

Asymmetric dyonic multicentered rotating black holes

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(Received 30 September 2025; accepted 8 November 2025; published 12 December 2025)

We construct an exact solution in four-dimensional Einstein-Maxwell-dilaton theory describing multicentered rotating black holes carrying both electric and magnetic charges obtained via dimensional reduction from five-dimensional Einstein gravity. This generalizes the Majumdar-Papapetrou solution to the rotating case and extends the recent multicentered rotating black hole solutions of Teo and Wan to configurations with unequal electric and magnetic charges. The resulting spacetimes are free of curvature singularities, conical defects, Dirac-Misner strings, and closed timelike curves, both on and outside the horizons, provided that the black holes have either aligned or antialigned spin orientations.

DOI: [10.1103/p824-5cpg](https://doi.org/10.1103/p824-5cpg)

I. INTRODUCTION

Exact solutions describing multi-black-hole systems are of considerable interest in both astrophysics and theoretical physics, as they provide valuable insights into the behavior of black hole binaries—systems that play a central role in gravitational wave astronomy. However, obtaining such solutions remains a formidable challenge due to the lack of symmetries and the inherently dynamical nature of the problem. Despite these difficulties, certain classes of static and stationary solutions describing black hole binaries have been identified. One of the earliest examples was presented by Israel and Khan [1], who found an exact solution representing static, axisymmetric configurations of multiple Schwarzschild black holes aligned along a common axis of rotation. However, the gravitational attraction between the black holes in these configurations necessitates the existence of conical singularities (or struts) between them to maintain equilibrium. As a rotational generalization of the double Schwarzschild solution, Kramer and Neugebauer [2] constructed the double Kerr solution describing the interaction of two rotating black holes.

The solution reflects the delicate balance between gravitational attraction and the spin-spin repulsion arising from their angular momenta. However, despite this interplay, conical singularities persist in the intermediate region, implying that exact vacuum solutions of static or stationary multi-black-hole systems necessarily contain singularities.

On the other hand, the inclusion of electric charge changes this situation. A well-known example is the Majumdar-Papapetrou solution [3,4], which provides an exact static multi-black-hole solution to the Einstein-Maxwell equations. In this configuration, the mutual gravitational attraction between the black holes is exactly balanced by the electrostatic repulsion due to their charges, allowing for static equilibrium without the need for conical singularities. Israel and Wilson [5], along with Perjés [6], extended the Majumdar-Papapetrou solution to include rotation, obtaining a class of stationary solutions within the Einstein-Maxwell theory. It was later shown, however, that these solutions represent configurations of naked singularities rather than genuine black holes. This result suggests that equilibrium configurations of rotating charged black holes cannot be realized within the framework of Einstein-Maxwell theory alone. More recently, Teo and Wan [7] succeeded in constructing a new class of exact regular solutions describing multicentered spinning black holes in 5D Kaluza-Klein theory. Upon dimensional reduction, these solutions correspond to balanced configurations of arbitrarily many dyonic rotating black holes in the 4D Einstein-Maxwell-dilaton theory. Each black hole in the solution is characterized by its own mass, angular momentum, equal electric and magnetic charges, and

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position. Furthermore, in the limit where all spin angular momenta vanish, the solution reduces to the Majumdar-Papapetrou solution, as the scalar (dilaton) field also vanishes in this limit.

The exact solutions for such Kaluza-Klein black holes were obtained by many researchers. Specifically, in 5D Kaluza-Klein theory, a black hole solution carrying a Kaluza-Klein (KK) electric charge but no magnetic charge—commonly referred to as the boosted Schwarzschild string—was first investigated by Chodos and Detweiler [8]. Shortly thereafter, Frolov *et al.* [9] extended this construction to include rotation. The first genuinely nontrivial configuration, in which a Kerr black hole is twisted along the compact Kaluza-Klein circle, was subsequently obtained by Dobiasch and Maison [10] through a transformation in the framework of the nonlinear sigma model. Their solution was later analyzed in detail by Gibbons and Wiltshire [11], Pollard [12], and Gibbons and Maeda [13]. A more general class of rotating Kaluza-Klein black hole solutions incorporating both KK electric and magnetic charges (interpreted as dyonic rotating black holes from the 4D viewpoint) was constructed by Rasheed [14], who employed an $SL(3, \mathbb{R})$ transformation acting on the Kerr string background characterized by a mass parameter and a rotation parameter. Independently, the same family of solutions was also obtained by Larsen [15,16].

As shown by Maison [17], the 5D vacuum Einstein theory with two commuting Killing vectors can be reduced to a 3D gravity-coupled nonlinear sigma model with an $SL(3, \mathbb{R})$ target space. In general, solving the resulting equations is challenging, as standard solitonic techniques such as the inverse scattering method cannot be applied due to the lack of a third Killing isometry. However, Clément [18,19] showed that a special class of solutions can be constructed from flat 3D Euclidean space and two harmonic functions. The slowly rotating extremal limit of the Rasheed-Larsen solution falls within this class. By generalizing the two harmonic functions to multicentered configurations, Teo and Wan [7] constructed a multicentered rotating Kaluza-Klein black hole solution that, after dimensional reduction to 4D, described an asymptotically flat rotating black hole with equal electric and magnetic charges. The purpose of this paper is to extend the Teo-Wan solution to the more general case with unequal electric and magnetic charges in 5D Kaluza-Klein theory. The resulting spacetime is free of curvature singularities, conical defects, Dirac-Misner strings, and closed timelike curves (CTCs), on and outside the horizons, provided that the black holes have either aligned or antialigned spin orientations.

We briefly outline the organization of this paper. In Sec. II, we review the multicentered rotating black hole solutions of Teo and Wan in 5D Kaluza-Klein theory. In Sec. III, we present the nonlinear sigma model framework

developed by Maison and show how the 5D Einstein equations with two commuting Killing vectors reduce to a 3D gravity-coupled sigma model. We also provide the necessary equations for our analysis, following the work of Clément. In Sec. IV, we see that the slowly rotating extremal limit of the Rasheed-Larsen solution falls within the class of solutions constructed from two harmonic functions with a single point source. By extending these harmonic functions from a single point source to multiple point sources, we construct a more general class of multicentered rotating black hole solutions, thereby extending the Teo-Wan solutions to the case of unequal electric and magnetic charges. In Sec. V, we analyze the properties of our solutions in detail, including the structure of the horizons, their asymptotic behavior, regularity conditions, and the absence of closed timelike curves. We also discuss the parameter region corresponding to physical solutions and examine various physical limits in which our solutions reduce to previously known configurations. Finally, Sec. VI is devoted to a summary and discussion.

II. THE TEO-WAN SOLUTIONS

Let us start with the 5D Kaluza-Klein theory, in which the metric can be written as

$$ds^2 = e^{-\frac{2\phi}{\sqrt{3}}}(dx^5 + \mathbf{A})^2 + e^{\frac{\phi}{\sqrt{3}}}g_{\mu\nu}dx^\mu dx^\nu, \quad (1)$$

where the function ϕ , the component A_μ of the 1-form $\mathbf{A} = A_\mu dx^\mu$, and the 4D metric $g_{\mu\nu}$ ($\mu, \nu = 0, \dots, 3$) do not depend on the fifth spatial coordinate x^5 which has a period of $2\pi R_{KK}$. As is well known, the dimensional reduction of the 5D Einstein theory leads to the 4D Einstein-Maxwell-dilaton theory described by the action

$$S = \int d^4x \sqrt{-g} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-\sqrt{3}\phi} F^2 \right), \quad (2)$$

where R is the Ricci scalar of the 4D metric, $g = \det(g_{\mu\nu})$, and $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. From this, the field equations, the Einstein equation for the 4D metric $g_{\mu\nu}$, the equations for the gauge potential A_μ , and the scalar fields ϕ can be written as, respectively,

$$R_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{-\sqrt{3}\phi} \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F^2 \right), \quad (3)$$

$$\nabla_\mu (e^{-\sqrt{3}\phi} F^{\mu\nu}) = 0, \quad (4)$$

$$\nabla_\mu \nabla^\mu \phi = -\frac{\sqrt{3}}{4} e^{-\sqrt{3}\phi} F^2. \quad (5)$$

The 4D metric, the gauge potential, and the scalar field of the multicentered rotating black hole solution constructed by Teo and Wan [7] can be written, respectively, as

$$ds_{(4)}^2 = g_{\mu\nu} dx^\mu dx^\nu = -(H_+ H_-)^{-\frac{1}{2}} (dt + \omega^0)^2 + (H_+ H_-)^{\frac{1}{2}} dx \cdot dx, \quad (6)$$

$$A = \frac{\sqrt{2}}{H_-} [-(1+f)f - 2g] dt + (1+f)\omega^0 + H_- \tilde{\omega}^5, \quad (7)$$

$$\phi = \frac{\sqrt{3}}{2} \ln \frac{H_+}{H_-}, \quad (8)$$

where $dx \cdot dx$ is the metric of the 3D Euclid space $\mathbb{E}^3$ with $\mathbf{x} = (x, y, z)$. The functions $H_\pm$, 1-forms $\omega^0, \tilde{\omega}^5$ on $\mathbb{E}_3$ are given by

$$H_\pm = (1+f)^2 \pm 2g, \quad (9)$$

$$\omega^0 = - \sum_{i=1}^N \frac{2J_i [(y-y_i)dx - (x-x_i)dy]}{|\mathbf{x} - \mathbf{x}_i|^3}, \quad (10)$$

$$\tilde{\omega}^5 = \sqrt{2} \sum_{i=1}^N \frac{M_i (z-z_i) [(y-y_i)dx - (x-x_i)dy]}{|\mathbf{x} - \mathbf{x}_i| [(x-x_i)^2 + (y-y_i)^2]}, \quad (11)$$

with the two harmonic functions f and g having point sources at the positions $\mathbf{x} = \mathbf{x}_i := (x_i, y_i, z_i)$ ($i = 1, \dots, N$) on $\mathbb{E}^3$,

$$f = \sum_{i=1}^N \frac{M_i}{|\mathbf{x} - \mathbf{x}_i|}, \quad g = \sum_{i=1}^N \frac{J_i (z - z_i)}{|\mathbf{x} - \mathbf{x}_i|^3}. \quad (12)$$

This solution describes an asymptotically flat, stationary multicentered rotating dyonic black holes, with each having an extremal horizon. The i th black hole at the position $\mathbf{x} = \mathbf{x}_i$ on $\mathbb{E}^3$ carries the mass M_i , spin angular momentum J_i , equal electric and magnetic charges Q_i and P_i , respectively, given by $Q_i = P_i = M_i/\sqrt{2}$. Furthermore, the regularity of the metric on the horizon requires the condition

$$|J_i| < \frac{M_i^2}{2}, \quad (13)$$

since the horizon area $4\pi\sqrt{M_i^4 - 4J_i^2}$ vanishes if this is saturated. Moreover, as shown in Ref. [7], under the condition (13), the spacetime is free from CTCs on and outside the horizon. At the limit as $J_i \rightarrow 0$ for all i , the scalar field ϕ vanishes, and consequently, the Majumdar-Papapetrou solution describing static multiple dyonic black holes is restored.

III. 5D KALUZA-KLEIN THEORY AND NONLINEAR SIGMA MODEL

Since we wish to consider stationary solutions, in addition to the Killing vector $\partial/\partial x^5$, we assume the existence of a timelike Killing vector field $\partial/\partial t$. With

these two commuting Killing vectors, the 5D Einstein equations reduce to a 3D system of gravity coupled to scalar fields [17]. We then review how this system of scalar fields can be described by a nonlinear sigma model. In general, solving the resulting equations is challenging because the scalar fields are coupled to 3D gravity. However, under the assumption that the 3D geometry is flat, Clément [18,19] showed that a special class of solutions can be constructed using two harmonic functions. We also present the necessary equations for our analysis, following Clément's formulation.

A. 5D Einstein equation with two commuting Killing vectors

Let ξ_a ($a = 0, 5$) be two mutually commuting Killing vector fields, so that $[\xi_a, \xi_b] = 0$, $\mathcal{L}_{\xi_a} g = 0$. Then, introducing the coordinates x^a as Killing parameters of ξ_a (so that $\xi_a = \partial/\partial x^a$), one can express the metric g as

$$ds^2 = \lambda_{ab} (dx^a + \omega^a_i dx^i) (dx^b + \omega^b_j dx^j) + |\tau|^{-1} h_{ij} dx^i dx^j, \quad (14)$$

where the functions $\tau := -\det(\lambda_{ab})$, ω^a_i , h_{ij} ($i = 1, 2, 3$) are independent of the coordinates x^a . We also introduce the twist 1-forms by

$$v_a = *(\xi_0 \wedge \xi_5 \wedge d\xi_a). \quad (15)$$

Then, we can write the exterior derivative of V_a as

$$dv_a = 2 * (\xi_0 \wedge \xi_5 \wedge R(\xi_a)), \quad (16)$$

where $R(\xi_a)$ is the Ricci 1-form. Therefore, since $R(\xi_a) = 0$ from the vacuum Einstein equation, there exists locally the twist potentials V_a that satisfy

$$dV_a = v_a, \quad (17)$$

which can be written as

$$\partial_k V_a = \tau \sqrt{|h|} \lambda_{ab} \varepsilon_{kij} h^{im} h^{jn} \partial_m \omega^b_n. \quad (18)$$

Then, the vacuum Einstein equations can be expressed as the field equations for the five scalar fields $\{\lambda_{ab}, V_a\}$,

$$\Delta_h \lambda_{ab} = \lambda^{cd} h^{ij} \frac{\partial \lambda_{ac}}{\partial x^i} \frac{\partial \lambda_{bd}}{\partial x^j} + \tau^{-1} h^{ij} \frac{\partial V_a}{\partial x^i} \frac{\partial V_b}{\partial x^j}, \quad (19)$$

$$\Delta_h V_a = \tau^{-1} h^{ij} \frac{\partial \tau}{\partial x^i} \frac{\partial V_a}{\partial x^j} + \lambda^{bc} h^{ij} \frac{\partial \lambda_{ab}}{\partial x^i} \frac{\partial V_c}{\partial x^j}, \quad (20)$$

and the Einstein equations for the 3D metric h_{ij} , which is coupled with the five scalar fields,

$$R_{ij}^h = \frac{1}{4} \lambda^{ab} \lambda^{cd} \frac{\partial \lambda_{ac}}{\partial x^i} \frac{\partial \lambda_{bd}}{\partial x^j} + \frac{1}{4} \tau^{-2} \frac{\partial \tau}{\partial x^i} \frac{\partial \tau}{\partial x^j} - \frac{1}{2} \tau^{-1} \lambda^{ab} \frac{\partial V_a}{\partial x^i} \frac{\partial V_b}{\partial x^j}, \quad (21)$$

where Δ_h is the Laplacian, and R_{ij}^h denotes the Ricci tensor with respect to h_{ij} , respectively.

B. $SL(3, \mathbb{R})$ nonlinear sigma model

As a consequence of the existence of two isometries, we have five scalar fields λ_{ab}, V_a ($a = 0, 5$), which we denote collectively by coordinates $\Phi^A = (\lambda_{ab}, V_a)$. Then, we can find that the equations of motion (19)–(21) are derived from the following action for sigma model Φ^A coupled with three-dimensional gravity with respect to the metric h_{ij} :

$$S = \int \left(R^h - G_{AB} \frac{\partial \Phi^A}{\partial x^i} \frac{\partial \Phi^B}{\partial x^j} h^{ij} \right) \sqrt{|h|} d^3x, \quad (22)$$

where the target space metric G_{AB} is given by

$$G_{AB} d\Phi^A d\Phi^B = \frac{1}{4} \text{Tr}(\lambda^{-1} d\lambda \lambda^{-1} d\lambda) + \frac{1}{4} \tau^{-2} d\tau^2 - \frac{1}{2} \tau^{-1} v^T \lambda^{-1} v, \quad (23)$$

with $\lambda = (\lambda_{ab})$, $v = (v_0, v_5)^T$, and $v = dV$. It can be actually shown that varying the action by h_{ij} derives Eq. (21),

$$R_{ij}^h = G_{AB} \frac{\partial \Phi^A}{\partial x^i} \frac{\partial \Phi^B}{\partial x^j}, \quad (24)$$

and also varying the action by Φ^A can derive Eqs. (19) and (20),

$$\Delta_h \Phi^A + h^{ij} \Gamma_{BC}^A \frac{\partial \Phi^B}{\partial x^i} \frac{\partial \Phi^C}{\partial x^j} = 0, \quad (25)$$

where Γ_{BC}^A is the Christoffel symbol with respect to G_{AB} .

C. Coset matrix

Maison [17] showed that the action (22) for the five scalar fields Φ^A is invariant under the global $SL(3, \mathbb{R})$ transformation, introducing the $SL(3, \mathbb{R})$ matrix χ defined by the 3×3 matrix:

$$\chi = \begin{pmatrix} \lambda_{ab} - \frac{V_a V_b^T}{\tau} & \frac{V_a}{\tau} \\ \frac{V_b^T}{\tau} & -\frac{1}{\tau} \end{pmatrix}, \quad (26)$$

which is symmetric $\chi^T = \chi$ and unimodular $\det(\chi) = 1$. Then, the action (22) can be expressed in terms of the matrix χ as

$$S = \int \left(R^h - \frac{1}{4} h^{ij} \text{tr}(\chi^{-1} \partial_i \chi \chi^{-1} \partial_j \chi) \right) \sqrt{|h|} d^3x, \quad (27)$$

which is invariant under the transformation

$$\chi \rightarrow \chi' = g \chi g^T, \quad h \rightarrow h, \quad (28)$$

with $g \in SL(3, \mathbb{R})$. The equations of motion (21), (19), and (20) can be written, in terms of χ , as

$$d\star_h(\chi^{-1} d\chi) = 0, \quad (29)$$

$$R_{ij}^h = \frac{1}{4} \text{tr}(\chi^{-1} \partial_i \chi \chi^{-1} \partial_j \chi). \quad (30)$$

Thus, the existence of two commuting Killing vector fields reduces the 5D vacuum Einstein theory to a 3D nonlinear sigma model with an $SL(3, \mathbb{R})$ target space symmetry. If both Killing vectors are spacelike, it is described by the $SL(3, \mathbb{R})/SO(3)$ sigma model coupled to gravity, while if one of the two Killing vectors is timelike, the symmetry is replaced with $SL(3, \mathbb{R})/SO(2, 1)$.

D. Asymptotically flat solutions of Clément

Since Eqs. (29) and (30) are mutually coupled, solving them is not straightforward. However, when the 3D metric h_{ij} is a Euclid space metric $\mathbb{E}^3$,

$$h_{ij} dx^i dx^j = d\mathbf{x} \cdot d\mathbf{x}, \quad (31)$$

with the position vector $\mathbf{x} = (x, y, z)$ on $\mathbb{E}^3$, they can be simplified to

$$\partial_i(\chi^{-1} \partial^i \chi) = 0, \quad (32)$$

$$\text{tr}(\chi^{-1} \partial_i \chi \chi^{-1} \partial_i \chi) = 0. \quad (33)$$

The general solution depending on two potentials originally obtained by Clément [18,19] is given by

$$\chi = \eta e^{fA} e^{gA^2}, \quad (34)$$

where f and g are harmonic functions on $\mathbb{E}^3$, and η and A are 3×3 constant matrices. To ensure that the 4D metric $ds_{(4)}^2 = g_{\mu\nu} dx^\mu dx^\nu$ is asymptotically flat, the matrix η must be chosen as

$$\eta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (35)$$

if the two harmonic functions vanish at infinity. If the matrix A satisfies

$$A^T = \eta A \eta, \quad \text{tr}(A) = 0, \quad \text{tr}(A^2) = 0, \quad (36)$$

then the matrix χ has the required properties: It is symmetric ($\chi^T = \chi$), unimodular ($\det \chi = 1$), and it satisfies the constraint (33).

IV. MULTICENTERED ROTATING BLACK HOLES WITH UNEQUAL MAGNETIC AND ELECTRIC CHARGES

Rasheed [14] and Larsen [15] constructed the exact solution describing the most general dyonic rotating black holes in 5D Kaluza-Klein theory. This solution possesses two distinct branches of the extremal limit: the slowly rotating extremal limit and the fast rotating extremal limit. In particular, the former falls within the class of Clément's solutions, whereas the latter does not. The slowly rotating extremal limit can be expressed in terms of the solution with two harmonic functions, Eq. (34), each associated with a single point source. In this section, by generalizing these harmonic functions to multiple point sources, we construct an exact solution describing multicentered rotating dyonic black holes with unequal electric and magnetic charges in 4D Einstein-Maxwell-dilaton theory. This extends the construction of Ref. [7], where the charges were taken to be equal.

A. The slowly rotating extremal limit of the Rasheed-Larsen solution

The 4D metric, gauge field, and scalar field of the Rasheed-Larsen solution [14,15] are given, respectively, by

$$ds_{(4)}^2 = -\frac{H_3}{\sqrt{H_1 H_2}}(dt + \mathbf{B})^2 + \sqrt{H_1 H_2} \left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{H_3} \sin^2 \theta d\phi^2 \right), \quad (37)$$

$$\mathbf{B} = \sqrt{pq} \frac{(pq + 4m^2)r - m(p - 2m)(q - 2m)}{2m(p + q)H_3} \times a \sin^2 \theta d\phi, \quad (38)$$

$$\begin{aligned} \mathbf{A} = & - \left[2Q \left(r + \frac{p-2m}{2} \right) + \sqrt{\frac{q^3(p^2-4m^2)}{4m^2(p+q)}} a \cos \theta \right] H_2^{-1} dt \\ & - \left[2P(H_2 + a^2 \sin^2 \theta) \cos \theta + \sqrt{\frac{p(q^2-4m^2)}{4m^2(p+q)^3}} \right. \\ & \times \{ (p+q)(pr - m(p-2m)) + q(p^2 - 4m^2) \} \\ & \left. \times a \sin^2 \theta \right] H_2^{-1} d\phi, \end{aligned} \quad (39)$$

$$e^{-2\phi} = \sqrt{\frac{H_2}{H_1}}, \quad (40)$$

where

$$H_1 = r^2 + a^2 \cos^2 \theta + r(p - 2m) + \frac{p(p - 2m)(q - 2m)}{2(p + q)} - \frac{p\sqrt{(q^2 - 4m^2)(p^2 - 4m^2)}}{2m(p + q)} a \cos \theta, \quad (41)$$

$$H_2 = r^2 + a^2 \cos^2 \theta + r(q - 2m) + \frac{q(p - 2m)(q - 2m)}{2(p + q)} + \frac{q\sqrt{(q^2 - 4m^2)(p^2 - 4m^2)}}{2m(p + q)} a \cos \theta, \quad (42)$$

$$H_3 = r^2 + a^2 \cos^2 \theta - 2mr, \quad (43)$$

$$\Delta = r^2 + a^2 - 2mr. \quad (44)$$

This solution is characterized by four parameters (m, a, q, p) , which are related to the physical mass M , angular momentum J , electric charge Q , and magnetic charge P through

$$\begin{aligned} M &= \frac{p + q}{4}, \quad J = \frac{\sqrt{pq}(pq + 4m^2)}{4m(p + q)} a, \\ Q^2 &= \frac{q(q^2 - 4m^2)}{4(p + q)}, \quad P^2 = \frac{p(p^2 - 4m^2)}{4(p + q)}. \end{aligned} \quad (45)$$

In order that the solutions have a regular horizon, they must always satisfy the following bound:

$$\left(\frac{P}{2M} \right)^{\frac{2}{3}} + \left(\frac{Q}{2M} \right)^{\frac{2}{3}} \leq 1. \quad (46)$$

The solution has two distinct branches with extremal horizons:

- (1) Slowly rotating extremal limit $|J| < |PQ|$

This limit is achieved by taking the limit $m \rightarrow 0$, $a \rightarrow 0$ with $j := a/m$ fixed. This solution is parametrized by the three parameters p , q , and j . The solution saturates the bound as

$$\left(\frac{P}{2M} \right)^{\frac{2}{3}} + \left(\frac{Q}{2M} \right)^{\frac{2}{3}} = 1. \quad (47)$$

Although the angular momentum J does not vanish, the angular velocity of the horizon vanishes, and therefore no ergoregion exist. The entropy is given by

$$S = 2\pi \sqrt{P^2 Q^2 - J^2}. \quad (48)$$

- (2) Fast rotating extremal limit $|J| > |PQ|$

This limit is obtained by taking $a \rightarrow m$. The resulting solution is parametrized by three quantities p , q , and m . Although the explicit relations among the physical charges M , J , P , and Q are not easy to express, they satisfy the inequality

$$\left(\frac{P}{2M}\right)^{\frac{2}{3}} + \left(\frac{Q}{2M}\right)^{\frac{2}{3}} < 1. \quad (49)$$

Unlike the slowly rotating limit, the angular velocity of the horizon does not vanish, and therefore an ergoregion exists around the horizon. The entropy is given by

$$S = 2\pi\sqrt{J^2 - P^2Q^2}. \quad (50)$$

The slowly rotating limit corresponds to the solution of Eqs. (32) and (33) with (31), whereas the fast rotating limit does not correspond to these solutions. Therefore, in what follows, we restrict our attention to the slowly rotating limit only. In the slowly rotating extremal limit, the 5D metric is simplified as

$$\begin{aligned} ds_{(5)}^2 = & \frac{H_2}{H_1} \left[dx^5 - \left\{ 2\left(r + \frac{p}{2}\right) - pj \cos \theta \right\} \frac{Q}{H_2} dt \right. \\ & - \left\{ 2H_2 \cos \theta - q\left(r + \frac{pq}{p+q}\right) j \sin^2 \theta \right\} \frac{P}{H_2} d\phi \left. \right]^2 \\ & - \frac{r^2}{H_2} \left(dt + \frac{2jPQ \sin^2 \theta d\phi}{r} \right)^2 \\ & + H_1 \left(\frac{dr^2}{r^2} + d\theta^2 + \sin^2 \theta d\phi^2 \right), \end{aligned} \quad (51)$$

with the two functions

$$H_1 = r^2 + pr + \frac{p^2 q (1 + j \cos \theta)}{2(p+q)}, \quad (52)$$

$$H_2 = r^2 + qr + \frac{q^2 p (1 - j \cos \theta)}{2(p+q)}. \quad (53)$$

The 4D metric, the gauge, and scalar fields are written as

$$\begin{aligned} ds_{(4)}^2 = & -\frac{r^2}{(H_1 H_2)^{1/2}} (dt + \omega^0)^2 \\ & + \frac{(H_1 H_2)^{1/2}}{r^2} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)], \end{aligned} \quad (54)$$

$$\begin{aligned} A = & -\left[2\left(r + \frac{p}{2}\right) - pj \cos \theta \right] \frac{Q}{H_2} dt \\ & - \left[2H_2 \cos \theta - q\left(r + \frac{pq}{p+q}\right) j \sin^2 \theta \right] \frac{P}{H_2} d\phi, \end{aligned} \quad (55)$$

$$e^{\frac{2\phi}{\sqrt{3}}} = \frac{H_2}{H_1}. \quad (56)$$

The physical charges (45) are written as

$$\begin{aligned} M &= \frac{p+q}{4}, \quad J = jPQ, \\ P^2 &= \frac{p^3}{4(p+q)}, \quad Q^2 = \frac{q^3}{4(p+q)}, \end{aligned} \quad (57)$$

which we can confirm saturate the bound (46).

This slowly rotating extremal solution belongs to the class of solutions (34) depending on two harmonic functions of Clément, where the corresponding harmonic functions f , g and the matrix A are given by

$$f = \frac{M}{r}, \quad g = \frac{JM^2 \cos \theta}{2PQ r^2}, \quad (58)$$

$$A = 4 \begin{pmatrix} -\frac{q}{p+q} & \left(\frac{q}{p+q}\right)^{\frac{3}{2}} & 0 \\ -\left(\frac{q}{p+q}\right)^{\frac{3}{2}} & -\frac{p-q}{p+q} & \left(\frac{p}{p+q}\right)^{\frac{3}{2}} \\ 0 & -\left(\frac{p}{p+q}\right)^{\frac{3}{2}} & \frac{p}{p+q} \end{pmatrix}. \quad (59)$$

Below, we show that under the assumption that the 3D metric is flat, $h_{ij} dx^i dx^j = d\mathbf{x} \cdot d\mathbf{x}$, the coset matrix $\chi = \eta e^{A^f} e^{A^g}$ with (58) and (59) reproduces the slowly rotating extremal limit of the Rasheed-Larsen solution. Since two parameters p and q (or P and Q) are not independent, it is useful to introduce a new parameter α defined by

$$\cos^2 \alpha = \frac{p}{p+q}, \quad \sin^2 \alpha = \frac{q}{p+q}, \quad (60)$$

where it is useful to know that (p, q) or (P, Q) can be expressed as

$$\begin{aligned} (p, q) &= (4M \cos^2 \alpha, 4M \sin^2 \alpha), \\ (P, Q) &= (2M \cos^3 \alpha, 2M \sin^3 \alpha). \end{aligned} \quad (61)$$

From (58) and (59), the coset matrix χ takes the form

$$\chi = \begin{pmatrix} -8s^4c^2f^2 + 4s^2f - 16c^6g + 32c^4g - 16c^2g - 1 & 4s^3(2c^2f^2 + 4c^2g - f) & -8c^3s^3(f^2 + 2g) \\ 4s^3(2c^2f^2 + 4c^2g - f) & -8s^2c^2f^2 + (4 - 8c^2)f + 16c^4g - 16c^2g + 1 & 4c^3(2s^2f^2 + 4s^2g + f) \\ -8c^3s^3(f^2 + 2g) & 4c^3(2s^2f^2 + 4s^2g + f) & 8c^6(f^2 + 2g) - 8c^4(f^2 + 2g) - 4c^2f - 1 \end{pmatrix} \quad (62)$$

with $(c, s) := (\cos \alpha, \sin \alpha)$. Hence, from Eq. (26), we can read off the conformal factor τ and the scalar fields (λ_{ab}, V_a) as

$$\tau = \frac{1}{8s^2c^4(f^2 + 2g) + 4c^2f + 1}, \quad (63)$$

$$\lambda_{00} = -\tau[-8s^2c^2f^2 + (8c^2 - 4)f + 16c^2s^2g + 1], \quad (64)$$

$$\lambda_{05} = -4\tau s^3(2c^2f^2 + f - 4c^2g), \quad (65)$$

$$\lambda_{55} = -\tau[-8s^4c^2f^2 - 4s^2f + 16c^2s^4g - 1], \quad (66)$$

$$V_0 = -8\tau c^3s^3(f^2 + 2g), \quad (67)$$

$$V_5 = 4\tau(2s^2f^2 + f + 4s^2g). \quad (68)$$

From Eqs. (67) and (68), the 1-forms ω^0 and ω^5 can be expressed as

$$\nabla \times \omega^0 = -2 \sin^3 2\alpha \nabla g, \quad (69)$$

$$\nabla \times \omega^5 = -4 \cos^3 \alpha (\nabla f + 4 \sin^2 \alpha \nabla g). \quad (70)$$

If we define $\tilde{\omega}^5 := \omega^5 - (\sin \alpha)^{-1} \omega^0$, then

$$\nabla \times \tilde{\omega}^5 = -4 \cos^3 \alpha \nabla f. \quad (71)$$

From Eq. (58), these can be solved as

$$\omega^0 = \frac{2J}{r} \sin^2 \theta d\phi = -\frac{2J}{r^3} [ydx - xdy], \quad (72)$$

$$\tilde{\omega}^5 = -2P \cos \theta d\phi = \frac{2Pz}{r} \frac{ydx - xdy}{x^2 + y^2}. \quad (73)$$

Thus, in terms of the spherical coordinates (r, θ, ϕ) defined by $(x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$, the 5D metric (14) can be obtained as

$$ds^2 = \frac{H_-}{H_+} \left[dx^5 - \frac{dt}{\sin \alpha} + \frac{1 + f \sin^2 2\alpha}{\sin \alpha H_-} \left(dt + \frac{2J}{r} \sin^2 \theta d\phi \right) - 2P \cos \theta d\phi \right]^2 - \frac{1}{H_-} \left(dt + \frac{2J}{r} \sin^2 \theta d\phi \right)^2 + H_+ [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)], \quad (74)$$

where the functions $H_{\pm}$ are given by

$$H_{\pm} = 1 + 2f + \sin^2 2\alpha f^2 + 2\sin^2 2\alpha \cos 2\alpha g \pm (2\sin^2 2\alpha g + 2\cos 2\alpha f + \cos 2\alpha \sin^2 2\alpha f^2). \quad (75)$$

This coincides with the metric (51) corresponding to the slowly rotating limit of the Rasheed-Larsen solution. It should be emphasized, however, that in contrast to this case, the fast rotating limit does not belong to Clément's class of solutions, since the three-dimensional metric h_{ij} is not flat.

B. Multicentered rotating black hole solutions

We replace the harmonic functions f and g in Eq. (58) with the following generalized forms with multicenters:

$$f = \sum_{i=1}^N \frac{M_i}{|\mathbf{x} - \mathbf{x}_i|}, \quad (76)$$

$$g = \sum_{i=1}^N \frac{J_i M_i^2 (z - z_i)}{2P_i Q_i |\mathbf{x} - \mathbf{x}_i|^3}, \quad (77)$$

where the constants $M_i, J_i, Q_i P_i (i = 1, \dots, N)$ denote the mass, angular momentum, and electric and magnetic charges of each black hole, with the following relation at each center of the magnetic and electric charges:

$$\frac{P_i}{2M_i} = \cos^3 \alpha, \quad \frac{Q_i}{2M_i} = \sin^3 \alpha (i = 1, 2, \dots, N). \quad (78)$$

Using Eqs. (69) and (71), we can show that the 1-forms ω^0 and $\tilde{\omega}^5$ take the following explicit forms:

$$\omega^0 = -\sum_{i=1}^N \frac{2J_i}{|\mathbf{x} - \mathbf{x}_i|^3} [(y - y_i)dx - (x - x_i)dy], \quad (79)$$

$$\tilde{\omega}^5 = \sum_{i=1}^N \frac{2P_i (z - z_i)}{|\mathbf{x} - \mathbf{x}_i|} \frac{(y - y_i)dx - (x - x_i)dy}{(x - x_i)^2 + (y - y_i)^2}. \quad (80)$$

Consequently, the 5D metric for the multicentered black hole solution takes the form

$$ds^2 = \frac{H_-}{H_+} \left[dx^5 - \frac{dt}{\sin \alpha} + \frac{1 + f \sin^2 2\alpha}{\sin \alpha H_-} (dt + \omega^0) + \tilde{\omega}^5 \right]^2 - \frac{1}{H_-} (dt + \omega^0)^2 + H_+ dx \cdot dx, \quad (81)$$

with Eqs. (75)–(77), (79), and (80). On the other hand, the 4D metric, gauge field, and scalar field after dimensional reduction are expressed as

$$ds_{(4)}^2 = -\frac{1}{\sqrt{H_+ H_-}} (dt + \omega^0)^2 + \sqrt{H_+ H_-} dx \cdot dx, \quad (82)$$

$$A = \left(-\frac{1}{\sin \alpha} + \frac{1 + f \sin^2 2\alpha}{\sin \alpha H_-} \right) dt + \frac{1 + f \sin^2 2\alpha}{\sin \alpha H_-} \omega^0 + \tilde{\omega}^5, \quad (83)$$

$$e^{\frac{2\phi}{\sqrt{3}}} = \frac{H_+}{H_-}. \quad (84)$$

From Eqs. (77) and (79), we see that each black hole has either an aligned or antialigned spin orientation along the z axis.

V. PROPERTIES OF THE MULTICENTERED ROTATING BLACK HOLE SOLUTION

In this section, we see that this solution is regular and describes asymptotically flat, multicentered rotating dyonic black holes, each possessing an extremal horizon. We demonstrate that curvature singularities are confined inside the horizons and do not occur on or outside them. Furthermore, we prove the absence of CTCs in the exterior region as well as on the horizons.

A. Near-horizon geometry

The 4D metric apparently diverges at the points $\mathbf{x} = \mathbf{x}_i$, but we show that they correspond to smooth Killing horizons, provided the slowly rotating conditions $|J_i| < |P_i Q_i|$ for all i . Introducing $r := |\mathbf{x} - \mathbf{x}_i|$ and the standard spherical coordinates $(x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$, we consider the limit of $r \rightarrow 0$. In this limit, the 4D metric, Maxwell gauge potential, and scalar field at $r \rightarrow 0$ behave as

$$ds_{(4)}^2 \simeq -\frac{r^2}{2\sqrt{P_i^2 Q_i^2 - J_i^2 \cos^2 \theta}} \left[dt + \frac{2J_i}{r} \sin^2 \theta d\phi \right]^2 + 2\sqrt{P_i^2 Q_i^2 - J_i^2 \cos^2 \theta} \left[\frac{dr^2}{r^2} + d\Omega^2 \right], \quad (85)$$

$$A \simeq -\frac{dt}{\sin \alpha} + 2P_i \left(-\cos \theta + \frac{J_i \sin^2 \theta}{P_i Q_i - J_i \cos \theta} \right) d\phi, \quad (86)$$

$$e^{-\frac{2\phi}{\sqrt{3}}} \simeq \tan^2 \alpha \frac{P_i Q_i - J_i \cos \theta}{P_i Q_i + J_i \cos \theta}, \quad (87)$$

where it should be noted that one can set $A_\theta = 0$ and $A_r = 0$ at $r = 0$ by performing the gauge transformation $A \rightarrow A - d\chi$, with $\chi = \int A_\theta(r=0) d\theta + A_r(r=0)r$ [here, $A_\theta(r=0) = 0$ from the outset]. The metric component g_{rr} diverges at $r = 0$, but we can show that this is apparent as follows: In terms of the new coordinates (v, ϕ') given by

$$dt = dv + \left(\frac{a_0}{r^2} + \frac{a_1}{r} \right) dr, \quad d\phi = d\phi' + \frac{b_0}{r} dr, \quad (88)$$

with

$$a_0 = \pm 2\sqrt{P_i^2 Q_i^2 - J_i^2}, \quad (89)$$

$$a_1 = \frac{2M_i^3 \sin^4 2\alpha [1 + (\sum_{j \neq i} M_j |\mathbf{x}_j|^{-1}) \sin^2 2\alpha]}{a_0}, \quad (90)$$

$$b_0 = \frac{2J_i}{a_0}, \quad (91)$$

the apparent divergence can be eliminated. It follows that the null surface $r = 0$ corresponds to the Killing horizon for the Killing vector field $\partial/\partial v$. The near-horizon limit is defined by $v \rightarrow v/\varepsilon$, $r \rightarrow \varepsilon r$, and $\varepsilon \rightarrow 0$, which leads to

$$ds_{(4)}^2 \simeq -\frac{2f(0)^2 \sin^2 \theta}{f(\theta)} \left[d\phi' - \frac{rJ_i}{2f(0)^2} dv \right]^2 - \frac{f(\theta)r^2}{2f(0)^2} dv^2 \mp \frac{2f(\theta)}{f(0)} dv dr + 2f(\theta) d\theta^2, \quad (92)$$

with $f(\theta) := \sqrt{P_i^2 Q_i^2 - J_i^2 \cos^2 \theta}$. This coincides with the near-horizon geometry of the slowly rotating extremal limit of the Rasheed-Larsen black hole with a single horizon.

B. Asymptotic structure

In terms of the standard spherical coordinates $(x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$, the functions $f, g, H_\pm$ and the 1-forms $\omega^0, \tilde{\omega}^5$ behave asymptotically as at $r \rightarrow \infty$,

$$f \simeq \frac{\sum_i M_i}{r} + \mathcal{O}(r^{-2}), \quad (93)$$

$$g \simeq \sum_i \frac{J_i M_i^2 \cos \theta}{2P_i Q_i r^2} + \mathcal{O}(r^{-3}), \quad (94)$$

$$H_{\pm} \simeq 1 + \frac{2(1 \pm \cos 2\alpha) \sum_i M_i}{r} + \mathcal{O}(r^{-2}), \quad (95)$$

and

$$\omega^0 \simeq \left(\frac{2 \sum_i J_i}{r} \sin^2 \theta + \mathcal{O}(r^{-2}) \right) d\phi, \quad (96)$$

$$\tilde{\omega}^5 \simeq \left(-2 \sum_i P_i \cos \theta + \mathcal{O}(r^{-1}) \right) d\phi. \quad (97)$$

The 5D metric at $r \rightarrow \infty$ behaves as

$$\begin{aligned} ds^2 \simeq & \left(1 - \frac{4 \cos 2\alpha \sum_i M_i}{r} \right) \left[dx^5 - 2 \left(\sum_i P_i \right) \cos \theta d\phi \right]^2 \\ & + \left(-1 + \frac{2 \sin^2 \alpha \sum_i M_i}{r} \right) \left[dt + \frac{2 \sum_i J_i}{r} \sin^2 \theta d\phi \right]^2 \\ & + \left(1 + \frac{2 \cos^2 \alpha \sum_i M_i}{r} \right) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)], \end{aligned} \quad (98)$$

indicating that the spacetime asymptotically approaches an S^1 fiber bundle over 4D Minkowski space. On the other hand, at $r \rightarrow \infty$, the dimensionally reduced 4D fields behave as follows:

$$\begin{aligned} ds_{(4)}^2 \simeq & \left(-1 + \frac{2 \sum_i M_i}{r} \right) \left[dt + \frac{2 \sum_i J_i}{r} \sin^2 \theta d\phi \right]^2 \\ & + \left(1 + \frac{2 \sum_i M_i}{r} \right) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)], \end{aligned} \quad (99)$$

$$A \simeq \frac{-2 \sum_i Q_i}{r} dt - 2 \left(\sum_i P_i \right) \cos \theta d\phi, \quad (100)$$

$$\phi \simeq \frac{2\sqrt{3} \cos 2\alpha \sum_i M_i}{r}. \quad (101)$$

Thus, the dimensionally reduced spacetime is asymptotically flat. Arnowitt-Deser-Misner (ADM) mass and ADM angular momentum, and the total electric and magnetic charges are given, respectively, by

$$M = \sum_{i=1}^N M_i, \quad J = \sum_{i=1}^N J_i, \quad (102)$$

$$P = \sum_{i=1}^N P_i = 2M \cos^3 \alpha, \quad Q = \sum_{i=1}^N Q_i = 2M \sin^3 \alpha, \quad (103)$$

where each individual electric and magnetic charge satisfies

$$(P_i, Q_i) = 2M_i (\cos^3 \alpha, \sin^3 \alpha), \quad (104)$$

leading to the identity

$$\left(\frac{P}{2M} \right)^{\frac{2}{3}} + \left(\frac{Q}{2M} \right)^{\frac{2}{3}} = 1. \quad (105)$$

C. Regularity

If curvature singularities exist on or outside the horizons, they appear at points where the metric or its inverse diverges, which happens only on the surfaces $H_+(x, y, z) = 0$ or $H_-(x, y, z) = 0$. Indeed, since the Kretschmann scalar can be calculated as

$$R_{\mu\nu\rho\lambda}^{(4)} R^{(4)\mu\nu\rho\lambda} \sim \frac{1}{(H_+ H_-)^3}, \quad (106)$$

curvature singularities arise precisely on the surfaces where either $H_+(x, y, z) = 0$ or $H_-(x, y, z) = 0$. We can show that such singularities do not exist on and outside the event horizons at $\mathbf{x} = \mathbf{x}_i$ provided that $|J_i| < |P_i Q_i|$. To demonstrate this, it is sufficient to verify that $H_{\pm} > 0$ on and outside the horizons, since asymptotically $H_{\pm} \rightarrow 1 > 0$ as $r \rightarrow \infty$. First, we note that under the assumptions $M_i > 0$, we have $f > 0$, while g can take both signs. Using this, we write for $(c_2, s_2) := (\cos 2\alpha, \sin 2\alpha)$, the functions $H_{\pm}$ as

$$\begin{aligned} H_{\pm} &= 1 + 2(1 \pm c_2)f + s_2^2(1 \pm c_2)f^2 + 2s_2^2(c_2 \pm 1)g \\ &> s_2^2(1 \pm c_2) + 2s_2^2(1 \pm c_2)f + s_2^2(1 \pm c_2)f^2 \\ &\quad + 2s_2^2(c_2 \pm 1)g \mp s_2^2 c_2 \\ &= s_2^2(1 \pm c_2)[(1 + f)^2 \pm 2g] \mp s_2^2 c_2 \\ &\geq s_2^2(1 \pm c_2)[(1 + f)^2 - 2|g|] \mp s_2^2 c_2 \\ &> s_2^2(1 \pm c_2) \mp s_2^2 c_2 \\ &= s_2^2 > 0, \end{aligned} \quad (107)$$

where we have used the inequality

$$\begin{aligned} (1 + f)^2 - 2|g| &= \left(1 + \sum_i f_i \right)^2 - 2 \left| \sum_i g_i \right| \\ &> 1 + \sum_i f_i^2 - 2 \sum_i |g_i| \\ &> 1 + \sum_i M_i^2 \left[\frac{1 - |J_i/P_i Q_i|}{|\mathbf{x} - \mathbf{x}_i|^2} \right] \\ &> 1, \end{aligned} \quad (108)$$

with

$$f_i := \frac{M_i}{|\mathbf{x} - \mathbf{x}_i|}, \quad g_i := \frac{J_i M_i^2 (z - z_i)}{2P_i Q_i |\mathbf{x} - \mathbf{x}_i|^3}. \quad (109)$$

D. Absence of CTCs

Here, we show the nonexistence of CTCs everywhere on and outside the horizons, provided that the inequality $|J_i| < |P_i Q_i|$ holds for each $i = 1, \dots, N$. The condition for the absence of CTCs is equivalent to requiring that the 2D metric $g_{(2)IJ} dx^I dx^J$ ($x^I, x^J = x, y$) is positive definite everywhere on and outside the horizons. This condition can be expressed in terms of its trace and determinant as follows:

$$\text{tr } g_{(2)} = g_{xx} + g_{yy} = \frac{2H_+ H_- - (\omega_x^0)^2 - (\omega_y^0)^2}{\sqrt{H_+ H_-}} > 0, \quad (110)$$

$$\det g_{(2)} = g_{xx} g_{yy} - g_{xy}^2 = H_+ H_- - (\omega_x^0)^2 - (\omega_y^0)^2 > 0. \quad (111)$$

Since $H_+ H_- > 0$, it is straightforward to verify that if the determinant condition (111) is satisfied, then the trace condition (110) is also automatically satisfied. Therefore, it is sufficient to consider only the inequality (111).

The determinant can be expressed as

$$\det g_{(2)} = 1 + 4f + s_2^2(6f^2 + 4c_2 g) + 4s_2^4 f^3 + s_2^6(f^4 - 4g^2) - (\omega_x^0)^2 - (\omega_y^0)^2 \quad (112)$$

$$\begin{aligned} &> 1 + \sum_i [s_2^6(f_i^4 - 4g_i^2) - (\omega_{xi}^0)^2 - (\omega_{yi}^0)^2] \\ &+ 2 \sum_{i \neq j} [s_2^6(3f_i^2 f_j^2 - 4g_i g_j) - (\omega_{xi}^0)(\omega_{xi}^0) - 2(\omega_{iy}^0)(\omega_{jy}^0)], \end{aligned} \quad (113)$$

with

$$\omega_{xi}^0 = \frac{2J_i(y - y_i)}{|\mathbf{x} - \mathbf{x}_i|^3}, \quad \omega_{yi}^0 = \frac{-2J_i(x - x_i)}{|\mathbf{x} - \mathbf{x}_i|^3}, \quad (114)$$

where the second and third summations can be shown to be positive under the conditions $|J_i| < |P_i Q_i|$ ($i = 1, \dots, N$). Indeed,

$$\begin{aligned} s_2^6(f_i^4 - 4g_i^2) - (\omega_{xi}^0)^2 - (\omega_{yi}^0)^2 &= s_2^6 \left(\frac{M_i^4}{|\mathbf{x} - \mathbf{x}_i|^4} - \frac{J_i^2 M_i^4 (z - z_i)^2}{P_i^2 Q_i^2 |\mathbf{x} - \mathbf{x}_i|^6} \right) - \frac{4J_i^2 (y - y_i)^2}{|\mathbf{x} - \mathbf{x}_i|^6} - \frac{4J_i^2 (x - x_i)^2}{|\mathbf{x} - \mathbf{x}_i|^6} \\ &= s_2^6 \left(\frac{M_i^4}{|\mathbf{x} - \mathbf{x}_i|^4} - \frac{J_i^2 M_i^4 (z - z_i)^2}{P_i^2 Q_i^2 |\mathbf{x} - \mathbf{x}_i|^6} \right) - s_2^6 \frac{J_i^2 M_i^4 [(x - x_i)^2 + (y - y_i)^2]}{P_i^2 Q_i^2 |\mathbf{x} - \mathbf{x}_i|^6} \\ &= s_2^6 \frac{M_i^4 (1 - |J_i/P_i Q_i|^2)}{|\mathbf{x} - \mathbf{x}_i|^4} > 0, \end{aligned} \quad (115)$$

and

$$\begin{aligned} &s_2^6(3f_i^2 f_j^2 - 4g_i g_j) - (\omega_{xi}^0)(\omega_{xj}^0) - (\omega_{yi}^0)(\omega_{yj}^0) \\ &= s_2^6 \left(\frac{3M_i^2 M_j^2}{|\mathbf{x} - \mathbf{x}_i|^2 |\mathbf{x} - \mathbf{x}_j|^2} - \frac{J_i J_j M_i^2 M_j^2 (z - z_i)(z - z_j)}{P_i P_j Q_i Q_j |\mathbf{x} - \mathbf{x}_i|^3 |\mathbf{x} - \mathbf{x}_j|^3} \right) - \frac{4J_i J_j [(x - x_i)(x - x_j) + (y - y_i)(y - y_j)]}{|\mathbf{x} - \mathbf{x}_i|^3 |\mathbf{x} - \mathbf{x}_j|^3} \\ &= s_2^6 \left(\frac{3M_i^2 M_j^2}{|\mathbf{x} - \mathbf{x}_i|^2 |\mathbf{x} - \mathbf{x}_j|^2} - \frac{J_i J_j M_i^2 M_j^2 (z - z_i)(z - z_j)}{P_i P_j Q_i Q_j |\mathbf{x} - \mathbf{x}_i|^3 |\mathbf{x} - \mathbf{x}_j|^3} \right) - s_2^6 \frac{J_i J_j M_i^2 M_j^2 [(x - x_i)(x - x_j) + (y - y_i)(y - y_j)]}{P_i P_j Q_i Q_j |\mathbf{x} - \mathbf{x}_i|^3 |\mathbf{x} - \mathbf{x}_j|^3} \\ &= s_2^6 \left(\frac{3M_i^2 M_j^2}{|\mathbf{x} - \mathbf{x}_i|^2 |\mathbf{x} - \mathbf{x}_j|^2} - \frac{J_i J_j M_i^2 M_j^2 (\mathbf{x} - \mathbf{x}_i) \cdot (\mathbf{x} - \mathbf{x}_j)}{P_i P_j Q_i Q_j |\mathbf{x} - \mathbf{x}_i|^3 |\mathbf{x} - \mathbf{x}_j|^3} \right) \\ &> s_2^6 \left(\frac{3M_i^2 M_j^2}{|\mathbf{x} - \mathbf{x}_i|^2 |\mathbf{x} - \mathbf{x}_j|^2} - \frac{|J_i| |J_j| M_i^2 M_j^2}{|P_i| |P_j| |Q_i| |Q_j| |\mathbf{x} - \mathbf{x}_i|^2 |\mathbf{x} - \mathbf{x}_j|^2} \right) \\ &> s_2^6 \frac{3M_i^2 M_j^2 (1 - |J_i/P_i Q_i| |J_j/P_j Q_j|)}{|\mathbf{x} - \mathbf{x}_i|^2 |\mathbf{x} - \mathbf{x}_j|^2} > 0. \end{aligned} \quad (116)$$

The positivity of these equations implies that $\det g_{(2)} > 1$. Therefore, no CTCs exist on or outside the horizons.

E. Parameter region for physical solutions

From Eq. (78) and the slowly rotating condition $|J_i| < |P_i Q_i|$, it follows that when $\alpha = 0$ or $\alpha = \pi/2$ —

corresponding to $Q_i = 0$ or $P_i = 0$, respectively—the angular momenta J_i must vanish. In these cases, the horizon areas shrink to zero and the solutions become singular. Therefore, we restrict our analysis to the range $0 < \alpha < \pi/2$.

As discussed in the subsections, the regularity and the absence of CTCs on and outside the horizons require that

the parameters must be subjected to

$$M_i > 0, \quad |J_i| < P_i Q_i, \quad 0 < \alpha < \frac{\pi}{2}. \quad (117)$$

When $\alpha \rightarrow \pi/4$, the electric and magnetic charges become equal $Q_i = P_i = \sqrt{2}M_i$ for all i , and then the solution exactly coincides with the Teo-Wan solution [7]. In addition, when $J_i = 0$ for all i , the scalar field vanishes, and the solution reduces to the static Majumdar-Papapetrou black hole solution with equal electric and magnetic charges in Einstein-Maxwell theory [20]. On the other hand, when $J_i = 0$ for all i but $\alpha \neq \pi/4$, the scalar field does not vanish, and the solution reduces to a static dyonic multi-black-hole configuration with a scalar field.

VI. SUMMARY AND DISCUSSION

We have constructed multi-Kaluza-Klein black hole solutions in 5D Einstein gravity that, upon dimensional reduction, yield multicentered rotating black holes with both electric and magnetic charges in 4D Einstein-Maxwell-dilaton theory. In this work, we have extended the multicentered rotating black hole solution of Teo and Wan to the case of unequal electric and magnetic charges. The resulting spacetime has remained regular—free from curvature singularities, conical defects, Dirac-Misner strings, and CTCs—both on or outside the horizons, provided the black holes have had either aligned or antialigned spin orientations. Furthermore, the solutions have important limits to physical solutions: When $J_i = 0$ for all i , the solution reduces to the static multi-black-holes with a scalar field; in addition, when $Q_i = P_i = M_i/\sqrt{2}$ for all i , the scalar field vanishes, and the solution reduces to the static Majumdar-Papapetrou black hole solution with equal electric and magnetic charges in Einstein-Maxwell theory [20].

We comment on the physical features of the present solution as follows: (1) Each black hole carries nonzero

angular momentum, whereas each horizon angular velocity vanishes. This should not be surprising, since the rotations of the Maxwell field and the scalar field around the horizons contribute to the angular momenta. (2) Interpreted as a 5D solution with a compact extra dimension, the geometry also describes 5D multicentered rotating black holes. The spatial cross section of each horizon can be regarded as a Hopf bundle, an S^1 fiber bundle over an S^2 base. The electric and magnetic charges correspond, respectively, to the momentum along the fifth dimension and to the twist between that direction and the 4D spacetime.

In this paper, we have focused on 5D Einstein theory with a compact dimension. However, we expect that this method can be applied more broadly to any theory whose coset matrix is symmetric, such as supergravity. As a future direction, we aim to derive extremal black hole solutions in a variety of 4D theories which, after Kaluza-Klein reduction, can be reformulated in terms of 3D gravity coupled to a sigma model with a symmetric target space [21]. For instance, in the bosonic sector of 5D minimal supergravity with two commuting Killing vectors (one timelike and one spacelike), the relevant coset space is $G_{2(2)}/[SL(2, \mathbb{R}) \times SL(2, \mathbb{R})]$. Since the most general black hole solution with six independent charges has been constructed in Ref. [22], it may be possible to further extend this framework to construct multicentered rotating black hole solutions.

ACKNOWLEDGMENTS

S. T. was supported by JSPS KAKENHI Grant No. 21K03560. R. S. was supported by JSPS KAKENHI Grant No. JP18K13541.

DATA AVAILABILITY

No data were created or analyzed in this study.

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