

Collinear structure of nonlinear small- x evolution

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We introduce a novel approach to high-energy QCD factorization of cross sections for processes involving a dilute projectile and a dense target. Our method preserves the factorization between “fast” and “slow” modes in the longitudinal momentum k^+ for a projectile moving along the positive light cone direction, making it compatible with higher-order loop computations done within the standard formulation of the color glass condensate (CGC) effective field theory. Moreover, it eliminates the anomalous double collinear logarithms that typically hinder the convergence of perturbation theory at small- x . Our scheme amounts to a change of basis in the space of CGC operators, introducing an arbitrary transverse scale dependence while leaving physical cross sections invariant. Implementing this scheme requires a modification of the high-energy renormalization group equation for the CGC operators, which we explicitly derive at next-to-leading order and in the nonlinear regime for the case of the dipole operator and the deep inelastic scattering impact factor. This general framework provides a consistent and systematic prescription for computing observables in the small- x regime of QCD, free from double logarithmic collinear instabilities.

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I. INTRODUCTION

Quantum chromodynamics (QCD) factorization provides the foundational framework for studying hadronic structure in high-energy processes [1]. A paradigmatic example is deep inelastic scattering (DIS), characterized by a large momentum transfer Q^2 and a fixed Bjorken variable $x_{\text{Bj}} = Q^2/s$, where s is of the order of the squared center-of-mass energy of the virtual photon-nucleon collision. In this kinematic regime, the leading power in Q^2 factorizes into the convolution of a hard matrix element—calculable in perturbative QCD—that encodes short-distance dynamics involving quark and gluon fields, with universal parton distribution functions, which capture the long-distance, nonperturbative structure of the hadron.

Higher precision is systematically achieved by incorporating corrections at higher orders in the strong coupling α_s .

This factorization picture has been remarkably successful in probing the hadron structure. However, it is expected to break down at small x_{Bj} (or equivalently, in the high-energy or Regge limit $s \gg Q^2$ [2]), where large logarithms of $\ln(1/x_{\text{Bj}})$ spoil the convergence of the perturbative series. Moreover, nonlinear effects become significant due to the rapid rise of gluon densities at small x_{Bj} , signaling the onset of gluon saturation [3–5].

To address this breakdown, alternative frameworks have been developed, notably k_t -factorization [6] and the dipole formalism in transverse coordinate space [7] (see e.g. Ref. [8] for a review). In these approaches, the central object is the unintegrated gluon distribution or, equivalently, the dipole S -matrix. Its evolution is governed by the Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation in the dilute regime [9–11] and by the nonlinear Balitsky–Kovchegov and Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner (BK–JIMWLK) equations in the dense regime [12–19]. These evolution equations incorporate saturation physics [3–5], a manifestation of unitarity at small x , and form the foundation of many

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phenomenological studies at HERA, RHIC, and the LHC (see e.g. Ref. [20] for a review).

A major milestone in this theoretical development was the derivation of the next-to-leading order (NLO) corrections to both the BK equation [21,22] and the DIS impact factor [23–30], aimed at improving the precision of small- x predictions. However, two related issues soon emerged, raising concerns about the consistency and reliability of the high-energy factorization framework: (i) the NLO BK equation was found to suffer from severe numerical instabilities [31], and (ii) the NLO impact factor for the total DIS cross section defined as the finite part after naively subtracting the projectile rapidity divergence, can become large [32] leading to unphysical results such as negative cross sections (see also Refs. [33,34] for a similar phenomenon in forward particle production in proton-nucleus collisions, as well as the subsequent literature addressing this issue [35–46]). For less inclusive processes, such as two-particle correlations in DIS, similar problems have been identified [47–49] in connection with the interplay between small- x and Sudakov resummation [50,51]. Finally, exclusive processes such as exclusive vector meson production in DIS [52] are also affected by large NLO corrections in the color glass condensate (CGC).

These instabilities have antecedents in the context of NLO BFKL evolution and are now understood to arise from an improper identification of rapidity logarithms $\ln(s/s_0)$, which require all-order resummation [53–61]. At NLO, this manifests in the appearance of large collinear double logarithms of the form $\ln^2(Q^2/Q_0^2)$, which compromise the convergence of the perturbative expansion, particularly when the energy scale is chosen as $s_0 = Q_0^2$ or $s_0 = Q_0 Q$ with $Q_0 \ll Q$ a characteristic nonperturbative scale of the target. In the context of DIS for instance, identifying the rapidity logarithm with the choice $s_0 = Q_0 Q$ [62] and relating it to $x_{\text{Bj}} = Q^2/s$ yields

$$\ln\left(\frac{s}{Q_0 Q}\right) = \ln\left(\frac{1}{x_{\text{Bj}}}\right) + \frac{1}{2} \ln\left(\frac{Q^2}{Q_0^2}\right), \quad (1)$$

highlighting the fact that a change in the rapidity variable induces spurious collinear logarithms that become problematic beyond leading order when $Q \gg Q_0$. This feature can be traced back to a violation of time ordering in gluon emissions that is not systematically accounted for in the standard high-energy factorization framework [32,63]. This underscores the need for a consistent rapidity-factorization scheme that minimizes the sensitivity of the perturbative series on the choice of rapidity variable. In the BFKL framework, such discrepancies are typically resolved by adjusting the energy scale s_0 . However, in the nonlinear BK case, the resolution is more subtle, as no fully analytic treatment is available.

In DIS, the transverse scale of the photon impact factor is $Q^2 \gg \Lambda_{\text{QCD}}^2$, or $\gg$ the saturation scale Q_s^2 , implying a large

phase space for strongly ordered gluon emissions:

$$Q_s \sim Q_0 \ll k_{\perp,1} \ll k_{\perp,2} \ll \cdots \ll Q. \quad (2)$$

This Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP)-like regime breaks the symmetry between Q and Q_0 underlying the Regge factorization, which deals solely with $\ln s$ resummation, and thus, requires the resummation of contributions of the form $(\bar{\alpha}_s \ln(Q^2/Q_0^2))^n$ to all orders in the coupling constant $\bar{\alpha}_s \equiv \alpha_s N_c/\pi$. In DIS, a more judicious choice for the rapidity variable is obtained by setting $s_0 = Q^2$, which ensures a proper resummation of the large collinear logarithms. In terms of rapidity ordering, this corresponds to evolution in k^- rather than k^+ [63]. In processes like $\gamma^* \gamma^*$ scattering, both collinear ($Q \gg Q_0$) and anticollinear ($Q \ll Q_0$) regimes must be treated symmetrically, motivating the scale choice $s_0 = Q Q_0$. Nevertheless, the shockwave formalism commonly adopts k^+ as the rapidity variable, which resums anticollinear logarithms but not collinear ones, making it suboptimal for DIS.

To illustrate this, consider the one-loop correction to DIS in the dipole frame [64–67], where the virtual photon propagates in the $+z$ direction. The natural evolution variable in this frame is the longitudinal momentum $k^+ \ll q^+$, where q^+ is the large light cone component of the photon momentum. At double logarithmic accuracy, the phase space integral for gluon radiation reads as

$$\begin{aligned} & \bar{\alpha}_s \int_{Q_0^2}^{Q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \int_{q_0^+}^{(k_{\perp}^2/Q^2)q^+} \frac{dk^+}{k^+} \\ &= \bar{\alpha}_s \ln\left(\frac{Q^2}{Q_0^2}\right) \ln\left(\frac{s}{Q_0 Q}\right) = \bar{\alpha}_s \ln\left(\frac{Q^2}{Q_0^2}\right) \\ & \times \ln\left(\frac{1}{x_{\text{Bj}}}\right) - \frac{\bar{\alpha}_s}{2} \ln^2\left(\frac{Q^2}{Q_0^2}\right), \end{aligned} \quad (3)$$

where q_0^+ is a target scale playing the role of a plus longitudinal momentum cutoff, and which can be taken as $Q_0^2/(2P^-)$, while $s = 2q^+ P^-$ is related the virtual photon-nucleon center-of-mass energy squared W^2 and the nucleon mass m by $W^2 = s - Q^2 + m^2$. Throughout, we implicitly work in the Regge kinematics, where one assumes that $W^2 \sim s$.

The first term in Eq. (3) is expected in the collinear limit from DGLAP resummation. However, the second collinear double logarithm—arising from a mistreatment of the collinear limit in high-energy factorization—is incorrect, and in fact incompatible with the renormalization group equations, which constrain the highest power of collinear logarithms to not exceed that of the coupling constant [56].

The k^+ integral in Eq. (3) is bounded from above by $(k_{\perp}^2/Q^2)q^+$ as consequence of the gluon lifetime $t_f \sim 1/k^-$ being smaller than that of the virtual photon $1/|q^-|$, a feature that is correctly taken into account by NLO

calculations in the CGC framework. However, CGC calculations in dilute-dense collisions typically neglect that k^- is also bounded from above by the target longitudinal momentum P^- . This is a consequence of taking the shock-wave approximation where the longitudinal size of the target $\sim 1/P^-$ is taken to be zero, such that the constraint $k^- \sim k^+/k_\perp^2 \ll P^-$ becomes irrelevant.

The kinematic constraint on k^+ ,

$$k^+ \gg \frac{k_\perp^2}{2P^-}, \quad (4)$$

ensures proper time ordering between the gluon and the longitudinal size of the target. Enforcing this constraint modifies the phase space to

$$\bar{\alpha}_s \int_{Q_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_{k_\perp^2/2P^-}^{k_\perp^2 q^+/Q^2} \frac{dk^+}{k^+} = \bar{\alpha}_s \ln\left(\frac{Q^2}{Q_0^2}\right) \ln\left(\frac{s}{Q^2}\right) = \bar{\alpha}_s \ln\left(\frac{Q^2}{Q_0^2}\right) \ln\left(\frac{1}{x_{\text{Bj}}}\right), \quad (5)$$

which differs from Eq. (3) by the argument of the rapidity logarithm. As compared to Eq. (3), Eq. (5) has the expected double logarithmic structure, without additional collinear double logarithmic.

In order to resum both collinear and anticollinear logarithms simultaneously in the BK equation, various strategies have been devised, including explicit resummation of leading double (and single) collinear logarithms and kinematically constrained evolutions [32,63,68], which enforce ordering in both k^+ and k^- .

While these methods alleviate numerical instabilities [63,69], they involve *ad hoc*, albeit physically motivated, modifications and do not provide a systematic prescription for higher-order calculations.

In this work and in the companion Letter [70], we propose a robust framework for high-energy factorization based on evolution in the commonly adopted rapidity variable $\ln(k^+/q^+)$. This framework does not rely on a new effective theory or on a new Wilsonian mode separation. Rather, we construct classes of scheme transformations—analogue to finite renormalization in the context of UV renormalization—embedded in the standard k^+ -ordered CGC effective field theory and implemented via an operator-level similarity transformation, such that the resulting high-energy renormalization-group equations are free of double collinear logarithms. Our approach exploits the factorization-scheme invariance of the cross section to redefine the dipole operator, its rapidity-evolution kernel, and the impact factor. This transformation introduces a new factorization scale μ that interpolates between collinear and rapidity logarithms, thereby reshuffling them among different components of the factorized cross section. Remarkably, the choice $\mu = Q$ matches the definition of the rapidity

variable that restores proper time ordering at NLO and provides a consistent path toward stable higher-order evolution as numerically illustrated in the companion letter [70]. This transformation ensures that collinear double logarithms are resummed to all orders. The remaining anticollinear double logarithms [71] are less severe as they show up at much higher rapidity [63].

We derive here a novel NLO BK equation given by Eq. (20) that systematically subtracts the collinear double logarithm order by order in perturbation theory and resums the proper physical large logarithm of the process at hand. In the companion letter [70], we demonstrate its numerical stability up to large rapidity where running-coupling resummation effects and next-to-next-to-leading order (NNLO) corrections start to be relevant. In concrete terms, the Letter [70] highlights the main construction of the scheme transformation in the Doi-Peliti language, the resulting transformed NLO BK equation, and its numerical consequences, whereas the present manuscript provides the full theoretical development, including detailed derivations of the transformed CGC operators, consistency checks, and a systematic analysis of the formal higher-order structure. In a future work, we plan to carry out a comprehensive numerical study of our novel NLO BK equation, which is free of double collinear logarithms, with particular emphasis on the choice of initial conditions and the implementation of running-coupling effects.

Our paper is organized as follows. Section II summarizes the main results of this work; it is designed to be self-contained, compiling all practical formulas to be used for the numerical implementation of our high-energy factorization framework in the specific case of the DIS structure functions (the generalization to other standard small- x processes depending on the dipole operator should be straightforward). Section III motivates the introduction of our composite dipole formulation of the high-energy operator product expansion (OPE) through an explicit computation of the one-loop correction to a generalized dipole operator, first introduced in [72,73]. This operator reduces to the standard dipole operator in the $x \rightarrow 0$ limit but also captures information about the longitudinal size of the shock wave, and hence about life-time ordering. In Sec. IV, we provide a formal treatment of the high-energy OPE in terms of this composite dipole, highlighting the structure of high-energy factorization and how it transforms under a change of basis between “bare” and “composite” operators. Sections V and VI respectively address how this new high-energy OPE resolves issues related to small- x impact factors and instabilities in the NLO renormalization group equation associated with collinear logarithms. In Sec. VII, we derive the NLO BK equation for the composite dipole in coordinate space, which constitutes the essential practical component of high-energy QCD factorization in the nonlinear regime. Finally, in Sec. VIII, we discuss the anticollinear regime of our new NLO BK equation and

outline how the present analysis can be extended to the running coupling case.

II. SUMMARY OF THE RESULTS

The main objective of this work is to amend the current formulation of high-energy factorization, which, in its present form, exhibits unstable evolution at NLO. The lack of convergence in the perturbative series has been extensively studied and is now understood to originate from large collinear logarithms, which stem from the absence of a proper time-ordering constraint. However, the commonly adopted resolution—imposing a kinematic constraint to restore time ordering—remains unsatisfactory from a formal standpoint, as it constitutes an *ad hoc* fix that may compromise the control over subleading logarithms at any given order in perturbation theory.

A recent study [63] has suggested that the problem arises from an inappropriate choice of rapidity variable, thereby casting doubt on the viability of the shockwave framework based on factorization in the k^+ variable. Nevertheless, a systematic framework for performing higher-order computations beyond the well-defined shockwave approach is still lacking.

In this work, we develop such a first-principles framework by

- (1) retaining the shockwave formalism with k^+ -ordering, along with its previously established results;
- (2) introducing a collinear factorization scale μ , that allows one to precisely define the rapidity phase space for high-energy evolution; and
- (3) exploiting the scheme dependence of rapidity factorization to reshuffle the large collinear logarithms (responsible for the instability of the rapidity evolution) from the evolution kernel to the initial condition and the impact factor.

To next to leading order the factorized DIS cross section, mediated by a longitudinally (L) or transversely (T) polarized virtual photon, reads schematically as

$$\sigma_{L/T}(x, Q^2) = [H_{\text{LO}}(Q^2) \otimes S_{\text{NLL}}(\zeta) + \Delta H_{\text{NLO}}(Q^2, \zeta/s) \otimes S_{\text{LL}}(\zeta)], \quad (6)$$

where $\zeta = 2\rho^+P^-$ denotes the rapidity factorization scale expressed in terms of the plus rapidity scale ρ^+ dividing the “slow” and the “fast” modes in the CGC and $s = 2q^+P^-$. In addition, S_{LL} and S_{NLL} obey the leading order (LO) [resumming leading logs (LL)] and NLO BK equations [resumming next-to-leading logs (NLL)], respectively. Explicit expressions ΔH_{NLO} can be found in [26,27]; we provide in Appendix A the formulas for the longitudinal DIS cross section $\sigma_L(x, Q^2)$, which is slightly simpler. The LO term is standard textbook material and reads as (see e.g.

Ref. [74])

$$\begin{aligned} H_{\text{LO}}(Q^2) \otimes S_{\text{NLL}}(\zeta) \\ = \sum_i \frac{4\alpha_{\text{em}} e_i^2 N_c S_\perp}{\pi^2} \int d^2\mathbf{x}_{12} \\ \int_0^1 dz_1 z_1 (1 - z_1) \bar{Q}^2 K_0^2(\bar{Q}x_{12}) (1 - S_{12}), \end{aligned} \quad (7)$$

with $\bar{Q}^2 = z_1(1 - z_1)Q^2$ and $S_\perp$ the transverse area of the target (we consider a homogeneous target in the transverse plane for simplicity). The LO contribution is expressed in terms of the dipole operator and defined as

$$S_{12} \equiv S(\mathbf{x}_{12}, \zeta) \equiv \frac{1}{N_c} \langle \text{Tr}[U_{\mathbf{x}_1} U_{\mathbf{x}_2}^\dagger] \rangle, \quad (8)$$

where $U_{\mathbf{x}_1}$ and $U_{\mathbf{x}_2}^\dagger$ denote infinite Wilson lines, which encode the color precession of the high-energy quark and antiquark forming the dipole as they traverse the target’s background field. Their explicit form is given in Eq. (29). Here, $\langle \dots \rangle$ denotes the ensemble average over target configurations.

The ζ -dependence arises from renormalization group evolution following the LL/NLL BK equation, as we shall see below. Throughout this paper, we use the shorthand notation $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ for the difference of two transverse coordinates. The sum over i runs over the n_f light quark flavors with fractional electric charge e_i . To LL accuracy, the dipole operator $S(\mathbf{x}_{12}, \zeta) \equiv S_{12}$, obeys the BK equation

$$\frac{\partial S_{12}}{\partial \ln \zeta} = \bar{\alpha}_s \int_3 K_{123} [S_{13} S_{32} - S_{12}], \quad (9)$$

where $\bar{\alpha}_s = \alpha_s N_c / \pi$ and the LO kernel reads as

$$K_{123} = \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2}. \quad (10)$$

We have used the shorthand notation for the transverse coordinate integrals

$$\int_3 \equiv \int \frac{d^2\mathbf{x}_3}{2\pi}. \quad (11)$$

The NLL evolution equation was derived in [21] and takes the following form in the large N_c limit:

$$\begin{aligned}
\frac{\partial S_{12}}{\partial \ln \zeta} &= \bar{\alpha}_s K_{\text{NLO}} \otimes S_{12} \\
&= \bar{\alpha}_s \int_3 K_{123}^{\text{NLO}} [S_{13} S_{32} - S_{12}] + \frac{\bar{\alpha}_s^2}{2} \int_{3,4} K_{1234} [S_{13} S_{34} S_{42} - S_{13} S_{32}] \\
&\quad + \frac{\bar{\alpha}_s^2 n_f}{2N_c} \int_{3,4} K_f S_{32} [S_{14} - S_{13}],
\end{aligned} \tag{12}$$

where the various kernels have the following definitions:

$$K_{123}^{\text{NLO}} = \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{\beta_0}{N_c} \ln(\mathbf{x}_{12}^2 \mu_R^2) - \frac{\beta_0}{N_c} \frac{\mathbf{x}_{13}^2 - \mathbf{x}_{32}^2}{\mathbf{x}_{12}^2} \ln \frac{\mathbf{x}_{13}^2}{\mathbf{x}_{32}^2} + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{n_f}{N_c} - 2 \ln \frac{\mathbf{x}_{13}^2}{\mathbf{x}_{12}^2} \ln \frac{\mathbf{x}_{32}^2}{\mathbf{x}_{12}^2} \right) \right], \tag{13}$$

$$K_{1234} = -\frac{2}{\mathbf{x}_{34}^4} + \left[\frac{\mathbf{x}_{13}^2 \mathbf{x}_{24}^2 + \mathbf{x}_{14}^2 \mathbf{x}_{32}^2 - 4\mathbf{x}_{12}^2 \mathbf{x}_{34}^2}{\mathbf{x}_{34}^4 (\mathbf{x}_{13}^2 \mathbf{x}_{24}^2 - \mathbf{x}_{14}^2 \mathbf{x}_{32}^2)} + \frac{\mathbf{x}_{12}^4}{\mathbf{x}_{13}^2 \mathbf{x}_{24}^2 (\mathbf{x}_{13}^2 \mathbf{x}_{24}^2 - \mathbf{x}_{14}^2 \mathbf{x}_{32}^2)} + \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{24}^2 \mathbf{x}_{34}^2} \right] \ln \frac{\mathbf{x}_{13}^2 \mathbf{x}_{24}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{32}^2}, \tag{14}$$

$$K_f = \frac{2}{\mathbf{x}_{34}^4} - \frac{\mathbf{x}_{14}^2 \mathbf{x}_{32}^2 + \mathbf{x}_{24}^2 \mathbf{x}_{13}^2 - \mathbf{x}_{12}^2 \mathbf{x}_{34}^2}{\mathbf{x}_{34}^4 (\mathbf{x}_{13}^2 \mathbf{x}_{24}^2 - \mathbf{x}_{14}^2 \mathbf{x}_{32}^2)} \ln \frac{\mathbf{x}_{13}^2 \mathbf{x}_{24}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{32}^2}. \tag{15}$$

In K_{123}^{NLO} , $\beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f$ and μ_R denotes the UV renormalization scale [75,76], not to be confused with the collinear scale μ discussed throughout this work. The double collinear logarithm—the last term in K_{123} given by Eq. (13),

$$2 \ln \frac{\mathbf{x}_{13}^2}{\mathbf{x}_{12}^2} \ln \frac{\mathbf{x}_{32}^2}{\mathbf{x}_{12}^2}, \tag{16}$$

is known to generate numerical instabilities and therefore must be properly treated to ensure the convergence of the perturbative series and maintain control over theoretical uncertainties.

The transformation we propose, which leaves the cross section invariant, introduces an arbitrary collinear scale μ , and results in the following modification of the dipole operator at NLO:

$$\begin{aligned}
S_{12}(\zeta) &\rightarrow \bar{S}_{12}(\zeta, \mu^2) = S_{12}(\zeta) - \bar{\alpha}_s \\
&\quad \times \int_3 K_{123} \ln \mathbf{r}_{123}^2 \mu^2 [S_{13} S_{32} - S_{12}](\zeta),
\end{aligned} \tag{17}$$

where $\mathbf{r}_{123}^2$ is a dynamical transverse scale that can be constructed from the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ and whose precise form does not affect the leading collinear double logarithms.

We note that Eq. (17) closely resembles the conformal dipole proposed in [22], with a key distinction: a factor of $1/2$, which plays a crucial role in fully canceling the double collinear logarithms. This subtle yet important difference, which directly links the coefficient of the logarithmic term $\ln \mu^2$ to the collinear divergences in the NLO BK equation, Eq. (12), had not been identified previously and constitutes one of the central insights of our approach. In the context of BFKL, this is associated with the difference between the so-called DIS and $\gamma^* \gamma^*$ schemes. In the DIS scheme, the scale hierarchy $Q \gg Q_s$ breaks Möbius invariance by spoiling the inversion symmetry $\mathbf{x} \rightarrow \mathbf{x}/\mathbf{x}^2$. This reflects the fact that the dipole operator exhibits a different behavior at short and long distances, with saturation effects governing the latter regime. Drawing on the similarity with Ref. [22], we adopt their terminology and denote $\bar{S}$ as the “composite dipole.”

Under the transformation, Eq. (17), the BK equation to NLO order takes the following form:

$$\begin{aligned}
\frac{\partial \bar{S}_{12}}{\partial \ln(\zeta)} &= \bar{\alpha}_s \bar{K}_{\text{NLO}} \otimes \bar{S}_{12} + \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{134} \ln \left(\frac{\mathbf{r}_{423}^2 \mathbf{r}_{134}^2}{\mathbf{r}_{123}^2 \mathbf{r}_{124}^2} \right) (\bar{S}_{14} \bar{S}_{43} - \bar{S}_{13}) \bar{S}_{32} \\
&\quad - \bar{\alpha}_s^2 \int_{3,4} K_{123} \left[K_{134} \ln \left(\frac{\mathbf{r}_{134}^2}{\mathbf{r}_{123}^2} \right) + K_{324} \ln \left(\frac{\mathbf{r}_{124}^2}{\mathbf{r}_{143}^2} \right) + K_{124} \ln \left(\frac{\mathbf{r}_{123}^2}{\mathbf{r}_{124}^2} \right) \right] (\bar{S}_{13} \bar{S}_{32} - \bar{S}_{12}).
\end{aligned} \tag{18}$$

We will show in detail, in particular in Sec. VI, that this form of the NLO BK equation is free of large collinear double logarithms. A convenient choice for the transverse vector $\mathbf{r}_{123}$ is

$$r_{123}^2 = |\mathbf{x}_{13}||\mathbf{x}_{32}|, \quad (19)$$

as it allows the cancellation of double logarithmic terms, individually, in both color structures $S^2 - S$ and $S^3 - S^2$. A formal justification for this judicious choice will be presented in the next section. This scheme is, in fact, related to the kinematic constraint that ensures $k^- \leq P^-$. The NLO BK equation can now be written explicitly as

$$\begin{aligned} \frac{\partial \bar{S}_{12}}{\partial \ln(\zeta)} &= \bar{\alpha}_s \tilde{K}_{\text{NLO}} \otimes \bar{S}_{12} + \frac{\bar{\alpha}_s^2}{2} \int_{3,4} K_{123} K_{324} \ln\left(\frac{\mathbf{x}_{34}^2}{\mathbf{x}_{14}^2}\right) (\bar{S}_{34} \bar{S}_{42} - \bar{S}_{32}) \bar{S}_{13} \\ &+ \frac{\bar{\alpha}_s^2}{2} \int_{3,4} K_{123} K_{134} \ln\left(\frac{\mathbf{x}_{34}^2}{\mathbf{x}_{24}^2}\right) (\bar{S}_{14} \bar{S}_{43} - \bar{S}_{13}) \bar{S}_{32}, \end{aligned} \quad (20)$$

where $\tilde{K}_{\text{NLO}}$ is the full NLO BK kernel *without* the double collinear logarithm shown in the last term of Eq. (13).

Under the composite dipole transformation the NLO cross section reads as

$$\begin{aligned} \sigma(x, Q^2) &= \bar{H}(Q^2, s/\zeta, \mu^2) \otimes \bar{S}(\zeta/\mu^2) \\ &= \bar{H}_{\text{LO}}(Q^2) \otimes \bar{S}_{\text{NLL}}(\zeta/\mu^2) + \Delta \bar{H}_{\text{NLO}}(Q^2, s/\zeta, \mu^2) \otimes \bar{S}_{\text{LL}}(\zeta/\mu^2) + \mathcal{O}(\bar{\alpha}_s^2), \end{aligned}$$

which differs from Eq. (21) by an additional dependence in the arbitrary collinear scale μ . More explicitly, the composite dipole transformation amounts to adding the following term to Eq. (A1):

$$\begin{aligned} \Delta \bar{H}_{\text{NLO}}(Q^2, s/\zeta, \mu^2) \otimes \bar{S}_{\text{LL}}(\zeta/\mu^2) &= \Delta H_{\text{NLO}}(Q^2, \zeta/s) \otimes \bar{S}_{\text{LL}}(\zeta) - \bar{\alpha}_s \sum_i \frac{4\alpha_{\text{em}} e_i^2 N_c S_\perp}{\pi^2} \\ &\times \int d^2 \mathbf{x}_{12} \int_0^1 dz_1 z_1 (1-z_1) \bar{Q}^2 K_0^2(\bar{Q} x_{12}) \int d^2 \mathbf{x}_3 K_{123} \ln r_{123}^2 \mu^2 [\bar{S}_{13} \bar{S}_{32} - \bar{S}_{12}](\zeta). \end{aligned} \quad (21)$$

It is interesting to note that such a counterterm in the NLO impact factor also appears when computing the DIS structure function with a rapidity regulator in the k^- variable instead of the standard k^+ variable [77,78]. Equations (17), (20), and (21) constitute the main results of this work.

Using the LO BK evolution equation, Eq. (9), together with Eq. (17), we can easily verify that

$$\frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \zeta} = -\frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \mu^2}, \quad (22)$$

which implies that $\bar{S}$ is a function of ζ and μ through the scaling variable ζ/μ^2 , that is,

$$\bar{S}(\zeta, \mu^2) \equiv \bar{S}(\zeta/\mu^2). \quad (23)$$

This property, which *a priori* holds at LO in α_s , will be generalized to all orders in Sec. IV. It entails three remarkable consequences:

- (1) It allows one to choose the initial condition to be independent of the specific value of μ^2 , thus,

$$\bar{S}(Y_0 = \ln(\zeta_0/\mu^2)) = \bar{S}(0), \quad (24)$$

for $\zeta_0 = \mu^2$. In particular, it explains why one can choose a μ -independent initial condition, such as the McLerran-Venugopalan model [5,79] for the composite dipole.

- (2) It provides precise control over the definition of the rapidity variable entering the BK evolution. In particular, one can choose

$$\mu = Q, \quad \zeta = s \Rightarrow Y = \ln\left(\frac{\zeta}{\mu^2}\right) = \ln\left(\frac{s}{Q^2}\right) = \ln\left(\frac{1}{x_{\text{Bj}}}\right), \quad (25)$$

which demonstrates that the change of the evolution rapidity variable can be systematically encoded within the shockwave formalism.

- (3) It offers guidance for building the composite dipole at NNLO, by enforcing the constraint Eq. (22) to be valid at NLO. Using this constraint together with Eq. (18), the composite dipole at $\mathcal{O}(\alpha_s^2)$ can be defined in terms of the original dipole as

$$\begin{aligned}\bar{S}_{12}(\zeta, \mu^2) = & S_{12} - \bar{\alpha}_s K_{\text{NLO}} \ln(r_{1234}^2 \mu^2) \otimes \bar{S}_{12} \\ & - \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{134} \left[\ln\left(\frac{r_{134}^2}{r_{123}^2}\right) \ln(r_{1234}^2 \mu^2) - \frac{1}{2} \ln^2(r_{134}^2 \mu^2) \right] (\bar{S}_{14} \bar{S}_{43} - \bar{S}_{13}) \bar{S}_{32} \\ & - \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{324} \left[\ln\left(\frac{r_{324}^2}{r_{123}^2}\right) \ln(r_{1234}^2 \mu^2) - \frac{1}{2} \ln^2(r_{324}^2 \mu^2) \right] (\bar{S}_{34} \bar{S}_{42} - \bar{S}_{32}) \bar{S}_{13} \\ & + \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{124} \left[\ln\left(\frac{r_{124}^2}{r_{123}^2}\right) \ln(r_{1234}^2 \mu^2) - \frac{1}{2} \ln^2(r_{124}^2 \mu^2) \right] (\bar{S}_{14} \bar{S}_{42} - \bar{S}_{12}),\end{aligned}\quad (26)$$

for some transverse coordinate r_{1234} defined in terms of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$. With this form of NNLO composite dipole, Eq. (20) also holds as an evolution equation in Y . Together with the recent calculation of the NNLO BK equation [80], this result paves the way toward the evaluation of the dipole operator at two loops without double logarithmic collinear instability.

III. THE INTERPLAY BETWEEN SMALL- x AND COLLINEAR LOGARITHMS

High-energy factorization in DIS is based on the separation between fast, $k^+ > \rho^+$, long-lived quantum fluctuations, which are part of the photon impact factor H and slow modes, $k^+ < \rho^+$ effectively included in the multipole operators $S^{(m)}$, that encode the target properties:

$$\begin{aligned}\sigma(x, Q^2) = & \sum_{m=2}^{\infty} \prod_{i=2}^m \int_{\mathbf{x}_i} H^{(m)}(Q^2, \{\mathbf{x}_i\}, \rho^+/q^+) S^{(m)}(\{\mathbf{x}_i\}, \rho^+ P^-).\end{aligned}\quad (27)$$

[See Eqs. (7) and (A1) for a concrete example of the structure given by Eq. (27) in the case of the longitudinal DIS structure function at LO and NLO, respectively.] Here ρ^+ is a rapidity factorization scale, in the frame where the photon is moving in the $+z$ direction with longitudinal momentum q^+ and the target is moving in the $-z$ direction with longitudinal momentum P^- . The sum runs over the number of high-energy partons that cross the shockwave. At leading order the operator $S^{(2)} \equiv S$ is defined as a correlator of two infinite Wilson lines evaluated at the transverse positions of the quark and antiquark, $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$S(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{N_c} \langle \text{Tr} U_{\mathbf{x}_1} U_{\mathbf{x}_2}^\dagger \rangle, \quad (28)$$

with the infinite Wilson lines defined as

$$U_x \equiv \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} dx^+ t \cdot A^-(x^+, \mathbf{x}) \right], \quad (29)$$

with $t \cdot A^- \equiv t^a A^{-,a}$ where t^a are the generators of $SU(N_c)$ in the fundamental representation.

Large rapidity logarithms are known to exponentiate [81]:

$$\begin{aligned}S^{(m)}(\{\mathbf{x}_i\}, \rho^+ P^-) = & \exp \left[\bar{\alpha}_s K(\{\mathbf{x}_i\}, \bar{\alpha}_s) \ln \frac{\rho^+}{\rho_0^+} \right]^{(mn)} \\ & \times S^{(n)}(\{\mathbf{x}_i\}, \rho_0^+ P^-),\end{aligned}\quad (30)$$

where the evolution operator K —in matrix form—is a functional differential operator. We may equivalently write the differential form

$$\frac{\partial S^{(m)}(\{\mathbf{x}_i\}, \zeta)}{\partial \ln \zeta} = \bar{\alpha}_s K^{(mn)}(\{\mathbf{x}_i\}, \bar{\alpha}_s) S^{(n)}(\{\mathbf{x}_i\}, \zeta), \quad (31)$$

which is essentially a formal rewriting of the Balitsky hierarchy as it results from the JIMWLK equation. At leading order, only two components contribute to the evolution of the dipole operator $S^{(2)}$, namely, K_{22} and K_{23} . The factorization scale $\zeta = 2\rho^+ P^-$ is bounded from above by the center-of-mass energy, $s = 2q^+ P^-$. However, there is an ambiguity on the lower bound, that may be Q^2 , or some nonperturbative scale Λ_{QCD}^2 . This ambiguity is the origin of the collinear anomaly that results in the emergence of large collinear logs of these two scales order by order in perturbation theory.

The extension of the time integration domain of the Wilson line reflects the shockwave approximation, which decouples the x^+ integrals in the form factor from those in the target operator S . In momentum space, this corresponds to a separation in k^- , which follows from the ordering in k^+ , modulo a transverse scale dependence. However, the condition $k^+ > \rho^+$ implies—for real gluons, at least—an

upper bound on k^- given by $k^- < k_\perp^2/\rho^+$. This dynamical bound must remain below P^- , yet the factorization scheme does not account for this kinematic constraint, leading to collinear logarithms arising in the unphysical region $k_\perp^2/\rho^+ > P^-$,

$$\ln \frac{k_\perp^2}{\rho^+ P^-}. \quad (32)$$

This issue, which arises only in NLO and beyond, can be addressed in various ways. The most common remedy is to impose a kinematic constraint on the phase space of the quantum corrections in an *ad hoc* manner. In this paper, we develop a systematic method that exploits the freedom in the choice of subtraction scheme to address this issue order by order in perturbation theory within the shockwave approach.

As a first step, we revisit the one-loop computation. To do so, we take a step back and recognize that the problem

described above originates from the extension of x^+ to infinity in the Wilson lines. To properly take the small- x limit, we introduce an x -dependent dipole operator whose limiting behavior yields the dipole operator defined in Eq. (28).

The one-loop result for S involving soft gluon emissions from the Wilson lines is both IR and UV divergent and is usually regulated with the help of a cutoff:

$$I = \int_0^{+\infty} \frac{dk^+}{k^+} \rightarrow \int_{\Lambda_0^+}^{\Lambda^+} \frac{dk^+}{k^+}. \quad (33)$$

However, while the upper limit indeed diverges for S and requires a regulator, the lower bound should be regulated by the dynamical size of the shock wave. To see that, it is convenient to temporarily relax the small $x = 0$ limit, to first carry out the integration over k^+ , then take formally the limit $x \rightarrow 0$. The operator that enables this operation is defined as [72,73]

$$S(x, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{N_c} \int_{-\infty}^{+\infty} dx^+ \int_{-\infty}^{x^+} dy^+ e^{ixP^-(x^+-y^+)} \frac{\partial^2}{\partial x^+ \partial y^+} \text{tr} \langle U_{x_1}(x^+, y^+) U_{x_2}^\dagger(x^+, y^+) \rangle, \quad (34)$$

where we have introduced the finite Wilson line in the fundamental representation

$$U_{\mathbf{x}}(x^+, y^+) \equiv [x^+, y^+]_{\mathbf{x}} = \mathcal{P} \exp \left[ig \int_{y^+}^{x^+} dz^+ A^-(z^+, \mathbf{x}) \right]. \quad (35)$$

It can easily be checked that for $x = 0$ the operator above reduces the dipole operator, Eq. (28).

Using a rapidity regulator η to regularize the $k^+ \rightarrow +\infty$ divergence, the one-loop correction to the above x -dependent dipole distribution reads as (see Appendix B for a detailed derivation)

$$\begin{aligned} \Delta S^{(2)}(x, \mathbf{x}_1, \mathbf{x}_2) &= \bar{\alpha}_s(\rho^+)^\eta \int_0^{+\infty} dx' \int_{\mathbf{k}, \mathbf{q}, z} K(\mathbf{x}_{13}, \mathbf{x}_{32}, \mathbf{k}, \mathbf{q}) \\ &\times \int_0^{+\infty} \frac{dk^+}{(k^+)^{1+\eta}} \delta \left(x' - x - \frac{(s\mathbf{k} + (1-s)\mathbf{q})^2}{2P^- k^+} \right) S^{(3)}(x', \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3). \end{aligned}$$

Here we have applied the partial twist expansion which is akin to a Wentzel-Kramers-Brillouin approximation for the propagation of the radiated gluon inside the target [72,73]. The variable s is arbitrary and will be chosen to be $s = 1$. The dependence on this choice is expected to be subleading in power counting.

Here, we have used the contextual notation for the unintegrated LO BK kernel

$$K(\mathbf{x}_{13}, \mathbf{x}_{32}, \mathbf{k}, \mathbf{q}) = (2\pi)^2 \frac{\mathbf{q} \cdot \mathbf{k}}{q^2 k^2} (e^{-ik \cdot \mathbf{x}_{13}} - e^{-ik \cdot \mathbf{x}_{23}}) (e^{iq \cdot \mathbf{x}_{13}} - e^{iq \cdot \mathbf{x}_{23}}), \quad (36)$$

and

$$S^{(3)}(x', \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{1}{N_c} \int_{-\infty}^{+\infty} dx^+ \int_{-\infty}^{x^+} dy^+ e^{ix'P^-(x^+-y^+)} \frac{\partial^2}{\partial x^+ \partial y^+} \text{tr} [\langle t^a U_{x_1}(x^+, y^+) t^b U_{x_2}^\dagger(x^+, y^+) \mathcal{U}_{x_3}^{ab}(x^+, y^+) \rangle], \quad (37)$$

where

$$\mathcal{U}_{x_3}^{ab}(x^+, y^+) = \mathcal{P} \exp \left[ig \int_{y^+}^{x^+} dz^+ T \cdot A^-(z^+, \mathbf{x}_3) \right]^{ab} \quad (38)$$

is a Wilson line in the adjoint representation.

For notational simplicity, we introduce the following shorthand:

$$\begin{aligned} S^{(2)}(x, \mathbf{x}_1, \mathbf{x}_2) &\equiv S_{12}(x), \\ S^{(3)}(x', \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &\equiv S_{123}(x'), \\ K(\mathbf{x}_{13}, \mathbf{x}_{32}, \mathbf{k}, \mathbf{q}) &\equiv K_{123}(\mathbf{k}, \mathbf{q}). \end{aligned} \quad (39)$$

The LO BK kernel is recovered by integrating over the transverse momenta $\mathbf{q}$ and $\mathbf{k}$ as follows:

$$K_{123} = \int_{\mathbf{q}, \mathbf{k}} K_{123}(\mathbf{k}, \mathbf{q}) = \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2}. \quad (40)$$

Again, similarly to the position space integrals the momentum space integral notation reads as

$$\int_{\mathbf{k}} \equiv \int \frac{d^2 \mathbf{k}}{(2\pi)^2}. \quad (41)$$

Hence, Eq. (36), for $s = 1$, becomes

$$\Delta S_{12}(x) = \bar{\alpha}_s(\rho^+)^{\eta} \int_0^{+\infty} dx' \int_{\mathbf{k}, \mathbf{q}} \int_3 K_{123}(\mathbf{k}, \mathbf{q}) \int_0^{+\infty} \frac{dk^+}{(k^+)^{1+\eta}} \delta\left(x' - x - \frac{\mathbf{k}^2}{2P^- k^+}\right) S_{123}(x'). \quad (42)$$

Note that neglecting the x and $\mathbf{k}^2/2k^+$ terms in the Dirac delta function, we can readily integrate over $\mathbf{k}$ and $\mathbf{q}$, and as a result we recover the BK kernel:

$$\Delta S_{12}(0) = \bar{\alpha}_s(\rho^+)^{\eta} \int_0^{+\infty} \frac{dk^+}{(k^+)^{1+\eta}} \int_3 \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2} S_{123}(0). \quad (43)$$

This result is the standard shockwave approximation; however, it does not fully reflect the proper small x limit which should include an additional subleading term which emerges when keeping $\mathbf{k}^2/2k^+$ as we shall shortly see.

Taking the $x \rightarrow 0$ limit of Eq. (36), we have

$$\begin{aligned} \Delta S^{(2)}(x=0) &= \lim_{x \rightarrow 0} \Delta S^{(2)}(x) = \lim_{x \rightarrow 0} (\rho^+)^{\eta} \int_0^{+\infty} dx' \int_{\mathbf{k}, \mathbf{q}} \int_3 K_{123}(\mathbf{k}, \mathbf{q}) \int_0^{+\infty} \frac{dk^+}{(k^+)^{1+\eta}} \delta\left(x' - x - \frac{\mathbf{k}^2}{2P^- k^+}\right) S^{(3)}(x'), \\ &= (\rho^+)^{\eta} \int_0^{+\infty} dx' \int_{\mathbf{k}, \mathbf{q}} \int_3 K_{123}(\mathbf{k}, \mathbf{q}) \int_0^{+\infty} \frac{dk^+}{(k^+)^{1+\eta}} \delta\left(x' - \frac{\mathbf{k}^2}{2k^+ P^-}\right) S^{(3)}(x'), \end{aligned} \quad (44)$$

$$= \int_{\mathbf{k}, \mathbf{q}} \int_3 K_{123}(\mathbf{k}, \mathbf{q}) \left(\frac{2\rho^+ P^-}{\mathbf{k}^2} \right)^{\eta} \int_0^{+\infty} \frac{dx'}{(x')^{1-\eta}} S^{(3)}(x'). \quad (45)$$

The delta function in x can be understood as encoding the kinematic constraint $xk^- < k^- < P^-$ which would be imposed by hand in other *ad hoc* schemes.

Since we are interested in the limit $\eta \rightarrow 0$, it is convenient to expand the integrand as

$$\frac{1}{(x')^{1-\eta}} = \frac{1}{\eta} \delta(x') + \left[\frac{1}{x'} \right]_+ + \eta \left[\frac{\ln x'}{x'} \right]_+ + \mathcal{O}(\eta^2), \quad (46)$$

where the plus distributions are defined through their action on a smooth test function $f(x')$. In particular,

$$\int_0^1 dx' \left[\frac{1}{x'} \right]_+ f(x') = \int_0^1 dx' \frac{f(x') - f(0)}{x'}. \quad (47)$$

Thus, we find

$$\begin{aligned} \Delta S^{(2)}(x=0) &\simeq \left(\frac{2\rho^+ P^-}{k^2} \right)^\eta \int_0^{+\infty} dx' \left\{ \frac{1}{\eta} \delta(x') + \left[\frac{1}{x'} \right]_+ \right\} S^{(3)}(x') \\ &\simeq \left(1 + \eta \ln \frac{2\rho^+ P^-}{k^2} \right) \left\{ \frac{1}{\eta} S^{(3)}(0) + \int_{-\infty}^{+\infty} \frac{dx'}{x'} (S(x') - S(0)) \right\} \\ &\simeq \left(\frac{1}{\eta} + \ln \frac{2\rho^+ P^-}{k^2} \right) S^{(3)}(0) + \int_0^{+\infty} \frac{dx'}{x'} (S^{(3)}(x') - S^{(3)}(0)) \\ &\simeq \left(\frac{1}{\eta} + \ln \frac{2\rho^+ P^-}{k^2} \right) S^{(3)}(x=0) + \dots \end{aligned} \quad (48)$$

The ellipsis denotes terms that are regular as $x \rightarrow 0$.

Interestingly, we obtain a logarithm of the ratio of the factorization scale and the dynamical scale $k^+ = k^2/2P^-$. This collinear log was missed from the standard shockwave approach and can be shown to be related to the so-called kinematic constraint. Although this may be expected when k^- ordering is enforced, what is less trivial is that the transverse logarithm multiplies the dipole operator evaluated at precisely $x = 0$. In addition, we obtain additional nondipole operators that encode physics at finite x which we shall discard. This demonstrates that the kinematic

constraint does not spoil the high-energy operator product expansion (OPE) formulated in terms of operators evaluated at $x = 0$. As such, there should be a way to formulate this OPE without formally breaking it by an *ad hoc* modification of the kernel for the high-energy evolution, as currently done in the literature.

To identify the proper OPE formulation of the kinematic constraint, we first relate the result Eq. (48) to the BK equation written in coordinate space. We first introduce a separation scale $\rho_0^+ \ll \rho^+$ and write

$$\Delta S^{(2)}(x=0) \simeq \left(\frac{1}{\eta} + \ln \frac{\rho^+}{\rho_0^+} - \ln \frac{k^2}{2\rho_0^+ P^-} \right) S^{(3)}(x=0) + \text{const.}, \quad (49)$$

$$\simeq \Delta S_{\text{BK}}^{(2)}(x=0) - \ln \frac{k^2}{2\rho_0^+ P^-} S^{(3)}(x=0) + \text{const.} \quad (50)$$

More precisely, setting

$$\mu^2 = 2\rho_0^+ P^-, \quad (51)$$

we have

$$\Delta S_{12}(x=0) \simeq \Delta S_{12}^{\text{BK}}(x=0) - \bar{\alpha}_s \int_3 L_{123}(\mu^2) S_{123}^{(3)}(x=0) + \text{const.}, \quad (52)$$

where

$$L_{123}(\mu^2) = \frac{(2\pi)^2}{2} \int_{q,k} \frac{\mathbf{q} \cdot \mathbf{k}}{q^2 k^2} (e^{-ik \cdot x_{13}} - e^{-ik \cdot x_{23}}) (e^{iq \cdot x_{13}} - e^{iq \cdot x_{23}}) \left(\ln \frac{k^2}{\mu^2} + \ln \frac{q^2}{\mu^2} \right), \quad (53)$$

which can be integrated using (cf. Appendix C)

$$\int_q \frac{q^i}{q^2} (e^{iq \cdot x_{13}} - e^{iq \cdot x_{23}}) = \frac{i}{2\pi} \left(\frac{x_{13}^i}{x_{13}^2} - \frac{x_{23}^i}{x_{23}^2} \right), \quad (54)$$

and

$$\int_k \frac{k^i}{k^2} (e^{-ik \cdot x_{13}} - e^{-ik \cdot x_{23}}) \ln \frac{k^2}{\mu^2} = -\frac{i}{(2\pi)} \left[\left(\ln \frac{|x_{13}| |x_{23}|}{\mu^2} + 2(\gamma_E - \ln(2)) \right) \left(\frac{x_{13}^i}{x_{13}^2} - \frac{x_{23}^i}{x_{23}^2} \right) + \left(\frac{x_{13}^i}{x_{13}^2} + \frac{x_{23}^i}{x_{23}^2} \right) \ln \frac{|x_{13}|}{|x_{23}|} \right]. \quad (55)$$

We thus obtain

$$\begin{aligned} L_{123}(\mu^2) &= \frac{x_{12}^2}{x_{13}^2 x_{23}^2} \ln(|x_{13}| |x_{23}| \mu^2) + \left(\frac{1}{x_{13}^2} - \frac{1}{x_{23}^2} \right) \ln \frac{|x_{13}|}{|x_{23}|} \\ &= \frac{x_{12}^2}{x_{13}^2 x_{23}^2} \left[\ln \left(\frac{|x_{13}| |x_{23}|}{|x_{12}| \mu^{-1}} \right) + \ln |x_{12}| \mu \right] + \left(\frac{1}{x_{13}^2} - \frac{1}{x_{23}^2} \right) \ln \frac{|x_{13}|}{|x_{23}|}, \end{aligned} \quad (56)$$

where we have absorbed the additive constant in a redefinition of the factorization scale

$$\mu^2 \rightarrow \mu^2 e^{-2(\gamma_E - \ln(2))}. \quad (57)$$

To sum up, the one-loop formula in coordinate space for the dipole operator reads as

$$\Delta S_{12} = \bar{\alpha}_s \ln \frac{\rho^+ P^-}{\mu^2} \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} (S_{13} S_{32} - S_{12}) + \bar{\alpha}_s \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \left(\ln \frac{|x_{13}| |x_{32}| \mu}{|x_{12}|} + \ln |x_{12}| \mu + \frac{x_{23}^2 - x_{13}^2}{x_{12}^2} \ln \frac{|x_{13}|}{|x_{32}|} \right) (S_{13} S_{32} - S_{12}). \quad (58)$$

We shall refer to the second term as the collinearly anomalous term. In this term, we have split the logarithm into two terms to highlight the connection with the “conformal dipole” formulation [22,23]. The first logarithm,

$$\ln \frac{|x_{13}| |x_{32}| \mu}{|x_{12}|}, \quad (59)$$

corresponds to the “conformal dipole” term. The second one,

$$\ln |x_{12}| \mu, \quad (60)$$

breaks conformal invariance. It is known that the conformal dipole prescription cancels the explicit double logarithms in the K_{123}^{NLO} kernel [cf. Eq. (13)] responsible for the instabilities in the rapidity evolution of the NLO BK equation.

However, it introduces a $\ln |x_{12}|$ term that generates instabilities of its own [31], because it generates a double collinear logarithm “hidden” in the K_{1234} kernel. We anticipate that our additional term will resolve this issue.

IV. RAPIDITY FACTORIZATION SCHEME TRANSFORMATION

We turn now to a more formal discussion of the high-energy OPE in terms of a our composite dipole, in close analogy with that of the conformal dipole introduced by Balitsky and Chirilli. Because the subtraction of the rapidity singularity is scheme dependent—up to terms that may depend on the transverse coordinates—the second term in Eq. (56) can be viewed as a different choice of subtraction scheme [81]. At one-loop order, this new subtraction scheme amounts to a redefinition of the dipole operator into a composite operator:

$$\bar{S}_{12} \equiv S_{12} - \bar{\alpha}_s \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \left(\ln(|x_{13}| |x_{32}| \mu^2) + \frac{x_{32}^2 - x_{13}^2}{x_{12}^2} \ln \frac{|x_{13}|}{|x_{32}|} \right) (S_{13} S_{32} - S_{12}) + \mathcal{O}(\alpha_s^2). \quad (61)$$

By construction the additional term cancels the collinearly anomalous term found in Eq. (58) such that, as we shall see, the evolution of the composite dipole will be free of double collinear logarithms.

Interestingly, the second term coincides with the result recently derived in the k^- scheme [77,78], pointing to an equivalence between our rotated k^+ scheme and the direct choice of k^- as the ordering variable.

This nonlinear transformation may seem obscure at first glance, but it can be naturally understood as a change of vector basis. Formally, it can be written to all orders as

$$\bar{S}^{(m)}(\{\mathbf{x}_i\}, \zeta, \mu^2) = \exp[-\bar{\alpha}_s L(\{\mathbf{x}_i\}, \bar{\alpha}_s, \mu^2)]^{(mn)} S^{(n)}(\{\mathbf{x}_i\}, \zeta). \quad (62)$$

This formulation is reminiscent of the Doi-Peliti [82,83] formalism which has been discussed in the companion letter using Dirac notations, e.g. $L^{(23)} \sim (\mathbf{x}_1 \mathbf{x}_2 | L | \mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_3)$ [70].

As a particular case, we have for the dipole operator at NLO [recall our shorthand notation, $S_{12} = S^{(2)}(\{\mathbf{x}_1, \mathbf{x}_2\}, \zeta)$]

$$\bar{S}_{12}(\zeta, \mu^2) = S_{12}(\zeta) - \bar{\alpha}_s \int_3 L_{123}(\mu^2) (S_{13} S_{32} - S_{12}),$$

$$L_{123} \equiv K_{123} \left(\ln |\mathbf{x}_{13}| |\mathbf{x}_{32}| \mu^2 + \frac{\mathbf{x}_{32}^2 - \mathbf{x}_{13}^2}{\mathbf{x}_{12}^2} \ln \frac{|\mathbf{x}_{13}|}{|\mathbf{x}_{32}|} \right), \quad (63)$$

after expanding to first order in α_s , the exponential in Eq. (62) and identifying the result to Eq. (61). As a result, both the evolution operator and the impact factor transform under this change of basis. Formally, the composite operators evolve according to [compare to Eq. (30)]

$$\bar{S}^{(m)}(\{\mathbf{x}_i\}, \zeta, \mu^2) = \exp \left[\bar{\alpha}_s \bar{K}(\{\mathbf{x}_i\}, \bar{\alpha}_s, \mu^2) \ln \frac{\zeta}{\zeta_0} \right]^{(mn)} \times \bar{S}^{(n)}(\{\mathbf{x}_i\}, \zeta_0, \mu^2), \quad (64)$$

where

$$\bar{K} = e^{-\bar{\alpha}_s L} K e^{+\bar{\alpha}_s L} = K + \bar{\alpha}_s [K, L] + \frac{\bar{\alpha}_s^2}{2!} [[K, L], L] + \mathcal{O}(\bar{\alpha}_s^3). \quad (65)$$

This result shows that the composite dipole evolves at NLO according to the NLO BK equation amended by an extra term given by the commutator between the kernels K and L . The explicit computation of this commutator in coordinate space will be performed in Sec. VII [84].

A priori, the composite operators depend on the factorization scale $\zeta = \rho^+ P^-$ and the collinear scale μ^2 . A remarkable feature of the NLO composite dipole is the following relation:

$$\frac{\partial \bar{S}_{12}}{\partial \ln \mu^2} = -\frac{\partial \bar{S}_{12}}{\partial \ln \zeta} + \mathcal{O}(\bar{\alpha}_s^2). \quad (66)$$

This becomes evident by taking the $\ln \mu^2$ derivative of Eq. (61), which exactly reproduces the LO BK equation given in Eq. (9). The identity in Eq. (66) implies that, at least to LO in α_s , $\bar{S}_{12}$ depends only on the ratio ζ/μ^2 . This property allows us to relate our composite dipole framework to the choice of evolution variable in high-energy evolution. For $\mu^2 \sim Q^2$ and $\rho^+ \sim q^+$, we find $\ln(\zeta/\mu^2) \sim \ln(1/x_{Bj})$, and therefore the composite dipole obeys the same evolution equation whether expressed as a function of $\ln(1/x_{Bj})$ (target rapidity ordering) or $\ln(1/\zeta)$ (projectile rapidity ordering).

We now aim to impose this property to all orders in α_s . Specifically, we require the composite operators defined by the general transformation, Eq. (62), to satisfy the constraint

$$\frac{\partial \bar{S}}{\partial \ln \mu^2} + \frac{\partial \bar{S}}{\partial \ln \zeta} = 0, \quad (67)$$

or equivalently, $\bar{S}(\zeta, \mu^2) = \bar{S}(\zeta/\mu^2)$. This constraint serves as our guiding principle for constructing the composite dipole order by order in perturbation theory, given the perturbative expansion of the kernel K . We note, however, that one could instead impose a different constraint than Eq. (67), for example $\frac{1}{2} \frac{\partial \bar{S}^{(m)}}{\partial \ln \mu^2} + \frac{\partial \bar{S}^{(m)}}{\partial \ln \zeta} = 0$, which corresponds to a symmetric ordering $\ln(\rho^+/\mu)$. This choice is related to the conformal dipole introduced in [22]. The derivative of $\bar{S}^{(m)}$ with respect to $\ln \zeta$ follows from the definition of the kernel $\bar{K}$,

$$\frac{\partial \bar{S}^{(m)}}{\partial \ln \zeta} = \bar{\alpha}_s \bar{K}^{(mn)}(\{\mathbf{x}_i\}, \bar{\alpha}_s, \mu^2) \bar{S}^{(n)}, \quad (68)$$

and the derivative with respect to $\ln \mu^2$ is obtained from the definition of the composite operators,

$$\frac{\partial \bar{S}}{\partial \ln \mu^2} = -\bar{\alpha}_s \int_0^1 du e^{-u \bar{\alpha}_s L} \frac{\partial L}{\partial \ln \mu^2} e^{u \bar{\alpha}_s L} \bar{S}, \quad (69)$$

where we have used Eq. (62) together with the identity,

$$\frac{d}{d \ln \mu^2} e^{-\bar{\alpha}_s L(\mu)} = -\bar{\alpha}_s \int_0^1 du e^{-(1-u) \bar{\alpha}_s L(\mu)} \frac{dL(\mu)}{d \ln \mu^2} e^{-u \bar{\alpha}_s L(\mu)}, \quad (70)$$

whose derivation is presented in Appendix D.

Imposing Eq. (67) for all $\bar{S}^{(m)}$, we find the identity

$$\bar{K} = \int_0^1 du e^{-u \bar{\alpha}_s L} \frac{\partial L}{\partial \ln \mu^2} e^{u \bar{\alpha}_s L}. \quad (71)$$

As a consequence of this identity, the kernel $\bar{K}$ for the high-energy evolution of the composite dipole is independent of

μ^2 , to all orders in α_s . Indeed,

$$\frac{\partial \bar{K}}{\partial \ln \mu^2} = \bar{\alpha}_s \int_0^1 du \left[\bar{K}, e^{-u\bar{\alpha}_s L} \frac{\partial L}{\partial \ln \mu^2} e^{u\bar{\alpha}_s L} \right] = 0. \quad (72)$$

This property is a consequence of the linear relation, Eq. (67), between the $\ln \zeta$ and $\ln \mu^2$ derivatives imposed on the composite dipole. It means that to all orders in α_s , the kernel of the Balitsky hierarchy for the composite operators has no explicit μ^2 dependence, a feature that we shall explicitly observe at NLO for the high-energy evolution of the composite dipole in Sec. VII.

The condition in Eq. (71) also imposes strong constraints on the composite dipole, allowing us to determine the form of the transformation given by Eq. (62) order by order in perturbation theory, provided the BK kernel for the “bare” dipole is known at the corresponding order. Let us illustrate this by explicit formal manipulations in order to obtain the $\mathcal{O}(\alpha_s^2)$ in Eq. (61). Up to $\mathcal{O}(\alpha_s^2)$ corrections, if we write

$$K = K_1 + \bar{\alpha}_s K_2, \quad L = L_1 + \bar{\alpha}_s L_2 = L_{10} + L_{11} \ln \mu^2 + \bar{\alpha}_s (L_{20} + L_{21} \ln \mu^2 + L_{22} \ln^2 \mu^2), \quad (73)$$

where K_1 and K_2 are the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ BK kernel respectively, then the condition [Eq. (71)] reads as

$$\bar{K}_{\text{NLO}} = \frac{\partial L_1}{\partial \ln \mu^2} + \bar{\alpha}_s \frac{\partial L_2}{\partial \ln \mu^2} + \frac{1}{2} \bar{\alpha}_s \left[\frac{\partial L_1}{\partial \ln \mu^2}, L_1 \right] = K_1 + \bar{\alpha}_s K_2 + \bar{\alpha}_s [K_1, L_1]. \quad (74)$$

Identifying term by term and solving order by order the corresponding elementary differential equation, we get

$$L_1 = L_{10} + K_1 \ln \mu^2, \quad (75)$$

$$L_2 = L_{20} + \left\{ K_2 + \frac{1}{2} [K_1, L_{10}] \right\} \ln \mu^2. \quad (76)$$

Given the $\mathcal{O}(\alpha_s^3)$ BK kernel K_3 , recently computed in [80], it is straightforward to obtain the formal expression for L_3 , the $\mathcal{O}(\alpha_s^3)$ term in the expansion of L ,

$$L_3 = L_{30} + \left\{ K_3 + \frac{1}{2} [K_1, L_{20}] + \frac{1}{2} [K_2, L_{10}] + \frac{1}{12} [[K_1, L_{10}], L_{10}] \right\} \ln \mu^2 + \frac{1}{12} [K_1, [K_1, L_{10}]] \ln^2 \mu^2, \quad (77)$$

and so forth for the higher order terms. In particular, using Eq. (65) and the above relations we can provide the modification of the BK kernel at NNLO, i.e., $K_{\text{NNLO}} = K_1 + \bar{\alpha}_s K_2 + \bar{\alpha}_s^3 K_3$,

$$\bar{K}_{\text{NNLO}} = K_{\text{NNLO}} + \bar{\alpha}_s [K_1, L_{10}] + \bar{\alpha}_s^2 [K_2, L_{10}] + \bar{\alpha}_s^2 [K_1, L_{20}] + \frac{\bar{\alpha}_s^2}{2} [[K_1, L_{10}], L_{10}]. \quad (78)$$

The two relations in Eq. (75) depend on arbitrary “constants” L_{10} and L_{20} , which reflect the freedom to choose the transverse coordinate scale that renders the argument of $\ln \mu^2$ dimensionless. In Sec. VI, we will provide another argument based on the Mellin space representation of the linearized NLO BK equation for the composite dipole demonstrating that indeed, what truly matters for canceling the double collinear logarithms in the NLO equation is the coefficient in front of the $\ln \mu^2$ dependence and not the precise form of the argument of the logarithm in the constant L_{10} . To connect with the composite dipole as defined in Eq. (17) or Eq. (61), we have $L_{10} S_{12} = \int_3 K_{123} \ln(r_{123}^2) (S_{13} S_{32} - S_{12})$ —the rule for applying L_{10}

on more complicated operators involving more than two Wilson lines follows, in the large N_c limit, from a “Leibniz rule”, i.e. $L_{10} [S_{ab} S_{cd}] = (L_{10} S_{ab}) S_{cd} + S_{ab} (L_{10} S_{cd})$. We thus find that imposing the constraint [Eq. (71)] naturally leads to the form of the composite dipole as it emerges from the NLO calculation of the generalized dipole performed in Sec. III. Since we have some freedom in choosing L_{10} , the simplest choice for L_{10} could be something like

$$L_{10} = \int_3 K_{123} \ln x_{12}^2. \quad (79)$$

We will explore other possibilities for the transverse coordinate scale accompanying μ in Sec. VII.

Let us finally conclude this section by a brief discussion of the impact factor which is also affected by the change of basis corresponding to the transformation, Eq. (62). Since the physical cross section $\sigma(x, Q^2)$ must be independent of the chosen basis, the impact factor must transform accordingly:

$$\bar{H}^{(n)}(Q^2, \{x_i\}, \rho^+/q^+, \mu^2) = H^{(m)}(Q^2, \{x_i\}, \rho^+/q^+) \exp[\bar{\alpha}_s L(\{x_i\}, \bar{\alpha}_s, \mu^2)]^{(mn)}. \quad (80)$$

Hence, at NLO the cross section takes the form

$$\begin{aligned} \sigma_{\text{NLO}}(x, Q^2) &= H_{\text{NLO}}(\rho^+/q^+)(1 + \bar{\alpha}_s L_1) \exp\left[\bar{\alpha}_s(K_1 + \bar{\alpha}_s[K_1, L_1]) \ln \frac{\rho^+}{\rho_0^+}\right] \bar{S}(\rho_0^+ P^-) + \mathcal{O}(\bar{\alpha}_s^2), \\ &= \bar{H}_{\text{NLO}}(\rho^+/q^+, \mu^2) \exp\left[\bar{\alpha}_s(K_1 + \bar{\alpha}_s[K_1, L_1]) \ln \frac{\rho^+}{\rho_0^+}\right] \bar{S}(\rho_0^+ P^-) + \mathcal{O}(\bar{\alpha}_s^2), \end{aligned} \quad (81)$$

where $\bar{H}_{\text{NLO}}(\rho^+/q^+, \mu^2) = H_{\text{NLO}}(\rho^+/q^+)(1 + \bar{\alpha}_s L_1(\mu^2)) = H_{\text{NLO}}(\rho^+/q^+) + \bar{\alpha}_s H_{\text{LO}} L_1(\mu^2)$. The explicit form for $\bar{H}_{\text{NLO}}$ (convoluted with the evolved composite dipole) in the case of the longitudinal structure function is shown in Eq. (21) of the summary section. Hence, although the μ -dependent terms in the operator L cancel out in the evolution, its exact form remains essential for determining the impact factor within this scheme. In the next sections, we will examine the effect of this transformation on the collinear double logarithms that arise at NLO.

V. HEURISTIC DISCUSSION OF THE ANOMALOUS COLLINEAR LOGARITHMS

It is not obvious from the structure of Eq. (12) that the pathological terms appearing in high-energy factorization beyond leading order will indeed cancel out. As briefly summarized in the Introduction, the pathology of naive high-energy factorization manifests itself in two places:

- (i) at the level of the impact factors computed order by order in perturbation theory, which become artificially large (such that the cross section can even become negative) if the factorization scheme “over-subtracts” the high-energy logarithms in the fixed order result, and
- (ii) in the renormalization group equation resumming these high-energy logarithms beyond leading order which displays large collinear logarithms in such naive factorization schemes.

How a composite dipole formulation based on Eq. (12) solves the issue (ii) will be the subject of the following sections. Here, we would like to address how it also generically addresses the issue (i). Instead of discussing the exact NLO impact factor for the DIS structure functions (given in Sec. II for longitudinally polarized virtual photons), we shall proceed with a heuristic argument based on a double logarithmic approximation of both the hard coefficient function and the dipole operator.

As first realized in [32] in the context of the NLO impact factors for the DIS structure functions at small x , the negativity problem of NLO cross sections in high energy factorization comes from the fact that the fixed order NLO results know about life-time ordering while the LO BK equation formulated with the evolution variable of the projectile does not. In DIS, the lifetime of the radiated gluon in the color dipole picture must be smaller than that of the virtual photon creating the $q\bar{q}$ pair. It is then not obvious at all to see how formula Eq. (12) knows about lifetime ordering. To convince ourselves that this is indeed the case, let us analyze the double logarithmic limit of gluon radiation at the level of the cross section, Eq. (27). At NLO the impact factor knows about one side of the ordered sequence; the gluon formation time is shorter than the photon decay time:

$$\frac{k^+}{k^2} \ll \frac{q^+}{Q^2} \Rightarrow k^+ \ll \left(\frac{k^2}{Q^2}\right) q^+. \quad (82)$$

This upper bound on k^+ is effectively encoded inside the NLO light cone wave function: mathematically, the Bessel function $K_0(QX_R)$ in Eq. (A1) suppress the phase space for high-energy evolution when $k^+ \gg q^+/(x_{13}^2 Q^2) \sim (k^2/Q^2)q^+$. Applying the above constraint to the hard coefficient function at NLO should generate the following term:

$$H_{\text{NLO}}(Q^2, \rho^+/q^+) \approx \bar{\alpha}_s \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \int_{\rho^+}^{(\frac{k^2}{Q^2})q^+} \frac{dk^+}{k^+} \quad (83)$$

$$= \bar{\alpha}_s \left[\ln \frac{Q^2}{Q_0^2} \ln \frac{q^+}{\rho^+} - \frac{1}{2} \ln^2 \frac{Q^2}{Q_0^2} \right], \quad (84)$$

where $Q_0 \ll Q^2$ is the characteristic (nonperturbative) scale of the target. The first term is *a priori* a small correction since we typically have $\rho^+ \sim q^+$: it features the double logarithm present in a single step of a *projectile* (k^+)

ordered high-energy evolution between ρ^+ and q^+ . On the other hand, the second term in this expression is a pure NLO correction from the point of view of the resummation of high-energy logarithms; yet it can be large when $Q^2 \gg Q_0^2$ such that H_{NLO} becomes negative. The dipole operator on the other hand yields

$$\begin{aligned} S(|\mathbf{r}| \sim Q^{-1}, \rho^+) &\approx \bar{\alpha}_s \int_{Q_0^2}^{Q^2} \frac{d\mathbf{k}^2}{k^2} \ln \frac{2\rho^+ P^-}{k^2} \\ &= \frac{\bar{\alpha}_s}{2} \left(\ln^2 \frac{2\rho^+ P^-}{Q_0^2} - \ln^2 \frac{2\rho^+ P^-}{Q^2} \right) \\ &= \bar{\alpha}_s \ln \frac{2\rho^+ P^-}{Q_0 Q} \ln \frac{Q^2}{Q_0^2}, \end{aligned} \quad (85)$$

which follows from Eq. (48) in momentum space after taking the double logarithmic limit. The k^2 dependence in the log is the additional term we found earlier as a consequence of the fact that the gluon lifetime must be shorter than the size of the target. As expected the ρ^+ dependence cancels out in the sum, and we find

$$\begin{aligned} \sigma_{\text{NLO}}(x_{\text{Bj}}, Q^2) &\approx \bar{\alpha}_s \left[\ln \frac{Q^2}{Q_0^2} \ln \frac{q^+}{\rho^+} - \frac{1}{2} \ln^2 \frac{Q^2}{Q_0^2} + \ln \frac{2\rho^+ P^-}{Q_0 Q} \ln \frac{Q^2}{Q_0^2} \right] \\ &= \bar{\alpha}_s \ln \frac{Q^2}{Q_0^2} \ln \frac{2q^+ P^-}{Q^2} = \bar{\alpha}_s \ln \frac{Q^2}{Q_0^2} \ln \frac{1}{x_{\text{Bj}}}. \end{aligned} \quad (86)$$

Although simple, this result is quite a satisfactory confirmation of consistency for our approach: it yields the correct double logarithmic structure at NLO, and only one collinear log appears at the $\mathcal{O}(\alpha_s)$ order as required by the renormalization group (RG) structure of QCD. In comparison, the “naive” shockwave approach which omits the collinearly anomalous term in the NLO correction to the dipole operator at small x yields an additional collinear double log:

$$\begin{aligned} \sigma_{\text{NLO,naive}}(x_{\text{Bj}}, Q^2) &\approx \bar{\alpha}_s \left[\ln \frac{Q^2}{Q_0^2} \ln \frac{q^+}{\rho^+} - \frac{1}{2} \ln^2 \frac{Q^2}{Q_0^2} + \ln \frac{\rho^+ P^-}{Q_0^2} \ln \frac{Q^2}{Q_0^2} \right] \\ &= \bar{\alpha}_s \ln \frac{Q^2}{Q_0^2} \ln \frac{1}{x_{\text{Bj}}} + \frac{\bar{\alpha}_s}{2} \ln^2 \frac{Q^2}{Q_0^2}, \end{aligned} \quad (87)$$

which is of course not the correct result because of the additional double collinear logarithm. Another way to see how the composite dipole transformation renders the NLO hard factor parametrically small (as it should) is to compute $\bar{H}_{\text{NLO}}$ directly in the double logarithmic approximation. To that aim, one simply has to add to Eq. (84) the counterterm given by $\bar{\alpha}_s L(\mu)$ resulting from the change of basis as shown in Eq. (81), which reads as, in the double logarithmic approximation

$$\begin{aligned} \bar{\alpha}_s L(\mu) &\approx \bar{\alpha}_s \int_{Q_0^2}^{Q^2} \frac{d\mathbf{k}^2}{k^2} \ln \left(\frac{\mu^2}{k^2} \right) \\ &= \frac{1}{2} \ln \left(\frac{Q Q_0}{\mu^2} \right) \ln \left(\frac{Q^2}{Q_0^2} \right), \end{aligned} \quad (88)$$

such that, combined with Eq. (84), we find that

$$\bar{H}_{\text{NLO}}(Q^2, \rho^+/q^+, \mu^2) \approx \bar{\alpha}_s \ln \left(\frac{Q^2}{Q_0^2} \right) \ln \left(\frac{\mu^2}{x_{\text{Bj}} \zeta} \right). \quad (89)$$

Since the factorization scale $\zeta = 2\rho^+ P^-$ is ultimately chosen of the order of $2q^+ P^-$ ($\rho^+ \sim q^+$ in a k^+ ordered factorization scheme) and $\mu^2 \sim Q^2$, we have $\mu^2/\zeta \sim x_{\text{Bj}}$ such that $\bar{H}_{\text{NLO}}$ is clearly a small $\mathcal{O}(\alpha_s)$ correction which does not contain either any high energy logarithm or any double collinear logarithm [unlike Eq. (84)]. In other words, the “change of basis” we propose eliminates the large negative double logarithms in the NLO impact factor originally given by Eq. (84).

VI. MELLIN SPACE ANALYSIS OF THE EVOLUTION EQUATION FOR THE COMPOSITE DIPOLE

Now that we have checked that the fixed order NLO result yields the correct double logarithmic structure, we want to demonstrate that the NLO high-energy evolution of the composite dipole defined by Eq. (12) (which would appear in the NNLO fixed order calculation) is also free of collinear logarithms. To that aim, it is sufficient to consider

the weak scattering limit where the evolution for the composite dipole linearizes in terms of $T \equiv 1 - S$. It is also convenient to work in Mellin space; we denote the Mellin transform of the dipole amplitude and its Mellin inverse as

$$\bar{T}(\gamma) = \mu_0^2 \int d^2 \mathbf{x}_{12} (\mu_0 |\mathbf{x}_{12}|)^{2(\gamma-1)} (1 - \bar{S}_{12}), \quad \bar{T}(\mathbf{x}_{12}) = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma}{2\pi i} \bar{T}(\gamma) (\mu_0 |\mathbf{x}_{12}|)^{2\gamma}, \quad (90)$$

with μ_0 an arbitrary transverse scale introduced to make the Mellin transform dimensionless, which can be thought of as the characteristic scale of the target.

We first compute from Eq. (61) the composite $\bar{T}$ matrix in Mellin space. To do so, we first express the composite dipole amplitude in the case where this amplitude is a pure power, $T(\mathbf{x}_{12}) = (\mu_0 |\mathbf{x}_{12}|)^{2\gamma}$:

$$\begin{aligned} \bar{T}(\mathbf{x}_{12}) = T(\mathbf{x}_{12}) + \frac{\alpha_s N_c}{4\pi^2} \int d^2 \mathbf{x}_3 \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2} & \left(\ln(\mathbf{x}_{13}^2 \mathbf{x}_{32}^2 \mu^4) + \frac{\mathbf{x}_{32}^2 - \mathbf{x}_{13}^2}{\mathbf{x}_{12}^2} \ln \frac{\mathbf{x}_{13}^2}{\mathbf{x}_{32}^2} \right) \\ & \times [(\mu_0 |\mathbf{x}_{12}|)^{2\gamma} - (\mu_0 |\mathbf{x}_{13}|)^{2\gamma} - (\mu_0 |\mathbf{x}_{32}|)^{2\gamma}]. \end{aligned} \quad (91)$$

The integral over $\mathbf{x}_3$ can be performed using the generalized result for the conformal “bubble” integral,

$$B(\mathbf{x}_{12}, a, b) \equiv \int d^D \mathbf{x}_3 \frac{1}{(\mathbf{x}_{13})^{2a} (\mathbf{x}_{32})^{2b}} = \pi^{D/2} \frac{\Gamma(a+b-D/2) \Gamma(D/2-a) \Gamma(D/2-b)}{\Gamma(a) \Gamma(b) \Gamma(D-a-b)} (\mathbf{x}_{12}^2)^{D/2-a-b}, \quad (92)$$

for arbitrary powers a and b . Taking the derivative with respect to a or b and setting $a = 0$ or $b = 0$ brings a logarithm of $\mathbf{x}_{13}$ or $\mathbf{x}_{32}$ such that all terms in Eq. (91) can be computed analytically in D dimension:

$$\begin{aligned} \bar{T}(\mathbf{x}_{12}) = T(\mathbf{x}_{12}) + \lim_{D \rightarrow 2} \left\{ \frac{\alpha_s N_c}{4\pi^2} (\mu_0 |\mathbf{x}_{12}|)^{2\gamma} \left[\mathbf{x}_{12}^2 \left(-2 \frac{\partial B}{\partial a} + 4 \ln(\mu) B \right) \Big|_{a=b=1} + 2 \left(\frac{\partial B}{\partial a} - \frac{\partial B}{\partial b} \right) \Big|_{a=0, b=1} \right] \right. \\ \left. - \frac{\alpha_s N_c}{2\pi^2} \mu_0^{2\gamma} \left[\mathbf{x}_{12}^2 \left(-\frac{\partial B}{\partial a} - \frac{\partial B}{\partial b} + 4 \ln(\mu) B \right) \Big|_{a=1, b=1-\gamma} + \left(\frac{\partial B}{\partial a} - \frac{\partial B}{\partial b} \right) \Big|_{a=0, b=1-\gamma} + \left(\frac{\partial B}{\partial b} - \frac{\partial B}{\partial a} \right) \Big|_{a=1, b=-\gamma} \right] \right\}, \end{aligned} \quad (93)$$

where we have used $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$ symmetry. In the end, we find

$$\bar{T}(\mathbf{x}_{12}) = \left[1 + \bar{\alpha}_s f_1(\gamma) - \bar{\alpha}_s \chi_1(\gamma) \left(\frac{d}{d\gamma} + \ln(\mu^2/\mu_0^2) \right) \right] (\mu_0 |\mathbf{x}_{12}|)^{2\gamma}, \quad (94)$$

which implies, after integrating by part,

$$\bar{T}(\gamma) = \left[1 + \bar{\alpha}_s (f_1(\gamma) + \chi_1'(\gamma)) + \bar{\alpha}_s \chi_1(\gamma) \left(\frac{d}{d\gamma} - \ln(\mu^2/\mu_0^2) \right) \right] T(\gamma). \quad (95)$$

In this expression, χ_1 is the LO BKFL eigenvalue, defined as

$$\chi_1(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1-\gamma), \quad (96)$$

with $\psi(\gamma)$ is the digamma function, $\psi(1) = -\gamma_E$ and f_1 is an analytically computable function, given by

$$\begin{aligned} f_1(\gamma) = -\frac{1}{2\gamma} [(2\gamma_E + \pi \cot(\pi\gamma))(2 + 2\gamma_E\gamma + \pi\gamma \cot(\pi\gamma)) + \pi^2 \gamma \cot^2(\pi\gamma) \\ + 4\psi(\gamma)(1 + 2\gamma_E\gamma + \pi\gamma \cot(\pi\gamma) + \gamma\psi(\gamma)) - 2\gamma\psi'(\gamma)], \end{aligned} \quad (97)$$

but whose precise form is actually of no relevance to our analysis. Indeed, the important structure arises from the derivative with respect to γ which is tied to the scale dependence (via μ) of our composite operator. In particular, in the conformal dipole case, the coefficient multiplying the $d/d\gamma$ operator is smaller by a factor of 2 because of the difference in the μ dependence between the conformal and our composite dipole. Using Eq. (95) and taking a derivative with respect to $\ln(\zeta)$, one easily gets the NLO evolution of the composite $\bar{T}$ matrix in Mellin space since

$$\begin{aligned} \frac{\partial \bar{T}(\gamma)}{\partial \ln(\xi)} &= \bar{\alpha}_s \chi_1(\gamma) \bar{T}(\gamma) + \bar{\alpha}_s^2 \chi_1(\gamma) \chi'_1(\gamma) \bar{T}(\gamma) \\ &+ \bar{\alpha}_s^2 \chi_2(\gamma) \bar{T}(\gamma) + \mathcal{O}(\alpha_s^3), \end{aligned} \quad (98)$$

where we have used that $[\chi_1(\gamma), f_1(\gamma)] = 0$ and

$$\left[\chi_1(\gamma), \chi_1(\gamma) \frac{d}{d\gamma} \right] = -\chi_1(\gamma) \chi'_1(\gamma). \quad (99)$$

In this expression χ_2 is the two-loop $\mathcal{O}(\alpha_s^2)$ BFKL eigenvalue, whose expression can be found in [87–89]. In an asymmetric scheme, the pathological collinear double logarithm behavior of the NLO BFKL equation manifests itself as a triple pole in $\gamma = 1$ in χ_2 :

$$\chi_2(\gamma) = -\frac{1}{(1-\gamma)^3} - \left(\frac{11}{12} + \frac{n_f}{6N_c^3} \right) \frac{1}{(1-\gamma)^2} + \mathcal{O}\left(\frac{1}{1-\gamma} \right). \quad (100)$$

On the other hand, the additional NLO term introduced by the scale dependence of the composite operator, i.e. $-\bar{\alpha}_s^2 \chi_1(\gamma) \chi'_1(\gamma)$, also yields a triple pole at $\gamma = 1$ which exactly cancels the collinear pole in the NLO BFKL kernel:

$$\chi_1(\gamma) \chi'_1(\gamma) = \frac{1}{(1-\gamma)^3} - 2\zeta(3) + \mathcal{O}((1-\gamma)^2). \quad (101)$$

This exact cancellation relies on the particular value of the coefficient in front of the $\chi_1(\gamma) \chi'_1(\gamma)$ term; in particular, the NLO evolution of the conformal dipole is not free of double collinear logarithms, which *a posteriori* explains why the numerical solution of the NLO BK equation for the conformal dipole is unstable [31].

However, the NLO evolution equation Eq. (98) contains spurious *anticollinear* logarithms which manifest in Mellin

space as poles when $\gamma \rightarrow 0$. In the asymmetric scheme, the two-loop BFKL eigenvalue behaves like

$$\chi_2(\gamma) = -\left(\frac{11}{12} + \frac{n_f}{6N_c^3} \right) \frac{1}{\gamma^2} + \mathcal{O}\left(\frac{1}{\gamma} \right), \quad (102)$$

while the $\chi_1(\gamma) \chi'_1(\gamma)$ counterterm introduces a triple pole in $\gamma = 0$:

$$\chi_1(\gamma) \chi'_1(\gamma) = -\frac{1}{\gamma^3} + 2\zeta(3) + \mathcal{O}(\gamma^2), \quad (103)$$

which is responsible, in physical space, to the additional anticollinear double logarithms in the NLO BFKL equation for the composite dipole.

The main point of this section is that the NLO BFKL equation for the composite dipole in the dilute limit is free from any collinear double logarithms. This property is largely independent of the specific form of the kernel L_{123} used in the basis change between the dipole and the composite dipole, as long as the dependence on the collinear factorization scale μ remains the same. In the next section, we will leverage this freedom to select an appropriate form for L_{123} , aiming to simplify the NLO equation for the composite dipole in the nonlinear regime as much as possible.

While a different choice of L_{123} would not affect the NLO BFKL equation for the composite dipole, it leads to a different form for the $f_1(\gamma)$ function which does appear in the NLO impact factor after performing the change of basis between T and $\bar{T}$. If, instead of Eq. (12), one uses a simplified version of the composite dipole with $L_{123} = K_{123} \ln(1/(|\mathbf{x}_{13}||\mathbf{x}_{32}|\mu^2))$, then the function f reads as

$$f_1(\gamma) = -\frac{1}{4} (4\gamma_E^2 + (4\gamma_E + \psi(\gamma) + \psi(1-\gamma))(\psi(\gamma) + \psi(1-\gamma)) + 3\psi'(1-\gamma) - 3\psi'(\gamma)). \quad (104)$$

These findings, together with those of the previous section, can be formalized at the level of the physical NLO DIS cross section. According to Eq. (95), the L operator [defined by Eq. (62)] in Mellin space reads as

$$\mathcal{M}\{L\} = -(f_1(\gamma) + \chi'_1(\gamma)) - \chi_1(\gamma) \left[\frac{d}{d\gamma} - \ln(\mu^2/\mu_0^2) \right], \quad (105)$$

at one loop [i.e. $L = L_1 + \mathcal{O}(\alpha_s)$ in the notations of Sec. IV]. After the change of basis, the resulting NLO cross section in Mellin space becomes [see also Eq. (81)]

$$\begin{aligned} \sigma_{\text{NLO}}(x, Q^2) &= \int \frac{d\gamma}{2\pi i} H_{\text{NLO}}(\rho^+/q^+, \gamma) \left[1 - \bar{\alpha}_s (f_1(\gamma) + \chi'_1(\gamma)) - \bar{\alpha}_s \chi_1(\gamma) \left(\frac{d}{d\gamma} - \ln(\mu^2/\mu_0^2) \right) \right] \\ &\times \exp \left[\bar{\alpha}_s (\chi_1(\gamma) + \bar{\alpha}_s \chi_1 \chi'_1(\gamma) + \bar{\alpha}_s \chi_2(\gamma)) \ln \frac{\rho^+}{\rho_0^+} \right] \bar{T}(\gamma, \rho_0^+ P^-) + \mathcal{O}(\alpha_s^2), \end{aligned}$$

where we have used that $\mathcal{M}\{[K, L]\} = [\chi_1(\gamma), \chi_1(\gamma) \frac{d}{d\gamma}] = -\chi_1(\gamma) \chi_1'(\gamma)$. Integrating by part the leftmost derivative we obtain

$$\sigma_{\text{NLO}}(x, Q^2) = \int \frac{d\gamma}{2\pi i} (\bar{H}_{\text{NLO}}(\rho^+/q^+, \gamma) + \mathcal{O}(\alpha_s^2)) \exp \left[\bar{\alpha}_s \bar{\chi}_{\text{NLO}}(\gamma) \ln \frac{\rho^+}{\rho_0^+} \right] \bar{T}(\gamma, \rho_0^+ P^-), \quad (106)$$

where the new NLO impact factor and NLO eigenvalue respectively read as

$$\bar{H}_{\text{NLO}}(\rho^+/q^+, \gamma) = H_{\text{NLO}}(\rho^+/q^+, \gamma) + \bar{\alpha}_s H_{\text{LO}}(\gamma) [\chi_1(\gamma) \ln(\mu^2/\mu_0^2) - f_1(\gamma)] + \bar{\alpha}_s \frac{d}{d\gamma} H_{\text{LO}}(\gamma), \quad (107)$$

$$\bar{\chi}_{\text{NLO}}(\gamma) = \chi_{\text{NLO}}(\gamma) + \bar{\alpha}_s \chi_1(\gamma) \chi_1'(\gamma) = \chi_1(\gamma) + \bar{\alpha}_s \chi_2(\gamma) + \bar{\alpha}_s \chi_1 \chi_1'(\gamma). \quad (108)$$

The NLO impact factor in Mellin space thus receives a new contribution coming from the composite dipole transformation, such that the physical cross section is invariant [up to $\mathcal{O}(\alpha_s^2)$ corrections]. In particular, one sees that it is sensitive to the function $f_1(\gamma)$ which depends on the specific choice of kernel L_{123} in the composite dipole [one can also choose $\mu_0 = \mu$ in the definition of the Mellin transform to eliminate the $\ln(\mu^2/\mu_0^2)$ term]. The derivative operator also brings a NLO correction proportional to the derivative of the LO impact factor.

Finally, the analysis of the collinear structure of the high-energy evolution of T carried out in Sec. IV enables one to obtain the NNLO evolution equation for the composite $\bar{T}$ matrix, given the three-loop BFKL eigenvalue $\chi_3(\gamma)$ which has been computed in [90,91] in $\mathcal{N} = 4$ supersymmetric Yang-Mills Theory (see also Ref. [92] for an approximate expression for QCD and note that the linearization of the results obtained in [80], together with the conformal and matter loop contributions, should provide the exact analytic result in QCD). Following Eq. (76), we define the $\mathcal{O}(\alpha_s^2)$ contribution to the L operator in Mellin space as

$$\mathcal{M}\{L_2\} = -f_2(\gamma) + \left\{ \chi_2(\gamma) + \frac{1}{2} \chi_1(\gamma) \chi_1'(\gamma) \right\} \left[-\frac{d}{d\gamma} + \ln(\mu^2/\mu_0^2) \right], \quad (109)$$

where the function f_2 is scheme dependent and does not matter for the NNLO BFKL kernel, while the $\ln(\mu^2)$ dependence (and thus, the factor in front of the $d/d\gamma$ operator) is constrained to be the operator $K_2 + \frac{1}{2}[K_1, L_{10}]$ in Mellin space, as demonstrated in Sec. IV. Combining this relation with Eq. (78), we find that the NNLO BFKL eigenvalue for the composite $\bar{T}$ matrix reads as

$$\bar{\chi}_{\text{NNLO}}(\gamma) = \chi_{\text{NNLO}}(\gamma) + \bar{\alpha}_s \chi_1(\gamma) \chi_1'(\gamma) + \bar{\alpha}_s^2 \left[\chi_2(\gamma) \chi_1'(\gamma) + \chi_2'(\gamma) \chi_1(\gamma) + \chi_1(\gamma) \chi_1'^2(\gamma) + \frac{1}{2} \chi_1^2(\gamma) \chi_1''(\gamma) \right], \quad (110)$$

with $\chi_{\text{NNLO}}(\gamma) = \chi_1(\gamma) + \bar{\alpha}_s \chi_2(\gamma) + \bar{\alpha}_s^2 \chi_3(\gamma)$. Equation (110) does not depend on f_2 nor $\ln \mu^2$ and is free of collinear poles associated with double collinear logarithms by construction. Equation (110) coincides with Eq. (46) of [93] upon the replacement $d/d\gamma \rightarrow -\frac{1}{2}d/d\gamma$, thereby confirming that our framework naturally incorporates the scale redefinition proposed in the BFKL context. The factor of 2 in front of the γ -derivative arises because our approach effectively corresponds to a change of variables from one asymmetric choice (k^+ -ordered) to another asymmetric choice (k^- -ordered), rather than from a symmetric to an asymmetric choice, as considered in [93].

VII. NLO BK EQUATION FOR THE COMPOSITE DIPOLE

In this section, we present the NLO BK equation for the composite dipole which becomes free of double collinear logarithms. From a formal standpoint, as discussed in Sec. III, we present here the calculation of the commutator between the LO BK kernel K_{123} and the kernel appearing in the composite dipole denoted as L_{123} . Unlike in the dilute BFKL regime though, the precise form of the composite dipole matters in the nonlinear regime. For this reason, we explore the dependence of the NLO BK equation on the choice of the transverse scale r_{123} accompanying the

arbitrary scale μ in the definition of the composite dipole. This general choice of composite dipole is given by Eq. (63), which we rewrite here as

$$\bar{S}_{12} = S_{12} + \bar{\alpha}_s \int_3 K_{123} \ln\left(\frac{1}{r_{123}^2 \mu^2}\right) [S_{13} S_{32} - S_{12}], \quad (111)$$

with r_{123} some transverse coordinate scale defined in terms of $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. Taking a derivative with respect to $\ln(\zeta)$ and working at NLO [we neglect $\mathcal{O}(\alpha_s^3)$ terms], the NLO BK equation for $\bar{S}_{12}$ reads as

$$\begin{aligned} \frac{\partial \bar{S}_{12}}{\partial \ln(\zeta)} = & \bar{\alpha}_s K_{\text{NLO}} \otimes \bar{S}_{12} + \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{134} \ln\left(\frac{r_{134}^2}{r_{123}^2}\right) (\bar{S}_{14} \bar{S}_{43} - \bar{S}_{13}) \bar{S}_{32} \\ & + \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{324} \ln\left(\frac{r_{324}^2}{r_{123}^2}\right) (\bar{S}_{34} \bar{S}_{42} - \bar{S}_{32}) \bar{S}_{13} - \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{124} \ln\left(\frac{r_{124}^2}{r_{123}^2}\right) (\bar{S}_{14} \bar{S}_{42} - \bar{S}_{12}). \end{aligned} \quad (112)$$

In this equality the second term (and similarly third and fourth) comes from the replacement of S_{13} (resp. S_{32} and S_{12}) in the LO BK equation by its expression in terms of $\bar{S}_{13}$ (resp. $\bar{S}_{32}$ and $\bar{S}_{12}$) and the $\ln(\zeta)$ derivative of S_{13} (resp. S_{32} and S_{12}) present in the definition of the composite dipole as given by the LO BK equation. To the accuracy considered here, one can then replace all the “bare” dipoles by the composite ones in the terms of order α_s^2 , in particular for the term given by the NLO BK kernel. After these manipulations, the NLO BK equation for the composite dipole is manifestly μ independent.

The simplest choice for r_{123} would be the parent dipole size, namely,

$$r_{123}^2 = x_{12}^2. \quad (113)$$

In this case, the above equation simplifies into

$$\begin{aligned} \frac{\partial \bar{S}_{12}}{\partial \ln(\zeta)} = & \bar{\alpha}_s K_{\text{NLO}} \otimes \bar{S}_{12} + \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{134} \ln\left(\frac{x_{13}^2}{x_{12}^2}\right) (\bar{S}_{14} \bar{S}_{43} - \bar{S}_{13}) \bar{S}_{32} \\ & + \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{324} \ln\left(\frac{x_{32}^2}{x_{12}^2}\right) (\bar{S}_{34} \bar{S}_{42} - \bar{S}_{32}) \bar{S}_{13}, \end{aligned} \quad (114)$$

which, after using $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$ symmetry, is identical to the equation obtained in [63] after the change of variable from projectile to target rapidity ordering. As discussed in [63], the collinear double logarithms from the NLO BK kernel are canceled by the two additional terms in Eq. (114), although not explicitly. Using another choice for r_{123} , one can make this cancellation explicit.

While Eq. (112) is general, it is not the most convenient form because, depending on the value of r_{123} , each term might contain UV divergences which only cancel in the sum. In order to obtain a NLO BK equation which is manifestly free of UV divergences term by term, we wish to reorganize Eq. (112) following the color structures present in the conventional NLO BK equation written in Eq. (12). To do so, we first isolate the two terms depending on the product of three dipoles, $\bar{S}_{34} \bar{S}_{42} \bar{S}_{13}$ and $\bar{S}_{14} \bar{S}_{43} \bar{S}_{32}$. By exchanging $\mathbf{r}_3 \leftrightarrow \mathbf{r}_4$ in the latter contribution, one realizes that these two triple dipole terms can be combined together; this yields a kernel of the form $K_{123} K_{324} \ln[r_{324}^2 r_{143}^2 / (r_{123}^2 r_{124}^2)]$. In Eq. (12), the color structure depending on the product of three dipoles is actually $(\bar{S}_{34} \bar{S}_{42} - \bar{S}_{32}) \bar{S}_{13}$; therefore, we further add and subtract a double dipole $\bar{S}_{13} \bar{S}_{32}$ with the same kernel. Finally, we group together all the remaining terms depending on the double dipoles. In the end, we find that the previous equation can be reorganized according to

$$\begin{aligned} \frac{\partial \bar{S}_{12}}{\partial \ln(\zeta)} = & \bar{\alpha}_s K_{\text{NLO}} \otimes \bar{S}_{12} + \bar{\alpha}_s^2 \int_{3,4} K_{123} K_{324} \ln\left(\frac{r_{324}^2 r_{143}^2}{r_{123}^2 r_{124}^2}\right) (\bar{S}_{34} \bar{S}_{42} - \bar{S}_{32}) \bar{S}_{13} \\ & - \bar{\alpha}_s^2 \int_{3,4} K_{123} \left[K_{134} \ln\left(\frac{r_{134}^2}{r_{123}^2}\right) + K_{324} \ln\left(\frac{r_{124}^2}{r_{143}^2}\right) + K_{124} \ln\left(\frac{r_{123}^2}{r_{124}^2}\right) \right] (\bar{S}_{13} \bar{S}_{32} - \bar{S}_{12}). \end{aligned} \quad (115)$$

Note that, by antisymmetry under $\mathbf{x}_3 \leftrightarrow \mathbf{x}_4$ exchange, we have

$$\int_{3,4} K_{123} \left[K_{134} \ln \left(\frac{r_{134}^2}{r_{123}^2} \right) + K_{324} \ln \left(\frac{r_{124}^2}{r_{143}^2} \right) + K_{124} \ln \left(\frac{r_{123}^2}{r_{124}^2} \right) \right] = 0, \quad (116)$$

such that one can add for free the single dipole term $\bar{S}_{12}$ in the subtraction term of the double dipole contribution depending on $\bar{S}_{13}\bar{S}_{32}$.

We now discuss the choices of $\mathbf{r}_{123}$ such that (i) the integral of $\mathbf{r}_4$ in the third term depending on the color structure $\bar{S}_{13}\bar{S}_{32} - \bar{S}_{12}$ can be performed analytically, and (ii) each term in Eq. (115) is separately UV finite. All the forms of $\mathbf{r}_{123}$ considered in this section lead to a NLL BK equation for the composite dipole which is free of collinear double logarithms; yet this is not necessarily manifest term by term in the evolution equation since depending on the choice of $\mathbf{r}_{123}$, double collinear logarithms can remain both in the triple dipole term depending on $\bar{S}_{13}(\bar{S}_{34}\bar{S}_{42} - \bar{S}_{32})$ and in the double dipole one, such that they only cancel in the sum.

For instance, if one uses a value for $\mathbf{r}_{123}$ inspired by the conformal dipole considered in [22],

$$\mathbf{r}_{123}^2 = \frac{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2}{\mathbf{x}_{12}^2}, \quad (117)$$

then, one can use the following integral computed in [22]:

$$\int_4 \left[\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{34}^2} \ln \left(\frac{\mathbf{x}_{14}^2 \mathbf{x}_{34}^2 \mathbf{x}_{12}^2}{\mathbf{x}_{13}^4 \mathbf{x}_{32}^2} \right) + \frac{\mathbf{x}_{32}^2}{\mathbf{x}_{34}^2 \mathbf{x}_{24}^2} \ln \left(\frac{\mathbf{x}_{14}^2 \mathbf{x}_{24}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{34}^2 \mathbf{x}_{12}^2} \right) + \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{24}^2} \ln \left(\frac{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{42}^2} \right) \right] = -\ln \left(\frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2} \right) \ln \left(\frac{\mathbf{x}_{12}^2}{\mathbf{x}_{32}^2} \right) - 4\zeta(3)\delta(\mathbf{x}_{13}). \quad (118)$$

The term proportional to $\delta(\mathbf{x}_{13})$ does not contribute to the remaining integral over $\mathbf{r}_3$ thanks to unitarity. Half of the double log on the right-hand side of Eq. (118) is canceled by the double log present in the K_{NLO} term, which reads [recall Eq. (13)] as

$$K_{\text{NLO}} \otimes \bar{S}_{12} = \int_3 K_{123} \left[1 - \frac{\bar{\alpha}_s}{2} \ln \left(\frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2} \right) \ln \left(\frac{\mathbf{x}_{12}^2}{\mathbf{x}_{32}^2} \right) \right] (\bar{S}_{13}\bar{S}_{32} - \bar{S}_{12}) + \dots, \quad (119)$$

while the other half gets canceled by the second term in Eq. (115) depending on the triple dipole color structure (this cancellation would be manifest in the linear regime, as shown by the Mellin space analysis).

For $\mathbf{r}_{123}^2 = \mathbf{x}_{12}^2$, in order to simplify the term in Eq. (115) depending on the color structure $\bar{S}_{13}\bar{S}_{32} - \bar{S}_{12}$, one must evaluate the integral

$$\frac{1}{\pi} \int d^2\mathbf{x}_4 \left[\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{34}^2} \ln \left(\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{12}^2} \right) + \frac{\mathbf{x}_{32}^2}{\mathbf{x}_{34}^2 \mathbf{x}_{24}^2} \ln \left(\frac{\mathbf{x}_{12}^2}{\mathbf{x}_{14}^2} \right) \right], \quad (120)$$

which is divergent in the UV as the logarithmic singularity in the limit $\mathbf{x}_4 \rightarrow \mathbf{x}_1$ of the first term does not cancel with the second term. This UV divergence gets canceled by a similar divergence in the term depending on the color structure $(\bar{S}_{34}\bar{S}_{42} - \bar{S}_{32})\bar{S}_{13}$; yet, this demonstrates that not all $\mathbf{r}_{123}$ scales are convenient from a numerical point of view, since it would be preferable that each term in Eq. (115) is separately UV and IR finite.

A sufficient condition on $\mathbf{r}_{123}$ for the integral over $\mathbf{x}_4$ to be UV finite is that it satisfies

$$\forall \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \quad \mathbf{r}_{134}^2 \underset{\mathbf{x}_4 \rightarrow \mathbf{x}_1}{\sim} \mathbf{r}_{124}^2, \quad (121)$$

which is indeed true for the choice given by Eq. (117) but obviously not for $\mathbf{r}_{123}^2 = \mathbf{x}_{12}^2$. Importantly, this condition also ensures that the second term in Eq. (115) is UV finite for $\mathbf{x}_3 \rightarrow \mathbf{x}_1$ (note that the other singular points $\mathbf{x}_4 \rightarrow \mathbf{x}_3$ and $\mathbf{x}_4 \rightarrow \mathbf{x}_2$ of the kernel $K_{123}K_{324}$ are cured by unitarity).

Possible expressions for $\mathbf{r}_{123}$ satisfying this condition are $\mathbf{r}_{123}^2 = \min(\mathbf{x}_{13}^2, \mathbf{x}_{23}^2)$, the choice given by Eq. (12) emerging from the $x \rightarrow 0$ limit of the x -dependent dipole distribution computed in Sec. III, which reads as

$$\mathbf{r}_{123}^2 = |\mathbf{x}_{13}| |\mathbf{x}_{32}| \exp \left[\frac{\mathbf{x}_{23}^2 - \mathbf{x}_{13}^2}{2\mathbf{x}_{12}^2} \ln \left(\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{23}^2} \right) \right], \quad (122)$$

or the generalization of Eq. (117) with an arbitrary power β :

$$\mathbf{r}_{123}^2 = \mathbf{x}_{13}^2 \left(\frac{\mathbf{x}_{32}^2}{\mathbf{x}_{12}^2} \right)^\beta. \quad (123)$$

In the two former cases, the drawback is that we did not manage to find a closed analytic expression for the $\mathbf{x}_4$ integral in the double dipole term. Among the latter class given by Eq. (123), there are two interesting cases. The first one is $\beta = -1$ since then, one can also easily perform the $\mathbf{x}_4$

integral using the standard result for the conformal bubble integral in d dimension:

$$\int_4 \left[\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{34}^2} \ln \left(\frac{\mathbf{x}_{14}^2 \mathbf{x}_{32}^2}{\mathbf{x}_{12}^2 \mathbf{x}_{34}^2} \right) + \frac{\mathbf{x}_{32}^2}{\mathbf{x}_{34}^2 \mathbf{x}_{24}^2} \ln \left(\frac{\mathbf{x}_{12}^2 \mathbf{x}_{34}^2}{\mathbf{x}_{24}^2 \mathbf{x}_{13}^2} \right) + \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{24}^2} \ln \left(\frac{\mathbf{x}_{13}^2 \mathbf{x}_{24}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{23}^2} \right) \right] = 0. \quad (124)$$

For this choice, the additional term depending on the color structure $\bar{S}_{13}\bar{S}_{32} - \bar{S}_{12}$ simply vanishes, and the double collinear log in the color structure $\bar{S}_{13}\bar{S}_{32} - \bar{S}_{12}$ of the NLO BK kernel remains as it stands, but gets canceled by a double collinear logarithm “hidden” in the triple dipole term of the NLO evolution of the composite dipole.

For $\beta = 0$, we have $\mathbf{r}_{123}^2 = \mathbf{x}_{13}^2$, and the integral of $\mathbf{x}_4$ one must compute involves not only bubble graphs but also a “triangle” graph with three UV singular points at $\mathbf{x}_4 = \mathbf{x}_3, \mathbf{x}_2$ and $\mathbf{x}_1$ from the second term of this expression:

$$\int_4 \left[\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{34}^2} \ln \left(\frac{\mathbf{x}_{14}^2}{\mathbf{x}_{13}^2} \right) + \frac{\mathbf{x}_{32}^2}{\mathbf{x}_{34}^2 \mathbf{x}_{24}^2} \ln \left(\frac{\mathbf{x}_{14}^2}{\mathbf{x}_{13}^2} \right) + \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{24}^2} \ln \left(\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{14}^2} \right) \right] = -\frac{1}{2} \ln \left(\frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2} \right) \ln \left(\frac{\mathbf{x}_{12}^2}{\mathbf{x}_{23}^2} \right). \quad (125)$$

Note the difference of factor of 2 with Eq. (118), corresponding to $\beta = 1$. The integral [Eq. (125)] is computed in Appendix C.

Therefore, interestingly, the case $\beta = 0$, i.e. $\mathbf{r}_{123}^2 = \mathbf{x}_{13}^2$, enables one to exactly cancel the double log contribution in the NLO BK kernel *color structure per color structure*. This choice of the composite dipole should be favored for numerical implementation of the NLO BK equation. With this choice the structure $(\bar{S}_{34}\bar{S}_{42} - \bar{S}_{32})\bar{S}_{13}$ in the NLO BK equation for the composite dipole is also quite simple and does not contain any double collinear logarithm as compared to the other values of β previously discussed; it is given by

$$\bar{\alpha}_s^2 \int_{3,4} K_{123} K_{324} \ln \left(\frac{\mathbf{x}_{34}^2}{\mathbf{x}_{14}^2} \right) (\bar{S}_{34}\bar{S}_{42} - \bar{S}_{32}) \bar{S}_{13} \quad (126)$$

and is also clearly UV finite.

The only drawback of Eq. (123) is that it breaks the $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$ symmetry when considering the evolution equation for the real part of the composite dipole, i.e. neglecting the Odderon component [94]. This symmetry can easily be restored by taking $\mathbf{r}_{123}^2 = |\mathbf{x}_{13}||\mathbf{x}_{32}|$, or equivalently by (i) symmetrizing the definition of the composite dipole when $\mathbf{r}_{123}$ is given by Eq. (123) and $\beta = 0$, namely,

$$\bar{S}_{12} = S_{12} + \frac{\bar{\alpha}_s}{2} \int_3 K_{123} \ln \left(\frac{1}{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2 \mu^4} \right) [S_{13}S_{32} - S_{12}], \quad (127)$$

and (ii) symmetrizing the manipulations between Eqs. (112) and (115). In the end, the NLO BK equation for the composite dipole Eq. (127) is

$$\begin{aligned} \frac{\partial \bar{S}_{12}}{\partial \ln(\zeta)} &= \bar{\alpha}_s \tilde{K}_{\text{NLO}} \otimes \bar{S}_{12} + \frac{\bar{\alpha}_s^2}{2} \int_{3,4} K_{123} K_{324} \ln \left(\frac{\mathbf{x}_{34}^2}{\mathbf{x}_{14}^2} \right) \bar{S}_{13} (\bar{S}_{34}\bar{S}_{42} - \bar{S}_{32}) \\ &\quad + \frac{\bar{\alpha}_s^2}{2} \int_{3,4} K_{123} K_{134} \ln \left(\frac{\mathbf{x}_{34}^2}{\mathbf{x}_{24}^2} \right) (\bar{S}_{14}\bar{S}_{43} - \bar{S}_{13}) \bar{S}_{32}, \end{aligned} \quad (128)$$

where $\tilde{K}_{\text{NLO}}$ is the NLO BK kernel *without* the double collinear logarithm in Eq. (119).

VIII. ANTICOLLINEAR LOGARITHMS AND RUNNING COUPLING

We conclude this work with a brief discussion of two aspects of the new NLO BK equation for the dipole $\bar{S}$, Eq. (128), that must be addressed to fully establish its practical usefulness: the presence of anticollinear logarithms

and the effects of the running coupling. This section is not intended to address these two aspects in detail, but rather to acknowledge some limitations of the equation proposed in this paper and to provide preliminary guidance toward their resolution. In particular, regarding running-coupling effects, our aim is merely to show that the NLO BK equation for $\bar{S}$, irrespective of the running-coupling scheme employed, is free from double collinear logarithms. Their practical implementation in combination with our scheme transformation remains an open problem at the numerical level,

and the present discussion should therefore be regarded as only indicative as far as numerical implementation is concerned.

A. The anticollinear regime

First, although the composite-dipole evolution is now free of double collinear logarithms, the transformation in Eq. (65) generates anticollinear double logarithms, as also observed in the BFKL regime in Mellin space in Sec. VI. In contrast with the collinear double logarithms, which spoil the convergence of the perturbative series at an early stage of the high-energy evolution, we have numerically observed [70] that their impact is delayed to larger rapidities. For example, for $\bar{\alpha}_s = 0.3$, the NLO-evolved dipole $\bar{S}$ exhibits a narrow window in $|\mathbf{x}_{12}|Q_s$ where the scattering amplitude $1 - \bar{S}$ becomes slightly negative for $Y \gtrsim 8$, before eventually returning to zero as $|\mathbf{x}_{12}| \rightarrow 0$. We have verified numerically that the terms responsible for this negativity are precisely those containing the double anticollinear logarithms, namely the last two terms in Eq. (128), which arise from the transformation from S to $\bar{S}$. Nevertheless, this behavior should not necessarily be interpreted as a failure of the NLO equation. Indeed, it is important to remember that we work within a perturbative framework, which entails an intrinsic limitation of Eq. (128) due to the truncation of the series at next-to-leading order. When $\alpha_s^3 Y \gtrsim 1$, NNLO corrections to the BK equation are expected to become sizeable, signaling the need for an NNLO treatment which could play a role in pushing the effect of the anticollinear double logarithms to even higher rapidities. The parametric criterion $Y \gtrsim 1/\alpha_s^3$ should be understood as a sufficient, rather than necessary, condition for the NLO BK equation to become unreliable due to missing higher-order corrections. In particular, when α_s is allowed to run (as briefly discussed below), the criterion clearly depends on the inverse transverse distance at which α_s is evaluated. Consequently, the aforementioned large- Y behavior of $\bar{S}$ is expected to be significantly affected by running-coupling improvements, and not uniformly across all dipole sizes. Of course, the same logic does not apply to collinear double logarithms which induce large NLO corrections at low rapidities and quickly render the evolution numerically unstable. We also note that an alternative strategy would be to resum these double anticollinear logarithms to all orders by modifying the LO BK kernel, following, for example, the approach proposed in [63]. Finally, since the dipole operator ultimately enters a full NLO cross section, it is the latter that must remain positive definite. In this regard, the convolution with the impact factor which contains an additional term induced by the rotation of the impact factor—currently missing from phenomenological studies—may play a quantitative role in suppressed higher-order corrections.

B. Running coupling

We have also restricted the present discussion to the fixed-coupling case, and one may therefore legitimately ask how the scheme transformation is defined once the running of the coupling is included, and whether this modifies our conclusion regarding the absence of a double collinear logarithm in the NLO equation satisfied by $\bar{S}$. In fact, the formal procedure outlined in Sec. IV can be straightforwardly extended to the running-coupling case. The scheme transformation from the operators S to $\bar{S}$, mediated by the operator L , then reads as

$$\bar{S}^{(m)}(\mathbf{x}_i, \zeta, \mu; \mu_R) = \exp[-\bar{\alpha}_s(\mu_R)L(\mathbf{x}_i, \bar{\alpha}_s(\mu_R), \mu_R, \mu)]^{(mn)} S^{(n)}(\mathbf{x}_i, \zeta; \mu_R). \quad (129)$$

Here, μ_R denotes the UV renormalization scale, not to be confused with the collinear scale μ . Note that both S and $\bar{S}$ are in fact independent of μ_R due the compensation between the explicit μ_R dependence [cf. Eq. (13) for a concrete example at NLO] and the μ_R dependence of α_s . With a sliding scale μ_R , the calculation of the commutator between the LO BK kernel and the L kernel proceeds exactly as in the fixed-coupling case, yielding the same equation as Eq. (128), with the substitution $\bar{\alpha}_s \rightarrow \bar{\alpha}_s(\mu_R)$. Therefore, including the running of the coupling does not modify our conclusion regarding the cancellation of the double collinear logarithm in the NLO equation satisfied by $\bar{S}$ after the scheme transformation. Once Eq. (128) is written with α_s evaluated at the scale μ_R , one may choose to fix μ_R to a scale determined by the transverse coordinates of either the parent or daughter dipoles in the NLO evolution. In general, fixing the scale μ_R allows one to resum, to all orders, a specific subset of higher-order loop corrections.

We stress that, at the accuracy of interest, this implementation of running-coupling effects is equivalent to fixing the scale μ_R in Eq. (129) *before* deriving the evolution equation satisfied by $\bar{S}$. If one proceeds in this way, however, the commutator between LO operator L_1 and the LO BK kernel K_1 must be recomputed. Moreover, if μ_R depends on internal transverse coordinates—such as in the Balitsky prescription [75] or in the minimal dipole prescription $\mu_R^2 = 1/\min(\mathbf{x}_{12}^2, \mathbf{x}_{13}^2, \mathbf{x}_{32}^2)$ —the result is no longer as simple as in Eq. (112), since the two factors of α_s appearing in the NLO terms are then evaluated at different scales. Nevertheless, up to NNLO corrections, these α_s factors can always be expanded around a common reference scale, thereby reducing the problem to the previous case in which μ_R was not fixed from the outset. It would be worthwhile to further investigate the differences in the resummation of higher-order corrections via the running of the coupling between these two approaches (either fixing the renormalization scale μ_R before or after the scheme transformation), especially since this question may overlap

with ongoing work on the running-coupling origin of certain contributions in the anticollinear regime of the NLO BK kernel [71,95]. As mentioned in the introductory paragraph of this section, we therefore defer a complete calculation, a numerical study of the corresponding NLO BK equation with running coupling, and a detailed analysis of the interplay between running-coupling effects and double anticollinear logarithms to future work.

IX. CONCLUSION AND OUTLOOK

In summary, we have developed a systematic operator-based framework for high-energy QCD factorization in dilute-dense collisions that eliminates the pathological collinear double logarithms responsible for numerical instabilities and negative cross sections at NLO. By redefining the dipole operator through a covariant transformation, we construct a composite dipole that restores time ordering and guarantees scheme-independent physical cross sections to all orders in perturbation theory.

Previous approaches to this problem relied on *ad hoc* prescriptions, such as direct resummation of logarithms, the imposition of kinematic constraints in the BK equation, or modifications of the rapidity-evolution variable. While partially effective, these methods depart from the systematic structure of the high-energy OPE and lack a transparent path toward generalization at higher orders. Moreover, they concentrate narrowly on BK evolution without addressing the underlying factorization scheme. A central insight of our framework is that the evolution of hadronic operators and the definition of the form factor are inseparably linked: modifying one necessarily requires a consistent redefinition of the other to preserve the invariance of physical cross sections. By introducing a covariant transformation of CGC operators and their form factors within the OPE, we provide a systematic method to track collinear double logarithmic terms order by order in perturbation theory.

Before presenting the general systematics of our approach in Sec. II, we revisited the high-energy limit that gave rise to the CGC framework, where operators are constructed from infinite Wilson-line correlators. To this end, we introduced a generalized dipole operator that explicitly retains its dependence on x —the conjugate variable to the lifetime of gluon fluctuations in the target—but reduces to the standard CGC dipole operator $\langle \text{Tr}(U_{x_1} U_{x_2}^\dagger) \rangle$ in the limit $x \rightarrow 0$. Preserving this explicit x dependence effectively accounts for the finite size of the shockwave, thereby restoring the time-ordering constraint absent in the conventional CGC setup, where the shockwave is assumed to be infinitely thin.

Analyzing the one-loop correction to the generalized dipole operator, we found that, in addition to the standard LO BK term, an extra contribution arises involving a transverse logarithm, $\ln(k^2/\mu^2)$, of the ratio between the transverse momentum of the radiated gluon and the

arbitrary collinear factorization scale μ . This logarithm persists in the $x \rightarrow 0$ limit and reflects the phase-space restriction on gluon radiation imposed by time ordering. This instructive result motivates the more general operator-based framework developed in this work.

Our central proposal is to perform a rotation in the space of CGC operators that transforms the naive dipole into a “composite dipole,” whose NLO expansion consists of the naive dipole (which obeys the BK equation) supplemented by the additional transverse-logarithm contribution identified above. In the limit $x \rightarrow 0$, this composite dipole becomes equivalent to the generalized dipole, not only at LO but also at one loop [up to nonlogarithmically enhanced $\mathcal{O}(\alpha_s)$ terms]. Importantly, the composite dipole retains the structural simplicity of CGC operators defined at $x = 0$: it is expressed solely in terms of infinite Wilson lines in the background field of the shockwave target, a feature that greatly simplifies higher-loop computations in the CGC.

This construction can be interpreted as a change of basis in the space of CGC operators. By design, physical observables remain invariant under such transformations. In this new basis, both the impact factor and the high-energy RG kernel acquire NLO corrections that cancel the large double collinear logarithms present in the original formulation. Effectively, the transverse logarithm is reshuffled into the form factor and the initial condition of the RG equation, which thus becomes μ -dependent. This mechanism is reminiscent of the QCD collinear anomaly introduced in [96,97] and discussed in [81] in the context of nonglobal logarithm resummation.

We define the composite dipole to all orders as

$$\bar{S}_{12}(\zeta, \mu^2) = \exp[-\bar{\alpha}_s L(\bar{\alpha}_s, \mu^2)] S_{12}(\zeta), \quad (130)$$

where $L(\bar{\alpha}_s, \mu^2)$ is a scale-dependent functional operator in the space of CGC operators, related to the JIMWLK Hamiltonian. Its action can be interpreted as a factorization scheme choice implemented within the standard k^+ ordered high-energy OPE and is constrained by requiring the scaling

$$\bar{S}_{12}(\zeta, \mu) = \bar{S}_{12}\left(\frac{\zeta}{\mu^2}\right), \quad (131)$$

with $\zeta = \rho^+ P^-$, where ρ^+ plays the role of a factorization scale associated with the $+$ -momentum component. This P^- -dependent scaling effectively corresponds to changing the ordering variable from k^+ to k^- , a choice often referred to in the literature as the DIS scheme. As a consequence of the above scaling, the evolution kernel must be μ -independent, i.e. $\bar{K}(\bar{\alpha}_s, \mu^2) \rightarrow \bar{K}(\bar{\alpha}_s)$. While the change-of-variable interpretation is physically motivated by this ζ/μ^2 scaling, it should be kept in mind that the underlying Wilsonian mode separation of the effective theory remains a separation in k^+ . From a Wilsonian

perspective, the action of L does not modify the degrees of freedom of the effective theory, but merely implements a scheme transformation in the space of CGC operators that systematically reshuffles overlapping rapidity and collinear logarithms.

Concerning the resolution of the collinear instability due to the NLO double collinear logarithm, the precise form of the composite dipole is not unique and reflects a choice of subtraction scheme. What ultimately matters is the dependence of the composite dipole on the collinear factorization scale μ . We have considered several variants, all sharing the same $\ln(\mu^2)$ -dependence but differing by finite $\mathcal{O}(\alpha_s)$ corrections, and computed their associated NLO high-energy RG equations in the nonlinear regime. By construction, these equations do not exhibit large double collinear logarithms and are therefore suitable for numerical evaluation. A particularly simple choice of composite dipole and RG equation is given by Eqs. (17) and (20). As a main application, we derived a new NLO BK equation (20) together with the associated transformed form factor for DIS cross sections, Eq. (21). A numerical study of Eq. (20) is presented in the companion Letter [70], where the stability of the solution is demonstrated, thereby providing further support for our proposal. Additional numerical investigations concerning the impact of the initial conditions, the choice of the running-coupling scheme, and the explicit form of the L operator are currently in progress and will be reported in a forthcoming long publication. Another medium-term, albeit challenging, objective is to perform a numerical evaluation of a physical observable—such as DIS cross sections—incorporating the modifications to both the evolution equation and the NLO impact factor, and to compare the results with experimental data.

Additional outlooks include deriving the collinear-safe evolution equation at NNLO by translating the recently obtained results into our factorization scheme [80], as well as extending the framework to other observables such as forward hadron production in proton-nucleus collisions and forward dijet production in electron-proton/nucleus scattering. We anticipate that this framework will provide the foundation for controlled higher-order calculations of hadronic observables at small x , particularly in view of the upcoming Electron-Ion Collider (EIC).

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: THE DIS IMPACT FACTOR AT NLO

In this appendix we give the explicit expression for the NLO correction to the longitudinal DIS cross section, which can be found in [26,27,98]

$$\begin{aligned}
 \Delta\sigma_L(x, Q^2) &= \Delta H_{\text{NLO}}(Q^2, \zeta/s) \otimes S_{\text{LL}}(\zeta) = \sum_i \frac{4\alpha_{\text{em}} e_i^2 N_c S_{\perp}}{\pi^2} \int d^2\mathbf{x}_{12} \int_0^1 dz_1 z_1 (1-z_1) \bar{Q}^2 K_0^2(\bar{Q}x_{12}) (1-S_{12}) \\
 &\times \frac{\alpha_s C_F}{2\pi} \left[\ln^2\left(\frac{z_1}{1-z_1}\right) + 5 - \frac{\pi^2}{3} \right] + \sum_i \frac{4\alpha_{\text{em}} e_i^2 N_c S_{\perp}}{\pi^2} \int d^2\mathbf{x}_{12} \int_0^1 dz_1 \frac{\alpha_s C_F}{\pi^2} \int_0^{1-z_1} \frac{dz_g}{(z_g)_{+,\zeta/s}} \int d^2\mathbf{x}_3 z_1^2 z_2^2 Q^2 \\
 &\times \left\{ \left(1 + \frac{z_g}{z_1} + \frac{z_g^2}{2z_1^2}\right) \frac{1}{\mathbf{x}_{13}^2} \left[K_0^2(QX_R) \left(1 - \frac{N_c}{2C_F} S_{13} S_{32} + \frac{S_{12}}{2N_c C_F}\right) - K_0^2(\bar{Q}x_{12}) e^{-\frac{x_{13}^2}{e^2 E x_{12}^2}} (1-S_{12}) \right] \right. \\
 &+ \left(1 + \frac{z_g}{z_2} + \frac{z_g^2}{2z_2^2}\right) \frac{1}{\mathbf{x}_{32}^2} \left[K_0^2(QX_R) \left(1 - \frac{N_c}{2C_F} S_{13} S_{32} + \frac{S_{12}}{2N_c C_F}\right) - K_0^2(\bar{Q}x_{12}) e^{-\frac{x_{32}^2}{e^2 E x_{12}^2}} (1-S_{12}) \right] \\
 &\left. - 2 \left(1 + \frac{z_g}{2z_1} + \frac{z_g}{2z_2}\right) \frac{\mathbf{x}_{13} \cdot \mathbf{x}_{32}}{\mathbf{x}_{13}^2 \mathbf{x}_{32}^2} K_0^2(QX_R) \left(1 - \frac{N_c}{2C_F} S_{13} S_{32} + \frac{S_{12}}{2N_c C_F}\right) \right\}, \tag{A1}
 \end{aligned}$$

where we use the notation

$$\int_0^a \frac{dz_g}{(z_g)_{+,z}} f(z_g) \equiv \int_0^a \frac{dz_g}{z_g} [f(z_g) - f(0)\Theta(z - z_g)] \quad (\text{A2})$$

for any function f of the NLO gluon longitudinal momentum fraction $z_g = k^+/q^+$ which is regular as $z_g \rightarrow 0$. In this expression, we use the shorthand notations $z_2 = 1 - z_1 - z_g$ and X_R for the effective size of the $q\bar{q}g$ dipole, which reads as $X_R^2 = z_1 z_2 \mathbf{x}_{12}^2 + z_1 z_g \mathbf{x}_{13}^2 + z_2 z_g \mathbf{x}_{32}^2$. We also note $\bar{Q}_R^2 = z_1 z_2 Q^2$ [which is not identical to $\bar{Q}^2 = z_1(1 - z_1)Q^2$ except when $z_g = 0$ since $z_2 + z_g = 1 - z_1$]. It is easy to check that Eq. (A1) is free of any UV or IR divergence such that it is suitable for direct numerical evaluation. In particular, there is no divergence as $z_g \rightarrow 0$ thanks to the $1/(z_g)_+$ prescription.

Clearly, $\sigma_L(x, Q^2)$ given by Eq. (6) must not depend on ζ . Taking a partial derivative of the right-hand side and using the general result

$$\frac{\partial}{\partial \ln(\zeta)} \left[\int_0^a \frac{dz_g}{(z_g)_{+, \zeta/s}} f(z_g) \right] = -f(0)\Theta(a - \zeta/s), \quad (\text{A3})$$

yields the LL renormalization group equation of $S(\mathbf{x}_{12}, \zeta)$ given by Eq. (9).

APPENDIX B: x -DEPENDENT GLUON DISTRIBUTION AT ONE LOOP

Let us consider the x -dependent gluon distribution in coordinate space:

$$S(x, \mathbf{x}_1, \mathbf{x}_2) \equiv \frac{1}{N_c} \int d\mathbf{z}_1^+ \int_{-\infty}^{z_1^+} d\mathbf{z}_2^+ e^{ixP^- z_{12}^+} \frac{\partial^2}{\partial z_1^+ \partial z_2^+} \text{tr}[z_1^+, z_2^+]_{\mathbf{x}_1} [z_2^+, z_1^+]_{\mathbf{x}_2}. \quad (\text{B1})$$

We will use the semiclassical background field methods to derive its evolution equation. Let us thus decompose the gluon field into its classical part A_{cl} and its quantum part a :

$$A^\mu = A_{\text{cl}}^\mu + a^\mu, \quad (\text{B2})$$

and expand the lines in powers of the quantum field a :

$$\begin{aligned} [z_1^+, z_2^+]_{\mathbf{x}_1} &= [z_1^+, z_2^+]_{\mathbf{x}_1}^{\text{cl}} + ig \int_{z_2^+}^{z_1^+} d\mathbf{z}_3^+ [z_1^+, z_3^+]_{\mathbf{x}_1}^{\text{cl}} a^-(z_3^+, \mathbf{x}_1) [z_3^+, z_2^+]_{\mathbf{x}_1}^{\text{cl}} \\ &\quad - g^2 \int_{z_2^+}^{z_1^+} d\mathbf{z}_3^+ \int_{z_2^+}^{z_3^+} d\mathbf{z}_4^+ [z_1^+, z_3^+]_{\mathbf{x}_1}^{\text{cl}} a^-(z_3^+, \mathbf{x}_1) [z_3^+, z_4^+]_{\mathbf{x}_1}^{\text{cl}} a^-(z_4^+, \mathbf{x}_1) [z_4^+, z_2^+]_{\mathbf{x}_1}^{\text{cl}}, \end{aligned} \quad (\text{B3})$$

and

$$\begin{aligned} [z_2^+, z_1^+]_{\mathbf{x}_2} &= [z_2^+, z_1^+]_{\mathbf{x}_2}^{\text{cl}} - ig \int_{z_2^+}^{z_1^+} d\mathbf{z}_4^+ [z_2^+, z_4^+]_{\mathbf{x}_2}^{\text{cl}} a^-(z_4^+, \mathbf{x}_2) [z_4^+, z_1^+]_{\mathbf{x}_2}^{\text{cl}} \\ &\quad - g^2 \int_{z_2^+}^{z_1^+} d\mathbf{z}_3^+ \int_{z_3^+}^{z_1^+} d\mathbf{z}_4^+ [z_2^+, z_3^+]_{\mathbf{x}_2}^{\text{cl}} a^-(z_3^+, \mathbf{x}_2) [z_3^+, z_4^+]_{\mathbf{x}_2}^{\text{cl}} a^-(z_4^+, \mathbf{x}_2) [z_4^+, z_1^+]_{\mathbf{x}_2}^{\text{cl}}. \end{aligned} \quad (\text{B4})$$

In the rhs of these two relations, the Wilson lines with the superscript “cl” are built solely from classical fields. From now on, we will omit this superscript and keep in mind that all contributions from quantum gluons are explicitly written. The quadratic term in a involved in our dipole operator reads as

$$\begin{aligned}
\frac{1}{g^2} \text{tr}[z_1^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2} &= \int_{z_2^+}^{z_1^+} dz_3^+ \int_{z_2^+}^{z_1^+} dz_4^+ \text{tr}[z_1^+, z_3^+]_{x_1} a^-(z_3^+, \mathbf{x}_1) [z_3^+, z_2^+]_{x_1} [z_2^+, z_4^+]_{x_2} a^-(z_4^+, \mathbf{x}_2) [z_4^+, z_1^+]_{x_2} \\
&\quad - \int_{z_2^+}^{z_1^+} dz_3^+ \int_{z_3^+}^{z_1^+} dz_4^+ \text{tr}[z_1^+, z_2^+]_{x_1} [z_2^+, z_3^+]_{x_2} a^-(z_3^+, \mathbf{x}_2) [z_3^+, z_4^+]_{x_2} a^-(z_4^+, \mathbf{x}_2) [z_4^+, z_1^+]_{x_2} \\
&\quad - \int_{z_2^+}^{z_1^+} dz_3^+ \int_{z_2^+}^{z_3^+} dz_4^+ \text{tr}[z_1^+, z_3^+]_{x_1} a^-(z_3^+, \mathbf{x}_1) [z_3^+, z_4^+]_{x_1} a^-(z_4^+, \mathbf{x}_1) [z_4^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2}. \quad (\text{B5})
\end{aligned}$$

This way, the difference $\Delta S(x, \mathbf{x}_1, \mathbf{x}_2)$ between the classical part of $S(x, \mathbf{x}_1, \mathbf{x}_2)$ and its one-loop correction with two quantum gluons reads as

$$\begin{aligned}
\Delta S(x, \mathbf{x}_1, \mathbf{x}_2) &= \frac{g^2}{N_c} \int dz_1^+ \int_{-\infty}^{z_1^+} dz_2^+ e^{ixP^- z_{12}^+} \frac{\partial^2}{\partial z_1^+ \partial z_2^+} \\
&\quad \times \left\{ \int_{z_2^+}^{z_1^+} dz_3^+ \int_{z_2^+}^{z_1^+} dz_4^+ a_a^-(z_3^+, \mathbf{x}_1) a_b^-(z_4^+, \mathbf{x}_2) \text{tr}([z_1^+, z_3^+]_{x_1} \mathbf{t}^a [z_3^+, z_2^+]_{x_1} [z_2^+, z_4^+]_{x_2} \mathbf{t}^b [z_4^+, z_1^+]_{x_2}) \right. \\
&\quad - \int_{z_2^+}^{z_1^+} dz_3^+ \int_{z_3^+}^{z_1^+} dz_4^+ a_a^-(z_3^+, \mathbf{x}_2) a_b^-(z_4^+, \mathbf{x}_2) \text{tr}([z_1^+, z_2^+]_{x_1} [z_2^+, z_3^+]_{x_2} \mathbf{t}^a [z_3^+, z_4^+]_{x_2} \mathbf{t}^b [z_4^+, z_1^+]_{x_2}) \\
&\quad \left. - \int_{z_2^+}^{z_1^+} dz_3^+ \int_{z_2^+}^{z_3^+} dz_4^+ a_a^-(z_3^+, \mathbf{x}_1) a_b^-(z_4^+, \mathbf{x}_1) \text{tr}([z_1^+, z_3^+]_{x_1} \mathbf{t}^a [z_3^+, z_4^+]_{x_1} \mathbf{t}^b [z_4^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2}) \right\}. \quad (\text{B6})
\end{aligned}$$

The propagator for the quantum gluons in the external classical field reads as

$$\langle a_a^-(z_3^+, \mathbf{x}) a_b^-(z_4^+, \mathbf{y}) \rangle = \frac{1}{4\pi} \int \frac{d\mathbf{k}^+}{(k^+)^3} \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} e^{i(k_1 \cdot \mathbf{x}) - i(k_2 \cdot \mathbf{y})} (\mathbf{k}_1 | 2ik^+ \delta(z_{34}^+) \delta^{ab} \mathbf{I} + (\mathbf{k}_1 \cdot \mathbf{k}_2) \mathcal{G}_{k^+}^{ab}(z_3^+, z_4^+) | \mathbf{k}_2), \quad (\text{B7})$$

where the reduced scalar propagator $\mathcal{G}_{k^+}$ satisfies the integrated Schrodinger equations

$$\mathcal{G}_{k^+}^{ab}(z_3^+, z_4^+) = \mathcal{G}_{k^+}^{(0)}(z_3^+, z_4^+) \mathbf{1} \delta^{ab} + ig \int_{z_4^+}^{z_3^+} dz_5^+ \mathcal{G}_{k^+}^{(0)}(z_3^+, z_5^+) [\hat{A}^-(z_5^+) \mathcal{G}_{k^+}(z_5^+, z_4^+)]^{ab}, \quad (\text{B8})$$

and

$$\mathcal{G}_{k^+}^{ab}(z_3^+, z_4^+) = \mathcal{G}_{k^+}^{(0)}(z_3^+, z_4^+) \mathbf{1} \delta^{ab} + ig \int_{z_4^+}^{z_3^+} dz_5^+ [\mathcal{G}_{k^+}(z_3^+, z_5^+) \hat{A}^-(z_5^+)]^{ab} \mathcal{G}_{k^+}^{(0)}(z_5^+, z_4^+). \quad (\text{B9})$$

Here and throughout, we will be using Schwinger bracket notations; see e.g. Ref. [99] for details. Plugging Eq. (B7) into Eq. (B6) yields

$$\begin{aligned}
\Delta S(x, \mathbf{x}_1, \mathbf{x}_2) &= \frac{\alpha_s}{N_c} \int dz_1^+ \int_{-\infty}^{z_1^+} dz_2^+ e^{ixP^- z_{12}^+} \frac{\partial^2}{\partial z_1^+ \partial z_2^+} \int \frac{d\mathbf{k}^+}{(k^+)^3} \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \\
&\quad \times \left\{ \int_{z_2^+}^{z_1^+} dz_3^+ \int_{z_2^+}^{z_1^+} dz_4^+ e^{i(k_1 \cdot \mathbf{x}_1) - i(k_2 \cdot \mathbf{x}_2)} \text{tr}[z_1^+, z_3^+]_{x_1} \mathbf{t}^a [z_3^+, z_2^+]_{x_1} [z_2^+, z_4^+]_{x_2} \mathbf{t}^b [z_4^+, z_1^+]_{x_2} \right. \\
&\quad - \int_{z_2^+}^{z_1^+} dz_3^+ \int_{z_3^+}^{z_1^+} dz_4^+ e^{i(k_1 \cdot \mathbf{x}_2) - i(k_2 \cdot \mathbf{x}_2)} \text{tr}[z_1^+, z_2^+]_{x_1} [z_2^+, z_3^+]_{x_2} \mathbf{t}^a [z_3^+, z_4^+]_{x_2} \mathbf{t}^b [z_4^+, z_1^+]_{x_2} \\
&\quad - \int_{z_2^+}^{z_1^+} dz_3^+ \int_{z_2^+}^{z_3^+} dz_4^+ e^{i(k_1 \cdot \mathbf{x}_1) - i(k_2 \cdot \mathbf{x}_1)} \text{tr}[z_1^+, z_3^+]_{x_1} \mathbf{t}^a [z_3^+, z_4^+]_{x_1} \mathbf{t}^b [z_4^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2} \left. \right\} \\
&\quad \times (\mathbf{k}_1 | 2ik^+ \delta(z_{34}^+) \delta^{ab} \mathbf{I} + (\mathbf{k}_1 \cdot \mathbf{k}_2) \mathcal{G}_{k^+}^{ab}(z_3^+, z_4^+) | \mathbf{k}_2). \quad (\text{B10})
\end{aligned}$$

Note that so-called instantaneous terms with $\delta(z_{34}^+)$ involve nonlogarithmic tadpole integrals, which are set to 0 in dimensional regularization. We will now detail the computation of the quark self energy diagram as an example, the others

being similar. First, one needs to rewrite the involved operators thanks to successive uses of the following relations which are direct consequences of the integrated Schrodinger equations (B8) and (B9): for any $x^+, y^+, z^+, \mathbf{x}, \mathbf{y}$,

$$[x^+, y^+]_y(\dots)[y^+, x^+]_x = (\dots) + \int_{y^+}^{x^+} dz^+ \partial_{z^+} \hat{\mathcal{I}}_{z^+} [z^+, y^+]_y(\dots)[y^+, z^+]_x, \quad (\text{B11})$$

$$[x^+, y^+]_y(\dots)[y^+, x^+]_x = (\dots) - \int_{y^+}^{x^+} dz^+ \hat{\mathcal{I}}_{z^+} [x^+, z^+]_y(\dots)[z^+, x^+]_x, \quad (\text{B12})$$

and

$$\mathcal{G}_{k^+}^{ab}(x^+, y^+)[x^+, y^+]_y \mathbf{t}^b [y^+, x^+]_x = \mathbf{t}^a \mathcal{G}_{k^+}^{(0)}(x^+, y^+) + \int_{y^+}^{x^+} dz^+ \hat{\mathcal{I}}_{z^+} \mathcal{G}_{k^+}^{ab}(x^+, z^+)[x^+, z^+]_y \mathbf{t}^b [z^+, x^+]_x \mathcal{G}_{k^+}^{(0)}(z^+, y^+), \quad (\text{B13})$$

$$[y^+, x^+]_y \mathbf{t}^a [x^+, y^+]_x \mathcal{G}_{k^+}^{ab}(x^+, y^+) = \mathbf{t}^b \mathcal{G}_{k^+}^{(0)}(x^+, y^+) - \int_{x^+}^{y^+} dz^+ \mathcal{G}_{k^+}^{(0)}(x^+, z^+) \hat{\mathcal{I}}_{z^+} \mathcal{G}_{k^+}^{ab}(z^+, y^+)[y^+, z^+]_y \mathbf{t}^a [z^+, y^+]_x, \quad (\text{B14})$$

where we introduced a gluon insertion operator $\hat{\mathcal{I}}_{z^+}$ which for all intents and purposes here, up to subeikonal corrections, will act as ∂_{z^+} .

Up to subeikonal corrections, and subtracting the disconnected contributions, the operator involved in the quark self-energy diagrams can be rewritten as

$$\begin{aligned} & \mathcal{G}_{k^+}^{ab}(z_3^+, z_4^+) \text{tr}[z_1^+, z_3^+]_{x_1} \mathbf{t}^a [z_3^+, z_4^+]_{x_1} \mathbf{t}^b [z_4^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2} \\ & \simeq -C_F \left(\int_{z_2^+}^{z_4^+} dz_5^+ \int_{z_2^+}^{z_5^+} dz_6^+ + \int_{z_3^+}^{z_1^+} dz_5^+ \int_{z_3^+}^{z_5^+} dz_6^+ \right) \mathcal{G}_{k^+}^{(0)}(z_3^+, z_4^+) \text{tr}\{\partial_{z_5^+} \partial_{z_6^+} [z_5^+, z_6^+]_{x_1} [z_6^+, z_5^+]_{x_2}\} \\ & + \int_{z_4^+}^{z_3^+} dz_5^+ \int_{z_4^+}^{z_5^+} dz_6^+ \mathcal{G}_{k^+}^{(0)}(z_3^+, z_5^+) \text{tr}\{\partial_{z_5^+} \partial_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_5^+, z_6^+) \mathbf{t}^a [z_5^+, z_6^+]_{x_1} \mathbf{t}^b [z_6^+, z_5^+]_{x_2}\} \mathcal{G}_{k^+}^{(0)}(z_6^+, z_4^+). \end{aligned} \quad (\text{B15})$$

It is possible to show that the first two terms, which correspond to the virtual corrections in the standard derivations of the BK evolution equation, are subleading in twist, so we can take their fully eikonal limit to reproduce their contribution to BK. This is known, so we will only detail the other term here, denoted with the RBK superscript. We can play the same game with all diagrams, to get

$$\begin{aligned} \Delta S^{\text{RBK}}(x, \mathbf{x}_1, \mathbf{x}_2) &= \frac{\alpha_s}{N_c} \int dz_1^+ \int_{-\infty}^{z_1^+} dz_2^+ e^{iXP^- z_{12}^+} \frac{\partial^2}{\partial z_1^+ \partial z_2^+} \int \frac{dk^+}{k^+} \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{(k^+)^2} \\ & \times \int dz_3^+ dz_4^+ dz_5^+ dz_6^+ \{ \theta(z_{13}^+) \theta(z_{35}^+) \theta(z_{56}^+) \theta(z_{64}^+) \theta(z_{42}^+) e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1) - i(\mathbf{k}_2 \cdot \mathbf{x}_2)} \\ & \times (\mathbf{k}_1 | \mathcal{G}_{k^+}^{(0)}(z_3^+, z_5^+) \text{tr}\{\hat{\mathcal{I}}_{z_5^+} \hat{\mathcal{I}}_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_5^+, z_6^+) [z_5^+, z_6^+]_{x_1} \mathbf{t}^b [z_6^+, z_5^+]_{x_2} \mathbf{t}^a\} \mathcal{G}_{k^+}^{(0)}(z_6^+, z_4^+) | \mathbf{k}_2) \\ & + \theta(z_{14}^+) \theta(z_{45}^+) \theta(z_{56}^+) \theta(z_{63}^+) \theta(z_{32}^+) e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1) - i(\mathbf{k}_2 \cdot \mathbf{x}_2)} \\ & \times (\mathbf{k}_1 | \mathcal{G}_{k^+}^{(0)}(z_3^+, z_6^+) \text{tr}\{\hat{\mathcal{I}}_{z_5^+} \hat{\mathcal{I}}_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_6^+, z_5^+) [z_5^+, z_6^+]_{x_1} \mathbf{t}^a [z_6^+, z_5^+]_{x_2} \mathbf{t}^b\} \mathcal{G}_{k^+}^{(0)}(z_5^+, z_4^+) | \mathbf{k}_2) \\ & - \theta(z_{14}^+) \theta(z_{45}^+) \theta(z_{56}^+) \theta(z_{63}^+) \theta(z_{32}^+) e^{i(\mathbf{k}_1 \cdot \mathbf{x}_2) - i(\mathbf{k}_2 \cdot \mathbf{x}_2)} \\ & \times (\mathbf{k}_1 | \mathcal{G}_{k^+}^{(0)}(z_3^+, z_6^+) \text{tr}\{\hat{\mathcal{I}}_{z_5^+} \hat{\mathcal{I}}_{z_6^+} \mathbf{t}^b \mathcal{G}_{k^+}^{ab}(z_6^+, z_5^+) [z_5^+, z_6^+]_{x_1} \mathbf{t}^a [z_6^+, z_5^+]_{x_2}\} \mathcal{G}_{k^+}^{(0)}(z_5^+, z_4^+) | \mathbf{k}_2) \\ & - \theta(z_{13}^+) \theta(z_{35}^+) \theta(z_{56}^+) \theta(z_{64}^+) \theta(z_{42}^+) e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1) - i(\mathbf{k}_2 \cdot \mathbf{x}_1)} \\ & \times (\mathbf{k}_1 | \mathcal{G}_{k^+}^{(0)}(z_3^+, z_5^+) \text{tr}\{\hat{\mathcal{I}}_{z_5^+} \hat{\mathcal{I}}_{z_6^+} \mathbf{t}^a [z_5^+, z_6^+]_{x_1} \mathbf{t}^b [z_6^+, z_5^+]_{x_2} \mathcal{G}_{k^+}^{ab}(z_5^+, z_6^+) \mathcal{G}_{k^+}^{(0)}(z_6^+, z_4^+) | \mathbf{k}_2) \}. \end{aligned} \quad (\text{B16})$$

The light cone time derivatives are trivial. Taking them and combining terms together yields

$$\begin{aligned}
\Delta S^{\text{RBK}}(x, \mathbf{x}_1, \mathbf{x}_2) &= \frac{\alpha_s}{N_c} \int dz_1^+ dz_2^+ e^{ixP^- z_{12}^+} \int \frac{dk^+}{k^+} \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{(k^+)^2} \\
&\times \int dz_5^+ dz_6^+ \theta(z_{15}^+) \theta(z_{56}^+) \theta(z_{62}^+) \{ e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1)} (e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_1)} - e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_2)}) \\
&\times (\mathbf{k}_1 | \mathcal{G}_{k^+}^{(0)}(z_1^+, z_5^+) \text{tr} \{ \hat{\mathcal{T}}_{z_5^+} \hat{\mathcal{T}}_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_5^+, z_6^+) [z_5^+, z_6^+]_{x_1} \mathbf{t}^b [z_6^+, z_5^+]_{x_2} \mathbf{t}^a \} \mathcal{G}_{k^+}^{(0)}(z_6^+, z_2^+) | \mathbf{k}_2) \\
&- e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_2)} (e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1)} - e^{i(\mathbf{k}_1 \cdot \mathbf{x}_2)}) \\
&\times (\mathbf{k}_1 | \mathcal{G}_{k^+}^{(0)}(z_2^+, z_6^+) \text{tr} \{ \hat{\mathcal{T}}_{z_5^+} \hat{\mathcal{T}}_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_6^+, z_5^+) [z_5^+, z_6^+]_{x_1} \mathbf{t}^a [z_6^+, z_5^+]_{x_2} \mathbf{t}^b \} \mathcal{G}_{k^+}^{(0)}(z_5^+, z_1^+) | \mathbf{k}_2) \}. \quad (\text{B17})
\end{aligned}$$

Going back to coordinate space with

$$\begin{aligned}
|\mathbf{k}_1| &= \int d^d \mathbf{z}_1 e^{-i(\mathbf{k}_1 \cdot \mathbf{z}_1)} |\mathbf{z}_1| \\
|\mathbf{k}_2| &= \int d^d \mathbf{z}_2 e^{i(\mathbf{k}_2 \cdot \mathbf{z}_2)} |\mathbf{z}_2| \\
1 &= \int d^d \mathbf{x}_3 d^d \mathbf{x}_4 |\mathbf{x}_3| (\mathbf{x}_3 | \mathbf{x}_4) (\mathbf{x}_4 |, \quad (\text{B18})
\end{aligned}$$

we obtain

$$\begin{aligned}
\Delta S^{\text{RBK}}(x, \mathbf{x}_1, \mathbf{x}_2) &= \frac{\alpha_s}{N_c} \int dz_1^+ dz_2^+ e^{ixP^- z_{12}^+} \int \frac{dk^+}{k^+} \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{(k^+)^2} \int d^d \mathbf{z}_1 d^d \mathbf{z}_2 d^d \mathbf{z}_3 d^d \mathbf{z}_4 \\
&\times \int dz_5^+ dz_6^+ \theta(z_{15}^+) \theta(z_{56}^+) \theta(z_{62}^+) e^{-i(\mathbf{k}_1 \cdot \mathbf{z}_1) + i(\mathbf{k}_2 \cdot \mathbf{z}_2)} \\
&\times \{ e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1)} (e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_1)} - e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_2)}) (\mathbf{z}_1 | \mathcal{G}_{k^+}^{(0)}(z_1^+, z_5^+) | \mathbf{x}_3) (\mathbf{x}_4 | \mathcal{G}_{k^+}^{(0)}(z_6^+, z_2^+) | \mathbf{z}_2) \\
&\times (\mathbf{x}_3 | \text{tr} \{ \hat{\mathcal{T}}_{z_5^+} \hat{\mathcal{T}}_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_5^+, z_6^+) [z_5^+, z_6^+]_{x_1} \mathbf{t}^b [z_6^+, z_5^+]_{x_2} \mathbf{t}^a \} | \mathbf{x}_4) \\
&- e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_2)} (e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1)} - e^{i(\mathbf{k}_1 \cdot \mathbf{x}_2)}) (\mathbf{z}_1 | \mathcal{G}_{k^+}^{(0)}(z_2^+, z_6^+) | \mathbf{x}_3) (\mathbf{x}_4 | \mathcal{G}_{k^+}^{(0)}(z_5^+, z_1^+) | \mathbf{z}_2) \\
&\times (\mathbf{x}_3 | \text{tr} \{ \hat{\mathcal{T}}_{z_5^+} \hat{\mathcal{T}}_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_6^+, z_5^+) [z_5^+, z_6^+]_{x_1} \mathbf{t}^a [z_6^+, z_5^+]_{x_2} \mathbf{t}^b \} | \mathbf{x}_4) \}. \quad (\text{B19})
\end{aligned}$$

Using the explicit expression for the free propagators,

$$(\mathbf{z}_i | \mathcal{G}_{k^+}^{(0)}(z_m^+, z_n^+) | \mathbf{z}_j) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i(\mathbf{k} \cdot \mathbf{z}_{ij})} e^{-i\frac{k^2 - i0}{2k^+} z_{mn}^+} [\theta(k^+) \theta(z_{mn}^+) - \theta(-k^+) \theta(z_{nm}^+)], \quad (\text{B20})$$

we can easily integrate with respect to z_1^+, z_2^+, z_1, z_2 and transverse momenta, which yields

$$\begin{aligned}
\Delta S^{\text{RBK}}(x, \mathbf{x}_1, \mathbf{x}_2) &= -\frac{4\alpha_s}{N_c} \int \frac{dk^+}{k^+} \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \int d^d \mathbf{x}_3 d^d \mathbf{x}_4 \int dz_5^+ \int_{-\infty}^{z_5^+} dz_6^+ e^{ixP^- z_{56}^+} \\
&\times \left\{ \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2) e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{13})} (e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{14})} - e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{24})})}{(k_1^2 - 2xP^- k^+ - i0)(k_2^2 - 2xP^- k^+ - i0)} \theta(k^+) \right. \\
&\times (\mathbf{x}_3 | \text{tr} \{ \hat{\mathcal{T}}_{z_5^+} \hat{\mathcal{T}}_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_5^+, z_6^+) [z_5^+, z_6^+]_{x_1} \mathbf{t}^b [z_6^+, z_5^+]_{x_2} \mathbf{t}^a \} | \mathbf{x}_4) \\
&- \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2) e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{24})} (e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{13})} - e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{23})})}{(k_1^2 + 2xP^- k^+ - i0)(k_2^2 + 2xP^- k^+ - i0)} \theta(-k^+) \\
&\left. \times (\mathbf{x}_3 | \text{tr} \{ \hat{\mathcal{T}}_{z_5^+} \hat{\mathcal{T}}_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_6^+, z_5^+) [z_5^+, z_6^+]_{x_1} \mathbf{t}^a [z_6^+, z_5^+]_{x_2} \mathbf{t}^b \} | \mathbf{x}_4) \right\}. \quad (\text{B21})
\end{aligned}$$

In the second term in the brackets, let us do $(k^+, \mathbf{k}_1, \mathbf{k}_2, \mathbf{x}_3, \mathbf{x}_4) \rightarrow (-k^+, -\mathbf{k}_2, -\mathbf{k}_1, \mathbf{x}_4, \mathbf{x}_3)$ after noting that

$$(\mathbf{x}_3|\mathcal{G}_{k^+}^{ab}(z_6^+, z_5^+)|\mathbf{x}_4) = -(\mathbf{x}_4|\mathcal{G}_{-k^+}^{ba}(z_5^+, z_6^+)|\mathbf{x}_3). \quad (\text{B22})$$

This allows one to recombine all terms with the same operator, so that

$$\begin{aligned} \Delta S^{\text{RBK}}(x, \mathbf{x}_1, \mathbf{x}_2) &= -\frac{4\alpha_s}{N_c} \int \frac{dk^+}{k^+} \theta(k^+) \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \int d^d \mathbf{x}_3 d^d \mathbf{x}_4 \int dz_5^+ \int_{-\infty}^{z_5^+} dz_6^+ e^{ixP^- z_{56}^+} \\ &\times \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)(e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{13})} - e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{23})})(e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{14})} - e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{24})})}{(\mathbf{k}_1^2 - 2xP^- k^+ - i0)(\mathbf{k}_2^2 - 2xP^- k^+ - i0)} \\ &\times (\mathbf{x}_3|\text{tr}\{\hat{\mathcal{T}}_{z_5^+} \hat{\mathcal{T}}_{z_6^+} \mathcal{G}_{k^+}^{ab}(z_5^+, z_6^+) \mathbf{t}^a[z_5^+, z_6^+]_{x_1} \mathbf{t}^b[z_6^+, z_5^+]_{x_2}\}|\mathbf{x}_4). \end{aligned} \quad (\text{B23})$$

The final step here consists of taking the so-called classical approximation:

$$\begin{aligned} (\mathbf{x}_3|\mathcal{G}_{k^+}^{ab}(z_5^+, z_6^+)|\mathbf{x}_4) &\simeq (\mathbf{x}_3|\mathcal{G}_{k^+}^{(0)}(z_5^+, z_6^+)|\mathbf{x}_4)[z_5^+, z_6^+]_{\bar{s}\mathbf{x}_3 + s\mathbf{x}_4}^{ab} \\ &= \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i(\mathbf{k} \cdot \mathbf{x}_{34})} e^{-i\frac{\mathbf{k}^2 - i0}{2k^+} z_{56}^+} [z_5^+, z_6^+]_{\bar{s}\mathbf{x}_3 + s\mathbf{x}_4}^{ab}, \end{aligned} \quad (\text{B24})$$

with $s \in [0, 1]$ an arbitrary parameter. This allows one to integrate with respect to $\mathbf{x}_{34}$ and to keep only the dependence on $\mathbf{x} \equiv \bar{s}\mathbf{x}_3 + s\mathbf{x}_4$:

$$\begin{aligned} \Delta S^{\text{RBK}}(x, \mathbf{x}_1, \mathbf{x}_2) &= -\frac{4\alpha_s}{N_c} \int \frac{dk^+}{k^+} \theta(k^+) \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \int d^d \mathbf{x} \int dz_5^+ \int_{-\infty}^{z_5^+} dz_6^+ e^{ixP^- z_{56}^+} \\ &\times \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)(e^{i\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x})} - e^{i\mathbf{k}_1 \cdot (\mathbf{x}_2 - \mathbf{x})})(e^{-i\mathbf{k}_2 \cdot (\mathbf{x}_1 - \mathbf{x})} - e^{-i\mathbf{k}_2 \cdot (\mathbf{x}_2 - \mathbf{x})})}{(\mathbf{k}_1^2 - 2xP^- k^+ - i0)(\mathbf{k}_2^2 - 2xP^- k^+ - i0)} \\ &\times e^{-i\frac{(s\mathbf{k}_1 + \bar{s}\mathbf{k}_2)^2 - i0}{2k^+} z_{56}^+} \text{tr}\{\hat{\mathcal{T}}_{z_5^+} \hat{\mathcal{T}}_{z_6^+} \mathbf{t}^a[z_5^+, z_6^+]_{x_1} \mathbf{t}^b[z_6^+, z_5^+]_{x_2} [z_5^+, z_6^+]_{\mathbf{x}}^{ab}\}. \end{aligned} \quad (\text{B25})$$

Renaming dummy variables, approximating $\hat{\mathcal{T}}$ by ∂ , we finally obtain

$$\begin{aligned} \Delta S^{\text{RBK}}(x, \mathbf{x}_1, \mathbf{x}_2) &= -\frac{4\alpha_s}{N_c} \int \frac{dk^+}{k^+} \theta(k^+) \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \int d^d \mathbf{x}_3 \\ &\times \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)(e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{13})} - e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{23})})(e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{13})} - e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{23})})}{(\mathbf{k}_1^2 - 2xP^- k^+ - i0)(\mathbf{k}_2^2 - 2xP^- k^+ - i0)} \\ &\times \int dz_1^+ \int_{-\infty}^{z_1^+} dz_2^+ e^{i(xP^- - \frac{(s\mathbf{k}_1 + \bar{s}\mathbf{k}_2)^2 - i0}{2k^+}) z_{12}^+} \text{tr}\{\partial_{z_1^+} \partial_{z_2^+} \mathbf{t}^a[z_1^+, z_2^+]_{x_1} \mathbf{t}^b[z_2^+, z_1^+]_{x_2} [z_1^+, z_2^+]_{\mathbf{x}_3}^{ab}\}. \end{aligned} \quad (\text{B26})$$

The virtual terms at $x = 0$ are known, and read as

$$\begin{aligned} \Delta S^{\text{V}}(x, \mathbf{x}_1, \mathbf{x}_2) &= -\frac{4\alpha_s}{N_c} \int \frac{dk^+}{k^+} \theta(k^+) \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \int d^d \mathbf{x}_3 \\ &\times \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)(e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{13})} - e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{23})})(e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{13})} - e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{23})})}{k_1^2 k_2^2} \\ &\times \int dz_1^+ \int_{-\infty}^{z_1^+} dz_2^+ (-2C_F) \text{tr}\{\partial_{z_1^+} \partial_{z_2^+} [z_1^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2}\}. \end{aligned} \quad (\text{B27})$$

Note that they cancel in the leading twist limit. It is thus possible to modify it by subeikonal corrections with no repercussion on the collinear part of this evolution equation, so we can add the Gaussian phase from Eq. (B26) and the x corrections to the denominator:

$$\begin{aligned}
\Delta S^V(x, \mathbf{x}_1, \mathbf{x}_2) &\simeq -\frac{4\alpha_s}{N_c} \int \frac{dk^+}{k^+} \theta(k^+) \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \int d^d \mathbf{x}_3 \\
&\times \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)(e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{13})} - e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{23})})(e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{13})} - e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{23})})}{(k_1^2 - 2xP^-k^+ - i0)(k_2^2 - 2xP^-k^+ - i0)} \\
&\times \int dz_1^+ \int_{-\infty}^{z_1^+} dz_2^+ e^{i(xP^- - \frac{(s\mathbf{k}_1 + \mathbf{k}_2)^2 - i0}{2k^+})z_{12}^+} C_F \text{tr}\{\partial_{z_1^+} \partial_{z_2^+} [z_1^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2}\}. \tag{B28}
\end{aligned}$$

Combining real and virtual terms, we can conclude that

$$\begin{aligned}
\Delta S(x, \mathbf{x}_1, \mathbf{x}_2) &= -\frac{4\alpha_s}{N_c} \int \frac{dk^+}{k^+} \theta(k^+) \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \int d^d \mathbf{x}_3 \\
&\times \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)(e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{13})} - e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{23})})(e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{13})} - e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{23})})}{(k_1^2 - 2xP^-k^+ - i0)(k_2^2 - 2xP^-k^+ - i0)} \\
&\times \int dz_1^+ \int_{-\infty}^{z_1^+} dz_2^+ e^{i(xP^- - \frac{(s\mathbf{k}_1 + \mathbf{k}_2)^2 - i0}{2k^+})z_{12}^+} \\
&\times [\text{tr}\{\partial_{z_1^+} \partial_{z_2^+} \mathbf{t}^a [z_1^+, z_2^+]_{x_1} \mathbf{t}^b [z_2^+, z_1^+]_{x_2} [z_1^+, z_2^+]_{x_3}^{ab}\} - C_F \text{tr}\{\partial_{z_1^+} \partial_{z_2^+} [z_1^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2}\}], \tag{B29}
\end{aligned}$$

which is our equation Eq.

APPENDIX C: TRANSVERSE INTEGRALS

1. Fourier transform integrals

The integrals appearing in Sec. III are evaluated using the following procedure:

$$I_1 = \int_q \frac{q^i}{q^2} (e^{iq \cdot \mathbf{x}_{13}} - e^{iq \cdot \mathbf{x}_{23}}) = \frac{-i}{(2\pi)^2} \partial_{x_3}^i \int \frac{d^2 q}{q^2} (e^{iq \cdot \mathbf{x}_{13}} - e^{iq \cdot \mathbf{x}_{23}}), \tag{C1}$$

$$= \frac{-i}{2\pi} \partial_{x_3}^i \ln \frac{|\mathbf{x}_{13}|}{|\mathbf{x}_{23}|} = \frac{i}{2\pi} \left(\frac{\mathbf{x}_{13}^i}{\mathbf{x}_{13}^2} - \frac{\mathbf{x}_{23}^i}{\mathbf{x}_{23}^2} \right) \tag{C2}$$

and

$$I_2 = \int_k \frac{\mathbf{k}^i}{k^2} (e^{-ik \cdot \mathbf{x}_{13}} - e^{-ik \cdot \mathbf{x}_{23}}) \ln \frac{k^2}{2\rho_0^+ P^-} = \frac{i}{(2\pi)^2} \partial_{x_3}^i \int \frac{d^2 k}{k^2} (e^{-ik \cdot \mathbf{x}_{13}} - e^{-ik \cdot \mathbf{x}_{23}}) \ln \frac{k^2}{2\rho_0^+ P^-}, \tag{C3}$$

$$= \frac{i}{2\pi} \partial_{x_3}^i \ln \frac{|\mathbf{x}_{13}|}{|\mathbf{x}_{23}|} \left(\ln \frac{|\mathbf{x}_{13}| |\mathbf{x}_{23}|}{2\rho_0^+ P^-} + 2(\gamma_E - \ln(2)) \right), \tag{C4}$$

$$= -\frac{i}{(2\pi)} \left[\left(\ln \frac{|\mathbf{x}_{13}| |\mathbf{x}_{23}|}{2\rho_0^+ P^-} + 2(\gamma_E - \ln(2)) \right) \left(\frac{\mathbf{x}_{13}^i}{\mathbf{x}_{13}^2} - \frac{\mathbf{x}_{23}^i}{\mathbf{x}_{23}^2} \right) + \left(\frac{\mathbf{x}_{13}^i}{\mathbf{x}_{13}^2} + \frac{\mathbf{x}_{23}^i}{\mathbf{x}_{23}^2} \right) \ln \frac{|\mathbf{x}_{13}|}{|\mathbf{x}_{23}|} \right]. \tag{C5}$$

2. Proof of Eq. (125)

To demonstrate the identity of Eq. (125), let us first compute in d dimension the second term, which involves three transverse coordinates, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. The trick is to decompose it into a finite term and terms which only depend on two transverse coordinates:

$$\int d^d \mathbf{x}_4 \frac{\mathbf{x}_{32}^2}{\mathbf{x}_{34}^2 \mathbf{x}_{24}^2} \ln \left(\frac{\mathbf{x}_{14}^2}{\mathbf{x}_{13}^2} \right) = \int d^d \mathbf{x}_{43} \left[\frac{2\mathbf{x}_{42} \cdot \mathbf{x}_{32}}{\mathbf{x}_{43}^2 \mathbf{x}_{42}^2} - \frac{1}{\mathbf{x}_{43}^2} + \frac{1}{\mathbf{x}_{42}^2} \right] \ln \left(\frac{(\mathbf{x}_{13} - \mathbf{x}_{43})^2}{\mathbf{x}_{13}^2} \right). \tag{C6}$$

Now, the first term is UV and IR finite in $d = 2$ and reads as

$$\int d^2 \mathbf{x}_{43} \left[\frac{2\mathbf{x}_{24} \cdot \mathbf{x}_{23}}{\mathbf{x}_{43}^2 \mathbf{x}_{42}^2} \right] \ln \left(\frac{(\mathbf{x}_{13} - \mathbf{x}_{43})^2}{\mathbf{x}_{13}^2} \right) = \pi \ln \left(\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{32}^2} \right) \ln \left(\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{12}^2} \right), \quad (\text{C7})$$

and the remaining d dimensional integrals can be computed using the standard bubble integral, Eq. (92):

$$\int d^d \mathbf{x}_4 \left[\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{34}^2} \ln \left(\frac{\mathbf{x}_{14}^2}{\mathbf{x}_{13}^2} \right) + \left(\frac{1}{\mathbf{x}_{42}^2} - \frac{1}{\mathbf{x}_{43}^2} \right) \ln \left(\frac{\mathbf{x}_{14}^2}{\mathbf{x}_{13}^2} \right) + \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{14}^2 \mathbf{x}_{24}^2} \ln \left(\frac{\mathbf{x}_{13}^2}{\mathbf{x}_{14}^2} \right) \right] = -\pi \ln^2 \left(\frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2} \right). \quad (\text{C8})$$

Combining the two previous equations, we find Eq. (125). We do not compute the terms involving δ functions since they do not contribute to the BK equation.

APPENDIX D: DERIVATIVE OF AN OPERATOR EXPONENTIAL

In this appendix, we provide a detailed derivation of Eq. (70). Setting $\bar{\alpha}_s = 1$ to alleviate the notations, we want to compute

$$\frac{d}{d \ln \mu^2} e^{L(\mu)}. \quad (\text{D1})$$

It is convenient to express the exponential as a path-ordered exponential,

$$e^{L(\mu)} \rightarrow \mathcal{P} \exp \left(\int_0^1 du L(u, \mu) \right) = \sum_{n=0}^{\infty} \int_{0 < u_1 < \dots < u_n < 1} du_1 \dots du_n L(u_n, \mu) \dots L(u_1, \mu), \quad (\text{D2})$$

where we keep $L(u, \mu)$ general for the manipulations below and only at the end specialize to the case $L(u, \mu) = L(\mu)$ (no u -dependence).

Differentiating the Dyson series term by term gives

$$\begin{aligned} & \frac{d}{d \ln \mu^2} \mathcal{P} \exp \left(\int_0^1 du L(u, \mu) \right) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^n \int_{0 < u_1 < \dots < u_n < 1} du_1 \dots du_n L(u_n, \mu) \dots L(u_{m+1}, \mu) \frac{dL(u_m, \mu)}{d \ln \mu^2} L(u_{m-1}, \mu) \dots L(u_1, \mu). \end{aligned} \quad (\text{D3})$$

Let $u \equiv u_m$ and set $r \equiv n - m \geq 0$. Using

$$\sum_{n=1}^{\infty} \sum_{m=1}^n \rightarrow \sum_{m=1}^{\infty} \sum_{r=0}^{\infty}, \quad (\text{D4})$$

we split the ordered product at u into a “left factor” $u < v_1 < \dots < v_r < 1$ and a “right factor” $0 < w_1 < \dots < w_{m-1} < u$, and obtain

$$\begin{aligned} \frac{d}{d \ln \mu^2} e^{L(\mu)} &= \int_0^1 du \left[\sum_{r=0}^{\infty} \int_{u < v_1 < \dots < v_r < 1} dv_1 \dots dv_r L(v_r, \mu) \dots L(v_1, \mu) \right] \frac{dL(u, \mu)}{d \ln \mu^2} \\ &\quad \times \left[\sum_{k=0}^{\infty} \int_{0 < w_1 < \dots < w_k < u} dw_1 \dots dw_k L(w_k, \mu) \dots L(w_1, \mu) \right]. \end{aligned} \quad (\text{D5})$$

The two bracketed series are a precisely path-ordered exponential on $[u, 1]$ and $[0, u]$, respectively, hence

$$\frac{d}{d \ln \mu^2} \mathcal{P} \exp \left(\int_0^1 du L(u, \mu) \right) = \int_0^1 du \mathcal{P} \exp \left(\int_u^1 dv L(v, \mu) \right) \frac{dL(u, \mu)}{d \ln \mu^2} \mathcal{P} \exp \left(\int_0^u dw L(w, \mu) \right). \quad (\text{D6})$$

If L is independent of u , i.e. $L(u, \mu) = L(\mu)$, the path ordering is trivial, and Eq. (D6) reduces to the standard Duhamel formula:

$$\frac{d}{d \ln \mu^2} e^{L(\mu)} = \int_0^1 du e^{(1-u)L(\mu)} \frac{dL(\mu)}{d \ln \mu^2} e^{uL(\mu)}. \quad (\text{D7})$$

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