h above satisfies

$$[B_1,h]\psi=(\imath\pi-1_{\mathcal{X}})\psi. \tag{C7}$$

for all $\psi\in\mathcal{X}$. To do so, it is useful to first study how the projector $\mathcal{P}$ commutes with the inverse of the derivatives. For any set $M\in\mathbb{Z}$, the following identities hold:

$$\mathcal{P}_{M\backslash\{0\}}\frac{1}{\partial_r}=\frac{1}{\partial_r}\mathcal{P}_{T_{-1}M\backslash\{-1\}}, \qquad \mathcal{P}_M\frac{1}{\partial_\alpha}=\frac{1}{\partial_\alpha}\mathcal{P}_M. \tag{C8}$$

The second equation is trivial, since ∂_α^{-1} leaves the r-expansion invariant. To prove the first one, we notice that for any integer $m\neq 0$

$$\begin{aligned}\mathcal{P}_m\frac{1}{\partial_r}f&=\sum_{n\in\mathbb{Z}}\frac{\delta_{m,n+1}}{n+1}r^{n+1}f^{(n)}=\frac{1}{m}r^m f^{(m-1)}\\ &=\frac{1}{\partial_r}(r^{m-1}f^{(m-1)})=\frac{1}{\partial_r}\mathcal{P}_{m-1}f,\end{aligned} \tag{C9}$$

where $\mathcal{P}_m$ is the projection defined in (A4). In particular, this implies that expressions of the form $\mathcal{P}_{M\setminus\{0\}}\partial_r^{-1}f$ are well defined. For instance, $h(A)$ in (C6) can be rewritten as

$$h(A) = -\mathcal{P}_{\mathbb{N}^+}\frac{1}{\partial_r}A_r = -\sum_{m\in\mathbb{N}^+}\frac{1}{m}\mathcal{P}_m(rA_r). \tag{C10}$$

Equation (C7) with $\psi = \Lambda \in X_{-1}$ reads

$$\begin{aligned}[B_1,h]\Lambda &= -\mathcal{P}_{\mathbb{N}^+}\frac{1}{\partial_r}\partial_r\Lambda = -\mathcal{P}_{\mathbb{N}^+}\Lambda \\ &= (\mathcal{P}_{\mathbb{Z}^{\leq 0}} - \mathcal{P}_{\mathbb{Z}})\Lambda = (\iota\pi - 1_{\mathcal{X}})\Lambda,\end{aligned} \tag{C11}$$

where we used (A2b) and the fact that $\mathcal{P}_{\mathbb{Z}^{\leq 0}}\Lambda = \iota\pi\Lambda$ and $\mathcal{P}_{\mathbb{Z}}\Lambda = \Lambda$. For $\psi = A_\mu \in X_0$ the right-hand side of (C7) has components

$$\begin{aligned}(\iota\pi - 1_{\mathcal{X}})A_r &= (\mathcal{P}_{\mathbb{Z}^{<0}\setminus\{-1\}} - \mathcal{P}_{\mathbb{Z}})A_r = -\mathcal{P}_{\mathbb{N}\cup\{-1\}}A_r \\ (\iota\pi - 1_{\mathcal{X}})A_\alpha &= (\mathcal{P}_{\mathbb{Z}^{\leq 0}} - \mathcal{P}_{\mathbb{Z}})A_\alpha = -\mathcal{P}_{\mathbb{N}^+}A_\alpha,\end{aligned} \tag{C12}$$

\while the components of the left-hand side are

$$\begin{aligned}([B_1,h]A)_r &= -\partial_r\mathcal{P}_{\mathbb{N}^+}\frac{1}{\partial_r}A_r + \frac{2}{\partial_{\bar z}}\mathcal{P}_{-1}\partial_{[r}A_{\bar z]} = -\mathcal{P}_{\mathbb{N}\cup\{-1\}}A_r \\ ([B_1,h]A)_u &= -\mathcal{P}_{\mathbb{N}^+}\frac{1}{\partial_r}\partial_u A_r \\ &\quad + \mathcal{P}_{\mathbb{N}^+}\frac{4}{\partial^2}\left[\partial_u\partial_{[r}A_{u]} - \frac{1}{r^2}\left(\frac{1}{\partial_r}\partial_u\partial_z\partial_{[r}A_{\bar z]} - \partial_{\bar z}\partial_{[u}A_{z]} - \partial_u\partial_{[z}A_{\bar z]}\right)\right] \\ &= \mathcal{P}_{\mathbb{N}^+}\left\{\frac{2}{\partial^2}\left[\left(-\partial_r\partial_u + \frac{1}{r^2}\partial_z\partial_{\bar z}\right)\left(\frac{1}{\partial_r}\partial_u A_r - A_u\right)\right] - \frac{1}{\partial_r}\partial_u A_r\right\} \\ &= -\mathcal{P}_{\mathbb{N}^+}A_u \\ ([B_1,h]A)_z &= \mathcal{P}_{\mathbb{N}^+}\left[\frac{1}{\partial_u}\left(\frac{1}{\partial_r}\partial_u\partial_z A_r - \partial_z A_u - 2\partial_{[u}A_{z]}\right) - \frac{1}{\partial_r}\partial_z A_r\right] = -\mathcal{P}_{\mathbb{N}^+}A_z \\ ([B_1,h]A)_{\bar z} &= -\mathcal{P}_{\mathbb{N}^+}\frac{1}{\partial_r}\partial_{\bar z}A_r - 2\mathcal{P}_{\mathbb{N}^+}\frac{1}{\partial_r}\partial_{[r}A_{\bar z]} = -\mathcal{P}_{\mathbb{N}^+}A_{\bar z}.\end{aligned} \tag{C13}$$

This proves that the homotopy map h in (C6) satisfies its defining Eq. (C7). ∎

2. Higher brackets

Lemma C.2.

$$\begin{array}{ccc}\mathcal{X}\times\mathcal{X} & \xrightarrow{B_2} & \mathcal{X} \\ {\scriptstyle \iota\times\iota}\uparrow & & \uparrow{\scriptstyle \iota} \\ \bar{\mathcal{X}}\times\bar{\mathcal{X}} & \xrightarrow{\bar B_2} & \bar{\mathcal{X}}\end{array} \quad \textit{commutes} \quad \Leftrightarrow \quad \mathrm{Jac}(\bar B_2) = \pi\mathrm{Jac}(B_2)\iota \tag{C14}$$

Moreover, if (C14) and the second condition $h\iota = 0$ are satisfied and $\mathrm{Jac}(B_2) = 0$, then $\bar B_3 = 0$.

Proof. The above diagram commutes iff $\iota\bar B_2 = B_2\iota$, which is equivalent to $\iota\pi B_2\iota = B_2\iota$ (recalling that by definition $\bar B_2 = \pi B_2\iota$). In other words, $\mathrm{Im}(B_2\iota) \subseteq \mathrm{Ker}[B_1,h]$. Thus

$$\begin{aligned}\mathrm{Jac}(\bar B_2)(\bar\psi_1,\bar\psi_2,\bar\psi_3) &= \bar B_2(\bar B_2(\bar\psi_1,\bar\psi_2),\bar\psi_3) + \ldots = \pi B_2(\iota\pi B_2(\iota\bar\psi_1,\iota\bar\psi_2),\iota\bar\psi_3) + \ldots \\ &= \pi B_2(B_2(\iota\bar\psi_1,\iota\bar\psi_2),\iota\bar\psi_3) + \ldots = \pi\mathrm{Jac}(B_2)(\iota\bar\psi_1,\iota\bar\psi_2,\iota\bar\psi_3)\end{aligned} \tag{C15}$$

for all $\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3 \in \bar{\mathcal{X}}$. From this, we see that if B_2 satisfies the Jacobi identity, then also $\bar{B}_2$ does. In this case $[\bar{B}_1, \bar{B}_3] = 0$, that does not imply, in general, $\bar{B}_3 = 0$. However, from

$$\begin{aligned}\bar{B}_3(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3) &= \pi B_2(hB_2(\iota\bar{\psi}_1, \iota\bar{\psi}_2)\iota\bar{\psi}_3) \\ &= \pi B_2(h\iota\pi B_2(\iota\bar{\psi}_1, \iota\bar{\psi}_2)\iota\bar{\psi}_3),\end{aligned} \tag{C16}$$

we deduce that $\bar{B}_3$ vanishes if the homotopy map satisfies the side condition $h\iota = 0$. ∎

Result C.3. $\bar{B}_3 = 0$

Proof. Thanks to the above lemma and since B_2 satisfies the Jacobi identity and the side condition $h\iota = 0$ is true, it is sufficient to prove that $\iota\pi B_2\iota = B_2\iota$. For example, $\forall\ \bar{\Lambda}^1, \bar{\Lambda}^2 \in \bar{X}_{-1}$

$$\begin{aligned}\iota\pi B_2(\iota\bar{\Lambda}^1, \iota\bar{\Lambda}^2) &= -\iota\mathcal{P}_{\mathbb{Z}^{\leq 0}}[\iota\bar{\Lambda}^1, \iota\bar{\Lambda}^2] \\ &= -\iota\mathcal{P}_{\mathbb{Z}^{\leq 0}}[\mathcal{P}_{\mathbb{Z}^{\leq 0}}\iota\bar{\Lambda}^1, \mathcal{P}_{\mathbb{Z}^{\leq 0}}\iota\bar{\Lambda}^2] \\ &= -[\mathcal{P}_{\mathbb{Z}^{\leq 0}}\iota\bar{\Lambda}^1, \mathcal{P}_{\mathbb{Z}^{\leq 0}}\iota\bar{\Lambda}^2] \\ &= -[\iota\bar{\Lambda}^1, \iota\bar{\Lambda}^2] = B_2(\iota\bar{\Lambda}^1, \iota\bar{\Lambda}^2)\end{aligned} \tag{C17}$$

where in the second equality we used $\bar{\Lambda}^i = \mathcal{P}_{\mathbb{Z}^{\leq 0}}\bar{\Lambda}^i$ while in the third we used (A2e) with $\mathbb{Z}^{\leq 0} + \mathbb{Z}^{\leq 0} = \mathbb{Z}^{\leq 0}$. All the other cases can be proved in the same way, recalling (A2b) and that $I_i + I_2 = I_2$ for $i = 1, 2$, with $I_1 = \mathbb{Z}^{\leq 0}$ and $I_2 = \mathbb{Z}^{\leq 0}\backslash\{-1\}$. In particular, $\bar{B}_2 = B_2|_{\bar{\mathcal{X}}}$ where $\bar{\mathcal{X}}$ is viewed as a subset of $\mathcal{X}$. ∎

3. Kinematic algebra from homotopy transfer

Since the operators in (7.9) are color blinded, the linear structure of the algebra $(\mathcal{K}, m_1, m_2)$ obtained by color stripping and the corresponding homotopy transfer are preserved.

$$\begin{array}{ccccccccc} 0 & \xrightarrow{m_1} & K_0 & \xrightarrow{m_1} & K_1 & \xrightarrow{m_1} & K_2 & \xrightarrow{m_1} & 0 \\ & & \iota\uparrow\downarrow\pi & & \iota\uparrow\downarrow\pi & & \iota\uparrow\downarrow\pi & & \\ 0 & \xrightarrow{\bar{m}_1} & \bar{K}_0 & \xrightarrow{\bar{m}_1} & \bar{K}_1 & \xrightarrow{\bar{m}_1} & \bar{K}_2 & \xrightarrow{\bar{m}_1} & 0 \end{array} \tag{C18}$$

where $\bar{m}_1 \coloneqq \pi m_1 \iota$. The above observation together with $\iota\bar{B}_2 = b_2\iota$ imply $\iota\bar{m}_2 = m_2\iota$. Moreover, we define

$$\bar{b} \coloneqq \pi b\iota, \tag{C19}$$

Lemma C.4. $\overline{\Box} = \pi\Box\iota$

Proof. From the homotopy transfer (C18) we know $\bar{m}_1\pi = \pi m_1$ and $\iota\bar{m}_1 = m_1\iota$, so

$$\begin{aligned}\overline{\Box} &= \bar{m}_1\bar{b} + \bar{b}\bar{m}_1 = \bar{m}_1\pi b\iota + \pi b\iota\bar{m}_1 \\ &= \pi m_1 b\iota + \pi b m_1\iota = \pi\Box\iota\end{aligned} \tag{C20}$$

∎

Lemma C.5.

$$\begin{array}{ccc} \mathcal{K} & \xleftarrow{b} & \mathcal{K} \\ \iota\uparrow & & \uparrow\iota \\ \bar{\mathcal{K}} & \xleftarrow{\bar{b}} & \bar{\mathcal{K}} \end{array} \quad \textit{commutes} \tag{C21}$$

Proof. The above diagram commutes if $\iota\bar{b} = b\iota$. From the definition of $\bar{b}$, this is equivalent to $\iota\pi b\iota = b\iota$, i.e. $\iota\pi = 1_{\mathcal{K}}$ on $\mathrm{Im}(b\iota)$. We make use of (A5) to conclude that $\forall\ \bar{A} \in \bar{K}_1$

$$\begin{aligned}\iota\pi b\iota\bar{A} &= -\mathcal{P}_{\mathbb{Z}^{\leq 0}}(\nabla^\mu\bar{A}_\mu) = \mathcal{P}_{\mathbb{Z}^{\leq 0}}(\nabla_{\{r}\bar{A}_{u\}} - r^{-2}\nabla_{\{z}\bar{A}_{\bar{z}\}}) \\ &= \mathcal{P}_{\mathbb{Z}^{\leq 0}}[\mathcal{P}_{\mathbb{Z}^{<}\backslash\{-1\}}(\nabla_{\{r}\bar{A}_{u\}} - r^{-2}\nabla_{\{z}\bar{A}_{\bar{z}\}}) \\ &\qquad + 2\mathcal{P}_{-1}(r^{-1}\bar{A}_u)] \\ &= \mathcal{P}_{\mathbb{Z}^{<}\backslash\{-1\}}(\nabla_{\{r}\bar{A}_{u\}} - r^{-2}\nabla_{\{z}\bar{A}_{\bar{z}\}}) + 2\mathcal{P}_{-1}(r^{-1}\bar{A}_u) \\ &= \nabla_{\{r}\bar{A}_{u\}} - r^{-2}\nabla_{\{z}\bar{A}_{\bar{z}\}} = b\iota\bar{A}\end{aligned} \tag{C22}$$

∎

Result C.6. $\bar{b}_2 = \pi b_2\iota$

Proof. Result (C.3) implies $\iota\bar{m}_2 = m_2\iota$, since color stripping does not affect the analogous property for B_2. Using the definition of $\bar{m}_2$ (C19), this relation translates to $\iota\pi m_2\iota = m_2\iota$. Recalling Lemma (C.5), we conclude that $\iota\pi = 1_{\mathcal{K}}$ on both $\mathrm{Im}(m_2\iota)$ and $\mathrm{Im}(b\iota)$. Thus, the bracket $\bar{b}_2$, that is defined from $\bar{b}$ and $\bar{m}_2$, is

$$\begin{aligned}\bar{b}_2(\bar{u}_1, \bar{u}_2) &\coloneqq \bar{b}\bar{m}_2(\bar{u}_1, \bar{u}_2) - \bar{m}_2(\bar{b}\bar{u}_1, \bar{u}_2) \\ &\quad - (-1)^{|\bar{u}_1|}\bar{m}_2(\bar{u}_1, \bar{b}\bar{u}_2) \\ &= \pi b\iota\pi m_2(\iota\bar{u}_1, \iota\bar{u}_2) - \pi m_2(\iota\pi b\iota\bar{u}_1, \iota\bar{u}_2) \\ &\quad - (-1)^{|\bar{u}_1|}\pi m_2(\iota\bar{u}_1, \iota\pi b\iota\bar{u}_2) \\ &= \pi b_2(\iota\bar{u}_1, \iota\bar{u}_2)\end{aligned} \tag{C23}$$

∎

Result C.7. $\bar{\theta}_3 = \pi\theta_3\iota$

Proof. The Poisson identity is

$$\begin{aligned}\overline{\mathrm{Poiss}}(\bar{u}_1, \bar{u}_2, \bar{u}_3) &\coloneqq \bar{b}_2(\bar{m}_2(\bar{u}_1, \bar{u}_2), \bar{u}_3) \\ &\quad - (-)^{\bar{u}_1(\bar{u}_2+\bar{u}_3)}\bar{m}_2(\bar{b}_2(\bar{u}_2, \bar{u}_3), \bar{u}_1) \\ &\quad - (-)^{\bar{u}_3(\bar{u}_1+\bar{u}_2)}\bar{m}_2(\bar{b}_2(\bar{u}_3, \bar{u}_1), \bar{u}_2) \\ &= [\bar{m}_1, \bar{\theta}_3](\bar{u}_1, \bar{u}_2, \bar{u}_3)\end{aligned} \tag{C24}$$

$\forall\ \bar{u}_1, \bar{u}_2, \bar{u}_3 \in \bar{\mathcal{K}}$. The first term in the first line above is

$$\bar{b}_2(\bar{m}_2(\bar{u}_1,\bar{u}_2),\bar{u}_3) = \pi b_2(\imath\pi m_2(\imath\bar{u}_1,\imath\bar{u}_2),\imath\bar{u}_3) = \pi b_2(m_2(\imath\bar{u}_1,\imath\bar{u}_2),\imath\bar{u}_3) \tag{C25}$$

where we used $\imath\pi = 1_{\mathcal{K}}$ on $\mathrm{Im}(m_2\imath)$. Note that we also have $\imath\pi = 1_{\mathcal{K}}$ on $\mathrm{Im}(b_2\imath)$, since

$$\begin{aligned}
\imath\pi b_2(\imath\bar{u}_1,\imath\bar{u}_2) &= \imath\pi b m_2(\imath\bar{u}_1,\imath\bar{u}_2) - \imath\pi m_2(b\imath\bar{u}_1,\imath\bar{u}_2) - (-)^{\bar{u}_1}\imath\pi m_2(\imath\bar{u}_1,b\imath\bar{u}_2)\\
&= \imath\pi b\imath\bar{m}_2(u_1,u_2) - \imath\pi m_2(\imath\bar{b}u_1,\imath\bar{u}_2) - (-)^{\bar{u}_1}\imath\pi m_2(\imath\bar{u}_1,\imath\bar{b}\bar{u}_2)\\
&= b\imath\bar{m}_2(u_1,u_2) - m_2(\imath\bar{b}u_1,\imath\bar{u}_2) - (-)^{\bar{u}_1} m_2(\imath\bar{u}_1,\imath\bar{b}\bar{u}_2)\\
&= b m_2(\imath\bar{u}_1,\imath\bar{u}_2) - m_2(b\imath\bar{u}_1,\imath\bar{u}_2) - (-)^{\bar{u}_1} m_2(\imath\bar{u}_1,b\imath\bar{u}_2)\\
&= b_2(\imath\bar{u}_1,\imath\bar{u}_2)
\end{aligned} \tag{C26}$$

$\forall\ \bar{u}_1,\bar{u}_2 \in \bar{\mathcal{K}}$, where we used $\imath\bar{m}_2 = m_2\imath$ and (C21). Thus, the second term in the first line of (C24) is

$$\bar{m}_2(\bar{b}_2(\bar{u}_2,\bar{u}_3),\bar{u}_1) = \pi m_2(\imath\pi b_2(\imath\bar{u}_2,\imath\bar{u}_3),\imath\bar{u}_1) = \pi m_2(b_2(\imath\bar{u}_2,\imath\bar{u}_3),\imath\bar{u}_1) \tag{C27}$$

The third term in the Poissonator can be treated analogously; putting everything together, we have

$$[\bar{m}_1,\bar{\theta}_3] = \overline{\mathrm{Poiss}} = \pi\mathrm{Poiss}\imath = \pi[m_1,\theta_3]\imath = \pi m_1\theta_3\imath + \pi\theta_3 m_1\imath = \bar{m}_1\pi\theta_3\imath + \pi\theta_3\imath\bar{m}_1 \tag{C28}$$

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