

Demystifying stringy miracles with eclectic flavor symmetries

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Effective field theories arising from string compactifications are subject to constraints originating from the duality transformations of string theory. Interpreting these so-called selection rules in terms of conventional symmetries has remained challenging. We show that particular selection rules in heterotic orbifolds can be explained from a subtle interplay between modular and traditional flavor symmetries within the eclectic flavor framework.

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I. INTRODUCTION

Symmetries play an important role in particle physics. The absence of certain couplings in an effective field theory is usually interpreted as a consequence of an underlying symmetry. As a “folklore theorem,” one can state that any coupling that is not explicitly forbidden by a symmetry is expected to appear in the action. This idea is frequently formulated as a so-called “naturalness criterion,” which aims to explain the absence (or smallness) of certain couplings.

Nonetheless, we are sometimes confronted with a situation where certain selection rules in a theory apparently cannot be explained by a symmetry and, thus, appear “miraculous.” Very often, however, a more careful inspection of the situations reveals the existence of a “hidden symmetry” that explains the selection rule, restores the naturalness argument, and demystifies the “miracle”.

In the present paper, we discuss a stringy selection rule which for more than 30 years has resisted an explanation in terms of a symmetry argument in the four-dimensional low-energy effective action. It originates in conformal field theory (CFT) correlators in orbifold compactifications of heterotic string theory [1,2], most notably in the $\mathbb{Z}_3$ orbifold. There, one encounters the so-called rule 4 [3],

a selection rule which states that certain couplings of twisted fields vanish when all the fields are located at the same fixed point. It was argued [4] that such a selection rule cannot be explained as a consequence of a symmetry of the effective action. As a main result of the present paper, we show that rule 4 can indeed be explained as a consequence of a hidden symmetry. The argument is based on so-called modular and eclectic symmetries [5] that originate from duality transformations in string theory. *A priori*, these are not conventional symmetries, as they map a theory not necessarily to itself but rather to its dual. In many cases, however, they lead to selection rules in the low-energy effective theory that can be understood in the eclectic symmetry framework [5]. This scheme combines discrete modular symmetries (that act nontrivially on a modulus field) with traditional flavor symmetries (which act as conventional discrete symmetries that leave the modulus invariant). They necessarily appear together and should not be considered in isolation, as these discrete modular symmetries are connected to the group of the outer automorphisms of the traditional flavor symmetry. At the heart of this construction is a hybrid $\mathbb{Z}_2^{\text{hybrid}}$ symmetry in the modular group $\text{SL}(2, \mathbb{Z})$ [not contained in $\text{PSL}(2, \mathbb{Z})$], which is intrinsically modular but does not transform the modulus and can, thus, be regarded as a traditional flavor symmetry, too. This describes the hybrid nature of $\mathbb{Z}_2^{\text{hybrid}}$.

As rule 4 is formulated in the framework of the $\mathbb{Z}_3$ orbifold, we shall exclusively consider this case here. Our results concerning the role of the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ symmetry and the eclectic scheme, however, shall also be relevant for more general cases to be discussed in future work. In the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold, we have the traditional flavor symmetry $\Delta(54)$ and the modular flavor symmetry T' . The hybrid $\mathbb{Z}_2^{\text{hybrid}}$ is contained in both of them. It extends $\Delta(27)$ to

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$\Delta(54)$ and A_4 to its double cover¹ T' . In the following, we shall show that rule 4 can be explained in a rather subtle way via the interplay of modular and traditional flavor symmetry.

We build our discussion on the earlier observation [6] that rule 4 was consistent with the appearance of specific trilinear couplings of twisted fields. Here, we reanalyze this result in detail and extract the basic reason for this fact. This insight will then allow us to prove rule 4 in the general case in terms of the eclectic flavor scheme. At the heart of this proof is the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ symmetry in combination with a $\mathbb{Z}_3^{\text{rot}}$ symmetry that rotates the three fixed points of the $\mathbb{Z}_3$ orbifold. It is a subgroup of $\Delta(27)$ in $\Delta(54)$. $\mathbb{Z}_2^{\text{hybrid}}$ and $\mathbb{Z}_3^{\text{rot}}$ combine to an S_3^R non-Abelian R -symmetry.² This shows that the eclectic scheme explains rule 4 and demystifies what was thought to be a “stringy miracle.” We thus see that, apart from understanding rule 4, we reveal a subtle interplay between modular and traditional flavor symmetries that might have important consequences for selection rules beyond the case of the $\mathbb{Z}_3$ orbifold.

The paper is organized as follows. Section II explains rule 4 as it was discussed in the framework of orbifold conformal field theory. In Sec. III, we present the concept of the eclectic flavor symmetry for the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold and the appearance of the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ as well as the representations of the twisted fields under $\Delta(54)$ and T' . Section IV recalls the explicit calculation of the trilinear couplings as provided earlier [6] and analyzes the implications of T' and $\Delta(54)$ separately. We will show that there are two dual explanations of rule 4 starting from either $\Delta(54)$ or T' . Armed with these observations, we give the general proof of rule 4 in these dual ways starting from either the traditional or the modular symmetry. We shall see that in both cases the appearance of $\mathbb{Z}_2^{\text{hybrid}}$ and $\mathbb{Z}_3^{\text{rot}}$ is crucial. We also point out a loophole in the earlier discussion [4], where one did not consider the possible role of non-Abelian discrete R -symmetries. Section V discusses some immediate consequences of the symmetry explanation of rule 4 in the presence of Wilson lines. Wilson lines typically break the degeneracy of the fixed points to allow for realistic spectra, as discussed in detail in Ref. [8]. Since the $\mathbb{Z}_3^{\text{rot}}$ is responsible for this degeneracy, it is broken in the presence of Wilson lines. As this symmetry is crucial for the proof of rule 4, one would thus expect that such a rule is no longer valid once a Wilson line is switched on. Some consequences of these observations are briefly discussed in Sec. VI. Section VII provides a summary and outlook on possible future investigations.

¹ T' and A_4 are finite modular groups arising from $\text{SL}(2, \mathbb{Z})$ and $\text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\mathbb{Z}_2$, respectively.

²A geometric construction of the S_3 group is given in detail in Ref. [7].

II. SELECTION RULE 4 IN HETEROTIC ORBIFOLDS

Couplings arising from orbifold compactification can be computed within the context of CFT [1,2]. They lead to selection rules that have been formulated in Ref. [3]. Of particular interest is a selection rule known as rule 4, as discussed in Ref. [4] and refined in Ref. [9]. Rule 4 states that certain local correlation functions of twisted fields vanish when all fields are located at the same fixed point. This has been explicitly discussed in the framework of the $\mathbb{Z}_3$ orbifold in Refs. [3,4].

Relevant for this rule is the presence of oscillator modes and derivatives in the corresponding vertex operators of the CFT [4]. When computing n -point Yukawa couplings with CFT methods, the picture-changing mechanism for correlation functions leads to $n - 3$ derivatives. In the $\mathbb{T}^2/\mathbb{Z}_3$ case, rule 4 states that couplings of twisted fields *at the same fixed point* are allowed only if the sum Σ of the number of these derivatives and the number of oscillator modes is equal to $0 \bmod 6$.

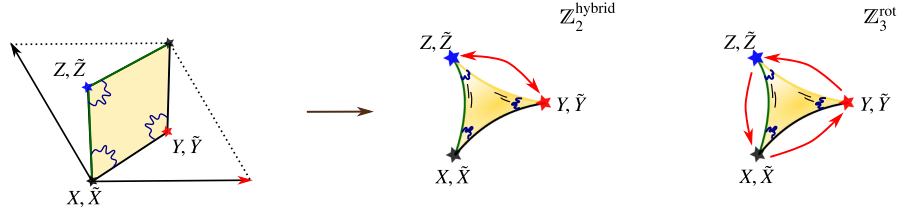
This rule appears somewhat peculiar, as there is no such rule for similar couplings of the same twisted fields located at different fixed points. This leads to the question whether such a selection rule can be understood in terms of the symmetries of the low-energy effective four-dimensional quantum field theory. A first analysis of the question [4] led to the conclusion that conventional symmetries were not able to explain rule 4. This would mean that certain selection rules of string theory might not be understandable through symmetries of the low-energy effective field theoretical approximation. In the following, we shall try to clarify this question.

III. ECLECTIC SCHEME IN $\mathbb{T}^2/\mathbb{Z}_3$

A. The origin of the eclectic symmetries

In the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold of the heterotic string, there are various symmetries of the associated low-energy effective field theory that are explained as symmetries of the toroidal orbifold compactification of the extra dimensions of a heterotic string in the Narain formalism [10]. First, a $\mathbb{T}^2$ of the heterotic string is characterized by two moduli: a complex structure U and a Kähler modulus T . The complex structure is geometrically stabilized at $\langle U \rangle = \omega := e^{2\pi i/3}$, so that the two-torus $\mathbb{T}^2$ exhibits a $\mathbb{Z}_3$ rotational symmetry compatible with the $\mathbb{Z}_3$ orbifold. In this case, the modular group $\text{SL}(2, \mathbb{Z})_U$ associated with the complex structure U is broken. Furthermore, one can identify two unbroken rotational outer automorphisms of the $\mathbb{T}^2/\mathbb{Z}_3$ Narain space group corresponding to the generators S and T of the modular group $\text{SL}(2, \mathbb{Z})_T$ for the Kähler modulus. These generators are subject to the constraints

$$S^4 = 1 = (ST)^3 \quad \text{and} \quad S^2T = TS^2. \quad (3.1)$$


 FIG. 1. The $\mathbb{T}^2/\mathbb{Z}_3$ orbifold and two of its symmetries in the absence of Wilson lines.

Furthermore, they act nontrivially on matter fields [11–13], yielding modular flavor symmetries.

On the other hand, there are two translational outer automorphisms of the $\mathbb{T}^2/\mathbb{Z}_3$ Narain space group, denoted A and B. Since translations leave all moduli invariant and do transform matter fields of the orbifold, they correspond to traditional flavor symmetries. Additionally, there exists a rotational outer automorphism C of the Narain space group, which leaves the moduli invariant and acts nontrivially on matter fields. Hence, C qualifies also as a traditional flavor symmetry. It turns out that, at the level of outer automorphisms of the Narain space group, one finds that

$$C = S^2, \quad (3.2)$$

showing that the generator C builds both a traditional and a modular flavor symmetry; i.e., it is a *hybrid symmetry generator*. Because of Eq. (3.1), S^2 describes a $\mathbb{Z}_2^{\text{hybrid}}$ symmetry.

B. Matter and the eclectic flavor group

The action of the various generators on matter fields depends on their localization in the extra dimensions: Bulk strings build singlets while twisted strings, localized at the three fixed points of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold (see Fig. 1), transform as (reducible) triplets. Under modular transformations matter fields carry a (rational) modular weight n and transform as

$$\Phi_n \xrightarrow{\gamma} (cT + d)^n \rho_s(\gamma) \Phi_n, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})_T, \quad (3.3)$$

where $(cT + d)^n$ is called automorphy factor and the $\text{SL}(2, \mathbb{Z})_T$ generators can be written as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (3.4)$$

Furthermore, bulk states transform with the representations $\rho_1(S) = \rho_1(T) = 1$ and have the modular weights $n = 0$ or $n = -1$. The three twisted states located at three fixed points of the orbifold are collected in two kinds of multiplets:

$$\Phi_{-2/3} = (X, Y, Z)^T \quad \text{without oscillator excitations,} \quad (3.5a)$$

$$\Phi_{-5/3} = (\tilde{X}, \tilde{Y}, \tilde{Z})^T$$

with one holomorphic oscillator excitation,

$$(3.5b)$$

which transform under S and T according to

$$\rho(S) = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.6)$$

This implies that S and T act on matter fields as a $\mathbb{Z}_4$ and a $\mathbb{Z}_3$ symmetry, respectively. Interestingly, these modular transformations build the $\mathbf{2}' \oplus \mathbf{1}$ irreducible representations of the finite modular group³ $\Gamma_3' \cong T' \cong \text{SL}(2, \mathbb{Z}_3) \cong [24, 3]$. According to the irreducible basis in Table III, the fields in Eq. (3.5) can be arranged as

$$\begin{aligned} \mathbf{2}': & \begin{pmatrix} \frac{1}{\sqrt{2}}(Y + Z) \\ -X \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}(\tilde{Y} + \tilde{Z}) \\ -\tilde{X} \end{pmatrix} \quad \text{and} \\ \mathbf{1}: & \frac{1}{\sqrt{2}}(Y - Z), \frac{1}{\sqrt{2}}(\tilde{Y} - \tilde{Z}). \end{aligned} \quad (3.7)$$

Under the generators of the traditional flavor symmetry, matter states $\Phi_{-2/3}$ build the $\Delta(54)$ triplet representation $\mathbf{3}_2$, while the multiplet $\Phi_{-5/3}$ builds the representation $\mathbf{3}_1$ of $\Delta(54)$. The representation matrices of $\mathbf{3}_2$ and $\mathbf{3}_1$ are given in terms of

$$\begin{aligned} \rho(A) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & \rho(B) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \\ \rho(C) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (3.8)$$

³[24, 3] corresponds to the GAP notation [14], where the first number is the order and the latter only a counter.

TABLE I. Various symmetries of the eclectic flavor group $\Omega(1) \cong \Delta(54) \cup T' \cong [648, 533]$ for a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold and their origin. The order-3 translational outer automorphisms A and B generate $\Delta(27)$. The rotational outer automorphism C is special as it belongs to both the traditional flavor symmetry $\Delta(54)$ and the finite modular symmetry T' . This table has been adapted from [18].

Nature of symmetry		Outer automorphism of Narain space group		Flavor groups		
Eclectic	Modular	Rotation $S \in \text{SL}(2, \mathbb{Z})_T$	$\mathbb{Z}_4$	T'		
	Traditional flavor	Rotation $T \in \text{SL}(2, \mathbb{Z})_T$	$\mathbb{Z}_3$	$\Delta(27)$ $\Delta(54)$		
		Translation A	$\mathbb{Z}_3^{\text{rot}}$			
		Translation B	$\mathbb{Z}_3^{\text{B}}$			
	Hybrid $\mathbb{Z}_2$	Rotation $C = S^2 \in \text{SL}(2, \mathbb{Z})_T$	$\mathbb{Z}_2^{\text{hybrid}}$			

TABLE II. T' and $\Delta(54)$ representations of (massless) matter fields Φ_n with modular weights n in the untwisted and first twisted sectors of $\mathbb{T}^2/\mathbb{Z}_3$ orbifolds [7, 17]. T' and $\Delta(54)$ combine nontrivially to the $\Omega(1) \cong [648, 533]$ eclectic flavor group [5], generated by $\rho_s(S)$, $\rho_s(T)$, $\rho_r(A)$, and $\rho_r(B)$. For $\rho_r(C)$, both $C = S^2$ and the modular weight n are important, as discussed in Eq. (3.9). Table adapted from [6].

Eclectic flavor group $\Omega(1)$									
Sector	Matter fields	Modular T' subgroup				Traditional $\Delta(54)$ subgroup			
	Φ_n	Irrep s	$\rho_s(S)$	$\rho_s(T)$	n	Irrep r	$\rho_r(A)$	$\rho_r(B)$	$\rho_r(C)$
Bulk	Φ_0	1	1	1	0	1	1	1	+1
	Φ_{-1}	1	1	1	-1	1'	1	1	-1
θ	$\Phi_{-2/3}$	2' $\oplus$ 1	$\rho(S)$	$\rho(T)$	-2/3	3₂	$\rho(A)$	$\rho(B)$	$-\rho(C)$
	$\Phi_{-5/3}$	2' $\oplus$ 1	$\rho(S)$	$\rho(T)$	-5/3	3₁	$\rho(A)$	$\rho(B)$	$+\rho(C)$
Super-potential	$\mathcal{W}$	1	1	1	-1	1'	1	1	-1

In particular, $\rho_{3_1}(A) = \rho_{3_2}(A) = \rho(A)$, $\rho_{3_1}(B) = \rho_{3_2}(B) = \rho(B)$, and $\rho_{3_1}(C) = -\rho_{3_2}(C) = \rho(C)$. We observe that $\rho(A)$ represents the order-3 cyclic symmetry $\mathbb{Z}_3^{\text{rot}}$ that exchanges all three matter states of the twisted multiplets Φ_n , as illustrated in Fig. 1. Similarly, the representation $\rho(C)$ of the hybrid symmetry generator builds the $\mathbb{Z}_2^{\text{hybrid}}$, which swaps $Y \leftrightarrow Z$ and $\tilde{Y} \leftrightarrow \tilde{Z}$.

The generators A and B yield the non-Abelian discrete group $\Delta(27)$. The inclusion of C enhances the traditional flavor group to $\Delta(54)$, which is recognized as the traditional flavor symmetry of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold [15]. Notably, C amounts to a π rotation in the compact dimensions [7], making it a remnant of the higher-dimensional Lorentz symmetry. As a result, $\Delta(54)$ can be interpreted as a non-Abelian discrete R symmetry of $\mathcal{N} = 1$ supersymmetry [16], under which the superpotential transforms as a non-trivial singlet **1'** of $\Delta(54)$ [17]. Furthermore, we note that the traditional generators A and C build a non-Abelian discrete $S_3^R \cong \mathbb{Z}_3^{\text{rot}} \rtimes \mathbb{Z}_2^{\text{hybrid}}$ symmetry.

Together, the traditional and modular symmetries build the eclectic group $\Omega(1) \cong \Delta(54) \cup T' \cong [648, 533]$, whose generators are given in Table I. Under the different components of the group, all matter fields of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold transform as described before and summarized in Table II.

Recall from Eq. (3.2) that $C = S^2$ and that this generator belongs to both the traditional and modular parts of $\Omega(1)$,

the eclectic flavor group.⁴ Hence, from Eq. (3.3), we obtain that

$$\rho_r(C) := (-1)^n (\rho_s(S))^2. \quad (3.9)$$

For example, the bulk fields Φ_0 and Φ_{-1} , which are trivial singlets under the modular group T' , exhibit a sign difference in their $\Delta(54)$ matrix representation $\rho_r(C)$. In contrast, for the twisted sector, the presence of fractional modular weights makes $(-1)^n$ multivalued, taking values in $\{1, \omega, \omega^2\}$. This factor gives rise to a traditional flavor symmetry associated with the point group $\mathbb{Z}_3^{(\text{PG})}$. Consequently, within the eclectic framework, the modular weights and the traditional flavor symmetry representation must be selected in a consistent way.

C. Yukawa couplings in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold

Since $\Gamma'_3 \cong T'$ is the modular flavor symmetry of the eclectic scheme of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold, all Yukawa couplings are given by T' vector-valued modular forms (VVMFs). All VVMFs $\hat{Y}_s^{(n_r)}(T)$ with modular weight

⁴This $\mathbb{Z}_2^{\text{hybrid}}$ overlap in $\Omega(1) \cong \Delta(54) \cup T'$ is precisely what the symbol $\cup$ implies. In principle, a more accurate group structure is $\Omega(1) \cong \Delta(27) \rtimes T'$ or, equivalently, $\Omega(1) \cong \Delta(54).A_4$; cf. Appendix B.2 in Ref. [19].

$n_Y \in \mathbb{N}$ transform under $\gamma \in \text{SL}(2, \mathbb{Z})_T$ according to

$$\hat{Y}_s^{(n_Y)}(T) \xrightarrow{\gamma} \hat{Y}_s^{(n_Y)}\left(\frac{aT+b}{cT+d}\right) = (cT+d)^{n_Y} \rho_s(\gamma) \hat{Y}_s^{(n_Y)}(T), \quad (3.10)$$

where $\rho_s(\gamma)$ is an s -dimensional representation of T' . These modular forms can be built from tensor products of the lowest weight ($n_Y = 1$) VVMF [20]

$$\hat{Y}_{2''}^{(1)} := \begin{pmatrix} -3\sqrt{2} \frac{\eta^3(3T)}{\eta(T)} \\ 3 \frac{\eta^3(3T)}{\eta(T)} + \frac{\eta^3(T/3)}{\eta(T)} \end{pmatrix}, \quad (3.11)$$

where $\eta(T)$ is the so-called Dedekind η function. Using the transformations under the modular generators S and T , it is easy to confirm that $\hat{Y}_{2''}^{(1)}(T)$ transforms as a doublet $2''$ of T' [6]. Modular forms of higher weight can be constructed by the tensor product of this lowest weight modular form doublet. For example, we know that, in general, $2'' \otimes 2'' = 3 \oplus 1'$, but $1'$ vanishes here for identical doublets in T' . Hence, $\hat{Y}_{2''}^{(1)} \otimes \hat{Y}_{2''}^{(1)}$ delivers a VVMF $\hat{Y}_3^{(2)}$.

As we mentioned earlier, the superpotential must have modular weight -1 ; this implies that trilinear couplings built by Φ_n multiplets require modular forms of weight $n_Y = -1 - 3n$. That is, $\Phi_{-2/3}^3$ would require the coupling given by $\hat{Y}_s^{(1)}$ while $\Phi_{-5/3}^3$ needs $\hat{Y}_s^{(4)}$. We observe that the former case includes T' doublets while the latter does not.

Finally, let us point out that T' is the double cover of A_4 . Therefore, the modular forms of T' include those of A_4 . The VVMFs $\hat{Y}_s^{(n_Y)}$ with even n_Y build irreducible representations of A_4 .

IV. RESULTS AND INSIGHTS FROM TRILINEAR COUPLINGS

To illustrate the power of the eclectic symmetry, we can use the allowed trilinear couplings of twisted fields in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold computed earlier [6]. The relevant fields are $\Phi_{-2/3}^i = (X_i, Y_i, Z_i)^T$ and $\Phi_{-5/3}^i = (\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i)^T$, where the index i labels different field multiplets. They both transform as a $1 \oplus 2'$ representation of the modular group T' but have different modular weight. The difference of the couplings for $\Phi_{-2/3}^i$ and $\Phi_{-5/3}^i$ will, therefore, be due to the appearance of the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ symmetry connected to the $S^2 = -1$ transformation in $\text{SL}(2, \mathbb{Z})$. This is not an element of $\text{PSL}(2, \mathbb{Z})$ and does not transform the modulus. Superficially, it can, thus, be understood as a traditional flavor symmetry, although it is intrinsically modular. This transformation $C = S^2$ is correlated with the modular weights as shown in Eq. (3.9). Under $\Delta(54)$, $\Phi_{-2/3}^i$ transforms as an irreducible 3_2 representation, while $\Phi_{-5/3}^i$

transforms as a 3_1 . The difference comes again from the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ that extends $\Delta(27)$ to $\Delta(54)$. In the following, we shall discuss the trilinear couplings from two different angles. First, in Sec. IV A, we start by imposing the modular flavor symmetry T' . Then, in Sec. IV B, we study the couplings based on the $\Delta(54)$ traditional flavor symmetry.

Our aim is to check the validity of rule 4 (see Sec. II), which predicts a different behavior for the trilinear couplings of $(\Phi_{-2/3})^3$ and $(\Phi_{-5/3})^3$. This is because $\Phi_{-5/3}$ contains an oscillator mode while $\Phi_{-2/3}$ does not. Rule 4 would then by definition require the absence of the trilinear couplings $\tilde{X}^3$, $\tilde{Y}^3$, and $\tilde{Z}^3$ while X^3 , Y^3 , and Z^3 should be allowed.

A. Restrictions from modular symmetry

Possible trilinear terms of twisted fields in the superpotential are given by

$$\mathcal{W} \supset \hat{Y}^{(1)}(T) \Phi_{-2/3}^1 \Phi_{-2/3}^2 \Phi_{-2/3}^3 + \hat{Y}^{(4)}(T) \Phi_{-5/3}^1 \Phi_{-5/3}^2 \Phi_{-5/3}^3, \quad (4.1)$$

with modular forms $\hat{Y}^{(n_Y)}$ of weight $n_Y = 1$ and $n_Y = 4$. As mentioned earlier in Sec. III C, the T' transformation of the modular forms can be deduced through products of the “fundamental” modular form $\hat{Y}_{2''}^{(1)}(T)$, which transforms as a $2''$ of T' . $\hat{Y}^{(4)}(T)$ as the fourth power of $2''$ then transforms as a $1 \oplus 1' \oplus 3$ -dimensional representation of T' . These couplings have been worked out in Ref. [6] and turn out to be

$$\begin{aligned} \mathcal{W}_1 = & \frac{1}{4} (\hat{Y}_2(T) (4X_1 X_2 X_3 + (Y_1 + Z_1)(Y_2 + Z_2)(Y_3 + Z_3)) \\ & - \sqrt{2} \hat{Y}_1(T) ((Y_1 + Z_1)(Y_2 + Z_2) X_3 \\ & + ((Y_1 + Z_1) X_2 + X_1 (Y_2 + Z_2))(Y_3 + Z_3))), \end{aligned} \quad (4.2a)$$

$$\mathcal{W}_2 = \frac{1}{4} (\sqrt{2} \hat{Y}_1(T) X_1 + \hat{Y}_2(T) (Y_1 + Z_1)) (Y_2 - Z_2) (Y_3 - Z_3), \quad (4.2b)$$

$$\mathcal{W}_3 = \frac{1}{4} (Y_1 - Z_1) (\sqrt{2} \hat{Y}_1(T) X_2 + \hat{Y}_2(T) (Y_2 + Z_2)) (Y_3 - Z_3), \quad (4.2c)$$

$$\mathcal{W}_4 = \frac{1}{4} (Y_1 - Z_1) (Y_2 - Z_2) (\sqrt{2} \hat{Y}_1(T) X_3 + \hat{Y}_2(T) (Y_3 + Z_3)), \quad (4.2d)$$

for the twisted matter fields $\Phi_{-2/3}^i = (X_i, Y_i, Z_i)^T$ and

$$\tilde{\mathcal{W}}_1 = \frac{1}{2\sqrt{2}} \hat{Y}_{1'}^{(4)}(T) (\tilde{X}_2 (\tilde{Y}_1 + \tilde{Z}_1) - \tilde{X}_1 (\tilde{Y}_2 + \tilde{Z}_2)) (\tilde{Y}_3 - \tilde{Z}_3), \quad (4.3a)$$

$$\tilde{\mathcal{W}}_2 = \frac{1}{2\sqrt{2}} \hat{Y}_{\mathbf{1}'}^{(4)}(T) (\tilde{X}_3(\tilde{Y}_1 + \tilde{Z}_1) - \tilde{X}_1(\tilde{Y}_3 + \tilde{Z}_3))(\tilde{Y}_2 - \tilde{Z}_2), \quad (4.3b)$$

$$\tilde{\mathcal{W}}_3 = \frac{1}{2\sqrt{2}} \hat{Y}_{\mathbf{1}'}^{(4)}(T) (\tilde{X}_3(\tilde{Y}_2 + \tilde{Z}_2) - \tilde{X}_2(\tilde{Y}_3 + \tilde{Z}_3))(\tilde{Y}_1 - \tilde{Z}_1), \quad (4.3c)$$

$$\tilde{\mathcal{W}}_4 = \frac{1}{2\sqrt{2}} \hat{Y}_{\mathbf{1}'}^{(4)}(T) (\tilde{Y}_1 - \tilde{Z}_1)(\tilde{Y}_2 - \tilde{Z}_2)(\tilde{Y}_3 - \tilde{Z}_3), \quad (4.3d)$$

$$\begin{aligned} \tilde{\mathcal{W}}_5 = & \frac{1}{2\sqrt{2}} (\tilde{Y}_3 - \tilde{Z}_3) [\tilde{X}_2(2\hat{Y}_{\mathbf{3},3}^{(4)}(T)\tilde{X}_1 + \hat{Y}_{\mathbf{3},2}^{(4)}(T)(\tilde{Y}_1 + \tilde{Z}_1)) \\ & + (\tilde{Y}_2 + \tilde{Z}_2)(\hat{Y}_{\mathbf{3},2}^{(4)}(T)\tilde{X}_1 + \hat{Y}_{\mathbf{3},1}^{(4)}(T)(\tilde{Y}_1 + \tilde{Z}_1))], \end{aligned} \quad (4.3e)$$

$$\begin{aligned} \tilde{\mathcal{W}}_6 = & \frac{1}{2\sqrt{2}} (\tilde{Y}_2 - \tilde{Z}_2) [\tilde{X}_3(2\hat{Y}_{\mathbf{3},3}^{(4)}(T)\tilde{X}_1 + \hat{Y}_{\mathbf{3},2}^{(4)}(T)(\tilde{Y}_1 + \tilde{Z}_1)) \\ & + (\tilde{Y}_3 + \tilde{Z}_3)(\hat{Y}_{\mathbf{3},2}^{(4)}(T)\tilde{X}_1 + \hat{Y}_{\mathbf{3},1}^{(4)}(T)(\tilde{Y}_1 + \tilde{Z}_1))], \end{aligned} \quad (4.3f)$$

$$\begin{aligned} \tilde{\mathcal{W}}_7 = & \frac{1}{2\sqrt{2}} (\tilde{Y}_1 - \tilde{Z}_1) [\tilde{X}_3(2\hat{Y}_{\mathbf{3},3}^{(4)}(T)\tilde{X}_2 + \hat{Y}_{\mathbf{3},2}^{(4)}(T)(\tilde{Y}_2 + \tilde{Z}_2)) \\ & + (\tilde{Y}_3 + \tilde{Z}_3)(\hat{Y}_{\mathbf{3},2}^{(4)}(T)\tilde{X}_2 + \hat{Y}_{\mathbf{3},1}^{(4)}(T)(\tilde{Y}_2 + \tilde{Z}_2))], \end{aligned} \quad (4.3g)$$

for the twisted fields $\Phi_{-5/3}^i = (\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i)^T$. These expressions differ significantly even though both fields $\Phi_{-2/3}^i$ and $\Phi_{-5/3}^i$ are in the same $\mathbf{1} \oplus \mathbf{2}'$ representation of T' . The difference is due to the different modular weights and, thus, the action of the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ symmetry [cf. Eq. (3.9)].

What can we learn from these results regarding the validity of rule 4? For the trilinear couplings of $\Phi_{-2/3}^i$, we see that the self-couplings X^3 , Y^3 , and Z^3 are all allowed. This is compatible with rule 4. The result for the self-couplings of $\Phi_{-5/3}^i$ is less transparent. A closer inspection of formula (4.3) reveals the fact that $\tilde{X}^3$ is forbidden while the couplings $\tilde{Y}^3$ and $\tilde{Z}^3$ are still allowed. What is the origin of this difference? A look at the T' representations (3.7) gives the answer. $\tilde{X}$ is exclusively a member of the doublet, while $\tilde{Y}$ and $\tilde{Z}$ appear both in the doublet and singlet representations of T' . Still, we see that the result does not respect rule 4 as $\tilde{Y}^3$ and $\tilde{Z}^3$ are still allowed.

But this is not yet the end of the story, as we still have traditional flavor symmetries at our disposal. These include a $\mathbb{Z}_3^{\text{rot}}$ symmetry that rotates the twisted fields as discussed in Fig. 2 in [7]. It can be understood as an outer automorphism of the space group of the orbifolded $\mathbb{Z}_3$ lattice. If we apply these restrictions on the couplings given in Eqs. (4.2) and (4.3), we obtain

$$\begin{aligned} \mathcal{W}(T, X_i, Y_i, Z_i) \supset & c^{(1)} [\hat{Y}_2(T)(X_1 X_2 X_3 + Y_1 Y_2 Y_3 + Z_1 Z_2 Z_3) \\ & - \frac{\hat{Y}_1(T)}{\sqrt{2}} (X_1 Y_2 Z_3 + X_1 Y_3 Z_2 + X_2 Y_1 Z_3 + X_3 Y_1 Z_2 + X_2 Y_3 Z_1 + X_3 Y_2 Z_1)] \end{aligned} \quad (4.4)$$

and

$$\tilde{\mathcal{W}}(T, \tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i) \supset c^{(4)} \hat{Y}_{\mathbf{1}'}^{(4)}(T) (\tilde{X}_1 \tilde{Y}_3 \tilde{Z}_2 - \tilde{X}_1 \tilde{Y}_2 \tilde{Z}_3 + \tilde{X}_2 \tilde{Y}_1 \tilde{Z}_3 - \tilde{X}_2 \tilde{Y}_3 \tilde{Z}_1 + \tilde{X}_3 \tilde{Y}_2 \tilde{Z}_1 - \tilde{X}_3 \tilde{Y}_1 \tilde{Z}_2), \quad (4.5)$$

respectively.⁵ Equations (4.4) and (4.5) are now compatible with rule 4. The difference between these equations is mainly a consequence of the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ symmetry $C = S^2$. In combination with the rotational $\mathbb{Z}_3^{\text{rot}}$ symmetry, it appears to provide the origin of rule 4 in the case of trilinear couplings. A geometrical explanation of both the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ and the rotational $\mathbb{Z}_3^{\text{rot}}$ symmetries has been given in Fig. 2 in [7], where the $\mathbb{Z}_2$ symmetry appears as a 180-degree rotation of the $\mathbb{Z}_3$ lattice. There, it was shown that these two symmetries combine to the non-Abelian R

symmetry S_3^R . Hence, this S_3^R is crucial for an explanation of rule 4.

B. Restrictions from the traditional flavor symmetry

Let us now analyze the restrictions from the traditional flavor symmetry $\Delta(27)$ and $\Delta(54)$ in more detail. Both twisted fields $\Phi_{-2/3}$ and $\Phi_{-5/3}$ are in the irreducible triplet representation of $\Delta(27)$ and should have identical trilinear couplings in the superpotential. Within $\Delta(54)$, however, $\Phi_{-2/3}$ and $\Phi_{-5/3}$ transform as different three-dimensional representations, $\mathbf{3}_2$ and $\mathbf{3}_1$, respectively. This leads to a decisive difference in the trilinear Yukawa couplings. The superpotential transforms as a $\mathbf{1}'$ representation of $\Delta(54)$. We, thus, have to consider possible $\mathbf{1}'$ representations in the product $\mathbf{3}_i \otimes \mathbf{3}_i \otimes \mathbf{3}_i$ for $i = 1, 2$. As shown explicitly in

⁵Note that this $\mathbb{Z}_3^{\text{rot}}$ symmetry is sufficient to obtain this result, and the $\mathbb{Z}_3^{\text{B}}$ symmetry is automatically satisfied. We do not need the full traditional flavor symmetry $\Delta(54)$ or $\Delta(27)$ for this result.

Appendix 2, tensor products for the triplets are given by

$$\mathbf{3}_1 \otimes \mathbf{3}_1 = \mathbf{3}_2 \otimes \mathbf{3}_2 = \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_2. \quad (4.6)$$

Nontrivial singlets are contained in the product of $\bar{\mathbf{3}}_1 \otimes \mathbf{3}_2$ and $\bar{\mathbf{3}}_2 \otimes \mathbf{3}_1$. For the product $(\Phi_{-2/3})^3 = \mathbf{3}_2 \otimes \mathbf{3}_2 \otimes \mathbf{3}_2$, we, thus, have two types of couplings $\bar{\mathbf{3}}_1 \otimes \mathbf{3}_2$, while for $(\Phi_{-5/3})^3 = \mathbf{3}_1 \otimes \mathbf{3}_1 \otimes \mathbf{3}_1$ there is only one, namely, $\bar{\mathbf{3}}_2 \otimes \mathbf{3}_1$. In an explicit calculation using the tensor products in Appendix A.2, we find that couplings of three $\mathbf{3}_1 \otimes \mathbf{3}_1 \otimes \mathbf{3}_1$ at the same fixed points are forbidden while those of the three $\mathbf{3}_2 \otimes \mathbf{3}_2 \otimes \mathbf{3}_2$ are allowed. Crucial for this selection rule is again the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ symmetry that, in this case, enhances $\Delta(27)$ to $\Delta(54)$ (which, in turn, can be identified with the symmetry $C = S^2$). We thus see that the traditional flavor symmetry forbids trilinear couplings for fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ at the same fixed point. It does not give, however, the full structure of formulas (4.4) and (4.5), which contain more information about the moduli dependence of the Yukawa couplings [which is not restricted by $\Delta(54)$].

As we have mentioned before, a geometric origin of this $\mathbb{Z}_2$ can also be found in the $\mathbb{Z}_2$ -outer automorphism of the space group selection rule of the $\mathbb{T}^2/\mathbb{Z}_3$ lattice (see Fig. 2 in [7]). This reflects the fact that the $\mathbb{Z}_3$ lattice has the same symmetries as the $\mathbb{Z}_6$ lattice. At the technical level, this is a consequence of the fact that the correlation functions involve a sum over sublattices, and the sublattice of the local coupling has an additional $\mathbb{Z}_2$ symmetry compared to the sublattice relevant for the nonlocal coupling.

C. Lessons from trilinear couplings

Explicit calculations show that rule 4 (in trilinear couplings) is the consequence of symmetries. A crucial role is played by a hybrid $\mathbb{Z}_2^{\text{hybrid}}$ and a rotational $\mathbb{Z}_3^{\text{rot}}$ symmetry that combine to the group S_3^R . Rule 4 can be reproduced in two dual ways: either by starting from the modular symmetry T' and invoking additionally the $\mathbb{Z}_3^{\text{rot}}$ symmetry or by considering the full $\Delta(54)$ traditional symmetry. In both cases, the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ and the rotational $\mathbb{Z}_3^{\text{rot}}$ symmetries play a central role. In this discussion, the particular properties of the fields X_i and $\tilde{X}_i$ play an important role.

V. GENERAL PROOF FOR HIGHER-ORDER COUPLINGS

We now proceed to study rule 4 for general n -point couplings in $\mathbb{Z}_3$ orbifolds. Let us consider higher-order couplings $(\Phi_{-2/3})^n$ and $(\Phi_{-5/3})^n$, $n > 3$, for the fields $\Phi_{-2/3}^i = (X_i, Y_i, Z_i)^T$ and $\Phi_{-5/3}^i = (\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i)^T$. $(\Phi_{-2/3})^n$ leads to $\Sigma = n - 3$ and, as stated in Sec. II, rule 4 requires

$\Sigma = 6\ell$ with integer ℓ . Thus, in this case, $n = 3 + 6\ell$ for the allowed local couplings X^n, Y^n, Z^n at the same fixed point. On the other hand, $\Phi_{-5/3}$ contains one oscillator excitation; hence, $(\Phi_{-5/3})^n$ leads to $\Sigma = n - 3 + n = 2n - 3$, which should be 0 mod 6 for allowed local couplings. As $2n - 3$ is always an odd integer, there is no solution for $2n - 3 = 6\ell$ (ℓ integer).

In order to arrive at a possible symmetry explanation of this outcome, we now follow the procedure applied for trilinear couplings; i.e., we start by relating it to the modular properties of the theory, and then we consider its possible origin within the $\Delta(54)$ traditional flavor symmetry.

A. Constraints from modular symmetry

The total modular weights of the couplings $(\Phi_{-2/3})^n$ and $(\Phi_{-5/3})^n$ are $-2n/3$ and $-5n/3$, respectively. Moreover, the modular weight of the superpotential must be -1 . This implies that the modular weights of the corresponding couplings strengths, which are VVMFs, should be $n_Y = -1 + 2n/3$ and $n_Y = -1 + 5n/3$. Since $n_Y \in \mathbb{Z}$ for the VVMFs of $\Gamma'_3 \cong T'$, this results in $n = 3k$ for integer k . We note in addition that this constraint also follows from the space and point-group selection rules (see, e.g., [9]).

To simplify the discussion, let us restrict ourselves to the local couplings of the twisted fields, as they are of relevance for rule 4. This leads to

$$\begin{aligned} \mathcal{W}_{\text{local}} = & \sum_{k_1, k_2 > 0} \hat{Y}_{\text{local}}^{(2k_1-1)}(T) (X^{3k_1} + Y^{3k_1} + Z^{3k_1}) \\ & + \tilde{Y}_{\text{local}}^{(5k_2-1)}(T) (\tilde{X}^{3k_2} + \tilde{Y}^{3k_2} + \tilde{Z}^{3k_2}). \end{aligned} \quad (5.1)$$

As we have seen before, the fields X and $\tilde{X}$ play a special role, as they belong exclusively to the doublet representation $\mathbf{2}'$ of T' . We, thus, consider

$$\mathcal{W}_{\text{local}} \supset \sum_{k_1, k_2 > 0} \hat{Y}_{\text{local}}^{(2k_1-1)} X^{3k_1} + \hat{Y}_{\text{local}}^{(5k_2-1)} \tilde{X}^{3k_2}. \quad (5.2)$$

The representations of the modular forms $\hat{Y}^{(n_Y)}$ of T' (cf. discussion in Sec. III.2 in [20]) fall into two classes with respect to the $\mathbb{Z}_2^{\text{hybrid}}$. For even modular weights, they correspond to the A_4 representations $\mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{3}$, which we will refer to as corresponding to the “even class” of $\mathbb{Z}_2^{\text{hybrid}}$. For odd modular weights, they belong to the doublet representations $\mathbf{2}, \mathbf{2}', \mathbf{2}''$ of T' , the double cover of A_4 , which we will refer to as corresponding to the “odd class” of $\mathbb{Z}_2^{\text{hybrid}}$. The products of fields and modular forms in the superpotential have to combine to the trivial singlet of T' , corresponding to the even class with respect to the hybrid $\mathbb{Z}_2^{\text{hybrid}}$.

X^{3k_1} has weight $-2k_1$. As the superpotential has weight -1 , the weight of $\hat{Y}^{(n_Y)}$ has to be $n_Y = 2k_1 - 1$, which is always in the odd class (only doublets). The class of X^{3k_1} is odd (even) if k_1 is odd (even). Thus, for odd k_1 the couplings are allowed, while for even k_1 they are forbidden. For the local couplings X^n , this reproduces the $n = 3 \bmod 6$ selection rule of rule 4.

$\tilde{X}^{3k_2}$ has weight $-5k_2$. For $\hat{Y}^{(n_Y)}$ we, thus, obtain weight $n_Y = 5k_2 - 1$. The product $\hat{Y}^{(5k_2-1)}\tilde{X}^{3k_2}$ has modular weight $8k_2 - 1$. This is always in the odd class of $\mathbb{Z}_2^{\text{hybrid}}$ and, therefore, not allowed as a term of the superpotential, in agreement with rule 4.

For the other twisted fields $Y, Z, \tilde{Y}, \tilde{Z}$, this argumentation does not hold, as they are not exclusively members of the $2'$ representation of T' (but also appear in the singlet representation). To complete the proof of rule 4, we, thus, again need the rotational $\mathbb{Z}_3^{\text{rot}}$ traditional flavor symmetry. Hybrid $\mathbb{Z}_2^{\text{hybrid}}$ and rotational $\mathbb{Z}_3^{\text{rot}}$ are the crucial symmetries to explain rule 4.

B. Constraints from traditional flavor symmetry

Let us again concentrate on local couplings. The fields $\Phi_{-2/3}$ and $\Phi_{-5/3}$ are both triplets of $\Delta(27)$ and differ only in their transformation properties under the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ generated by $C = S^2$. If we impose $\Delta(27)$, we obtain

$$\mathcal{W}_{\text{local}} = \sum_{k_1, k_2 > 0} a_{k_1} (X^{3k_1} + Y^{3k_1} + Z^{3k_1}) + b_{k_2} (\tilde{X}^{3k_2} + \tilde{Y}^{3k_2} + \tilde{Z}^{3k_2}), \quad (5.3)$$

with general coefficients a_{k_1} and b_{k_2} . $\Delta(27)$ includes the rotational $\mathbb{Z}_3^{\text{rot}}$ symmetry, and this manifests itself in the fact that the coefficients for the terms with X, Y, Z and $\tilde{X}, \tilde{Y}, \tilde{Z}$ are universal. We remark that the powers $3k_1$ and $3k_2$ here arise from the $\mathbb{Z}_3^B$ (or $\mathbb{Z}_3^{A^2B^2AB}$) factor of $\Delta(27)$. Under the generator C of the hybrid $\mathbb{Z}_2^{\text{hybrid}}$, the superpotential changes sign. In addition, the field $\Phi_{-2/3}$ changes sign, while $\Phi_{-5/3}$ does not. It follows that

$$\mathcal{W}_{\text{local}} \xrightarrow{C} \sum_{k_1, k_2 > 0} (-1)^{3k_1} a_{k_1} (X^{3k_1} + Z^{3k_1} + Y^{3k_1}) + b_{k_2} (\tilde{X}^{3k_2} + \tilde{Z}^{3k_2} + \tilde{Y}^{3k_2}) \quad (5.4)$$

$$\stackrel{!}{=} -\mathcal{W}_{\text{local}} = \sum_{k_1, k_2 > 0} -a_{k_1} (X^{3k_1} + Y^{3k_1} + Z^{3k_1}) - b_{k_2} (\tilde{X}^{3k_2} + \tilde{Y}^{3k_2} + \tilde{Z}^{3k_2}). \quad (5.5)$$

This implies that the b coefficients have to vanish and that $3k_1$ has to be odd. This reproduces rule 4. Again, the hybrid $\mathbb{Z}_2^{\text{hybrid}}$ plays a crucial role. The rotational $\mathbb{Z}_3^{\text{rot}}$ (as discussed in the last section) is relevant again, as it is included in $\Delta(27)$. Hybrid $\mathbb{Z}_2^{\text{hybrid}}$ and rotational $\mathbb{Z}_3^{\text{rot}}$ are at the heart of

this solution. This is of truly eclectic nature as an interplay of the traditional $\mathbb{Z}_3$ symmetry and the modular $\mathbb{Z}_2$ symmetry. They combine to the non-Abelian group S_3 . As the superpotential transforms nontrivially under $\mathbb{Z}_2$, this is an R symmetry S_3^R . This explains why this solution has been missed in the earlier discussion [4], as there only Abelian R symmetries had been considered.

VI. SOME IMMEDIATE CONSEQUENCES

As long as rule 4 was considered as a stringy miracle, its range of validity was not clear. In many applications, it was, thus, assumed that this rule 4 would also hold under more general circumstances as, for example, in the presence of background fields such as Wilson lines. Now that we know that rule 4 has its origin in a non-Abelian R symmetry, we can analyze more general situations.

Properties of Wilson lines have been first discussed in Ref. [8]. There, it was shown that such background fields not only break gauge symmetries (in the heterotic orbifold picture), but also lift the degeneracy of the fixed points. This second property was important for the construction of models with three families of quarks and leptons. In our previous discussion, the degeneracy of the fixed points was the result of the rotational $\mathbb{Z}_3^{\text{rot}}$ symmetry. Thus, Wilson lines break this symmetry [and with it $\Delta(54)$]. An analysis of the modular flavor symmetry reveals the fact that also $\text{SL}(2, \mathbb{Z})$ is generically broken by Wilson lines [21]. This would then imply that the finite modular flavor symmetry T' is broken as well. As both $\mathbb{Z}_3^{\text{rot}}$ and T' have been crucial for the proof of rule 4, we would then have to worry that such a rule might not hold in the presence of Wilson lines. It seems that the full beauty of the eclectic flavor symmetry needs the consideration of models like those constructed in Refs. [22,23], where one of the three two-tori $\mathbb{T}^2$ (in six-dimensional) compactified space does not feel the Wilson line background fields.

VII. SUMMARY AND OUTLOOK

We have seen that the presence of duality transformations in string theory can manifest itself in a variety of restrictions in the low-energy effective action. Some of them can be understood through the appearance of certain modular symmetries derived from the modular group $\text{SL}(2, \mathbb{Z})$. Among these are

- (i) the appearance of a (nonlinearly realized) discrete modular symmetry;
- (ii) it is accompanied by a discrete linearly realized traditional flavor symmetry;
- (iii) the discrete modular group can be understood in a bottom-up approach through the outer automorphisms of the traditional flavor group;
- (iv) both combine to form an eclectic flavor group; and
- (v) at some specific points in moduli space, there are enhancements of the linearly realized discrete symmetry.

Apart from these somewhat obvious restrictions, we are led to some surprises. The full power of the modular symmetry is encoded in the modular forms that appear as coefficients in the Yukawa couplings. These constraints are indirect and appear somewhat intransparent from a general field theoretic point of view. Technically, the rules can be easily implemented through a construction of the modular forms as products of the basic modular forms of lowest weight [around Eq. (5.8) in [24]]. In the $\mathbb{Z}_3$ case, the lowest modular form at weight one builds a $2''$ representation of T' . Its products lead to a 3 at weight two, but the singlet is missing. The absence of certain representations continues at higher weight. The direct consequences of these observations are not yet fully understood and need further investigations.

In the present paper, we have discussed in detail the explanation of an enigmatic selection rule in string compactifications, Rule 4, in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold through modular and conventional flavor symmetries. The important lesson is not just the explanation of this selection rule, but the mechanism how traditional and modular symmetries combine to lead to this result. In particular, we showed that rule 4 can be explained in two different ways through a non-Abelian R symmetry. In the first case, rule 4 can be explained through the T' modular group (including the $\mathbb{Z}_2^{\text{hybrid}}$ symmetry), the point group $\mathbb{Z}_3^{(\text{PG})}$, and the traditional flavor symmetry $\mathbb{Z}_3^{\text{rot}}$. In the second case, rule 4 is explained solely by the traditional flavor group $\Delta(54)$. In both cases, the key component to explain rule 4 is the non-Abelian R symmetry given by $S_3^R \cong \mathbb{Z}_3^{\text{rot}} \rtimes \mathbb{Z}_2^{\text{hybrid}}$. This explains why rule 4 could not be obtained from conventional symmetries in [4], which did not consider non-Abelian R symmetries.

This seems to be just the tip of the iceberg, though. Many questions are still open. The modular forms “know” that the nonlinearly realized modular symmetries get enhanced at some specific points in moduli space.⁶ What is the role of these symmetries in the interior of moduli space? Is there something specific in the nonlinear realization of modular symmetries that is not shared by nonlinearly realized symmetries that appear from a spontaneous breakdown of a linearly realized symmetry via a Higgs mechanism?

One of the important aspects of the system seems to be the role of outer automorphisms of various symmetry groups. In the orbifold construction, outer automorphisms of the lattice lead to the space-group selection rule and outer automorphisms of the Narain space group complete the eclectic flavor group. From a bottom-up approach, the discrete modular flavor group is connected to the outer automorphisms of the traditional discrete flavor group. It should be further analyzed how these properties can

influence the presence or absence of couplings in the low-energy effective theory.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: SOME GROUP THEORETICAL DETAILS

1. Group $T' \cong [24, 3]$

The finite modular group $\Gamma'_3 \cong T' \cong Q_8 \rtimes \mathbb{Z}_3 \cong [24, 3]$ has 24 elements, which can be generated by the generators S and T , satisfying the presentation

$$T' = \langle S, T | S^4 = (ST)^3 = T^3 = S^2TS^{-2}T^{-1} = 1 \rangle. \quad (\text{A1})$$

Its irreducible representations are a triplet 3 , three doublets $2, 2', 2''$, and three singlets $1, 1', 1''$. The corresponding representation matrices are shown in Table III below.

The nontrivial tensor product decomposition and Clebsch-Gordan coefficients of T' (in the basis chosen in this work) are given by

TABLE III. The irreducible representation matrices for finite group T' .

Irrep	$\rho_r(S)$	$\rho_r(T)$
1	1	1
$1'$	1	ω
$1''$	1	ω^2
2	$\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\omega \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$
$2'$	$\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$
$2''$	$\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$
3	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

⁶The explicit form of the superpotential at these specific points can be found in, e.g., Table 4 and Eq. (3.59) in Ref. [19].

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_2 &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2'} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{2''} = (x_1 y_2 - x_2 y_1)_1 \oplus \begin{pmatrix} x_1 y_2 + x_2 y_1 \\ \sqrt{2} x_2 y_2 \\ -\sqrt{2} x_1 y_1 \end{pmatrix}_3, \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2'} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{2'} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{2''} = (x_1 y_2 - x_2 y_1)_{1''} \oplus \begin{pmatrix} \sqrt{2} x_2 y_2 \\ -\sqrt{2} x_1 y_1 \\ x_1 y_2 + x_2 y_1 \end{pmatrix}_3, \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2''} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{2''} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{2'} = (x_1 y_2 - x_2 y_1)_{1'} \oplus \begin{pmatrix} -\sqrt{2} x_1 y_1 \\ x_1 y_2 + x_2 y_1 \\ \sqrt{2} x_2 y_2 \end{pmatrix}_3, \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_3 &= \begin{pmatrix} x_1 y_1 + \sqrt{2} x_2 y_3 \\ \sqrt{2} x_1 y_2 - x_2 y_1 \end{pmatrix}_2 \oplus \begin{pmatrix} x_1 y_2 + \sqrt{2} x_2 y_1 \\ \sqrt{2} x_1 y_3 - x_2 y_2 \end{pmatrix}_{2'} \oplus \begin{pmatrix} x_1 y_3 + \sqrt{2} x_2 y_2 \\ \sqrt{2} x_1 y_1 - x_2 y_3 \end{pmatrix}_{2''}, \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2'} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_3 &= \begin{pmatrix} x_1 y_3 + \sqrt{2} x_2 y_2 \\ \sqrt{2} x_1 y_1 - x_2 y_3 \end{pmatrix}_2 \oplus \begin{pmatrix} x_1 y_1 + \sqrt{2} x_2 y_3 \\ \sqrt{2} x_1 y_2 - x_2 y_1 \end{pmatrix}_{2'} \oplus \begin{pmatrix} x_1 y_2 + \sqrt{2} x_2 y_1 \\ \sqrt{2} x_1 y_3 - x_2 y_2 \end{pmatrix}_{2''}, \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2''} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_3 &= \begin{pmatrix} x_1 y_2 + \sqrt{2} x_2 y_1 \\ \sqrt{2} x_1 y_3 - x_2 y_2 \end{pmatrix}_2 \oplus \begin{pmatrix} x_1 y_3 + \sqrt{2} x_2 y_2 \\ \sqrt{2} x_1 y_1 - x_2 y_3 \end{pmatrix}_{2'} \oplus \begin{pmatrix} x_1 y_1 + \sqrt{2} x_2 y_3 \\ \sqrt{2} x_1 y_2 - x_2 y_1 \end{pmatrix}_{2''}, \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_3 &= (x_1 y_1 + x_2 y_3 + x_3 y_2)_1 \oplus (x_1 y_2 + x_2 y_1 + x_3 y_3)_{1'} \oplus (x_1 y_3 + x_2 y_2 + x_3 y_1)_{1''} \\
&\oplus \begin{pmatrix} 2x_1 y_1 - x_2 y_3 - x_3 y_2 \\ 2x_3 y_3 - x_1 y_2 - x_2 y_1 \\ 2x_2 y_2 - x_1 y_3 - x_3 y_1 \end{pmatrix}_{3_S} \oplus \begin{pmatrix} x_3 y_2 - x_2 y_3 \\ x_2 y_1 - x_1 y_2 \\ x_1 y_3 - x_3 y_1 \end{pmatrix}_{3_A}. \tag{A2}
\end{aligned}$$

Here, the subscripts “S” and “A” denote symmetric and antisymmetric contractions, respectively.

A very common and useful quotient group for T' is $A_4 \cong \mathbb{Z}_2^2 \rtimes \mathbb{Z}_3 \cong T'/\mathbb{Z}_2$ (GAP ID [12, 3]). A_4 has 12 elements, which can be generated by the generators S and T, satisfying the presentation

$$A_4 = \langle S, T | S^2 = (ST)^3 = T^3 = 1 \rangle. \tag{A3}$$

Its irreducible representations are a triplet **3** and three singlets **1**, **1'**, **1''**. Note that T' is the double cover of A_4 .

2. Group $\Delta(54) \cong [54, 8]$

The finite group $\Delta(54) \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes S_3 \cong [54, 8]$ has 54 elements, which can be generated by three generators A, B, and C satisfying the presentation

$$\Delta(54) = \langle A, B, C | A^3 = B^3 = C^2 = (AB)^3 = (AB^2)^3 = (AC)^2 = (BC)^2 = 1 \rangle. \tag{A4}$$

Its irreducible representations are two singlets, four doublets, and two triplets plus their complex conjugates. The corresponding representation matrices are shown in Table IV below. It is useful to list the nontrivial tensor products of $\Delta(54)$ irreducible representations:

$$\begin{aligned}
\mathbf{1}' \otimes \mathbf{1}' &= \mathbf{1}, & \mathbf{1}' \otimes \mathbf{2}_k &= \mathbf{2}_k, & \mathbf{1}' \otimes \mathbf{3}_1 &= \mathbf{3}_2, & \mathbf{1}' \otimes \mathbf{3}_2 &= \mathbf{3}_1, & \mathbf{1}' \otimes \bar{\mathbf{3}}_1 &= \bar{\mathbf{3}}_2, & \mathbf{1}' \otimes \bar{\mathbf{3}}_2 &= \bar{\mathbf{3}}_1, \\
\mathbf{2}_k \otimes \mathbf{2}_k &= \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}_k, & \mathbf{2}_k \otimes \mathbf{2}_\ell &= \mathbf{2}_m \oplus \mathbf{2}_n \quad \text{with } k \neq \ell \neq m \neq n, k, \ell, m, n = 1, \dots, 4, \\
\mathbf{2}_k \otimes \mathbf{3}_\ell &= \mathbf{3}_1 \oplus \mathbf{3}_2, & \mathbf{2}_k \otimes \bar{\mathbf{3}}_\ell &= \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_2 \quad \text{for all } k = 1, \dots, 4, \quad \ell = 1, 2, \\
\mathbf{3}_\ell \otimes \mathbf{3}_\ell &= \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_2 \oplus \bar{\mathbf{3}}_3, & \mathbf{3}_1 \otimes \mathbf{3}_2 &= \bar{\mathbf{3}}_2 \oplus \bar{\mathbf{3}}_3 \oplus \bar{\mathbf{3}}_1, & \mathbf{3}_1 \otimes \bar{\mathbf{3}}_1 &= \mathbf{1} \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{2}_4, \\
\mathbf{3}_1 \otimes \bar{\mathbf{3}}_2 &= \mathbf{3}_2 \otimes \bar{\mathbf{3}}_1 = \mathbf{1}' \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{2}_4, \\
\mathbf{3}_2 \otimes \bar{\mathbf{3}}_2 &= \mathbf{1} \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{2}_4, & \bar{\mathbf{3}}_\ell \otimes \bar{\mathbf{3}}_\ell &= \mathbf{3}_1 \oplus \mathbf{3}_1 \oplus \mathbf{3}_2, & \bar{\mathbf{3}}_1 \otimes \bar{\mathbf{3}}_2 &= \mathbf{3}_2 \oplus \mathbf{3}_2 \oplus \mathbf{3}_1.
\end{aligned} \tag{A5}$$

Some useful Clebsch-Gordan coefficients (in the basis chosen in this work) are as follows [25]:

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_\ell} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_\ell} &= \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ x_3 y_3 \end{pmatrix}_{\bar{\mathbf{3}}_1} \oplus \begin{pmatrix} x_2 y_3 + x_3 y_2 \\ x_3 y_1 + x_1 y_3 \\ x_1 y_2 + x_2 y_1 \end{pmatrix}_{\bar{\mathbf{3}}_1} \oplus \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}_{\bar{\mathbf{3}}_2}, \quad \ell = 1, 2, \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_2} &= (x_1 y_1 + x_2 y_2 + x_3 y_3)_{\mathbf{1}'} \oplus \begin{pmatrix} x_1 y_1 + \omega^2 x_2 y_2 + \omega x_3 y_3 \\ -\omega x_1 y_1 - \omega^2 x_2 y_2 - x_3 y_3 \end{pmatrix}_{\mathbf{2}_1} \oplus \begin{pmatrix} x_1 y_2 + \omega^2 x_2 y_3 + \omega x_3 y_1 \\ -\omega x_1 y_3 - \omega^2 x_2 y_1 - x_3 y_2 \end{pmatrix}_{\mathbf{2}_2} \\
&\oplus \begin{pmatrix} x_1 y_3 + \omega^2 x_2 y_1 + \omega x_3 y_2 \\ -\omega x_1 y_2 - \omega^2 x_2 y_3 - x_3 y_1 \end{pmatrix}_{\mathbf{2}_3} \oplus \begin{pmatrix} x_1 y_3 + x_2 y_1 + x_3 y_2 \\ -x_1 y_2 - x_2 y_3 - x_3 y_1 \end{pmatrix}_{\mathbf{2}_4}, \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\bar{\mathbf{3}}_1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_2} &= (x_1 y_1 + x_2 y_2 + x_3 y_3)_{\mathbf{1}'} \oplus \begin{pmatrix} x_1 y_1 + \omega^2 x_2 y_2 + \omega x_3 y_3 \\ -\omega x_1 y_1 - \omega^2 x_2 y_2 - x_3 y_3 \end{pmatrix}_{\mathbf{2}_1} \oplus \begin{pmatrix} x_1 y_3 + \omega^2 x_2 y_1 + \omega x_3 y_2 \\ -\omega x_1 y_2 - \omega^2 x_2 y_3 - x_3 y_1 \end{pmatrix}_{\mathbf{2}_2} \\
&\oplus \begin{pmatrix} x_1 y_2 + \omega^2 x_2 y_3 + \omega x_3 y_1 \\ -\omega x_1 y_3 - \omega^2 x_2 y_1 - x_3 y_2 \end{pmatrix}_{\mathbf{2}_3} \oplus \begin{pmatrix} x_1 y_2 + x_2 y_3 + x_3 y_1 \\ -x_1 y_3 - x_2 y_1 - x_3 y_2 \end{pmatrix}_{\mathbf{2}_4}.
\end{aligned} \tag{A6}$$

TABLE IV. Irreducible representation matrices for the finite group $\Delta(54)$.

Irrep	$\rho_r(\text{A})$	$\rho_r(\text{B})$	$\rho_r(\text{C})$
1	1	1	1
1'	1	1	-1
2₁	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
2₂	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
2₃	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
2₄	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3₁	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\bar{3}_1$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
3₂	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$\bar{3}_2$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

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- [1] S. Hamidi and C. Vafa, *Nucl. Phys.* **B279**, 465 (1987).
[2] L. J. Dixon, D. Friedan, E. J. Martinec, and S. H. Shenker, *Nucl. Phys.* **B282**, 13 (1987).
[3] A. Font, L. E. Ibáñez, H. P. Nilles, and F. Quevedo, *Nucl. Phys.* **B307**, 109 (1988); **B310**, 764(E) (1988).
[4] A. Font, L. E. Ibáñez, H. P. Nilles, and F. Quevedo, *Phys. Lett. B* **213**, 274 (1988).
[5] H. P. Nilles, S. Ramos-Sánchez, and P. K. Vaudrevange, *J. High Energy Phys.* **02** (2020) 045.
[6] H. P. Nilles, S. Ramos-Sánchez, and P. K. S. Vaudrevange, *Nucl. Phys.* **B957**, 115098 (2020).
[7] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, *Phys. Lett. B* **795**, 7 (2019).
[8] L. E. Ibáñez, H. P. Nilles, and F. Quevedo, *Phys. Lett. B* **187**, 25 (1987).
[9] T. Kobayashi, S. L. Parameswaran, S. Ramos-Sánchez, and I. Zavala, *J. High Energy Phys.* **05** (2012) 008.
[10] S. Groot Nibbelink and P. K. S. Vaudrevange, *J. High Energy Phys.* **04** (2017) 030.
[11] J. Lauer, J. Mas, and H. P. Nilles, *Phys. Lett. B* **226**, 251 (1989).
[12] W. Lerche, D. Lust, and N. P. Warner, *Phys. Lett. B* **231**, 417 (1989).
[13] E. J. Chun, J. Mas, J. Lauer, and H. P. Nilles, *Phys. Lett. B* **233**, 141 (1989).
[14] The GAP Group, GAP—Groups, Algorithms, and Programming, Version 4.15.1 (2025), <https://www.gap-system.org>.
[15] T. Kobayashi, H. P. Nilles, F. Plöger, S. Raby, and M. Ratz, *Nucl. Phys.* **B768**, 135 (2007).
[16] M.-C. Chen, M. Ratz, and A. Trautner, *J. High Energy Phys.* **09** (2013) 096.
[17] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, *Nucl. Phys.* **B947**, 114737 (2019).
[18] H. P. Nilles, S. Ramos-Sánchez, and P. K. S. Vaudrevange, *Phys. Lett. B* **808**, 135615 (2020).
[19] X. Li, X.-G. Liu, H. P. Nilles, M. Ratz, and A. Stewart, *J. High Energy Phys.* **09** (2025) 026.

-
- [20] X.-G. Liu and G.-J. Ding, *J. High Energy Phys.* **08** (2019) 134.
- [21] D. Bailin, A. Love, W. A. Sabra, and S. Thomas, *Mod. Phys. Lett. A* **09**, 1229 (1994).
- [22] B. Carballo-Pérez, E. Peinado, and S. Ramos-Sánchez, *J. High Energy Phys.* **12** (2016) 131.
- [23] A. Baur, H. P. Nilles, S. Ramos-Sánchez, A. Trautner, and P. K. S. Vaudrevange, *J. High Energy Phys.* **09** (2022) 224.
- [24] X.-G. Liu and G.-J. Ding, *J. High Energy Phys.* **03** (2022) 123.
- [25] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, *Prog. Theor. Phys. Suppl.* **183**, 1 (2010).