Soft and virtual corrections to semi-inclusive DIS up to four loops in QCD

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We develop the threshold resummation formalism for semi-inclusive deep-inelastic scattering (SIDIS) using their factorization property and renormalization group invariance. This allows us to obtain the soft and virtual (SV) and next-to SV (NSV) corrections for the SIDIS cross section beyond second order in QCD as well as to systematically resum threshold logarithms to all orders in Mellin space. Thanks to the recent results on next-to-next-to-leading order (NNLO) QCD corrections for the SIDIS and four loop results on the vector form factor and Altarelli-Parisi splitting functions in QCD, we obtain all the SV corrections at fourth order in strong coupling constant a_s except a pure delta function term resulting from the real emissions. The numerical analysis on the impact of resummed contributions at next-to-next-to leading logarithmic ($\overline{\text{NNLL}}$) accuracy matched with the exact fixed order NNLO results on the cross section demonstrates very good perturbative stability of the perturbation prediction, including the reduced dependence on the renormalization and factorization scales μ_R and μ_F.

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I. INTRODUCTION

The semi-inclusive deep-inelastic scattering (SIDIS) process [1], which involves the scattering of a lepton off an initial hadron P and the production of an identified hadron H′ in the final state, is a vital experimental tool in probing the internal structure of hadrons. This process is particularly important for studying parton distribution functions (PDFs) and fragmentation functions (FFs) through quantum chromodynamics (QCD).

On the theoretical side, the earliest next-to-leading order (NLO) QCD results for SIDIS are from a seminal paper [2], see also [3–6]. Recently, the QCD results at next-to-next-to-leading order (NNLO) accuracy for both spin-independent and spin-dependent SIDIS cross sections were obtained independently by two groups [7–12]. In QCD factorization, which provides the backbone of the perturbative framework, the fixed order results for the coefficient functions (CFs), parametrizing the hard scattering, often provide reliable predictions for a wide range of kinematic variables. However, they fail in the threshold regions, because of the presence of large logarithms. In the case of SIDIS, these logarithms originate from the partonic configurations where partons are soft and/or collinear to each other. Although the CFs in SIDIS are infrared safe, meaning that soft and collinear divergences cancel out, these cancellations lead to certain distributions in CFs represented by distributions such as $\delta(1-\xi)$ and

$$\mathcal{D}^j_{\bar{\xi}} = \left[\frac{\ln^j(1-\xi)}{1-\xi}\right]_+, \qquad j = 0, 1, \cdots \tag{1}$$

where $\bar{\xi} = 1-\xi$ and $\xi = x', z'$ are appropriate scaling variables. Here subscript $+$ means that $\mathcal{D}^j_{\bar{\xi}}$ is a plus distribution defined as

$$\int_0^1 d\xi\, f(\xi)\mathcal{D}^j_{\bar{\xi}} = \int_0^1 d\xi\, (f(\xi) - f(1))\frac{\ln^j(1-\xi)}{1-\xi} \tag{2}$$

where $f(\xi)$ is regular at $\xi = 1$. We call them soft plus virtual (SV) contributions, and their importance was recognized early on, leading to several improvements in QCD perturbation theory, see [13–16]. Currently, approximations at (next-to-)^3-leading order (N^{3}LO) accuracy taking into account only threshold contributions, also

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called SV and threshold resummation to (next-to-)^3-leading logarithmic (N^{3}LL) accuracy are available [15,17,18], improving upon earlier results [19–21] at next-to-leading logarithmic (NLL) accuracy. These achievements were possible thanks to an intriguing connection between SIDIS and the rapidity distributions of pair of leptons in the Drell-Yan (DY) process, [15,22,23]. Beyond the SV contributions, one encounters terms with logarithmic enhancement of the type $\ln^j(1-\xi)$, with $\xi = x', z'$, for $j = 0, 1, \ldots$, which are classified as next-to-leading power (NLP) corrections or next-to-SV (NSV) terms. Threshold resummation provides a framework to resum these large logarithms, i.e., the SV terms in Eq. (1) and NSV terms, to all orders in perturbation theory to obtain improved phenomenological predictions. In this paper, we push threshold resummation for the SIDIS CFs to N^{4}LL except N-independent terms coming from real emission processes and provide their approximations at N^{4}LO in QCD, accounting for both SV as well as NSV terms in a systematic procedure.

The outline of the article is as follows: In Sec. II, we present the theory framework for the SIDIS process, recall the basics of mass factorization in QCD, and set up the formalism, that allows us to resum the threshold enhanced SV and NSV contributions. The new fixed-order approximate SV and NSV results for the coefficient functions at N^{3}LO and N^{4}LO are presented in Sec. III. The explicit form of the exponential for the resummation of the threshold logarithms is presented in Sec. IV in Mellin space as a function of complex valued $N_i, i = 1, 2$, conjugate to x', z' respectively. An alternative approach to the threshold expansion through the physical evolution kernel is presented in Sec. V, while phenomenological studies are performed with Sec. VI. We summarize and present our conclusions in Sec. VII. Relevant formulas for the threshold resummation are collected in the appendixes.

II. THEORY

The SIDIS process is given by

$$l(k_l) + \mathrm{P}(P) \to l(k_l') + \mathrm{H}'(P_H) + \mathrm{X}, \tag{3}$$

where the incoming (outgoing) lepton l has momentum $k_l(k_l')$, and P(H) is the incoming (outgoing) hadron with momentum $P(P_H)$. The remaining set of final-state particles is denoted by X.

We consider two types of differential observables namely, the spin-independent cross section where we average over the spins of incoming particles and sum the spins of final-state particles:

$$d\sigma = \frac{1}{4} \sum_{s_l, S, s_l', S_H} d\sigma_{s_l, S}^{s_l', S_H}, \tag{4}$$

and the corresponding spin-dependent cross section,

$$d\Delta\sigma = \frac{1}{2} \sum_{s_l', S_H} \left(d\sigma_{s_l=\frac{1}{2}, S=\frac{1}{2}}^{s_l', S_H} - d\sigma_{s_l=\frac{1}{2}, S=-\frac{1}{2}}^{s_l', S_H} \right), \tag{5}$$

where s_l, S are the spins of incoming lepton and hadron, and s_l', S_H the spins of the outgoing ones, respectively. The cross section depends on the spacelike momentum transfer $Q^2 = -q^2$ where $q = k_l - k_l'$, the Bjorken variable $x = Q^2/2P \cdot q$, the inelasticity $y = P \cdot q / P \cdot k_l$, and the scaling variable $z = P \cdot P_H / P \cdot q$, the fraction of the initial energy transferred to the final-state hadron.

We restrict ourselves to scattering events where there is only photon exchange between the incoming lepton and the hadron. We find that the SIDIS cross section factorizes into a leptonic part and a hadronic part. The leptonic part is simple to calculate, whereas the hadronic part encapsulates the physics of the strong interactions and is challenging due to the compositeness of both the incoming and outgoing hadrons. However, using parity and time reversal invariance and electromagnetic current conservation, we can parametrize the hadronic part of the SIDIS cross section in terms of structure functions. For example, for the spin averaged case, we have two structure functions, $F_J, J = 1, 2$. The structure functions are Lorentz invariant, depend on the hadronic scaling variables x, z, and, when higher order quantum corrections are included, also on the scale Q^2 of the virtual momentum transferred. We find

$$\frac{d^3\sigma}{dxdydz} = \frac{4\pi\alpha_e^2}{Q^2} \left[yF_1(x,z,Q^2) + \frac{(1-y)}{y} F_2(x,z,Q^2) \right], \tag{6}$$

where $\alpha_e = e^2/(4\pi)$ is the fine-structure constant and e is the electric charge. In the literature, one often parametrizes the hadronic tensor in terms of structure functions denoted by W_J $(J = 1, 2)$ and the F_J are related to W_J through $W_1 = F_1/M$ and $W_2 = F_2/(yE)$ with the energy E of the incoming lepton and M being mass of the incoming hadron. The spin-dependent cross section for longitudinally polarized hadrons in the initial state depends only on one structure function, g_1,

$$\frac{d^3\Delta\sigma}{dxdydz} = \frac{4\pi\alpha_e^2}{Q^2} (2-y) g_1(x,z,Q^2). \tag{7}$$

In perturbative QCD, the structure functions factorize as a convolution of (spin-dependent) PDFs, $(\Delta)f_{a/\mathrm{P}}(x_1, \mu_F^2)$ and FFs, $D_{\mathrm{H}'/b}(z_1, \mu_F^2)$ with (spin-dependent) CFs, $(\Delta)\mathcal{C}_{J,ab}(x/x_1, z/z_1, Q^2, \mu_F^2)$, the latter being calculable in perturbation theory,

$$(g_1)F_J(x,z,Q^2) = \sum_{a,b=q,\bar{q},g} \int_x^1 \frac{dx_1}{x_1} (\Delta) f_{a/\mathrm{P}}(x_1,\mu_F^2) \int_z^1 \frac{dz_1}{z_1} D_{\mathrm{H}'/b}(z_1,\mu_F^2) (\Delta)\mathcal{C}_{J,ab}\left(\frac{x}{x_1},\frac{z}{z_1},Q^2,\mu_F^2\right). \tag{8}$$

Here, μ_F is the factorization scale, x_1 is the incoming parton's momentum fraction with respect to the hadron P, i.e., $x_1 = p_a/P$, and z_1 is the momentum fraction of the final-state parton b carried away by the outgoing hadron H′, i.e., $z_1 = P_H/p_b$. Note also that the spin-dependent PDFs are given by $\Delta f_{a/\mathrm{P}} = f_{a(\uparrow)/\mathrm{P}(\uparrow)} - f_{a(\downarrow)/\mathrm{P}(\uparrow)}$, where arrows denote the polarizations of respective particles with respect to beam direction.

SIDIS coefficient functions are derived from parton subprocesses via mass factorization, neglecting quark mass effects. Collinear divergences arise when massless partons become collinear, such as an incoming or outgoing parton with others produced in the hard scattering. Thanks to QCD factorization, the bare (spin-dependent) CFs $(\Delta)\hat{\mathcal{C}}_{J,ab}(x',z',\varepsilon)$ in dimensional regularization $(d = 4+\varepsilon)$ factorize through Altarelli-Parisi kernels at the scale μ_F,

$$(\Delta)\hat{\mathcal{C}}_{J,ab}(x',z',\varepsilon) = (\Delta)\Gamma_{c\leftarrow a}(x',\mu_F^2,\varepsilon) \otimes (\Delta)\mathcal{C}_{J,cd}(x',z',\mu_F^2,\varepsilon)\tilde{\otimes}\tilde{\Gamma}_{b\leftarrow d}(z',\mu_F^2,\varepsilon). \tag{9}$$

The kernels $\Gamma_{c\leftarrow a}$ and $\tilde{\Gamma}_{b\leftarrow d}$ are given in terms of spacelike and timelike splitting functions and summarize the collinear divergences through poles in ε, see, e.g., [11]. The partonic scaling variables are $x' = x/x_1$ and $z' = z/z_1$ and the convolutions $\otimes$ and $\tilde{\otimes}$ between the functions $A(x')$, $C(x',z')$, and $B(z')$ are defined as follows:

$$\begin{aligned} &A(x') \otimes C(x',z')\tilde{\otimes}B(z') \\ &= \int_{x'}^1 \frac{dx_1}{x_1} \int_{z'}^1 \frac{dz_1}{z_1} A(x_1) C\left(\frac{x'}{x_1},\frac{z'}{z_1}\right) B(z_1). \end{aligned} \tag{10}$$

The collinear finite CFs $(\Delta)\mathcal{C}_{J,ab}$ in Eq. (9) are expanded in powers of strong coupling constant $a_s = \alpha_s/(4\pi)$ at the renormalization scale μ_R,

$$\begin{aligned} (\Delta)\mathcal{C}_{J,ab}(x',z',\mu_F^2) &= \delta_{ab}\delta(1-x')\delta(1-z') \\ &\quad + \sum_{i=1}^{\infty} a_s^i(\mu_R^2)(\Delta)\mathcal{C}_{J,ab}^{(i)}(x',z',\mu_F^2,\mu_R^2). \end{aligned} \tag{11}$$

The CFs $(\Delta)\mathcal{C}_{J,ab}^{(i)}$ in general contain distributions $\delta(1-\xi)$ and $\mathcal{D}_{\bar{\xi}}^j$, cf. Eq. (1), where $\xi = x', z'$ and regular functions of x', z'. We can write $(\Delta)\mathcal{C}_{J,ab}$ as

$$(\Delta)\mathcal{C}_{J,ab} = \sum_r (\Delta)C_{J,ab}^r h_r(x',z') + \sum_\beta ((\Delta)C_{J,ab,z'}^\beta(z')X_\beta(x') + (\Delta)C_{J,ab,x'}^\beta(x')Z_\beta(z')) + (\Delta)R_{J,ab}(x',z'), \tag{12}$$

where $r = \{\delta_x\delta_z, \delta_x j, j\delta_z, jk\}$ with $j, k = 0, 1, 2$ etc. and

$$\begin{aligned} &h_{\delta_x\delta_z} = \delta(1-x')\delta(1-z'), \qquad h_{\delta_x j} = \delta(1-x')\mathcal{D}_{\bar{z}'}^j, \\ &h_{j\delta_z} = \mathcal{D}_{\bar{x}'}^j\delta(1-z'), \qquad h_{jk} = \mathcal{D}_{\bar{x}'}^j\mathcal{D}_{\bar{z}'}^k \end{aligned} \tag{13}$$

and $\beta = \{\delta_x, \delta_z, j\}$ with $j = 0, 1, 2$ etc.,

$$\begin{aligned} &X_{\delta_x} = \delta(1-x'), \qquad Z_{\delta_z} = \delta(1-z'), \\ &X_{\delta_z} = 0, \qquad Z_{\delta_x} = 0, \\ &X_j = \mathcal{D}_{\bar{x}'}^j, \qquad Z_j = \mathcal{D}_{\bar{z}'}^j. \end{aligned} \tag{14}$$

The functions $(\Delta)R_{J,ab}(x',z')$ contain all remaining terms that are regular (no distributions). Furthermore, if we expand the CFs $(\Delta)C_{J,ab,x'}^\beta$ and $(\Delta)C_{J,ab,z'}^\beta$ in Eq. (12) around $x' = 1$ and $z' = 1$, we obtain a series of the form $(1-\xi)^i \ln^j(1-\xi), \xi = x', z'$ for $i, j = 0, 1$ etc. The leading terms in this series $(i = 0)$ are the so-called NSV terms, while the rest with powers $i > 0$ are denoted as beyond-NSV (BNSV) terms. In this notation, the CFs in Eq. (12) read

$$(\Delta)\mathcal{C}_{J,ab} = \delta_{ab}((\Delta)\mathcal{C}_J^{\mathrm{SV}}) + (\Delta)\mathcal{C}_{J,ab}^{\mathrm{NSV}} + (\Delta)\mathcal{C}_{J,ab}^{\mathrm{BNSV}}, \tag{15}$$

where the SV term does not receive any contributions if $a = b = g$. Each term in the above equation can be expanded in powers series of a_s as

$$(\Delta)\mathcal{C}_J^{\rm SV} = (\Delta)C_{J,qq}^{(0),\delta_x\delta_z} h_{\delta_x\delta_z} + \sum_{n=1}^{\infty} a_s^n \left[(\Delta)C_{J,qq}^{(n),\delta_x\delta_z} h_{\delta_x\delta_z} + \sum_{j=0}^{2n-1} (\Delta)C_{J,qq}^{(n),\delta_x j} h_{\delta_x j} + \sum_{j=0}^{2n-1} (\Delta)C_{J,qq}^{(n),j\delta_z} h_{j\delta_z} + \sum_{j=0}^{2n-2} \sum_{k=0}^{2n-2-j} (\Delta)C_{J,qq}^{(n),jk} h_{jk} \right],$$

$$(\Delta)\mathcal{C}_{J,ab}^{\rm NSV} = \sum_{n=1}^{\infty} a_s^n \left[\sum_{j=0}^{2n-1} \left((\Delta)C_{J,ab}^{(n),\delta_x j} X_{\delta_x} L_j(z') + (\Delta)C_{J,ab}^{(n),j\delta_z} L_j(x') Z_{\delta_z} \right) + \sum_{j=0}^{2n-2} \sum_{k=0}^{2n-2-j} \left((\Delta)C_{J,ab}^{(n),jk} X_j L_k(z') + (\Delta)C_{J,ab}^{(n),jk} L_k(x') Z_j \right) \right], \tag{16}$$

where $L_j(\xi) = \ln^j(1-\xi), \xi = x', z'$. Schematically, $(\Delta)\mathcal{C}_J^{\rm SV}$ contains double distributions (h_r) given in Eq. (13), $(\Delta)\mathcal{C}_{J,ab}^{\rm NSV}$ contains single distribution in x' or z' (X_β, Z_β) each multiplied by powers of $\ln^j(1-z')$ or $\ln^j(1-x')$ respectively, where $j = 0, 1, 2$ etc. Any regular terms in x', z' that do not contain any distributions fall within the BNSV category. The NNLO results in [7–12] for the CFs $(\Delta)\mathcal{C}_{J,ab}^{\rm SV}$ are in complete agreement with SV predictions for unpolarized structure functions $F_{1,2}$ [17], see also [24,25]. The leading terms of the NSV contributions agree as well with Ref. [17], where the dominant NLP terms have been presented.

The first principle's derivation of the threshold enhanced logarithms in Eq. (16) exploits the renormalization group equations and the factorization of soft and collinear singularities to all orders in QCD, building on well-known developments for inclusive cross sections such as DIS, DY, and Higgs boson production (or, more generally, colorless n-particle production), see e.g., [26–37]. For observables in differential kinematics, where the threshold limit is approached in terms of two different scaling variables, such as the rapidity distributions of lepton pairs in the DY process or Higgs boson production, threshold logarithms have also been systematically resumed to all orders in perturbation theory, see [23–25,27,38–45]. In addition, both single as well as double variable resummations were extended for flavor diagonal channels to include NSV contributions [46–66].

The kinematics in these cases is similar to the SIDIS process, where the limits $x' \to 1$ and $z' \to 1$ are considered. In the following we develop the formalism to resum both SV as well as NSV logarithms of flavor diagonal contributions, namely $(\Delta)\mathcal{C}_{J,qq}$ to all orders both in (x', z') space as well in their corresponding Mellin space (N_1, N_2). The reason to consider only flavor diagonal channels is because of their simple collinear factorization structure. The starting point is the mass factorized form of the bare CFs in Eq. (9) in $d = 4 + \varepsilon$ dimensions, expanding the sum over the parton channels,

$$\begin{aligned}(\Delta)\hat{\mathcal{C}}_{J,qq}(x', z', \varepsilon) = {} & (\Delta)\Gamma_{q\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,qq} \widetilde{\otimes} \tilde{\Gamma}_{q\leftarrow q} + (\Delta)\Gamma_{q\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,q\bar{q}} \widetilde{\otimes} \tilde{\Gamma}_{q\leftarrow \bar{q}} + (\Delta)\Gamma_{q\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,qg} \widetilde{\otimes} \tilde{\Gamma}_{q\leftarrow g} \\ & + (\Delta)\Gamma_{\bar{q}\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,\bar{q}q} \widetilde{\otimes} \tilde{\Gamma}_{q\leftarrow q} + (\Delta)\Gamma_{\bar{q}\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,\bar{q}\,\bar{q}} \widetilde{\otimes} \tilde{\Gamma}_{q\leftarrow \bar{q}} + (\Delta)\Gamma_{\bar{q}\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,\bar{q}g} \widetilde{\otimes} \tilde{\Gamma}_{q\leftarrow g} \\ & + (\Delta)\Gamma_{g\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,gq} \widetilde{\otimes} \tilde{\Gamma}_{q\leftarrow q} + (\Delta)\Gamma_{g\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,g\bar{q}} \widetilde{\otimes} \tilde{\Gamma}_{q\leftarrow \bar{q}} + (\Delta)\Gamma_{g\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,gg} \widetilde{\otimes} \tilde{\Gamma}_{q\leftarrow g}. \end{aligned} \tag{17}$$

The convolutions with the spacelike and timelike Altarelli-Parisi (AP) kernals $\Gamma_{a\leftarrow b}$ and $\tilde{\Gamma}_{a\leftarrow b}$ can be distinguished as follows: (i) terms involving only diagonal combinations, such as $(\Delta)\Gamma_{q\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,qq} \tilde{\otimes} \tilde{\Gamma}_{q\leftarrow q}$; (ii) contributions involving a product of one diagonal with a pair of off-diagonal terms, for example $(\Delta)\Gamma_{q\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,qg} \tilde{\otimes} \tilde{\Gamma}_{q\leftarrow g}$. The terms in (i) contain SV plus NSV terms when the convolutions have been performed while the latter ones in case (ii) will be composed beyond NSV terms only. Note that the diagonal AP kernels $(\Delta)\Gamma_{q\leftarrow q}$ and $\tilde{\Gamma}_{q\leftarrow q}$ contain convolutions among diagonal splitting functions, $(\Delta)P_{qq}$ and $\tilde{P}_{qq}$ respectively, or one diagonal and a pair of off-diagonal ones, $(\Delta)P_{ab}, \tilde{P}_{ab}, a \neq b$. We drop those terms in $(\Delta)\Gamma_{q\leftarrow q}$ $(\tilde{\Gamma}_{q\leftarrow q})$, that contain a pair of off-diagonal splitting functions $(\Delta)P_{ab}$ $(\tilde{P}_{ab})$, as they contribute only beyond NSV accuracy. In addition, we drop all the BNSV terms in the diagonal splitting functions as well. This leads to

$$(\Delta)\hat{\mathcal{C}}_{J,qq}^{\rm SV+NSV} = (\Delta)\Gamma_{q\leftarrow q}^{\rm SV+NSV} \otimes (\Delta)\mathcal{C}_{J,qq}^{\rm SV+NSV} \tilde{\otimes} \tilde{\Gamma}_{q\leftarrow q}^{\rm SV+NSV}. \tag{18}$$

Here, the superscript on the AP kernels indicates their truncation to SV and NSV terms only. In this way we can write the bare CFs in Eq. (9) in terms of only diagonal quantities, i.e., $(\Delta)\hat{\mathcal{C}}_{J,qq}$, $(\Delta)\mathcal{C}_{J,qq}$ and AP kernels $(\Delta)\Gamma_{q\leftarrow q}$, $\tilde{\Gamma}_{q\leftarrow q}$, where BNSV terms and the sum ab over the parton channels are dropped.

The bare flavor off-diagonal CFs instead contain both diagonal and off-diagonal terms from $(\Delta)\mathcal{C}_{J,ab}$ and AP kernels. For instance $(\Delta)\hat{\mathcal{C}}_{J,qg}$ reads

$$\begin{aligned}(\Delta)\hat{\mathcal{C}}_{J,qg}(x',z',\varepsilon) = {}& (\Delta)\Gamma_{q\leftarrow q}\otimes(\Delta)\mathcal{C}_{J,qq}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow q} + (\Delta)\Gamma_{q\leftarrow q}\otimes(\Delta)\mathcal{C}_{J,q\bar{q}}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow\bar{q}} + (\Delta)\Gamma_{q\leftarrow q}\otimes(\Delta)\mathcal{C}_{J,qg}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow g}\\ &+ (\Delta)\Gamma_{\bar{q}\leftarrow q}\otimes(\Delta)\mathcal{C}_{J,\bar{q}q}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow q} + (\Delta)\Gamma_{\bar{q}\leftarrow q}\otimes(\Delta)\mathcal{C}_{J,\bar{q}\bar{q}}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow\bar{q}} + (\Delta)\Gamma_{\bar{q}\leftarrow q}\otimes(\Delta)\mathcal{C}_{J,\bar{q}g}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow g}\\ &+ (\Delta)\Gamma_{g\leftarrow q}\otimes(\Delta)\mathcal{C}_{J,gq}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow q} + (\Delta)\Gamma_{g\leftarrow q}\otimes(\Delta)\mathcal{C}_{J,g\bar{q}}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow\bar{q}} + (\Delta)\Gamma_{g\leftarrow q}\otimes(\Delta)\mathcal{C}_{J,gg}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow g}.\end{aligned} \tag{19}$$

Unlike the diagonal case in Eq. (17), no single term provides pure SV contributions in the threshold limit. Hence, the mass factorized result for the off-diagonal channel contains NSV terms only and those beyond, where the former ones come from contributions that contain at least two diagonal quantities, either from $(\Delta)\mathcal{C}_{J,ab}$ or $(\Delta)\Gamma_{a\leftarrow b},\tilde{\Gamma}_{a\leftarrow b}$. Dropping all terms that contain more than two off-diagonal quantities in the mass factorization formula, we obtain

$$\begin{aligned}(\Delta)\hat{\mathcal{C}}^{\mathrm{NSV}}_{J,qg} = {}& (\Delta)\Gamma^{\mathrm{SV+NSV}}_{q\leftarrow q}\otimes(\Delta)\mathcal{C}^{\mathrm{SV+NSV}}_{J,qq}\tilde{\otimes}\tilde{\Gamma}^{\mathrm{NSV}}_{g\leftarrow q}\\ &+(\Delta)\Gamma^{\mathrm{SV+NSV}}_{q\leftarrow q}\otimes(\Delta)\mathcal{C}^{\mathrm{NSV}}_{J,qg}\tilde{\otimes}\tilde{\Gamma}^{\mathrm{SV+NSV}}_{g\leftarrow g},\end{aligned} \tag{20}$$

which receives contribution from both CFs, $(\Delta)\mathcal{C}_{J,qg}$, and $(\Delta)\mathcal{C}_{J,qq}$, unlike the diagonal channel in Eq. (18) which is directly proportional to $(\Delta)\hat{\mathcal{C}}_{J,qq}$.

The simple structure in the diagonal channel makes it easy for us to study its all-order structure of SV and NSV logarithms. We achieve this by setting up homogeneous differential equations for CFs in terms of the soft and collinear anomalous dimensions. The presence of inhomogeneous terms in corresponding differential equations for the off-diagonal channel poses technical challenges when solving them. Hence, in the rest of the paper, we will focus only on diagonal partonic channels.

We begin with the Eq. (18) for the diagonal CFs and rewrite it as

$$\begin{aligned}(\Delta)\mathcal{C}^{\mathrm{SV+NSV}}_{J,qq} = {}& |\hat{F}_q(Q^2,\varepsilon)|^2((\Delta)\Gamma)^{-1}_{q\leftarrow q}(x',\mu_F^2,\varepsilon)\\ &\otimes(\Delta)\hat{\mathcal{S}}_{J,q}(Q^2,x',z',\varepsilon)\tilde{\otimes}(\tilde{\Gamma})^{-1}_{q\leftarrow q}(z',\mu_F^2,\varepsilon)\end{aligned} \tag{21}$$

where $\hat{F}_q(Q^2,\varepsilon)$ is the massless quark form factor (FF) in dimensional regularization and $(\Delta)\hat{\mathcal{S}}_{J,q}(Q^2,x',z',\varepsilon)$, called soft function, is defined by

$$(\Delta)\hat{\mathcal{S}}_{J,q}(Q^2,x',z',\varepsilon) = \frac{(\Delta)\hat{\mathcal{C}}_{J,qq}(Q^2,x',z',\varepsilon)}{|\hat{F}_q(Q^2,\varepsilon)|^2}. \tag{22}$$

Note that in the absence of any real radiation, the soft function is constrained by kinematics and takes the simple form $(\Delta)\hat{\mathcal{S}}_{J,q}|_{\text{no radiation}} = \delta(1-x')\delta(1-z')$.

The vector form factor $\hat{F}_q$ has been subject to extensive studies, see, e.g., [30,67–72]. In massless QCD and within dimensional regularization it exhibits IR and UV singularities as poles in ε. The UV divergences can be removed by renormalization of the strong coupling, while the soft and collinear, i.e., IR singularities, factorizes as

$$\hat{F}_q(Q^2,\varepsilon) = Z_{\hat{F}_q}(Q^2,\mu_s^2,\varepsilon)F_{q,\mathrm{fin}}(Q^2,\mu_s^2,\varepsilon), \tag{23}$$

where $Z_{\hat{F}_q}$ is IR singular and $F_{q,\mathrm{fin}}$ is IR finite, an the scale μ_s is IR factorization scale. Renormalization group invariance implies that

$$\mu_s^2\frac{d}{d\mu_s^2}\ln Z_{\hat{F}_q}(\mu_s^2) = \gamma_{\hat{F}_q}(\mu_s^2), \tag{24}$$

where $\gamma_{\hat{F}_q}$ to all orders in a_s takes the form

$$\gamma_{\hat{F}_q} = \frac{1}{2}\left(-A^q(\mu_s^2)\ln\left(\frac{Q^2}{\mu_s^2}\right) + 2B^q(\mu_s^2) + f^q(\mu_s^2)\right). \tag{25}$$

Here, A^q, B^q, f^q denote the cusp (A^q), the quark virtual B^q, and the eikonal anomalous dimensions; they all have expansions in powers of the strong coupling as $A^q(\mu_s^2) = \sum_{i=1}^{\infty} a_s^i(\mu_s^2)A_i^q$ etc. They are currently known to four-loop accuracy in QCD [73–80]. Thus, the solution to Eq. (24) for $Z_{\hat{F}_q}$ in the $\overline{\mathrm{MS}}$ scheme contains only poles in ε whose coefficients are completely controlled by universal anomalous dimensions A^q, B^q, and f^q and the coefficients of QCD β function, the latter being known even to five loops [81–84].

The bare form factor $\hat{F}_q$ is subject to a differential equation with respect to Q^2,

$$Q^2\frac{d}{dQ^2}\ln\hat{F}_q(Q^2,\varepsilon) = \Gamma_{\hat{F}_q}(Q^2,\varepsilon), \tag{26}$$

where

$$\Gamma_{\hat{F}_q}(Q^2,\varepsilon) = \frac{1}{2}(K_d^q(\mu_s^2,\varepsilon) + G_d^q(Q^2,\mu_s^2,\varepsilon)). \tag{27}$$

The kernel $K_d^q = 2d\ln Z_{\hat{F}_q}/d\ln(Q^2)$ contains entire IR singular terms while the function $G_d^q = 2d\ln F_{q,\mathrm{fin}}/d\ln(Q^2)$ is IR finite by definition. Since we have defined the form factor

in dimensional regularization, the solution to the $K+G$ equation takes the simple form

$$\hat{F}_q(Q^2,\varepsilon)=\exp\left(\int_0^{Q^2}\frac{d\lambda^2}{\lambda^2}\Gamma_{\hat{F}_q}(\lambda^2,\varepsilon)\right), \tag{28}$$

with $\hat{F}_q(Q^2=0,\varepsilon)=1$ as boundary condition. Since all IR poles in ε are contained in K_d^q only and G_d^q is finite, the solution reveals the remarkable structure for the form factor, namely, at given order a_s^i, all the poles except the single pole in ε can be predicted from knowledge of K_d^q and G_d^q at previous orders up to a_s^{i-1}. Explicit results for K_d^q and G_d^q are presented in [28,30] and also in Appendix A.

Next, we briefly review the structure of the AP kernels in Eq. (21) and the associated splitting functions, controlling their evolution as a function of the factorization scale μ_F. We denote spacelike (unpolarized and polarized) and timelike kernels collectively by $\mathbb{I}_{c\leftarrow c}$ and their respective splitting functions by $\mathbb{P}_{c\leftarrow c}$. In the threshold limit only the diagonal splitting functions $\mathbb{P}_{c\leftarrow c}, c=q,\bar{q}$ contribute,

$$\mu_F^2\frac{d}{d\mu_F^2}\mathbb{I}_{c\leftarrow c}(\mu_F^2,\xi)=\frac{1}{2}\mathbb{P}_{c\leftarrow c}(\mu_F^2,\xi)\otimes\mathbb{I}_{c\leftarrow c}(\mu_F^2,\xi), \tag{29}$$

where $\xi=x',z'$. The splitting functions are expanded in powers of $a_s(\mu_F^2)$ as $\mathbb{P}_{c\leftarrow c}(\mu_F^2)=\sum_{i=1}^{\infty}a_s^i(\mu_F^2)\mathbb{P}_{i-1}^c$ and we have suppressed any dependence on ε in both $\mathbb{I}_{c\leftarrow c}$ and $\mathbb{P}_{c\leftarrow c}$. Due to its simple structure the all-order solution of Eq. (29) takes the simple form

$$\mathbb{I}_{c\leftarrow c}(\mu_F^2,\xi)=\mathcal{C}\exp\left(\frac{1}{2}\int_0^{\mu_F^2}\frac{d\lambda^2}{\lambda^2}\mathbb{P}_{c\leftarrow c}(\lambda^2,\xi)\right), \tag{30}$$

where the symbol $\mathcal{C}$ denotes an ordered exponential, appropriately defined through iterated convolutions with respect to x' or z', see [30]. The relevant splitting functions are known to third order in perturbative QCD [85–89] and their expansion in the soft and collinear limit reads

$$\mathbb{P}_{c\leftarrow c}(\mu_F^2,\xi)=2\left(\frac{A^c(\mu_F^2)}{(1-\xi)_+}+B^c(\mu_F^2)\delta(1-\xi)+\mathbb{C}^c(\mu_F^2)\ln(1-\xi)+\mathbb{D}^c(\mu_F^2)\right), \tag{31}$$

in terms of the cusp and the virtual anomalous dimensions A^c and B^c already encountered in the form factor above. The NSV terms, denoted as $\mathbb{C}^c$ and $\mathbb{D}^c$, are determined from A^c, B^c, and coefficients of the QCD β function at lower orders. There is distinction between spacelike and timelike kinematics, i.e., $\mathbb{C}^c=C^q,\mathbb{D}^c=D^q$ for spacelike and $\mathbb{C}^c=\tilde{C}^q,\mathbb{D}^c=\tilde{D}^q$ for timelike splitting functions. Explicit results are listed in Appendix G.

The final ingredient in the threshold factorization formulas in Eq. (21) is the soft function $(\Delta)\hat{\mathcal{S}}_{J,q}$, hereafter denoted by $\hat{\mathcal{S}}_q$. A differential equation for $\hat{\mathcal{S}}_q$ in dimensional regularization, organizing its all-order perturbative structure, can be obtained with Eq. (21) and the differential equation for the form factor, see Eq. (26),

$$Q^2\frac{d}{dQ^2}\mathcal{C}\ln\hat{\mathcal{S}}_q(Q^2,\varepsilon)=\Gamma_{\hat{\mathcal{S}}_q}(Q^2,\varepsilon), \tag{32}$$

where we have suppressed the dependence on x',z' and

$$\Gamma_{\hat{\mathcal{S}}_q}(Q^2,\varepsilon)=Q^2\frac{d}{dQ^2}(\mathcal{C}\ln(\Delta)\mathcal{C}_{J,qq}(Q^2,\mu_R^2,\mu_F^2,x',z',\varepsilon)-\ln|\hat{F}_q(Q^2,\varepsilon)|^2\delta(1-x')\delta(1-z')). \tag{33}$$

The symbol $\mathcal{C}$ represents that each product of x'- (z'-) dependent functions in the expansion of $\ln(\Delta)\mathcal{C}_{J,qq}$ in a_s is treated as convolution with respect to x' (z'). The fact that $\hat{\mathcal{S}}_q$ is renormalization group invariant with respect to μ_R and μ_F implies that the Q^2 derivative of $(\Delta)\mathcal{C}_{J,qq}$ in the first term in Eq. (33) has to be a function of only Q^2 and x',z'. This implies that $\Gamma_{\hat{\mathcal{S}}_q}$ is independent of μ_R and μ_F. Observe that the first term is finite and the second term contains both singular and finite parts through K_d^q and G_d^q via $K+G$ equation of $\Gamma_{\hat{F}_q}$, see Eq. (26). This means that we can express the kernel $\Gamma_{\hat{\mathcal{S}}_q}$ in Eq. (33) as a sum of singular term $\bar{K}_d^q$ and finite $\bar{G}_d^q$ pieces to all orders in perturbation theory, similar to the Q^2 evolution equation of the form factor, i.e., $\Gamma_{\hat{\mathcal{S}}_q}=(\bar{K}_d^q(\mu_s^2,x',z')+\bar{G}_d^q(Q^2,\mu_s^2,x',z'))/2$. Since the singular part of the kernel comes from K_d^q of the form factor, the term $\bar{K}_d^q$ can depend only on μ_s and process-independent anomalous dimension, A^q. The finite $\bar{G}_d^q(Q^2,\mu_s^2,x',z')$ will contain G_d^q from the form factor and the other process-dependent terms from CF $(\Delta)\mathcal{C}_{J,qq}$.

The fact that the kernel admits the decomposition of singular and finite parts as it happens for the form factor, implies that the $\hat{\mathcal{S}}_q$ is also factorizable, i.e., we can write $\hat{\mathcal{S}}_q(Q^2,x',z')=Z_q(Q^2,\mu_s^2,x',z')\otimes\mathcal{S}_{q,\mathrm{fin}}(Q^2,\mu_s^2,x',z')$, where $\mathcal{S}_{q,\mathrm{fin}}$ is IR finite, Z_q is IR singular, and identify the IR singular $\bar{K}_d^q=2d\ln Z_q/d\ln(Q^2)$ and IR finite $\bar{G}_d^q=2d\ln\mathcal{S}_{q,\mathrm{fin}}/d\ln(Q^2)$. Since $\bar{K}_d^q(\mu_s^2)$ is independent of Q^2, we can fix only the $\ln(Q^2)$ terms in Z_q. The factorization of $\hat{\mathcal{S}}_q$ in terms of Z_q and $\mathcal{S}_{q,\mathrm{fin}}$ at arbitrary scale μ_s leads to the renormalization group equation,

$$\mu_s^2\frac{dZ_q(Q^2,\mu_s^2)}{d\mu_s^2}=\gamma_{\hat{\mathcal{S}}_q}(Q^2,\mu_s^2)\otimes Z_q(Q^2,\mu_s^2),$$

where we have suppressed any dependence on x',z' in Z_q and $\gamma_{\hat{\mathcal{S}}_q}$. From our two-loop results we find that $\gamma_{\hat{\mathcal{S}}_q}$ takes the remarkable structure, namely $\gamma_{\hat{\mathcal{S}}_q}(Q^2,\mu_s^2)=\xi_1(\mu_s^2,x',z')\ln(Q^2/\mu_s^2)+\xi_2(\mu_s^2,x',z')$. This is expected to

continue to all orders in perturbation theory because it results from the IR pole structure of $\hat{\mathcal{S}}_q$. In other words, the singular part of $\hat{\mathcal{S}}_q$, namely, Z_q has to contain precisely those IR poles to cancel against the ones from the form factor and AP kernels leaving $(\Delta)\mathcal{C}_{J,qq}$ finite and this pole structure determines the structure of $\gamma_{\hat{\mathcal{S}}_q}$. Since the poles of the form factor and AP kernels are controlled by the cusp, virtual, and eikonal anomalous dimensions, $\gamma_{\hat{\mathcal{S}}_q}$ can be expressed as follows:

$$\gamma_{\hat{\mathcal{S}}_q} = \delta(\bar{x}')\delta(\bar{z}')A^q(\mu_s^2)\ln\left(\frac{Q^2}{\mu_s^2}\right) + P'_{q\leftarrow q}(\mu_s, \bar{x}', \bar{z}'), \tag{34}$$

where

$$\begin{aligned} P'_{q\leftarrow q} = &\left[-\frac{1}{2}\delta(\bar{x}')\delta(\bar{z}')f^q(\mu_s^2) + \delta(\bar{z}')\frac{A^q(\mu_s^2)}{(\bar{x}')_+}\right. \\ &\left. + \delta(\bar{z}')(C^q(\mu_s^2)\ln(\bar{x}') + D^q(\mu_s^2))\right] \\ &+ (\bar{x}' \leftrightarrow \bar{z}', C^q \leftrightarrow \tilde{C}^q, D^q \leftrightarrow \tilde{D}^q), \end{aligned} \tag{35}$$

and we use the shorthand notation $\bar{x}' = 1 - x', \bar{z}' = 1 - z'$.

With the boundary $Q^2 = 0$, $\hat{\mathcal{S}}_q(Q^2 = 0, x', z', \varepsilon) = \delta(1 - x')\delta(1 - z')$ to all orders in a_s we obtain the solution to Eq. (32) as

$$\begin{aligned} \hat{\mathcal{S}}_q(Q^2, x', z', \varepsilon) &= \mathcal{C}\exp\left(\int_0^{Q^2}\frac{d\lambda^2}{\lambda^2}\Gamma_{\hat{\mathcal{S}}_q}(\lambda^2, x', z', \varepsilon)\right) \\ &= \mathcal{C}\exp\left(2\Phi_{d,q}(Q^2, x', z', \varepsilon)\right). \end{aligned} \tag{36}$$

The exponent $\Phi_{d,q}$ can be decomposed into an IR singular part $\Phi_{d,q,s}$ and a finite part $\Phi_{d,q,f}$, i.e., $\Phi_{d,q} = \Phi_{d,q,s} + \Phi_{d,q,f}$. Using this decomposition, we obtain $Z_q = \mathcal{C}\exp(2\Phi_{d,q,s})$ and $\mathcal{S}_{q,\text{fin}} = \mathcal{C}\exp(2\Phi_{d,q,f})$. The singular part is controlled by $\gamma_{\hat{\mathcal{S}}_q}$ given in Eq. (34). A careful inspection of the one- and two-loop results for $\hat{\mathcal{S}}_q$, i.e., using the results presented in [7,9], provides hints to its all order structure. However, before we proceed, we split $\Phi_{d,q}$ into the sum of two parts. The SV part denoted by $\Phi_{d,q}^{\text{SV}}$ contains only distributions such as $\delta(1-\xi)$ and $\mathcal{D}_{\bar{\xi},i}$, while the NSV part $\Phi_{d,q}^{\text{NSV}}$ summarizes all terms that are products of distributions and $\ln^i(1-\xi)$ with $\xi = x', z'$. We find that $\Phi_{d,q}^{\text{SV}}$ is symmetric under the interchange of x' and z' while $\Phi_{d,q}^{\text{NSV}}$ is not. Our observations for $\Phi_{d,q}$ read as follows:

(i) For the SV terms, at order $\hat{a}_s$, the leading pole in ε starts at order two; at $\hat{a}_s^2$ it is three and it is expected to increase by one unit at every order in $\hat{a}_s$. Hence, at order $\hat{a}_s^i$, the leading pole in ε will be of order $i+1$. As we had already mentioned, this pole structure is completely dictated by the pole structure of $\ln|\hat{F}_q|^2$ and the AP kernels. For the NSV terms, we find that at order $\hat{a}_s$ the leading pole in ε is of order one and at two loops, it is two etc. This pattern indicates that the increment of one unit for the leading poles is expected to continue with the order of perturbation and hence at order $\hat{a}_s^i$, the leading pole will be of order i.

(ii) At every order in $\hat{a}_s$, for a given color factor, the combination of ε and the leading logarithm shows uniform transcendentality weight. If we assign n_ε weight for ε^{-n} and $n_\mathcal{D}$ for $\mathcal{D}_{\bar{\xi}}^n$, n_L for $\ln^n(1-\xi)$ and n_ζ for ζ_n then the highest weight at every order in ε shows uniform transcendentality $\omega = n_\varepsilon + n_\mathcal{D} + n_L + n_\zeta$. For example, in SV, at one loop, we find $\omega = 2$ at every order of ε and at two loops it is three ($\omega = 3$) etc. For NSV, $\omega = 1$ at one loop and $\omega = 2$ at two loops at every order in ε etc.

The above observations on the fixed-order results and the resemblance of Eq. (26) with Eq. (32), the similarity in the pole structure of the respective kernels Eqs. (27) and (33) and the renormalization group invariance reveal an all-order structure for $\Phi_{d,q}^{\text{SV}}$ of the form

$$\begin{aligned} \Phi_{d,q}^{\text{SV}} = &\sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{Q^2(1-x')(1-z')}{\mu^2}\right)^{i\frac{\varepsilon}{2}} \\ &\times S_\varepsilon^i\left[\frac{(i\varepsilon)^2}{4(1-x')(1-z')}\hat{\phi}_{d,q}^{\text{SV},(i)}(\varepsilon)\right], \end{aligned} \tag{37}$$

where

$$\hat{\phi}_{d,q}^{\text{SV},(i)}(\varepsilon) = \frac{1}{i\varepsilon}[\bar{K}_d^{q(i)}(\varepsilon) + \bar{G}_{d,\text{SV}}^{q(i)}(\varepsilon)], \tag{38}$$

and for the NSV part

$$\begin{aligned} \Phi_{d,q}^{\text{NSV}} = &\sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{Q^2(1-x')(1-z')}{\mu^2}\right)^{i\frac{\varepsilon}{2}} \\ &\times S_\varepsilon^i\left[\frac{i\varepsilon}{4(1-x')}\hat{\varphi}_{d,z',q}^{\text{NSV},(i)}(z',\varepsilon)\right. \\ &\left. + \frac{i\varepsilon}{4(1-z')}\hat{\varphi}_{d,x',q}^{\text{NSV},(i)}(x',\varepsilon)\right]. \end{aligned} \tag{39}$$

Here $S_\varepsilon = \exp\left(\frac{\varepsilon}{2}[\gamma_E - \ln 4\pi]\right)$ with γ_E being the Euler Mascheroni constant. For the reader's convenience, we have listed the results of $\bar{K}_d^{q(i)}$ and $\bar{G}_{d,\text{SV}}^{q(i)}$ in the Appendix B. After substituting them the constants $\hat{\phi}_{d,q}^{\text{SV},(i)}(\varepsilon)$ are found to be

$$\hat{\phi}^{\mathrm{SV},(1)}_{d,q}(\varepsilon) = \frac{1}{\varepsilon^2}(2A^q_1) + \frac{1}{\varepsilon}(\bar{\mathcal{G}}^q_{d,1}(\varepsilon)),$$
$$\hat{\phi}^{\mathrm{SV},(2)}_{d,q}(\varepsilon) = \frac{1}{\varepsilon^3}(-\beta_0 A^q_1) + \frac{1}{\varepsilon^2}\left(\frac{1}{2}A^q_2 - \beta_0\bar{\mathcal{G}}^q_{d,1}(\varepsilon)\right) + \frac{1}{2\varepsilon}\bar{\mathcal{G}}^q_{d,2}(\varepsilon),$$
$$\hat{\phi}^{\mathrm{SV},(3)}_{d,q}(\varepsilon) = \frac{1}{\varepsilon^4}\left(\frac{8}{9}\beta_0^2 A^q_1\right) + \frac{1}{\varepsilon^3}\left(-\frac{2}{9}\beta_1 A^q_1 - \frac{8}{9}\beta_0 A^q_2 - \frac{4}{3}\beta_0^2\bar{\mathcal{G}}^q_{d,1}(\varepsilon)\right) + \frac{1}{\varepsilon^2}\left(\frac{2}{9}A^c_3 - \frac{1}{3}\beta_1\bar{\mathcal{G}}^q_{d,1}(\varepsilon) - \frac{4}{3}\beta_0\bar{\mathcal{G}}^q_{d,2}(\varepsilon)\right) + \frac{1}{\varepsilon}\left(\frac{1}{3}\bar{\mathcal{G}}^q_{d,3}(\varepsilon)\right),$$
$$\hat{\phi}^{\mathrm{SV},(4)}_{d,q}(\varepsilon) = \frac{1}{\varepsilon^5}(-\beta_0^3 A^q_1) + \frac{1}{\varepsilon^4}\left(\frac{2}{3}\beta_0\beta_1 A^q_1 + \frac{3}{2}\beta_0^2 A^q_2 - 2\beta_0^3\bar{\mathcal{G}}^q_{d,1}(\varepsilon)\right) - \frac{1}{\varepsilon^3}\left(\frac{1}{12}\beta_2 A^q_1 - \frac{1}{4}\beta_1 A^q_2 - \frac{3}{4}\beta_0 A^q_3\right.$$
$$\left. + \frac{4}{3}\beta_0\beta_1\bar{\mathcal{G}}^q_{d,1}(\varepsilon) + 3\beta_0^2\bar{\mathcal{G}}^q_{d,2}(\varepsilon)\right) + \frac{1}{\varepsilon^2}\left(\frac{1}{8}A^q_4 - \frac{1}{6}\beta_2\bar{\mathcal{G}}^q_{d,1}(\varepsilon) - \frac{1}{2}\beta_1\bar{\mathcal{G}}^q_{d,2}(\varepsilon) - \frac{3}{2}\beta_0\bar{\mathcal{G}}^q_{d,3}(\varepsilon)\right) + \frac{1}{\varepsilon}\left(\frac{1}{4}\bar{\mathcal{G}}^q_{d,4}(\varepsilon)\right). \tag{40}$$

The cusp anomalous dimensions A^q_i in the above equations carry the right sign to cancel all the leading singularities against the form factor contributions in CFs. The ε-dependent terms $\bar{\mathcal{G}}^q_{d,i}$ originate from $|\hat{F}_q|^2$ as well as the CFs $(\Delta)\mathcal{C}_{J,qq}$ in Eq. (33). Using renormalization group invariance and demanding the cancellation of single poles in Eq. (40) against the form factor and the AP kernels, we can parametrize $\bar{\mathcal{G}}^q_{d,i}$ as

$$\bar{\mathcal{G}}^q_{d,i}(\varepsilon) = -f^q_i + \bar{\chi}^q_{d,i} + \sum_{j=1}^{\infty}\varepsilon^j\bar{\mathcal{G}}^{q,(j)}_{d,i}, \tag{41}$$

where

$$\bar{\chi}^q_{d,1} = 0, \qquad \bar{\chi}^q_{d,2} = -2\beta_0\bar{\mathcal{G}}^{q,(1)}_{d,1}, \qquad \bar{\chi}^q_{d,3} = -2\beta_1\bar{\mathcal{G}}^{q,(1)}_{d,1} - 2\beta_0\left(\bar{\mathcal{G}}^{q,(1)}_{d,2} + 2\beta_0\bar{\mathcal{G}}^{q,(2)}_{d,1}\right),$$
$$\bar{\chi}^q_{d,4} = -2\beta_2\bar{\mathcal{G}}^{q,(1)}_{d,1} - 2\beta_1\left(\bar{\mathcal{G}}^{q,(1)}_{d,2} + 4\beta_0\bar{\mathcal{G}}^{q,(2)}_{d,1}\right) - 2\beta_0\left(\bar{\mathcal{G}}^{q,(1)}_{d,3} + 2\beta_0\bar{\mathcal{G}}^{q,(2)}_{d,2} + 4\beta_0^2\bar{\mathcal{G}}^{q,(3)}_{d,1}\right), \tag{42}$$

$$\bar{\mathcal{G}}^{q,(1)}_{d,1} = C_F\{-\zeta_2\}, \qquad \bar{\mathcal{G}}^{q,(2)}_{d,1} = C_F\left\{\frac{1}{3}\zeta_3\right\},$$
$$\bar{\mathcal{G}}^{q,(3)}_{d,1} = C_F\left\{\frac{1}{80}\zeta_2^2\right\}, \qquad \bar{\mathcal{G}}^{q,(4)}_{d,1} = C_F\left\{\frac{1}{20}\zeta_5 - \frac{1}{24}\zeta_2\zeta_3\right\},$$
$$\bar{\mathcal{G}}^{q,(5)}_{d,1} = C_F\left\{\frac{1}{144}\zeta_3^2 - \frac{1}{896}\zeta_2^3\right\},$$
$$\bar{\mathcal{G}}^{q,(1)}_{d,2} = C_F n_f\left\{-\frac{328}{81} + \frac{8}{3}\zeta_3 + \frac{10}{3}\zeta_2\right\} + C_F C_A\left\{\frac{2428}{81} - \frac{44}{3}\zeta_3 - \frac{67}{3}\zeta_2 - 4\zeta_2^2\right\},$$
$$\bar{\mathcal{G}}^{q,(2)}_{d,2} = C_F n_f\left\{\frac{976}{243} - \frac{70}{27}\zeta_3 - \frac{28}{9}\zeta_2 + \frac{29}{60}\zeta_2^2\right\} - C_F C_A\left\{\frac{7288}{243} - 43\zeta_5 - \frac{469}{27}\zeta_3 - \frac{202}{9}\zeta_2 + \frac{71}{3}\zeta_2\zeta_3 + \frac{319}{120}\zeta_2^2\right\},$$
$$\bar{\mathcal{G}}^{q,(1)}_{d,3} = C_F n_f^2\left\{\frac{11584}{2187} - \frac{320}{81}\zeta_3 - \frac{316}{27}\zeta_2 + \frac{152}{45}\zeta_2^2\right\} - C_F^2 n_f\left\{\frac{42727}{324} - \frac{112}{3}\zeta_5 - \frac{1672}{27}\zeta_3 - \frac{275}{6}\zeta_2 + 40\zeta_2\zeta_3 - \frac{152}{15}\zeta_2^2\right\}$$
$$+ C_F C_A^2\left\{\frac{7135981}{8748} - \frac{1430}{3}\zeta_5 - 936\zeta_3 + \frac{536}{3}\zeta_3^2 - \frac{379417}{486}\zeta_2 + \frac{4136}{9}\zeta_2\zeta_3 + \frac{1538}{45}\zeta_2^2 + \frac{17392}{315}\zeta_2^3\right\}$$
$$- C_F C_A n_f\left\{\frac{716509}{4374} - \frac{148}{3}\zeta_5 - \frac{12356}{81}\zeta_3 - \frac{51053}{243}\zeta_2 + \frac{392}{9}\zeta_2\zeta_3 + \frac{1372}{45}\zeta_2^2\right\}. \tag{43}$$

Here, $C_A = N_c$ and $C_F \equiv \frac{N_c^2-1}{2N_c}$ are the quadratic Casimirs of the adjoint and fundamental representations of $SU(N_c)$, respectively, while n_f denoted the number of quark flavors. Note, that the constants in Eq. (43) are also related to the threshold exponent $\mathbf{D}^q(a_s(q^2(1-z)^2))$ via Eq. (46) of [31].

The results for $\bar{\mathcal{G}}^{q,(j)}_{d,i}$ for $i = 1, 2$ have been obtained from the NLO and NNLO CFs expanded to sufficient order in ε [2,7,8]. We find that these results are identical to the corresponding ones for the rapidity distribution of pairs of leptons in DY production [23–25,38,39,61,64,90–93], see also [15,22,23], where a conjecture on the relation between these two

processes in the threshold limit had been proposed. Assuming that this conjecture holds true beyond NNLO, we have used $\bar{\mathcal{G}}^{q,(1)}_{d,3}$ of the rapidity distribution for the DY process also for SIDIS at N^{3}LO.

The all-order result given in Eq. (37) is useful to obtain predictions for the SV part of the CFs order by order a_s. However, to study the numerical impact of the all-order result, we need to perform a resummation in Mellin space. This can be achieved by expressing the summations given in Eq. (37) in the integral form. The integral representation for $\Phi^{\rm SV}_{d,q}$ for the DY process is given in [24] and the same can be used for SIDIS in Eq. (37):

$$\begin{aligned}\Phi^{\rm SV}_{d,q}(\hat{a}_s, Q^2, \mu^2, x', z', \varepsilon) = {}& \frac{1}{2}\delta(\bar{z}')\left(\frac{1}{\bar{x}'}\left\{\int_{\mu_F^2}^{Q^2\bar{x}'}\frac{d\lambda^2}{\lambda^2}A^q(a_s(\lambda^2)) + \bar{G}^q_d(a_s(Q^2\bar{x}'),\varepsilon)\right\}\right)_+ \\ & + Q^2\frac{d}{dQ^2}\left[\left(\frac{1}{4\bar{x}'\bar{z}'}\left\{\int_{\mu_F^2}^{Q^2\bar{x}'\bar{z}'}\frac{d\lambda^2}{\lambda^2}A^q(a_s(\lambda^2)) + \bar{G}^q_d(a_s(Q^2\bar{x}'\bar{z}'),\varepsilon)\right\}\right)_+\right] \\ & + \frac{1}{2}\delta(\bar{x}')\delta(\bar{z}')\sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{Q^2}{\mu^2}\right)^{i\frac{\varepsilon}{2}}S^i_\varepsilon\hat{\phi}^{{\rm SV},(i)}_{d,q}(\varepsilon) + \frac{1}{2}\delta(\bar{z}')\left(\frac{1}{\bar{x}'}\right)_+\sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{\mu_F^2}{\mu^2}\right)^{i\frac{\varepsilon}{2}}S^i_\varepsilon\bar{K}^{q,(i)}_d(\varepsilon) + (x'\leftrightarrow z').\end{aligned} \tag{44}$$

In complete analogy to the SV part, using the renormalization group, the coefficients $\hat{\varphi}^{{\rm NSV},(i)}_{d,\xi,q}$ in the NSV solution $\Phi^{\rm NSV}_{d,q}$, Eq. (39) can be expanded as follows, for $\xi = x', z'$:

$$\begin{aligned}\hat{\varphi}^{{\rm NSV},(1)}_{d,\xi,q}(\xi,\varepsilon) &= \frac{1}{\varepsilon}\bar{\mathcal{G}}^q_{d,\xi,1}(\xi,\varepsilon), \\ \hat{\varphi}^{{\rm NSV},(2)}_{d,\xi,q}(\xi,\varepsilon) &= \frac{1}{\varepsilon^2}(-\beta_0\bar{\mathcal{G}}^q_{d,\xi,1}(\xi,\varepsilon)) + \frac{1}{2\varepsilon}\bar{\mathcal{G}}^q_{d,\xi,2}(\xi,\varepsilon), \\ \hat{\varphi}^{{\rm NSV},(3)}_{d,\xi,q}(\xi,\varepsilon) &= \frac{1}{\varepsilon^3}\left(\frac{4}{3}\beta_0^2\bar{\mathcal{G}}^q_{d,\xi,1}(\xi,\varepsilon)\right) + \frac{1}{\varepsilon^2}\left(-\frac{1}{3}\beta_1\bar{\mathcal{G}}^q_{d,\xi,1}(\xi,\varepsilon) - \frac{4}{3}\beta_0\bar{\mathcal{G}}^q_{d,\xi,2}(\xi,\varepsilon)\right) + \frac{1}{3\varepsilon}\bar{\mathcal{G}}^q_{d,\xi,3}(\xi,\varepsilon), \\ \hat{\varphi}^{{\rm NSV},(4)}_{d,\xi,q}(\xi,\varepsilon) &= \frac{1}{\varepsilon^4}(-2\beta_0^3\bar{\mathcal{G}}^q_{d,\xi,1}(\xi,\varepsilon)) + \frac{1}{\varepsilon^3}\left(\frac{4}{3}\beta_0\beta_1\bar{\mathcal{G}}^q_{d,\xi,1}(\xi,\varepsilon) + 3\beta_0^2\bar{\mathcal{G}}^q_{d,\xi,2}(\xi,\varepsilon)\right) \\ &\quad + \frac{1}{\varepsilon^2}\left(-\frac{1}{6}\beta_2\bar{\mathcal{G}}^q_{d,\xi,1}(\xi,\varepsilon) - \frac{1}{2}\beta_1\bar{\mathcal{G}}^q_{d,\xi,2}(\xi,\varepsilon) - \frac{3}{2}\beta_0\bar{\mathcal{G}}^q_{d,\xi,3}(\xi,\varepsilon)\right) + \frac{1}{4\varepsilon}\bar{\mathcal{G}}^q_{d,\xi,4}(\xi,\varepsilon).\end{aligned} \tag{45}$$

In order to cancel the collinear poles from the AP kernels, the quantities $\bar{\mathcal{G}}^q_{d,\xi,i}(\xi,\varepsilon)$ can be further decomposed as

$$\begin{aligned}\bar{\mathcal{G}}^q_{d,x',i}(x',\varepsilon) &= 2L^q_i(x') + \bar{\chi}^q_{d,x',i}(x') + \sum_{j=1}^{\infty}\varepsilon^j\bar{\mathcal{G}}^{q,(j)}_{d,x',i}(x'), \\ \bar{\mathcal{G}}^q_{d,z',i}(z',\varepsilon) &= 2\tilde{L}^q_i(z') + \bar{\chi}^q_{d,z',i}(z') + \sum_{j=1}^{\infty}\varepsilon^j\bar{\mathcal{G}}^{q,(j)}_{d,z',i}(z'),\end{aligned} \tag{46}$$

with $L^q_i(x') = C^q_i\ln(1-x') + D^q_i$ and $\tilde{L}^q_i(z') = \tilde{C}^q_i\ln(1-z') + \tilde{D}^q_i$ and

$$\bar{\chi}^q_{d,\xi,i}(\xi) = \bar{\chi}^q_{d,i}\big|_{(\bar{\mathcal{G}}^{q,(j)}_{d,i}\to\bar{\mathcal{G}}^{q,(j)}_{d,\xi,i}(\xi))}, \tag{47}$$

where $\bar{\chi}^q_{d,i}$ is given in Eq. (42). The coefficients $C^q_i(\tilde{C}^q_i)$ and $D^q_i(\tilde{D}^q_i)$ are related to the cusp A^q_i and virtual B^q_i anomalous dimensions in the following way [85,94]. For spacelike splitting functions one has

$$\begin{aligned}&D^q_1 = -A^q_1, \qquad D^q_2 = -A^q_2 + A^q_1(B^q_1 - \beta_0), \qquad D^q_3 = -A^q_3 - A^q_1(-B^q_2 + \beta_1) - A^q_2(-B^q_1 + \beta_0), \\ &C^q_1 = 0, \qquad C^q_2 = (A^q_1)^2, \qquad C^q_3 = 2A^q_1A^q_2,\end{aligned} \tag{48}$$

and for the timelike ones

$$\tilde{D}_1^q = -A_1^q, \qquad \tilde{D}_2^q = -A_2^q + A_1^q(-B_1^q - \beta_0),$$
$$\tilde{D}_3^q = -A_3^q - A_1^q(B_2^q + \beta_1) - A_2^q(B_1^q + \beta_0),$$
$$\tilde{C}_1^q = 0, \qquad \tilde{C}_2^q = -(A_1^q)^2, \qquad \tilde{C}_3^q = -2A_1^q A_2^q. \quad (49)$$

The renormalized coefficients $\bar{\mathcal{G}}^{q,(j)}_{d,\xi,i}(\xi)$ in the above equations are parametrized in terms of $\ln^k(1-\xi), k = 0, 1, \ldots$ and all the terms vanishing as $\xi \to 1$ are dropped,

$$\bar{\mathcal{G}}^{q,(j)}_{d,\xi,i}(\xi) = \sum_{k=0}^{i+j-1} \bar{\mathcal{G}}^{q,(j,k)}_{d,\xi,i} \ln^k(1-\xi). \quad (50)$$

The leading term in $\ln(1-\xi)$ at every order in a_s depends on the power of a_s, but also the power of ε. The leading power of $\ln(1-\xi)$ can be determined by expanding the partonic cross sections to high accuracy in ε at every order in a_s. One can also use the mass factorized results such as the CFs to determine them. In the former approach, we have computed the partonic cross section at NLO to an accuracy of ε up to third order and at NNLO to zeroth power in ε. We expect that the observed pattern for those logarithms persists at higher orders and we can extrapolate to all orders in $\hat{a}_s$ and ε by fixing the highest power for $\ln(1-\xi)$ to be $i+j-1$ in Eq. (50).

For a resummation to all orders an integral form for $\Phi^{\rm NSV}_{d,q}$ is needed to study its numerical impact. In order to do this, we decompose $\Phi^{\rm NSV}_{d,q}$ as a sum of a singular part and a finite part. The singular part can be obtained by setting all the terms to zero in Eq. (46) except the first one, $2L_i^q$ and $2\tilde{L}_i^q$. It contains the correct terms that will cancel against NSV terms of the AP kernels. We write $\Phi^{\rm NSV}_{d,q} = \Phi^{\rm NSV}_{d,q,s} + \Phi^{\rm NSV}_{d,q,f}$, where the first term is singular as $\varepsilon \to 0$ and contains only L_i^q as well as $\tilde{L}_i^q$ and the second one contains the finite remainder. This decomposition facilitates to express $\Phi^{\rm NSV}_{d,q}$ in an integral form as follows:

$$\Phi^{\rm NSV}_{d,q} = \frac{1}{2}\delta(\bar{x}')\int_{\mu_F^2}^{Q^2\bar{z}'} \frac{d\lambda^2}{\lambda^2}\tilde{L}^q(a_s(\lambda^2), z') + \frac{1}{2}\left(\frac{1}{\bar{x}'}\tilde{L}^q(a_s(Q^2\bar{x}'\bar{z}'), z')\right)_+ + \frac{1}{2}\delta(\bar{x}')\varphi^q_{d,z',f}(a_s(Q^2\bar{z}'), z')$$
$$+ \frac{1}{2}\left(\frac{1}{\bar{x}'}Q^2\frac{d}{dQ^2}\varphi^q_{d,z',f}(a_s(Q^2\bar{x}'\bar{z}'), z')\right)_+ + \frac{1}{2}\delta(\bar{x}')\sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{\mu_F^2}{\mu^2}\right)^{i\frac{\varepsilon}{2}} S^i_\varepsilon \varphi^{(i)}_{d,z',s}(z', \varepsilon) + (\tilde{L}^q \leftrightarrow L^q, x' \leftrightarrow z'). \quad (51)$$

In the above equation, the term containing $\varphi^{(i)}_{d,\xi,s}$ is singular while the remaining ones are all finite as $\varepsilon \to 0$. The fact that $\Phi^{\rm NSV}_{d,q}$ is renormalization group invariant implies that $\varphi^q_{d,\xi,s}(a_s(\mu_F^2), \xi) = \sum_{i=1}^{\infty}\hat{a}_s^i(\frac{\mu_F^2}{\mu^2})^{i\frac{\varepsilon}{2}} S^i_\varepsilon \varphi^{(i)}_{d,\xi,s}(\xi, \varepsilon)$ satisfies the equations

$$\mu_F^2\frac{d}{d\mu_F^2}\varphi^q_{d,x',s}(a_s(\mu_F^2), x') = L^q(a_s(\mu_F^2), x'),$$
$$\mu_F^2\frac{d}{d\mu_F^2}\varphi^q_{d,z',s}(a_s(\mu_F^2), z') = \tilde{L}^q(a_s(\mu_F^2), z'), \quad (52)$$

where

$$L^q(a_s(\mu_F^2), x') = \sum_{i=1}^{\infty} a_s^i(\mu_F^2) L_i^q(x'),$$
$$\tilde{L}^q(a_s(\mu_F^2), z') = \sum_{i=1}^{\infty} a_s^i(\mu_F^2)\tilde{L}_i^q(z'). \quad (53)$$

Having fixed the divergent part of $\Phi^{\rm NSV}_{d,q}$ completely, we need to understand the all-order structure of the finite piece $\varphi^q_{d,\xi,f}$. Using Eq. (50), we can expand this functions in powers of renormalized coupling a_s:

$$\varphi^q_{d,\xi,f}(a_s(\lambda^2), \xi) = \sum_{i=1}^{\infty} a_s^i(\lambda^2)\sum_{k=0}^{i}\bar{\varphi}^{(k)}_{q,\xi,i}\ln^k(1-\xi), \quad (54)$$

where the highest power of $\ln(1-\xi)$ aligns at each order in a_s with that in Eq. (50). The coefficients $\bar{\varphi}^{(k)}_{q,\xi,i}$ can be expressed in terms of their unrenormalized counter part $\bar{\mathcal{G}}^{q,(j,k)}_{d,\xi,i}$'s in Eq. (50) as

$$\bar{\varphi}^{(k)}_{q,\xi,1} = \bar{\mathcal{G}}^{q,(1,k)}_{d,\xi,1}, \quad k = 0,1$$
$$\bar{\varphi}^{(k)}_{q,\xi,2} = \left(\frac{1}{2}\bar{\mathcal{G}}^{q,(1,k)}_{d,\xi,2} + \beta_0\bar{\mathcal{G}}^{q,(2,k)}_{d,\xi,1}\right), \quad k = 0,1,2$$
$$\bar{\varphi}^{(k)}_{q,\xi,3} = \left(\frac{1}{3}\bar{\mathcal{G}}^{q,(1,k)}_{d,\xi,3} + \frac{2}{3}\beta_1\bar{\mathcal{G}}^{q,(2,k)}_{d,\xi,1} + \frac{2}{3}\beta_0\bar{\mathcal{G}}^{q,(2,k)}_{d,\xi,2} + \frac{4}{3}\beta_0^2\bar{\mathcal{G}}^{q,(3,k)}_{d,\xi,1}\right), \quad k = 0,1,2,3$$
$$\bar{\varphi}^{(k)}_{q,\xi,4} = \left(\frac{1}{4}\bar{\mathcal{G}}^{q,(1,k)}_{d,\xi,4} + \frac{1}{2}\beta_2\bar{\mathcal{G}}^{q,(2,k)}_{d,\xi,1} + \frac{1}{2}\beta_1\bar{\mathcal{G}}^{q,(2,k)}_{d,\xi,2} + \frac{1}{2}\beta_0\bar{\mathcal{G}}^{q,(2,k)}_{d,\xi,3} + 2\beta_0\beta_1\bar{\mathcal{G}}^{q,(3,k)}_{d,\xi,1} + \beta_0^2\bar{\mathcal{G}}^{q,(3,k)}_{d,\xi,2} + 2\beta_0^3\bar{\mathcal{G}}^{q,(4,k)}_{d,\xi,1}\right), \quad k = 0,1,2,3,4. \quad (55)$$

The structure of divergent and finite pieces of $\Phi^{\rm NSV}_{d,q}$ allows us to determine the coefficients $\bar{\mathcal{G}}^{q,(j,k)}_{d,\xi,i}$. Since our final result depends on $\bar{\varphi}^{(k)}_{q,\xi,i}$ we can directly determine them by comparison with the fixed-order results for the CFs, known to NNLO level [7–12] and they are listed below:

$$\begin{aligned}
&\bar{\varphi}^{(0)}_{q,x',1} = C_F\{-4\}, \qquad \bar{\varphi}^{(1)}_{q,x',1} = 0,\\
&\bar{\varphi}^{(0)}_{q,x',2} = C_F C_A\left\{\frac{1108}{27} - 28\zeta_3 - \frac{44}{3}\zeta_2\right\} - C_F n_f\left\{\frac{52}{27} - \frac{8}{3}\zeta_2\right\} - C_F^2\{8\zeta_2\},\\
&\bar{\varphi}^{(1)}_{q,x',2} = C_F C_A\{-10\} + C_F^2\{10\}, \qquad \bar{\varphi}^{(2)}_{q,x',2} = C_F^2\{4\},
\end{aligned} \tag{56}$$

$$\begin{aligned}
&\bar{\varphi}^{(0)}_{q,z',1} = C_F\{4\}, \qquad \bar{\varphi}^{(1)}_{q,z',1} = 0,\\
&\bar{\varphi}^{(0)}_{q,z',2} = C_F C_A\left\{\frac{1492}{27} - 28\zeta_3 - \frac{68}{3}\zeta_2\right\} - C_F n_f\left\{\frac{364}{27} - \frac{8}{3}\zeta_2\right\} - C_F^2\{16\zeta_2\},\\
&\bar{\varphi}^{(1)}_{q,z',2} = C_F C_A\{10\} - C_F^2\{10\}, \qquad \bar{\varphi}^{(2)}_{q,z',2} = C_F^2\{-4\},
\end{aligned} \tag{57}$$

Note that the constants $\bar{\varphi}^{(k)}_{q,x',i}$ are different from $\bar{\varphi}^{(k)}_{q,z',i}$ unlike those appearing in rapidity distributions of the DY process, see [61].

Having obtained the general structure of the soft function, the final step is the substitution of the integral representations for $\Phi^{\rm SV}_{d,q}$ in Eq. (44), for $\Phi^{\rm NSV}_{d,q}$ in Eq. (51), the solutions to the form factor in Eq. (28), and the AP kernels in Eq. (30) into Eq. (21). We find that all the soft and collinear singularities cancel among themselves leaving SV and NSV parts of the CFs IR finite,

$$(\Delta)\mathcal{C}^{\rm SV+NSV}_{J,qq} = \mathcal{C}\exp(\Psi^q_d(Q^2,\mu_F^2,x',z',\varepsilon))|_{\varepsilon=0}, \tag{58}$$

where the IR finite expression Ψ^q_d takes a simple form

$$\begin{aligned}
\Psi^q_d = {}& \frac{\delta(\bar{x}')}{2}\left(\int_{\mu_F^2}^{Q^2\bar{z}'}\frac{d\lambda^2}{\lambda^2}\tilde{\mathcal{P}}^q(a_s(\lambda^2),\bar{z}') + \mathcal{Q}^q(a_s(Q_2^2),\bar{z}')\right)_+\\
&+\frac{1}{4}\left(\frac{1}{\bar{x}'}\left\{\tilde{\mathcal{P}}^q(a_s(Q_{12}^2),\bar{z}') + 2\tilde{L}^q(a_s(Q_{12}^2),\bar{z}')\right.\right.\\
&\left.\left.+ Q^2\frac{d}{dQ^2}(\mathcal{Q}^q(a_s(Q_2^2),\bar{z}') + 2\varphi^q_{d,z',f}(a_s(Q_2^2),\bar{z}'))\right\}\right)_+\\
&+\frac{1}{2}\delta(\bar{x}')\delta(\bar{z}')\ln(g^q_{d,0}(a_s(\mu_F^2))) + (x'\leftrightarrow z').
\end{aligned} \tag{59}$$

Here, we use the shorthand $\bar{x}' = (1-x')$ and $\bar{z}' = (1-z')$, $Q_1^2 = Q^2\bar{x}', Q_2^2 = Q^2\bar{z}'$ and $Q_{12}^2 = Q^2\bar{x}'\bar{z}'$. The subscript $+$ indicates the standard plus distribution. In the above equation, $\mathcal{P}^q, \tilde{\mathcal{P}}^q$ are related diagonal splitting functions given in Eq. (31),

$$\begin{aligned}
&\mathcal{P}^q(a_s,\bar{x}') = P_{q\leftarrow q}(\mu_s^2,\bar{x}') - 2B^q(a_s)\delta(\bar{x}'),\\
&\tilde{\mathcal{P}}^q(a_s,\bar{z}') = \tilde{P}_{q\leftarrow q}(\mu_s^2,\bar{z}') - 2B^q(a_s)\delta(\bar{z}'),
\end{aligned} \tag{60}$$

and $\mathcal{Q}^q_d(a_s,\bar{z}') = \frac{2}{\bar{z}'}G^q_d(a_s) + 2\,\varphi^q_{d,z',f}(a_s,\bar{z}')$ and similar for x'. Note that $g^q_{d,0}$ is process dependent and receives contributions only from the form factor and finite parts of $\Phi^{\rm SV}_{d,q}$.

The result for $(\Delta)\mathcal{C}^{\rm SV+NSV}_{J,qq}$ in Eq. (59) is valid to all orders in a_s. The exponent Ψ^q_d contains universal (process-independent) quantities such as the QCD β function and anomalous dimensions A^q, B^q, f^q etc. and process-dependent quantities G^q_d, $\varphi^q_{d,\xi,f}$, and $g^q_{d,0}$. Each of them can be expanded in powers of a_s, hence Ψ^q_d is expanded as

$$\Psi^q_d(Q^2,\mu_F^2) = \sum_{i=1}^{\infty} a_s^i(\mu_R^2)\Psi^{q,(i)}_d(Q^2,\mu_F^2,\mu_R^2), \tag{61}$$

where we have suppressed the dependence on x', z' in Ψ^q_d and $\Psi^{q,(i)}_d$.

The result in Eq. (58) is the main result of the paper. It sums up towers of SV and NSV terms to all orders in a_s in those CFs, which dominate near threshold. The particular logarithmic structure of Ψ^q_d given in Eq. (59) allows us to predict certain SV and NSV terms to any order in a_s, i.e., a_s^n for the diagonal CFs for all $n \geq 1$, provided Ψ^q_d is known up to order a_s^m with $n > m$. The achievable accuracy of those predictions for $\Psi^{q,(i)}_d$ s for $i = 1, 2, 3$ and for general $i = n$ is summarized in Tables I and II. For example, if we know only $\Psi^{q,(1)}_d$, we can predict the following terms for $(\Delta)\mathcal{C}^{(2)}_{J,cc}$, i.e., for the SV part, $\{\delta_{\bar{x}'}\mathcal{D}^3_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{D}^3_{\bar{x}'}\}$ and $\{\mathcal{D}^i_{\bar{x}'}\mathcal{D}^j_{\bar{z}'}\}$ for $i+j=2$ and, likewise, for the NSV one $\{\delta_{\bar{x}'}\mathcal{L}^3_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{L}^3_{\bar{x}'}\}$, and $\{\mathcal{D}^i_{\bar{x}'}\mathcal{L}^j_{\bar{z}'}, \mathcal{D}^i_{\bar{z}'}\mathcal{L}^j_{\bar{x}'}\}$ for $i+j=2$. If, in addition, we allow for terms present in $\Psi^{q,(2)}_d$ which depend only on information about $\Psi^{q,(1)}_d$, then, because of the renormalization group

TABLE I. Towers of SV distributions $(\delta_{\bar{x}'}, \delta_{\bar{z}'}, \mathcal{D}_{\bar{x}'}, \mathcal{D}_{\bar{z}'})$ that can be predicted for $(\Delta)\mathcal{C}^{(i)}_{J,cc}$ for $c = q, \bar{q}$ given $\Psi^{q,(n)}_d$, $j,\ k \geq 0$ and $\delta_{\bar{\xi}} = \delta(1-\xi)$, $\mathcal{D}^j_{\bar{\xi}} = [\frac{\ln^j(1-\xi)}{1-\xi}]_+$ with $\xi = x', z'$.

Given				Preductuibs for SV		
$\Psi^{q,(1)}_d$	$\Psi^{q,(2)}_d$	$\Psi^{q,(3)}_d$	$\Psi^{q,(n)}_d$	$(\Delta)\mathcal{C}^{(2)}_{J,cc}$	$(\Delta)\mathcal{C}^{(3)}_{J,cc}$	$(\Delta)\mathcal{C}^{(i)}_{J,cc}$
$\delta_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}$ $\mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=0$ $j'=1,0$				$\mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}$, $\mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2,1$ $j'=3,2$	$\mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}$, $\mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=4,3$ $j'=5,4$	$\mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2i-2, 2i-3$ $j'=2i-1, 2i-2$
	$\delta_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}$ $\mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2,1,0$ $j'=3,\ldots,0$				$\mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}$, $\mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=4,3,2,1$ $j'=5,4,3,2$	$\mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2i-2,\ldots,2i-4,2i-5$ $j'=2i-1,\ldots,2i-3,2i-4$
		$\delta_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}$ $\mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=4,\ldots,0$ $j'=5,\ldots,0$				$\mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2i-2,\ldots,2i-5,\ldots,2i-7$ $j'=2i-1,\ldots,2i-4,\ldots,2i-6$
			$\delta_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}$ $\mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2n-2,\ldots,0$ $j'=2n-1,\ldots,0$			$\mathcal{D}^j_{\bar{x}'}\mathcal{D}^k_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{D}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2i-2,\ldots,2i-n-2,\ldots,2i-2n-1$ $j'=2i-1,\ldots,2i-n-1,\ldots,2i-2n$

TABLE II. Towers of NSV logarithms that can be predicted for $(\Delta)\mathcal{C}^{(i)}_{J,cc}$ for $c = q, \bar{q}$ given $\Psi^{q,(n)}_d$, $j,\ k \geq 0$ and $\delta_{\bar{\xi}} = \delta(1-\xi)$, $\mathcal{D}^j_{\bar{\xi}} = [\frac{\ln^j(1-\xi)}{1-\xi}]_+$ and $\mathcal{L}^j_{\bar{\xi}} = \ln^j(1-\xi)$ with $\xi = x', z'$.

Given, SV +				Predictions for NSV		
$\Psi^{q,(1)}_d$	$\Psi^{q,(2)}_d$	$\Psi^{q,(3)}_d$	$\Psi^{q,(n)}_d$	$(\Delta)\mathcal{C}^{(2)}_{J,cc}$	$(\Delta)\mathcal{C}^{(3)}_{J,cc}$	$(\Delta)\mathcal{C}^{(i)}_{J,cc}$
$\mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}$, $\mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}$ $j+k=0$ $j'=1,0$				$\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}, \mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}$ $\mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2$ $j'=3$	$\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}, \mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}$ $\mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=4$ $j'=5$	$\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}, \mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}, \mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2i-2$ $j'=2i-1$
	$\mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}$, $\mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}$ $j+k=2,1,0$ $j'=3,\ldots,0$				$\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}, \mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}$ $\mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=4,3,2$ $j'=5,4$	$\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}, \mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}, \mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2i-2,2i-3,2i-4$ $j'=2i-1,2i-2$
		$\mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}, \mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}$ $j+k=4,\ldots,0$ $j'=5,\ldots,0$				$\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}, \mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}, \mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2i-2,\ldots,2i-5,2i-6$ $j'=2i-1,\ldots,2i-3$
			$\mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}, \mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}$ $j+k=2n-2,\ldots,0$ $j'=2n-1,\ldots,0$			$\mathcal{D}^j_{\bar{x}'}\mathcal{L}^k_{\bar{z}'}, \mathcal{D}^j_{\bar{z}'}\mathcal{L}^k_{\bar{x}'}, \mathcal{L}^{j'}_{\bar{x}'}\delta_{\bar{z}'}, \mathcal{L}^{j'}_{\bar{z}'}\delta_{\bar{x}'}$ $j+k=2i-2,\ldots,2i-n-2,\ldots,2i-2n$ $j'=2i-1,\ldots,2i-n$

equations, we can predict, for the SV part also $\{\delta_{\bar{x}'}\mathcal{D}^2_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{D}^2_{\bar{x}'}\}$ and $\{\mathcal{D}^i_{\bar{x}'}\mathcal{D}^j_{\bar{z}'}\}$ for $i+j=1$, and for the NSV part $\{\mathcal{D}^1_{\bar{x}'}\mathcal{L}^0_{\bar{z}'}, \mathcal{D}^1_{\bar{z}'}\mathcal{L}^0_{\bar{x}'}\}$. Knowledge on higher-order corrections to the anomalous dimensions leads to predictions for the SV terms $\{\delta_{\bar{x}'}\mathcal{D}^i_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{D}^i_{\bar{x}'}\}$ for $i=0$, 1 and $\{\mathcal{D}^0_{\bar{x}'}\mathcal{D}^0_{\bar{z}'}\}$ and for the NSV terms $\{\mathcal{D}^0_{\bar{x}'}\mathcal{L}^i_{\bar{z}'}, \mathcal{D}^0_{\bar{z}'}\mathcal{L}^i_{\bar{x}'}\}$ for $i=0$, 1. We elaborate on this further below when we make predictions at third and fourth order in a_s using the $\Psi^{q,(i)}_d$ s at $i=1$, 2 and partial results on $\Psi^{q,(i)}_d$ s for $i>2$.

III. RESULTS OF THE SV AND NSV COEFFICIENT FUNCTIONS

In this section, we present the analytic results of the SV and NSV CFs, $(\Delta)\mathcal{C}^{\mathrm{SV}}_{J,cc}$, and $(\Delta)\mathcal{C}^{\mathrm{NSV}}_{J,cc}$ for $c=\{q,\bar{q}\}$ respectively. We present complete results at NLO and NNLO and partial results up to N^{4}LO in the threshold region for the CFs, expanded in a_s as $(\Delta)\mathcal{C}^{(\mathrm{N})\mathrm{SV}}_{J,cc}=\sum_{i=0}^{\infty}a_s^i(\mu_R^2)(\Delta)\mathcal{C}^{(i),(\mathrm{N})\mathrm{SV}}_{J,cc}$. Their normalization at LO reads $(\Delta)\mathcal{C}^{(0),\mathrm{SV}}_{J,cc}=\delta(1-x')\delta(1-z')$ and $(\Delta)\mathcal{C}^{(0),\mathrm{NSV}}_{J,cc}=0$, and we have set $\mu_F^2=\mu_R^2=Q^2$ in the sequel.

A. SV coefficient functions

For the perturbative expansion of the SV part of the CFs, we define in addition to C_A and C_F further group invariants of the color $SU(N_c)$ gauge group. We also define

$$\frac{d_F^{abcd}d_F^{abcd}}{N_F}=\frac{(N_c^2-1)(N_c^4-6N_c^2+18)}{96N_c^3},$$
$$\frac{d_F^{abcd}d_A^{abcd}}{N_F}=\frac{(N_c^2-1)(N_c^2+6)}{48},\qquad N_4=\frac{(N_c^2-4)}{N_c}. \tag{62}$$

Also, below we will use n_{fv}, which is defined as the charge weighted sum of quark flavors. At NLO, i.e., at $\mathcal{O}(a_s)$,

$$(\Delta)\mathcal{C}^{(1),\mathrm{SV}}_{J,cc}=\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{z}'}[C_F\{4\}]-\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\delta}_{\bar{z}'}[C_F\{8\}]+\boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{z}'}[C_F\{2\}]+(\boldsymbol{x}'\leftrightarrow\boldsymbol{z}'). \tag{63}$$

The NNLO CF at SV accuracy read

$$\begin{aligned}(\Delta)\mathcal{C}^{(2),\mathrm{SV}}_{J,cc}=&\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{3}}_{\bar{z}'}[C_F^2\{8\}]-\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{z}'}\left[C_FC_A\left\{\frac{22}{3}\right\}-C_Fn_f\left\{\frac{4}{3}\right\}\right]+\boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{z}'}[C_F^2\{24\}]+\boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{z}'}[C_F^2\{24\}]\\
&-\boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{z}'}\left[C_FC_A\left\{\frac{44}{3}\right\}-C_Fn_f\left\{\frac{8}{3}\right\}\right]+\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{z}'}\left[C_FC_A\left\{\frac{268}{9}-8\zeta_2\right\}-C_Fn_f\left\{\frac{40}{9}\right\}\right.\\
&\left.-C_F^2\{64+32\zeta_2\}\right]-\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\delta}_{\bar{z}'}\left[C_FC_A\left\{\frac{1535}{24}-\frac{86}{3}\zeta_3+\frac{269}{9}\zeta_2-\frac{34}{5}\zeta_2^2\right\}-C_F^2\left\{\frac{511}{8}-30\zeta_3+29\zeta_2-\frac{28}{5}\zeta_2^2\right\}\right.\\
&\left.-C_Fn_f\left\{\frac{127}{12}+\frac{4}{3}\zeta_3+\frac{38}{9}\zeta_2\right\}\right]+\boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{z}'}\left[C_FC_A\left\{\frac{134}{9}-4\zeta_2\right\}-C_Fn_f\left\{\frac{20}{9}\right\}-C_F^2\{32+16\zeta_2\}\right]\\
&-\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{z}'}\left[C_FC_A\left\{\frac{808}{27}-28\zeta_3-\frac{44}{3}\zeta_2\right\}-C_Fn_f\left\{\frac{112}{27}-\frac{8}{3}\zeta_2\right\}-C_F^2\{32\zeta_3\}\right]+(\boldsymbol{x}'\leftrightarrow\boldsymbol{z}').\end{aligned} \tag{64}$$

In the following we predict the CF at third order in a_s using the available information on the perturbative results:

(i) the knowledge of $\Psi^{q,(1)}_d$ and $\Psi^{q,(2)}_d$ can be used to predict the coefficients of $\{\delta_{\bar{x}'}\mathcal{D}^i_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{D}^i_{\bar{x}'}\}$ for $i=5$, 4, 3, 2 and $\{\mathcal{D}^i_{\bar{x}'}\mathcal{D}^j_{\bar{z}'}\}$ for $i+j=4$, 3, 2, 1 in $(\Delta)\mathcal{C}^{(3),\mathrm{SV}}_{J,cc}$;

(ii) the results on the cusp and eikonal anomalous dimensions (A^q_3, f^q_3) can be used to obtain the coefficients of $\{\delta_{\bar{x}'}\mathcal{D}^i_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{D}^i_{\bar{x}'}\}$ for $i=1$, 0 and $\{\mathcal{D}^0_{\bar{x}'}\mathcal{D}^0_{\bar{z}'}\}$;

(iii) using $\mathcal{G}^{q,(1)}_{d,3}$ from the three-loop form factors and $\bar{\mathcal{G}}^{q,(1)}_{d,3}$ from the three-loop soft functions, we can obtain the coefficient of $\delta_{\bar{x}'}\delta_{\bar{z}'}$.

The results for the SV part of the CFs at third order obtained in this way read

$$\begin{aligned}(\Delta)\mathcal{C}^{(3),\mathrm{SV}}_{J,cc}=&\boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{z}'}[C_F^3\{120\}]+\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{5}}_{\bar{z}'}[C_F^3\{8\}]-\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{4}}_{\bar{z}'}\left[C_F^2C_A\left\{\frac{220}{9}\right\}-C_F^2n_f\left\{\frac{40}{9}\right\}\right]\\
&+\boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{3}}_{\bar{z}'}\left[C_FC_A^2\left\{\frac{484}{27}\right\}-C_Fn_fC_A\left\{\frac{176}{27}\right\}+C_Fn_f^2\left\{\frac{16}{27}\right\}+C_F^2C_A\left\{\frac{1072}{9}-32\zeta_2\right\}-C_F^2n_f\left\{\frac{160}{9}\right\}\right.\end{aligned}$$

$$
\begin{aligned}
&- C_F^3\{128+128\zeta_2\}] - \boldsymbol{\delta_{\bar{x}'}\mathcal{D}^2_{\bar{z}'}}\left[C_F C_A^2\left\{\frac{3560}{27}-\frac{88}{3}\zeta_2\right\} - C_F C_A n_f\left\{\frac{1156}{27}-\frac{16}{3}\zeta_2\right\} + C_F n_f^2\left\{\frac{80}{27}\right\}\right.\\
&\left. + C_F^2 C_A\left\{\frac{560}{9}-168\zeta_3-264\zeta_2\right\} - C_F^2 n_f\left\{\frac{68}{9}-48\zeta_2\right\} - C_F^3\{320\zeta_3\}\right] + \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{D}^3_{\bar{z}'}}[C_F^3\{160\}]\\
&- \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{D}^2_{\bar{z}'}}\left[C_F^2 C_A\left\{\frac{880}{3}\right\} - C_F^2 n_f\left\{\frac{160}{3}\right\}\right] + \boldsymbol{\delta_{\bar{x}'}\mathcal{D}^1_{\bar{z}'}}\left[C_F C_A^2\left\{\frac{31006}{81}-176\zeta_3-\frac{680}{3}\zeta_2+\frac{176}{5}\zeta_2^2\right\}\right.\\
&- C_F C_A n_f\left\{\frac{8204}{81}-\frac{512}{9}\zeta_2\right\} + C_F n_f^2\left\{\frac{400}{81}-\frac{32}{9}\zeta_2\right\} - C_F^2 C_A\left\{\frac{8893}{9}+\frac{368}{3}\zeta_3+\frac{5288}{9}\zeta_2-\frac{912}{5}\zeta_2^2\right\}\\
&\left. + C_F^2 n_f\left\{\frac{1072}{9}+\frac{320}{3}\zeta_3+\frac{944}{9}\zeta_2\right\} + C_F^3\left\{511-240\zeta_3+744\zeta_2+\frac{288}{5}\zeta_2^2\right\}\right] + \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{D}^4_{\bar{z}'}}[C_F^3\{40\}]\\
&+ \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{D}^3_{\bar{z}'}}\left[-C_F^2 C_A\left\{\frac{880}{9}\right\} + C_F^2 n_f\left\{\frac{160}{9}\right\}\right] + \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{D}^1_{\bar{z}'}}[C_F C_A^2\left\{\frac{484}{9}\right\} - C_F C_A n_f\left\{\frac{176}{9}\right\} + C_F n_f^2\left\{\frac{16}{9}\right\}\\
&+ C_F^2 C_A\left\{\frac{1072}{3}-96\zeta_2\right\} - C_F^2 n_f\left\{\frac{160}{3}\right\} - C_F^3\{384+384\zeta_2\}\Big] - \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{D}^1_{\bar{z}'}}\left[C_F C_A^2\left\{\frac{7120}{27}-\frac{176}{3}\zeta_2\right\}\right.\\
&- C_F C_A n_f\left\{\frac{2312}{27}-\frac{32}{3}\zeta_2\right\} + C_F n_f^2\left\{\frac{160}{27}\right\} + C_F^2 C_A\left\{\frac{1120}{9}-336\zeta_3-528\zeta_2\right\} - C_F^2 n_f\left\{\frac{136}{9}-96\zeta_2\right\}\\
&- C_F^3\{640\zeta_3\}] - \boldsymbol{\delta_{\bar{x}'}\mathcal{D}^0_{\bar{z}'}}\left[C_F n_f^2\left\{\frac{1856}{729}+\frac{32}{27}\zeta_3-\frac{160}{27}\zeta_2\right\} + C_F^2 n_f\left\{3+\frac{944}{9}\zeta_3-\frac{40}{27}\zeta_2-\frac{224}{15}\zeta_2^2\right\}\right.\\
&- C_F C_A n_f\left\{\frac{62626}{729}-\frac{536}{9}\zeta_3-\frac{7760}{81}\zeta_2+\frac{208}{15}\zeta_2^2\right\} - C_F^2 C_A\left\{\frac{12928}{27}+\frac{256}{9}\zeta_3+\frac{128}{27}\zeta_2-352\zeta_2\zeta_3-\frac{352}{3}\zeta_2^2\right\}\\
&\left. + C_F C_A^2\left\{\frac{297029}{729}+192\zeta_5-\frac{14264}{27}\zeta_3-\frac{27752}{81}\zeta_2+\frac{176}{3}\zeta_2\zeta_3+\frac{616}{15}\zeta_2^2\right\} - C_F^3\{384\zeta_5-512\zeta_3-512\zeta_2\zeta_3\}\right]\\
&+ \boldsymbol{\delta_{\bar{x}'}\delta_{\bar{z}'}}\left[C_F N_4 n_{fv}\left\{4-\frac{80}{3}\zeta_5+\frac{14}{3}\zeta_3+10\zeta_2-\frac{2}{5}\zeta_2^2\right\} - C_F n_f^2\left\{\frac{7081}{486}+\frac{152}{81}\zeta_3+\frac{1028}{81}\zeta_2+\frac{64}{135}\zeta_2^2\right\}\right.\\
&- C_F C_A^2\left\{\frac{1505881}{1944}+102\zeta_5-\frac{125105}{162}\zeta_3+\frac{200}{3}\zeta_3^2+\frac{124895}{162}\zeta_2-234\zeta_2\zeta_3-\frac{23539}{270}\zeta_2^2+\frac{4456}{315}\zeta_2^3\right\}\\
&+ C_F C_A n_f\left\{\frac{110651}{486}-4\zeta_5-\frac{9944}{81}\zeta_3+\frac{5696}{27}\zeta_2-\frac{8}{3}\zeta_2\zeta_3-\frac{1054}{135}\zeta_2^2\right\} + C_F^2 C_A\left\{\frac{74321}{72}-\frac{3812}{9}\zeta_5-\frac{2986}{3}\zeta_3\right.\\
&+ \frac{632}{3}\zeta_3^2+\frac{33451}{27}\zeta_2-\frac{2480}{9}\zeta_2\zeta_3+\frac{1304}{135}\zeta_2^2-\frac{3896}{63}\zeta_2^3\Big\} - C_F^2 n_f\left\{\frac{421}{6}+\frac{112}{9}\zeta_5-180\zeta_3+\frac{3844}{27}\zeta_2\right.\\
&\left.\left. + \frac{448}{9}\zeta_2\zeta_3+\frac{2504}{135}\zeta_2^2\right\} - C_F^3\left\{\frac{5599}{12}-664\zeta_5+230\zeta_3-\frac{368}{3}\zeta_3^2+\frac{1598}{3}\zeta_2-280\zeta_2\zeta_3+\frac{442}{5}\zeta_2^2-\frac{20816}{315}\zeta_3^2\right\}\right]\\
&+ \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{D}^2_{\bar{z}'}}\left[C_F C_A^2\left\{\frac{484}{9}\right\} - C_F C_A n_f\left\{\frac{176}{9}\right\} + C_F n_f^2\left\{\frac{16}{9}\right\} + C_F^2 C_A\left\{\frac{1072}{3}-96\zeta_2\right\} - C_F^2 n_f\left\{\frac{160}{3}\right\}\right.\\
&\left. - C_F^3\{384+384\zeta_2\}\right] + \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{D}^0_{\bar{z}'}}\left[C_F C_A^2\left\{\frac{15503}{81}-88\zeta_3-\frac{340}{3}\zeta_2+\frac{88}{5}\zeta_2^2\right\} - C_F C_A n_f\left\{\frac{4102}{81}-\frac{256}{9}\zeta_2\right\}\right.\\
&+ C_F n_f^2\left\{\frac{200}{81}-\frac{16}{9}\zeta_2\right\} - C_F^2 C_A\left\{\frac{8893}{18}+\frac{184}{3}\zeta_3+\frac{2644}{9}\zeta_2-\frac{456}{5}\zeta_2^2\right\} + C_F^2 n_f\left\{\frac{536}{9}+\frac{160}{3}\zeta_3+\frac{472}{9}\zeta_2\right\}\\
&\left.\left. + C_F^3\left\{\frac{511}{2}-120\zeta_3+372\zeta_2+\frac{144}{5}\zeta_2^2\right\}\right] + (\boldsymbol{x'}\leftrightarrow\boldsymbol{z'}). \qquad (65)
\end{aligned}
$$

Similarly at fourth order we can predict various terms as follows:

(i) Using $\Psi_d^{q,(1)}$, $\Psi_d^{q,(2)}$, and the three-loop cusp and eikonal anomalous dimensions (A_3^q, f_3^q), we can obtain the coefficients of $\{\delta_{\bar{x}'}\boldsymbol{\mathcal{D}}^i_{\bar{z}'}, \delta_{\bar{z}'}\boldsymbol{\mathcal{D}}^i_{\bar{x}'}\}$ for $i = 7, 6, 5, 4, 3, 2$ and $\{\boldsymbol{\mathcal{D}}^i_{\bar{x}'}\boldsymbol{\mathcal{D}}^j_{\bar{z}'}\}$ for $i+j = 6, 5, 4, 3, 2, 1$, in $(\Delta)\mathcal{C}^{(4),\mathrm{SV}}_{J,cc}$.

(ii) Using $\bar{\mathcal{G}}^{q,(1)}_{d,3}$ from the three-loop soft functions as given in Eq. (43), along with higher order anomalous dimensions $(A_3^q, A_4^q, B_3^q, f_3^q)$, we obtain the coefficients of $\{\delta_{\bar{x}'}\boldsymbol{\mathcal{D}}^1_{\bar{z}'}, \delta_{\bar{z}'}\boldsymbol{\mathcal{D}}^1_{\bar{x}'}\}$ and $\{\boldsymbol{\mathcal{D}}^0_{\bar{x}'}\boldsymbol{\mathcal{D}}^0_{\bar{z}'}\}$.

(iii) Finally, thanks to the recent computation [80] on four-loop eikonal anomalous dimension (f_4) the coefficients of $\{\delta_{\bar{x}'}\boldsymbol{\mathcal{D}}^0_{\bar{z}'}, \delta_{\bar{z}'}\boldsymbol{\mathcal{D}}^0_{\bar{x}'}\}$ in $(\Delta)\mathcal{C}^{(4),\mathrm{SV}}_{J,cc}$ can now be predicted.

However, the coefficient of $\delta_{\bar{x}'}\delta_{\bar{z}'}$ cannot be predicted with the available information. Thus, the SV part of the CFs at fourth order is given by

$$
\begin{aligned}
(\Delta)\mathcal{C}^{(4),\mathrm{SV}}_{J,cc} = {} & \boldsymbol{\mathcal{D}}^{\mathbf{3}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{3}}_{\bar{z}'}\left[C_F^4\left\{\frac{1120}{3}\right\}\right] + \boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{4}}_{\bar{z}'}[C_F^4\{560\}] - \boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{3}}_{\bar{z}'}\left[C_F^3C_A\left\{\frac{6160}{3}\right\} - C_F^3n_f\left\{\frac{1120}{3}\right\}\right] \\
& + \boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{z}'}\left[C_F^2C_A^2\left\{\frac{9680}{9}\right\} - C_F^2n_fC_A\left\{\frac{3520}{9}\right\} + C_F^2n_f^2\left\{\frac{320}{9}\right\} + C_F^3C_A\{2680 - 720\zeta_2\right\} - C_F^3n_f\{400\} \\
& - C_F^4\{1920 + 2880\zeta_2\}\right] + \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{7}}_{\bar{z}'}\left[C_F^4\left\{\frac{16}{3}\right\}\right] - \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{6}}_{\bar{z}'}\left[C_F^3C_A\left\{\frac{308}{9}\right\} - C_F^3n_f\left\{\frac{56}{9}\right\}\right] \\
& + \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{5}}_{\bar{z}'}\left[C_F^2C_A^2\left\{\frac{1936}{27}\right\} - C_F^2n_fC_A\left\{\frac{704}{27}\right\} + C_F^2n_f^2\left\{\frac{64}{27}\right\} + C_F^3C_A\left\{\frac{536}{3} - 48\zeta_2\right\} - C_F^3n_f\left\{\frac{80}{3}\right\}\right. \\
& \left. - C_F^4\{128 + 192\zeta_2\}\right] + \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{4}}_{\bar{z}'}\left[C_Fn_fC_A^2\left\{\frac{242}{9}\right\} - C_FC_A^3\left\{\frac{1331}{27}\right\} - C_Fn_f^2C_A\left\{\frac{44}{9}\right\} + C_Fn_f^3\left\{\frac{8}{27}\right\}\right. \\
& + C_F^4\left\{\frac{2240}{3}\zeta_3\right\} - C_F^2C_A^2\left\{\frac{16780}{27} - \frac{440}{3}\zeta_2\right\} + C_F^2n_fC_A\left\{\frac{5480}{27} - \frac{80}{3}\zeta_2\right\} - C_F^3n_f\left\{\frac{440}{27} + \frac{1520}{9}\zeta_2\right\} \\
& \left. - C_F^2n_f^2\left\{\frac{400}{27}\right\} + C_F^3C_A\left\{\frac{2480}{27} + 280\zeta_3 + \frac{8360}{9}\zeta_2\right\}\right] + \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{3}}_{\bar{z}'}\left[C_F^3n_f\left\{\frac{2764}{9} + \frac{5696}{9}\zeta_3 + \frac{4448}{9}\zeta_2\right\}\right. \\
& - C_Fn_f^3\left\{\frac{160}{81}\right\} + C_F^2C_A^2\left\{\frac{481216}{243} - \frac{8800}{9}\zeta_3 - \frac{56080}{27}\zeta_2 + \frac{864}{5}\zeta_2^2\right\} - C_Fn_fC_A^2\left\{\frac{7324}{27} - \frac{352}{9}\zeta_2\right\} \\
& + C_Fn_f^2C_A\left\{\frac{1144}{27} - \frac{32}{9}\zeta_2\right\} - C_F^2n_fC_A\left\{\frac{133988}{243} - \frac{448}{9}\zeta_3 - \frac{16256}{27}\zeta_2\right\} + C_F^2n_f^2\left\{\frac{7768}{243} - \frac{1216}{27}\zeta_2\right\} \\
& - C_F^3C_A\left\{\frac{26362}{9} + \frac{19808}{9}\zeta_3 + \frac{25424}{9}\zeta_2 - \frac{4384}{5}\zeta_2^2\right\} + C_F^4\{1022 - 480\zeta_3 + 2512\zeta_2 + \frac{3136}{5}\zeta_2^2\right\} \\
& \left. + C_FC_A^3\left\{\frac{43648}{81} - \frac{968}{9}\zeta_2\right\}\right] + \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{z}'}\left[C_F^3C_A\left\{\frac{34849}{18} + \frac{14696}{3}\zeta_3 - \frac{15988}{9}\zeta_2 - 4608\zeta_2\zeta_3 - \frac{36784}{15}\zeta_2^2\right\}\right. \\
& + C_Fn_fC_A^2\left\{\frac{10277}{9} - 176\zeta_3 - \frac{5096}{9}\zeta_2 + \frac{176}{5}\zeta_2^2\right\} - C_FC_A^3\left\{\frac{206440}{81} - 968\zeta_3 - \frac{4012}{3}\zeta_2 + \frac{968}{5}\zeta_2^2\right\} \\
& - C_Fn_f^2C_A\left\{\frac{3947}{27} - \frac{688}{9}\zeta_2\right\} + C_Fn_f^3\left\{\frac{400}{81} - \frac{32}{9}\zeta_2\right\} + C_F^3n_f\left\{\frac{259}{3} - \frac{4048}{3}\zeta_3 + \frac{2728}{9}\zeta_2 + \frac{6112}{15}\zeta_2^2\right\} \\
& + C_F^2n_fC_A\left\{\frac{42140}{243} - \frac{13712}{9}\zeta_3 - \frac{21376}{9}\zeta_2 + \frac{976}{3}\zeta_2^2\right\} + C_F^4\{5376\zeta_5 - 5120\zeta_3 - 7680\zeta_2\zeta_3\} \\
& - C_F^2C_A^2\left\{\frac{356573}{486} + 1152\zeta_5 - \frac{57296}{9}\zeta_3 - \frac{202868}{27}\zeta_2 + 688\zeta_2\zeta_3 + \frac{23672}{15}\zeta_2^2\right\} \\
& \left. - C_F^2n_f^2\left\{\frac{3718}{243} - \frac{320}{3}\zeta_3 - \frac{4624}{27}\zeta_2\right\}\right] + \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{z}'}\left[\frac{d_F^{abcd}d_A^{abcd}}{N_F}\left\{\frac{3520}{3}\zeta_5 + \frac{128}{3}\zeta_3 - 384\zeta_3^2 - 128\zeta_2 - \frac{7936}{35}\zeta_2^3\right\}\right. \\
& - n_f\frac{d_F^{abcd}d_F^{abcd}}{N_F}\left\{\frac{1280}{3}\zeta_5 + \frac{256}{3}\zeta_3 - 256\zeta_2\right\} + C_F^2N_4n_{fv}\left\{32 - \frac{640}{3}\zeta_5 + \frac{112}{3}\zeta_3 + 80\zeta_2 - \frac{16}{5}\zeta_2^2\right\} \\
& + C_FC_A^3\left\{\frac{4520317}{729} + \frac{15400}{9}\zeta_5 - \frac{51032}{9}\zeta_3 - 16\zeta_3^2 - \frac{140200}{27}\zeta_2 + 528\zeta_2\zeta_3 + \frac{3520}{3}\zeta_2^2 - \frac{20032}{105}\zeta_2^3\right\} \\
& \left. - C_Fn_fC_A^2\left\{\frac{571387}{243} + \frac{1360}{9}\zeta_5 - \frac{3128}{3}\zeta_3 - \frac{58000}{27}\zeta_2 - 32\zeta_2\zeta_3 + \frac{3872}{15}\zeta_2^2\right\} - C_Fn_f^3\left\{\frac{4000}{729} - \frac{320}{27}\zeta_2\right\}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- C_F^2 C_A^2 \left\{ \frac{11106458}{729} + 816\zeta_5 - \frac{6164}{81}\zeta_3 - \frac{752}{3}\zeta_3^2 + \frac{1049066}{81}\zeta_2 - 7984\zeta_2\zeta_3 - \frac{796252}{135}\zeta_2^2 + \frac{287648}{315}\zeta_2^3 \right\} \\
&+ C_F^3 C_A \left\{ 12062 - \frac{110752}{9}\zeta_5 - \frac{214592}{27}\zeta_3 + \frac{15808}{3}\zeta_3^2 + \frac{602990}{27}\zeta_2 + \frac{86240}{9}\zeta_2\zeta_3 + \frac{193264}{135}\zeta_2^2 - \frac{91648}{63}\zeta_2^3 \right\} \\
&- C_F^4 \left\{ \frac{11198}{3} - 5312\zeta_5 + 1840\zeta_3 - \frac{18304}{3}\zeta_3^2 + \frac{25048}{3}\zeta_2 - 4160\zeta_2\zeta_3 + \frac{21008}{5}\zeta_2^2 - \frac{48512}{63}\zeta_2^3 \right\} \\
&+ C_F^2 n_f C_A \left\{ \frac{2300107}{729} + 128\zeta_5 + \frac{234368}{81}\zeta_3 + \frac{37180}{9}\zeta_2 - \frac{2560}{3}\zeta_2\zeta_3 - \frac{182576}{135}\zeta_2^2 \right\} - C_F^3 n_f \left\{ \frac{4310}{9} \right. \\
&\left. - \frac{10816}{9}\zeta_5 - \frac{29456}{27}\zeta_3 + \frac{70880}{27}\zeta_2 + \frac{30464}{9}\zeta_2\zeta_3 + \frac{95872}{135}\zeta_2^2 \right\} + C_F n_f^2 C_A \left\{ \frac{58045}{243} - \frac{208}{9}\zeta_3 - \frac{7616}{27}\zeta_2 \right. \\
&\left. + \frac{64}{5}\zeta_2^2 \right\} - C_F^2 n_f^2 \left\{ \frac{72590}{729} + \frac{30688}{81}\zeta_3 + \frac{22000}{81}\zeta_2 - \frac{8512}{135}\zeta_2^2 \right\} \Big] + \boldsymbol{\mathcal{D}^1_{\vec{x}'}\mathcal{D}^5_{\vec{z}'}}[C_F^4\{224\}] \\
&- \boldsymbol{\mathcal{D}^1_{\vec{x}'}\mathcal{D}^4_{\vec{z}'}} \left[C_F^3 C_A \left\{ \frac{3080}{3} \right\} - C_F^3 n_f \left\{ \frac{560}{3} \right\} \right] + \boldsymbol{\mathcal{D}^1_{\vec{x}'}\mathcal{D}^3_{\vec{z}'}} \left[C_F^2 C_A^2 \left\{ \frac{38720}{27} \right\} + C_F^3 C_A \left\{ \frac{10720}{3} - 960\zeta_2 \right\} \right. \\
&\left. - C_F^2 n_f C_A \left\{ \frac{14080}{27} \right\} - C_F^4 \{2560 + 3840\zeta_2\} + C_F^2 n_f^2 \left\{ \frac{1280}{27} \right\} - C_F^3 n_f \left\{ \frac{1600}{3} \right\} \right] \\
&+ \boldsymbol{\mathcal{D}^1_{\vec{x}'}\mathcal{D}^2_{\vec{z}'}} \left[C_F n_f C_A^2 \left\{ \frac{968}{3} \right\} - C_F C_A^3 \left\{ \frac{5324}{9} \right\} + C_F^2 n_f C_A \left\{ \frac{21920}{9} - 320\zeta_2 \right\} - C_F^2 C_A^2 \left\{ \frac{67120}{9} - 1760\zeta_2 \right\} \right. \\
&- C_F n_f^2 C_A \left\{ \frac{176}{3} \right\} + C_F n_f^3 \left\{ \frac{32}{9} \right\} - C_F^3 n_f \left\{ \frac{1760}{9} + \frac{6080}{3}\zeta_2 \right\} + C_F^3 C_A \left\{ \frac{9920}{9} + 3360\zeta_3 + \frac{33440}{3}\zeta_2 \right\} \\
&\left. - C_F^2 n_f^2 \left\{ \frac{1600}{9} \right\} + C_F^4 \{8960\zeta_3\} \right] + \boldsymbol{\mathcal{D}^1_{\vec{x}'}\mathcal{D}^1_{\vec{z}'}} \left[C_F C_A^3 \left\{ \frac{43648}{27} - \frac{968}{3}\zeta_2 \right\} - C_F n_f C_A^2 \left\{ \frac{7324}{9} - \frac{352}{3}\zeta_2 \right\} \right. \\
&- C_F^2 n_f C_A \left\{ \frac{133988}{81} - \frac{448}{3}\zeta_3 - \frac{16256}{9}\zeta_2 \right\} + C_F^2 C_A^2 \left\{ \frac{481216}{81} - \frac{8800}{3}\zeta_3 - \frac{56080}{9}\zeta_2 + \frac{2592}{5}\zeta_2^2 \right\} \\
&- C_F n_f^3 \left\{ \frac{160}{27} \right\} + C_F^2 n_f^2 \left\{ \frac{7768}{81} - \frac{1216}{9}\zeta_2 \right\} - C_F^3 C_A \left\{ \frac{26362}{3} + \frac{19808}{3}\zeta_3 + \frac{25424}{3}\zeta_2 - \frac{13152}{5}\zeta_2^2 \right\} \\
&+ C_F^3 n_f \left\{ \frac{2764}{3} + \frac{5696}{3}\zeta_3 + \frac{4448}{3}\zeta_2 \right\} + C_F^4 \left\{ 3066 - 1440\zeta_3 + 7536\zeta_2 + \frac{9408}{5}\zeta_2^2 \right\} \\
&\left. + C_F n_f^2 C_A \left\{ \frac{1144}{9} - \frac{32}{3}\zeta_2 \right\} \right] + \boldsymbol{\mathcal{D}^0_{\vec{x}'}\mathcal{D}^6_{\vec{z}'}} \left[C_F^4 \left\{ \frac{112}{3} \right\} \right] - \boldsymbol{\mathcal{D}^0_{\vec{x}'}\mathcal{D}^5_{\vec{z}'}} \left[C_F^3 C_A \left\{ \frac{616}{3} \right\} - C_F^3 n_f \left\{ \frac{112}{3} \right\} \right] \\
&+ \boldsymbol{\mathcal{D}^0_{\vec{x}'}\mathcal{D}^4_{\vec{z}'}} \left[C_F^2 C_A^2 \left\{ \frac{9680}{27} \right\} - C_F^2 n_f C_A \left\{ \frac{3520}{27} \right\} + C_F^3 C_A \left\{ \frac{2680}{3} - 240\zeta_2 \right\} - C_F^4 \{640 + 960\zeta_2 \right\} \\
&+ C_F^2 n_f^2 \left\{ \frac{320}{27} \right\} - C_F^3 n_f \left\{ \frac{400}{3} \right\} \Big] + \boldsymbol{\mathcal{D}^0_{\vec{x}'}\mathcal{D}^3_{\vec{z}'}} \left[C_F^2 n_f C_A \left\{ \frac{21920}{27} - \frac{320}{3}\zeta_2 \right\} - C_F^2 C_A^2 \left\{ \frac{67120}{27} - \frac{1760}{3}\zeta_2 \right\} \right. \\
&+ C_F n_f C_A^2 \left\{ \frac{968}{9} \right\} - C_F C_A^3 \left\{ \frac{5324}{27} \right\} - C_F n_f^2 C_A \left\{ \frac{176}{9} \right\} + C_F n_f^3 \left\{ \frac{32}{27} \right\} - C_F^3 n_f \left\{ \frac{1760}{27} + \frac{6080}{9}\zeta_2 \right\} \\
&- C_F^2 n_f^2 \left\{ \frac{1600}{27} \right\} + C_F^3 C_A \left\{ \frac{9920}{27} + 1120\zeta_3 + \frac{33440}{9}\zeta_2 \right\} + C_F^4 \left\{ \frac{8960}{3}\zeta_3 \right\} \Big] \\
&+ \boldsymbol{\mathcal{D}^0_{\vec{x}'}\mathcal{D}^2_{\vec{z}'}} \left[C_F C_A^3 \left\{ \frac{43648}{27} - \frac{968}{3}\zeta_2 \right\} + C_F^2 C_A^2 \left\{ \frac{481216}{81} - \frac{8800}{3}\zeta_3 - \frac{56080}{9}\zeta_2 + \frac{2592}{5}\zeta_2^2 \right\} - C_F n_f^3 \left\{ \frac{160}{27} \right\} \right. \\
&\left. - C_F n_f C_A^2 \left\{ \frac{7324}{9} - \frac{352}{3}\zeta_2 \right\} + C_F n_f^2 C_A \left\{ \frac{1144}{9} - \frac{32}{3}\zeta_2 \right\} - C_F^2 n_f C_A \left\{ \frac{133988}{81} - \frac{448}{3}\zeta_3 - \frac{16256}{9}\zeta_2 \right\} \right.
\end{aligned}
$$

$$
\begin{aligned}
&- C_F^3 C_A \left\{ \frac{26362}{3} + \frac{19808}{3}\zeta_3 + \frac{25424}{3}\zeta_2 - \frac{13152}{5}\zeta_2^2 \right\} + C_F^3 n_f \left\{ \frac{2764}{3} + \frac{5696}{3}\zeta_3 + \frac{4448}{3}\zeta_2 \right\} \\
&+ C_F^2 n_f^2 \left\{ \frac{7768}{81} - \frac{1216}{9}\zeta_2 \right\} + C_F^4 \left\{ 3066 - 1440\zeta_3 + 7536\zeta_2 + \frac{9408}{5}\zeta_2^2 \right\} \Bigg] + \boldsymbol{\mathcal{D}}_{\vec{x}'}^{\mathbf{0}} \boldsymbol{\mathcal{D}}_{\vec{z}'}^{\mathbf{1}} \Bigg[C_F n_f^3 \left\{ \frac{800}{81} - \frac{64}{9}\zeta_2 \right\} \\
&- C_F C_A^3 \left\{ \frac{412880}{81} - 1936\zeta_3 - \frac{8024}{3}\zeta_2 + \frac{1936}{5}\zeta_2^2 \right\} + C_F n_f C_A^2 \left\{ \frac{20554}{9} - 352\zeta_3 - \frac{10192}{9}\zeta_2 + \frac{352}{5}\zeta_2^2 \right\} \\
&- C_F n_f^2 C_A \left\{ \frac{7894}{27} - \frac{1376}{9}\zeta_2 \right\} + C_F^3 C_A \left\{ \frac{34849}{9} + \frac{29392}{3}\zeta_3 - \frac{31976}{9}\zeta_2 - 9216\zeta_2\zeta_3 - \frac{73568}{15}\zeta_2^2 \right\} \\
&- C_F^2 C_A^2 \left\{ \frac{356573}{243} + 2304\zeta_5 - \frac{114592}{9}\zeta_3 - \frac{405736}{27}\zeta_2 + 1376\zeta_2\zeta_3 + \frac{47344}{15}\zeta_2^2 \right\} \\
&+ C_F^2 n_f C_A \left\{ \frac{84280}{243} - \frac{27424}{9}\zeta_3 - \frac{42752}{9}\zeta_2 + \frac{1952}{3}\zeta_2^2 \right\} + C_F^3 n_f \left\{ \frac{518}{3} - \frac{8096}{3}\zeta_3 + \frac{5456}{9}\zeta_2 + \frac{12224}{15}\zeta_2^2 \right\} \\
&- C_F^2 n_f^2 \left\{ \frac{7436}{243} - \frac{640}{3}\zeta_3 - \frac{9248}{27}\zeta_2 \right\} + C_F^4 \{ 10752\zeta_5 - 10240\zeta_3 - 15360\zeta_2\zeta_3 \} \Bigg] \\
&+ \boldsymbol{\mathcal{D}}_{\vec{x}'}^{\mathbf{0}} \boldsymbol{\mathcal{D}}_{\vec{z}'}^{\mathbf{0}} \Bigg[\frac{d_F^{abcd} d_A^{abcd}}{N_F} \left\{ \frac{1760}{3}\zeta_5 + \frac{64}{3}\zeta_3 - 192\zeta_3^2 - 64\zeta_2 - \frac{3968}{35}\zeta_2^3 \right\} - n_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} \left\{ \frac{640}{3}\zeta_5 + \frac{128}{3}\zeta_3 \right. \\
&\left. - 128\zeta_2 \right\} + C_F C_A^3 \left\{ \frac{4520317}{1458} + \frac{7700}{9}\zeta_5 - \frac{25516}{9}\zeta_3 - 8\zeta_3^2 - \frac{70100}{27}\zeta_2 + 264\zeta_2\zeta_3 + \frac{1760}{3}\zeta_2^2 - \frac{10016}{105}\zeta_2^3 \right\} \\
&- C_F n_f C_A^2 \left\{ \frac{571387}{486} + \frac{680}{9}\zeta_5 - \frac{1564}{3}\zeta_3 - \frac{29000}{27}\zeta_2 - 16\zeta_2\zeta_3 + \frac{1936}{15}\zeta_2^2 \right\} - C_F n_f^3 \left\{ \frac{2000}{729} - \frac{160}{27}\zeta_2 \right\} \\
&+ C_F n_f^2 C_A \left\{ \frac{58045}{486} - \frac{104}{9}\zeta_3 - \frac{3808}{27}\zeta_2 + \frac{32}{5}\zeta_2^2 \right\} + C_F^2 N_4 n_{fv} \left\{ 16 - \frac{320}{3}\zeta_5 + \frac{56}{3}\zeta_3 + 40\zeta_2 - \frac{8}{5}\zeta_2^2 \right\} \\
&- C_F^2 C_A^2 \left\{ \frac{5553229}{729} + 408\zeta_5 - \frac{3082}{81}\zeta_3 - \frac{376}{3}\zeta_3^2 + \frac{524533}{81}\zeta_2 - 3992\zeta_2\zeta_3 - \frac{398126}{135}\zeta_2^2 + \frac{143824}{315}\zeta_2^3 \right\} \\
&+ C_F^3 C_A \left\{ 6031 - \frac{55376}{9}\zeta_5 - \frac{107296}{27}\zeta_3 + \frac{7904}{3}\zeta_3^2 + \frac{301495}{27}\zeta_2 + \frac{43120}{9}\zeta_2\zeta_3 + \frac{96632}{135}\zeta_2^2 - 45824/63\zeta_2^3 \right\} \\
&+ C_F^2 n_f C_A \left\{ \frac{2300107}{1458} + 64\zeta_5 + \frac{117184}{81}\zeta_3 + \frac{18590}{9}\zeta_2 - \frac{1280}{3}\zeta_2\zeta_3 - \frac{91288}{135}\zeta_2^2 \right\} - C_F^2 n_f^2 \left\{ \frac{36295}{729} \right. \\
&\left. + \frac{15344}{81}\zeta_3 + \frac{11000}{81}\zeta_2 - \frac{4256}{135}\zeta_2^2 \right\} - C_F^3 n_f \left\{ \frac{2155}{9} - \frac{5408}{9}\zeta_5 - \frac{14728}{27}\zeta_3 + \frac{35440}{27}\zeta_2 + \frac{15232}{9}\zeta_2\zeta_3 \right. \\
&\left. + \frac{47936}{135}\zeta_2^2 \right\} - C_F^4 \left\{ \frac{5599}{3} - 2656\zeta_5 + 920\zeta_3 - \frac{9152}{3}\zeta_3^2 + \frac{12524}{3}\zeta_2 - 2080\zeta_2\zeta_3 + \frac{10504}{5}\zeta_2^2 - \frac{24256}{63}\zeta_2^3 \right\} \Bigg] \\
&+ \boldsymbol{\delta}_{\boldsymbol{x}} \boldsymbol{\mathcal{D}}_{\boldsymbol{z}}^{\mathbf{0}} \Bigg[n_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} \left\{ \frac{1600}{9}\zeta_5 - \frac{640}{9}\zeta_3 + \frac{320}{3}\zeta_3^2 - 256\zeta_2 + \frac{64}{5}\zeta_2^2 + \frac{1280}{21}\zeta_2^3 \right\} - \frac{d_F^{abcd} d_A^{abcd}}{N_F} \left\{ 192 \right. \\
&\left. - 2b_{4,FA}^q - \frac{1840}{9}\zeta_5 + 3484\zeta_7 - \frac{7808}{9}\zeta_3 - \frac{3344}{3}\zeta_3^2 - \frac{2176}{3}\zeta_2 + 1024\zeta_2\zeta_5 - 1792\zeta_2\zeta_3 + \frac{224}{15}\zeta_2^2 \right. \\
&\left. - \frac{736}{5}\zeta_2^2\zeta_3 + \frac{27808}{315}\zeta_2^3 \right\} - C_F C_A^3 \left\{ \frac{28290079}{4374} + \frac{1}{12}b_{4,FA}^q + \frac{74990}{27}\zeta_5 - \frac{11071}{6}\zeta_7 - \frac{258224}{27}\zeta_3 \right. \\
&\left. + \frac{7414}{9}\zeta_3^2 - \frac{1634851}{243}\zeta_2 - \frac{1376}{3}\zeta_2\zeta_5 + \frac{31444}{9}\zeta_2\zeta_3 + \frac{15632}{9}\zeta_2^2 - \frac{4228}{15}\zeta_2^2\zeta_3 - \frac{27808}{189}\zeta_2^3 \right\} \\
&+ C_F n_f C_A^2 \left\{ \frac{11551831}{5832} - \frac{3532}{27}\zeta_5 - \frac{150988}{81}\zeta_3 - \frac{2276}{9}\zeta_3^2 - \frac{645476}{243}\zeta_2 + \frac{4328}{9}\zeta_2\zeta_3 + \frac{29624}{45}\zeta_2^2 - \frac{7288}{105}\zeta_2^3 \right\}
\end{aligned}
$$

$$
\begin{aligned}
&- C_F n_f^2 C_A \left\{\frac{898033}{5832} - \frac{304}{3}\zeta_5 - \frac{2456}{81}\zeta_3 - \frac{75718}{243}\zeta_2 - \frac{80}{9}\zeta_2\zeta_3 + \frac{3104}{45}\zeta_2^2\right\} + C_F n_f^3 \left\{\frac{5216}{2187} + \frac{80}{81}\zeta_3 - \frac{800}{81}\zeta_2\right.\\
&\left. + \frac{16}{9}\zeta_2^2\right\} + C_F^2 C_A^2 \left\{\frac{7543094}{729} + \frac{58624}{9}\zeta_5 - \frac{134305}{27}\zeta_3 - 2032\zeta_3^2 - \frac{384365}{729}\zeta_2 + 1536\zeta_2\zeta_5 - \frac{361240}{27}\zeta_2\zeta_3\right.\\
&- \frac{382312}{81}\zeta_2^2 + \frac{5968}{3}\zeta_2^2\zeta_3 + \frac{2288}{3}\zeta_2^3\Big\} - C_F^2 n_f C_A \left\{\frac{1792393}{2916} + \frac{15176}{9}\zeta_5 + \frac{55924}{27}\zeta_3 - 536\zeta_3^2 + \frac{202685}{729}\zeta_2\right.\\
&- \frac{31040}{9}\zeta_2\zeta_3 - \frac{96664}{81}\zeta_2^2 + \frac{4064}{21}\zeta_2^3\Big\} - C_F^2 n_f^2 \left\{\frac{142769}{1458} - \frac{1168}{9}\zeta_5 - \frac{8552}{27}\zeta_3 - \frac{26980}{729}\zeta_2 + \frac{5888}{27}\zeta_2\zeta_3\right.\\
&\left. + \frac{2240}{27}\zeta_2^2\right\} - C_F^3 C_A \left\{\frac{103222}{27} - 8576\zeta_5 + \frac{57095}{9}\zeta_3 + 5008\zeta_3^2 + \frac{33233}{9}\zeta_2 + 2304\zeta_2\zeta_5 + \frac{62744}{9}\zeta_2\zeta_3\right.\\
&- \frac{34456}{15}\zeta_2^2 - \frac{19552}{5}\zeta_2^2\zeta_3 - \frac{1056}{5}\zeta_2^3\Big\} - C_F^3 n_f \left\{\frac{73309}{108} + \frac{2240}{3}\zeta_5 - \frac{9256}{9}\zeta_3 - 1616\zeta_3^2 - \frac{1510}{27}\zeta_2 - \frac{21664}{9}\zeta_2\zeta_3\right.\\
&\left. + \frac{4492}{15}\zeta_2^2 - \frac{2048}{35}\zeta_2^3\right\} - C_F^4 \left\{6144\zeta_5 - 7680\zeta_7 - 4088\zeta_3 + 1920\zeta_3^2 + 9216\zeta_2\zeta_5 - 10048\zeta_2\zeta_3 - \frac{12544}{5}\zeta_2^2\zeta_3\right\}\Big]\\
&+ (\boldsymbol{x}' \leftrightarrow \boldsymbol{z}'). \qquad (66)
\end{aligned}
$$

In the above equation, the constant $b^q_{4,FA} = 998.02 \pm 0.02$ [79,80].

B. NSV coefficient functions

The NSV contributions to the CFs are next. At NLO, i.e., at $\mathcal{O}(a_s)$, we have

$$
\begin{aligned}
(\Delta)\mathcal{C}^{(1),\mathrm{NSV}}_{J,cc} = {}& \boldsymbol{\delta}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{x}'}[C_F\{4\}] - \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{z}'}[C_F\{4\}] - \boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{z}'}[C_F\{4\}]\\
&- \boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{x}'}[C_F\{4\}] - \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{1}}_{\bar{z}'}[C_F\{4\}]. - \boldsymbol{\delta}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{1}}_{\bar{x}'}[C_F\{4\}]. \qquad (67)
\end{aligned}
$$

The NSV terms of the CFs at NNLO are given by

$$
\begin{aligned}
(\Delta)\mathcal{C}^{(2),\mathrm{NSV}}_{J,cc} = {}& \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{z}'}\left[C_F C_A\left\{\frac{1108}{27} - 28\zeta_3 - \frac{44}{3}\zeta_2\right\} - C_F n_f\left\{\frac{52}{27} - \frac{8}{3}\zeta_2\right\} + C_F^2\{64 - 32\zeta_3 - 24\zeta_2\}\right]\\
&+ \boldsymbol{\delta}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{x}'}\left[C_F C_A\left\{\frac{1492}{27} - 28\zeta_3 - \frac{68}{3}\zeta_2\right\} - C_F n_f\left\{\frac{364}{27} - \frac{8}{3}\zeta_2\right\} - C_F^2\{64 + 32\zeta_3 + 32\zeta_2\}\right]\\
&- \boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{z}'}\left[C_F C_A\left\{\frac{268}{9} - 8\zeta_2\right\} - C_F n_f\left\{\frac{40}{9}\right\} - C_F^2\{52 + 32\zeta_2\}\right] - \boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{x}'}\left[C_F C_A\left\{\frac{532}{9} - 8\zeta_2\right\}\right.\\
&\left. - C_F n_f\left\{\frac{88}{9}\right\} - C_F^2\{76 + 32\zeta_2\}\right] + \boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{1}}_{\bar{z}'}\left[C_F C_A\left\{\frac{44}{3}\right\} - C_F n_f\left\{\frac{8}{3}\right\} - C_F^2\{16\}\right]\\
&+ \boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{1}}_{\bar{x}'}\left[C_F C_A\left\{\frac{44}{3}\right\} - C_F n_f\left\{\frac{8}{3}\right\} + C_F^2\{48\}\right] - \boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{2}}_{\bar{z}'}[C_F^2\{24\}] - \boldsymbol{\mathcal{D}}^{\mathbf{0}}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{2}}_{\bar{x}'}[C_F^2\{24\}]\\
&- \boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{1}}_{\bar{z}'}[C_F^2\{48\}] - \boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{1}}_{\bar{x}'}[C_F^2\{48\}] + \boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{z}'}\left[C_F C_A\left\{\frac{44}{3}\right\} - C_F n_f\left\{\frac{8}{3}\right\}\right]\\
&+ \boldsymbol{\mathcal{D}}^{\mathbf{1}}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{x}'}\left[C_F C_A\left\{\frac{44}{3}\right\} - C_F n_f\left\{\frac{8}{3}\right\} + C_F^2\{32\}\right] - \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{1}}_{\bar{z}'}\left[C_F C_A\left\{\frac{358}{9} - 8\zeta_2\right\} - C_F n_f\left\{\frac{40}{9}\right\}\right.\\
&\left. - C_F^2\{62 + 32\zeta_2\}\right] - \boldsymbol{\delta}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{1}}_{\bar{x}'}\left[C_F C_A\left\{\frac{442}{9} - 8\zeta_2\right\} - C_F n_f\left\{\frac{88}{9}\right\} - C_F^2\{66 + 32\zeta_2\}\right]\\
&+ \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{2}}_{\bar{z}'}\left[C_F C_A\left\{\frac{22}{3}\right\} - C_F n_f\left\{\frac{4}{3}\right\} - C_F^2\{12\}\right] + \boldsymbol{\delta}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{2}}_{\bar{x}'}\left[C_F C_A\left\{\frac{22}{3}\right\} - C_F n_f\left\{\frac{4}{3}\right\} + C_F^2\{28\}\right]\\
&- \boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{z}'}[C_F^2\{24\}] - \boldsymbol{\mathcal{D}}^{\mathbf{2}}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{0}}_{\bar{x}'}[C_F^2\{24\}] - \boldsymbol{\delta}_{\bar{x}'}\boldsymbol{\mathcal{L}}^{\mathbf{3}}_{\bar{z}'}[C_F^2\{8\}] - \boldsymbol{\delta}_{\bar{z}'}\boldsymbol{\mathcal{L}}^{\mathbf{3}}_{\bar{x}'}[C_F^2\{8\}]. \qquad (68)
\end{aligned}
$$

At third order, only partial result for NSV contributions can be obtained with the presently known ingredients, unlike the SV case. Using the complete second-order results for $\Psi_d^{q,(1)}$ and $\Psi_d^{q,(2)}$ along with third-order results $(A_3^q, C_3^q, \tilde{C}_3^q, D_3^q, \tilde{D}_3^q)$ and an intriguing relation $\bar{\varphi}^{(3)}_{q,z',3} = -\bar{\varphi}^{(3)}_{q,x',3} = -16/9\beta_0 C_F^2$, see [95], we can predict the coefficients of $\{\delta_{\bar{x}'}\mathcal{L}^i_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{L}^i_{\bar{x}'}\}$ for $i = 5, 4, 3$ and of $\{\mathcal{D}^i_{\bar{x}'}\mathcal{L}^j_{\bar{z}'}, \mathcal{D}^i_{\bar{z}'}\mathcal{L}^j_{\bar{x}'}\}$ for $i + j = 4, 3, 2, 1, 0$. The coefficients of $\{\delta_{\bar{x}'}\mathcal{L}^i_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{L}^i_{\bar{x}'}\}$ for $i = 2, 1, 0$ cannot be predicted at third order, as they require explicit information at that order, namely $\bar{\varphi}^{(i)}_{q,\xi,3}$ for $i = 0, 1, 2$ and $\xi = x', z'$. Thus, the available partial NSV results at $\mathcal{O}(a_s^3)$ are

$$
\begin{aligned}
(\Delta)\mathcal{C}^{(3),\mathrm{NSV}}_{J,cc} = {} & \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{L}^0_{\bar{z}'}}\left[C_F n_f C_A\left\{\frac{11012}{81} - \frac{560}{9}\zeta_2\right\} - C_F n_f^2\left\{\frac{400}{81} - \frac{32}{9}\zeta_2\right\} - C_F^2 n_f\left\{\frac{940}{9} + \frac{320}{3}\zeta_3 + \frac{1136}{9}\zeta_2\right\}\right. \\
& - C_F C_A^2\left\{\frac{46450}{81} - 176\zeta_3 - 256\zeta_2 + \frac{176}{5}\zeta_2^2\right\} + C_F^2 C_A\left\{\frac{7987}{9} + \frac{512}{3}\zeta_3 + \frac{6920}{9}\zeta_2 - \frac{912}{5}\zeta_2^2\right\} \\
& \left. - C_F^3\left\{325 - 112\zeta_3 + 640\zeta_2 + \frac{288}{5}\zeta_2^2\right\}\right] + \boldsymbol{\mathcal{D}^0_{\bar{z}'}\mathcal{L}^0_{\bar{x}'}}\left[C_F^2 C_A\left\{\frac{35605}{27} + \frac{896}{3}\zeta_3 + \frac{10520}{9}\zeta_2 - \frac{912}{5}\zeta_2^2\right\}\right. \\
& + C_F n_f C_A\left\{\frac{21572}{81} - \frac{656}{9}\zeta_2\right\} - C_F n_f^2\left\{\frac{1648}{81} - \frac{32}{9}\zeta_2\right\} - C_F^2 n_f\left\{\frac{4588}{27} + \frac{320}{3}\zeta_3 + \frac{2000}{9}\zeta_2\right\} \\
& \left. - C_F C_A^2\left\{\frac{62242}{81} - 176\zeta_3 - \frac{944}{3}\zeta_2 + \frac{176}{5}\zeta_2^2\right\} - C_F^3\{697 - 880\zeta_3 + 848\zeta_2 + \frac{288}{5}\zeta_2^2\}\right] \\
& + \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{L}^1_{\bar{z}'}}\left[C_F C_A^2\left\{\frac{9100}{27} - \frac{176}{3}\zeta_2\right\} - C_F n_f C_A\left\{\frac{2672}{27} - \frac{32}{3}\zeta_2\right\} + C_F^2 C_A\left\{\frac{580}{9} - 336\zeta_3 - 496\zeta_2\right\}\right. \\
& \left. + C_F n_f^2\left\{\frac{160}{27}\right\} + C_F^2 n_f\left\{\frac{80}{9} + 96\zeta_2\right\} + C_F^3\{256 - 640\zeta_3\}\right] + \boldsymbol{\mathcal{D}^0_{\bar{z}'}\mathcal{L}^1_{\bar{x}'}}\left[C_F C_A^2\left\{\frac{10948}{27} - \frac{176}{3}\zeta_2\right\}\right. \\
& - C_F n_f C_A\left\{\frac{4064}{27} - \frac{32}{3}\zeta_2\right\} + C_F^2 C_A\left\{\frac{2420}{3} - 336\zeta_3 - 720\zeta_2\right\} - C_F^3\{768 + 640\zeta_3 + 736\zeta_2\} \\
& \left. + C_F n_f^2\left\{\frac{352}{27}\right\} - C_F^2 n_f\left\{\frac{416}{3} - 96\zeta_2\right\}\right] + \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{L}^2_{\bar{z}'}}\left[C_F n_f C_A\left\{\frac{176}{9}\right\} - C_F C_A^2\left\{\frac{484}{9}\right\} - C_F n_f^2\left\{\frac{16}{9}\right\}\right. \\
& \left. - C_F^2 C_A\{368 - 96\zeta_2\} + C_F^2 n_f\{48\} + C_F^3\{352 + 384\zeta_2\}\right] + \boldsymbol{\mathcal{D}^0_{\bar{z}'}\mathcal{L}^2_{\bar{x}'}}\left[C_F n_f C_A\left\{\frac{176}{9}\right\} - C_F n_f^2\left\{\frac{16}{9}\right\}\right. \\
& \left. - C_F C_A^2\left\{\frac{484}{9}\right\} - C_F^2 C_A\left\{\frac{2096}{3} - 96\zeta_2\right\} + C_F^2 n_f\left\{\frac{368}{3}\right\} + C_F^3\{416 + 384\zeta_2\}\right] \\
& + \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{L}^3_{\bar{z}'}}\left[C_F^2 C_A\left\{\frac{880}{9}\right\} - C_F^2 n_f\left\{\frac{160}{9}\right\} - C_F^3\{48\}\right] + \boldsymbol{\mathcal{D}^0_{\bar{z}'}\mathcal{L}^3_{\bar{x}'}}\left[C_F^2 C_A\left\{\frac{880}{9}\right\} - C_F^2 n_f\left\{\frac{160}{9}\right\}\right. \\
& \left. + C_F^3\{176\}\right] - \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{L}^4_{\bar{z}'}}[C_F^3\{40\}] - \boldsymbol{\mathcal{D}^0_{\bar{z}'}\mathcal{L}^4_{\bar{x}'}}[C_F^3\{40\}] + \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{L}^0_{\bar{z}'}}\left[C_F C_A^2\left\{\frac{7120}{27} - \frac{176}{3}\zeta_2\right\}\right. \\
& + C_F n_f^2\left\{\frac{160}{27}\right\} - C_F n_f C_A\left\{\frac{2312}{27} - \frac{32}{3}\zeta_2\right\} + C_F^2 C_A\{376 - 336\zeta_3 - 560\zeta_2\} - C_F^3\{640\zeta_3 + 96\zeta_2\} \\
& \left. - C_F^2 n_f\{40 - 96\zeta_2\}\right] + \boldsymbol{\mathcal{D}^1_{\bar{z}'}\mathcal{L}^0_{\bar{x}'}}\left[C_F C_A^2\left\{\frac{12928}{27} - \frac{176}{3}\zeta_2\right\} + C_F n_f^2\left\{\frac{352}{27}\right\} - C_F n_f C_A\left\{\frac{4424}{27} - \frac{32}{3}\zeta_2\right\}\right. \\
& + C_F^2 C_A\left\{\frac{4456}{9} - 336\zeta_3 - 656\zeta_2\right\} - C_F^2 n_f\left\{\frac{808}{9} - 96\zeta_2\right\} - C_F^3\{512 + 640\zeta_3 + 640\zeta_2\}\Big] \\
& + \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{L}^1_{\bar{z}'}}\left[C_F n_f C_A\left\{\frac{352}{9}\right\} - C_F C_A^2\left\{\frac{968}{9}\right\} - C_F n_f^2\left\{\frac{32}{9}\right\} - C_F^2 C_A\left\{\frac{2264}{3} - 192\zeta_2\right\} + C_F^2 n_f\left\{\frac{320}{3}\right\}\right. \\
& \left. + C_F^3\{664 + 768\zeta_2\}\right] + \boldsymbol{\mathcal{D}^1_{\bar{z}'}\mathcal{L}^1_{\bar{x}'}}\left[C_F n_f C_A\left\{\frac{352}{9}\right\} - C_F C_A^2\left\{\frac{968}{9}\right\} - C_F n_f^2\left\{\frac{32}{9}\right\} + C_F^2 n_f\left\{\frac{704}{3}\right\}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- C_F^2 C_A\left\{\frac{4136}{3} - 192\zeta_2\right\} + C_F^3\{872 + 768\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{L}^2_{\bar{z}'}}\Big[C_F^2 C_A\left\{\frac{880}{3}\right\} - C_F^2 n_f\left\{\frac{160}{3}\right\} - C_F^3\{80\}\Big] \\
&+ \boldsymbol{\mathcal{D}^1_{\bar{z}'}\mathcal{L}^2_{\bar{x}'}}\Big[C_F^2 C_A\left\{\frac{880}{3}\right\} - C_F^2 n_f\left\{\frac{160}{3}\right\} + C_F^3\{464\}\Big] - \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{L}^3_{\bar{z}'}}[C_F^3\{160\}] - \boldsymbol{\mathcal{D}^1_{\bar{z}'}\mathcal{L}^3_{\bar{x}'}}[C_F^3\{160\}] \\
&+ \boldsymbol{\mathcal{D}^2_{\bar{x}'}\mathcal{L}^0_{\bar{z}'}}\Big[C_F n_f C_A\left\{\frac{176}{9}\right\} - C_F C_A^2\left\{\frac{484}{9}\right\} - C_F n_f^2\left\{\frac{16}{9}\right\} - C_F^2 C_A\{416 - 96\zeta_2\} + C_F^2 n_f\{64\} \\
&+ C_F^3\{312 + 384\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^2_{\bar{z}'}\mathcal{L}^0_{\bar{x}'}}\Big[C_F n_f C_A\left\{\frac{176}{9}\right\} - C_F C_A^2\left\{\frac{484}{9}\right\} - C_F n_f^2\left\{\frac{16}{9}\right\} + C_F^2 n_f\left\{\frac{320}{3}\right\} \\
&- C_F^2 C_A\left\{\frac{1952}{3} - 96\zeta_2\right\} + C_F^3\{456 + 384\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^2_{\bar{x}'}\mathcal{L}^1_{\bar{z}'}}\Big[C_F^2 C_A\left\{\frac{880}{3}\right\} - C_F^2 n_f\left\{\frac{160}{3}\right\}\Big] \\
&+ \boldsymbol{\mathcal{D}^2_{\bar{z}'}\mathcal{L}^1_{\bar{x}'}}\Big[C_F^2 C_A\left\{\frac{880}{3}\right\} - C_F^2 n_f\left\{\frac{160}{3}\right\} + C_F^3\{384\}\Big] - \boldsymbol{\mathcal{D}^2_{\bar{x}'}\mathcal{L}^2_{\bar{z}'}}[C_F^3\{240\}] - \boldsymbol{\mathcal{D}^2_{\bar{z}'}\mathcal{L}^2_{\bar{x}'}}[C_F^3\{240\}] \\
&+ \boldsymbol{\delta_{\bar{x}'}\mathcal{L}^3_{\bar{z}'}}\Big[C_F n_f C_A\left\{\frac{176}{27}\right\} - C_F C_A^2\left\{\frac{484}{27}\right\} - C_F n_f^2\left\{\frac{16}{27}\right\} - C_F^2 C_A\left\{\frac{3316}{27} - 32\zeta_2\right\} + C_F^2 n_f\left\{\frac{400}{27}\right\} \\
&+ C_F^3\{124 + 128\zeta_2\}\Big] + \boldsymbol{\delta_{\bar{z}'}\mathcal{L}^3_{\bar{x}'}}\Big[C_F n_f C_A\left\{\frac{176}{27}\right\} - C_F C_A^2\left\{\frac{484}{27}\right\} - C_F n_f^2\left\{\frac{16}{27}\right\} + C_F^2 n_f\left\{\frac{1136}{27}\right\} \\
&- C_F^2 C_A\left\{\frac{6284}{27} - 32\zeta_2\right\} + C_F^3\{132 + 128\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^3_{\bar{x}'}\mathcal{L}^0_{\bar{z}'}}\Big[C_F^2 C_A\left\{\frac{880}{9}\right\} - C_F^2 n_f\left\{\frac{160}{9}\right\} + C_F^3\{32\}\Big] \\
&+ \boldsymbol{\mathcal{D}^3_{\bar{z}'}\mathcal{L}^0_{\bar{x}'}}\Big[C_F^2 C_A\left\{\frac{880}{9}\right\} - C_F^2 n_f\left\{\frac{160}{9}\right\} + C_F^3\{96\}\Big] - \boldsymbol{\mathcal{D}^3_{\bar{x}'}\mathcal{L}^1_{\bar{z}'}}[C_F^3\{160\}] - \boldsymbol{\mathcal{D}^3_{\bar{z}'}\mathcal{L}^1_{\bar{x}'}}[C_F^3\{160\}] \\
&+ \boldsymbol{\delta_{\bar{x}'}\mathcal{L}^4_{\bar{z}'}}\Big[C_F^2 C_A\left\{\frac{220}{9}\right\} - C_F^2 n_f\left\{\frac{40}{9}\right\} - C_F^3\{16\}\Big] + \boldsymbol{\delta_{\bar{z}'}\mathcal{L}^4_{\bar{x}'}}\Big[C_F^2 C_A\left\{\frac{220}{9}\right\} - C_F^2 n_f\left\{\frac{40}{9}\right\} + C_F^3\{48\}\Big] \\
&- \boldsymbol{\mathcal{D}^4_{\bar{x}'}\mathcal{L}^0_{\bar{z}'}}[C_F^3\{40\}] - \boldsymbol{\mathcal{D}^4_{\bar{z}'}\mathcal{L}^0_{\bar{x}'}}[C_F^3\{40\}] - \boldsymbol{\delta_{\bar{x}'}\mathcal{L}^5_{\bar{z}'}}[C_F^3\{8\}] - \boldsymbol{\delta_{\bar{z}'}\mathcal{L}^5_{\bar{x}'}}[C_F^3\{8\}]. \qquad (69)
\end{aligned}
$$

In analogy to the third order, many terms at the fourth order can be predicted from the knowledge of $\Psi_d^{q,(1)}$, $\Psi_d^{q,(2)}$ and other higher order results $(A_3^q, C_3^q, \tilde{C}_3^q, D_3^q, \tilde{D}_3^q)$ and the relation $\bar{\varphi}^{(3)}_{q,z',3} = -\bar{\varphi}^{(3)}_{q,x',3} = -16/9\beta_0 C_F^2$. For example, we can predict the coefficients of $\{\delta_{\bar{x}'}\mathcal{L}^i_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{L}^i_{\bar{x}'}\}$ for $i = 7, 6, 5$ and $\{\mathcal{D}^i_{\bar{x}'}\mathcal{L}^j_{\bar{z}'}, \mathcal{D}^i_{\bar{z}'}\mathcal{L}^j_{\bar{x}'}\}$ for $i + j = 6$, 5, 4 and in addition the following terms: $\{\mathcal{D}^i_{\bar{x}'}\mathcal{L}^j_{\bar{z}'}, \mathcal{D}^i_{\bar{z}'}\mathcal{L}^j_{\bar{x}'}\}$ for $i + j = 3$ with constraint $i \geq 2$. However, with the present information available we cannot predict $\{\delta_{\bar{x}'}\mathcal{L}^i_{\bar{z}'}, \delta_{\bar{z}'}\mathcal{L}^i_{\bar{x}'}\}$ for $i = 4, 3, 2, 1, 0$ and $\{\mathcal{D}^i_{\bar{x}'}\mathcal{L}^j_{\bar{z}'}, \mathcal{D}^i_{\bar{z}'}\mathcal{L}^j_{\bar{x}'}\}$ for $i = 1$, 0 and $j = 2, 1, 0$. They require explicit third- and fourth-order results for $\bar{\varphi}^{(i)}_{q,\xi,3}$ for $i = 0, 1, 2$ and $\bar{\varphi}^{(i)}_{q,\xi,4}$ for $i = 0, 1, 2, 3, 4$ where $\xi = x', z'$. The partial results at $\mathcal{O}(a_s^4)$ are presented below:

$$
\begin{aligned}
(\Delta)\mathcal{C}^{(4),\mathrm{NSV}}_{J,cc} = &\boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{L}^4_{\bar{z}'}}\Big[C_F^2 n_f C_A\left\{\frac{3520}{27}\right\} - C_F^2 C_A^2\left\{\frac{9680}{27}\right\} - C_F^3 C_A\left\{\frac{22144}{27} - 240\zeta_2\right\} + C_F^4\{600 + 960\zeta_2\} \\
&- C_F^2 n_f^2\left\{\frac{320}{27}\right\} + C_F^3 n_f\left\{\frac{2848}{27}\right\}\Big] + \boldsymbol{\mathcal{D}^0_{\bar{z}'}\mathcal{L}^4_{\bar{x}'}}\Big[C_F^2 n_f C_A\left\{\frac{3520}{27}\right\} - C_F^2 C_A^2\left\{\frac{9680}{27}\right\} - C_F^2 n_f^2\left\{\frac{320}{27}\right\} \\
&- C_F^3 C_A\left\{\frac{55136}{27} - 240\zeta_2\right\} + C_F^3 n_f\left\{\frac{9632}{27}\right\} + C_F^4\{680 + 960\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{L}^5_{\bar{z}'}}\Big[C_F^3 C_A\left\{\frac{616}{3}\right\} \\
&- C_F^3 n_f\left\{\frac{112}{3}\right\} - C_F^4\{64\}\Big] + \boldsymbol{\mathcal{D}^0_{\bar{z}'}\mathcal{L}^5_{\bar{x}'}}\Big[C_F^3 C_A\left\{\frac{616}{3}\right\} - C_F^3 n_f\left\{\frac{112}{3}\right\} + C_F^4\{256\}\Big] \\
&- \boldsymbol{\mathcal{D}^0_{\bar{x}'}\mathcal{L}^6_{\bar{z}'}}\Big[C_F^4\left\{\frac{112}{3}\right\}\Big] - \boldsymbol{\mathcal{D}^0_{\bar{z}'}\mathcal{L}^6_{\bar{x}'}}\Big[C_F^4\left\{\frac{112}{3}\right\}\Big] + \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{L}^3_{\bar{z}'}}\Big[C_F^2 n_f C_A\left\{\frac{14080}{27}\right\} - C_F^2 C_A^2\left\{\frac{38720}{27}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&- C_F^3 C_A \left\{\frac{94336}{27} - 960\zeta_2\right\} - C_F^2 n_f^2 \left\{\frac{1280}{27}\right\} + C_F^3 n_f \left\{\frac{12832}{27}\right\} + C_F^4 \{2320 + 3840\zeta_2\}\Big] \\
&+ \boldsymbol{\mathcal{D}^1_{\bar{z}'}\mathcal{L}^3_{\bar{x}'}}\Big[C_F^2 n_f C_A \left\{\frac{14080}{27}\right\} - C_F^2 C_A^2 \left\{\frac{38720}{27}\right\} - C_F^2 n_f^2 \left\{\frac{1280}{27}\right\} - C_F^3 C_A \left\{\frac{214784}{27} - 960\zeta_2\right\} \\
&+ C_F^3 n_f \left\{\frac{37088}{27}\right\} + C_F^4 \{2800 + 3840\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{L}^4_{\bar{z}'}}\Big[C_F^3 C_A \left\{\frac{3080}{3}\right\} - C_F^3 n_f \left\{\frac{560}{3}\right\} - C_F^4 \{224\}\Big] \\
&+ \boldsymbol{\mathcal{D}^1_{\bar{z}'}\mathcal{L}^4_{\bar{x}'}}\Big[C_F^3 C_A \left\{\frac{3080}{3}\right\} - C_F^3 n_f \left\{\frac{560}{3}\right\} + C_F^4 \{1184\}\Big] - \boldsymbol{\mathcal{D}^1_{\bar{x}'}\mathcal{L}^5_{\bar{z}'}}[C_F^4\{224\}] - \boldsymbol{\mathcal{D}^1_{\bar{z}'}\mathcal{L}^5_{\bar{x}'}}[C_F^4\{224\}] \\
&+ \boldsymbol{\mathcal{D}^2_{\bar{x}'}\mathcal{L}^0_{\bar{z}'}}\Big[C_F n_f C_A^2 \left\{\frac{7324}{9} - \frac{352}{3}\zeta_2\right\} - C_F C_A^3 \left\{\frac{43648}{27} - \frac{968}{3}\zeta_2\right\} - C_F n_f^2 C_A \left\{\frac{1144}{9} - \frac{32}{3}\zeta_2\right\} \\
&- C_F^2 C_A^2 \left\{\frac{758188}{81} - \frac{8800}{3}\zeta_3 - \frac{61360}{9}\zeta_2 + \frac{2592}{5}\zeta_2^2\right\} + C_F n_f^3 \left\{\frac{160}{27}\right\} - C_F^2 n_f^2 \left\{\frac{12184}{81} - \frac{1216}{9}\zeta_2\right\} \\
&+ C_F^2 n_f C_A \left\{\frac{210284}{81} - \frac{448}{3}\zeta_3 - \frac{17216}{9}\zeta_2\right\} + C_F^3 C_A \left\{\frac{70810}{9} + \frac{22688}{3}\zeta_3 + \frac{33872}{3}\zeta_2 - \frac{13152}{5}\zeta_2^2\right\} \\
&- C_F^3 n_f \left\{\frac{7132}{9} + \frac{5696}{3}\zeta_3 + \frac{5696}{3}\zeta_2\right\} - C_F^4 \left\{1950 - 1696\zeta_3 + 6336\zeta_2 + \frac{9408}{5}\zeta_2^2\right\}\Big] \\
&+ \boldsymbol{\mathcal{D}^2_{\bar{z}'}\mathcal{L}^0_{\bar{x}'}}\Big[C_F n_f C_A^2 \left\{\frac{13132}{9} - \frac{352}{3}\zeta_2\right\} - C_F C_A^3 \left\{\frac{75592}{27} - \frac{968}{3}\zeta_2\right\} - C_F n_f^2 C_A \left\{\frac{2200}{9} - \frac{32}{3}\zeta_2\right\} \\
&+ C_F n_f^3 \left\{\frac{352}{27}\right\} - C_F^2 C_A^2 \left\{\frac{974548}{81} - \frac{8800}{3}\zeta_3 - \frac{70336}{9}\zeta_2 + \frac{2592}{5}\zeta_2^2\right\} - C_F^2 n_f^2 \left\{\frac{23128}{81} - \frac{1216}{9}\zeta_2\right\} \\
&+ C_F^2 n_f C_A \left\{\frac{316412}{81} - \frac{448}{3}\zeta_3 - \frac{18848}{9}\zeta_2\right\} + C_F^3 C_A \left\{12466 + \frac{24992}{3}\zeta_3 + \frac{55168}{3}\zeta_2 - \frac{13152}{5}\zeta_2^2\right\} \\
&- C_F^3 n_f \left\{1484 + \frac{5696}{3}\zeta_3 + \frac{10144}{3}\zeta_2\right\} - C_F^4 \left\{4182 - 8864\zeta_3 + 8736\zeta_2 + \frac{9408}{5}\zeta_2^2\right\}\Big] \\
&+ \boldsymbol{\mathcal{D}^2_{\bar{x}'}\mathcal{L}^1_{\bar{z}'}}\Big[C_F C_A^3 \left\{\frac{5324}{9}\right\} - C_F n_f C_A^2 \left\{\frac{968}{3}\right\} + C_F n_f^2 C_A \left\{\frac{176}{3}\right\} - C_F n_f^3 \left\{\frac{32}{9}\right\} + C_F^2 n_f^2 \left\{\frac{1696}{9}\right\} \\
&+ C_F^2 C_A^2 \left\{\frac{74644}{9} - 1760\zeta_2\right\} - C_F^2 n_f C_A \left\{\frac{23816}{9} - 320\zeta_2\right\} + C_F^3 C_A \left\{\frac{6964}{9} - 3360\zeta_3 - \frac{34016}{3}\zeta_2\right\} \\
&- C_F^3 n_f \left\{\frac{184}{9} - \frac{6080}{3}\zeta_2\right\} - C_F^4 \{8960\zeta_3 + 768\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^2_{\bar{z}'}\mathcal{L}^1_{\bar{x}'}}\Big[C_F C_A^3 \left\{\frac{5324}{9}\right\} - C_F n_f C_A^2 \left\{\frac{968}{3}\right\} \\
&+ C_F n_f^2 C_A \left\{\frac{176}{3}\right\} - C_F n_f^3 \left\{\frac{32}{9}\right\} + C_F^2 C_A^2 \left\{\frac{111868}{9} - 1760\zeta_2\right\} - C_F^2 n_f C_A \left\{\frac{39032}{9} - 320\zeta_2\right\} \\
&+ C_F^3 C_A \left\{\frac{58252}{9} - 3360\zeta_3 - \frac{40352}{3}\zeta_2\right\} - C_F^4 \{6144 + 8960\zeta_3 + 9792\zeta_2\} - C_F^3 n_f \left\{\frac{9352}{9} - \frac{6080}{3}\zeta_2\right\} \\
&+ C_F^2 n_f^2 \left\{\frac{3232}{9}\right\}\Big] + \boldsymbol{\mathcal{D}^2_{\bar{x}'}\mathcal{L}^2_{\bar{z}'}}\Big[C_F^2 n_f C_A \left\{\frac{7040}{9}\right\} - C_F^2 C_A^2 \left\{\frac{19360}{9}\right\} - C_F^3 C_A \{5688 - 1440\zeta_2\} \\
&- C_F^2 n_f^2 \left\{\frac{640}{9}\right\} + C_F^3 n_f \{816\} + C_F^4 \{3360 + 5760\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^2_{\bar{z}'}\mathcal{L}^2_{\bar{x}'}}\Big[C_F^2 n_f C_A \left\{\frac{7040}{9}\right\} - C_F^2 n_f^2 \left\{\frac{640}{9}\right\} \\
&- C_F^2 C_A^2 \left\{\frac{19360}{9}\right\} - C_F^3 C_A \left\{\frac{34456}{3} - 1440\zeta_2\right\} + C_F^3 n_f \left\{\frac{5872}{3}\right\} + C_F^4 \{4320 + 5760\zeta_2\}\Big] \\
&+ \boldsymbol{\mathcal{D}^2_{\bar{x}'}\mathcal{L}^3_{\bar{z}'}}\Big[C_F^3 C_A \left\{\frac{6160}{3}\right\} - C_F^3 n_f \left\{\frac{1120}{3}\right\} - C_F^4 \{224\}\Big] + \boldsymbol{\mathcal{D}^2_{\bar{z}'}\mathcal{L}^3_{\bar{x}'}}\Big[C_F^3 C_A \left\{\frac{6160}{3}\right\} - C_F^3 n_f \left\{\frac{1120}{3}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ C_F^4\{2144\}\Big] - \boldsymbol{\mathcal{D}^2_{\vec{x}'}\mathcal{L}^4_{\vec{z}'}}[C_F^4\{560\}] - \boldsymbol{\mathcal{D}^2_{\vec{z}'}\mathcal{L}^4_{\vec{x}'}}[C_F^4\{560\}] + \boldsymbol{\mathcal{D}^3_{\vec{x}'}\mathcal{L}^0_{\vec{z}'}}\Big[C_F^2C_A^2\left\{\frac{75832}{27} - \frac{1760}{3}\zeta_2\right\} \\
&+ C_FC_A^3\left\{\frac{5324}{27}\right\} - C_Fn_fC_A^2\left\{\frac{968}{9}\right\} + C_Fn_f^2C_A\left\{\frac{176}{9}\right\} - C_Fn_f^3\left\{\frac{32}{27}\right\} - C_F^2n_fC_A\left\{\frac{25088}{27} - \frac{320}{3}\zeta_2\right\} \\
&+ C_F^3C_A\left\{\frac{30880}{27} - 1120\zeta_3 - \frac{35744}{9}\zeta_2\right\} - C_F^4\left\{512 + \frac{8960}{3}\zeta_3 + 704\zeta_2\right\} - C_F^3n_f\left\{\frac{3904}{27} - \frac{6080}{9}\zeta_2\right\} \\
&+ C_F^2n_f^2\left\{\frac{1888}{27}\right\}\Big] + \boldsymbol{\mathcal{D}^3_{\vec{z}'}\mathcal{L}^0_{\vec{x}'}}\Big[C_FC_A^3\left\{\frac{5324}{27}\right\} - C_Fn_fC_A^2\left\{\frac{968}{9}\right\} + C_Fn_f^2C_A\left\{\frac{176}{9}\right\} - C_Fn_f^3\left\{\frac{32}{27}\right\} \\
&+ C_F^2C_A^2\left\{\frac{110680}{27} - \frac{1760}{3}\zeta_2\right\} - C_F^2n_fC_A\left\{\frac{37760}{27} - \frac{320}{3}\zeta_2\right\} + C_F^3C_A\left\{\frac{34336}{27} - 1120\zeta_3 - \frac{38624}{9}\zeta_2\right\} \\
&+ C_F^2n_f^2\left\{\frac{3040}{27}\right\} - C_F^3n_f\left\{\frac{5632}{27} - \frac{6080}{9}\zeta_2\right\} - C_F^4\left\{1536 + \frac{8960}{3}\zeta_3 + 2816\zeta_2\right\}\Big] \\
&+ \boldsymbol{\mathcal{D}^3_{\vec{x}'}\mathcal{L}^1_{\vec{z}'}}\Big[C_F^2n_fC_A\left\{\frac{14080}{27}\right\} - C_F^2C_A^2\left\{\frac{38720}{27}\right\} - C_F^3C_A\left\{\frac{37808}{9} - 960\zeta_2\right\} + C_F^4\{2160 + 3840\zeta_2\} \\
&- C_F^2n_f^2\left\{\frac{1280}{27}\right\} + C_F^3n_f\left\{\frac{5696}{9}\right\}\Big] + \boldsymbol{\mathcal{D}^3_{\vec{z}'}\mathcal{L}^1_{\vec{x}'}}\Big[C_F^2n_fC_A\left\{\frac{14080}{27}\right\} - C_F^2C_A^2\left\{\frac{38720}{27}\right\} - C_F^2n_f^2\left\{\frac{1280}{27}\right\} \\
&- C_F^3C_A\{7248 - 960\zeta_2\} + C_F^3n_f\{1216\} + C_F^4\{2960 + 3840\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^3_{\vec{x}'}\mathcal{L}^2_{\vec{z}'}}\Big[C_F^3C_A\left\{\frac{6160}{3}\right\} \\
&- C_F^3n_f\left\{\frac{1120}{3}\right\} + C_F^4\{32\}\Big] + \boldsymbol{\mathcal{D}^3_{\vec{z}'}\mathcal{L}^2_{\vec{x}'}}\Big[C_F^3C_A\left\{\frac{6160}{3}\right\} - C_F^3n_f\left\{\frac{1120}{3}\right\} + C_F^4\{1888\}\Big] \\
&- \boldsymbol{\mathcal{D}^3_{\vec{x}'}\mathcal{L}^3_{\vec{z}'}}\Big[C_F^4\left\{\frac{2240}{3}\right\}\Big] - \boldsymbol{\mathcal{D}^3_{\vec{z}'}\mathcal{L}^3_{\vec{x}'}}\Big[C_F^4\left\{\frac{2240}{3}\right\}\Big] + \boldsymbol{\mathcal{D}^4_{\vec{x}'}\mathcal{L}^0_{\vec{z}'}}\Big[C_F^2n_fC_A\left\{\frac{3520}{27}\right\} - C_F^3C_A\left\{\frac{3560}{3} - 240\zeta_2\right\} \\
&- C_F^2C_A^2\left\{\frac{9680}{27}\right\} - C_F^2n_f^2\left\{\frac{320}{27}\right\} + C_F^3n_f\left\{\frac{560}{3}\right\} + C_F^4\{520 + 960\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^4_{\vec{z}'}\mathcal{L}^0_{\vec{x}'}}\Big[C_F^2n_fC_A\left\{\frac{3520}{27}\right\} \\
&- C_F^2C_A^2\left\{\frac{9680}{27}\right\} - C_F^2n_f^2\left\{\frac{320}{27}\right\} - C_F^3C_A\left\{\frac{15080}{9} - 240\zeta_2\right\} + C_F^3n_f\left\{\frac{2480}{9}\right\} + C_F^4\{760 + 960\zeta_2\}\Big] \\
&+ \boldsymbol{\mathcal{D}^4_{\vec{x}'}\mathcal{L}^1_{\vec{z}'}}\Big[C_F^3C_A\left\{\frac{3080}{3}\right\} - C_F^3n_f\left\{\frac{560}{3}\right\} + C_F^4\{160\}\Big] - \boldsymbol{\mathcal{D}^4_{\vec{x}'}\mathcal{L}^2_{\vec{z}'}}[C_F^4\{560\}] - \boldsymbol{\mathcal{D}^4_{\vec{z}'}\mathcal{L}^2_{\vec{x}'}}[C_F^4\{560\}] \\
&+ \boldsymbol{\mathcal{D}^4_{\vec{z}'}\mathcal{L}^1_{\vec{x}'}}\Big[C_F^3C_A\left\{\frac{3080}{3}\right\} - C_F^3n_f\left\{\frac{560}{3}\right\} + C_F^4\{800\}\Big] + \boldsymbol{\delta_{\vec{x}'}\mathcal{L}^5_{\vec{z}'}}\Big[C_F^2n_fC_A\left\{\frac{704}{27}\right\} - C_F^2C_A^2\left\{\frac{1936}{27}\right\} \\
&- C_F^2n_f^2\left\{\frac{64}{27}\right\} - C_F^3C_A\left\{\frac{4220}{27} - 48\zeta_2\right\} + C_F^3n_f\left\{\frac{512}{27}\right\} + C_F^4\{124 + 192\zeta_2\}\Big] \\
&+ \boldsymbol{\delta_{\vec{z}'}\mathcal{L}^5_{\vec{x}'}}\Big[C_F^2n_fC_A\left\{\frac{704}{27}\right\} - C_F^2C_A^2\left\{\frac{1936}{27}\right\} - C_F^2n_f^2\left\{\frac{64}{27}\right\} - C_F^3C_A\left\{\frac{11236}{27} - 48\zeta_2\right\} + C_F^3n_f\left\{\frac{1984}{27}\right\} \\
&+ C_F^4\{132 + 192\zeta_2\}\Big] + \boldsymbol{\mathcal{D}^5_{\vec{x}'}\mathcal{L}^0_{\vec{z}'}}\Big[C_F^3C_A\left\{\frac{616}{3}\right\} - C_F^3n_f\left\{\frac{112}{3}\right\} + C_F^4\{64\}\Big] - \boldsymbol{\mathcal{D}^5_{\vec{x}'}\mathcal{L}^1_{\vec{z}'}}[C_F^4\{224\}] \\
&- \boldsymbol{D^5_{\vec{z}'}\mathcal{L}^1_{\vec{x}'}}[C_F^4\{224\}] + \boldsymbol{D^5_{\vec{z}'}\mathcal{L}^0_{\vec{x}'}}\Big[C_F^3C_A\left\{\frac{616}{3}\right\} - C_F^3n_f\left\{\frac{112}{3}\right\} + C_F^4\{128\}\Big] - \boldsymbol{D^6_{\vec{x}'}\mathcal{L}^0_{\vec{z}'}}\Big[C_F^4\left\{\frac{112}{3}\right\}\Big] \\
&- \boldsymbol{D^6_{\vec{z}'}\mathcal{L}^0_{\vec{x}'}}\Big[C_F^4\left\{\frac{112}{3}\right\}\Big] - \boldsymbol{\delta_{\vec{x}'}\mathcal{L}^7_{\vec{z}'}}\Big[C_F^4\left\{\frac{16}{3}\right\}\Big] + \boldsymbol{\delta_{\vec{x}'}\mathcal{L}^6_{\vec{z}'}}\Big[C_F^3C_A\left\{\frac{308}{9}\right\} - C_F^3n_f\left\{\frac{56}{9}\right\} - C_F^4\left\{\frac{40}{3}\right\}\Big] \\
&- \boldsymbol{\delta_{\vec{z}'}\mathcal{L}^7_{\vec{x}'}}\Big[C_F^4\left\{\frac{16}{3}\right\}\Big] + \boldsymbol{\delta_{\vec{z}'}\mathcal{L}^6_{\vec{x}'}}\Big[C_F^3C_A\left\{\frac{308}{9}\right\} - C_F^3n_f\left\{\frac{56}{9}\right\} + C_F^4\left\{\frac{136}{3}\right\}\Big]. \qquad (70)
\end{aligned}
$$

IV. RESUMMATION

The formalism for the SV and NSV terms applied in the previous section is useful for obtaining results in (x', z') space at a given order a_s^n, provided that the exponent Ψ_d^q is known to order a_s^m, where $m = 1, \ldots, n-1$. It is impractical, though, for studying the numerical impact at all orders in a_s, because it involves iterated convolutions among distributions as well as among distributions and powers of $\ln(1-\xi)$ terms, which become difficult to evaluate at higher orders in a_s. All-order resummation of our result in Eq. (58) is therefore carried out in Mellin space spanned by two complex numbers $N_i, i = 1, 2$, taking double Mellin moments with respect to $N_i, i = 1, 2$ of CF given in Eq. (58). In Mellin space, the convolution $\mathcal{C}$ of the exponential in Eq. (58) becomes an exponential of the double Mellin moment of the exponent Ψ_d^q, i.e.,

$$\begin{aligned}(\Delta)\mathcal{C}^{\vec{N}}_{J,qq} &= \mathcal{N}\int_0^1 dx' x'^{N_1-1}\int_0^1 dz' z'^{N_2-1}(\Delta)\mathcal{C}^{\mathrm{SV+NSV}}_{qq}(x',z') \\ &= \exp\left(\mathcal{N}\int_0^1 dx' x'^{N_1-1}\int_0^1 dz' z'^{N_2-1}\Psi^q_d(x',z')\right) \\ &= \exp(G^{\vec{N}}_{d,q}),\end{aligned} \tag{71}$$

where we have suppressed the dependence on the scales Q^2 and μ_F^2. In Mellin $\vec{N}$-space, $\vec{N} = (N_1, N_2)$, the threshold limits of the (x', z') space, namely $\{x', z'\} \to \{1, 1\}$, correspond to taking both $\{N_1, N_2\} \to \{\infty, \infty\}$ and the limits on N_1 and N_2 are applied simultaneously. The symbol $\mathcal{N}$ in Eq. (71) indicates that we retain only the (N_1, N_2) space versions of SV and NSV terms, which are denoted as $\overline{\mathrm{SV}}$ and $\overline{\mathrm{NSV}}$, respectively. Specifically, $\overline{\mathrm{SV}}$ terms are of the form $\ln^i(N_1)\ln^j(N_2)$ with $i, j = 0, \ldots, \infty$. $\overline{\mathrm{NSV}}$ terms, on the other hand, include corrections of the type $1/N_i$. They are of the form $\ln^i(N_1)/N_1 \ln^j(N_2)$ or $\ln^i(N_1)\ln^j(N_2)/N_2$ where $i, j = 0, \ldots, \infty$. All other terms are classified as $\overline{\mathrm{BNSV}}$. We use the overline notations to distinguish the $\vec{N}$-space approximations for those directly obtained in (x', z') space, because the inverse Mellin transform of $\overline{\mathrm{SV}}$ terms in (x', z') space yields SV terms as well as NSV and BNSV terms. Similarly, the inverse Mellin transform of $\overline{\mathrm{NSV}}$ terms will produce NSV and BNSV terms in (x', z') space. The computation $G^{\vec{N}}_{d,q}$ is described in the Appendix C.

The Mellin moments of Eq. (44) contain terms of the type $\omega = a_s(\mu_R^2)\beta_0 \ln(N_1 N_2)$ as well as single logarithms $\omega_i = a_s(\mu_R^2)\beta_0 \ln(N_i)$, where large logarithms $\ln(N_i)$ and the strong coupling $a_s(\mu_R^2)$ at moderate values combine into an effective expansion parameter of $\mathcal{O}(1)$, spoiling the reliability of perturbative predictions, if truncated at specific order in a_s. A systematic reorganization of the $\vec{N}$-space in the exponent sums all the terms containing ω, ω_i etc. using the resummed expression for strong coupling a_s as described in Appendix E.

By rewriting $G^{\vec{N}}_{d,q} = \bar{G}^{\vec{N}}_{d,q} + G^{\vec{N}=\vec{1}}_{d,q}$ and defining $\tilde{g}^q_{d,0} = \exp(G^{\vec{N}=\vec{1}}_{d,q})$, we obtain

$$(\Delta)\mathcal{C}^{\vec{N}}_{J,qq} = \tilde{g}^q_{d,0}(Q^2, \mu_F^2)\exp\left(\bar{G}^{\vec{N}}_{d,q}(Q^2, \mu_F^2)\right). \tag{72}$$

We expand these $\vec{N}$-independent constants as $\tilde{g}^q_{d,0} = \sum_{i=0}^{\infty} a_s^i(\mu_R^2)\tilde{g}^q_{d,0,i}(Q^2, \mu_F^2, \mu_R^2)$, so that the resummed result takes the following form:

$$\begin{aligned}\bar{G}^{\vec{N}}_{d,q} &= g^q_{d,1}(\omega)\ln N_1 + \sum_{i=0}^{\infty} a_s^i\left(\frac{1}{2}g^q_{d,i+2}(\omega) + \frac{1}{N_1}\bar{g}^q_{d,i}(\omega)\right) \\ &\quad + \frac{1}{N_1}\left(h^q_{d,0}(\omega, N_1) + \sum_{i=1}^{\infty} a_s^i h^q_{d,i}(\omega, \omega_1, N_1)\right) \\ &\quad + (N_1 \leftrightarrow N_2, \omega_1 \leftrightarrow \omega_2),\end{aligned} \tag{73}$$

where

$$\begin{aligned}h^q_{d,0}(\omega, N_l) &= h^q_{d,0,0}(\omega) + h^q_{d,0,1}(\omega)\ln N_l, \\ h^q_{d,i}(\omega, \omega_l, N_l) &= \sum_{k=0}^{i-1} h^q_{d,i,k}(\omega, \omega_l)\ln^k N_l + \tilde{h}^q_{d,i,i}(\omega, \omega_l)\ln^i N_l.\end{aligned} \tag{74}$$

The functions $g^q_{d,i}$ and $\bar{g}^q_{d,i}$ are derived from $\Phi^{\mathrm{SV}}_{d,q}$, as given in Eq. (44), and $h^q_{d,0}$ and $h^q_{d,i}$ originate from $\Phi^{\mathrm{NSV}}_{d,q}$, as given in Eq. (51). These functions are listed in Appendix H.

V. PHYSICAL EVOLUTION KERNEL

The ingredients of SIDIS structure functions, CFs as well as the PDFs and FFs are defined in a factorization scheme, but structure functions themselves, being physical observables, are scheme independent. This inherent scheme independence allows one to derive a scheme-invariant combination of CFs and splitting functions by constructing a system of differential equations among the structure functions [4,96–98]. Such physical evolution equations have been employed previously in Refs. [99–101] to obtain certain NSV terms for the CFs of inclusive DIS. With an ansatz based on the known results for the CFs and the AP kernels from fixed-order computations, predictions for the second- and third-order CFs of structure functions in inclusive DIS were made [101], finding agreement for some of the color factors. A systematic construction of physical evolution kernels (PEK), relying on the observation about the logarithmic structure of, namely the single-logarithmic enhancement in $\ln(1-z)$ at large z to all orders, has been performed for the CFs of inclusive DIS, semi-inclusive e^+e^- annihilation and the DY process in [54] (also see Refs. [55,76]). By conjecturing that this behavior holds true at every order in a_s, constraints on the

structure of the corresponding leading $\ln(1-z)$ terms in the kernel could be derived and predictions for power suppressed contributions, including NSV logarithms at higher orders in a_s, were obtained, in agreement with known third-order results for DY and Higgs production [102,103].

Following Ref. [54], we have established physical evolution equations for the structure functions F_1, F_2, and g_1 to analyze the PEK in the SV and NSV limits. In these limits, we have found that the evolution equations simplify significantly, with only diagonal splitting functions and diagonal CFs contributing to the PEK. Setting $\mu_F^2 = Q^2$ for convenience, we find

$$\begin{aligned} Q^2\frac{d}{dQ^2}(g_1)F_J &= \left\{\frac{1}{2}(\Delta)P_{q\leftarrow q}(a_s) + \frac{1}{2}\tilde{P}_{q\leftarrow q}(a_s) + \beta(a_s)\frac{d(\Delta)\mathcal{C}_{J,qq}(a_s)}{da_s} \otimes ((\Delta)\mathcal{C}_{J,qq}(a_s))^{-1}\right\} \otimes (g_1)F_J \\ &= (\Delta)K_J \otimes (g_1)F_J = \sum_{l=0}^{\infty} a_s^{l+1}(\Delta)K_{J,l} \otimes (g_1)F_J, \end{aligned} \tag{75}$$

where $a_s = a_s(Q^2)$. Note that the kernels $K_{J,l}$ in the differential equations contain scheme-dependent CFs and splitting functions. However, because of the scheme invariance of the structure functions, the kernels also have to be invariant under scheme transformations. Using Eq. (11), and expanding the QCD β function and the splitting functions in powers of a_s, we get

$$\begin{aligned} K_0 &= \frac{1}{2}P_0^q + \frac{1}{2}\tilde{P}_0^q, \\ K_1 &= \frac{1}{2}P_1^q + \frac{1}{2}\tilde{P}_1^q - \beta_0\mathcal{C}_q^{(1)}, \\ K_2 &= \frac{1}{2}P_2^q + \frac{1}{2}\tilde{P}_2^q - \beta_1\mathcal{C}_q^{(1)} - 2\beta_0\mathcal{C}_q^{(2)} + \beta_0\mathcal{C}_q^{(1)} \otimes^2, \\ K_3 &= \frac{1}{2}P_3^q + \frac{1}{2}\tilde{P}_3^q - \beta_2\mathcal{C}_q^{(1)} - 2\beta_1\mathcal{C}_q^{(2)} - 3\beta_0\mathcal{C}_q^{(3)} + 3\beta_0\mathcal{C}_q^{(2)} \otimes \mathcal{C}_q^{(1)} + \beta_1\mathcal{C}_q^{(1)} \otimes^2 - \beta_0\mathcal{C}_q^{(1)} \otimes^3, \\ K_4 &= \frac{1}{2}P_4^q + \frac{1}{2}\tilde{P}_4^q - \beta_3\mathcal{C}_q^{(1)} - 2\beta_2\mathcal{C}_q^{(2)} - 3\beta_1\mathcal{C}_q^{(3)} - 4\beta_0\mathcal{C}_q^{(4)} + 4\beta_0\mathcal{C}_q^{(3)} \otimes \mathcal{C}_q^{(1)} + 2\beta_0\mathcal{C}_q^{(2)} \otimes^2 \\ &\quad + 3\beta_1\mathcal{C}_q^{(2)} \otimes \mathcal{C}_q^{(1)} - 4\beta_0\mathcal{C}_q^{(2)} \otimes \mathcal{C}_q^{(1)} \otimes^2 + \beta_2\mathcal{C}_q^{(1)} \otimes^2 - \beta_1\mathcal{C}_q^{(1)} \otimes^3 + \beta_0\mathcal{C}_q^{(1)} \otimes^4 . \end{aligned} \tag{76}$$

In the above, we have suppressed Δ and the index J, and used a shorthand notation $\mathcal{C}_q^{(l)}$ for $\mathcal{C}_{J,qq}^{(l)}$ and $\mathcal{C}_q^{(l)} \otimes^i$ for the $(i-1)$-fold convolution of $\mathcal{C}_q^{(l)}$ with itself, i.e., $\mathcal{C}_q^{(1)} \otimes^2 = \mathcal{C}_q^{(1)} \otimes \mathcal{C}_q^{(1)}$ etc. Following the approach of Ref. [54] and using CFs known to second order [7–12] in a_s, we obtain K_l for $l = 0, 1, 2$ in $\vec{N}$-space, as

$$\begin{aligned} K_0 &= \boldsymbol{L_1^0 L_2^0}[C_F\{6-8\gamma_E\}] - \frac{\boldsymbol{L_1^0 L_2^0}}{\boldsymbol{N_1}}[C_F\{2\}] - \frac{\boldsymbol{L_1^0 L_2^0}}{\boldsymbol{N_2}}[C_F\{2\}] - \boldsymbol{L_1^0 L_2^1}[C_F\{4\}] - \boldsymbol{L_1^1 L_2^0}[C_F\{4\}] \\ K_1 &= \boldsymbol{L_1^0 L_2^0}\left[-C_F n_f\left\{\frac{34}{3} + \frac{8}{3}\zeta_2 - \frac{80}{9}\gamma_E - \frac{16}{3}\gamma_E^2\right\} + C_F C_A\left\{\frac{193}{3} - 24\zeta_3 + \frac{44}{3}\zeta_2 - \frac{536}{9}\gamma_E + 16\gamma_E\zeta_2 - \frac{88}{3}\gamma_E^2\right\}\right. \\ &\quad \left. + C_F^2\{3 + 48\zeta_3 - 24\zeta_2\}\right] + \frac{\boldsymbol{L_1^0 L_2^0}}{\boldsymbol{N_1}}\left[C_F n_f\left\{\frac{56}{9} + \frac{8}{3}\gamma_E\right\} - C_F C_A\left\{\frac{332}{9} - 4\zeta_2 + \frac{44}{3}\gamma_E\right\} + C_F^2\{12 - 16\gamma_E\}\right] \\ &\quad + \frac{\boldsymbol{L_1^0 L_2^0}}{\boldsymbol{N_2}}\left[C_F n_f\left\{\frac{8}{9} + \frac{8}{3}\gamma_E\right\} - C_F C_A\left\{\frac{68}{9} - 4\zeta_2 + \frac{44}{3}\gamma_E\right\} - C_F^2\{12 - 16\gamma_E\}\right] + \boldsymbol{L_1^0 L_2^1}\left[C_F n_f\left\{\frac{40}{9} + \frac{16}{3}\gamma_E\right\}\right. \\ &\quad \left. + C_F C_A\left\{-\frac{268}{9} + 8\zeta_2 - \frac{88}{3}\gamma_E\right\}\right] + \frac{\boldsymbol{L_1^0 L_2^1}}{\boldsymbol{N_2}}\left[C_F n_f\left\{\frac{4}{3}\right\} - C_F C_A\left\{\frac{22}{3}\right\} + C_F^2\{16\}\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{\boldsymbol{L_1^0 L_2^1}}{\boldsymbol{N_1}}\left[C_F n_f\left\{\frac{4}{3}\right\}-C_F C_A\left\{\frac{22}{3}\right\}\right]+\boldsymbol{L_1^0 L_2^2}\left[C_F n_f\left\{\frac{4}{3}\right\}-C_F C_A\left\{\frac{22}{3}\right\}\right]+\boldsymbol{L_1^2 L_2^0}\left[C_F n_f\left\{\frac{4}{3}\right\}-C_F C_A\left\{\frac{22}{3}\right\}\right]\\
&+\boldsymbol{L_1^1 L_2^0}\left[C_F n_f\left\{\frac{40}{9}+\frac{16}{3}\gamma_E\right\}-C_F C_A\left\{\frac{268}{9}-8\zeta_2+\frac{88}{3}\gamma_E\right\}\right]+\boldsymbol{L_1^1 L_2^1}\left[C_F n_f\left\{\frac{8}{3}\right\}-C_F C_A\left\{\frac{44}{3}\right\}\right]\\
&+\frac{\boldsymbol{L_1^1 L_2^0}}{\boldsymbol{N_1}}\left[C_F n_f\left\{\frac{4}{3}\right\}-C_F C_A\left\{\frac{22}{3}\right\}-C_F^2\{16\}\right]+\frac{\boldsymbol{L_1^1 L_2^0}}{\boldsymbol{N_2}}\left[C_F n_f\left\{\frac{4}{3}\right\}-C_F C_A\left\{\frac{22}{3}\right\}\right]\\
K_2=&\boldsymbol{L_1^0 L_2^0}\left[C_F n_f^2\left\{\frac{220}{9}-\frac{64}{27}\zeta_3+\frac{304}{27}\zeta_2-\frac{800}{81}\gamma_E-\frac{320}{27}\gamma_E^2-\frac{128}{27}\gamma_E^3\right\}-C_F C_A n_f\left\{\frac{3052}{9}-\frac{3440}{27}\zeta_3+\frac{4184}{27}\zeta_2\right.\right.\\
&-\frac{136}{15}\zeta_2^2-\frac{16408}{81}\gamma_E+\frac{320}{9}\gamma_E\zeta_2-\frac{4624}{27}\gamma_E^2+\frac{64}{3}\gamma_E^2\zeta_2-\frac{1408}{27}\gamma_E^3\Big\}+C_F C_A^2\left\{\frac{3082}{3}+80\zeta_5-\frac{22600}{27}\zeta_3\right.\\
&+\frac{13708}{27}\zeta_2-\frac{676}{15}\zeta_2^2-\frac{62012}{81}\gamma_E+352\gamma_E\zeta_3+\frac{2144}{9}\gamma_E\zeta_2-\frac{352}{5}\gamma_E\zeta_2^2-\frac{14240}{27}\gamma_E^2+\frac{352}{3}\gamma_E^2\zeta_2-\frac{3872}{27}\gamma_E^3\Big\}\\
&-C_F^2 n_f\left\{\frac{235}{3}+\frac{512}{3}\zeta_3-\frac{296}{3}\zeta_2-\frac{112}{15}\zeta_2^2-\frac{220}{3}\gamma_E+64\gamma_E\zeta_3-16\gamma_E^2\right\}+C_F^2 C_A\left\{\frac{232}{3}+240\zeta_5+\frac{3008}{3}\zeta_3\right.\\
&\left.\left.-\frac{2096}{3}\zeta_2+32\zeta_2\zeta_3-\frac{8}{3}\zeta_2^2\right\}+C_F^3\left\{29-480\zeta_5+136\zeta_3+36\zeta_2-64\zeta_2\zeta_3+\frac{576}{5}\zeta_2^2\right\}\right]\\
&+\frac{\boldsymbol{L_1^0 L_2^0}}{\boldsymbol{N_1}}\left[C_F C_A n_f\left\{\frac{14002}{81}-\frac{176}{9}\zeta_2+\frac{5120}{27}\gamma_E-\frac{32}{3}\gamma_E\zeta_2+\frac{352}{9}\gamma_E^2\right\}-C_F n_f^2\left\{\frac{1208}{81}+\frac{448}{27}\gamma_E+\frac{32}{9}\gamma_E^2\right\}\right.\\
&-C_F C_A^2\left\{\frac{36059}{81}-88\zeta_3-\frac{1064}{9}\zeta_2+\frac{88}{5}\zeta_2^2+\frac{13852}{27}\gamma_E-\frac{176}{3}\gamma_E\zeta_2+\frac{968}{9}\gamma_E^2\right\}+C_F^3\{6+96\zeta_3-48\zeta_2\}\\
&\left.+C_F^2 C_A\left\{\frac{302}{3}-48\zeta_3+64\zeta_2-\frac{1220}{9}\gamma_E+64\gamma_E\zeta_2-\frac{616}{3}\gamma_E^2\right\}+C_F^2 n_f\left\{\frac{47}{3}-16\zeta_3-16\zeta_2+\frac{224}{9}\gamma_E+\frac{112}{3}\gamma_E^2\right\}\right]\\
&+\frac{\boldsymbol{L_1^0 L_2^0}}{\boldsymbol{N_2}}\left[C_F n_f^2\left\{\frac{40}{81}-\frac{64}{27}\gamma_E-\frac{32}{9}\gamma_E^2\right\}+C_F C_A n_f\left\{\frac{3442}{81}-\frac{80}{9}\zeta_2+\frac{1616}{27}\gamma_E-\frac{32}{3}\gamma_E\zeta_2+\frac{352}{9}\gamma_E^2\right\}\right.\\
&-C_F C_A^2\left\{\frac{20267}{81}-88\zeta_3-\frac{536}{9}\zeta_2+\frac{88}{5}\zeta_2^2+\frac{6196}{27}\gamma_E-\frac{176}{3}\gamma_E\zeta_2+\frac{968}{9}\gamma_E^2\right\}-C_F^3\{6+96\zeta_3-48\zeta_2\}\\
&\left.-C_F^2 C_A\left\{\frac{302}{3}-48\zeta_3-112\zeta_2-\frac{1220}{9}\gamma_E+64\gamma_E\zeta_2-\frac{616}{3}\gamma_E^2\right\}+C_F^2 n_f\left\{29-16\zeta_3-16\zeta_2-\frac{80}{9}\gamma_E-\frac{112}{3}\gamma_E^2\right\}\right]\\
&+\boldsymbol{L_1^0 L_2^1}\left[C_F C_A n_f\left\{\frac{8204}{81}-\frac{160}{9}\zeta_2+\frac{4624}{27}\gamma_E-\frac{64}{3}\gamma_E\zeta_2+\frac{704}{9}\gamma_E^2\right\}-C_F n_f^2\left\{\frac{400}{81}+\frac{320}{27}\gamma_E+\frac{64}{9}\gamma_E^2\right\}\right.\\
&\left.-C_F C_A^2\left\{\frac{31006}{81}-176\zeta_3-\frac{1072}{9}\zeta_2+\frac{176}{5}\zeta_2^2+\frac{14240}{27}\gamma_E-\frac{352}{3}\gamma_E\zeta_2+\frac{1936}{9}\gamma_E^2\right\}+C_F^2 n_f\left\{\frac{110}{3}-32\zeta_3+16\gamma_E\right\}\right]\\
&+\frac{\boldsymbol{L_1^0 L_2^1}}{\boldsymbol{N_1}}\left[C_F C_A n_f\left\{\frac{2740}{27}-\frac{16}{3}\zeta_2+\frac{352}{9}\gamma_E\right\}-C_F n_f^2\left\{\frac{224}{27}+\frac{32}{9}\gamma_E\right\}-C_F C_A^2\left\{\frac{7916}{27}-\frac{88}{3}\zeta_2+\frac{968}{9}\gamma_E\right\}\right.\\
&\left.-C_F^2 n_f\left\{12-\frac{64}{3}\gamma_E\right\}+C_F^2 C_A\left\{88-\frac{352}{3}\gamma_E\right\}\right]-\frac{\boldsymbol{L_1^0 L_2^1}}{\boldsymbol{N_2}}\left[C_F n_f^2\left\{\frac{32}{27}+\frac{32}{9}\gamma_E\right\}+C_F^2 n_f\left\{\frac{260}{9}+\frac{160}{3}\gamma_E\right\}\right.\\
&\left.-C_F C_A n_f\left\{\frac{988}{27}-\frac{16}{3}\zeta_2+\frac{352}{9}\gamma_E\right\}+C_F C_A^2\left\{\frac{4088}{27}-\frac{88}{3}\zeta_2+\frac{968}{9}\gamma_E\right\}-C_F^2 C_A\left\{\frac{2012}{9}-64\zeta_2+\frac{880}{3}\gamma_E\right\}\right]\\
&+\boldsymbol{L_1^0 L_2^2}\left[C_F C_A n_f\left\{\frac{1156}{27}-\frac{16}{3}\zeta_2+\frac{352}{9}\gamma_E\right\}-C_F n_f^2\left\{\frac{80}{27}+\frac{32}{9}\gamma_E\right\}-C_F C_A^2\left\{\frac{3560}{27}-\frac{88}{3}\zeta_2+\frac{968}{9}\gamma_E\right\}\right.\\
&\left.+C_F^2 n_f\{4\}\right]+\frac{\boldsymbol{L_1^0 L_2^2}}{\boldsymbol{N_1}}\left[C_F C_A n_f\left\{\frac{88}{9}\right\}-C_F n_f^2\left\{\frac{8}{9}\right\}-C_F C_A^2\left\{\frac{242}{9}\right\}\right]+\frac{\boldsymbol{L_1^0 L_2^2}}{\boldsymbol{N_2}}\left[C_F C_A n_f\left\{\frac{88}{9}\right\}-C_F n_f^2\left\{\frac{8}{9}\right\}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- C_F C_A^2\left\{\frac{242}{9}\right\} - C_F^2 n_f\{16\} + C_F^2 C_A\{88\}\Big] + \boldsymbol{L_1^0 L_2^3}\Big[C_F C_A n_f\left\{\frac{176}{27}\right\} - C_F n_f^2\left\{\frac{16}{27}\right\} - C_F C_A^2\left\{\frac{484}{27}\right\}\Big] \\
&+ \boldsymbol{L_1^1 L_2^0}\Big[C_F C_A n_f\left\{\frac{8204}{81} - \frac{160}{9}\zeta_2 + \frac{4624}{27}\gamma_E - \frac{64}{3}\gamma_E\zeta_2 + \frac{704}{9}\gamma_E^2\right\} - C_F n_f^2\left\{\frac{400}{81} + \frac{320}{27}\gamma_E + \frac{64}{9}\gamma_E^2\right\} \\
&- C_F C_A^2\left\{\frac{31006}{81} - 176\zeta_3 - \frac{1072}{9}\zeta_2 + \frac{176}{5}\zeta_2^2 + \frac{14240}{27}\gamma_E - \frac{352}{3}\gamma_E\zeta_2 + \frac{1936}{9}\gamma_E^2\right\} + C_F^2 n_f\left\{\frac{110}{3} - 32\zeta_3 + 16\gamma_E\right\}\Big] \\
&+ \frac{\boldsymbol{L_1^1 L_2^0}}{\boldsymbol{N_1}}\Big[C_F C_A n_f\left\{\frac{2380}{27} - \frac{16}{3}\zeta_2 + \frac{352}{9}\gamma_E\right\} - C_F n_f^2\left\{\frac{224}{27} + \frac{32}{9}\gamma_E\right\} - C_F C_A^2\left\{\frac{5936}{27} - \frac{88}{3}\zeta_2 + \frac{968}{9}\gamma_E\right\} \\
&+ C_F^2 n_f\left\{\frac{332}{9} + \frac{160}{3}\gamma_E\right\} - C_F^2 C_A\left\{\frac{2012}{9} - 64\zeta_2 + \frac{880}{3}\gamma_E\right\}\Big] + \frac{\boldsymbol{L_1^1 L_2^0}}{\boldsymbol{N_2}}\Big[C_F C_A n_f\left\{\frac{628}{27} - \frac{16}{3}\zeta_2 + \frac{352}{9}\gamma_E\right\} \\
&- C_F n_f^2\left\{\frac{32}{27} + \frac{32}{9}\gamma_E\right\} - C_F C_A^2\left\{\frac{2108}{27} - \frac{88}{3}\zeta_2 + \frac{968}{9}\gamma_E\right\} + C_F^2 n_f\left\{20 - \frac{64}{3}\gamma_E\right\} - C_F^2 C_A\left\{88 - \frac{352}{3}\gamma_E\right\}\Big] \\
&+ \boldsymbol{L_1^1 L_2^1}\Big[C_F C_A n_f\left\{\frac{2312}{27} - \frac{32}{3}\zeta_2 + \frac{704}{9}\gamma_E\right\} - C_F n_f^2\left\{\frac{160}{27} + \frac{64}{9}\gamma_E\right\} - C_F C_A^2\left\{\frac{7120}{27} - \frac{176}{3}\zeta_2 + \frac{1936}{9}\gamma_E\right\} \\
&+ C_F^2 n_f\{8\}\Big] + \frac{\boldsymbol{L_1^1 L_2^1}}{\boldsymbol{N_1}}\Big[C_F C_A n_f\left\{\frac{176}{9}\right\} - C_F n_f^2\left\{\frac{16}{9}\right\} - C_F C_A^2\left\{\frac{484}{9}\right\} + C_F^2 n_f\left\{\frac{64}{3}\right\} - C_F^2 C_A\left\{\frac{352}{3}\right\}\Big] \\
&+ \frac{\boldsymbol{L_1^1 L_2^1}}{\boldsymbol{N_2}}\Big[C_F C_A n_f\left\{\frac{176}{9}\right\} - C_F n_f^2\left\{\frac{16}{9}\right\} - C_F C_A^2\left\{\frac{484}{9}\right\} - C_F^2 n_f\left\{\frac{64}{3}\right\} + C_F^2 C_A\left\{\frac{352}{3}\right\}\Big] \\
&+ \boldsymbol{L_1^1 L_2^2}\Big[C_F C_A n_f\left\{\frac{176}{9}\right\} - C_F n_f^2\left\{\frac{16}{9}\right\} - C_F C_A^2\left\{\frac{484}{9}\right\}\Big] + \boldsymbol{L_1^2 L_2^0}\Big[C_F C_A n_f\left\{\frac{1156}{27} - \frac{16}{3}\zeta_2 + \frac{352}{9}\gamma_E\right\} \\
&- C_F n_f^2\left\{\frac{80}{27} + \frac{32}{9}\gamma_E\right\} - C_F C_A^2\left\{\frac{3560}{27} - \frac{88}{3}\zeta_2 + \frac{968}{9}\gamma_E\right\} + C_F^2 n_f\{4\}\Big] + \frac{\boldsymbol{L_1^2 L_2^0}}{\boldsymbol{N_1}}\Big[C_F C_A n_f\left\{\frac{88}{9}\right\} - C_F n_f^2\left\{\frac{8}{9}\right\} \\
&- C_F C_A^2\left\{\frac{242}{9}\right\} + C_F^2 n_f\{16\} - C_F^2 C_A\{88\}\Big] + \frac{\boldsymbol{L_1^2 L_2^0}}{\boldsymbol{N_2}}\Big[C_F C_A n_f\left\{\frac{88}{9}\right\} - C_F n_f^2\left\{\frac{8}{9}\right\} - C_F C_A^2\left\{\frac{242}{9}\right\}\Big] \\
&+ \boldsymbol{L_1^2 L_2^1}\Big[C_F C_A n_f\left\{\frac{176}{9}\right\} - C_F n_f^2\left\{\frac{16}{9}\right\} - C_F C_A^2\left\{\frac{484}{9}\right\}\Big] + \boldsymbol{L_1^3 L_2^0}\Big[C_F C_A n_f\left\{\frac{176}{27}\right\} - C_F n_f^2\left\{\frac{16}{27}\right\} - C_F C_A^2\left\{\frac{484}{27}\right\}\Big],
\end{aligned}
\tag{77}
$$

where $\boldsymbol{L_j^i} = \ln^i(N_j)$, for $j = 1, 2$. From the above results for $K_i, i = 0, 1, 2$, we observe that

(i) in the SV part of $K_{J,l}$, the highest power of logarithms N_i is constrained, for example for the logarithms $\ln^i(N_1)\ln^j(N_2)$, we find $0 \le i + j \le l + 1$;
(ii) in the NSV part of $K_{J,l}$, the power of highest logarithms is constrained, for example, for the logarithms $\frac{1}{N_1}\ln^i(N_1)\ln^j(N_2)$ or $\frac{1}{N_2}\ln^i(N_1)\ln^j(N_2)$, we have the limit on i, j as $0 \le i + j \le l$.

Thus, also the SIDIS PEK shows a single-logarithmic enhancement, both in (x', z') space as well as in $\vec{N}$ space, similar to what was observed in Ref. [54] in the case of PEKs from inclusive cross-sections. If we conjecture that this logarithm structure persists at every order in a_s for $K_{J,l}$, then we can determine certain SV and NSV logarithms at a given order l for $\mathcal{C}_q^{(l)}$ from the knowledge of AP kernels and CFs from previous order in a_s, i.e., results for all orders up to a_s^k where $k < l$.

The validation of the SIDIS PEK predictions against the known second-order CFs can be readily performed. We use K_0 and K_1, i.e., the PEKs known from the NLO CFs and NLO splitting functions. Then, for SV terms, the maximum sum of logarithmic powers is $(i + j)|_{\max} = 3$ for terms of the form $\ln^i(N_1)\ln^j(N_2)$. For NSV terms, the constraint is $(i + j)|_{\max} = 2$ for logarithms of the form $\frac{1}{N_1}\ln^i(N_1)\ln^j(N_2)$ or $\frac{1}{N_2}\ln^i(N_1)\ln^j(N_2)$. By substituting explicit results for P_2^q, $\tilde{P}_2^q$, and $\mathcal{C}_q^{(1)}$ in K_2 of Eq. (76), and applying the general structure of SV and NSV terms specified in Eq. (16), we impose the condition that all terms with $i + j > 3$ for SV and $i + j > 2$ for NSV must vanish. This process yields a set of 13 equations relating the second-order coefficients $C_{J,ab}^{(2),\xi_1\xi_2}$ in Eq. (16). The solutions to these equations provide the NNLO coefficients for terms such as $\delta_{\bar{x}'}\{\mathcal{D}_{\bar{z}'}^i, \mathcal{L}_{\bar{z}'}^i\}$ with $i = 3$ and $\{\mathcal{D}_{\bar{x}'}^i\mathcal{D}_{\bar{z}'}^j, \mathcal{D}_{\bar{x}'}^i\mathcal{L}_{\bar{z}'}^j\}$ with $i + j = 2$, along with similar terms where $x' \leftrightarrow z'$. We find

that these predictions are in complete agreement with the results obtained from explicit computations [7–12].

Next, by using known NNLO results, we can predict the third- and fourth-order results for CFs, i.e., classes of logarithms for $\mathcal{C}_q^{(3)}$ and $\mathcal{C}_q^{(4)}$. For SV terms in K_3, we consider logarithms of the form $\ln^i(N_1)\ln^j(N_2)$ with the constraint $(i+j)|_{\max}=4$, and for NSV logarithms $\frac{1}{N_1}\ln^i(N_1)\ln^j(N_2)$ or $\frac{1}{N_2}\ln^i(N_1)\ln^j(N_2)$, the constraint is $(i+j)|_{\max}=3$. By imposing that there are no SV logarithms for $i+j>4$ and no NSV logarithms exist for $i+j>3$, we obtain a system of 35 linear equations that relate the coefficients $C_{J,ab}^{(3),\xi_1\xi_2}$. Solving for these coefficients, we can predict 35 different logarithms at N^{3}LO level. For example, at third order, for the CF $\mathcal{C}_q^{(3)}$ in (x',z')-space, we can predict terms such as $\delta_{\tilde{x}'}\{\mathcal{D}^i_{\tilde{z}'},\mathcal{L}^i_{\tilde{z}'}\}$ for $i=4$, 5 and $\{\mathcal{D}^i_{\tilde{x}'}\mathcal{D}^j_{\tilde{z}'},\mathcal{D}^i_{\tilde{x}'}\mathcal{L}^j_{\tilde{z}'}\}$ for $i+j=3$, 4 and similarly for $x'\leftrightarrow z'$. We have found that all these predictions are in complete agreement with those obtained using the resummation approach.

Similarly, we can determine the coefficients of highest logarithms in $\mathcal{C}_q^{(4)}$ by observing that the SV logarithms of the form $\ln^i(N_1)\ln^j(N_2)$ are constrained by $(i+j)|_{\max}=5$ and for NSV logarithms, $\frac{1}{N_1}\ln^i(N_1)\ln^j(N_2)$ or $\frac{1}{N_2}\ln^i(N_1)\ln^j(N_2)$, the constraint is $(i+j)|_{\max}=4$. These constraints yield a system of 66 equations, with which we can predict 47 SV and NSV logarithms in $\mathcal{C}_q^{(4)}$, namely terms of the form $\delta_{\tilde{x}'}\{\mathcal{D}^i_{\tilde{z}'},\mathcal{L}^i_{\tilde{z}'}\}$ for $i=6$, 7 and $\{\mathcal{D}^i_{\tilde{x}'}\mathcal{D}^j_{\tilde{z}'},\mathcal{D}^i_{\tilde{x}'}\mathcal{L}^j_{\tilde{z}'}\}$ for $i+j=5$, 6, including analogous terms where $x'\leftrightarrow z'$. These predictions are also in agreement with the ones obtained via the resummation approach. However, the remaining 19 equations require information from $\mathcal{C}_q^{(3)}$ and hence we cannot predict them using this current approach.

VI. PHENOMENOLOGY

We are now in the position to illustrate the impact of resummed SV and NSV contributions on the structure functions F_1 and g_1 for the EIC at a center-of-mass energy of $\sqrt{s}=140$ GeV. In the following, we perform resummation at various logarithmic accuracies, namely $\overline{\text{LL}}$, $\overline{\text{NLL}}$, and $\overline{\text{NNLL}}$, where the bar denotes the inclusion of both SV and NSV logarithms. As shown in the first row of Table III, the resummed result at $\overline{\text{LL}}$ accuracy requires the resummed exponents $\{\tilde{g}^q_{d,0,0},g^q_{d,1},\bar{g}^q_{d,0},h^q_{d,0}\}$. They systematically resum SV logarithms of the form $\{\ln^i N_1\ln^j N_2\}$ (where $i+j=2n$) and NSV logarithms like $\{\frac{\ln^i N_1}{N_1}\ln^j N_2,\frac{\ln^i N_2}{N_2}\ln^j N_1\}$ (where $i+j=2n-1$) at nth order in a_s, i.e., a_s^n for all $n>1$:

$$(\Delta)\mathcal{C}^{\overline{\text{LL}}}_{J,qq,\vec{N}}=(\tilde{g}^q_{d,0,0})\exp\Big[g^q_{d,1}(\omega)\ln N_1+\frac{1}{N_1}(\bar{g}^q_{d,0}(\omega)+h^q_{d,0}(\omega,N_1))+(N_1\leftrightarrow N_2)\Big]. \tag{78}$$

Similarly, for the resummed result at $\overline{\text{NLL}}$ accuracy, as detailed in the second row of Table III, we require the additional resummation exponents $\{\tilde{g}^q_{d,0,1},g^q_{d,2},\bar{g}^q_{d,1},h^q_{d,1}\}$ alongside the $\overline{\text{LL}}$ exponents. This suffices to sum up the next-to-leading SV logarithms and next-to-highest NSV logarithms to all orders. Specifically, this includes SV logarithms of the form $\{\ln^i N_1\ln^j N_2\}$ (where $i+j=\{2n-1,2n-2\}$) and NSV logarithms of the form

TABLE III. The set of resummed exponents required to predict the tower of NSV logarithm in $\mathcal{C}^{(n)}_{J,qq,\vec{N}}$, $L_1^i=\ln^i(N_1)$, $L^i_{N_1}=\frac{\ln^i(N_1)}{N_1}$, $L_2^j=\ln^j(N_2)$, $L^j_{N_2}=\frac{\ln^j(N_2)}{N_2}$.

Given	Predictions: SV and NSV logarithms					
Resummed exponents up to	$(\Delta)\mathcal{C}^{(2)}_{J,qq,\vec{N}}$	$(\Delta)\mathcal{C}^{(3)}_{J,qq,\vec{N}}$	$(\Delta)\mathcal{C}^{(4)}_{J,qq,\vec{N}}$	…	$(\Delta)\mathcal{C}^{(n)}_{J,qq,\vec{N}}$	Logarithmic accuracy
$\tilde{g}^q_{d,0,0}, g^q_{d,1}, \bar{g}^q_{d,0}, h^q_{d,0}$	$L_1^iL_2^j\vert_{i+j=4}$ $L^i_{N_1}L_2^j\vert_{i+j=3}$ $L_1^iL^j_{N_2}\vert_{i+j=3}$	$L_1^iL_2^j\vert_{i+j=6}$ $L^i_{N_1}L_2^j\vert_{i+j=5}$ $L_1^iL^j_{N_2}\vert_{i+j=5}$	$L_1^iL_2^j\vert_{i+j=8}$ $L^i_{N_1}L_2^j\vert_{i+j=7}$ $L_1^iL^j_{N_2}\vert_{i+j=7}$	…	$L_1^iL_2^j\vert_{i+j=2n}$ $L^i_{N_1}L_2^j\vert_{i+j=2n-1}$ $L_1^iL^j_{N_2}\vert_{i+j=2n-1}$	$\overline{\text{LL}}$
$\tilde{g}^q_{d,0,1}, g^q_{d,2}$ $\bar{g}^q_{d,1}, h^q_{d,1}$		$L_1^iL_2^j\vert_{i+j=6,5,4}$ $L^i_{N_1}L_2^j\vert_{i+j=5,4}$ $L_1^iL^j_{N_2}\vert_{i+j=5,4}$	$L_1^iL_2^j\vert_{i+j=8,7,6}$ $L^i_{N_1}L_2^j\vert_{i+j=7,6}$ $L_1^iL^j_{N_2}\vert_{i+j=7,6}$	…	$L_1^iL_2^j\vert_{i+j=2n,\ldots,2n-2}$ $L^i_{N_1}L_2^j\vert_{i+j=2n-1,2n-2}$ $L_1^iL^j_{N_2}\vert_{i+j=2n-1,2n-2}$	$\overline{\text{NLL}}$
$\tilde{g}^q_{d,0,2}, g^q_{d,3}$ $\bar{g}^q_{d,2}, h^q_{d,2}$			$L_1^iL_2^j\vert_{i+j=8,\ldots,4}$ $L^i_{N_1}L_2^j\vert_{i+j=7,\ldots,5}$ $L_1^iL^j_{N_2}\vert_{i+j=7,\ldots,5}$	…	$L_1^iL_2^j\vert_{i+j=2n,\ldots,2n-4}$ $L^i_{N_1}L_2^j\vert_{i+j=2n-1,\ldots,2n-3}$ $L_1^iL^j_{N_2}\vert_{i+j=2n-1,\ldots,2n-3}$	$\overline{\text{NNLL}}$

$\{\frac{\ln^i N_1}{N_1}\ln^j N_2, \frac{\ln^i N_2}{N_2}\ln^j N_1\}$ (where $i+j=2n-2$ with $n>2$), i.e.,

$$(\Delta)\mathcal{C}^{\overline{\mathrm{NLL}}}_{J,qq,\vec{N}} = (\tilde{g}^q_{d,0,0} + a_s\tilde{g}^q_{d,0,1})\exp\Big[g^q_{d,1}(\omega)\ln N_1 + \frac{1}{2}g^q_{d,2}(\omega) + \frac{1}{N_1}(\bar{g}^q_{d,0}(\omega) + a_s\bar{g}^q_{d,1}(\omega) + h^q_{d,0}(\omega, N_1) + a_s h^q_{d,1}(\omega,\omega_1,N_1)) + (N_1 \leftrightarrow N_2, \omega_1 \leftrightarrow \omega_2)\Big]. \tag{79}$$

According to the third row of Table III, we include the resummation exponents $\{\tilde{g}^q_{d,0,2}, g^q_{d,3}, \bar{g}^q_{d,2}, h^q_{d,2}\}$ alongside the $\overline{\mathrm{LL}}$ and $\overline{\mathrm{NLL}}$ terms to achieve $\overline{\mathrm{NNLL}}$ accuracy. This set resums SV logarithms of the form $\{\ln^i N_1 \ln^j N_2\}$ (where $i+j=\{2n-3, 2n-4\}$) and NSV logarithms of the form $\{\frac{\ln^i N_1}{N_1}\ln^j N_2, \frac{\ln^i N_2}{N_2}\ln^j N_1\}$ with $i+j=2n-3, n>3$,

$$(\Delta)\mathcal{C}^{\overline{\mathrm{NNLL}}}_{J,qq,\vec{N}} = (\tilde{g}^q_{d,0,0} + a_s\tilde{g}^q_{d,0,1} + a_s^2\tilde{g}^q_{d,0,2})\exp\Big[g^q_{d,1}(\omega)\ln N_1 + \frac{1}{2}(g^q_{d,2}(\omega) + a_s g^q_{d,3}(\omega)) + \frac{1}{N_1}(\bar{g}^q_{d,0}(\omega) + a_s\bar{g}^q_{d,1}(\omega) + a_s^2\bar{g}^q_{d,2}(\omega) + h^q_{d,0}(\omega,N_1) + a_s h^q_{d,1}(\omega,\omega_1,N_1) + a_s^2 h^q_{d,2}(\omega,\omega_1,N_1)) + (N_1 \leftrightarrow N_2, \omega_1 \leftrightarrow \omega_2)\Big]. \tag{80}$$

The resummed results are matched to the respective fixed-order results as follows:

$$(g_1)F_J^{\mathrm{N^nLO}+\overline{\mathrm{N^nLL}}} = (g_1)F_J^{\mathrm{N^nLO}} + \sum_{cc}\int_{c_1-i\infty}^{c_1+i\infty}\frac{dN_1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{dN_2}{2\pi i}(\Delta)f_{\frac{c}{P},N_1}(\mu_F^2)D_{\frac{H'}{c},N_2}(\mu_F^2)\times((\Delta)\mathcal{C}^{\overline{\mathrm{N^nLL}}}_{J,cc,\vec{N}} - (\Delta)\mathcal{C}_{J,cc,\vec{N}}|_{\mathrm{tr}^n}). \tag{81}$$

The first term in Eq. (81) represents the fixed-order contributions up to $\mathrm{N}^n\mathrm{LO}$ accuracy. The second term corresponds to the resummed result at $\overline{\mathrm{N^nLL}}$ accuracy. This term is derived by subtracting its fixed-order expansion, truncated up to order a_s^n, from the full resummed result, which precisely avoids double-counting of threshold logarithms. Therefore, the latter encapsulates the all-order SV and NSV resummed contributions in perturbation theory starting from a_s^{n+1} onward.

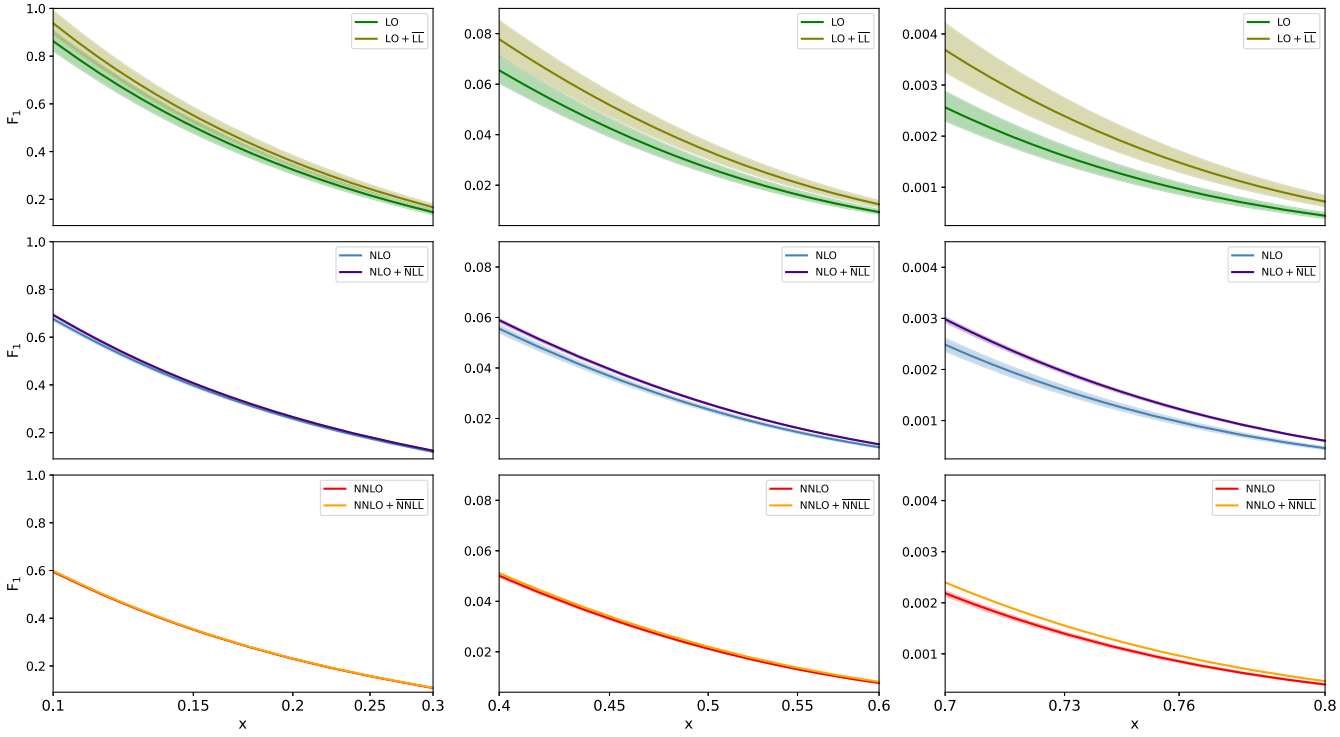


FIG. 1. Seven-point scale variation of the structure function F_1 as a function of x (log-scale) for the EIC at $\sqrt{s} = 140$ GeV.

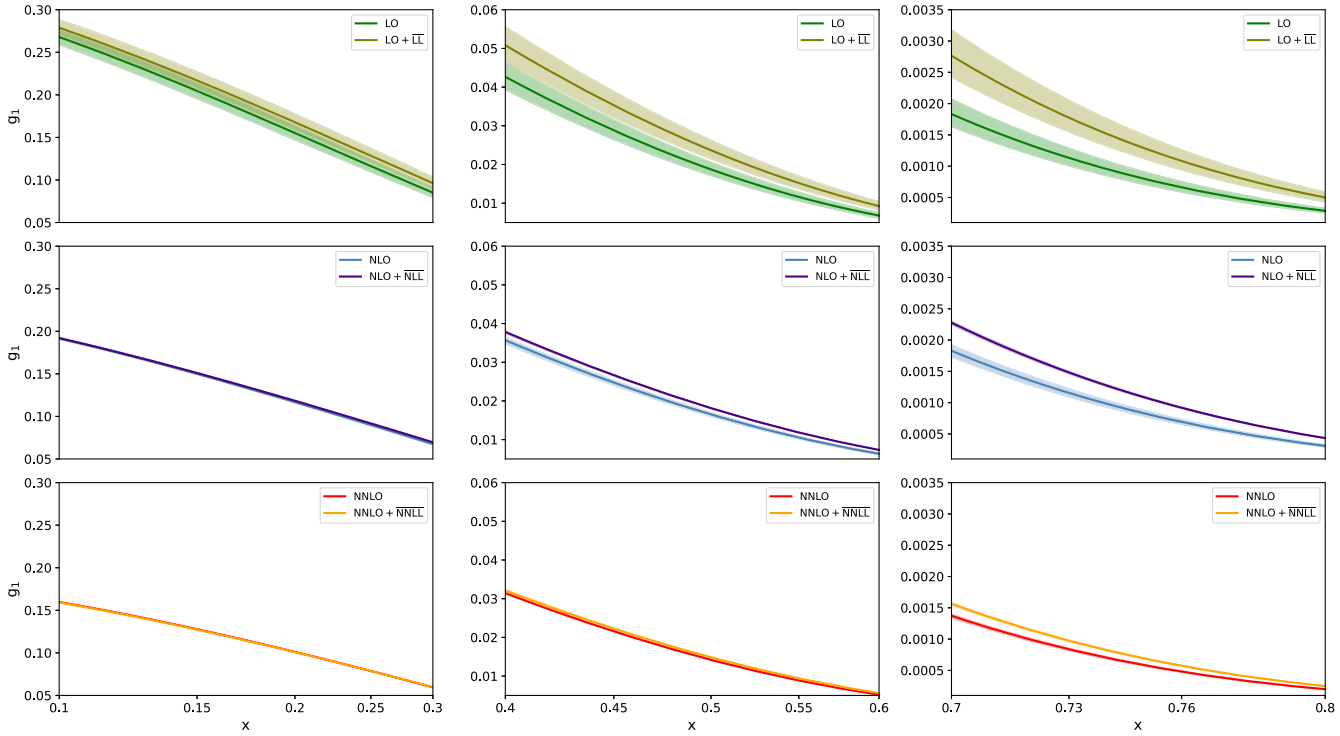


FIG. 2. Seven-point scale variation of structure function g_1 as a function x (log-scale) for the EIC at $\sqrt{s} = 140$ GeV.

The subscript "tr^n" denotes the nth-order "truncated" part of the resummed result which is already contained in the fixed-order expressions. The CFs required for $(g_1)F_J^{\mathrm{N^nLO}}$ in the first term are known up to NNLO [2,7–12].

For the unpolarized structure function (F_1), we have employed the ABMP16 PDF sets [104] and NNFF10PIp FF sets [105] at their respective orders, except for the LO where we use NLO PDF set. For the polarized structure

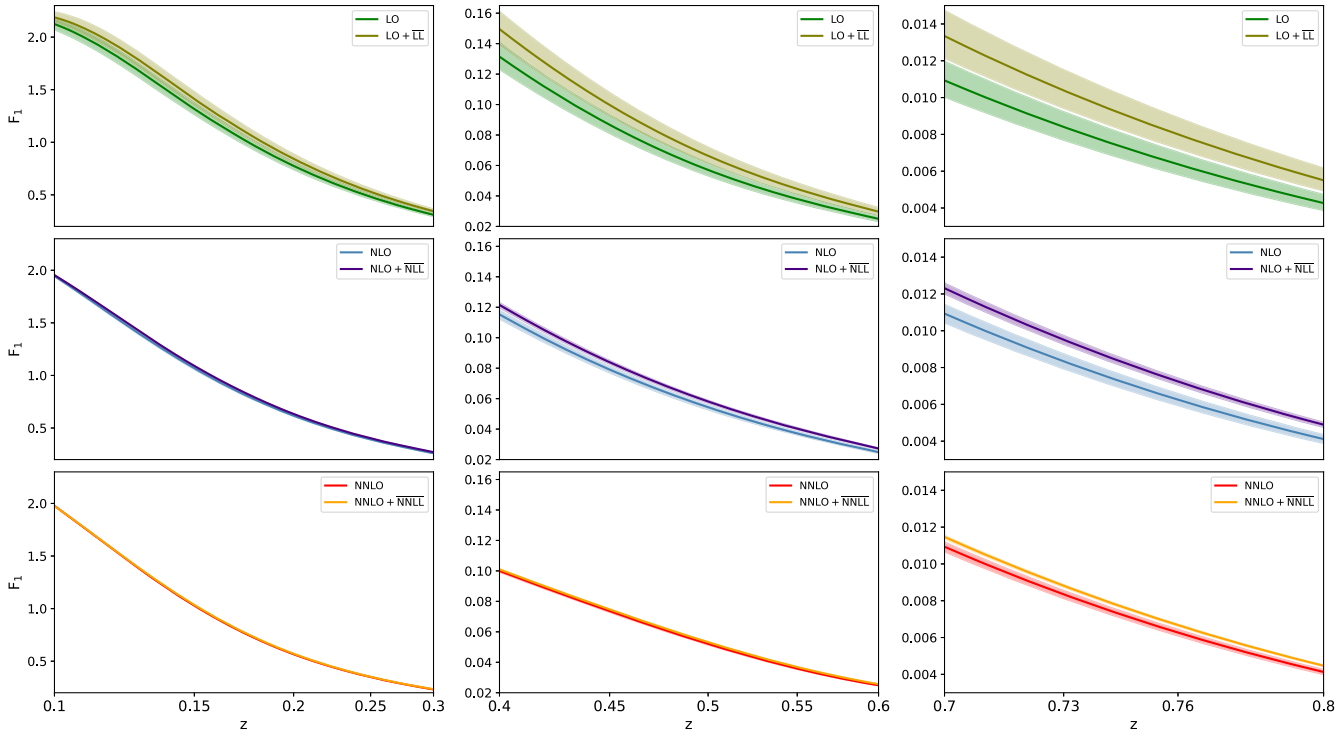


FIG. 3. Seven-point scale variation of the structure function F_1 as a function of z (log-scale) for the EIC at $\sqrt{s} = 140$ GeV.

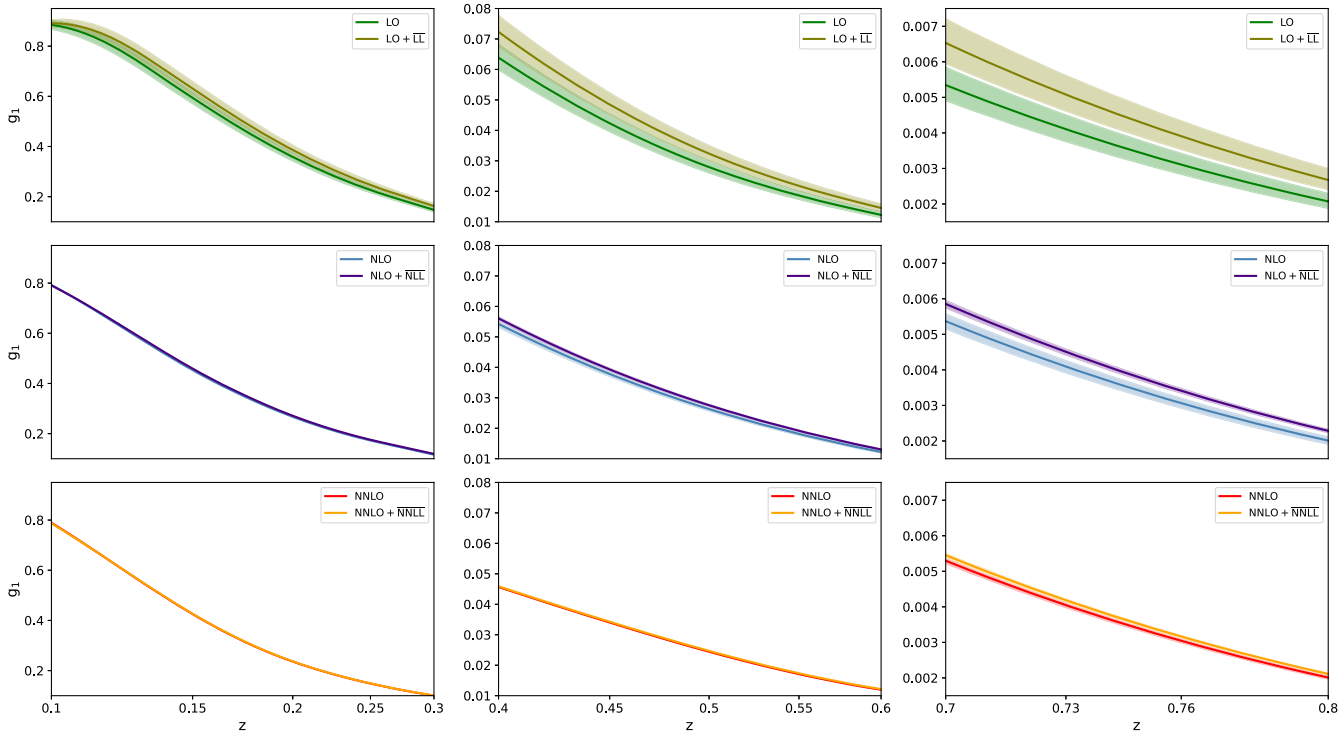


FIG. 4. Seven-point scale variation of structure function g_1 for z (log scale) for the EIC at $\sqrt{s} = 140$ GeV.

function (g_1), we have used the polarized BDSSV24NLO PDFs at LL and NLL accuracies, and BDSSV24NNLO PDFs at NNLL accuracy. The strong coupling a_s is taken as provided through the LHAPDF interface [106] with $n_f = 3$ active massless quark flavors throughout. We have used an in-house FORTRAN code to perform the double Mellin inversion for the resummed contributions, and we have adopted the so-called minimal prescription [107] to handle the Landau pole in the Mellin inversion routines.

In Figs. 1 and 2, we present the structure functions F_1 and g_1 for the EIC at $\sqrt{s} = 140$ GeV center-of-mass energy as a function of x after integrating over the kinematic ranges $y \in [0.5, 0.9]$ and $z \in [0.2, 0.85]$. The displayed curves correspond to fixed-order QCD predictions up to NNLO accuracy and the resummed cross sections to NNLO + $\overline{\text{NNLL}}$ accuracy. The associated bands indicate the variation of the μ_R^2 and μ_F^2 scales independently by a factor of two, i.e. $\mu_R^2, \mu_F^2 \in [Q^2_{\text{avg}}/2, 2Q^2_{\text{avg}}]$ with the constraint $1/2 \leq \mu_R^2/\mu_F^2 \leq 2$ in the so-called seven-point scale variation, where $Q^2_{\text{avg}} = xy_{\text{avg}}s$.

Figures 3 and 4 display again F_1 and g_1 for the EIC at $\sqrt{s} = 140$ GeV, now as a function of z after integrating over the kinematic ranges $x \in [0.1, 0.8]$ and $y \in [0.5, 0.9]$,

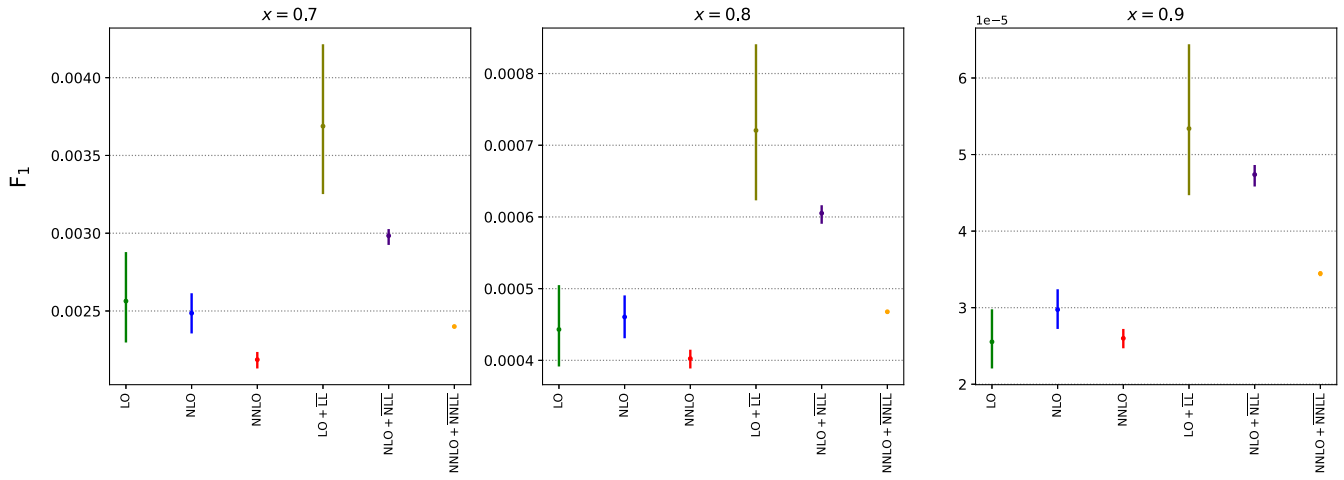


FIG. 5. Contributions from various perturbative orders to F_1 for three different x values for the EIC at $\sqrt{s} = 140$ GeV.

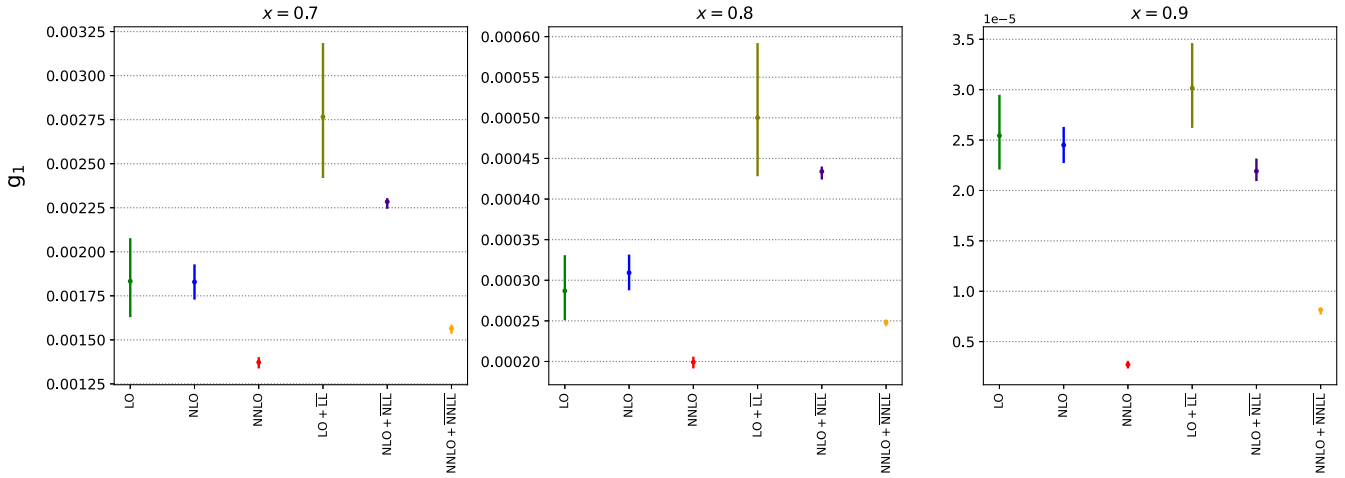


FIG. 6. Plots of the structure function g_1 against various perturbative orders at different x values, $x = \{0.7, 0.8, 0.9\}$ for the EIC at $\sqrt{s} = 140$ GeV. The y axis of the rightmost plot (for $x = 0.9$) is scaled by 10^5.

and again showing predictions both at fixed-order NNLO and at NNLO + $\overline{\text{NNLL}}$ accuracies. The corresponding bands illustrate the uncertainty from a seven-point scale variation, where μ_R^2 and μ_F^2 are independently varied by a factor of two, i.e., μ_R^2, $\mu_F^2 \in [Q^2_{\text{avg}}/2, 2Q^2_{\text{avg}}]$, where $Q^2_{\text{avg}} = x_{\text{avg}} y_{\text{avg}} s$.

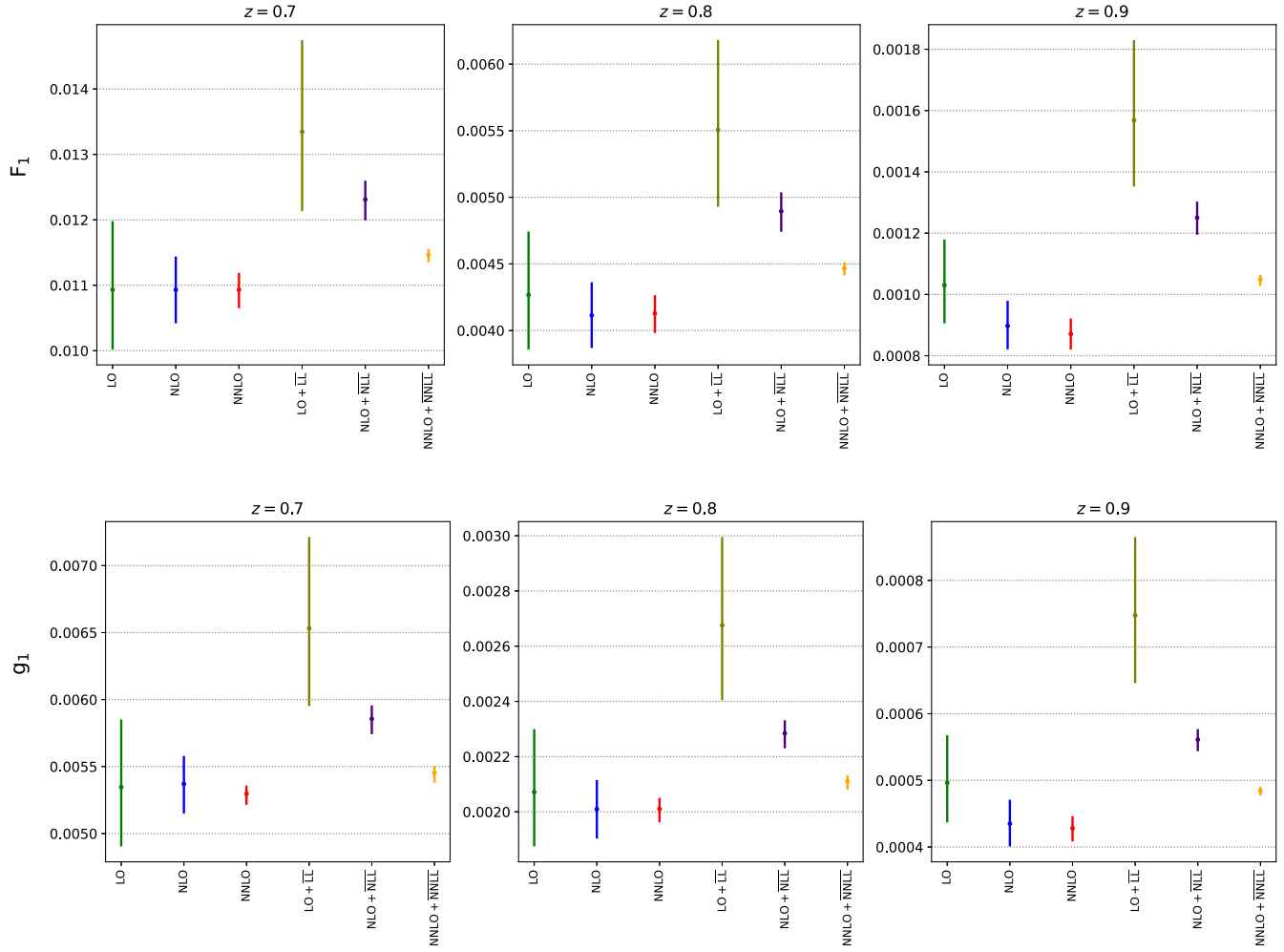


FIG. 7. Various perturbative contributions to F_1 (upper) and g_1 (lower) at different z values for the EIC at $\sqrt{s} = 140$ GeV.

TABLE IV. Values of resummed structure function F_1 at various orders in comparison to the fixed-order results at the central scale $\mu_R^2 = \mu_F^2 = Q_{\rm avg}^2$ as a function of x (see text for integration ranges of y, z).

	F_1					
x	LO	NLO	NNLO	LO + $\overline{\rm LL}$	NLO + $\overline{\rm NLL}$	NNLO + $\overline{\rm NNLL}$
0.15	$0.5044^{+5.971\%}_{-5.389\%}$	$0.3980^{+1.237\%}_{-1.557\%}$	$0.3526^{+0.570\%}_{-0.700\%}$	$0.5518^{+6.393\%}_{-5.740\%}$	$0.4077^{+0.715\%}_{-0.934\%}$	$0.3541^{+0.475\%}_{-0.357\%}$
0.25	$0.2168^{+7.160\%}_{-6.375\%}$	$0.1751^{+1.758\%}_{-2.104\%}$	$0.1569^{+0.719\%}_{-0.899\%}$	$0.2433^{+7.698\%}_{-6.809\%}$	$0.1807^{+0.919\%}_{-1.189\%}$	$0.1579^{+0.484\%}_{-0.397\%}$
0.45	$0.0426^{+9.307\%}_{-8.104\%}$	$0.0368^{+2.984\%}_{-3.301\%}$	$0.0331^{+1.180\%}_{-1.452\%}$	$0.0518^{+10.294\%}_{-8.862\%}$	$0.0395^{+1.121\%}_{-1.492\%}$	$0.0341^{+0.244\%}_{-0.360\%}$
0.6	$0.0094^{+10.991\%}_{-9.400\%}$	$0.0087^{+4.157\%}_{-4.360\%}$	$0.0077^{+1.738\%}_{-2.044\%}$	$0.0125^{+12.509\%}_{-10.521\%}$	$0.0098^{+1.269\%}_{-1.740\%}$	$0.0082^{+0.123\%}_{-0.347\%}$
0.75	$0.0012^{+13.053\%}_{-10.968\%}$	$0.0012^{+5.773\%}_{-5.787\%}$	$0.0010^{+2.618\%}_{-2.957\%}$	$0.0017^{+15.374\%}_{-12.620\%}$	$0.0014^{+1.604\%}_{-2.160\%}$	$0.0011^{+0.349\%}_{-0.465\%}$

Figures 1 and 3 (2 and 4) demonstrate an increase of the cross sections that incorporate the SV + NSV resummed results together with the fixed-order predictions as the logarithmic accuracy improves. In addition, we find that the uncertainty band, obtained from variations of the renormalization and factorization scales, shrinks considerably from LO + $\overline{\rm LL}$ to NNLO + $\overline{\rm NNLL}$ accuracy. This improvement in scale independence is more significant in the large-x or large-z regions e.g., $x > 0.7$ or $z > 0.7$, due to dominance of the threshold resummation.

The impact of resummation on F_1 and g_1 is illustrated for three distinct values of x ($x = \{0.7, 0.8, 0.9\}$) in Figs. 5 and 6, respectively, when compared to fixed-order perturbative QCD predictions, and keeping all other parameters consistent with those in Fig. 1, e.g., y and z are integrated over the kinematic ranges $y \in [0.5, 0.9]$ and $z \in [0.2, 0.85]$. Similarly, in Fig 7, we plot F_1 and g_1 for three distinct values of z ($z = \{0.7, 0.8, 0.9\}$) and compare to the different fixed perturbative orders, while y and x have been integrated as in the Fig. 3. In all these plots (Figs. 5–7), the vertical lines represent a seven-point μ_R^2, μ_F^2 scale variation. Note, that there is a significant scale dependence in the LO + $\overline{\rm LL}$ predictions in Figs. 5–7, which arises because the LO contribution inherently lacks renormalization scale dependence. It is then introduced solely by the $\overline{\rm LL}$ terms without counteracting contributions. These plots clearly demonstrate how, in the large x and/or large z regions, the contribution of resummed terms is significant. Furthermore, they illustrate how resummed predictions substantially reduce the theoretical uncertainties arising from the choice of μ_R and μ_F.

Finally, in Tables IV and V, we present the explicit results for F_1 at a few values of x and z after integrating y and z or x, respectively, and setting $\mu_R^2 = \mu_F^2 = Q_{\rm avg}^2$ as was done for the Figs. 1 and 3. A similar table can be constructed for the polarized case; however, we omit it for brevity.

VII. CONCLUSIONS

Threshold-enhanced logarithms, present in the fixed-order predictions, need to be resummed in order to obtain reliable phenomenological predictions in the relevant kinematic range. SIDIS is characterized by the scaling variables x' or z', which tend to unity in the threshold region. Fixed-order perturbative predictions of the CFs are then dominated by the plus and delta function distributions (SV terms) as well as $\log^j(1-x')$ or $\log^j(1-z')$ (NSV terms) with $j = 0, 1, 2, \ldots$. This is a situation similar to the case of rapidity distributions in the DY process. Based on the insights there, we have developed a formalism, using collinear factorization and renormalization group equations, to achieve the resummation of these threshold logarithms (SV as well as NSV) both in (x', z') space as well as in the space of the conjugate Mellin variables

TABLE V. Values of resummed structure function F_1 at various orders in comparison to the fixed-order results at the central scale $\mu_R^2 = \mu_F^2 = Q_{\rm avg}^2$ as function of z (see text for integration ranges of x, y).

	F_1					
z	LO	NLO	NNLO	LO + $\overline{\rm LL}$	NLO + $\overline{\rm NLL}$	NNLO + $\overline{\rm NNLL}$
0.15	$1.3176^{+4.645\%}_{-4.259\%}$	$1.0682^{+1.051\%}_{-1.263\%}$	$1.0282^{+0.441\%}_{-0.554\%}$	$1.4127^{+4.927\%}_{-4.497\%}$	$1.0908^{+0.522\%}_{-0.669\%}$	$1.0350^{+0.214\%}_{-0.176\%}$
0.25	$0.4817^{+5.822\%}_{-5.261\%}$	$0.3909^{+1.479\%}_{-1.712\%}$	$0.3476^{+0.609\%}_{-0.744\%}$	$0.5286^{+6.257\%}_{-5.619\%}$	$0.4020^{+0.759\%}_{-0.950\%}$	$0.3507^{+0.270\%}_{-0.304\%}$
0.45	$0.0866^{+7.581\%}_{-6.726\%}$	$0.0790^{+2.765\%}_{-2.945\%}$	$0.0741^{+1.096\%}_{-1.291\%}$	$0.0997^{+8.307\%}_{-7.299\%}$	$0.0838^{+1.494\%}_{-1.715\%}$	$0.0752^{+0.439\%}_{-0.540\%}$
0.6	$0.0251^{+8.644\%}_{-7.598\%}$	$0.0251^{+3.797\%}_{-3.905\%}$	$0.0247^{+1.790\%}_{-2.003\%}$	$0.0299^{+9.544\%}_{-8.297\%}$	$0.0275^{+1.972\%}_{-2.208\%}$	$0.0255^{+0.680\%}_{-0.817\%}$
0.75	$0.0070^{+10.248\%}_{-8.889\%}$	$0.0069^{+5.253\%}_{-5.220\%}$	$0.0069^{+2.755\%}_{-2.980\%}$	$0.0087^{+11.255\%}_{-9.656\%}$	$0.0079^{+2.588\%}_{-2.828\%}$	$0.0073^{+0.877\%}_{-1.093\%}$

$\vec{N} = (N_1, N_2)$. In Mellin space, these threshold logarithms show up as $\ln^i(N_1)\ln^j(N_2)$ for SV as well as $\ln^i(N_1)\ln^j(N_2)/N_2$, $\ln^i(N_1)\ln^j(N_2)/N_1$ where $\{i, j\} = 0, 1, 2, \ldots$ and they can be systematically resummed to all orders in a_s.

The compact expressions for the resummed results encapsulate both SV and NSV logarithms in $\vec{N}$-space and allow for phenomenological studies with EIC kinematics. Upon expansion in a_s the resummed result generates predictions for the full SV and partial NSV terms for the CFs $(\Delta)\mathcal{C}^{(3),\mathrm{SV}}_{J,qq}$, $(\Delta)\mathcal{C}^{(3),\mathrm{NSV}}_{J,qq}$ third order in (x', z') space. At fourth order, using the recently obtained results for four-loop quark virtual and eikonal anomalous dimensions B^q_4 and f^q_4, we have predicted all SV terms in $(\Delta)\mathcal{C}^{(4),\mathrm{SV}}_{J,qq}$ except the coefficient of $\delta_{\bar{x}'}\delta_{\bar{z}'}$ The derivation of the NSV terms has been corroborated by an independent approach that uses PEKs for the structure functions, and we have confirmed our predictions both at third and fourth order. Finally, we have demonstrated the numerical importance of the resummed threshold logarithms for structure functions F_1 and g_1 at EIC energies. We have shown that at each logarithmic order the resummed contributions are larger than the corresponding fixed-order ones. Moreover, including these logarithms to all orders through resummation reduces the dependence on the renormalization and factorization scales and hence improves the reliability of our predictions.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: COEFFICIENTS IN $\Gamma_{\hat{F}_q}$: K^q_d AND G^q_d

The coefficients K^q_d to fourth order in $\hat{a}_s$ are given below:

$$K^q_d(\mu^2_R, \varepsilon) = \sum_{i=0}^{\infty} \hat{a}^i_s \left(\frac{\mu^2_R}{\mu^2}\right)^{i\frac{\varepsilon}{2}} S^i_\varepsilon K^{q(i)}_d(\varepsilon), \tag{A1}$$

where

$$\begin{aligned}
K^{q(1)}_d(\varepsilon) &= \frac{1}{\varepsilon}\{-2A^q_1\},\\
K^{q(2)}_d(\varepsilon) &= \frac{1}{\varepsilon^2}\{2\beta_0 A^q_1\} + \frac{1}{\varepsilon}\{-A^q_2\},\\
K^{q(3)}_d(\varepsilon) &= \frac{1}{\varepsilon^3}\left\{-\frac{8}{3}\beta^2_0 A^q_1\right\} + \frac{1}{\varepsilon^2}\left\{\frac{2}{3}\beta_1 A^q_1 + \frac{8}{3}\beta_0 A^q_2\right\} + \frac{1}{\varepsilon}\left\{-\frac{2}{3}A^q_3\right\},\\
K^{q(4)}_d(\varepsilon) &= \frac{1}{\varepsilon^4}\left\{4\beta^3_0 A^q_1\right\} + \frac{1}{\varepsilon^3}\left\{-\frac{8}{3}\beta_0\beta_1 A^q_1 - 6\beta^2_0 A^q_2\right\} + \frac{1}{\varepsilon^2}\left\{\frac{1}{3}\beta_2 A^q_1 + \beta_1 A^q_2 + 3\beta_0 A^q_3\right\} + \frac{1}{\varepsilon}\left\{-\frac{1}{2}A^q_4\right\}.
\end{aligned} \tag{A2}$$

The constants A^q_i are the a^i_s coefficients of the cusp anomalous dimension A^q:

$$A^q(a_s(\mu^2_R)) = \sum_i a^i_s(\mu^2_R) A^q_i. \tag{A3}$$

The finite terms G^q_d can be written as

$$G^q_d(\hat{a}_s, Q^2, \mu^2_R, \varepsilon) = G^q_d(a_s(Q^2), \varepsilon) + \sum_{i=1}^{\infty} \hat{a}^i_s \left(\frac{Q^2}{\mu^2}\right)^{i\frac{\varepsilon}{2}} S^i_\varepsilon K^{q,(i)}_d(\varepsilon) - K^q_d(\mu^2_R, \varepsilon). \tag{A4}$$

The first term is expanded as $G^q_d(a_s(Q^2), \varepsilon) = \sum_{i=1}^{\infty} a^i_s(Q^2) G^q_{d,i}(\varepsilon)$, where

$$G^q_{d,i}(\varepsilon) = 2B^q_i + f^q_i + \chi^q_{d,i} + \sum_{j=1}^{\infty} \varepsilon^j \mathcal{G}^{q,(j)}_{d,i}, \tag{A5}$$

with

$$\chi^q_{d,1}=0,\chi^q_{d,2}=-2\beta_0\mathcal{G}^{q,(1)}_{d,1},\chi^q_{d,3}=-2\beta_1\mathcal{G}^{q,(1)}_{d,1}-2\beta_0(\mathcal{G}^{q,(1)}_{d,2}+2\beta_0\mathcal{G}^{q,(2)}_{d,1}),$$
$$\chi^q_{d,4}=-2\beta_2\mathcal{G}^{q,(1)}_{d,1}-2\beta_1(\mathcal{G}^{q,(1)}_{d,2}+4\beta_0\mathcal{G}^{q,(2)}_{d,1})-2\beta_0(\mathcal{G}^{q,(1)}_{d,3}+2\beta_0\mathcal{G}^{q,(2)}_{d,2}+4\beta_0^2\mathcal{G}^{q,(3)}_{d,1}). \tag{A6}$$

Next, the results for $\mathcal{G}^{q,(j)}_{d,i}$ are given below:

$$\mathcal{G}^{q,(1)}_{d,1}=C_F\{-8+\zeta_2\},\qquad \mathcal{G}^{q,(2)}_{d,1}=C_F\left\{8-\frac{7}{3}\zeta_3-\frac{3}{4}\zeta_2\right\},\qquad \mathcal{G}^{q,(3)}_{d,1}=C_F\left\{-8+\frac{7}{4}\zeta_3+\zeta_2+\frac{47}{80}\zeta_2^2\right\},$$
$$\mathcal{G}^{q,(4)}_{d,1}=C_F\left\{8-\frac{31}{20}\zeta_5-\frac{7}{3}\zeta_3-\zeta_2+\frac{7}{24}\zeta_2\zeta_3-\frac{141}{320}\zeta_2^2\right\},$$
$$\mathcal{G}^{q,(5)}_{d,1}=C_F\left\{-8+\frac{93}{80}\zeta_5+\frac{7}{3}\zeta_3-\frac{49}{144}\zeta_3^2+\zeta_2-\frac{7}{32}\zeta_2\zeta_3+\frac{47}{80}\zeta_2^2+\frac{949}{4480}\zeta_2^3\right\},$$
$$\mathcal{G}^{q,(1)}_{d,2}=C_Fn_f\left\{\frac{5813}{162}-\frac{8}{3}\zeta_3+\frac{37}{9}\zeta_2\right\}-C_FC_A\left\{\frac{70165}{324}-\frac{260}{3}\zeta_3+\frac{575}{18}\zeta_2-\frac{88}{5}\zeta_2^2\right\}-C_F^2\left\{\frac{1}{4}+60\zeta_3-58\zeta_2+\frac{88}{5}\zeta_2^2\right\},$$
$$\mathcal{G}^{q,(2)}_{d,2}=C_Fn_f\left\{-\frac{129389}{1944}+\frac{301}{27}\zeta_3-\frac{425}{54}\zeta_2+\frac{7}{12}\zeta_2^2\right\}-C_F^2\left\{\frac{109}{16}-12\zeta_5-184\zeta_3+\frac{437}{4}\zeta_2+28\zeta_2\zeta_3-\frac{108}{5}\zeta_2^2\right\}$$
$$+C_FC_A\left\{\frac{1547797}{3888}-51\zeta_5-\frac{12479}{54}\zeta_3+\frac{7297}{108}\zeta_2+\frac{89}{3}\zeta_2\zeta_3-\frac{653}{24}\zeta_2^2\right\},$$
$$\mathcal{G}^{q,(3)}_{d,2}=C_Fn_f\left\{\frac{2628821}{23328}-6\zeta_5-\frac{4085}{162}\zeta_3+\frac{8405}{648}\zeta_2-\frac{11}{6}\zeta_2\zeta_3-\frac{1873}{720}\zeta_2^2\right\}-C_FC_A\left\{\frac{31174909}{46656}-105\zeta_5-\frac{154405}{324}\zeta_3\right.$$
$$\left.+\frac{569}{6}\zeta_3^2+\frac{155701}{1296}\zeta_2+\frac{239}{12}\zeta_2\zeta_3-\frac{100907}{1440}\zeta_2^2+\frac{809}{140}\zeta_2^3\right\}+C_F^2\left\{\frac{1287}{64}-9\zeta_5-\frac{1807}{4}\zeta_3+134\zeta_3^2+\frac{2991}{16}\zeta_2\right.$$
$$\left.+13\zeta_2\zeta_3-\frac{127}{2}\zeta_2^2+\frac{858}{35}\zeta_2^3\right\},$$
$$\mathcal{G}^{q,(4)}_{d,2}=C_Fn_f\left\{-\frac{50947325}{279936}+\frac{3109}{180}\zeta_5+\frac{91985}{1944}\zeta_3-\frac{289}{108}\zeta_3^2-\frac{160493}{7776}\zeta_2+\frac{791}{216}\zeta_2\zeta_3+\frac{12781}{2160}\zeta_2^2+\frac{953}{480}\zeta_2^3\right\}$$
$$+C_FC_A\left\{\frac{599567077}{559872}+93\zeta_7-\frac{96503}{360}\zeta_5-\frac{3643249}{3888}\zeta_3+\frac{32339}{216}\zeta_3^2+\frac{3110773}{15552}\zeta_2+\frac{497}{20}\zeta_2\zeta_5+\frac{16739}{432}\zeta_2\zeta_3\right.$$
$$\left.-\frac{619829}{4320}\zeta_2^2+\frac{7103}{120}\zeta_2^2\zeta_3-\frac{7073}{1344}\zeta_2^3\right\}-C_F^2\left\{\frac{10901}{256}+\frac{675}{2}\zeta_7-99\zeta_5-\frac{50219}{48}\zeta_3+\frac{373}{2}\zeta_3^2+\frac{19405}{64}\zeta_2+3\zeta_2\zeta_5\right.$$
$$\left.+\frac{235}{6}\zeta_2\zeta_3-\frac{24453}{160}\zeta_2^2+\frac{2597}{30}\zeta_2^2\zeta_3+\frac{123}{4}\zeta_2^3\right\},$$
$$\mathcal{G}^{q,(1)}_{d,3}=C_FN_4n_{fv}\left\{12-80\zeta_5+14\zeta_3+30\zeta_2-\frac{6}{5}\zeta_2^2\right\}-C_Fn_f^2\left\{\frac{258445}{2187}-\frac{536}{81}\zeta_3+\frac{3466}{81}\zeta_2+\frac{40}{9}\zeta_2^2\right\}$$
$$-C_FC_A^2\left\{\frac{48902713}{8748}-\frac{688}{3}\zeta_5-\frac{85883}{18}\zeta_3+\frac{1136}{3}\zeta_3^2+\frac{1083305}{486}\zeta_2-\frac{1786}{9}\zeta_2\zeta_3-\frac{37271}{90}\zeta_2^2+\frac{6152}{63}\zeta_2^3\right\}$$
$$+C_F^2C_A\left\{\frac{230}{3}-\frac{3020}{3}\zeta_5-\frac{23402}{9}\zeta_3+296\zeta_3^2+\frac{55499}{18}\zeta_2-\frac{3448}{3}\zeta_2\zeta_3+\frac{2432}{45}\zeta_2^2-\frac{15448}{105}\zeta_2^3\right\}$$
$$-C_F^3\left\{\frac{1527}{4}-1992\zeta_5+2130\zeta_3-48\zeta_3^2+206\zeta_2-840\zeta_2\zeta_3+534\zeta_2^2-\frac{21584}{105}\zeta_2^3\right\}$$
$$+C_FC_An_f\left\{\frac{3702974}{2187}-72\zeta_5-\frac{68660}{81}\zeta_3+\frac{155008}{243}\zeta_2+\frac{392}{9}\zeta_2\zeta_3-\frac{1298}{45}\zeta_2^2\right\}$$
$$+C_F^2n_f\left\{\frac{73271}{162}-\frac{368}{3}\zeta_5+\frac{19700}{27}\zeta_3-\frac{7541}{18}\zeta_2-\frac{152}{3}\zeta_2\zeta_3-\frac{704}{45}\zeta_2^2\right\}. \tag{A7}$$

APPENDIX B: COEFFICIENTS IN $\Phi_{d,q}^{\mathrm{SV}}$: $\bar{K}_d^q$ AND $\bar{G}_d^q$

Here we present the SV coefficients in $\Phi_{d,q}$. We expand $\bar{K}_d^q(\mu_R^2, x', z', \varepsilon) = \delta(1-x')\delta(1-z') \sum_{i=0}^{\infty} \hat{a}_s^i (\frac{\mu_R^2}{\mu^2})^{i\frac{\varepsilon}{2}} S_\varepsilon^i \bar{K}_d^{q(i)}(\varepsilon)$ to fourth order in a_s, where

$$\begin{aligned}
\bar{K}_d^{q(1)}(\varepsilon) &= \frac{1}{\varepsilon}\{2A_1^q\},\\
\bar{K}_d^{q(2)}(\varepsilon) &= \frac{1}{\varepsilon^2}\{-2\beta_0 A_1^q\} + \frac{1}{\varepsilon}\{A_2^q\},\\
\bar{K}_d^{q(3)}(\varepsilon) &= \frac{1}{\varepsilon^3}\left\{\frac{8}{3}\beta_0^2 A_1^q\right\} + \frac{1}{\varepsilon^2}\left\{-\frac{2}{3}\beta_1 A_1^q - \frac{8}{3}\beta_0 A_2^q\right\} + \frac{1}{\varepsilon}\left\{\frac{2}{3}A_3^q\right\},\\
\bar{K}_d^{q(4)}(\varepsilon) &= \frac{1}{\varepsilon^4}\{-4\beta_0^3 A_1^q\} + \frac{1}{\varepsilon^3}\left\{\frac{8}{3}\beta_0\beta_1 A_1^q + 6\beta_0^2 A_2^q\right\} + \frac{1}{\varepsilon^2}\left\{-\frac{1}{3}\beta_2 A_1^q - \beta_1 A_2^q - 3\beta_0 A_3^q\right\} + \frac{1}{\varepsilon}\left\{\frac{1}{2}A_4^q\right\}.
\end{aligned} \tag{B1}$$

The finite terms $\bar{G}_{d,\mathrm{SV}}^{q(i)}(\varepsilon)$ are related to its renormalized counter parts $\bar{\mathcal{G}}_{d,i}^q(\varepsilon)$ in the following way:

$$\sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{Q^2(1-x')(1-z')}{\mu^2}\right)^{i\frac{\varepsilon}{2}} S_\varepsilon^i \bar{G}_{d,\mathrm{SV}}^{q,(i)}(\varepsilon) = \sum_{i=1}^{\infty} a_s^i (Q^2(1-x')(1-z'))\bar{\mathcal{G}}_{d,i}^q(\varepsilon), \tag{B2}$$

we find

$$\begin{aligned}
\bar{G}_{d,\mathrm{SV}}^{q(1)}(\varepsilon) &= \bar{\mathcal{G}}_{d,1}^q(\varepsilon),\\
\bar{G}_{d,\mathrm{SV}}^{q(2)}(\varepsilon) &= \frac{1}{\varepsilon}(-2\beta_0 \bar{\mathcal{G}}_{d,1}^q(\varepsilon)) + \bar{\mathcal{G}}_{d,2}^q(\varepsilon),\\
\bar{G}_{d,\mathrm{SV}}^{q(3)}(\varepsilon) &= \frac{1}{\varepsilon^2}(4\beta_0^2 \bar{\mathcal{G}}_{d,1}^q(\varepsilon)) + \frac{1}{\varepsilon}(-\beta_1 \bar{\mathcal{G}}_{d,1}^q(\varepsilon) - 4\beta_0 \bar{\mathcal{G}}_{d,2}^q(\varepsilon)) + \bar{\mathcal{G}}_{d,3}^q(\varepsilon),\\
\bar{G}_{d,\mathrm{SV}}^{q(4)}(\varepsilon) &= \frac{1}{\varepsilon^3}(-8\beta_0^3 \bar{\mathcal{G}}_{d,1}^q(\varepsilon)) + \frac{1}{\varepsilon^2}\left(\frac{16}{3}\beta_0\beta_1 \bar{\mathcal{G}}_{d,1}^q(\varepsilon) + 12\beta_0^2 \bar{\mathcal{G}}_{d,2}^q(\varepsilon)\right) + \frac{1}{\varepsilon}\left(-\frac{2}{3}\beta_2 \bar{\mathcal{G}}_{d,1}^q(\varepsilon) - 2\beta_1 \bar{\mathcal{G}}_{d,2}^q(\varepsilon) - 6\beta_0 \bar{\mathcal{G}}_{d,3}^q(\varepsilon)\right) + \bar{\mathcal{G}}_{d,4}^q(\varepsilon).
\end{aligned} \tag{B3}$$

APPENDIX C: DETAILS OF MELLIN MOMENT

Here we describe how to obtain the Mellin moment of Ψ_d^q given in Eq. (59). We decompose the Mellin moment of Ψ_d^q as

$$\begin{aligned}
G_{d,q}^{\vec{N}} &= \mathcal{N} \int_0^1 dx'\, x'^{N_1-1} \int_0^1 dz'\, z'^{N_2-1} \Psi_d^q(x', z') = \Psi_d^{q,\vec{N}} + \ln(g_{d,0}^q(a_s(\mu_F^2)))\\
&= \left[\left(\sum_{i=1}^{6} \psi_i^d\right) + (N_1 \leftrightarrow N_2, L^q \leftrightarrow \tilde{L}^q)\right] + \ln(g_{d,0}^q(a_s(\mu_F^2))),
\end{aligned} \tag{C1}$$

where the symbol $\mathcal{N}$ indicates we need to keep only SV and NSV terms. The terms ψ_i^d in the above equation are given below:

$$\begin{aligned}
\psi_1^d(N_1) &= \int_0^1 dx'\, x'^{N_1-1} \int_{\mu_F^2}^{Q^2(1-x')} \frac{d\lambda^2}{\lambda^2}\left(\frac{A^q(a_s(\lambda^2))}{(1-x')_+} + L^q(a_s(\lambda^2), (1-x'))\right),\\
\psi_2^d(N_1) &= \int_0^1 dx'\, x'^{N_1-1}\left(\frac{G_d^q(a_s(Q^2(1-x')))}{(1-x')_+} + \varphi_{d,q,f}^q(a_s(Q^2(1-x')), (1-x'))\right),\\
\psi_3^d(N_1, N_2) &= \frac{1}{2}\int_0^1 dx'\, x'^{N_1-1} \int_0^1 dz'\, z'^{N_2-1}\left(\frac{A^q(a_s(Q^2(1-x')(1-z')))}{(1-x')_+(1-z')_+}\right),
\end{aligned}$$

$$\psi_4^d(N_1,N_2)=\frac{1}{2}\int_0^1 dx'\,x'^{N_1-1}\int_0^1 dz'\,z'^{N_2-1}Q^2\frac{d}{dQ^2}\left(\frac{G_d^q(a_s(Q^2(1-x')(1-z')))}{(1-x')_+(1-z')_+}\right),$$
$$\psi_5^d(N_1,N_2)=\int_0^1 dx'\,x'^{N_1-1}\int_0^1 dz'\,z'^{N_2-1}\left(\frac{L^q(a_s(Q^2(1-x')(1-z')),(1-z'))}{(1-x')_+}\right),$$
$$\psi_6^d(N_1,N_2)=\int_0^1 dx'\,x'^{N_1-1}\int_0^1 dz'\,z'^{N_2-1}Q^2\frac{d}{dQ^2}\left(\frac{\varphi_{d,q,f}^q(a_s(Q^2(1-x')(1-z')),(1-z'))}{(1-x')_+}\right).$$

Following [46], we replace

$$\int_0^1 d\xi\left(\frac{\xi^{N-1}-1}{1-\xi}\right)\to\hat{\Gamma}_A\left(N\frac{d}{dN}\right)\int_0^1\frac{d\xi}{1-\xi}\theta\left(1-\frac{1}{N}-\xi\right),\tag{C2}$$

$$\int_0^1 d\xi\xi^{N-1}\to\frac{1}{N}\hat{\Gamma}_B\left(N\frac{d}{dN}\right)\int_0^1\frac{d\xi}{1-\xi}\theta\left(1-\frac{1}{N}-\xi\right).\tag{C3}$$

Here, ξ can be x' or z' and corresponding N can be N_1 or N_2. See Appendix D for the expansion of $\hat{\Gamma}_A(N\frac{d}{dN})$ and $\hat{\Gamma}_B(N\frac{d}{dN})$ operators. We need to apply $\hat{\Gamma}_A$ and $\hat{\Gamma}_B$ on the integrals through their expansions in terms of $N\frac{d}{dN}$. This can be done only after making an appropriate change of variables and interchange of the order of the integrals. For type-ψ_1^d,

$$\begin{aligned}\psi_1^d(N)&=\int_0^1 d\xi\,\xi^{N-1}\int_{\mu_F^2}^{Q^2(1-\xi)}\frac{d\lambda^2}{\lambda^2}\left(\frac{F_1^d(a_s(\lambda^2))}{(1-\xi)_+}+\tilde{F}_1^d(a_s(\lambda^2),(1-\xi))\right)\\&=\int_0^1 d\xi\left(\frac{\xi^{N-1}-1}{1-\xi}\right)\int_{\mu_F^2}^{Q^2(1-\xi)}\frac{d\lambda^2}{\lambda^2}F_1^d(a_s(\lambda^2))+\int_0^1 d\xi\xi^{N-1}\int_{\mu_F^2}^{Q^2(1-\xi)}\frac{d\lambda^2}{\lambda^2}\tilde{F}_1^d(a_s(\lambda^2),(1-\xi))\\&\underset{N\to\infty}{=}\hat{\Gamma}_A\left(N\frac{d}{dN}\right)\int_0^{1-\frac{1}{N}}\frac{d\xi}{1-\xi}\int_{\mu_F^2}^{Q^2(1-\xi)}\frac{d\lambda^2}{\lambda^2}F_1^d(a_s(\lambda^2))\\&\quad+\frac{1}{N}\hat{\Gamma}_B\left(N\frac{d}{dN}\right)\int_0^{1-\frac{1}{N}}\frac{d\xi}{1-\xi}\int_{\mu_F^2}^{Q^2(1-\xi)}\frac{d\lambda^2}{\lambda^2}\tilde{F}_1^d(a_s(\lambda^2),(1-\xi)).\end{aligned}\tag{C4}$$

To simplify further we set $\mu_F^2=Q^2$ and perform a change of variables by introducing $\mu^2=Q^2(1-\xi)$

$$\int_0^{1-\frac{1}{N}}\frac{d\xi}{1-\xi}\int_{Q^2}^{Q^2(1-\xi)}\frac{d\lambda^2}{\lambda^2}\to-\int_{\frac{Q^2}{N}}^{Q^2}\frac{d\mu^2}{\mu^2}\int_{\mu^2}^{Q^2}\frac{d\lambda^2}{\lambda^2},$$

and interchange the order of integrations,

$$\int_{\frac{Q^2}{N}}^{Q^2}\frac{d\mu^2}{\mu^2}\int_{\mu^2}^{Q^2}\frac{d\lambda^2}{\lambda^2}=\int_{\frac{Q^2}{N}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N}}^{\lambda^2}\frac{d\mu^2}{\mu^2}.$$

These manipulations lead to

$$\psi_1^d(N)\underset{N\to\infty}{=}-\hat{\Gamma}_A\left(N\frac{d}{dN}\right)\int_{\frac{Q^2}{N}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N}}^{\lambda^2}\frac{d\mu^2}{\mu^2}F_1^d(a_s(\lambda^2))-\frac{1}{N}\hat{\Gamma}_B\left(N\frac{d}{dN}\right)\int_{\frac{Q^2}{N}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N}}^{\lambda^2}\frac{d\mu^2}{\mu^2}\tilde{F}_1^d\left(a_s(\lambda^2),\frac{\mu^2}{Q^2}\right).\tag{C5}$$

For type-ψ_2^d,

$$\psi_2^d(N) = \int_0^1 d\xi \xi^{N-1}\left(\frac{F_2^d(a_s(Q^2(1-\xi)))}{(1-\xi)_+} + \tilde{F}_2^d(a_s(Q^2(1-\xi)),(1-\xi))\right)$$
$$= \int_0^1 d\xi\left(\frac{\xi^{N-1}-1}{1-\xi}\right)F_2^d(a_s(Q^2(1-\xi))) + \int_0^1 d\xi\xi^{N-1}\tilde{F}_2^d(a_s(Q^2(1-\xi)),(1-\xi))$$
$$\underset{N\to\infty}{=} \hat{\Gamma}_A\left(N\frac{d}{dN}\right)\int_0^{1-\frac{1}{N}}\frac{d\xi}{1-\xi}F_2^d(a_s(Q^2(1-\xi))) + \frac{1}{N}\hat{\Gamma}_B\left(N\frac{d}{dN}\right)\int_0^{1-\frac{1}{N}}\frac{d\xi}{1-\xi}\tilde{F}_2^d(a_s(Q^2(1-\xi)),(1-\xi)). \quad \text{(C6)}$$

Changing the integration variable from $\xi \to \lambda^2 = Q^2(1-\xi)$ we obtain

$$\psi_2^d(N) \underset{N\to\infty}{=} \hat{\Gamma}_A\left(N\frac{d}{dN}\right)\int_{\frac{Q^2}{N}}^{Q^2}\frac{d\lambda^2}{\lambda^2}F_2^d(a_s(\lambda^2)) + \frac{1}{N}\hat{\Gamma}_B\left(N\frac{d}{dN}\right)\int_{\frac{Q^2}{N}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\tilde{F}_2^d\left(a_s(\lambda^2),\frac{\lambda^2}{Q^2}\right) \quad \text{(C7)}$$

Note that in ψ_4^d and ψ_6^d after replacing $Q^2\frac{d}{dQ^2}$ by $(\beta(a_s(Q^2(1-x')(1-z'))) \times \frac{d}{da_s})$, it becomes similar to ψ_3^d and ψ_5^d respectively.

For type-ψ_3^d,

$$\psi_3^d(N_1,N_2) = \int_0^1 dx'\, x'^{N_1-1}\int_0^1 dz'\, z'^{N_2-1}\left(\frac{F_3^d(a_s(Q^2(1-x')(1-z')))}{(1-x')_+(1-z')_+}\right)$$
$$\underset{\substack{N_1\to\infty\\N_2\to\infty}}{=} \hat{\Gamma}_A\left(N_1\frac{d}{dN_1}\right)\hat{\Gamma}_A\left(N_2\frac{d}{dN_2}\right)\int_0^{1-\frac{1}{N_1}}\frac{dx'}{1-x'}\int_0^{1-\frac{1}{N_2}}\frac{dz'}{1-z'}F_3^d(a_s(Q^2(1-x')(1-z'))). \quad \text{(C8)}$$

After an appropriate change of variables we get

$$\psi_3^d(N_1,N_2) \underset{\substack{N_1\to\infty\\N_2\to\infty}}{=} \hat{\Gamma}_A\left(N_1\frac{d}{dN_1}\right)\hat{\Gamma}_A\left(N_2\frac{d}{dN_2}\right)\left[\int_{\frac{Q^2}{N_1N_2}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N_1N_2}}^{\lambda^2}\frac{d\mu^2}{\mu^2}F_3^d(a_s(\lambda^2)) - \int_{\frac{Q^2}{N_2}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N_1N_2}}^{\lambda^2}\frac{d\mu^2}{\mu^2}F_3^d(a_s(\lambda^2))\right.$$
$$\left. - \int_{\frac{Q^2}{N_1}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N_1}}^{\lambda^2}\frac{d\mu^2}{\mu^2}F_3^d(a_s(\lambda^2)) + \int_{\frac{Q^2}{N_2}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N_1}}^{Q^2}\frac{d\mu^2}{\mu^2}F_3^d(a_s(\lambda^2))\right]. \quad \text{(C9)}$$

For type-ψ_5^d,

$$\psi_5^d(N_1,N_2) = \int_0^1 dx'\, x'^{N_1-1}\int_0^1 dz'\, z'^{N_2-1}\left(\frac{F_5^d(a_s(Q^2(1-x')(1-z')),(1-z'))}{(1-x')_+}\right)$$
$$\underset{\substack{N_1\to\infty\\N_2\to\infty}}{=} \frac{1}{N_2}\hat{\Gamma}_A\left(N_1\frac{d}{dN_1}\right)\hat{\Gamma}_B\left(N_2\frac{d}{dN_2}\right)\int_0^{1-\frac{1}{N_1}}\frac{dx'}{1-x'}\int_0^{1-\frac{1}{N_2}}\frac{dz'}{1-z'}F_5^d(a_s(Q^2(1-x')(1-z')),(1-z')). \quad \text{(C10)}$$

After a change of variables we get

$$\psi_5^d(N_1,N_2) \underset{\substack{N_1\to\infty\\N_2\to\infty}}{=} \frac{1}{N_2}\hat{\Gamma}_A\left(N_1\frac{d}{dN_1}\right)\hat{\Gamma}_B\left(N_2\frac{d}{dN_2}\right)\left[\int_{\frac{Q^2}{N_1N_2}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N_1N_2}}^{\lambda^2}\frac{d\mu^2}{\mu^2}F_5^d\left(a_s(\lambda^2),\frac{\lambda^2}{\mu^2N_2}\right)\right.$$
$$- \int_{\frac{Q^2}{N_2}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N_1N_2}}^{\lambda^2}\frac{d\mu^2}{\mu^2}F_5^d\left(a_s(\lambda^2),\frac{\lambda^2}{\mu^2N_2}\right) - \int_{\frac{Q^2}{N_1}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N_1}}^{\lambda^2}\frac{d\mu^2}{\mu^2}F_5^d\left(a_s(\lambda^2),\frac{\mu^2}{Q^2}\right)$$
$$\left. + \int_{\frac{Q^2}{N_2}}^{Q^2}\frac{d\lambda^2}{\lambda^2}\int_{\frac{Q^2}{N_1}}^{Q^2}\frac{d\mu^2}{\mu^2}F_5^d\left(a_s(\lambda^2),\frac{\lambda^2}{\mu^2}\right)\right]. \quad \text{(C11)}$$

For all ψ_i^d, note that the μ^2 integral is easy to perform. Before carrying out the λ^2 integral we apply $\hat{\Gamma}_A$ and $\hat{\Gamma}_B$ using Leibniz integral rule, for e.g.,

$$N_1\frac{d}{dN_1}\int_{g_l(N_1,N_2)}^{g_u(N_1,N_2)}\frac{d\lambda^2}{\lambda^2}F(a_s(\lambda^2),N_1,N_2)=\frac{d\ln(g_u)}{d\ln N_1}F(a_s(g_u),N_1,N_2)-\frac{d\ln(g_l)}{d\ln N_1}F(a_s(g_l),N_1,N_2)+\int_{g_l(N_1,N_2)}^{g_u(N_1,N_2)}\frac{d\lambda^2}{\lambda^2}\frac{\partial}{\partial\ln N_1}F(a_s(\lambda^2),N_1,N_2),$$

and generally

$$\left(N_1\frac{d}{dN_1}\right)^n\int_{g_l(N_1,N_2)}^{g_u(N_1,N_2)}\frac{d\lambda^2}{\lambda^2}F(a_s(\lambda^2),N_1,N_2)=\sum_{j=1}^{n}\left[D_{u,1}^{n-j}\left(\frac{d\ln(g_u)}{d\ln N_1}\times\frac{\partial^{j-1}}{\partial(\ln N_1)^{j-1}}F(a_s(\lambda^2),N_1,N_2)\Big|_{\lambda^2=g_u}\right)-D_{l,1}^{n-j}\left(\frac{d\ln(g_l)}{d\ln N_1}\times\frac{\partial^{j-1}}{\partial(\ln N_1)^{j-1}}F(a_s(\lambda^2),N_1,N_2)\Big|_{\lambda^2=g_l}\right)\right]+\int_{g_l(N_1,N_2)}^{g_u(N_1,N_2)}\frac{d\lambda^2}{\lambda^2}\frac{\partial^n}{\partial(\ln N_1)^n}F(a_s(\lambda^2),N_1,N_2),\tag{C12}$$

where $D_{k,1}^i$ for $k=l,u$ is defined as

$$D_{k,1}^i=\left(\frac{\partial}{\partial\ln N_1}-\frac{d\ln(g_k)}{d\ln N_1}\beta(a_s(g_k))\frac{\partial}{\partial a_s(g_k)}\right)^i,$$

which is used to deal with N_1 dependence appearing in the argument of $a_s(g_k(N_1,N_2))$. The integral over λ^2 is done after replacing $a_s(\lambda^2)$ by Eq. (E1) and changing variables from $\lambda^2\to l=1+\beta_0 a_s(\mu_R^2)\ln(\lambda^2/\mu_R^2)$. While substituting the limits we have used ω,ω_1,ω_2 variables where, $\omega=a_s(\mu_R^2)\beta_0\ln(N_1N_2)$ and $\omega_l=a_s(\mu_R^2)\beta_0\ln N_l$ for $l=1,2$.

APPENDIX D: EXPANSION OF $\hat{\Gamma}_A(x)$ AND $\hat{\Gamma}_B(x)$

In this section we present the expansion of $\hat{\Gamma}_A(x)$ and $\hat{\Gamma}_B(x)$ used in Appendix C,

$$\hat{\Gamma}_A(x)=-\sum_{k=0}^{\infty}\gamma_k^A(x)^k,\tag{D1}$$

where the coefficients γ_k^A are given as

$$\gamma_k^A=\frac{\Gamma_k(N)}{k!}(-1)^k.\tag{D2}$$

For the definition of $\Gamma_k(N)$, see Eq. (25) of [46]. We get

$$\begin{aligned}
\gamma_0^A&=1,\\
\gamma_1^A&=\gamma_E-\frac{1}{2N},\\
\gamma_2^A&=\frac{1}{2}(\gamma_E^2+\zeta_2)-\frac{1}{2N}(1+\gamma_E),\\
\gamma_3^A&=\frac{1}{6}\gamma_E^3+\frac{1}{2}(\gamma_E\zeta_2)+\frac{1}{3}\zeta_3-\frac{1}{4N}(\gamma_E^2+2\gamma_E+\zeta_2),\\
\gamma_4^A&=\frac{1}{24}\gamma_E^4+\frac{1}{4}(\gamma_E^2\zeta_2)+\frac{9}{40}\zeta_2^2+\frac{1}{3}(\gamma_E\zeta_3)-\frac{1}{12N}(\gamma_E^3+3\gamma_E^2+3\zeta_2+3\gamma_E\zeta_2+2\zeta_3),
\end{aligned}$$

$$
\begin{aligned}
\gamma_5^A &= \frac{1}{120}\gamma_E^5 + \frac{1}{12}(\gamma_E^3\zeta_2) + \frac{1}{40}(9\gamma_E\zeta_2^2) + \frac{1}{6}(\gamma_E^2\zeta_3) + \frac{1}{6}(\zeta_2\zeta_3) + \frac{1}{5}\zeta_5 \\
&\quad - \frac{1}{240N}(20\gamma_E^3 + 5\gamma_E^4 + 30\gamma_E^2\zeta_2 + 27\zeta_2^2 + 40\zeta 3 + 20\gamma_E(3\zeta_2 + 2\zeta_3)), \\
\gamma_6^A &= \frac{1}{720}\gamma_E^6 + \frac{1}{48}(\gamma_E^4\zeta_2) + \frac{9}{80}(\gamma_E^2\zeta_2^2) + \frac{61}{560}\zeta_2^3 + \frac{1}{18}(\gamma_E^3\zeta_3) + \frac{1}{6}(\gamma_E\zeta_2\zeta_3) \\
&\quad + \frac{1}{18}\zeta_3^2 + \frac{1}{5}\gamma_E\zeta_5 - \frac{1}{240N}(5\gamma_E^4 + \gamma_E^5 + 10\gamma_E^3\zeta_2 + 27\zeta_2^2 + 20\zeta_2\zeta_3 + 10\gamma_E^2(3\zeta_2 + 2\zeta_3) \\
&\quad + \gamma_E(27\zeta_2^2 + 40\zeta_3) + 24\zeta_5), \\
\gamma_7^A &= \frac{1}{5040}\gamma_E^7 + \frac{1}{240}(\gamma_E^5\zeta_2) + \frac{3}{80}(\gamma_E^3\zeta_2^2) + \frac{61}{560}(\gamma_E\zeta_2^3) + \frac{1}{72}(\gamma_E^4\zeta_3) \\
&\quad + \frac{1}{12}(\gamma_E^2\zeta_2\zeta_3) + \frac{3}{40}(\zeta_2^2\zeta_3) + \frac{1}{18}(\gamma_E\zeta_3^2) + \frac{1}{10}(\gamma_E^2\zeta_5) + \frac{1}{10}(\zeta_2\zeta_5) + \frac{1}{7}\zeta_7 \\
&\quad - \frac{1}{10080N}(42\gamma_E^5 + 7\gamma_E^6 + 105\gamma_E^4\zeta_2 + 549\zeta_2^3 + 840\zeta_2\zeta_3 + 140\gamma_E^3(3\zeta_2 + 2\zeta_3) \\
&\quad + 21\gamma_E^2(27\zeta_2^2 + 40\zeta_3) + 56(5\zeta_3^2 + 18\zeta_5) + 42\gamma_E(27\zeta_2^2 + 20\zeta_2\zeta_3 + 24\zeta_5)), \\
\gamma_8^A &= \frac{1}{40320}\gamma_E^8 + \frac{1}{1440}(\gamma_E^6\zeta_2) + \frac{1}{360}(\gamma_E^5\zeta_3) + \frac{3}{320}(\gamma_E^4\zeta_2^2) + \gamma_E^3\left(\frac{\zeta_2\zeta_3}{36} + \frac{\zeta_5}{30}\right) + \gamma_E^2\left(\frac{61\zeta_2^3}{1120} + \frac{\zeta_3^2}{36}\right) \\
&\quad + \gamma_E\left(\frac{3}{40}\zeta_3\zeta_2^2 + \frac{\zeta_5\zeta_2}{10} + \frac{\zeta_7}{7}\right) + \frac{1261}{22400}(\zeta_2^4) + \frac{1}{36}(\zeta_2\zeta_3^2) + \frac{1}{15}(\zeta_3\zeta_5) - \frac{1}{10080N}(\gamma_E^7 + 7\gamma_E^6 \\
&\quad + 21\gamma_E^5\zeta_2 + 35\gamma_E^4(3\zeta_2 + 2\zeta_3) + 7\gamma_E^3(27\zeta_2^2 + 40\zeta_3) + 21\gamma_E^2(27\zeta_2^2 + 20\zeta_3\zeta_2 + 24\zeta_5) \\
&\quad + \gamma_E(549\zeta_2^3 + 840\zeta_3\zeta_2 + 280\zeta_3^2 + 1008\zeta_5) + 549\zeta_2^3 + 280\zeta_3^2 + 378\zeta_2^2\zeta_3 + 504\zeta_2\zeta_5 + 720\zeta_7).
\end{aligned} \tag{D3}
$$

Here γ_E is the Euler-Mascheroni constant. Similarly $\hat{\Gamma}_B(x)$ is given by

$$
\hat{\Gamma}_B(x) = \sum_{k=0}^{\infty} \gamma_k^B (x)^{k+1}, \tag{D4}
$$

where γ_k^B are given as

$$
\gamma_k^B = \frac{\Gamma^k(1)}{k!}(-1)^k. \tag{D5}
$$

Here, $\Gamma^k(1)$ is the kth derivative of the Euler gamma function. We find explicitly

$$
\begin{aligned}
\gamma_0^B &= 1, \\
\gamma_1^B &= \gamma_E, \\
\gamma_2^B &= \frac{1}{2}(\gamma_E^2 + \zeta_2), \\
\gamma_3^B &= \frac{1}{6}\gamma_E^3 + \frac{1}{2}(\gamma_E\zeta_2) + \frac{1}{3}\zeta_3, \\
\gamma_4^B &= \frac{1}{24}\gamma_E^4 + \frac{1}{4}(\gamma_E^2\zeta_2) + \frac{9}{40}\zeta_2^2 + \frac{1}{3}(\gamma_E\zeta_3), \\
\gamma_5^B &= \frac{1}{120}\gamma_E^5 + \frac{1}{12}(\gamma_E^3\zeta_2) + \frac{1}{40}(9\gamma_E\zeta_2^2) + \frac{1}{6}(\gamma_E^2\zeta_3) + \frac{1}{6}(\zeta_2\zeta_3) + \frac{1}{5}\zeta_5, \\
\gamma_6^B &= \frac{1}{720}\gamma_E^6 + \frac{1}{48}(\gamma_E^4\zeta_2) + \frac{9}{80}(\gamma_E^2\zeta_2^2) + \frac{61}{560}\zeta_2^3 + \frac{1}{18}(\gamma_E^3\zeta_3) + \frac{1}{6}(\gamma_E\zeta_2\zeta_3) + \frac{1}{18}\zeta_3^2 + \frac{1}{5}\gamma_E\zeta_5,
\end{aligned}
$$

$$\gamma_7^B = \frac{1}{5040}\gamma_E^7 + \frac{1}{240}(\gamma_E^5\zeta_2) + \frac{3}{80}(\gamma_E^3\zeta_2^2) + \frac{61}{560}(\gamma_E\zeta_2^3) + \frac{1}{72}(\gamma_E^4\zeta_3) + \frac{1}{12}(\gamma_E^2\zeta_2\zeta_3) + \frac{3}{40}(\zeta_2^2\zeta_3) + \frac{1}{18}(\gamma_E\zeta_3^2) + \frac{1}{10}(\gamma_E^2\zeta_5) + \frac{1}{10}(\zeta_2\zeta_5) + \frac{1}{7}\zeta_7,$$

$$\gamma_8^B = \frac{1}{40320}\gamma_E^8 + \frac{1}{1440}(\gamma_E^6\zeta_2) + \frac{1}{360}(\gamma_E^5\zeta_3) + \frac{3}{320}(\gamma_E^4\zeta_2^2) + \frac{1}{36}(\gamma_E^3\zeta_2\zeta_3) + \frac{1}{30}(\gamma_E^3\zeta_5) + \frac{61}{1120}(\gamma_E^2\zeta_2^3) + \frac{1}{36}(\gamma_E^2\zeta_3^2) + \frac{3}{40}(\gamma_E\zeta_2^2\zeta_3) + \frac{1}{10}(\gamma_E\zeta_2\zeta_5) + \frac{1}{7}(\gamma_E\zeta_7) + \frac{1261}{22400}\zeta_2^4 + \frac{1}{36}(\zeta_2\zeta_3^2) + \frac{1}{15}(\zeta_3\zeta_5). \tag{D6}$$

APPENDIX E: RESUMMED COUPLING CONSTANT

For performing the integration over λ^2 used in Appendix C, we replace $a_s(\lambda^2)$ by

$$\begin{aligned} a_s(\lambda^2) = \left(\frac{a_s(\mu_R^2)}{l}\right)\Bigg[1 - \frac{a_s(\mu_R^2)}{l}\left\{\frac{\beta_1}{\beta_0}\log(l)\right\} + \left(\frac{a_s(\mu_R^2)}{l}\right)^2\left\{\frac{\beta_1^2}{\beta_0^2}(\log^2(l) - \log(l) + l - 1) - \frac{\beta_2}{\beta_0}(l-1)\right\} \\ + \left(\frac{a_s(\mu_R^2)}{l}\right)^3\left\{\frac{\beta_1^3}{\beta_0^3}\left(2(1-l)\log(l) + \frac{5}{2}\log^2(l) - \log^3(l) - \frac{1}{2} + l - \frac{1}{2}l^2\right) + \frac{\beta_3}{2\beta_0}(1-l^2)\right. \\ \left. + \frac{\beta_1\beta_2}{\beta_0^2}(2l\log(l) - 3\log(l) - l(1-l))\right\} \\ + \left(\frac{a_s(\mu_R^2)}{l}\right)^4\left\{\frac{\beta_1^4}{\beta_0^4}\left(\frac{7}{6} - 2l + \frac{l^2}{2} + \frac{l^3}{3} + 4\log(l) - 5l\log(l) + l^2\log(l) - \frac{3}{2}\log(l)^2 + 3l\log(l)^2\right.\right. \\ \left.\left. - \frac{13}{3}\log(l)^3 + \log(l)^4\right) + \frac{\beta_4}{3\beta_0}(1-l^3) + \left(\frac{\beta_2}{\beta_0}\right)^2\left(\frac{5}{3} - 3l + l^2 + \frac{l^3}{3}\right)\right. \\ + \frac{\beta_1\beta_3}{\beta_0^2}\left(-\frac{1}{6} - \frac{l^2}{2} + \frac{2}{3}l^3 - 2\log(l) + l^2\log(l)\right) + \frac{\beta_1^2\beta_2}{\beta_0^3}(-3 + 5l - l^2 - l^3 - 3\log(l) + 5l\log(l) \\ \left. - 2l^2\log(l) + 6\log(l)^2 - 3l\log(l)^2)\right\}\Bigg], \end{aligned} \tag{E1}$$

where $l = 1 + \beta_0 a_s(\mu_R^2)\ln(\lambda^2/\mu_R^2)$ and β_i are the coefficients of QCD β function $\beta(a_s) = -\sum_{i=0}^{\infty} a_s^{i+2}\beta_i$, known to five loops [81–84].

APPENDIX F: QCD β FUNCTIONS

$$\begin{aligned} \beta_0 &= \frac{11}{3}C_A - \frac{2}{3}n_f, \\ \beta_1 &= \frac{34}{3}C_A^2 - 2n_fC_F - \frac{10}{3}n_fC_A, \\ \beta_2 &= \frac{2857}{54}C_A^3 - \frac{1415}{54}C_A^2n_f + \frac{79}{54}C_An_f^2 + \frac{11}{9}C_Fn_f^2 - \frac{205}{18}C_FC_An_f + C_F^2n_f. \end{aligned} \tag{F1}$$

APPENDIX G: ANOMALOUS DIMENSIONS

$$\begin{aligned} f_1^q &= 0, \\ f_2^q &= C_FC_A\left\{\frac{808}{27} - 28\zeta_3 - \frac{22}{3}\zeta_2\right\} - C_Fn_f\left\{\frac{112}{27} - \frac{4}{3}\zeta_2\right\}, \\ f_3^q &= -C_F^2n_f\left\{\frac{1711}{27} - \frac{304}{9}\zeta_3 - 4\zeta_2 - \frac{32}{5}\zeta_2^2\right\} + C_FC_A^2\left\{\frac{136781}{729} + 192\zeta_5 - \frac{1316}{3}\zeta_3 - \frac{12650}{81}\zeta_2 + \frac{176}{3}\zeta_2\zeta_3 + \frac{352}{5}\zeta_2^2\right\} \\ &\quad - C_FC_An_f\left\{\frac{11842}{729} - \frac{728}{27}\zeta_3 - \frac{2828}{81}\zeta_2 + \frac{96}{5}\zeta_2^2\right\} - C_Fn_f^2\left\{\frac{2080}{729} - \frac{112}{27}\zeta_3 + \frac{40}{27}\zeta_2\right\}, \end{aligned} \tag{G1}$$

$$
\begin{aligned}
A_1^q &= 4C_F,\\
A_2^q &= C_A C_F\left\{\frac{268}{9}-8\zeta_2\right\}-C_F n_f\left\{\frac{40}{9}\right\},\\
A_3^q &= C_F C_A^2\left\{\frac{490}{3}+\frac{88}{3}\zeta_3-\frac{1072}{9}\zeta_2+\frac{176}{5}\zeta_2^2\right\}-C_F^2 n_f\left\{\frac{110}{3}-32\zeta_3\right\}-C_F C_A n_f\left\{\frac{836}{27}+\frac{112}{3}\zeta_3-\frac{160}{9}\zeta_2\right\}-C_F n_f^2\left\{\frac{16}{27}\right\},\\
A_4^q &= \frac{d_F^{abcd}d_A^{abcd}}{N_c}\left\{\frac{3520}{3}\zeta_5+\frac{128}{3}\zeta_3-384\zeta_3^2-128\zeta_2-\frac{7936}{35}\zeta_2^3\right\}-n_f\frac{d_F^{abcd}d_F^{abcd}}{N_c}\left\{\frac{1280}{3}\zeta_5+\frac{256}{3}\zeta_3-256\zeta_2\right\}\\
&\quad -C_F n_f^3\left\{\frac{32}{81}-\frac{64}{27}\zeta_3\right\}-C_F C_A^2 n_f\left\{\frac{24137}{81}-\frac{2096}{9}\zeta_5+\frac{23104}{27}\zeta_3-\frac{20320}{81}\zeta_2-\frac{448}{3}\zeta_2\zeta_3+\frac{352}{15}\zeta_2^2\right\}\\
&\quad +C_F C_A^3\left\{\frac{84278}{81}-\frac{3608}{9}\zeta_5+\frac{20944}{27}\zeta_3-16\zeta_3^2-\frac{88400}{81}\zeta_2-\frac{352}{3}\zeta_2\zeta_3+\frac{3608}{5}\zeta_2^2-\frac{20032}{105}\zeta_2^3\right\}\\
&\quad +C_F C_A n_f^2\left\{\frac{923}{81}+\frac{2240}{27}\zeta_3-\frac{608}{81}\zeta_2-\frac{224}{15}\zeta_2^2\right\}+C_F^2 n_f^2\left\{\frac{2392}{81}-\frac{640}{9}\zeta_3+\frac{64}{5}\zeta_2^2\right\}\\
&\quad -C_F^2 C_A n_f\left\{\frac{34066}{81}-160\zeta_5-\frac{3712}{9}\zeta_3-\frac{440}{3}\zeta_2+128\zeta_2\zeta_3+\frac{352}{5}\zeta_2^2\right\}+C_F^3 n_f\left\{\frac{572}{9}-320\zeta_5+\frac{592}{3}\zeta_3\right\},
\end{aligned}
\tag{G2}
$$

$$
\begin{aligned}
B_1^q &= 3C_F,\\
B_2^q &= C_F^2\left\{\frac{3}{2}+24\zeta_3-12\zeta_2\right\}+C_F C_A\left\{\frac{17}{6}-12\zeta_3+\frac{44}{3}\zeta_2\right\}-C_F n_f\left\{\frac{1}{3}+\frac{8}{3}\zeta_2\right\},\\
B_3^q &= C_F^3\left\{\frac{29}{2}-240\zeta_5+68\zeta_3+18\zeta_2-32\zeta_2\zeta_3+\frac{288}{5}\zeta_2^2\right\}-C_F^2 n_f\left\{23+\frac{136}{3}\zeta_3-\frac{20}{3}\zeta_2-\frac{232}{15}\zeta_2^2\right\}\\
&\quad +C_F^2 C_A\left\{\frac{151}{4}+120\zeta_5+\frac{844}{3}\zeta_3-\frac{410}{3}\zeta_2+16\zeta_2\zeta_3-\frac{988}{15}\zeta_2^2\right\}-C_F C_A^2\left\{\frac{1657}{36}-40\zeta_5+\frac{1552}{9}\zeta_3-\frac{4496}{27}\zeta_2+2\zeta_2^2\right\}\\
&\quad +C_F C_A n_f\left\{20+\frac{200}{9}\zeta_3-\frac{1336}{27}\zeta_2+\frac{4}{5}\zeta_2^2\right\}-C_F n_f^2\left\{\frac{17}{9}+\frac{16}{9}\zeta_3-\frac{80}{27}\zeta_2\right\},
\end{aligned}
\tag{G3}
$$

$$
\begin{aligned}
C_1^q &= 0,\\
C_2^q &= 16C_F^2,\\
C_3^q &= C_F^2 C_A\left\{\frac{2144}{9}-64\zeta_2\right\}-C_F^2 n_f\left\{\frac{320}{9}\right\},
\end{aligned}
\tag{G4}
$$

$$
\begin{aligned}
\tilde{C}_1^q &= 0,\\
\tilde{C}_2^q &= -16C_F^2,\\
\tilde{C}_3^q &= -C_F^2 C_A\left\{\frac{2144}{9}-64\zeta_2\right\}+C_F^2 n_f\left\{\frac{320}{9}\right\},
\end{aligned}
\tag{G5}
$$

$$
\begin{aligned}
D_1^q &= -4C_F,\\
D_2^q &= C_F^2\{12\}-C_F C_A\left\{\frac{400}{9}-8\zeta_2\right\}+C_F n_f\left\{\frac{64}{9}\right\},\\
D_3^q &= C_F^3\left\{6+96\zeta_3-48\zeta_2\right\}+C_F^2 C_A\left\{\frac{302}{3}-48\zeta_3+\frac{104}{3}\zeta_2\right\}+C_F^2 n_f\left\{30-32\zeta_3-\frac{32}{3}\zeta_2\right\}\\
&\quad -C_F C_A^2\left\{\frac{8582}{27}+\frac{88}{3}\zeta_3-\frac{1336}{9}\zeta_2+\frac{176}{5}\zeta_2^2\right\}+C_F C_A n_f\left\{\frac{724}{9}+\frac{112}{3}\zeta_3-\frac{208}{9}\zeta_2\right\}-C_F n_f^2\left\{\frac{64}{27}\right\},
\end{aligned}
\tag{G6}
$$

$$\tilde{D}_1^q = -4C_F,$$
$$\tilde{D}_2^q = -C_F^2\{12\} - C_F C_A\left\{\frac{400}{9} - 8\zeta_2\right\} + C_F n_f\left\{\frac{64}{9}\right\},$$
$$\tilde{D}_3^q = -C_F^3\{6 + 96\zeta_3 - 48\zeta_2\} - C_F^2 C_A\left\{\frac{302}{3} - 48\zeta_3 + \frac{104}{3}\zeta_2\right\} + C_F^2 n_f\left\{\frac{178}{3} - 32\zeta_3 + \frac{32}{3}\zeta_2\right\}$$
$$- C_F C_A^2\left\{\frac{8582}{27} + \frac{88}{3}\zeta_3 - \frac{1336}{9}\zeta_2 + \frac{176}{5}\zeta_2^2\right\} + C_F C_A n_f\left\{\frac{724}{9} + \frac{112}{3}\zeta_3 - \frac{208}{9}\zeta_2\right\} - C_F n_f^2\left\{\frac{64}{27}\right\}. \quad \text{(G7)}$$

APPENDIX H: RESUMMATION EXPONENTS

In this appendix we present resummation exponents used in Ψ_d^q given in Eq. (59) and exponents used in the N-space result in Eqs. (72) and (73). First, we present the process-dependent $g_{d,0}^q = \sum_{i=0} a_s^i(\mu_R^2) g_{d,0,i}^q$ given in Eq. (59),

$$g_{d,0,0}^q = 1,$$
$$g_{d,0,1}^q = C_F\{-16 + 6(L_{qr} - L_{fr})\},$$
$$g_{d,0,2}^q = C_F^2\left\{\frac{511}{4} + 18(L_{qr}^2 + L_{fr}^2) + (48\zeta_3 - 24\zeta_2 - 93)(L_{qr} - L_{fr}) - 36L_{fr}L_{qr} - 60\zeta_3 + 58\zeta_2 - \frac{88}{5}\zeta_2^2\right\}$$
$$- C_F C_A\left\{\frac{1535}{12} + 11(L_{qr}^2 - L_{fr}^2) + \left(24\zeta_3 - \frac{88}{3}\zeta_2\right)(L_{qr} - L_{fr}) - \frac{193}{3}L_{qr} + \frac{17}{3}L_{fr} - \frac{172}{3}\zeta_3 + \frac{538}{9}\zeta_2 - \frac{68}{5}\zeta_2^2\right\}$$
$$+ C_F n_f\left\{\frac{127}{6} + 2(L_{qr}^2 - L_{fr}^2) - \frac{16}{3}\zeta_2(L_{qr} - L_{fr}) - \frac{34}{3}L_{qr} + \frac{2}{3}L_{fr} + \frac{8}{3}\zeta_3 + \frac{76}{9}\zeta_2\right\} \quad \text{(H1)}$$

The $\vec{N}$-independent $\tilde{g}_{d,0}^q = \sum_{i=0} a_s^i(\mu_R^2)\tilde{g}_{d,0,i}^q$ given in Eq. (72) are found to be as follows:

$$\tilde{g}_{d,0,0}^q = 1,$$
$$\tilde{g}_{d,0,1}^q = C_F\{-16 + (6 - 8\gamma_E)(L_{qr} - L_{fr}) + 8\gamma_E^2 + 4\zeta_2\},$$
$$\tilde{g}_{d,0,2}^q = C_F^2\left\{\frac{511}{4} + (18 - 48\gamma_E + 32\gamma_E^2)(L_{qr}^2 + L_{fr}^2) - (93 - 128\gamma_E - 48\gamma_E^2 + 64\gamma_E^3 - 48\zeta_3 + 32\zeta_2\gamma_E)(L_{qr} - L_{fr})\right.$$
$$\left. - (36 - 96\gamma_E + 64\gamma_E^2)L_{fr}L_{qr} - 128\gamma_E^2 + 32\gamma_E^4 - 60\zeta_3 - 6\zeta_2 + 32\zeta_2\gamma_E^2 - \frac{48}{5}\zeta_2^2\right\}$$
$$- C_F C_A\left\{\frac{1535}{12} + \left(11 - \frac{44}{3}\gamma_E\right)(L_{qr}^2 - L_{fr}^2) + \left(\frac{536}{9}\gamma_E - 16\zeta_2\gamma_E + 24\zeta_3\right)(L_{qr} - L_{fr}) - \frac{193}{3}L_{qr} + \frac{17}{3}L_{fr}\right.$$
$$\left. + \frac{88}{3}\gamma_E^2 L_{qr} - \frac{44}{3}\zeta_2(L_{qr} - 2L_{fr}) - \frac{176}{9}\gamma_E^3 - \frac{536}{9}\gamma_E^2 - \frac{1616}{27}\gamma_E - \frac{604}{9}\zeta_3 + 56\zeta_3\gamma_E + 30\zeta_2 + 16\zeta_2\gamma_E^2 - \frac{28}{5}\zeta_2^2\right\}$$
$$+ C_F n_f\left\{\frac{127}{6} + \left(2 - \frac{8}{3}\gamma_E\right)(L_{qr}^2 - L_{fr}^2) + \frac{80}{9}\gamma_E(L_{qr} - L_{fr}) + \frac{16}{3}\gamma_E^2 L_{qr} - \frac{34}{3}L_{qr} + \frac{2}{3}L_{fr}\right.$$
$$\left. - \frac{8}{3}\zeta_2(L_{qr} - 2L_{fr}) - \frac{32}{9}\gamma_E^3 - \frac{80}{9}\gamma_E^2 - \frac{224}{27}\gamma_E + \frac{8}{9}\zeta_3 + 4\zeta_2\right\}. \quad \text{(H2)}$$

The SV resummation exponents $g_{d,i}^q(\omega)$ given in Eq. (73) are presented below.

$$g_{d,1}^q(\omega) = C_F\frac{1}{\beta_0}\left\{4 + \frac{4}{\omega}L_\omega - 4L_\omega\right\},$$
$$g_{d,2}^q(\omega) = C_F\frac{\beta_1}{\beta_0^3}\{4L_\omega + 2L_\omega^2 + 4\omega\} - C_F\frac{1}{\beta_0}\{8\gamma_E L_\omega - 4\omega L_{fr} - 4L_{qr}L_\omega\} + C_F n_f\frac{1}{\beta_0^2}\left\{\frac{40}{9}L_\omega + \frac{40}{9}\omega\right\}$$
$$- C_F C_A\frac{1}{\beta_0^2}\left\{\frac{268}{9}L_\omega + \frac{268}{9}\omega - 8\zeta_2 L_\omega - 8\zeta_2\omega\right\},$$

$$
\begin{aligned}
g^q_{d,3}(\omega) &= C_F\frac{1}{(1-\omega)}\frac{\beta_1^2}{\beta_0^4}\{2L_\omega^2+4\omega L_\omega+2\omega^2\}+C_F\frac{\omega}{(1-\omega)}\frac{\beta_2}{\beta_0^3}\{4-2\omega\}+C_F\frac{\beta_2}{\beta_0^3}\{4L_\omega\}\\
&-C_F\frac{1}{(1-\omega)}\frac{\beta_1}{\beta_0^2}\{8\gamma_E L_\omega-4L_{qr}L_\omega+8\omega\gamma_E-4\omega L_{qr}\}+C_F\frac{\omega}{(1-\omega)}\{4\zeta_2+8\gamma_E^2-8L_{qr}\gamma_E+2\omega L_{qr}^2\}\\
&+C_F\{2\omega(L_{qr}^2-L_{fr}^2)\}+C_F n_f\frac{1}{\beta_0}\left\{\frac{40}{9}\omega(L_{qr}-L_{fr})\right\}+C_F n_f\frac{1}{(1-\omega)}\frac{\beta_1}{\beta_0^3}\left\{\frac{40}{9}L_\omega+\frac{40}{9}\omega+\frac{20}{9}\omega^2\right\}\\
&-C_F n_f\frac{\omega}{(1-\omega)}\frac{1}{\beta_0}\left\{\frac{112}{27}-\frac{8}{3}\zeta_2+\frac{80}{9}\gamma_E-\frac{40}{9}\omega L_{qr}\right\}-C_F n_f^2\frac{\omega^2}{(1-\omega)}\frac{1}{\beta_0^2}\left\{\frac{8}{27}\right\}\\
&-C_F C_A\frac{1}{(1-\omega)}\frac{\beta_1}{\beta_0^3}\left\{\frac{268}{9}L_\omega-8\zeta_2 L_\omega+\frac{268}{9}\omega-8\omega\zeta_2+\frac{134}{9}\omega^2-4\omega^2\zeta_2\right\}\\
&+C_F C_A\frac{\omega}{(1-\omega)}\frac{1}{\beta_0}\left\{\frac{808}{27}-28\zeta_3-\frac{44}{3}\zeta_2+\frac{536}{9}\gamma_E-16\gamma_E\zeta_2-\frac{268}{9}\omega L_{qr}+8\omega L_{qr}\zeta_2\right\}\\
&-C_F C_A n_f\frac{\omega^2}{(1-\omega)}\frac{1}{\beta_0^2}\left\{\frac{418}{27}+\frac{56}{3}\zeta_3-\frac{80}{9}\zeta_2\right\}+C_F C_A^2\frac{\omega^2}{(1-\omega)}\frac{1}{\beta_0^2}\left\{\frac{245}{3}+\frac{44}{3}\zeta_3-\frac{536}{9}\zeta_2+\frac{88}{5}\zeta_2^2\right\}\\
&-C_F^2 n_f\frac{\omega^2}{(1-\omega)}\frac{1}{\beta_0^2}\left\{\frac{55}{3}-16\zeta_3\right\}-C_F C_A\frac{1}{\beta_0}\left\{\left(\frac{268}{9}\omega-8\omega\zeta_2\right)(L_{qr}-L_{fr})\right\}. \qquad \text{(H3)}
\end{aligned}
$$

The NSV resummation exponents $\bar{g}^q_{d,i}(\omega)$ given in Eq. (73) read

$$
\begin{aligned}
\bar{g}^q_{d,0}(\omega) &= C_F\frac{1}{\beta_0}\{2L_\omega\},\\
\bar{g}^q_{d,1}(\omega) &= C_F\frac{1}{(1-\omega)}\frac{\beta_1}{\beta_0^2}\{2L_\omega+2\omega\}-C_F\frac{1}{(1-\omega)}\{2+4\gamma_E-2L_{qr}\}-C_F\{2(L_{fr})\}\\
&+C_F n_f\frac{\omega}{(1-\omega)}\frac{1}{\beta_0}\left\{\frac{20}{9}\right\}-C_F C_A\frac{\omega}{(1-\omega)}\frac{1}{\beta_0}\left\{\frac{134}{9}-4\zeta_2\right\},\\
\bar{g}^q_{d,2}(\omega) &= C_F\frac{1}{(1-\omega)^2}\frac{\beta_1^2}{\beta_0^3}\{\omega^2-L_\omega^2\}-C_F\frac{\omega^2}{(1-\omega)^2}\frac{\beta_2}{\beta_0^2}\{1\}+C_F\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0}\{(2+4\gamma_E-2L_{qr})L_\omega\}\\
&-C_F\frac{1}{(1-\omega)^2}\beta_0\{4\gamma_E+4\gamma_E^2-(1-\omega)^2L_{fr}^2-2L_{qr}-4L_{qr}\gamma_E+L_{qr}^2+2\zeta_2\}\\
&-C_F n_f\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0^2}\left\{\frac{20}{9}L_\omega+\frac{20}{9}\omega-\frac{10}{9}\omega^2\right\}+C_F n_f\frac{1}{(1-\omega)^2}\left\{\frac{116}{27}+\frac{40}{9}\gamma_E-\frac{20}{9}L_{qr}-\frac{4}{3}\zeta_2\right\}\\
&+C_F n_f\left\{\frac{20}{9}(L_{fr})\right\}+C_F n_f^2\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{\frac{8}{27}-\frac{4}{27}\omega\right\}+C_F C_A\left\{\left(\frac{134}{9}-4\zeta_2\right)(L_{fr})\right\}\\
&+C_F C_A\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0^2}\left\{(2L_\omega+2\omega-\omega^2)\left(\frac{67}{9}-2\zeta_2\right)\right\}+C_F C_A\frac{1}{(1-\omega)^2}\left\{\left(\frac{134}{9}-4\zeta_2\right)L_{qr}-\frac{806}{27}\right.\\
&\left.-\frac{268}{9}\gamma_E+8\zeta_2\gamma_E+14\zeta_3+\frac{34}{3}\zeta_2\right\}-C_F C_A^2\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{(2-\omega)\left(\frac{245}{6}-\frac{268}{9}\zeta_2+\frac{44}{5}\zeta_2^2+\frac{22}{3}\zeta_3\right)\right\}\\
&+C_F^2 n_f\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{(2-\omega)\left(\frac{55}{6}-8\zeta_3\right)\right\}+C_F C_A n_f\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{(2-\omega)\left(\frac{209}{27}-\frac{40}{9}\zeta_2+\frac{28}{3}\zeta_3\right)\right\}. \qquad \text{(H4)}
\end{aligned}
$$

The NSV resummation exponents $h^q_{d,i,j}$ and $\tilde{h}^q_{d,i,i}$ given in Eq. (74) are listed next:

$$
\begin{aligned}
h^q_{d,0,0}(\omega) &= C_F \frac{1}{\beta_0}\{-4L_\omega\},\\
h^q_{d,0,1}(\omega) &= 0,\\
h^q_{d,1,0}(\omega,\omega_l) &= -C_F\frac{1}{(1-\omega)}\frac{\beta_1}{\beta_0^2}\{4L_\omega+4\omega\} - C_F\frac{1}{(1-\omega)}\{(-1)^l 4 - 8\gamma_E - 4L_{fr} + 4\omega L_{fr} + 4L_{qr}\}\\
&\quad - C_F n_f \frac{\omega}{(1-\omega)}\frac{1}{\beta_0}\left\{\frac{64}{9}\right\} + C_F C_A \frac{\omega}{(1-\omega)}\frac{1}{\beta_0}\left\{\frac{400}{9} - 8\zeta_2\right\} + C_F^2\frac{\omega}{(1-\omega)}\frac{1}{\beta_0}\{(-1)^l(12-16\gamma_E)\},\\
\tilde{h}^q_{d,1,1}(\omega,\omega_l) &= (-1)^l\left[C_F^2\frac{1}{(1-\omega)^2}\frac{1}{\beta_0}\{4\omega_l\} - C_F^2\frac{\omega}{(1-\omega)}\frac{1}{\beta_0}\{16\}\right],\\
h^q_{d,2,0}(\omega,\omega_1) &= h^q_{d,2,0}(\omega,\omega_2) + \left[C_F\frac{1}{(1-\omega)^2}\beta_0\{16\gamma_E - 8L_{qr}\} - C_F\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0}\{8L_\omega\}\right.\\
&\quad + C_F C_A\frac{1}{(1-\omega)^2}\left\{\frac{128}{9} - 8\zeta_2 - 20\gamma_E\right\} + C_F^2\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0^2}\{(12-16\gamma_E)(2L_\omega+2\omega-\omega^2)\}\\
&\quad + C_F^2\frac{1}{(1-\omega)^2}\{16\zeta_2 - 28\gamma_E + 56\gamma_E^2 + (24-32\gamma_E)L_{qr}\} - C_F n_f\frac{1}{(1-\omega)^2}\left\{\frac{104}{9}\right\}\\
&\quad - C_F^2\{(24-32\gamma_E)(L_{fr})\} + C_F^2 n_f\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{(2-\omega)\left(\frac{44}{3} - \frac{320}{9}\gamma_E + \frac{32}{3}\zeta_2\right)\right\}\\
&\quad - C_F^2 C_A\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{(2-\omega)\left(\frac{302}{3} - \frac{2144}{9}\gamma_E + \frac{104}{3}\zeta_2 + 64\gamma_E\zeta_2 - 48\zeta_3\right)\right\}\\
&\quad \left. - C_F^3\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\{(2-\omega)(6-48\zeta_2+96\zeta_3)\}\right],\\
h^q_{d,2,0}(\omega,\omega_2) &= C_F\frac{1}{(1-\omega)^2}\frac{\beta_1^2}{\beta_0^3}\{2L_\omega^2 - 2\omega^2\} + C_F\frac{\omega^2}{(1-\omega)^2}\frac{\beta_2}{\beta_0^2}\{2\} + C_F\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0}\{(4-8\gamma_E+4L_{qr})L_\omega\}\\
&\quad - C_F\frac{1}{(1-\omega)^2}\beta_0\{8\gamma_E - 8\gamma_E^2 - 4L_{qr} + 8L_{qr}\gamma_E - 2L_{qr}^2 - 4\zeta_2\} - C_F\beta_0\{2L_{fr}^2\}\\
&\quad + C_F n_f\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0^2}\left\{\frac{64}{9}L_\omega + \frac{64}{9}\omega - \frac{32}{9}\omega^2\right\} - C_F n_f\left\{\frac{64}{9}(L_{fr})\right\}\\
&\quad - C_F n_f\frac{1}{(1-\omega)^2}\left\{\frac{52}{27} + \frac{128}{9}\gamma_E - \frac{64}{9}L_{qr} - \frac{8}{3}\zeta_2\right\} + C_F n_f^2\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{\frac{64}{27} - \frac{32}{27}\omega\right\}\\
&\quad - C_F C_A\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0^2}\left\{(2L_\omega+2\omega-\omega^2)\left(\frac{200}{9} - 4\zeta_2\right)\right\} + C_F C_A\left\{\left(\frac{400}{9} - 8\zeta_2\right)(L_{fr})\right\}\\
&\quad + C_F C_A\frac{1}{(1-\omega)^2}\left\{\frac{1108}{27} + \frac{890}{9}\gamma_E - \left(\frac{400}{9} - 8\zeta_2\right)L_{qr} - 28\zeta_3 - \frac{44}{3}\zeta_2 - 16\zeta_2\gamma_E\right\}\\
&\quad - C_F C_A n_f\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{(2-\omega)\left(\frac{362}{9} - \frac{104}{9}\zeta_2 + \frac{56}{3}\zeta_3\right)\right\} + C_F^2\{(12-16\gamma_E)(L_{fr})\}\\
&\quad - C_F^2\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0^2}\{(6-8\gamma_E)(2L_\omega+2\omega-\omega^2)\} + C_F^2\frac{1}{(1-\omega)^2}\{14\gamma_E - 28\gamma_E^2 - 20\zeta_2\\
&\quad - (12-16\gamma_E)L_{qr}\} + C_F C_A^2\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{(2-\omega)\left(\frac{4291}{27} - \frac{668}{9}\zeta_2 + \frac{88}{5}\zeta_2^2 + \frac{44}{3}\zeta_3\right)\right\}\\
&\quad + C_F^2 C_A\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{(2-\omega)\left(\frac{151}{3} - \frac{1072}{9}\gamma_E + \frac{52}{3}\zeta_2 + 32\gamma_E\zeta_2 - 24\zeta_3\right)\right\}\\
&\quad - C_F^2 n_f\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{(2-\omega)\left(\frac{89}{3} - \frac{160}{9}\gamma_E + \frac{16}{3}\zeta_2 - 16\zeta_3\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ C_F^3 \frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\{(2-\omega)(3-24\zeta_2+48\zeta_3)\},\\
h^q_{d,2,1}(\omega,\omega_l) &= (-1)^l\Bigg[C_F C_A \frac{1}{(1-\omega)^2}\{10\} + C_F^2\frac{1}{(1-\omega)^2}\frac{\beta_1}{\beta_0^2}\{16L_\omega+16\omega-8\omega^2\} - C_F^2\{16(L_{fr})\}\\
&- C_F^2\frac{1}{(1-\omega)^2}\{10+24\gamma_E-16L_{qr}\} + C_F^2 n_f\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{\frac{160}{9}(2-\omega)\right\}\\
&- C_F^2 C_A\frac{\omega}{(1-\omega)^2}\frac{1}{\beta_0}\left\{\frac{16}{9}(2-\omega)(67-18\zeta_2)\right\}\Bigg],\\
\tilde{h}^q_{d,2,2}(\omega,\omega_l) &= (-1)^l\left[C_F^2 n_f\frac{1}{(1-\omega)^3}\frac{1}{\beta_0}\left\{\frac{32}{27}\omega_l\right\} - C_F^2 C_A\frac{1}{(1-\omega)^3}\frac{1}{\beta_0}\left\{\frac{176}{27}\omega_l\right\}\right],
\end{aligned}
\tag{H5}
$$

where γ_E is the Euler-Mascheroni constant, $\omega = a_s\beta_0\ln(N_1N_2)$, $\omega_l = a_s\beta_0\ln(N_l)$ for $l=1,\ 2$, $L_\omega = \ln(1-\omega)$ and $L_{qr} = \ln(\frac{Q^2}{\mu_R^2})$, $L_{fr} = \ln(\frac{\mu_F^2}{\mu_R^2})$. Here $\tilde{h}^q_{d,2,2}(\omega,\omega_l)$ is written using $\varphi^{(3)}_{q,x',3} = -\varphi^{(3)}_{q,z',3}$ and $\varphi^{(3)}_{q,z',3} = -16/9\beta_0 C_F^2$, see the main text.

APPENDIX I: CONVOLUTIONS

Relevant formulas for convolutions of distributions $\delta_{\bar{\xi}} = \delta(1-\xi)$, $\mathcal{D}_{\bar{\xi},i} = [\frac{\ln^i(1-\xi)}{1-\xi}]_+$ and logarithms $\mathcal{L}_{\bar{\xi},i} = \ln^i(1-\xi)$ in (N)SV limits are listed below.

$$
\begin{aligned}
\boldsymbol{\delta}_{\bar{\xi}} \otimes \boldsymbol{\delta}_{\bar{\xi}} &= \mathbf{1},\\
\boldsymbol{\delta}_{\bar{\xi}} \otimes \boldsymbol{D}_{\bar{\xi},i} &= \boldsymbol{D}_{\bar{\xi},i},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{0}} &= \boldsymbol{D}_{\bar{\xi},\mathbf{1}}\{2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}} - \boldsymbol{\delta}_{\bar{\xi}}\{\zeta_2\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{1}} &= \boldsymbol{D}_{\bar{\xi},\mathbf{0}}\{-\zeta_2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{2}}\left\{\frac{3}{2}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}} + \boldsymbol{\delta}_{\bar{\xi}}\{\zeta_3\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{2}} &= \boldsymbol{D}_{\bar{\xi},\mathbf{0}}\{2\zeta_3\} - \boldsymbol{D}_{\bar{\xi},\mathbf{1}}\{2\zeta_2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{3}}\left\{\frac{4}{3}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}} - \boldsymbol{\delta}_{\bar{\xi}}\left\{\frac{4}{5}\zeta_2^2\right\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{3}} &= \boldsymbol{D}_{\bar{\xi},\mathbf{0}}\left\{-\frac{12\zeta_2^2}{5}\right\} + \boldsymbol{D}_{\bar{\xi},\mathbf{1}}\{6\zeta_3\} - \boldsymbol{D}_{\bar{\xi},\mathbf{2}}\{3\zeta_2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{4}}\left\{\frac{5}{4}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}} + \boldsymbol{\delta}_{\bar{\xi}}\{6\zeta_5\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{4}} &= \boldsymbol{D}_{\bar{\xi},\mathbf{0}}\{24\zeta_5\} - \boldsymbol{D}_{\bar{\xi},\mathbf{1}}\left\{\frac{48\zeta_2^2}{5}\right\} + \boldsymbol{D}_{\bar{\xi},\mathbf{2}}\{12\zeta_3\} - \boldsymbol{D}_{\bar{\xi},\mathbf{3}}\{4\zeta_2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{5}}\left\{\frac{6}{5}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}} - \boldsymbol{\delta}_{\bar{\xi}}\left\{\frac{192}{35}\zeta_2^3\right\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{5}} &= \boldsymbol{D}_{\bar{\xi},\mathbf{0}}\left\{-\frac{192\zeta_2^3}{7}\right\} + \boldsymbol{D}_{\bar{\xi},\mathbf{1}}\{120\zeta_5\} - \boldsymbol{D}_{\bar{\xi},\mathbf{2}}\{24\zeta_2^2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{3}}\{20\zeta_3\} - \boldsymbol{D}_{\bar{\xi},\mathbf{4}}\{5\zeta_2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{6}}\left\{\frac{7}{6}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}} + \boldsymbol{\delta}_{\bar{\xi}}\{120\zeta_7\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{1}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{1}} &= \boldsymbol{D}_{\bar{\xi},\mathbf{0}}\{2\zeta_3\} - \boldsymbol{D}_{\bar{\xi},\mathbf{1}}\{2\zeta_2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{3}} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}} - \boldsymbol{\delta}_{\bar{\xi}}\left\{\frac{1}{10}\zeta_2^2\right\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{1}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{2}} &= \boldsymbol{D}_{\bar{\xi},\mathbf{0}}\{-\zeta_2^2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{1}}\{6\zeta_3\} - \boldsymbol{D}_{\bar{\xi},\mathbf{2}}\{3\zeta_2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{4}}\left\{\frac{5}{6}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{2\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{2\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}} + \boldsymbol{\delta}_{\bar{\xi}}\{4\zeta_5 - 2\zeta_2\zeta_3\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{1}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{3}} &= \boldsymbol{D}_{\bar{\xi},\mathbf{0}}\{18\zeta_5 - 6\zeta_2\zeta_3\} - \boldsymbol{D}_{\bar{\xi},\mathbf{1}}\left\{\frac{27}{5}\zeta_2^2\right\} + \boldsymbol{D}_{\bar{\xi},\mathbf{2}}\{12\zeta_3\} - \boldsymbol{D}_{\bar{\xi},\mathbf{3}}\{4\zeta_2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{5}}\left\{\frac{3}{4}\right\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{\frac{12}{5}\zeta_2^2\right\}\\
&\quad + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{6\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{3\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}} + \boldsymbol{\delta}_{\bar{\xi}}\left\{3\zeta_3^2 - \frac{36\zeta_2^3}{35}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{D}_{\bar{\xi},1}\otimes\boldsymbol{D}_{\bar{\xi},4} &= \boldsymbol{D}_{\bar{\xi},0}\left\{12\zeta_3^2-\frac{48\zeta_2^3}{5}\right\}+\boldsymbol{D}_{\bar{\xi},1}\{96\zeta_5-24\zeta_2\zeta_3\}+\boldsymbol{D}_{\bar{\xi},3}\{20\zeta_3\}+\boldsymbol{D}_{\bar{\xi},6}\left\{\frac{7}{10}\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},0}\{24\zeta_5\}\\
&\quad-\boldsymbol{D}_{\bar{\xi},2}\left\{\frac{78}{5}\zeta_2^2\right\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},1}\left\{\frac{48}{5}\zeta_2^2\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},2}\{12\zeta_3\}-\boldsymbol{D}_{\bar{\xi},4}\{5\zeta_2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},3}\{4\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},5}\\
&\quad-\boldsymbol{\delta}_{\bar{\xi}}\left\{\frac{48}{5}\zeta_2^2\zeta_3+24\zeta_2\zeta_5-72\zeta_7\right\},\\
\boldsymbol{D}_{\bar{\xi},1}\otimes\boldsymbol{D}_{\bar{\xi},5} &= \boldsymbol{D}_{\bar{\xi},0}\{480\zeta_7-120\zeta_5\zeta_2-48\zeta_2^2\zeta_3\}+\boldsymbol{D}_{\bar{\xi},1}\left\{60\zeta_3^2-\frac{528}{7}\zeta_2^3\right\}+\boldsymbol{D}_{\bar{\xi},2}\{300\zeta_5-60\zeta_3\zeta_2\}+\boldsymbol{D}_{\bar{\xi},4}\{30\zeta_3\}\\
&\quad+\boldsymbol{D}_{\bar{\xi},7}\left\{\frac{2}{3}\right\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},0}\left\{\frac{192}{7}\zeta_2^3\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},1}\{120\zeta_5\}-\boldsymbol{D}_{\bar{\xi},3}\{34\zeta_2^2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},2}\{24\zeta_2^2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},3}\{20\zeta_3\}\\
&\quad-\boldsymbol{D}_{\bar{\xi},5}\{6\zeta_2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},4}\{5\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},6}-\boldsymbol{\delta}_{\bar{\xi}}\left\{\frac{144}{7}\zeta_2^4-120\zeta_3\zeta_5\right\},\\
\boldsymbol{D}_{\bar{\xi},2}\otimes\boldsymbol{D}_{\bar{\xi},2} &= \boldsymbol{D}_{\bar{\xi},0}\{16\zeta_5-8\zeta_2\zeta_3\}+\boldsymbol{D}_{\bar{\xi},2}\{12\zeta_3\}+\boldsymbol{D}_{\bar{\xi},5}\left\{\frac{2}{3}\right\}-\boldsymbol{D}_{\bar{\xi},1}\{4\zeta_2^2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},0}\left\{\frac{2}{5}\zeta_2^2\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},1}\{8\zeta_3\}\\
&\quad-\boldsymbol{D}_{\bar{\xi},3}\{4\zeta_2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},2}\{4\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},4}+\boldsymbol{\delta}_{\bar{\xi}}\left\{4\zeta_3^2-\frac{46\zeta_2^3}{35}\right\},\\
\boldsymbol{D}_{\bar{\xi},2}\otimes\boldsymbol{D}_{\bar{\xi},3} &= \boldsymbol{D}_{\bar{\xi},0}\{18\zeta_3^2-6\zeta_2^3\}+\boldsymbol{D}_{\bar{\xi},1}\{84\zeta_5-36\zeta_2\zeta_3\}-\boldsymbol{D}_{\bar{\xi},2}\left\{\frac{57}{5}\zeta_2^2\right\}+\boldsymbol{D}_{\bar{\xi},3}\{20\zeta_3\}-\boldsymbol{D}_{\bar{\xi},4}\{5\zeta_2\}\\
&\quad+\boldsymbol{D}_{\bar{\xi},6}\left\{\frac{7}{12}\right\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},0}\{12\zeta_2\zeta_3-24\zeta_5\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},1}\{6\zeta_2^2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},2}\{18\zeta_3\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},3}\{6\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},5}\\
&\quad-\boldsymbol{\delta}_{\bar{\xi}}\{6\zeta_2^2\zeta_3+24\zeta_2\zeta_5-60\zeta_7\},\\
\boldsymbol{D}_{\bar{\xi},2}\otimes\boldsymbol{D}_{\bar{\xi},4} &= \boldsymbol{D}_{\bar{\xi},0}\left\{384\zeta_7-144\zeta_2\zeta_5-\frac{216}{5}\zeta_2^2\zeta_3\right\}+\boldsymbol{D}_{\bar{\xi},1}\left\{96\zeta_3^2-\frac{216}{5}\zeta_2^3\right\}+\boldsymbol{D}_{\bar{\xi},2}\{264\zeta_5-96\zeta_2\zeta_3\}\\
&\quad+\boldsymbol{D}_{\bar{\xi},4}\{30\zeta_3\}+\boldsymbol{D}_{\bar{\xi},7}\left\{\frac{8}{15}\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},0}\left\{24\zeta_3^2-\frac{288}{35}\zeta_2^3\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},1}\{144\zeta_5-48\zeta_2\zeta_3\}-\boldsymbol{D}_{\bar{\xi},3}\left\{\frac{128}{5}\zeta_2^2\right\}\\
&\quad-\boldsymbol{\mathcal{L}}_{\bar{\xi},2}\left\{\frac{108}{5}\zeta_2^2\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},3}\{32\zeta_3\}-\boldsymbol{D}_{\bar{\xi},5}\{6\zeta_2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},4}\{8\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},6}+\boldsymbol{\delta}_{\bar{\xi}}\left\{144\zeta_3\zeta_5-24\zeta_2\zeta_3^2-\frac{2928\zeta_2^4}{175}\right\},\\
\boldsymbol{D}_{\bar{\xi},2}\otimes\boldsymbol{D}_{\bar{\xi},5} &= \boldsymbol{D}_{\bar{\xi},0}\left\{960\zeta_3\zeta_5-\frac{624}{5}\zeta_2^4-120\zeta_2\zeta_3^2\right\}-\boldsymbol{D}_{\bar{\xi},1}\{312\zeta_2^2\zeta_3+960\zeta_2\zeta_5-2880\zeta_7\}+\boldsymbol{D}_{\bar{\xi},2}\left\{300\zeta_3^2-\frac{1284}{7}\zeta_2^3\right\}\\
&\quad-\boldsymbol{D}_{\bar{\xi},3}\{200\zeta_2\zeta_3-640\zeta_5\}-\boldsymbol{D}_{\bar{\xi},4}\{49\zeta_2^2\}+\boldsymbol{D}_{\bar{\xi},5}\{42\zeta_3\}-\boldsymbol{D}_{\bar{\xi},6}\{7\zeta_2\}+\boldsymbol{D}_{\bar{\xi},8}\left\{\frac{1}{2}\right\}\\
&\quad-\boldsymbol{\mathcal{L}}_{\bar{\xi},0}\{96\zeta_2^2\zeta_3+240\zeta_2\zeta_5-720\zeta_7\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},1}\{96\zeta_2^3-120\zeta_3^2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},2}\{120\zeta_2\zeta_3-480\zeta_5\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},3}\{52\zeta_2^2\}\\
&\quad+\boldsymbol{\mathcal{L}}_{\bar{\xi},4}\{50\zeta_3\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},5}\{10\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},7}-\boldsymbol{\delta}_{\bar{\xi}}\{96\zeta_2^3\zeta_3+216\zeta_2^2\zeta_5+720\zeta_2\zeta_7-40\zeta_3^3-2240\zeta_9\},\\
\boldsymbol{D}_{\bar{\xi},3}\otimes\boldsymbol{D}_{\bar{\xi},3} &= \boldsymbol{D}_{\bar{\xi},0}\{360\zeta_7-36\zeta_2^2\zeta_3-144\zeta_2\zeta_5\}-\boldsymbol{D}_{\bar{\xi},1}\{36\zeta_2^3-108\zeta_3^2\}-\boldsymbol{D}_{\bar{\xi},2}\{108\zeta_2\zeta_3-252\zeta_5\}-\boldsymbol{D}_{\bar{\xi},3}\left\{\frac{114\zeta_2^2}{5}\right\}\\
&\quad+\boldsymbol{D}_{\bar{\xi},4}\{30\zeta_3\}-\boldsymbol{D}_{\bar{\xi},5}\{6\zeta_2\}+\boldsymbol{D}_{\bar{\xi},7}\left\{\frac{1}{2}\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},0}\left\{36\zeta_3^2-\frac{414\zeta_2^3}{35}\right\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},1}\{72\zeta_2\zeta_3-144\zeta_5\}\\
&\quad-\boldsymbol{\mathcal{L}}_{\bar{\xi},2}\{18\zeta_2^2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},2}\{36\zeta_3\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},4}\{9\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},6}+\boldsymbol{\delta}_{\bar{\xi}}\left\{144\zeta_3\zeta_5-36\zeta_2\zeta_3^2-\frac{4491\zeta_2^4}{350}\right\},\\
\boldsymbol{D}_{\bar{\xi},3}\otimes\boldsymbol{D}_{\bar{\xi},4} &= \boldsymbol{D}_{\bar{\xi},0}\left\{1008\zeta_3\zeta_5-216\zeta_2\zeta_3^2-\frac{2538\zeta_2^4}{25}\right\}-\boldsymbol{D}_{\bar{\xi},1}\left\{\frac{1368}{5}\zeta_2^2\zeta_3+1008\zeta_2\zeta_5-2592\zeta_7\right\}\\
&\quad+\boldsymbol{D}_{\bar{\xi},2}\left\{360\zeta_3^2-\frac{684\zeta_2^3}{5}\right\}-\boldsymbol{D}_{\bar{\xi},3}\{240\zeta_2\zeta_3-600\zeta_5\}-\boldsymbol{D}_{\bar{\xi},4}\{42\zeta_2^2\}+\boldsymbol{D}_{\bar{\xi},5}\{42\zeta_3\}-\boldsymbol{D}_{\bar{\xi},6}\{7\zeta_2\}
\end{aligned}
$$

$$
\begin{aligned}
&+\boldsymbol{D}_{\bar{\xi},\mathbf{8}}\left\{\frac{9}{20}\right\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{72\zeta_2^2\zeta_3+288\zeta_2\zeta_5-720\zeta_7\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{72\zeta_2^3-216\zeta_3^2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{216\zeta_2\zeta_3-504\zeta_5\}\\
&-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\left\{\frac{228\zeta_2^2}{5}\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{60\zeta_3\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{12\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{7}}\\
&-\boldsymbol{\delta}_{\bar{\xi}}\left\{72\zeta_2^3\zeta_3+\frac{1008}{5}\zeta_2^2\zeta_5+720\zeta_2\zeta_7-72\zeta_3^3+2016\zeta_9\right\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{3}}\otimes\boldsymbol{D}_{\bar{\xi},\mathbf{5}}=&\boldsymbol{D}_{\bar{\xi},\mathbf{0}}\{480\zeta_3^3+16800\zeta_9-648\zeta_2^3\zeta_3-1656\zeta_2^2\zeta_5-5760\zeta_2\zeta_7\}-\boldsymbol{D}_{\bar{\xi},\mathbf{1}}\{882\zeta_2^4+1440\zeta_2\zeta_3^2-7920\zeta_3\zeta_5\}\\
&-\boldsymbol{D}_{\bar{\xi},\mathbf{2}}\{1152\zeta_2^2\zeta_3+3960\zeta_2\zeta_5-10800\zeta_7\}+\boldsymbol{D}_{\bar{\xi},\mathbf{3}}\left\{900\zeta_3^2-\frac{2880\zeta_2^3}{7}\right\}-\boldsymbol{D}_{\bar{\xi},\mathbf{4}}\{450\zeta_2\zeta_3-1230\zeta_5\}\\
&-\boldsymbol{D}_{\bar{\xi},\mathbf{5}}\left\{\frac{357\zeta_2^2}{5}\right\}+\boldsymbol{D}_{\bar{\xi},\mathbf{6}}\{56\zeta_3\}-\boldsymbol{D}_{\bar{\xi},\mathbf{7}}\{8\zeta_2\}+\boldsymbol{D}_{\bar{\xi},\mathbf{9}}\left\{\frac{5}{12}\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{2160\zeta_3\zeta_5-360\zeta_2\zeta_3^2-\frac{8784\zeta_2^4}{35}\right\}\\
&-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{648\zeta_2^2\zeta_3+2160\zeta_2\zeta_5-5760\zeta_7\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{324\zeta_2^3-720\zeta_3^2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{480\zeta_2\zeta_3-1320\zeta_5\}\\
&-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{96\zeta_2^2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{90\zeta_3\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\{15\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{8}}\\
&-\boldsymbol{\delta}_{\bar{\xi}}\left\{\frac{20016}{55}\zeta_2^5+324\zeta_2^2\zeta_3^2+2160\zeta_2\zeta_3\zeta_5-2880\zeta_5^2-5760\zeta_3\zeta_7\right\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{4}}\otimes\boldsymbol{D}_{\bar{\xi},\mathbf{4}}=&\boldsymbol{D}_{\bar{\xi},\mathbf{0}}\left\{576\zeta_3^3+16128\zeta_9-576\zeta_2^3\zeta_3-\frac{8064}{5}\zeta_2^2\zeta_5-5760\zeta_2\zeta_7\right\}-\boldsymbol{D}_{\bar{\xi},\mathbf{1}}\left\{\frac{20304}{25}\zeta_2^4+1728\zeta_2\zeta_3^2-8064\zeta_3\zeta_5\right\}\\
&-\boldsymbol{D}_{\bar{\xi},\mathbf{2}}\left\{\frac{5472}{5}\zeta_2^2\zeta_3+4032\zeta_2\zeta_5-10368\zeta_7\right\}-\boldsymbol{D}_{\bar{\xi},\mathbf{3}}\left\{\frac{1824}{5}\zeta_2^3-960\zeta_3^2\right\}-\boldsymbol{D}_{\bar{\xi},\mathbf{4}}\{480\zeta_2\zeta_3-1200\zeta_5\}\\
&-\boldsymbol{D}_{\bar{\xi},\mathbf{5}}\left\{\frac{336}{5}\zeta_2^2\right\}+\boldsymbol{D}_{\bar{\xi},\mathbf{6}}\{56\zeta_3\}-\boldsymbol{D}_{\bar{\xi},\mathbf{7}}\{8\zeta_2\}+\boldsymbol{D}_{\bar{\xi},\mathbf{9}}\left\{\frac{2}{5}\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{2304\zeta_3\zeta_5-576\zeta_2\zeta_3^2-\frac{35928\zeta_2^4}{175}\right\}\\
&-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{576\zeta_2^2\zeta_3+2304\zeta_2\zeta_5-5760\zeta_7\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{288\zeta_2^3-864\zeta_3^2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{576\zeta_2\zeta_3-1344\zeta_5\}\\
&-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\left\{\frac{456}{5}\zeta_2^2\right\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{96\zeta_3\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\{16\zeta_2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{8}}\\
&+\boldsymbol{\delta}_{\bar{\xi}}\left\{2880\zeta_5^2+5760\zeta_3\zeta_7-288\zeta_2^2\zeta_3^2-2304\zeta_2\zeta_3\zeta_5-\frac{96552\zeta_2^5}{275}\right\},\\
\boldsymbol{D}_{\bar{\xi},\mathbf{4}}\otimes\boldsymbol{D}_{\bar{\xi},\mathbf{5}}=&\boldsymbol{D}_{\bar{\xi},\mathbf{0}}\left\{25920\zeta_5^2+51840\zeta_3\zeta_7-2736\zeta_2^2\zeta_3^2-20160\zeta_2\zeta_3\zeta_5-\frac{16056}{5}\zeta_2^5\right\}-\boldsymbol{D}_{\bar{\xi},\mathbf{1}}\{5472\zeta_2^3\zeta_3-4800\zeta_3^3\\
&+14688\zeta_2^2\zeta_5+51840\zeta_2\zeta_7-147840\zeta_9\}+\boldsymbol{D}_{\bar{\xi},\mathbf{10}}\left\{\frac{11}{30}\right\}-\boldsymbol{D}_{\bar{\xi},\mathbf{2}}\left\{\frac{18972}{5}\zeta_2^4+7200\zeta_2\zeta_3^2-36000\zeta_3\zeta_5\right\}\\
&-\boldsymbol{D}_{\bar{\xi},\mathbf{3}}\{3360\zeta_2^2\zeta_3+12000\zeta_2\zeta_5-31680\zeta_7\}+\boldsymbol{D}_{\bar{\xi},\mathbf{4}}\left\{2100\zeta_3^2-\frac{6072\zeta_2^3}{7}\right\}-\boldsymbol{D}_{\bar{\xi},\mathbf{5}}\{840\zeta_2\zeta_3-2184\zeta_5\}\\
&-\boldsymbol{D}_{\bar{\xi},\mathbf{6}}\left\{\frac{518}{5}\zeta_2^2\right\}+\boldsymbol{D}_{\bar{\xi},\mathbf{7}}\{72\zeta_3\}-\boldsymbol{D}_{\bar{\xi},\mathbf{8}}\{9\zeta_2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{1440\zeta_2^3\zeta_3+4032\zeta_2^2\zeta_5+14400\zeta_2\zeta_7-1440\zeta_3^3\\
&-40320\zeta_9\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\left\{\frac{10152}{5}\zeta_2^4+4320\zeta_2\zeta_3^2-20160\zeta_3\zeta_5\right\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{2736\zeta_2^2\zeta_3+10080\zeta_2\zeta_5-25920\zeta_7\}\\
&-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{912\zeta_2^3-2400\zeta_3^2\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{1200\zeta_2\zeta_3-3000\zeta_5\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{168\zeta_2^2\}+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\{140\zeta_3\}-\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{7}}\{20\zeta_2\}\\
&+\boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{9}}+\boldsymbol{\delta}_{\bar{\xi}}\left\{120960\zeta_{11}-\frac{10152}{5}\zeta_2^4\zeta_3-4032\zeta_2^3\zeta_5+10080\zeta_3^2\zeta_5-10656\zeta_2^2\zeta_7-1440\zeta_2\zeta_3^3-40320\zeta_2\zeta_9\right\},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{D}_{\bar{\xi},\mathbf{5}} \otimes \boldsymbol{D}_{\bar{\xi},\mathbf{5}} = {} & \boldsymbol{D}_{\bar{\xi},\mathbf{0}}\{1209600\zeta_{11} - 20304\zeta_2^4\zeta_3 - 40320\zeta_2^3\zeta_5 + 100800\zeta_3^2\zeta_5 - 106560\zeta_2^2\zeta_7 - 14400\zeta_2\zeta_3^3 - 403200\zeta_2\zeta_9\} \\
& - \boldsymbol{D}_{\bar{\xi},\mathbf{1}}\{32112\zeta_2^5 + 27360\zeta_2^2\zeta_3^2 + 201600\zeta_2\zeta_3\zeta_5 - 259200\zeta_5^2 - 518400\zeta_3\zeta_7\} + \boldsymbol{D}_{\bar{\xi},\mathbf{11}}\left\{\frac{1}{3}\right\} \\
& - \boldsymbol{D}_{\bar{\xi},\mathbf{2}}\{27360\zeta_2^3\zeta_3 - 24000\zeta_3^3 + 73440\zeta_2^2\zeta_5 + 259200\zeta_2\zeta_7 - 739200\zeta_9\} - \boldsymbol{D}_{\bar{\xi},\mathbf{3}}\{12648\zeta_2^4 + 24000\zeta_2\zeta_3^2 \\
& - 120000\zeta_3\zeta_5\} - \boldsymbol{D}_{\bar{\xi},\mathbf{4}}\{8400\zeta_2^2\zeta_3 + 30000\zeta_2\zeta_5 - 79200\zeta_7\} + \boldsymbol{D}_{\bar{\xi},\mathbf{5}}\left\{4200\zeta_3^2 - \frac{12144\zeta_2^3}{7}\right\} \\
& - \boldsymbol{D}_{\bar{\xi},\mathbf{6}}\{1400\zeta_2\zeta_3 - 3640\zeta_5\} - \boldsymbol{D}_{\bar{\xi},\mathbf{7}}\{148\zeta_2^2\} + \boldsymbol{D}_{\bar{\xi},\mathbf{8}}\{90\zeta_3\} - \boldsymbol{D}_{\bar{\xi},\mathbf{9}}\{10\zeta_2\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{\frac{96552}{11}\zeta_2^5 + 7200\zeta_2^2\zeta_3^2\right. \\
& \left. + 57600\zeta_2\zeta_3\zeta_5 - 72000\zeta_5^2 - 144000\zeta_3\zeta_7\right\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{14400\zeta_2^3\zeta_3 + 40320\zeta_2^2\zeta_5 + 144000\zeta_2\zeta_7 - 14400\zeta_3^3 \\
& - 403200\zeta_9\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{10}} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{10152\zeta_2^4 + 21600\zeta_2\zeta_3^2 - 100800\zeta_3\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{9120\zeta_2^2\zeta_3 + 33600\zeta_2\zeta_5 \\
& - 86400\zeta_7\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{2280\zeta_2^3 - 6000\zeta_3^2\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{2400\zeta_2\zeta_3 - 6000\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\{280\zeta_2^2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{7}}\{200\zeta_3\} \\
& - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{8}}\{25\zeta_2\} + \boldsymbol{\delta}_{\bar{\xi}}\left\{3600\zeta_3^4 + 374400\zeta_5\zeta_7 + 403200\zeta_3\zeta_9 - 7200\zeta_2^3\zeta_3^2 - 40320\zeta_2^2\zeta_3\zeta_5 - 72000\zeta_2\zeta_5^2\right. \\
& \left. - 144000\zeta_2\zeta_3\zeta_7 - \frac{509128524}{35035}\zeta_2^6\right\},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\delta}_{\bar{\xi}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\boldsymbol{i}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\boldsymbol{i}}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{-\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{2\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{2\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{-\frac{12}{5}\zeta_2^2\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{6\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{3\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{24\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\left\{\frac{48\zeta_2^2}{5}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{12\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{4\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{0}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{-\frac{192}{7}\zeta_2^3\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{120\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{24\zeta_2^2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{20\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{5\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{1}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\left\{\frac{1}{2}\right\}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{1}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\left\{\frac{1}{2}\right\}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{1}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{-\frac{1}{5}\zeta_2^2\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{4\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{2\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\left\{\frac{1}{2}\right\}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{1}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{12\zeta_5 - 6\zeta_2\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{3\zeta_2^2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{9\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{3\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\left\{\frac{1}{2}\right\}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{1}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{12\zeta_3^2 - \frac{144\zeta_2^3}{35}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{72\zeta_5 - 24\zeta_2\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\left\{\frac{54\zeta_2^2}{5}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{16\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{4\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\left\{\frac{1}{2}\right\}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{1}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{360\zeta_7 - 48\zeta_2^2\zeta_3 - 120\zeta_2\zeta_5\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{60\zeta_3^2 - 48\zeta_2^3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{60\zeta_2\zeta_3 - 240\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{26\zeta_2^2\} \\
&\quad + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{25\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{5\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{7}}\left\{\frac{1}{2}\right\}, \\
\boldsymbol{D}_{\bar{\xi},\mathbf{2}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}} &= \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\left\{\frac{1}{3}\right\},
\end{aligned}
$$

$$\boldsymbol{D}_{\bar{\xi},\mathbf{2}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}} = \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{-\frac{4}{5}\zeta_2^2\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{2\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\left\{\frac{1}{3}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},\mathbf{2}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}} = \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{8\zeta_5 - 4\zeta_2\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{2\zeta_2^2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{6\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{2\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\left\{\frac{1}{3}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},\mathbf{2}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}} = \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{12\zeta_3^2 - \frac{138\zeta_2^3}{35}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{48\zeta_5 - 24\zeta_2\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{6\zeta_2^2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{12\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{3\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\left\{\frac{1}{3}\right\},$$

$$\begin{aligned}\boldsymbol{D}_{\bar{\xi},\mathbf{2}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}} = {} & \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{240\zeta_7 - 24\zeta_2^2\zeta_3 - 96\zeta_2\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{24\zeta_2^3 - 72\zeta_3^2\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{72\zeta_2\zeta_3 - 168\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\left\{\frac{76\zeta_2^2}{5}\right\} \\ & + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{20\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{4\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{7}}\left\{\frac{1}{3}\right\},\end{aligned}$$

$$\begin{aligned}\boldsymbol{D}_{\bar{\xi},\mathbf{2}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}} = {} & \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{720\zeta_3\zeta_5 - 120\zeta_2\zeta_3^2 - \frac{2928\zeta_2^4}{35}\right\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{216\zeta_2^2\zeta_3 + 720\zeta_2\zeta_5 - 1920\zeta_7\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{108\zeta_2^3 - 240\zeta_3^2\} \\ & - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{160\zeta_2\zeta_3 - 440\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{32\zeta_2^2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{30\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\{5\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{8}}\left\{\frac{1}{3}\right\},\end{aligned}$$

$$\boldsymbol{D}_{\bar{\xi},\mathbf{3}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}} = \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\left\{\frac{1}{4}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},\mathbf{3}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}} = \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{6\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\left\{\frac{12\zeta_2^2}{5}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{3\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\left\{\frac{1}{4}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},\mathbf{3}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}} = \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{6\zeta_3^2 - \frac{72\zeta_2^3}{35}\right\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{12\zeta_2\zeta_3 - 36\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\left\{\frac{27\zeta_2^2}{5}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{8\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{2\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\left\{\frac{1}{4}\right\},$$

$$\begin{aligned}\boldsymbol{D}_{\bar{\xi},\mathbf{3}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}} = {} & \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{180\zeta_7 - 18\zeta_2^2\zeta_3 - 72\zeta_2\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{18\zeta_2^3 - 54\zeta_3^2\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{54\zeta_2\zeta_3 - 126\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\left\{\frac{57\zeta_2^2}{5}\right\} \\ & + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{15\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{3\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{7}}\left\{\frac{1}{4}\right\},\end{aligned}$$

$$\begin{aligned}\boldsymbol{D}_{\bar{\xi},\mathbf{3}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}} = {} & \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{576\zeta_3\zeta_5 - 144\zeta_2\zeta_3^2 - \frac{8982\zeta_2^4}{175}\right\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{144\zeta_2^2\zeta_3 + 576\zeta_2\zeta_5 - 1440\zeta_7\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{72\zeta_2^3 - 216\zeta_3^2\} \\ & - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{144\zeta_2\zeta_3 - 336\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\left\{\frac{114\zeta_2^2}{5}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{24\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\{4\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{8}}\left\{\frac{1}{4}\right\},\end{aligned}$$

$$\begin{aligned}\boldsymbol{D}_{\bar{\xi},\mathbf{3}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}} = {} & \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\{360\zeta_3^3 + 10080\zeta_9 - 360\zeta_2^3\zeta_3 - 1008\zeta_2^2\zeta_5 - 3600\zeta_2\zeta_7\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\left\{\frac{2538}{5}\zeta_2^4 + 1080\zeta_2\zeta_3^2 - 5040\zeta_3\zeta_5\right\} \\ & - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{684\zeta_2^2\zeta_3 + 2520\zeta_2\zeta_5 - 6480\zeta_7\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{600\zeta_3^2 - 228\zeta_2^3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{300\zeta_2\zeta_3 - 750\zeta_5\} \\ & - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{42\zeta_2^2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\{35\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{7}}\{5\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{9}}\left\{\frac{1}{4}\right\},\end{aligned}$$

$$\boldsymbol{D}_{\bar{\xi},\mathbf{4}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}} = \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\left\{\frac{1}{5}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},\mathbf{4}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}} = \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{-\frac{192}{35}\zeta_2^3\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\{24\zeta_5\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\left\{\frac{24\zeta_2^2}{5}\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\{4\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{6}}\left\{\frac{1}{5}\right\},$$

$$\begin{aligned}\boldsymbol{D}_{\bar{\xi},\mathbf{4}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}} = {} & \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{-\frac{96}{5}\zeta_2^2\zeta_3 - 48\zeta_2\zeta_5 + 144\zeta_7\right\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\left\{\frac{96}{5}\zeta_2^3 - 24\zeta_3^2\right\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\{24\zeta_2\zeta_3 - 96\zeta_5\} \\ & - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}}\left\{\frac{52}{5}\zeta_2^2\right\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{4}}\{10\zeta_3\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{5}}\{2\zeta_2\} + \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{7}}\left\{\frac{1}{5}\right\},\end{aligned}$$

$$\boldsymbol{D}_{\bar{\xi},\mathbf{4}} \otimes \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{3}} = \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{0}}\left\{432\zeta_3\zeta_5 - 72\zeta_2\zeta_3^2 - \frac{8784\zeta_2^4}{175}\right\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{1}}\left\{\frac{648}{5}\zeta_2^2\zeta_3 + 432\zeta_2\zeta_5 - 1152\zeta_7\right\} - \boldsymbol{\mathcal{L}}_{\bar{\xi},\mathbf{2}}\left\{\frac{324}{5}\zeta_2^3 - 144\zeta_3^2\right\}$$

$$+\mathcal{L}_{\bar{\xi},3}\{264\zeta_5-96\zeta_2\zeta_3\}-\mathcal{L}_{\bar{\xi},4}\left\{\frac{96}{5}\zeta_2^2\right\}+\mathcal{L}_{\bar{\xi},5}\{18\zeta_3\}-\mathcal{L}_{\bar{\xi},6}\{3\zeta_2\}+\mathcal{L}_{\bar{\xi},8}\left\{\frac{1}{5}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},4}\otimes\mathcal{L}_{\bar{\xi},4}=\mathcal{L}_{\bar{\xi},0}\left\{288\zeta_3^3+8064\zeta_9-288\zeta_2^3\zeta_3-\frac{4032}{5}\zeta_2^2\zeta_5-2880\zeta_2\zeta_7\right\}-\mathcal{L}_{\bar{\xi},1}\left\{\frac{10152}{25}\zeta_2^4+864\zeta_2\zeta_3^2-4032\zeta_3\zeta_5\right\}$$
$$-\mathcal{L}_{\bar{\xi},2}\left\{\frac{2736}{5}\zeta_2^2\zeta_3+2016\zeta_2\zeta_5-5184\zeta_7\right\}+\mathcal{L}_{\bar{\xi},3}\left\{480\zeta_3^2-\frac{912\zeta_2^3}{5}\right\}-\mathcal{L}_{\bar{\xi},4}\{240\zeta_2\zeta_3-600\zeta_5\}$$
$$-\mathcal{L}_{\bar{\xi},5}\left\{\frac{168\zeta_2^2}{5}\right\}+\mathcal{L}_{\bar{\xi},6}\{28\zeta_3\}-\mathcal{L}_{\bar{\xi},7}\{4\zeta_2\}+\mathcal{L}_{\bar{\xi},9}\left\{\frac{1}{5}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},4}\otimes\mathcal{L}_{\bar{\xi},5}=\mathcal{L}_{\bar{\xi},0}\left\{14400\zeta_5^2-\frac{96552}{55}\zeta_2^5-1440\zeta_2^2\zeta_3^2-11520\zeta_2\zeta_3\zeta_5+22800\zeta_3\zeta_7\right\}-\mathcal{L}_{\bar{\xi},1}\{2880\zeta_2^3\zeta_3+8064\zeta_2^2\zeta_5$$
$$+28800\zeta_2\zeta_7-2880\zeta_3^3-80640\zeta_9\}-\mathcal{L}_{\bar{\xi},2}\left\{\frac{10152}{5}\zeta_2^4+4320\zeta_2\zeta_3^2-20160\zeta_3\zeta_5\right\}-\mathcal{L}_{\bar{\xi},3}\{1824\zeta_2^2\zeta_3$$
$$+6720\zeta_2\zeta_5-17280\zeta_7\}-\mathcal{L}_{\bar{\xi},4}\{456\zeta_2^3-1200\zeta_3^2\}-\mathcal{L}_{\bar{\xi},5}\{480\zeta_2\zeta_3-1200\zeta_5\}-\mathcal{L}_{\bar{\xi},6}\{56\zeta_2^2\}$$
$$+\mathcal{L}_{\bar{\xi},7}\{40\zeta_3\}-\mathcal{L}_{\bar{\xi},8}\{5\zeta_2\}+\mathcal{L}_{\bar{\xi},10}\left\{\frac{1}{5}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},5}\otimes\mathcal{L}_{\bar{\xi},0}=\mathcal{L}_{\bar{\xi},6}\left\{\frac{1}{6}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},5}\otimes\mathcal{L}_{\bar{\xi},1}=\mathcal{L}_{\bar{\xi},0}\{120\zeta_7\}-\mathcal{L}_{\bar{\xi},1}\left\{\frac{192\zeta_2^3}{7}\right\}+\mathcal{L}_{\bar{\xi},2}\{60\zeta_5\}-\mathcal{L}_{\bar{\xi},3}\{8\zeta_2^2\}+\mathcal{L}_{\bar{\xi},4}\{5\zeta_3\}-\mathcal{L}_{\bar{\xi},5}\{\zeta_2\}+\mathcal{L}_{\bar{\xi},7}\left\{\frac{1}{6}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},5}\otimes\mathcal{L}_{\bar{\xi},2}=\mathcal{L}_{\bar{\xi},0}\left\{240\zeta_3\zeta_5-\frac{288}{7}\zeta_2^4\right\}-\mathcal{L}_{\bar{\xi},1}\{96\zeta_2^2\zeta_3+240\zeta_2\zeta_5-960\zeta_7\}+\mathcal{L}_{\bar{\xi},2}\left\{60\zeta_3^2-\frac{528\zeta_2^3}{7}\right\}$$
$$-\mathcal{L}_{\bar{\xi},3}\{40\zeta_2\zeta_3-200\zeta_5\}-\mathcal{L}_{\bar{\xi},4}\{17\zeta_2^2\}+\mathcal{L}_{\bar{\xi},5}\{12\zeta_3\}-\mathcal{L}_{\bar{\xi},6}\{2\zeta_2\}+\mathcal{L}_{\bar{\xi},8}\left\{\frac{1}{6}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},5}\otimes\mathcal{L}_{\bar{\xi},3}=\mathcal{L}_{\bar{\xi},0}\{120\zeta_3^3+6720\zeta_9-288\zeta_2^3\zeta_3-648\zeta_2^2\zeta_5-2160\zeta_2\zeta_7\}-\mathcal{L}_{\bar{\xi},1}\left\{\frac{1872}{5}\zeta_2^4+360\zeta_2\zeta_3^2-2880\zeta_3\zeta_5\right\}$$
$$-\mathcal{L}_{\bar{\xi},2}\{468\zeta_2^2\zeta_3+1440\zeta_2\zeta_5-4320\zeta_7\}+\mathcal{L}_{\bar{\xi},3}\left\{300\zeta_3^2-\frac{1284\zeta_2^3}{7}\right\}-\mathcal{L}_{\bar{\xi},4}\{150\zeta_2\zeta_3-480\zeta_5\}$$
$$-\mathcal{L}_{\bar{\xi},5}\left\{\frac{147\zeta_2^2}{5}\right\}+\mathcal{L}_{\bar{\xi},6}\{21\zeta_3\}-\mathcal{L}_{\bar{\xi},7}\{3\zeta_2\}+\mathcal{L}_{\bar{\xi},9}\left\{\frac{1}{6}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},5}\otimes\mathcal{L}_{\bar{\xi},4}=\mathcal{L}_{\bar{\xi},0}\left\{11520\zeta_5^2+23040\zeta_3\zeta_7-\frac{80064}{55}\zeta_2^5-1296\zeta_2^2\zeta_3^2-8640\zeta_2\zeta_3\zeta_5\right\}-\mathcal{L}_{\bar{\xi},1}\{2592\zeta_2^3\zeta_3+6624\zeta_2^2\zeta_5$$
$$+23040\zeta_2\zeta_7-1920\zeta_3^3-67200\zeta_9\}-\mathcal{L}_{\bar{\xi},2}\{1764\zeta_2^4+2880\zeta_2\zeta_3^2-15840\zeta_3\zeta_5\}-\mathcal{L}_{\bar{\xi},3}\{1536\zeta_2^2\zeta_3$$
$$+5280\zeta_2\zeta_5-14400\zeta_7\}+\mathcal{L}_{\bar{\xi},4}\left\{900\zeta_3^2-\frac{2880\zeta_2^3}{7}\right\}+\mathcal{L}_{\bar{\xi},5}\{984\zeta_5-360\zeta_2\zeta_3\}-\mathcal{L}_{\bar{\xi},6}\left\{\frac{238\zeta_2^2}{5}\right\}$$
$$+\mathcal{L}_{\bar{\xi},7}\{32\zeta_3\}-\mathcal{L}_{\bar{\xi},8}\{4\zeta_2\}+\mathcal{L}_{\bar{\xi},10}\left\{\frac{1}{6}\right\},$$

$$\boldsymbol{D}_{\bar{\xi},5}\otimes\mathcal{L}_{\bar{\xi},5}=\mathcal{L}_{\bar{\xi},0}\{604800\zeta_{11}-10152\zeta_2^4\zeta_3-20160\zeta_2^3\zeta_5+50400\zeta_3^2\zeta_5-53280\zeta_2^2\zeta_7-7200\zeta_2\zeta_3^3-201600\zeta_2\zeta_9\}$$
$$-\mathcal{L}_{\bar{\xi},1}\{16056\zeta_2^5+13680\zeta_2^2\zeta_3^2+100800\zeta_2\zeta_3\zeta_5-129600\zeta_5^2-259200\zeta_3\zeta_7\}-\mathcal{L}_{\bar{\xi},2}\{13680\zeta_2^3\zeta_3-12000\zeta_3^3$$
$$+36720\zeta_2^2\zeta_5+129600\zeta_2\zeta_7-369600\zeta_9\}-\mathcal{L}_{\bar{\xi},3}\{6324\zeta_2^4+12000\zeta_2\zeta_3^2-60000\zeta_3\zeta_5\}$$
$$-\mathcal{L}_{\bar{\xi},4}\{4200\zeta_2^2\zeta_3+15000\zeta_2\zeta_5-39600\zeta_7\}+\mathcal{L}_{\bar{\xi},5}\left\{2100\zeta_3^2-\frac{6072\zeta_2^3}{7}\right\}-\mathcal{L}_{\bar{\xi},6}\{700\zeta_2\zeta_3-1820\zeta_5\}$$
$$-\mathcal{L}_{\bar{\xi},7}\{74\zeta_2^2\}+\mathcal{L}_{\bar{\xi},8}\{45\zeta_3\}-\mathcal{L}_{\bar{\xi},9}\{5\zeta_2\}+\mathcal{L}_{\bar{\xi},11}\left\{\frac{1}{6}\right\}.$$

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