

First insight into transverse-momentum-dependent fragmentation physics at photon-photon colliders

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(Received 29 April 2025; accepted 9 June 2025; published 7 July 2025)

Future planned lepton colliders, both in the circular and linear configurations, can effectively work as virtual and quasireal photon-photon colliders and are expected to stimulate an intense physics program in the next few years. In this paper, we suggest to consider photon-photon scattering as a useful source of information on transverse-momentum-dependent fragmentation functions (TMD FFs), complementing semi-inclusive deeply inelastic scattering and e^+e^- annihilation processes, which provide most of the present phenomenological information on TMD FFs. As a first illustrative example, we study two-hadron azimuthal asymmetries around the jet thrust axis in the process $\ell^+\ell^- \rightarrow \gamma^*\gamma \rightarrow q\bar{q} \rightarrow h_1h_2 + X$, in which in a circular lepton collider one tagged, deeply virtual photon scatters off an untagged quasireal photon, both originating from the initial lepton beams, producing inclusively an almost back-to-back light-hadron pair with large transverse momentum, in the $\gamma^*\gamma$ center-of-mass frame. Similar processes, in a more complicated environment due to the presence of initial hadronic states, can also be studied in ultraperipheral collisions at the LHC and the planned future hadron colliders.

DOI: 10.1103/p6w2-dn9z

I. INTRODUCTION

Nucleon tomography and the three-dimensional structure of hadrons are nowadays among the most exciting topics in hadron physics. In this context, a relevant role is played by the so-called transverse-momentum-dependent partonic distribution and fragmentation functions (TMD PDFs and FFs, respectively, also named TMDs collectively). Compared to the usual collinear PDFs and FFs, where the transverse component of the parton (hadron) intrinsic momenta with respect to the parent hadrons (partons) is integrated over, leading to the scale dependence of the distributions, TMDs explicitly retain their dependence on intrinsic transverse momenta. As a consequence, they account for a much richer structure involving correlations among the spin, or polarization state, and the intrinsic motion of partons and hadrons. These correlations manifest themselves in observable spin and azimuthal asymmetries in inclusive and semi-inclusive hadronic processes. For an up-to-date and exhaustive review on

the formalism and phenomenology of TMDs see, e.g., Ref. [1] and references therein.

QCD factorization theorems within the TMD approach have been proven for three fundamental processes involving two well distinct energy scales: a perturbative one, related to a large momentum transfer in the process, and a much smaller second scale comparable to the QCD scale, Λ_{QCD} , and related to the intrinsic parton motion inside hadrons [2–5]: (1) Semi-inclusive deeply inelastic scattering (SIDIS), $\ell p(N) \rightarrow \ell' h + X$; (2) the Drell-Yan (DY) process for inclusive dilepton production in hadronic collisions, $AB \rightarrow \gamma^*, Z \rightarrow \ell^+\ell^- + X$; and (3) two almost back-to-back hadron production in lepton-antilepton annihilations, $\ell^+\ell^- \rightarrow \gamma^*, Z \rightarrow h_1h_2 + X$, also known as semi-inclusive $\ell^+\ell^-$ annihilation (SIA).

Indeed, most of the available experimental information on quark TMDs come from these three processes. Concerning phenomenology, the full (leading and, in some cases, next-to-leading twist) structure of azimuthal dependences for particle production in (un)polarized processes has been derived in the TMD approach for SIDIS [6–8] and DY [9] processes, as well as for hadron-pair SIA production [10,11]. In particular, Refs. [8,11] present, respectively, for the SIDIS and SIA cases, an independent derivation within the TMD approach and the helicity formalism that will be adopted also in this paper. Other processes, like Higgs [12,13], photon pair [14], J/ψ pair [15], J/ψ + photon [16] and C-even quarkonium production [17,18] in hadronic collisions, where the dominance of

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the color-singlet quarkonium formation mechanism is assumed, represent another source of valuable information on TMDs, in particular, the almost unknown gluon ones. For these reactions, TMD factorization is not fully proven yet and there are some indications for possible factorization-breaking effects for processes involving four hadrons, see, e.g., Ref. [19].

Concerning quark TMD fragmentation functions, available information mainly comes from SIDIS and e^+e^- annihilation, for which, however, quark flavor separation is not easy. Moreover, TMD PDFs and FFs always appear coupled in SIDIS, making phenomenology even more complicated, while in e^+e^- collisions, where only fragmentation processes play a role, experimental information is presently relatively scarce. It is therefore clear that more processes and observables sensible to intrinsic parton motion effects, for which TMD factorization is guaranteed, would be very helpful in improving the knowledge and phenomenology of TMD fragmentation functions.

A new class of processes for which TMD factorization is expected to hold, given the clearness of the initial electromagnetic state and the presence of final state interactions only, is inclusive hadron-pair production in photon-photon collisions. These can be studied at present in ultraperipheral collisions at hadron colliders, like the Large Hadron Collider (LHC) at CERN and the Relativistic Heavy Ion Collider at Brookhaven National Laboratory, although in these experiments the intricate hadronic environment can hinder the study of TMD observables. Photon-photon collisions are also actively investigated as a source of information for Higgs and heavy-boson properties and decays, as well as light-by-light scattering. In fact, the $\gamma\gamma$ collider operational mode is considered in all major proposals for future circular and linear lepton colliders (for more information on photon-photon physics, see, e.g., Refs. [20–24]). Therefore, in this paper we propose to consider photon-photon scattering as a future tool for gaining new complementary and clean information on quark TMD fragmentation functions. One interesting point is that in photon-photon scattering flavor separation would be more effective, as compared to SIDIS or $\ell^+\ell^-$ annihilations, since the contributions of d , s , b quarks should be suppressed by a relative factor $1/16$ (coming from their fractional electric charge to the fourth power) with respect to those of u , c , quarks. This would certainly help in better determining quark TMD FFs and disentangling different flavor contributions. Another advantage is that, while $\ell^+\ell^-$ colliders operate at some fixed energy scale [the center-of-mass (c.m.) energy of the two leptons], photon-photon collisions allow one to vary the perturbative energy scale (related to the photon virtualities). Therefore, one can study the scale dependence of the TMD FFs in the same process and experimental setup.

One may, however, wonder about the reachable luminosity in the photon-photon collision mode at lepton colliders, and the attainable c.m. energy. Indeed, all these aspects need to be carefully considered while developing proposals for future lepton colliders and the related detectors. At this stage, we suggest to take into account the possibility of undertaking a fructuous and complementary analysis of the 3D structure of hadrons and TMDs, together with the main physics cases considered. The main motivation of this paper is in fact to stimulate the study of TMD physics at future photon-photon colliders at the time the nuclear and particle physics communities keep discussing perspectives for future large-scale hadron and lepton colliders. As a first illustrative example, we will consider the azimuthal distribution of a pseudoscalar hadron pair, inclusively produced in opposite hemispheres with respect to the final jet thrust axis, in photon-photon collisions, $\ell^+\ell^- \rightarrow \gamma^*\gamma \rightarrow q\bar{q} \rightarrow h_1h_2 + X$, with one deeply virtual and one quasireal photon. To this end, we will adopt the TMD approach at leading order and leading twist, complemented by the helicity formalism, which allows us to follow step by step, in the physical process, the role of the spin and polarization state of the particles involved.

The plan of the paper is the following: In Sec. II we describe the formalism adopted and the kinematics of the process; Sec. III is devoted to the detailed derivation of the differential cross section of the process and the measurable azimuthal asymmetries and to a discussion of their physical content. Our final conclusions and remarks are given in Sec. IV. More technical details on the kinematics, the virtual photon helicity density matrix and the hard-scattering helicity amplitudes are presented in the Appendixes.

II. KINEMATICS AND FORMALISM

In this section we provide the main analytic expressions and kinematical details required to derive the differential cross section for the process

$$\ell^+(l_+) + \ell^-(l_-) \rightarrow \ell^+(l'_+) + h_1(P_1) + h_2(P_2) + X, \quad (1)$$

where the four-momenta of the particles involved are shown within brackets. At leading order in the electromagnetic and strong coupling constants, α and α_s , respectively, the dominant channel of this reaction is

$$\begin{aligned} \gamma_1^*(q_1) + \gamma_2(q_2) &\rightarrow q(K_q) + \bar{q}(K_{\bar{q}}) \\ &\rightarrow h_1(P_1) + h_2(P_2) + X, \end{aligned} \quad (2)$$

that is the production of a quark-antiquark pair by two-photon fusion, and their subsequent fragmentation into two light unpolarized or scalar mesons (we mainly have in mind

pion and kaon mesons here). In Eq. (1) we are considering the single-tagged configuration for a leptonic circular collider ($\ell = e, \mu$) where, for instance, the final antilepton ℓ'^+ with four-momentum l'_+ is detected and the virtuality $q_1^2 = (l_+ - l'_+)^2 \equiv -Q_1^2$ of γ_1 is known, while the final lepton ℓ'^- is undetected and γ_2 can be effectively considered as (quasi)real, $q_2^2 \simeq 0$, and described by a collinear Weizsäcker-Williams distribution inside the parent beam lepton [25,26]. For completeness, we will also consider the case in which the lepton beams can be longitudinally polarized. Furthermore, we note that the two final hadrons are produced almost back to back (in the partonic c.m. frame), with a large transverse momentum with respect to the $\gamma^*\gamma$ axis. Intrinsic transverse momentum effects in the fragmentation process, encoded in the quark TMD fragmentation functions, lead to an observable noncollinearity of the two final hadrons around the $q\bar{q}$ axis. This in turn manifests itself as azimuthal correlations in the two-hadron angular distribution around the jet thrust axis.

Before going into more details, some comments are in order:

- (a) As mentioned in the Introduction, we adopt here a TMD factorization approach within the helicity formalism. The simple electromagnetic initial state should guarantee the validity of the approach as in $\ell^+\ell^-$ SIA processes. Moreover, it has been proven that TMD fragmentation functions are universal and process independent (see, e.g., Refs. [27–29]), so that we can consider the process in Eq. (1) as a useful additional tool for TMD FF phenomenology.
- (b) From the theoretical point of view, the two-hadron azimuthal distribution around the jet thrust axis is the cleanest possible observable to consider; from the experimental point of view, however, it requires a good determination of the thrust axis that can be difficult to achieve. In fact, experimental results for SIA processes are often presented for the azimuthal distribution of one final hadron around the direction of motion of the second one. In this paper, mainly devoted to illustrate a first application of the TMD approach in photon-photon collisions, we keep on working in the thrust-axis configuration. The formalism adopted here has been already worked out also for the second kinematical configuration and could be easily implemented in our case, see Ref. [11] and references therein for more details.
- (c) TMD evolution with the energy scale has been formulated within the Collins-Soper-Sterman approach [4,30–32] and the soft collinear effective theory, see, e.g., Refs. [33,34]. In this paper we present results in a simplified framework valid at a fixed energy scale. Full implementation of scale evolution of TMD FFs, crucial when experimental data will become available, has been already performed, using our same framework, for the $e^+e^- \rightarrow h_1 h_2 + X$ process in Refs. [35,36] and can

be directly applied to the process under study. While making the analytical expressions of the quantities considered more involved, it does not modify the general structure of the azimuthal modulations which are the main subject of this paper.

- (d) A possible competing contribution to our observable comes from the gluon distribution $f_{g/\ell}(\xi)$ inside the second lepton, coupled to the hard process $\gamma^*g \rightarrow q\bar{q}$. However, this contribution should be suppressed with respect to the photon one, since one needs first to produce a $q\bar{q}$ pair by a primary photon inside the lepton to generate a gluon. This is only partially compensated by the order $\alpha\alpha_s(Q^2)$ of the cross section for the hard $\gamma^*g \rightarrow q\bar{q}$ process, as compared to the $\gamma^*\gamma \rightarrow q\bar{q}$ one of order α^2 . Moreover, in principle, this contribution can be distinguished experimentally by the presence of additional hadronic production along the second lepton beam, which is suppressed in photon-photon scattering. Anyway, the γ -gluon contribution could be easily added to the γ - γ one by simply adapting the results presented below, implementing the required changes in the couplings and replacing the photon distribution in the second lepton, $f_{\gamma/\ell}$, by the gluon one. In this case the results are very similar to those discussed in the literature for the SIDIS case, see, e.g., Ref. [37] and references therein. In fact, we have verified that, with due changes, our results agree with those of Ref. [37] for the common parts.

Concerning the kinematics of the process, different c.m. frames enter into play: the lepton-beam $\ell^+\ell^-$ c.m. frame, the $\gamma_1^*\ell^-$ (analogous to the γ^*p frame commonly adopted in SIDIS) and the $\gamma_1^*\gamma_2$ ones. We will summarize in this section the derivation of the differential cross section of the process under investigation. The azimuthal distributions of the two final hadrons around the final jet axis, which in our lowest-order (in the strong coupling constant) analysis coincides with the $q\bar{q}$ axis, will be discussed in the following section. Useful kinematical relations in the different c.m. frames, required for the calculations, are collected and discussed in Appendix A.

To summarize what has been said so far, we will adopt a leading-order and leading twist TMD factorization approach within the helicity formalism, which clearly describes, for each step of the process, the polarization state of the particles involved and their role in the measured azimuthal distributions. Given the simplicity of the initial (electromagnetic) state, the process under consideration is on an equal footing with direct lepton-antilepton annihilation, for which TMD factorization has been fully proven. Therefore, we are confident that the same approach can be applied to photon-photon collisions.

Within the above described framework, the differential cross section for the process in Eq. (1) can be written as follows:

$$\begin{aligned}
d\sigma^{\ell^+\ell^-\rightarrow\ell'^+h_1h_2X} &= \frac{1}{4(l_+ \cdot l_-)} \frac{d^3l'_+}{2(2\pi)^3 l'_+{}^0} \frac{d^3K_q}{2(2\pi)^3 K_q^0} \frac{d^3K_{\bar{q}}}{2(2\pi)^3 K_{\bar{q}}^0} (2\pi)^4 \delta^{(4)}(q_1 + q_2 - K_q - K_{\bar{q}}) \\
&\times \sum_q \sum_{\{\lambda_i\}} \tilde{\rho}_{\lambda_1, \lambda'_1}(\gamma_1^*) \rho_{\lambda_2, \lambda'_2}(\gamma_2) f_{\gamma/\ell^-, \mathcal{P}_{\xi_-}^{\ell^-}}(\xi) \frac{d\xi}{\xi} \hat{H}_{\lambda_q, \lambda_{\bar{q}}; \lambda_1, \lambda_2} \hat{H}_{\lambda'_q, \lambda'_{\bar{q}}; \lambda'_1, \lambda'_2}^* \\
&\times \hat{D}_{\lambda_q, \lambda'_q}^{h_1}(z_1, \mathbf{p}_{\perp 1}) dz_1 d^2\mathbf{p}_{\perp 1} \hat{D}_{\lambda_{\bar{q}}, \lambda'_{\bar{q}}}^{h_2}(z_2, \mathbf{p}_{\perp 2}) dz_2 d^2\mathbf{p}_{\perp 2}.
\end{aligned} \tag{3}$$

In this equation, the first line contains the kinematical terms related to the initial flux factor, the ℓ'^+ Lorentz-invariant phase space factor (LIPS), which will be expressed in the $\ell^+\ell^-$ c.m. frame, the LIPS for the quark and the antiquark produced in the $\gamma_1^*\gamma_2$ annihilation, as well as the Dirac delta imposing momentum conservation in the hard process. The last two lines refer to the dynamical kernel of the cross section, including the hard-scattering amplitudes and the parton hadronization process into the two final hadrons, through the TMD fragmentation functions. According to the factorization approach, in a reference frame where the initial photons move collinearly and in opposite directions, this kernel is given in terms of the distributions of the initial photons inside the parent leptons, the hard-scattering amplitudes for the process $\gamma_1^*\gamma_2 \rightarrow q\bar{q}$, and the fragmentation functions of the final quark and antiquark ($q, \bar{q}$) into the observed hadrons (h_1, h_2). Let us illustrate all the ingredients entering this expression in more detail:

- (1) The first sum over (light) quark flavors extends to $q = u, \bar{u}, d, \bar{d}, s, \bar{s}$; it can be generalized to include heavy (c, b) flavors.
- (2) In the second sum $\{\lambda_i\}$ stays for a sum over all involved helicity indices.
- (3) $\tilde{\rho}(\gamma_1^*)$ and $\rho(\gamma_2)$ are, respectively, the helicity density matrices of the virtual photon γ_1^* and the quasireal one, γ_2 , whose expression will be given in the following.
- (4) $f_{\gamma/\ell^-, \mathcal{P}_{\xi_-}^{\ell^-}}(\xi)$ is the Weizsäcker-Williams distribution for the quasireal photon inside the initial unpolarized ($\mathcal{P}_{\xi_-}^{\ell^-} = 0$) or longitudinally polarized ($\mathcal{P}_{\xi_-}^{\ell^-} = \pm 1$) ℓ^- lepton and ξ the corresponding light-cone momentum fraction.
- (5) The $\hat{H}_{\lambda_q, \lambda_{\bar{q}}; \lambda_1, \lambda_2}$'s are the helicity scattering amplitudes for the hard partonic process $\gamma_1^*(\lambda_1) + \gamma_2(\lambda_2) \rightarrow q(\lambda_q) + \bar{q}(\lambda_{\bar{q}})$.
- (6) Finally, the $\hat{D}_{\lambda_q, \lambda'_q}^h(z, \mathbf{p}_{\perp})$'s are the transverse-momentum-dependent fragmentation functions encoding the soft fragmentation process of quark q into the final hadron h carrying a light-cone momentum fraction z of the parent quark momentum and an intrinsic transverse momentum $\mathbf{p}_{\perp}$ with respect to its direction of motion.

An energy scale dependence of parton distribution and fragmentation functions is implied throughout this paper.

Let us now summarize, referring to Appendix A for more details, some useful standard manipulations on the kinematical factors in the first line of Eq. (3). We will adopt the usual invariant variables for deeply inelastic scattering (DIS),

$$\begin{aligned}
s &= (l_+ + l_-)^2 = 2l_+ \cdot l_-, \quad x_B = \frac{Q^2}{2q_1 \cdot l_-}, \\
y &= \frac{q_1 \cdot l_-}{l_+ \cdot l_-},
\end{aligned} \tag{4}$$

with $Q^2 = -q_1^2 = x_B y s$. Notice that we will neglect lepton, quark, and hadron masses in the following. From Eq. (4) it is easy to derive the flux factor in Eq. (3), $4l_+ \cdot l_- = 2s$. The Lorentz-invariant phase space for the final tagged positron can be written, in the $\ell^+\ell^-$ c.m. frame, as follows:

$$\frac{d^3l'_+}{2(2\pi)^3 l'_+{}^0} = \frac{1}{16\pi^2} s y dx_B dy, \tag{5}$$

where the angular dependence has been integrated over. This can be seen explicitly by looking, e.g., at the expression of l'_+ in Eq. (A1) and evaluating from there the Jacobian for the change of variables $d(l'_+)^1 d(l'_+)^2 d(l'_+)^3 = |J| dx_B dy d\phi_{\ell'}$. Concerning the four-dimensional Dirac delta in Eq. (3), we can write

$$\begin{aligned}
\delta^{(4)}(q_1 + q_2 - K_q - K_{\bar{q}}) &= \delta(q_1^+ + q_2^+ - K_q^+ - K_{\bar{q}}^+) \\
&\times \delta(q_1^- + q_2^- - K_q^- - K_{\bar{q}}^-) \\
&\times \delta^{(2)}(-\mathbf{K}_{qT} - \mathbf{K}_{\bar{q}T}),
\end{aligned} \tag{6}$$

where we have switched temporarily to light-cone 4-vector components, $a^\mu = (a^+, a^-, \mathbf{a}_T)$, with $a^\pm = (a^0 \pm a^3)/\sqrt{2}$. The two-dimensional Dirac delta on the transverse momenta fixes $\mathbf{K}_{qT} \equiv \mathbf{K}_T = -\mathbf{K}_{\bar{q}T}$. Moreover, by using the results of Appendix A, we find

$$\begin{aligned}
&\delta(q_1^+ + q_2^+ - K_q^+ - K_{\bar{q}}^+) \delta(q_1^- + q_2^- - K_q^- - K_{\bar{q}}^-) \\
&= \frac{2}{ys} \delta(1 - \zeta_q - \zeta_{\bar{q}}) \delta\left(\xi - x_B - \frac{\mathbf{K}_T^2}{\zeta_q \zeta_{\bar{q}} ys}\right),
\end{aligned} \tag{7}$$

where the additional invariants $\zeta_{q, \bar{q}}$ are defined as

$$\zeta_q = \frac{K_q \cdot l_-}{q_1 \cdot l_-}, \quad \zeta_{\bar{q}} = \frac{K_{\bar{q}} \cdot l_-}{q_1 \cdot l_-}, \quad (8)$$

see Appendix A for more details. The LIPS for the final quark-antiquark pair can be further manipulated,

$$\frac{d^3 K_q}{2(2\pi)^3 K_q^0} \frac{d^3 K_{\bar{q}}}{2(2\pi)^3 K_{\bar{q}}^0} \delta^{(2)}(-\mathbf{K}_{qT} - \mathbf{K}_{\bar{q}T}) = \frac{1}{4(2\pi)^6} \frac{dK_q^3}{K_q^0} d^2 \mathbf{K}_T \frac{dK_{\bar{q}}^3}{K_{\bar{q}}^0}. \quad (9)$$

By using again the results of Appendix A we find that

$$\frac{dK_q^3}{K_q^0} = d\eta_q = \frac{d\zeta_q}{\zeta_q}, \quad (10)$$

where η_q is the quark pseudorapidity and similar for $K_{\bar{q}}$. Furthermore, $d^2 \mathbf{K}_T = K_T dK_T d\phi_q = (1/2) dK_T^2 d\phi_q$.

Inserting all these results into Eq. (3) and collecting the constant factors we finally get

$$\begin{aligned} \frac{d\sigma^{\ell^+ \ell^- \rightarrow \ell'^+ h_1 h_2 X}}{dx_B dy d\zeta dK_T^2 d\phi_q d\xi dz_1 d^2 \mathbf{p}_{\perp 1} dz_2 d^2 \mathbf{p}_{\perp 2}} &= \frac{1}{2^9 \pi^4} \frac{1}{\zeta(1-\zeta)\xi s} \sum_q \sum_{\{\lambda_i\}} \tilde{\rho}_{\lambda_1, \lambda'_1}(\gamma_1^*) \rho_{\lambda_2, \lambda'_2}(\gamma_2) f_{\gamma/\ell^-, \mathcal{P}_{\hat{z}_-}^{\ell^-}}(\xi) \hat{H}_{\lambda_q, \lambda_{\bar{q}}; \lambda_1, \lambda_2} \hat{H}_{\lambda'_q, \lambda'_{\bar{q}}; \lambda'_1, \lambda'_2}^* \\ &\times \hat{D}_{\lambda_q, \lambda'_q}^{h_1}(z_1, \mathbf{p}_{\perp 1}) \hat{D}_{\lambda_{\bar{q}}, \lambda'_{\bar{q}}}^{h_2}(z_2, \mathbf{p}_{\perp 2}) \delta\left(\xi - x_B - \frac{K_T^2}{\zeta(1-\zeta)ys}\right), \end{aligned} \quad (11)$$

where $\zeta_{\bar{q}} = \zeta$, $\zeta_q = 1 - \zeta$, and the remaining Dirac delta can be used either to fix ξ in terms of K_T^2 or vice versa. The various ingredients of the dynamical kernel in the above equation are described below.

The expression of the helicity density matrix for the tagged virtual photon γ_1^* in the deeply inelastic scattering regime, $\rho_{\lambda_1, \lambda'_1}(\gamma_1^*)$, properly normalized to unity, has been derived and discussed in detail, e.g., in Refs. [20,38,39]. For completeness, we summarize its derivation in Appendix B. Its expression in terms of the DIS invariants, in the photon helicity frame, where the photon moves along the $+\hat{z}$ axis and the leptonic $\ell^+ - \ell'^+$ plane spans an azimuthal angle ϕ_ℓ with respect to the $\hat{x} - \hat{z}$ plane, reads

$$\begin{aligned} \rho(\gamma_1^*) &= \frac{1}{2(2-y)^2} \\ &\times \begin{pmatrix} 1 + (1-y)^2 + \mathcal{P}_{\hat{z}_+}^{\ell^+} y(2-y) & -e^{-i\phi_\ell} \sqrt{2(1-y)}[(2-y) + \mathcal{P}_{\hat{z}_+}^{\ell^+} y] & -e^{-i2\phi_\ell} 2(1-y) \\ -e^{i\phi_\ell} \sqrt{2(1-y)}[(2-y) + \mathcal{P}_{\hat{z}_+}^{\ell^+} y] & 4(1-y) & e^{-i\phi_\ell} \sqrt{2(1-y)}[(2-y) - \mathcal{P}_{\hat{z}_+}^{\ell^+} y] \\ -e^{i2\phi_\ell} 2(1-y) & e^{i\phi_\ell} \sqrt{2(1-y)}[(2-y) - \mathcal{P}_{\hat{z}_+}^{\ell^+} y] & 1 + (1-y)^2 - \mathcal{P}_{\hat{z}_+}^{\ell^+} y(2-y) \end{pmatrix}, \end{aligned} \quad (12)$$

where $\mathcal{P}_{\hat{z}_+}^{\ell^+} = 0, \pm 1$ for unpolarized or longitudinally polarized initial leptons ℓ^+ , respectively, and, without loss of generality, in the following we will take $\phi_\ell = 0$. Notice that the normalized helicity density matrix ($\rho = \tilde{\rho}/\text{Tr}[\tilde{\rho}]$, so that $\text{Tr}[\rho] = 1$) in Eq. (12) has to be used in normalized observables related to $d\sigma/\sigma$, while in the differential cross section $d\sigma$, Eq. (11), one has to reinstate the appropriate normalization factor, using $\tilde{\rho} = \rho \text{Tr}[\tilde{\rho}]$, where

$$\text{Tr}[\tilde{\rho}] = \frac{2e^2(2-y)^2}{Q^2 y^2} \equiv \frac{2e^2(2-y)^2}{x_B y^3 s}. \quad (13)$$

Concerning the untagged quasireal photon γ_2 , since it can only have $\lambda_2 = \pm 1$ helicities, its helicity density matrix can be effectively written as a 2×2 matrix,

$$\rho(\gamma_2) = \frac{1}{2} \begin{pmatrix} 1 + \mathcal{P}_{\hat{z}_2}^{\gamma_2} & 0 \\ 0 & 1 - \mathcal{P}_{\hat{z}_2}^{\gamma_2} \end{pmatrix}, \quad (14)$$

where $\mathcal{P}_{\hat{z}_2}^{\gamma_2}$ is the longitudinal component of the γ_2 polarization (pseudo)vector along its direction of motion, $\mathcal{P}_{\hat{z}_2}^{\gamma_2} = \pm 1$. Notice that in our partonic reference frame γ_1^* and γ_2 move back to back along the $\hat{z}$ axis, so that $\hat{z}_2 = \hat{z}_- = -\hat{z}_1 \equiv -\hat{z}$. As a consequence of Eq. (14), only two distinct combinations play a role in Eq. (11),

$$[\rho_{++}(\gamma_2) + \rho_{--}(\gamma_2)]f_{\gamma/\ell^-, \mathcal{P}_{\hat{z}_-}^{\ell^-}}(\xi) = f_{\gamma, +/\ell^-, \mathcal{P}_{\hat{z}_-}^{\ell^-}}(\xi) + f_{\gamma, -/\ell^-, \mathcal{P}_{\hat{z}_-}^{\ell^-}}(\xi) = f_{\gamma/\ell}(\xi), \quad (15)$$

$$[\rho_{++}(\gamma_2) - \rho_{--}(\gamma_2)]f_{\gamma/\ell^-, \mathcal{P}_{\hat{z}_-}^{\ell^-}}(\xi) = \mathcal{P}_{\hat{z}_2}^{\gamma_2} f_{\gamma/\ell^-, \mathcal{P}_{\hat{z}_-}^{\ell^-}}(\xi) = f_{\gamma, +/\ell^-, \mathcal{P}_{\hat{z}_-}^{\ell^-}}(\xi) - f_{\gamma, -/\ell^-, \mathcal{P}_{\hat{z}_-}^{\ell^-}}(\xi) = \mathcal{P}_{\hat{z}_-}^{\ell^-} \Delta_L f_{\gamma/\ell}(\xi), \quad (16)$$

where $f_{\gamma/\ell}(\xi) = f_{\gamma, \pm/\ell^-, \pm} + f_{\gamma, \mp/\ell^-, \pm}$ and $\Delta_L f_{\gamma/\ell}(\xi) = f_{\gamma, \pm/\ell^-, \pm} - f_{\gamma, \mp/\ell^-, \pm}$ are, respectively, the unpolarized and longitudinally polarized Weizsäcker-Williams parton distributions for γ_2 inside lepton ℓ^- .

The next ingredients in the dynamical kernel of Eq. (11) are the helicity amplitudes $\hat{H}_{\lambda_q, \lambda_{\bar{q}}; \lambda_1, \lambda_2}$ for the hard-scattering process $\gamma_1^*(\lambda_1) + \gamma_2(\lambda_2) \rightarrow q(\lambda_q) + \bar{q}(\lambda_{\bar{q}})$. Their explicit leading-order expression in the $\gamma_1^*\gamma_2$ c.m. frame is given in Appendix C. Notice that since we are considering the production of light quarks, neglecting their masses, due to helicity conservation in the photon-quark vertices, the only nonvanishing amplitudes have opposite values of the quark and antiquark helicities, which helps in simplifying the expression of the kernel in Eq. (11). Moreover, using parity conservation, one can see that there are only six independent amplitudes.

The last step of the scattering process consists in the independent fragmentation of the quark and the antiquark (produced exactly back to back in their c.m. frame in the leading-order approach considered here, along the jet-thrust-axis direction) in the final observed hadrons. This nonperturbative process is embodied into the transverse-momentum-dependent fragmentation functions $D_{\lambda_q, \lambda_q'}^{h_1}(z_1, \mathbf{p}_{\perp 1})$ for $q \rightarrow h_1 + X$ (and analogously for the antiquark fragmentation). As a result of the explicit account of intrinsic transverse motion effects, the two observed hadrons are no longer exactly back to back in the partonic c.m. frame. This generates possible azimuthal asymmetries in their distribution around the jet axis, which are the main subject of this study.

TMD fragmentation functions into unpolarized (or spinless) and spin-1/2 hadrons within the helicity formalism have been discussed in detail in Refs. [11, 40]. Here we only summarize some relations useful for the evaluation of the kernel. TMD FFs for the process $a(s_a) \rightarrow h + X$, where a is a quark or an antiquark, can be written as

$$\hat{D}_{\lambda_a, \lambda_a'}^{h/a}(z, \mathbf{p}_{\perp}) = \sum_{\lambda_h} \sum_{X, \lambda_X} \hat{\mathcal{D}}_{\lambda_h, \lambda_X; \lambda_a}(z, \mathbf{p}_{\perp}) \hat{\mathcal{D}}_{\lambda_h, \lambda_X; \lambda_a'}^*(z, \mathbf{p}_{\perp}), \quad (17)$$

where the $\hat{\mathcal{D}}_{\lambda_h, \lambda_X; \lambda_a}(z, \mathbf{p}_{\perp})$'s are soft, nonperturbative helicity amplitudes for the process $a(\lambda_a) \rightarrow h(\lambda_h) + X(\lambda_X)$ and the symbol $\sum_{X, \lambda_X}$ stands for the helicity sum and

phase-space integration for the final unobserved remnants, collectively named X , in the fragmentation process. Using parity symmetry of strong interactions it is easy to see that for quark fragmentation into spinless or unpolarized final hadrons there are only two independent, leading twist TMD FFs: the unpolarized one,

$$\hat{D}_{++}^{h/a}(z, \mathbf{p}_{\perp}) = \hat{D}_{--}^{h/a}(z, \mathbf{p}_{\perp}) = D_a^h(z, p_{\perp}), \quad (18)$$

where $p_{\perp} = |\mathbf{p}_{\perp}|$, and the Collins fragmentation function [41], describing the fragmentation of a transversely polarized quark into an unpolarized hadron,

$$\hat{D}_{+-}^{h/a}(z, \mathbf{p}_{\perp}) = D_{+-}^{h/a}(z, p_{\perp}) e^{i\phi_a^h}, \quad (19)$$

where ϕ_a^h is the azimuthal angle of the hadron h momentum in the fragmenting parton helicity frame. It is also easy to see that

$$\hat{D}_{-+}^{h/a}(z, \mathbf{p}_{\perp}) = -[\hat{D}_{+-}^{h/a}(z, \mathbf{p}_{\perp})]^* = -D_{+-}^{h/a}(z, p_{\perp}) e^{-i\phi_a^h}. \quad (20)$$

Common notations adopted in the literature for the $p_{\perp}$ -dependent term of the quark Collins FF are

$$\Delta^N D_{a\uparrow}^h(z, p_{\perp}) = \frac{2p_{\perp}}{zm_h} H_1^{\perp, a}(z, p_{\perp}) = -i2D_{+-}^{h/a}(z, p_{\perp}), \quad (21)$$

with m_h the mass of hadron h , while the $\uparrow$ arrow specifies that the quark a is transversely polarized with respect to the plane containing the quark itself and the hadron. Notice also that

$$\int d^2\mathbf{p}_{\perp} D_a^h(z, p_{\perp}) = D_a^h(z), \quad (22)$$

the usual collinear unpolarized fragmentation function. Moreover, for future use, we also define the lowest transverse moment of the Collins function,

$$\begin{aligned} \int d^2\mathbf{p}_{\perp} \Delta^N D_{a\uparrow}^h(z, p_{\perp}) &\equiv \int d^2\mathbf{p}_{\perp} \frac{2p_{\perp}}{zm_h} H_1^{\perp, a}(z, p_{\perp}) \\ &= 2\pi \int dp_{\perp} p_{\perp} \Delta^N D_{a\uparrow}^h(z, p_{\perp}) \\ &= \Delta^N D_{a\uparrow}^h(z) = 4H_1^{\perp(1/2)a}(z). \end{aligned} \quad (23)$$

III. CROSS SECTION AND AZIMUTHAL ASYMMETRIES

Inserting Eqs. (12)–(21) into Eq. (11) and using symmetry considerations, after some lengthy but straightforward calculations one finally finds the explicit expression of the differential cross section (in the sequel we will use $\mathcal{P}_\pm$ for $\mathcal{P}_{\hat{z}_\pm}^\pm$ for shortness),

$$\begin{aligned} \frac{d\sigma^{\ell^+\ell^-\rightarrow\ell'^+h_1h_2X}(\mathcal{P}_+, \mathcal{P}_-)}{dx_B dy d\zeta d\phi_q d\xi dz_1 d^2\mathbf{p}_{\perp 1} dz_2 d^2\mathbf{p}_{\perp 2}} &= \frac{3\alpha^3}{4\pi} \frac{1}{x_B y^2 \xi^3 s} \sum_q e_q^4 \left\{ \left[A_U + \mathcal{P}_+ \mathcal{P}_- A_L + \left(A_U^{\cos\phi_q} + \mathcal{P}_+ \mathcal{P}_- A_L^{\cos\phi_q} \right) \cos\phi_q \right. \right. \\ &\quad + A_U^{\cos 2\phi_q} \cos 2\phi_q \left. \right] D_q^{h_1}(z_1, p_{\perp 1}) D_{\bar{q}}^{h_2}(z_2, p_{\perp 2}) + \left[\left(B_U^{\cos\phi_{12}} + \mathcal{P}_+ \mathcal{P}_- B_L^{\cos\phi_{12}} \right) \cos\phi_{12} \right. \\ &\quad + \left(B_U^{\cos(\phi_q - \phi_{12})} + \mathcal{P}_+ \mathcal{P}_- B_L^{\cos(\phi_q - \phi_{12})} \right) \cos(\phi_q - \phi_{12}) \\ &\quad + \left(B_U^{\cos(\phi_q + \phi_{12})} + \mathcal{P}_+ \mathcal{P}_- B_L^{\cos(\phi_q + \phi_{12})} \right) \cos(\phi_q + \phi_{12}) + B_U^{\cos(2\phi_q - \phi_{12})} \cos(2\phi_q - \phi_{12}) \\ &\quad \left. \left. + B_U^{\cos(2\phi_q + \phi_{12})} \cos(2\phi_q + \phi_{12}) \right] \Delta^N D_{q^\dagger}^{h_1}(z_1, p_{\perp 1}) \Delta^N D_{\bar{q}^\dagger}^{h_2}(z_2, p_{\perp 2}) \right\}, \end{aligned} \quad (24)$$

where we have used the remaining Dirac delta in Eq. (11) to fix $K_T^2 = \zeta(1-\zeta)(\xi-x_B)ys = \zeta(1-\zeta)\hat{s}$. In Eq. (24), ϕ_q is the azimuthal angle of the scattering plane of the process $\gamma_1^* \gamma_2 \rightarrow q\bar{q}$ with respect to the leptonic plane $\ell^+ - \ell'^+$ for the tagged photon. Moreover, we have introduced the angle $\phi_{12} = \phi_q^{h_1} - \phi_q^{h_2}$, where, as can be seen from Eq. (19), $\phi_q^{h_1}$ ($\phi_q^{h_2}$) is the azimuthal angle of hadron h_1 (h_2) around the direction of motion of the parent quark(antiquark). The subscripts U and L in the A, B coefficients refer to configurations where both lepton beams are unpolarized, that is $\mathcal{P}_+ = \mathcal{P}_- = 0$, or longitudinally polarized, $\mathcal{P}_+ = \pm 1$ and $\mathcal{P}_- = \pm 1$, respectively. Using also the results of Appendix C, the A coefficients read

$$\begin{aligned} A_U &= 2 \left\{ [1 + (1-y)^2][x_B^2 + (\xi - x_B)^2] \frac{1 - 2\zeta(1-\zeta)}{\zeta(1-\zeta)} + 16(1-y)x_B(\xi - x_B) \right\} f_{\gamma/\ell}(\xi), \\ A_U^{\cos\phi_q} &= -8(2-y)\sqrt{1-y}(\xi - 2x_B)\sqrt{x_B(\xi - x_B)} \frac{1 - 2\zeta}{\sqrt{\zeta(1-\zeta)}} f_{\gamma/\ell}(\xi), \\ A_U^{\cos 2\phi_q} &= 16(1-y)x_B(\xi - x_B) f_{\gamma/\ell}(\xi), \\ A_L &= -2y(2-y)\xi(\xi - 2x_B) \frac{1 - 2\zeta(1-\zeta)}{\zeta(1-\zeta)} \Delta_L f_{\gamma/\ell}(\xi), \\ A_L^{\cos\phi_q} &= 8y\sqrt{1-y}\xi\sqrt{x_B(\xi - x_B)} \frac{1 - 2\zeta}{\sqrt{\zeta(1-\zeta)}} \Delta_L f_{\gamma/\ell}(\xi). \end{aligned} \quad (25)$$

Similarly, for the B terms we obtain

$$\begin{aligned} B_U^{\cos\phi_{12}} &= \{[1 + (1-y)^2][x_B^2 + (\xi - x_B)^2] - 8(1-y)x_B(\xi - x_B)\} f_{\gamma/\ell}(\xi), \\ B_U^{\cos(\phi_q - \phi_{12})} &= -2(2-y)\sqrt{1-y}(\xi - 2x_B)\sqrt{x_B(\xi - x_B)} \sqrt{\frac{\zeta}{1-\zeta}} f_{\gamma/\ell}(\xi), \\ B_U^{\cos(\phi_q + \phi_{12})} &= 2(2-y)\sqrt{1-y}(\xi - 2x_B)\sqrt{x_B(\xi - x_B)} \sqrt{\frac{1-\zeta}{\zeta}} f_{\gamma/\ell}(\xi), \\ B_U^{\cos(2\phi_q - \phi_{12})} &= 2(1-y)x_B(\xi - x_B) \frac{\zeta}{1-\zeta} f_{\gamma/\ell}(\xi), \\ B_U^{\cos(2\phi_q + \phi_{12})} &= 2(1-y)x_B(\xi - x_B) \frac{1-\zeta}{\zeta} f_{\gamma/\ell}(\xi), \\ B_L^{\cos\phi_{12}} &= -y(2-y)\xi(\xi - 2x_B) \Delta_L f_{\gamma/\ell}(\xi), \end{aligned}$$

$$\begin{aligned}
B_L^{\cos(\phi_q - \phi_{12})} &= 2y\sqrt{1-y}\xi\sqrt{x_B(\xi - x_B)}\sqrt{\frac{\zeta}{1-\zeta}}\Delta_L f_{\gamma/\ell}(\xi), \\
B_L^{\cos(\phi_q + \phi_{12})} &= -2y\sqrt{1-y}\xi\sqrt{x_B(\xi - x_B)}\sqrt{\frac{1-\zeta}{\zeta}}\Delta_L f_{\gamma/\ell}(\xi).
\end{aligned} \tag{26}$$

In the above equations we have given all the expressions in terms of the variables x_B , y , ζ , and ξ in which the cross section is differential. Using the results of Appendix A one can easily find the same quantities in terms of the partonic Mandelstam variables. To collect more statistics one can first perform the change of variables $(\phi_q^{h_1}, \phi_q^{h_2}) \rightarrow (\phi_q^{h_1}, \phi_q^{h_1} - \phi_q^{h_2} \equiv \phi_{12})$, then integrate Eq. (24) over the azimuthal angle $\phi_q^{h_1}$ and the moduli of the intrinsic transverse momenta, $p_{\perp 1}$, and $p_{\perp 2}$. By using Eqs. (22) and (23), one finally gets

$$\begin{aligned}
\frac{d\sigma^{\ell^+\ell^-\rightarrow\ell'^+h_1h_2X}(\mathcal{P}_+, \mathcal{P}_-)}{dx_B dy d\zeta d\phi_q d\xi dz_1 dz_2 d\phi_{12}} &= \frac{3\alpha^3}{8\pi^2} \frac{1}{x_B y^2 \xi^3 s} \sum_q e_q^4 \left\{ \left[A_U + \mathcal{P}_+ \mathcal{P}_- A_L + \left(A_U^{\cos\phi_q} + \mathcal{P}_+ \mathcal{P}_- A_L^{\cos\phi_q} \right) \cos\phi_q \right. \right. \\
&\quad + A_U^{\cos 2\phi_q} \cos 2\phi_q \left. \right] D_q^{h_1}(z_1) D_{\bar{q}}^{h_2}(z_2) + \left[\left(B_U^{\cos\phi_{12}} + \mathcal{P}_+ \mathcal{P}_- B_L^{\cos\phi_{12}} \right) \cos\phi_{12} \right. \\
&\quad + \left(B_U^{\cos(\phi_q - \phi_{12})} + \mathcal{P}_+ \mathcal{P}_- B_L^{\cos(\phi_q - \phi_{12})} \right) \cos(\phi_q - \phi_{12}) \\
&\quad + \left(B_U^{\cos(\phi_q + \phi_{12})} + \mathcal{P}_+ \mathcal{P}_- B_L^{\cos(\phi_q + \phi_{12})} \right) \cos(\phi_q + \phi_{12}) + B_U^{\cos(2\phi_q - \phi_{12})} \cos(2\phi_q - \phi_{12}) \\
&\quad \left. \left. + B_U^{\cos(2\phi_q + \phi_{12})} \cos(2\phi_q + \phi_{12}) \right] \Delta^N D_q^{h_1}(z_1) \Delta^N D_{\bar{q}}^{h_2}(z_2) \right\}. \tag{27}
\end{aligned}$$

From Eqs. (24) and (27), and using the shorthand notation $d\sigma(\mathcal{P}_+, \mathcal{P}_-)$ for the differential cross section, we see that

$$\begin{aligned}
d\sigma(0, 0) &= d\sigma(1, 0) = d\sigma(0, 1) = d\sigma^{\text{unp}}, \\
d\sigma(1, 1) &= d\sigma(-1, -1), \\
d\sigma(1, -1) &= d\sigma(-1, 1).
\end{aligned} \tag{28}$$

Additionally,

$$\begin{aligned}
d\sigma^{\text{unp}} &= \frac{1}{4} [d\sigma(1, 1) + d\sigma(1, -1) + d\sigma(-1, 1) + d\sigma(-1, -1)] \\
&= \frac{1}{2} [d\sigma(1, 1) + d\sigma(1, -1)],
\end{aligned} \tag{29}$$

$$\Delta_L \sigma = d\sigma(1, 1) - d\sigma(1, -1) = d\sigma(-1, -1) - d\sigma(-1, +1). \tag{30}$$

We can therefore define the longitudinal spin asymmetry

$$A_{LL} = \frac{d\sigma(1, 1) - d\sigma(1, -1)}{d\sigma(1, 1) + d\sigma(1, -1)} = \frac{\Delta_L \sigma}{2d\sigma^{\text{unp}}}. \tag{31}$$

To isolate the factors associated with the different azimuthal modulations appearing in Eqs. (24), (27), and (31), it is common to define appropriate azimuthal moments of the unpolarized cross section $d\sigma^{\text{unp}}$ and the longitudinal spin asymmetry A_{LL} . In our case their general form can be given as follows:

$$\langle d\sigma^{\text{unp}} | n_q; m_{12} \rangle = 2 \frac{\int d\phi_q d\phi_{12} d\sigma^{\text{unp}}(\phi_q, \phi_{12}) \cos[n_q \phi_q + m_{12} \phi_{12}]}{\int d\phi_q d\phi_{12} d\sigma^{\text{unp}}(\phi_q, \phi_{12})}, \tag{32}$$

and

$$\langle A_{LL}|n_q; m_{12}\rangle = 2 \frac{\int d\phi_q d\phi_{12} A_{LL} d\sigma^{\text{unp}}(\phi_q, \phi_{12}) \cos[n_q \phi_q + m_{12} \phi_{12}]}{\int d\phi_q d\phi_{12} d\sigma^{\text{unp}}(\phi_q, \phi_{12})}, \quad (33)$$

where $n_q = 0, 1, 2$ and $m_{12} = 0, \pm 1$. Notice that in the case of Eq. (32) the trivial case $n_q = m_{12} = 0$ will not be considered anymore. The relevant azimuthal moments are summarized in Table I.

For clarity, in Table I we have simplified all the overall prefactors appearing in Eqs. (24) and (27); this is correct if we are considering the fully differential cross section, without any integration over the kinematical variables x_B , y , and ξ ; however, if we want to integrate over some of the variables in order to gather further statistics, we must explicitly reinstate the corresponding prefactors in the numerator and denominator of the moments. The azimuthal moments in the first three rows of Table I do not depend (at fixed z_1 and z_2) on the fragmentation functions, therefore they do not provide any information on the Collins FF. However, they can be useful in testing the approach as far as the initial state of the process is concerned. The azimuthal moments in the rest of Table I are the ones carrying information on the Collins FF. The complete prefactor ratios multiplying the ratio of products of (Collins and unpolarized) fragmentation functions can be separately integrated over some (or even all) of the other variables (x_B , y , ζ , ξ). In doing so, the kinematical constraints described in Appendix A must be taken into

account in order to stay in the regime of validity of the approach.

Similar reasoning applies also to the more differential expression of the cross section in Eq. (24), where the dependence on the moduli of the hadron transverse momenta with respect to the parent quarks(antiquarks) is still explicit.

Let us also recall that the two angles entering the azimuthal moments, ϕ_q and ϕ_{12} , are measured around different, well distinct, (light-cone) directions, the first one being the azimuthal angle of the q - $\bar{q}$ plane (relative to the leptonic plane) around the direction of motion of the two colliding photons in their c.m. frame, while the second one is the difference among the azimuthal angles of the two final hadrons around the parent quark(antiquark) direction of motion (coinciding with the jet thrust axis in our leading-order approach).

From the phenomenological point of view, it is important to stress that the ratio of combinations of Collins and unpolarized fragmentation functions appearing in Table I is the same one involved in the e^+e^- SIA process, $e^+e^- \rightarrow h_1 h_2 + X$, with the important difference that the fractional charge relative weight among different flavors is 16:1:1 in our case and 4:1:1 in the SIA case, respectively, for u , d , s flavors. Therefore, the analysis of the process considered

TABLE I. Summary of the relevant azimuthal moments of the unpolarized cross section $d\sigma^{\text{unp}}$ and of the longitudinal azimuthal asymmetry A_{LL} according to Eqs. (32) and (33).

n_q	m_{12}	$\langle d\sigma^{\text{unp}} n_q; m_{12}\rangle$	$\langle A_{LL} n_q; m_{12}\rangle$
0	0		$\frac{A_L}{A_U}$
± 1	0	$\frac{A_U^{\cos \phi_q}}{A_U}$	$\frac{A_L^{\cos \phi_q}}{A_U}$
± 2	0	$\frac{A_U^{\cos 2\phi_q}}{A_U}$	0
0	± 1	$\frac{B_U^{\cos \phi_{12}}}{A_U} \frac{\sum_q e_q^4 \Delta^N D_{q^\dagger}^{h_1} \Delta^N D_{\bar{q}^\dagger}^{h_2}}{\sum_q e_q^4 D_q^{h_1} D_{\bar{q}}^{h_2}}$	$\frac{B_L^{\cos \phi_{12}}}{A_U} \frac{\sum_q e_q^4 \Delta^N D_{q^\dagger}^{h_1} \Delta^N D_{\bar{q}^\dagger}^{h_2}}{\sum_q e_q^4 D_q^{h_1} D_{\bar{q}}^{h_2}}$
1	± 1	$\frac{B_U^{\cos(\phi_q \pm \phi_{12})}}{A_U} \frac{\sum_q e_q^4 \Delta^N D_{q^\dagger}^{h_1} \Delta^N D_{\bar{q}^\dagger}^{h_2}}{\sum_q e_q^4 D_q^{h_1} D_{\bar{q}}^{h_2}}$	$\frac{B_L^{\cos(\phi_q \pm \phi_{12})}}{A_U} \frac{\sum_q e_q^4 \Delta^N D_{q^\dagger}^{h_1} \Delta^N D_{\bar{q}^\dagger}^{h_2}}{\sum_q e_q^4 D_q^{h_1} D_{\bar{q}}^{h_2}}$
-1	∓ 1	"	"
2	± 1	$\frac{B_U^{\cos(2\phi_q \pm \phi_{12})}}{A_U} \frac{\sum_q e_q^4 \Delta^N D_{q^\dagger}^{h_1} \Delta^N D_{\bar{q}^\dagger}^{h_2}}{\sum_q e_q^4 D_q^{h_1} D_{\bar{q}}^{h_2}}$	0
-2	∓ 1	"	"

here would contribute to better disentangle the flavor dependence of the Collins FF, especially for u quarks. In fact, this is one of the main motivations for considering this observable in future photon-photon colliders. A second important benefit, once the already available full TMD scale evolution scheme will be implemented, is to study the scale dependence of the TMD Collins FF within the same process and experimental setup by varying the virtuality of the DIS photon.

IV. CONCLUSIONS

In this paper we have investigated the azimuthal asymmetries around the jet thrust axis of light-hadron pairs inclusively produced in the collision of two photons. To this aim, we have adopted the helicity formalism and assumed TMD factorization, which requires the two hadrons to be almost back to back and have large transverse momenta in the photon-photon c.m. frame. At the leading-order approximation considered here, the reaction proceeds through the channel $\ell^+\ell^- \rightarrow \gamma^*\gamma \rightarrow q\bar{q} \rightarrow h_1h_2 + X$, where one tagged, deeply virtual photon scatters off an untagged quasireal photon, both of them originating from the initial lepton beams.

The main motivation of the suggested measurements is to stimulate the study of TMD physics in photon-photon scattering at a time when future lepton colliders are starting to be discussed by the physics community. In particular, our proposal would certainly help in better determining quark TMD fragmentation functions, allowing for a more effective flavor separation than SIDIS or $\ell^+\ell^-$ annihilation processes, because the contributions of d , s , b quarks should be suppressed by a charge weight factor of $1/16$ (due to their fractional electric charge) with respect to those of u , c quarks. Another advantage is that, while $\ell^+\ell^-$ colliders operate at some fixed energy scale, that is the center-of-mass energy of the two leptons, in photon-photon scattering the perturbative energy scale, related to the photon virtualities, can be varied. One can therefore study the scale dependence of the TMD fragmentation function within the same process and experimental setup.

The azimuthal asymmetries discussed in the paper and their moments, summarized in Table I, involve (ratios of) the quark unpolarized and Collins TMD FFs, with different combinations of prefactors depending on the kinematics and polarization state of the initial photon-photon pair and on the dynamics of the hard process. Studying the relative weight of these prefactors in different DIS and SIDIS kinematical configurations is of interest by itself for the validation of the approach.

The same formalism could also be applied to photon-photon scattering in linear lepton colliders, where the photons are generated by Compton backscattering of laser beams off the initial leptons. It would also be interesting to consider the production of hadron pairs formed by either two spin-1/2 particles or by a spin zero (a pion or a kaon) and a spin-1/2 (e.g., a Λ hyperon) particle. This should

allow one to study the interesting phenomenon of spontaneous Λ polarization observed in different processes, like, e.g., proton-proton collisions, e^+e^- SIA processes, and specifically the role of quark flavor.

It still remains to be seen in detail whether the reachable luminosities in the photon-photon collision mode at lepton colliders will be sufficient for the proposed analysis, as well as the achievable range of center-of-mass energies. All these aspects should be carefully considered while developing proposals for future lepton colliders. To conclude, we note that similar processes, in a more complicated environment due to the presence of initial hadronic states, can also be studied in ultraperipheral collisions at the LHC and the planned future hadron colliders.

ACKNOWLEDGMENTS

This work is supported by the European Union “Next Generation EU” program through the Italian Progetti di ricerca di Rilevante Interesse Nazionale (PRIN 2022), Grant No. 20225ZHA7W.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: KINEMATICS

For completeness, we collect here the explicit expressions of the four-momenta of the particles involved in the process of interest, in the different reference frames considered. We also summarize several useful kinematical relations.

1. The lepton-beam or laboratory c.m. frame

We first consider the laboratory, or $\ell^+\ell^-$ c.m. frame, where the tagged lepton ℓ^+ moves along the $+\hat{z}$ axis and the untagged one ℓ^- , in the opposite direction; for massless leptons, we have

$$\begin{aligned} l_+ &= \frac{\sqrt{s}}{2}(1, 0, 0, 1), \\ l_- &= q_2/\xi = \frac{\sqrt{s}}{2}(1, 0, 0, -1), \\ l'_+ &= \frac{\sqrt{s}}{2}(1 - (1 - x_B)y, 2\sqrt{x_By(1-y)}, 0, 1 - (1 + x_B)y), \\ q_1 &= \frac{\sqrt{s}}{2}((1 - x_B)y, -2\sqrt{x_By(1-y)}, 0, (1 + x_B)y), \end{aligned} \quad (\text{A1})$$

where in the second line ξ is the light-cone momentum fraction of the quasireal photon γ_2 inside the untagged lepton ℓ^- . Without loss of generality, we have chosen the leptonic plane as the x - z plane of our reference frame. If necessary, one can easily reinstate the most general dependence on the azimuthal angle of the leptonic plane ϕ_ℓ , with respect to the x - z plane of a general arbitrary frame. Let us also remind the reader of the definition of the usual kinematical invariants adopted in deeply inelastic scattering,

$$s = (l_+ + l_-)^2 = 2l_+ \cdot l_-, \quad Q^2 = -q_1^2 = -(l_+ - l'_+)^2 = 2l_+ \cdot l'_+,$$

$$x_B = \frac{Q^2}{2l_- \cdot q_1}, \quad y = \frac{l_- \cdot q_1}{l_- \cdot l_+}, \quad Q^2 = x_B y s. \quad (\text{A2})$$

By defining as K_q and $K_{\bar{q}}$ the four-momenta of the quark and antiquark produced in the two-photon collision ($q = u, \bar{u}, d, \bar{d}, s, \bar{s}$), we can also introduce two additional invariants,

$$\zeta_q = \frac{K_q \cdot l_-}{q_1 \cdot l_-}, \quad \zeta_{\bar{q}} = \frac{K_{\bar{q}} \cdot l_-}{q_1 \cdot l_-}. \quad (\text{A3})$$

Notice that we neglect light quark masses, therefore $K_q^2 = K_{\bar{q}}^2 = 0$.

2. The virtual photon–untagged lepton $\gamma_1^* \ell^-$ c.m. frame

This frame is analogous to the γ^* -nucleon reference frame usually adopted in semi-inclusive deeply inelastic

scattering processes, with the untagged lepton ℓ^- and the quasireal photon γ_2 playing, respectively, the role of the target nucleon and the collinear struck parton. In this frame γ_1^* moves along the $+\hat{z}$ axis, and the untagged lepton ℓ^- and γ_2 in the opposite direction. The c.m. energy, usually named W , is given by

$$W^2 = (q_1 + l_-)^2 = (1 - x_B)ys = \frac{1 - x_B}{x_B} Q^2. \quad (\text{A4})$$

The four-momenta of the particles involved are

$$l_+ = \frac{\sqrt{s}}{2} \frac{1}{\sqrt{(1 - x_B)y}} (1 - x_B y, 2\sqrt{x_B(1 - x_B)(1 - y)}, 0, 1 - 2x_B + x_B y),$$

$$l_- = q_2/\xi = \frac{\sqrt{s}}{2} \frac{1}{\sqrt{(1 - x_B)y}} (y, 0, 0, -y),$$

$$l'_+ = \frac{\sqrt{s}}{2} \frac{1}{\sqrt{(1 - x_B)y}} (1 - y + x_B y, 2\sqrt{x_B(1 - x_B)(1 - y)}, 0, 1 - y - 2x_B + x_B y),$$

$$q_1 = \frac{\sqrt{s}}{2} \frac{1}{\sqrt{(1 - x_B)y}} ((1 - 2x_B)y, 0, 0, y). \quad (\text{A5})$$

The four-momenta of the final quark and antiquark can be written as

$$K_q = K_T (\cosh \eta_q, \cos \phi_q, \sin \phi_q, \sinh \eta_q),$$

$$K_{\bar{q}} = K_T (\cosh \eta_{\bar{q}}, -\cos \phi_q, -\sin \phi_q, \sinh \eta_{\bar{q}}), \quad (\text{A6})$$

where we have introduced the pseudorapidities of the quark and antiquark, η_q and $\eta_{\bar{q}}$, that in the massless limit coincide with their rapidities.

It is easy then to verify that in this frame the invariants $\zeta_{q,\bar{q}}$ introduced in Eq. (A3) are given by

$$\zeta_q = \frac{k_T}{W} e^{\eta_q}, \quad \zeta_{\bar{q}} = \frac{k_T}{W} e^{\eta_{\bar{q}}}. \quad (\text{A7})$$

3. The virtual photon–quasireal photon $\gamma_1^* \gamma_2$ c.m. frame

This is the c.m. frame of the hard partonic scattering process $\gamma_1^* \gamma_2 \rightarrow q\bar{q}$, again with γ_1^* moving along the $+\hat{z}$ axis and the untagged lepton and γ_2 moving in the opposite direction. The final quark and antiquark are produced back to back in a plane forming an angle ϕ_q with the $\ell^+ \ell^-$ leptonic plane, that we have assumed to be the $\hat{x}\text{--}\hat{z}$ plane. This frame is related to the previous one in Appendix A2 by a Lorentz boost along the $\hat{z}$ axis specified by the Lorentz factors

$$\beta = \frac{1 - \xi}{1 - 2x_B + \xi}, \quad \gamma = \frac{1 - 2x_B + \xi}{2\sqrt{(1 - x_B)(\xi - x_B)}}. \quad (\text{A8})$$

The squared c.m. energy in this frame is

$$\hat{s} = (q_1 + q_2)^2 = (\xi - x_B)ys = \frac{\xi - x_B}{x_B} Q^2, \quad (\text{A9})$$

which also implies the constraint $\xi > x_B$. The four-momenta of the particles involved are

$$\begin{aligned} l_+ &= \frac{\sqrt{s}}{2} \frac{1}{\sqrt{(\xi - x_B)y}} (\xi - x_B y, 2\sqrt{x_B(\xi - x_B)(1 - y)}, 0, \xi - 2x_B + x_B y), \\ l_- &= q_2/\xi = \frac{\sqrt{s}}{2} \sqrt{\frac{y}{\xi - x_B}} (1, 0, 0, -1), \\ l'_+ &= \frac{\sqrt{s}}{2} \frac{1}{\sqrt{(\xi - x_B)y}} (\xi - \xi y + x_B y, 2\sqrt{x_B(\xi - x_B)(1 - y)}, 0, \xi - \xi y - 2x_B + x_B y), \\ q_1 &= \frac{\sqrt{s}}{2} \sqrt{\frac{y}{\xi - x_B}} (\xi - 2x_B, 0, 0, \xi). \end{aligned} \quad (\text{A10})$$

As for the final quark-antiquark pair, we can write in general

$$K_{q,\bar{q}} = \frac{\sqrt{\hat{s}}}{2} (1, \pm \sin \theta_q \cos \phi_q, \pm \sin \theta_q \sin \phi_q, \pm \cos \theta_q), \quad (\text{A11})$$

where θ_q and ϕ_q are, respectively, the polar and azimuthal angle of the quark direction of motion in the $\gamma_1^* - \gamma_2$ c.m. frame. It is again more convenient to consider the modulus of the $q, \bar{q}$ transverse momentum, K_T , and their pseudorapidities $\hat{\eta}_{q,\bar{q}}$. In fact, the first one is invariant, while the second ones are simply additive under boosts along the $\hat{z}$ axis. In the $\gamma_1^* - \gamma_2$ frame (which is also the $q - \bar{q}$ c.m. frame) we clearly have $\hat{\eta}_q = -\hat{\eta}_{\bar{q}} = \hat{\eta}$. We can therefore write the quark and antiquark four-momenta as

$$K_{q,\bar{q}} = K_T (\cosh \hat{\eta}, \pm \cos \phi_q, \pm \sin \phi_q, \pm \sinh \hat{\eta}), \quad (\text{A12})$$

and the transverse and longitudinal components of the quark three-momentum with respect to the $\hat{z}$ axis are

$$K_T = \frac{\sqrt{\hat{s}}}{2} \frac{1}{\cosh \hat{\eta}}, \quad K_L = \frac{\sqrt{\hat{s}}}{2} \tanh \hat{\eta}. \quad (\text{A13})$$

From this relation we can also see another useful property,

$$\xi = x_B + \frac{4K_T^2 \cosh^2 \hat{\eta}}{ys}. \quad (\text{A14})$$

Using Eq. (A8) it is easy to see that, moving back to the $\gamma_1^* - \ell^-$ c.m. frame, by an inverse Lorentz boost with rapidity

$$y_b = \frac{1}{2} \log \left(\frac{\xi - x_B}{1 - x_B} \right) = \log \left(\frac{\sqrt{\hat{s}}}{W} \right), \quad (\text{A15})$$

the pseudorapidities of the quark and antiquark will become

$$\eta_q = \hat{\eta} - y_b, \quad \eta_{\bar{q}} = -\hat{\eta} - y_b, \quad (\text{A16})$$

and

$$\begin{aligned} \eta_q + \eta_{\bar{q}} &= -2y_b = \log \left(\frac{1 - x_B}{\xi - x_B} \right) = \log \left(\frac{W^2}{\hat{s}} \right), \\ \eta_q - \eta_{\bar{q}} &= 2\hat{\eta}. \end{aligned} \quad (\text{A17})$$

Following Ref. [37], we can also write the invariant variables $\zeta_{q,\bar{q}}$ introduced in Eq. (A3) in terms of $\eta_{q,\bar{q}}$,

$$\begin{aligned} \zeta_{\bar{q}} \equiv \zeta &= \frac{1}{e^{\eta_q - \eta_{\bar{q}}} + 1} = \frac{e^{-\hat{\eta}}}{2 \cosh \hat{\eta}} = \frac{K_T}{\sqrt{\hat{s}}} e^{-\hat{\eta}}, \\ \zeta_q \equiv 1 - \zeta &= \frac{1}{e^{\eta_{\bar{q}} - \eta_q} + 1} = \frac{e^{\hat{\eta}}}{2 \cosh \hat{\eta}} = \frac{K_T}{\sqrt{\hat{s}}} e^{\hat{\eta}}, \end{aligned} \quad (\text{A18})$$

from which it is also clear that $\zeta_q + \zeta_{\bar{q}} = 1$, and in the last equality we have used Eq. (A13).

Using these results, we can also write

$$\begin{aligned} K_{q,\bar{q}} &= \frac{1}{2} \sqrt{(\xi - x_B)ys} \left(1, \pm 2\sqrt{\zeta(1 - \zeta)} \cos \phi_q, \right. \\ &\quad \left. \pm 2\sqrt{\zeta(1 - \zeta)} \sin \phi_q, \pm (1 - 2\zeta) \right). \end{aligned} \quad (\text{A19})$$

It is also useful to give the relation between these invariant variables and the Mandelstam invariants in the two-photon c.m. frame,

$$\hat{s} = (\xi - x_B)ys, \quad \hat{t} = -\zeta \xi ys, \quad \hat{u} = -(1 - \zeta) \xi ys. \quad (\text{A20})$$

Let us also remind the reader that there are a few additional kinematical constraints that must be fulfilled in order to guarantee the validity of the formalism adopted: (1) First of all, to stay in the deeply virtual and factorization regime we require $Q^2 \geq Q_0^2 \gg \Lambda_{\text{QCD}}^2$, with Q_0^2 an arbitrary fixed hard scale of at least a few GeV^2 . (2) Second, in order to clearly distinguish the two light-cone directions

considered, the $\gamma^*\gamma$ axis and the jet thrust axis, and keep staying in the validity regime of the factorization approach adopted, we also require $|\mathbf{K}_{qT}| = K_T \geq Q_0$. From Eq. (A18) it is also easy to see that

$$K_T^2 = \zeta(1 - \zeta)\hat{s}. \quad (\text{A21})$$

Therefore, putting a lower limit on the value of K_T , as required for the applicability of our factorization scheme, implies a lower cut on $\hat{s}$ and a limited range of allowed values for ζ : Since $\text{Max}[\zeta(1 - \zeta)] = 1/4$, $\hat{s} \geq 4Q_0^2$ and, at fixed $\hat{s}$,

$$\frac{1}{2} \left\{ 1 - \sqrt{1 - \frac{4Q_0^2}{\hat{s}}} \right\} \leq \zeta \leq \frac{1}{2} \left\{ 1 + \sqrt{1 - \frac{4Q_0^2}{\hat{s}}} \right\}. \quad (\text{A22})$$

For a generic $Q^2 \geq Q_0^2$, requiring that $\hat{s} = (\xi - x_B)Q^2 / x_B \geq 4Q_0^2$ implies

$$\xi \geq x_B \left(1 + \frac{4Q_0^2}{Q^2} \right). \quad (\text{A23})$$

Since $\xi \leq 1$, one finds

$$x_B \leq x_B^{\max}(Q^2) = \frac{Q^2}{Q^2 + 4Q_0^2} < 1. \quad (\text{A24})$$

Given that $x_B^{\min}(Q^2) = Q^2/s$ (for $y = 1$), imposing that $x_B^{\min}(Q^2) \leq x_B^{\max}(Q^2)$, one also finds an upper limit for Q^2 , $Q^2 \leq s - 4Q_0^2$. Furthermore,

$$y^{\min}(Q^2) = \frac{Q^2}{x_B^{\max}(Q^2)s} = \frac{Q^2 + 4Q_0^2}{s}, \quad (\text{A25})$$

and, from Eq. (A23),

$$\xi^{\min}(Q^2) = x_B^{\min}(Q^2) \left(1 + \frac{4Q_0^2}{Q^2} \right) = y^{\min}(Q^2). \quad (\text{A26})$$

Summarizing, then, we have the constraints

$$\begin{aligned} Q_0^2 &\leq Q^2 \leq s - 4Q_0^2, \\ \frac{Q^2}{s} &\leq x_B \leq \frac{Q^2}{Q^2 + 4Q_0^2}, \\ \frac{Q^2 + 4Q_0^2}{s} &\leq y, \xi \leq 1. \end{aligned} \quad (\text{A27})$$

APPENDIX B: THE VIRTUAL PHOTON HELICITY DENSITY MATRIX IN DIS PROCESSES

The derivation of the virtual photon helicity density matrix was given in detail, although with a notation different from the one adopted in this paper, in Ref. [38]. It was also briefly discussed, within the same notation adopted here, in Ref. [39]. We therefore believe it can be useful to summarize

here the main steps leading to the explicit expression of Eq. (12).

Let us start from the helicity amplitudes for a generic process in which a lepton ℓ interacts with a given initial system M by exchanging a single virtual photon, $\ell(l, \lambda_\ell) + M(P_i, \Lambda_i) \rightarrow \ell'(l', \lambda_{\ell'}) + M(P_f, \Lambda_f)$. For shortness, $P_{i,f}, \Lambda_{i,f}$ represents the full set of initial and final moments and helicities of M . This system can be, e.g., a second lepton, like in our case, or a proton/nucleon target like in SIDIS.

The helicity amplitude for this process can be written as follows:

$$\begin{aligned} H_{\lambda_{\ell'}, \Lambda_f; \lambda_\ell, \Lambda_i} &= e \bar{u}_{\lambda_{\ell'}} \gamma^\mu u_{\lambda_\ell} \left(-i \frac{g_{\mu\nu}}{q^2} \right) M_{\Lambda_f; \Lambda_i}^\nu(P_f, P_i) \\ &= e \bar{u}_{\lambda_{\ell'}} \gamma^\mu u_{\lambda_\ell} \left[-i \frac{g_{\mu\nu}}{q^2} + \frac{q_\mu q_\nu}{q^4} \right] M_{\Lambda_f; \Lambda_i}^\nu(P_f, P_i) \\ &= e \frac{1}{q^2} \bar{u}_{\lambda_{\ell'}} \gamma^\mu u_{\lambda_\ell} \left[\sum_{\lambda_\gamma} (-1)^{\lambda_\gamma+1} \epsilon_{\mu, \lambda_\gamma}^*(q) \epsilon_{\nu, \lambda_\gamma}(q) \right] \\ &\quad \times M_{\Lambda_f; \Lambda_i}^\nu(P_f, P_i), \end{aligned} \quad (\text{B1})$$

where in the second line we have used Ward identity and in the third line well-known properties of the polarization vectors of a virtual photon.

At the same time, we can separately define the helicity amplitudes for the processes $\ell(l, \lambda_\ell) \rightarrow \ell'(l', \lambda_{\ell'}) + \gamma^*(q, \lambda_\gamma)$,

$$\mathcal{H}_{\lambda_{\ell'}, \lambda_\gamma; \lambda_\ell} = e \bar{u}_{\lambda_{\ell'}} \gamma^\mu u_{\lambda_\ell} \epsilon_{\mu, \lambda_\gamma}^*(q), \quad (\text{B2})$$

and $\gamma^*(q, \lambda_\gamma) + M(P_i, \Lambda_i) \rightarrow M(P_f, \Lambda_f)$,

$$\mathcal{H}_{\Lambda_f; \lambda_\gamma, \Lambda_i} = \epsilon_{\nu, \lambda_\gamma}(q) M_{\Lambda_f; \Lambda_i}^\nu(P_f, P_i). \quad (\text{B3})$$

Let us now consider the differential cross section for the full process $\ell(l, \lambda_\ell) + M(P_i, \Lambda_i) \rightarrow \ell'(l', \lambda_{\ell'}) + M(P_f, \Lambda_f)$. Assuming in general that the initial lepton ℓ is in a polarization state described by its helicity density matrix $\rho_{\lambda_\ell, \lambda'_\ell}^{(s_\ell)}$, where s_ℓ specifies the lepton ℓ spin state, and using Eqs. (B1)–(B3), we can write

$$\begin{aligned} d\sigma &\propto \sum_{\lambda_\ell, \lambda'_\ell, \lambda_{\ell'}} \sum_{\Lambda_i, \Lambda_f} \rho_{\lambda_\ell, \lambda'_\ell}^{(s_\ell)} H_{\lambda_{\ell'}, \Lambda_f; \lambda_\ell, \Lambda_i} H_{\lambda_{\ell'}, \Lambda_f; \lambda'_\ell, \Lambda_i}^* \\ &= \sum_{\lambda_\ell, \lambda'_\ell, \lambda_{\ell'}} \sum_{\Lambda_i, \Lambda_f} \rho_{\lambda_\ell, \lambda'_\ell}^{(s_\ell)} \left[\frac{1}{q^2} \sum_{\lambda_\gamma} (-1)^{\lambda_\gamma+1} \mathcal{H}_{\lambda_{\ell'}, \lambda_\gamma; \lambda_\ell} \mathcal{H}_{\Lambda_f; \lambda_\gamma, \Lambda_i} \right] \\ &\quad \times \left[\frac{1}{q^2} \sum_{\lambda'_\gamma} (-1)^{\lambda'_\gamma+1} \mathcal{H}_{\lambda_{\ell'}, \lambda'_\gamma; \lambda'_\ell}^* \mathcal{H}_{\Lambda_f; \lambda'_\gamma, \Lambda_i}^* \right] \\ &= \sum_{\lambda_\gamma, \lambda'_\gamma} \tilde{\rho}_{\lambda_\gamma, \lambda'_\gamma}(\gamma^*) \sum_{\Lambda_i, \Lambda_f} \mathcal{H}_{\Lambda_f; \lambda_\gamma, \Lambda_i} \mathcal{H}_{\Lambda_f; \lambda'_\gamma, \Lambda_i}^*, \end{aligned} \quad (\text{B4})$$

where we have defined the (non-normalized) helicity density matrix for the virtual photon γ^* ,

$$\tilde{\rho}_{\lambda_\gamma, \lambda'_\gamma}(\gamma^*) = \frac{1}{q^4} (-1)^{\lambda_\gamma + \lambda'_\gamma} \sum_{\lambda_\ell, \lambda'_\ell, \lambda_{\ell'}} \rho_{\lambda_\ell, \lambda'_\ell}^{\ell(s_\ell)} \mathcal{H}_{\lambda_{\ell'}, \lambda_\ell; \lambda_\gamma} \mathcal{H}_{\lambda_{\ell'}, \lambda'_\ell; \lambda'_\gamma}^* \quad (\text{B5})$$

This is the expression we were looking for.

The helicity density matrix for the initial lepton can be written in general as

$$\rho_{\lambda_\ell, \lambda'_\ell}^{\ell(s_\ell)} = \frac{1}{2} (I + \mathcal{P}^\ell \cdot \boldsymbol{\sigma}) = \frac{1}{2} \begin{pmatrix} 1 + \mathcal{P}_z^\ell & \mathcal{P}_x^\ell + i\mathcal{P}_y^\ell \\ \mathcal{P}_x^\ell - i\mathcal{P}_y^\ell & 1 - \mathcal{P}_z^\ell \end{pmatrix}, \quad (\text{B6})$$

where $\mathcal{P}^\ell$ is the polarization (pseudo)vector of the lepton. In this paper, we will only consider the case of unpolarized

or fully longitudinally polarized lepton beams ($\mathcal{P}_z^\ell = 0$ or ± 1 , respectively), taking $\mathcal{P}_x^\ell = \mathcal{P}_y^\ell = 0$.

Inserting Eqs. (B2) and (B6) into Eq. (B5) and performing some traces we finally get

$$\begin{aligned} \tilde{\rho}_{\lambda_\gamma, \lambda'_\gamma}(\gamma^*) &= \frac{e^2}{q^4} (-1)^{\lambda_\gamma + \lambda'_\gamma} [4l \cdot \epsilon_{\lambda'_\gamma}^*(q) l \cdot \epsilon_{\lambda_\gamma}(q) \\ &\quad + q^2 \epsilon_{\lambda'_\gamma}^*(q) \cdot \epsilon_{\lambda_\gamma}(q) \\ &\quad + 2i\mathcal{P}_z^\ell \epsilon^{\alpha\beta\mu\nu} l_\alpha \epsilon_{\beta, \lambda_\gamma}(q) l_\mu \epsilon_{\nu, \lambda'_\gamma}(q)]. \end{aligned} \quad (\text{B7})$$

By assuming that the virtual photon moves along the positive $\hat{z}$ axis of our reference frame, and the leptonic ℓ - ℓ' plane forms an angle ϕ_ℓ with the $\hat{x}$ - $\hat{z}$ plane, and using the results of Appendix A in the γ^* - ℓ_2 frame, we get

$$\begin{aligned} l &= \frac{\sqrt{s}}{2} \frac{1}{\sqrt{(1-x_B)y}} (1-x_B y, 2\sqrt{x_B(1-x_B)(1-y)} \cos \phi_\ell, 2\sqrt{x_B(1-x_B)(1-y)} \sin \phi_\ell, 1-2x_B+x_B y) \\ q &= \frac{\sqrt{s}}{2} \frac{1}{\sqrt{(1-x_B)y}} ((1-2x_B)y, 0, 0, y). \end{aligned} \quad (\text{B8})$$

Furthermore, one has

$$\begin{aligned} \epsilon_{\lambda_\gamma=\pm 1}(q) &= \frac{1}{\sqrt{2}} (0, -\lambda_\gamma, -i, 0) \\ \epsilon_{\lambda_\gamma=0}(q) &= \frac{1}{2} \frac{1}{\sqrt{x_B(1-x_B)}} (1, 0, 0, 1-2x_B). \end{aligned} \quad (\text{B9})$$

Inserting these expressions into Eq. (B7), after some straightforward calculation we finally get the result of Eq. (13).

Notice that the normalized helicity density matrix of the virtual photon is entirely given in terms of the DIS invariant y . The only kinematical frame-dependent quantity is the azimuthal angle ϕ_ℓ of the leptonic plane, which is, however, invariant under boosts along the $\hat{z}$ axis.

Let us finally observe that the virtual photon helicity density matrix is directly related to the usual leptonic tensor $L^{\mu\nu}$.

APPENDIX C: HELICITY AMPLITUDES FOR THE PROCESS $\gamma_1^*(\lambda_1)\gamma_2(\lambda_2) \rightarrow q(\lambda_q)\bar{q}(\lambda_{\bar{q}})$

We list here for reference the nonvanishing helicity amplitudes $\hat{H}_{\lambda_q, \lambda_{\bar{q}}; \lambda_1, \lambda_2}$ for the process $\gamma_1^*(\lambda_1)\gamma_2(\lambda_2) \rightarrow q(\lambda_q)\bar{q}(\lambda_{\bar{q}})$, for real photon γ_2 and massless quarks, in terms of Q^2 , the virtuality of photon 1, the partonic Mandelstam invariants $\hat{s}, \hat{t}, \hat{u}$ and the azimuthal angle ϕ_q between the q - $\bar{q}$ plane and the leptonic plane for the virtual photon 1 (assumed to be, without loss of generality, the $\hat{x}$ - $\hat{z}$ plane). We also give their expression in terms of the invariants x_B, ξ , and ζ , that have been used in order to evaluate the $A_{U,L}$ and $B_{U,L}$ factors in Eqs. (25) and (26), respectively,

$$\begin{aligned} \hat{H}_{+-;1,1} &= -\hat{H}_{-+;-1,-1} = -2\sqrt{3}e^2e_q^2 \frac{Q^2}{\hat{s}+Q^2} \sqrt{\frac{\hat{u}}{\hat{t}}} = -2\sqrt{3}e^2e_q^2 \frac{x_B}{\xi} \sqrt{\frac{1-\zeta}{\zeta}}, \\ \hat{H}_{+-;1,-1} &= -\hat{H}_{-+;-1,1}^* = -2\sqrt{3}e^2e_q^2 e^{i2\phi_q} \frac{\hat{s}}{\hat{s}+Q^2} \sqrt{\frac{\hat{u}}{\hat{t}}} = -2\sqrt{3}e^2e_q^2 e^{i2\phi_q} \frac{\xi-x_B}{\xi} \sqrt{\frac{1-\zeta}{\zeta}}, \\ \hat{H}_{+-;-1,1} &= -\hat{H}_{-+;1,-1}^* = 2\sqrt{3}e^2e_q^2 e^{-i2\phi_q} \frac{\hat{s}}{\hat{s}+Q^2} \sqrt{\frac{\hat{t}}{\hat{u}}} = \sqrt{3}e^2e_q^2 e^{-i2\phi_q} \frac{\xi-x_B}{\xi} \sqrt{\frac{\zeta}{1-\zeta}}, \\ \hat{H}_{+-;-1,-1} &= -\hat{H}_{-+;1,1} = 2\sqrt{3}e^2e_q^2 \frac{Q^2}{\hat{s}+Q^2} \sqrt{\frac{\hat{t}}{\hat{u}}} = 2\sqrt{3}e^2e_q^2 \frac{x_B}{\xi} \sqrt{\frac{\zeta}{1-\zeta}}, \\ \hat{H}_{+-;0,\pm 1} &= -\hat{H}_{-+;0,\pm 1} = \pm 2\sqrt{6}e^2e_q^2 e^{\mp i\phi_q} \frac{\sqrt{\hat{s}}Q}{\hat{s}+Q^2} = \pm 2\sqrt{6}e^2e_q^2 e^{\mp i\phi_q} \frac{\sqrt{x_B(\xi-x_B)}}{\xi}. \end{aligned} \quad (\text{C1})$$

- [1] R. Boussarie *et al.*, TMD handbook, [arXiv:2304.03302](#).
- [2] X.-d. Ji, J.-P. Ma, and F. Yuan, QCD factorization for spin-dependent cross sections in DIS and Drell-Yan processes at low transverse momentum, *Phys. Lett. B* **597**, 299 (2004).
- [3] X.-d. Ji, J.-p. Ma, and F. Yuan, QCD factorization for semi-inclusive deep-inelastic scattering at low transverse momentum, *Phys. Rev. D* **71**, 034005 (2005).
- [4] J. Collins, *Foundations of Perturbative QCD* (Cambridge University Press, Cambridge, England, 2011), Vol. 32.
- [5] M. G. Echevarria, A. Idilbi, and I. Scimemi, Factorization theorem for Drell-Yan at low q_T and transverse momentum distributions on-the-light-cone, *J. High Energy Phys.* **07** (2012) 002.
- [6] P. J. Mulders and R. D. Tangerman, The complete tree-level result up to order $1/Q$ for polarized deep-inelastic lepton production, *Nucl. Phys.* **B461**, 197 (1996).
- [7] A. Bacchetta, M. Diehl, K. Goeke, A. Metz, P. J. Mulders, and M. Schlegel, Semi-inclusive deep inelastic scattering at small transverse momentum, *J. High Energy Phys.* **02** (2007) 093.
- [8] M. Anselmino, M. Boglione, U. D'Alesio, S. Melis, F. Murgia, E. R. Nocera, and A. Prokudin, General helicity formalism for polarized semi-inclusive deep inelastic scattering, *Phys. Rev. D* **83**, 114019 (2011).
- [9] S. Arnold, A. Metz, and M. Schlegel, Dilepton production from polarized hadron hadron collisions, *Phys. Rev. D* **79**, 034005 (2009).
- [10] D. Boer, R. Jakob, and P. J. Mulders, Asymmetries in polarized hadron production in e^+e^- annihilation up to order $1/Q$, *Nucl. Phys.* **B504**, 345 (1997).
- [11] U. D'Alesio, F. Murgia, and M. Zacccheddu, General helicity formalism for two-hadron production in e^+e^- annihilation within a TMD approach, *J. High Energy Phys.* **10** (2021) 078.
- [12] D. Boer, W. J. den Dunnen, C. Pisano, M. Schlegel, and W. Vogelsang, Linearly polarized gluons and the Higgs transverse momentum distribution, *Phys. Rev. Lett.* **108**, 032002 (2012).
- [13] P. Sun, B.-W. Xiao, and F. Yuan, Gluon distribution functions and Higgs boson production at moderate transverse momentum, *Phys. Rev. D* **84**, 094005 (2011).
- [14] J.-W. Qiu, M. Schlegel, and W. Vogelsang, Probing gluonic spin-orbit correlations in photon pair production, *Phys. Rev. Lett.* **107**, 062001 (2011).
- [15] F. Scarpa, D. Boer, M. G. Echevarria, J.-P. Lansberg, C. Pisano, and M. Schlegel, Studies of gluon TMDs and their evolution using quarkonium-pair production at the LHC, *Eur. Phys. J. C* **80**, 87 (2020).
- [16] W. J. den Dunnen, J. P. Lansberg, C. Pisano, and M. Schlegel, Accessing the transverse dynamics and polarization of gluons inside the proton at the LHC, *Phys. Rev. Lett.* **112**, 212001 (2014).
- [17] D. Boer and C. Pisano, Polarized gluon studies with charmonium and bottomonium at LHCb and AFTER, *Phys. Rev. D* **86**, 094007 (2012).
- [18] N. Kato, L. Maxia, and C. Pisano, Spin asymmetries for C-even quarkonium production as a probe of gluon distributions, *Phys. Rev. D* **110**, 034038 (2024).
- [19] T. C. Rogers and P. J. Mulders, No generalized TMD-factorization in hadro-production of high transverse momentum hadrons, *Phys. Rev. D* **81**, 094006 (2010).
- [20] V. M. Budnev, I. F. Ginzburg, G. V. Meledin, and V. G. Serbo, The two photon particle production mechanism. Physical problems. Applications. Equivalent photon approximation, *Phys. Rep.* **15**, 181 (1975).
- [21] I. F. Ginzburg, G. L. Kotkin, V. G. Serbo, and V. I. Telnov, Colliding ge and gg beams based on the single pass e^+e^- colliders (VLEPP type). *Nucl. Instrum. Methods Phys. Res.* **205**, 47 (1983).
- [22] I. F. Ginzburg, G. L. Kotkin, S. L. Panfil, V. G. Serbo, and V. I. Telnov, Colliding γe and $\gamma\gamma$ beams based on the single pass e^+e^- accelerators. 2. Polarization effects. Monochromatization improvement, *Nucl. Instrum. Methods Phys. Res., Sect. A* **219**, 5 (1984).
- [23] H. Kolanoski, *Two-Photon Physics at e^+e^- Storage Rings* (Springer-Verlag, Berlin, 1984), Vol. 105.
- [24] L. Schoeffel, C. Baldenegro, H. Hamdaoui, S. Hassani, C. Royon, and M. Saimpert, Photon-photon physics at the LHC and laser beam experiments, present and future, *Prog. Part. Nucl. Phys.* **120**, 103889 (2021).
- [25] C. F. v. Weizsäcker, Radiation emitted in collisions of very fast electrons, *Z. Phys.* **88**, 612 (1934).
- [26] E. J. Williams, Nature of the high-energy particles of penetrating radiation and status of ionization and radiation formulae, *Phys. Rev.* **45**, 729 (1934).
- [27] J. C. Collins and A. Metz, Universality of soft and collinear factors in hard-scattering factorization, *Phys. Rev. Lett.* **93**, 252001 (2004).
- [28] F. Yuan, Azimuthal asymmetric distribution of hadrons inside a jet at hadron collider, *Phys. Rev. Lett.* **100**, 032003 (2008).
- [29] L. P. Gamberg, A. Mukherjee, and P. J. Mulders, A model independent analysis of gluonic pole matrix elements and universality of TMD fragmentation functions, *Phys. Rev. D* **83**, 071503 (2011).
- [30] J. C. Collins and D. E. Soper, Back-to-back jets in QCD, *Nucl. Phys.* **B193**, 381 (1981).
- [31] J. C. Collins and D. E. Soper, Back-to-back jets: Fourier transform from b to k_T , *Nucl. Phys.* **B197**, 446 (1982).
- [32] J. C. Collins, D. E. Soper, and G. Sterman, Transverse momentum distribution in Drell-Yan pair and W and Z boson production, *Nucl. Phys.* **B250**, 199 (1985).
- [33] M. G. Echevarria, A. Idilbi, A. Schäfer, and I. Scimemi, Model-independent evolution of transverse momentum dependent distribution functions (TMDs) at NNLL, *Eur. Phys. J. C* **73**, 2636 (2013).
- [34] M. G. Echevarria, A. Idilbi, and I. Scimemi, Soft and collinear factorization and transverse momentum dependent parton distribution functions, *Phys. Lett. B* **726**, 795 (2013).
- [35] U. D'Alesio, L. Gamberg, F. Murgia, and M. Zacccheddu, Transverse Λ polarization in e^+e^- processes within a TMD factorization approach and the polarizing fragmentation function, *J. High Energy Phys.* **12** (2022) 074.
- [36] U. D'Alesio, L. Gamberg, F. Murgia, and M. Zacccheddu, Transverse Λ polarization in e^+e^- annihilations and in SIDIS processes at the EIC within TMD factorization, *Phys. Rev. D* **108**, 094004 (2023).

- [37] C. Pisano, D. Boer, S. J. Brodsky, M. G. A. Buffing, and P. J. Mulders, Linear polarization of gluons and photons in unpolarized collider experiments, *J. High Energy Phys.* **10** (2013) 024.
- [38] K. Schilling and G. Wolf, How to analyze vector meson production in inelastic lepton scattering, *Nucl. Phys.* **B61**, 381 (1973).
- [39] M. Anselmino, M. Bertini, F. Murgia, and B. Pire, Off diagonal helicity density matrix elements for heavy vector mesons inclusively produced in NN , γN and ℓN interactions, *Phys. Lett. B* **438**, 347 (1998).
- [40] M. Anselmino, M. Boglione, U. D'Alesio, E. Leader, S. Melis, and F. Murgia, The general partonic structure for hadronic spin asymmetries, *Phys. Rev. D* **73**, 014020 (2006).
- [41] J. C. Collins, Fragmentation of transversely polarized quarks probed in transverse momentum distributions, *Nucl. Phys.* **B396**, 161 (1993).