

Analytic approximations for fermionic preheating

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Nonthermal fermions can be produced nonperturbatively in the early Universe during coherent oscillations of a scalar field. We explore fermion production in $\lambda\phi^4$ inflation through this mechanism and analyze the momentum spectrum of the fermions produced, which depends on a coupling parameter q . For $q \gtrsim 0.01$, the main contribution to the total number density comes from an approximately half-filled Fermi sphere as a result of nonadiabaticity. For $q \lesssim 0.01$, we find that the major contributions instead come from resonance peaks at higher momentum values. We find a simple relation to predict the momentum values corresponding to resonance peaks for any q . We observe power-law behavior for the total number density of fermions as a function of q and find numerical approximations for the integrals describing the total number density. We find that it is proportional to $q^{1/2}$ for $q \lesssim 0.01$ and proportional to $q^{3/4}$ for $q \gtrsim 10$. If fermions produced by this mechanism make up the entirety of dark matter, we estimate lower bounds on their mass.

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I. INTRODUCTION

Particles can be produced nonperturbatively during coherent oscillations of a scalar field before it decays. If the scalar field is the inflaton, the mechanism is called preheating [1–4]. We explore a case of the production of fermions through this mechanism, resulting in the production of particles out of thermal equilibrium [5]. The production of particles during the oscillatory phase of the inflaton field was first described by perturbative decays of the inflaton field through a process termed reheating [6]. However, it was found that particle production can occur explosively before the perturbative decay of the inflaton through the process of preheating [1,7].¹

The description of reheating is through individual inflaton quanta decaying into produced particles, whereas preheating is production from an oscillating inflaton field. The mechanism of preheating is described by an oscillator-like differential equation (called the ‘mode equation’) and the number density per mode of the fermions produced is described by a model-independent equation. The method of

calculating it involves techniques, such as Bogoliubov transformations [9], that are well-defined in the context of particle production from time-dependent background fields [10] and are also used in gravitational particle production [11] and the production of particles from an oscillating axion field [12,13].

Particle production through preheating in the early Universe affects the expansion history of the Universe. It leads to different kinetic distributions and number densities of particles in the early Universe than what is expected from perturbative reheating, which may lead to a different value of the reheating temperature (T_{rh}) [7]. Unlike reheating, preheating also allows the production of particles with a mass greater than the mass of the inflaton [14]. Massive particles produced during preheating may play an important role in baryogenesis [15] and leptogenesis [16] scenarios to explain the observed baryon asymmetry of the Universe. Significant gravitational waves may also be generated as a consequence of the preheating mechanism [17]. Preheating can also lead to the production of particles out of thermal equilibrium that may act as a viable candidate for dark matter [18,19]. For a general review of preheating scenarios and their consequences, see [20,21].

In the momentum spectrum of fermions produced through preheating, there is a high occupation of particles per mode at low momentum (which we call the ‘bulk region’) as a consequence of the nonadiabaticity condition for fermion mass oscillations [7]. The condition essentially implies that the energy cost of particle production becomes zero when the effective mass of the fermions arising from the coupling with the inflaton becomes zero. For larger momentum values outside the bulk region, high occupation

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¹For a recent discussion of fermion production after inflation, see [8].

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of particles per mode occurs only at specific discrete momentum values, giving rise to resonance peaks in the momentum spectrum of produced fermions.

In this paper, we analyze particle production assuming a $\lambda\phi^4$ quartic potential for the inflaton [22], following from [5,23,24]. Even though $\lambda\phi^4$ inflation is strongly disfavored due to constraints on slow-roll parameters [25], we focus on describing only the oscillations of the inflaton field about its minimum by a $\lambda\phi^4$ potential. We analyze fermion production during the oscillatory phase of the inflaton, which is decoupled from the dynamics and constraints of the slow-roll phase. We do so under the assumption that the bare mass of the fermions m_ψ is small compared to the scale of inflaton oscillations. As done before in [5,23,24], the number density and the momentum spectrum can be obtained through numerical solutions of the mode equation and are dependent on a coupling parameter $q = h^2/\lambda$, which represents the strength of the coupling between the inflaton and the fermions. Here h is the Yukawa coupling between the inflaton and fermions, and λ is the quartic coupling for the inflaton. Fermion production through preheating can also occur with other inflation potentials, such as a hybrid inflation potential [26].² Backreaction effects can also be relevant for such a production mechanism [16], but we will assume that q is sufficiently small that they can be neglected.

In [5], it was found that the momentum values of the resonance peaks in the momentum spectrum can be predicted by a simple relation for $q \ll 1$. Although the mechanism is nonperturbative, we show this relation to be a consequence of energy conservation in a perturbative approximation for the description of resonant fermion production. We also generalize the idea to a semianalytic relation that predicts the momentum values corresponding to resonance peaks in the momentum spectrum for any q , without the need to numerically solve the mode equation.

In [5,23,26], it was shown that the momentum spectrum of the fermions produced during preheating is not degenerate. For large q , the major contribution to the total number density of fermions comes from the bulk region, which was shown earlier in [16]. However, for small q ($\lesssim 0.01$), we find that the major contributions actually come from resonance peaks instead of the bulk region. We observe power-law behavior for the total number density of fermions as a function of q and find numerical approximations for the integrals describing the total number density, accounting for contributions from the full momentum distribution. We find good approximations in two regimes; the total number density is proportional to $q^{1/2}$ for $q \lesssim 0.01$ and proportional to $q^{3/4}$ for $q \gtrsim 10$. We also give a detailed review on the derivation of the number density per mode of

fermions and show how different descriptions in the literature (from [5,24,26]) are equivalent.

Fermions produced during preheating can be a promising dark matter candidate [19,27–30]. Fermionic dark matter can also be produced nonperturbatively from a coupling to a coherently oscillating field that is not the inflaton [31–33]. Even if the fermions are produced with zero temperature, fermion degeneracy can make the fermions relativistic if the Fermi momentum is greater than their mass. One can then place cosmological constraints, for example from structure formation, on these fermions that will impose a lower bound on their mass m_ψ . We adapt the bounds from [19] to our model for approximately degenerate fermions and find that for small q ($\lesssim 0.01$), the bound on m_ψ is mildly strengthened. For larger q , we obtain the same bound on m_ψ as in [19].

The organization of this paper is as follows. In Sec. II, we describe fermionic preheating, the mechanism through which particles are produced during the oscillation the inflaton, and give expressions for the number density of the fermions produced. In Sec. III, we find a simple semianalytic relation describing the momentum values that correspond to resonance peaks in the momentum distribution of the fermions. In Sec. IV, we calculate the contributions of various components of the momentum distribution of fermions to their total number density and find approximations for the contributions of both the bulk region and the resonance peaks to the total number density of fermions. In Sec. V, we discuss the possibility of fermions produced by the aforementioned mechanism making up the observed dark matter density and state our conclusions. Various technical details and derivations are given in the Appendixes.

II. FERMIONIC PREHEATING

Fermionic particles can be produced nonperturbatively and out of thermal equilibrium through a coupling with a coherently oscillating scalar field. If the scalar driving this fermion production is the inflaton, then the process is called fermionic preheating [5,16,23]. We choose to study the mechanism of production of fermions through preheating by considering inflation to be described by a simple $\lambda\phi^4$ quartic potential [24].

A. Inflaton oscillations

The evolution of the inflaton (ϕ) field is described by the following Klein-Gordon equation:

$$\phi''(t) + 3H(t)\phi'(t) + \frac{\partial V}{\partial \phi} = 0, \quad (2.1)$$

where $'$ indicates derivatives with respect to time t . Here, V represents the inflaton potential and $H(t) \equiv a'(t)/a(t)$ is the Hubble parameter (and a is the scale factor), which is given by the Friedmann equation [34],

²For any case of particle production from a coherently oscillating field with a scalar coupling to fermions, the description of the mechanism will be similar to $\lambda\phi^4$ preheating.

$$H^2 = \left(\frac{a'(t)}{a(t)} \right)^2 = \frac{1}{3M_{\text{Pl}}^2} \left(\frac{\phi'(t)^2}{2} + V(\phi) \right), \quad (2.2)$$

where M_{Pl} is the reduced Planck mass (i.e., $M_{\text{Pl}} = 1/\sqrt{8\pi G_N} \approx 2.44 \times 10^{18}$ GeV). For our paper, we take $V(\phi) = \lambda\phi^4/4$.

Inflation occurs as a ‘slow-roll’ phase followed by inflaton oscillations. Particle production occurs only during the oscillatory phase of the inflaton evolution and not when it is undergoing slow-roll. The oscillations of the inflaton field have an attractor solution, which means that irrespective of the initial amplitude, the inflaton will evolve towards the same pattern of oscillations. Hence, it is possible to describe the oscillations of the inflaton field using a function that is independent of the initial amplitude. In order to do this, first the Klein-Gordon equation for the inflaton from Eq. (2.1) is written in terms of conformal variables, $\varphi = a\phi$, $\eta = \int dt/a(t)$, to remove the damping term,

$$\phi''(t) + 3H(t)\phi'(t) + \lambda\phi(t)^3 = 0 \rightarrow \varphi'' + \lambda\varphi^3 \approx 0, \quad (2.3)$$

where $'$ now represents derivatives with respect to the conformal time η . We have neglected the $-a''\varphi/a$ term from the equation in conformal variables, as we can approximate that the scale factor a grows linearly with η during the oscillatory phase, which implies $a''(\eta) \approx 0$. The conformal variables can then be rescaled using the initial amplitude φ_0 ,

$$f(\tau) = \varphi(\tau)/\varphi_0, \quad \tau = \sqrt{\lambda}\varphi_0\eta. \quad (2.4)$$

Using $\dot{}$ to represent derivatives with respect to τ , we can now describe the inflaton oscillations in dimensionless variables by

$$\ddot{f} + f^3 = 0. \quad (2.5)$$

The solution of this equation is a Jacobi elliptic cosine³ of the form [24],

$$f(\tau) = cn(\tau - \tau_1, m); \quad m = 1/2. \quad (2.6)$$

We set τ_1 to 0 to ensure that $f(\tau)$ is symmetric around $\tau = 0$. This is important for the derivation of equations needed [10] to study the late-time behavior of the produced fermions, and we elaborate on this in Appendix E. The general $cn(u, m)$ function can be represented as a series expansion [36]. For the case of $m = 1/2$, it is written as [24]

³We have used the notation from Wolfram Mathematica [35] for the Jacobi elliptic cosine, while some references like [24] use the notation $cn(u, k)$ where $k = \sqrt{m}$.

$$cn(u, 1/2) = \frac{8\pi\sqrt{2}}{T} \sum_{n=1}^{\infty} \frac{e^{-\pi(n-1/2)}}{1 + e^{-2\pi(n-1/2)}} \cos\left(\frac{2\pi(2n-1)u}{T}\right). \quad (2.7)$$

The Jacobi elliptic function is periodic with its period for $m = 1/2$ given by an elliptic integral as

$$T = 4 \int_0^{\pi/2} \frac{d\theta}{(1 - \frac{1}{2}\sin^2 \theta)^{1/2}} \approx 7.4163. \quad (2.8)$$

The oscillatory phase of the inflaton dynamics can thus be described by

$$\phi_{\text{Osc}}(t) = \left(\frac{3M_{\text{Pl}}^2}{\lambda} \right)^{1/4} \frac{cn((48\lambda M_{\text{Pl}}^2)^{1/4} t^{1/2})}{t^{1/2}}. \quad (2.9)$$

B. Production and number density of fermions

Fermions (ψ) are assumed to have a $h\phi\bar{\psi}\psi$ Yukawa coupling to the inflaton field, along with a mass term $m_\psi\bar{\psi}\psi$. As in the previous section, we take the expansion of the Universe into account by first using the following conformal transformations of the relevant quantities:

$$\psi \rightarrow \Psi = a^{3/2}\psi, \quad \phi \rightarrow \varphi = a\phi, \quad t \rightarrow \eta = \int \frac{dt}{a(t)}. \quad (2.10)$$

Then, the fermion field satisfies the Dirac equation,

$$(i\gamma^\mu \partial_\mu - h\varphi(\eta) - m_\psi a(\eta))\Psi = 0, \quad (2.11)$$

where ∂_μ is in terms of conformal variables. The solution for this equation can be found by considering an auxiliary field $X(\mathbf{x}, \eta)$, separated into components (with comoving momentum $\mathbf{k}$ and position $\mathbf{x}$) as follows:

$$\begin{aligned} X(\mathbf{x}, \eta) &= e^{i\mathbf{k}\cdot\mathbf{x}} X_k(\eta) R_\pm(\mathbf{k}), \quad \text{or} \\ X(\mathbf{x}, \eta) &= e^{(-i\mathbf{k}\cdot\mathbf{x})} X_k(\eta) \bar{R}_\pm(\mathbf{k}), \end{aligned} \quad (2.12)$$

where $R_\pm, \bar{R}_\pm$ are eigenvectors of γ^0 and of the helicity operator $\hat{\mathbf{k}} \cdot \boldsymbol{\Sigma}$ (see Appendix A). The relevant ansatz to solve the Dirac equation is then written as follows:

$$\Psi = (i\gamma^\mu \partial_\mu + h\varphi(\eta) + m_\psi a(\eta))X(\mathbf{x}, \eta). \quad (2.13)$$

Plugging this ansatz in Eq. (2.11) and using Eq. (2.12) yields a differential equation:

$$\begin{aligned} X_k''(\eta) + [k^2 + (h\varphi(\eta) + m_\psi a(\eta))^2 - i(h\varphi'(\eta) \\ + m_\psi a'(\eta))]X_k(\eta) = 0, \end{aligned} \quad (2.14)$$

where $'$ represents derivatives with respect to the conformal time η . This is an oscillatorlike differential equation and we

can represent its ‘frequency’ as

$$\Omega_k^2(\eta) = k^2 + (h\varphi(\eta) + m_\psi a(\eta))^2. \quad (2.15)$$

This ‘mode equation’ of Eq. (2.14) is independent of the inflaton potential. In fact, it will be valid in the study of fermion production through a Yukawa coupling to any coherently oscillating scalar field.

In order to study fermion production, the field can also be written as

$$\Psi(\eta, \mathbf{x}) = \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3} (\hat{a}_{k,s} u_{k,s}(\eta) e^{+i\mathbf{k}\cdot\mathbf{x}} + \hat{b}_{k,s}^\dagger v_{k,s}(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}), \quad (2.16)$$

where $\hat{a}_{k,s}^\dagger, \hat{a}_{k,s}$ and $\hat{b}_{k,s}^\dagger, \hat{b}_{k,s}$ are fermion creation and annihilation operators for particles and antiparticles and $u_{k,s}(\eta)$ and $v_{k,s}(\eta)$ are, respectively, the positive- and negative-frequency eigenspinors of the Dirac equation, Eq. (2.11) (see Appendix A).

The Hamiltonian for the system is written in conformal variables as follows [24]:

$$H_D = \int d^3x (i\Psi^\dagger \partial_\eta \Psi). \quad (2.17)$$

H_D is diagonal in the creation and annihilation operators at an initial time $\eta = \eta_0$ but not diagonal at later times. This breaking of the time-translational symmetry of the Hamiltonian is a signature of particle creation. The number density per mode of the produced fermions, i.e., $n_k(\eta)$, can then be calculated after diagonalizing H_D at time η using a Bogoliubov transformation. A Bogoliubov transformation [9] is a rotation of the creation and annihilation operators that preserves anticommutators for fermions (or commutators for bosons) and diagonalizes the Hamiltonian in terms of the new operators.

First, we use Eq. (2.16) to evaluate the Hamiltonian expression of Eq. (2.17) in terms of the fermion creation and annihilation operators (see Appendix A for some details used in the derivation),

$$H_D(\eta) = \int \frac{d^3k}{(2\pi)^3} [E(\eta)(\hat{a}_{k,+}^\dagger \hat{a}_{k,+} - \hat{b}_{k,+} \hat{b}_{k,+}^\dagger + \hat{a}_{k,-}^\dagger \hat{a}_{k,-} - \hat{b}_{k,-} \hat{b}_{k,-}^\dagger) + F(\eta)(\hat{a}_{k,+}^\dagger \hat{b}_{-k,-}^\dagger + \hat{a}_{k,-}^\dagger \hat{b}_{-k,+}^\dagger) + F^*(\eta)(\hat{b}_{-k,+} \hat{a}_{k,-} + \hat{b}_{-k,-} \hat{a}_{k,+})]. \quad (2.18)$$

Here we have defined:

$$E(\eta) = (h\varphi(\eta) + m_\psi a(\eta))(|X'_k(\eta)|^2 + \Omega_k^2(\eta)|X_k(\eta)|^2) + 2\Omega_k^2(\eta) \text{Im}(X_k(\eta)X_k^*(\eta)), \\ F(\eta) = k(X_k^*(\eta))^2 + k\Omega_k^2(\eta)(X_k^*(\eta))^2. \quad (2.19)$$

For any two solutions $f(\eta)$ and $g(\eta)$ of the mode equation in Eq. (2.14), including the case where $f(\eta) = g(\eta)$, the following equation will hold [10]:

$$f'(\eta)g^*(\eta) + \Omega_k^2(\eta)f(\eta)g^*(\eta) + i(m_\psi a(\eta) + h\varphi(\eta))(f'(\eta)g^*(\eta) - f(\eta)g'^*(\eta)) = C, \quad (2.20)$$

where C is a constant. This can be easily shown by taking the derivative of both sides of Eq. (2.20) with respect to η and then substituting the mode equation from Eq. (2.14). We can then use the particular case of Eq. (2.20) for which $f(\eta) = g(\eta) \equiv X_k(\eta)$, in order to show that (Appendix B)

$$E(\eta)^2 + |F(\eta)|^2 = C^2 \Omega_k^2(\eta). \quad (2.21)$$

This is a generalization of a similar equation in [2], where C is taken to be 1. However, C actually depends on the initial conditions of the mode Eq. (2.14).

The initial conditions should ensure that H_D is diagonal at $\eta = \eta_0$. One such choice of initial conditions is as follows [26]:

$$X_k(\eta_0) = \{2\Omega_k(\eta_0)[\Omega_k(\eta_0) + (h\varphi(\eta_0) + m_\psi a(\eta_0))]\}^{-1/2}, \\ X'_k(\eta_0) = -i\Omega_k(\eta_0)X_k(\eta_0). \quad (2.22)$$

This choice of $X'_k(\eta_0)$ ensures that $H_D(\eta_0)$ is diagonal in the creation and annihilation operators because $F(\eta_0) = 0$. We have chosen $X_k(\eta_0)$ so that $E(\eta_0) = \Omega_k(\eta_0)$ and therefore $C = 1$. We will see that this choice also ensures consistency with the initial condition for the number density per mode, i.e., $n_k(\eta_0) = 0$, from the expression we will derive for $n_k(\eta)$. We can find the expression for the number density of fermions after applying a Bogoliubov transformation and diagonalizing the

transformed $H_D(\eta)$ by setting the nondiagonal coefficients to 0. The relevant Bogoliubov transformation is as follows [2]:

$$\begin{aligned} \hat{a}_{k,+} &= \alpha_1^* \hat{c}_{k,+} - \beta_1 \hat{d}_{k,+}^\dagger, & \hat{a}_{k,-} &= \alpha_2^* \hat{c}_{k,-} - \beta_2 \hat{d}_{k,-}^\dagger, \\ \hat{b}_{-k,+} &= \alpha_2^* \hat{d}_{k,-}^\dagger + \beta_2 \hat{c}_{k,-}, & \hat{b}_{-k,-} &= \alpha_1^* \hat{d}_{k,+}^\dagger + \beta_1 \hat{c}_{k,+}. \end{aligned} \quad (2.23)$$

Preserving the canonical anticommutation relations $\{\hat{a}_{k,s}, \hat{a}_{k,s}^\dagger\} = 1$ and the same for the other three sets of operators implies that the Bogoliubov coefficients α_j, β_j have the following relation:

$$\begin{aligned} H_D(\eta) &= \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \{ \hat{c}_{k,s}^\dagger \hat{c}_{k,s} (E(|\alpha|^2 - |\beta|^2) + F\alpha\beta^* + F^*\alpha^*\beta) - \hat{d}_{k,s} \hat{d}_{k,s}^\dagger (E(|\alpha|^2 - |\beta|^2) + F\alpha\beta^* + F^*\alpha^*\beta) \\ &\quad + \hat{c}_{k,s}^\dagger \hat{d}_{k,s}^\dagger (F\alpha^2 - F^*\beta^2 - 2E\alpha\beta) + \hat{d}_{k,s} \hat{c}_{k,s} (F^*(\alpha^*)^2 - F(\beta^*)^2 - 2E\alpha^*\beta^*) \}. \end{aligned} \quad (2.25)$$

If we apply the number density operator to the ground state of the Hamiltonian and use the Bogoliubov transformation from Eq. (2.23), we can see that the number density per k mode of particles is given by $n_k(\eta) = |\beta|^2$. We shall henceforth call $n_k(\eta)$ the number density per mode and show how to compute it in later sections. We can evaluate $|\beta|^2$ from Eq. (2.25) (Appendix C) and get

$$n_k(\eta) = |\beta|^2 = \frac{C\Omega_k(\eta) - E(\eta)}{2C\Omega_k(\eta)}. \quad (2.26)$$

We have used Eq. (2.21) here to simplify the expression. Clearly, for the choice of initial conditions Eq. (2.22), the resulting values of $C = 1$ and $E(\eta_0) = \Omega_k(\eta_0)$ ensure that $n_k(\eta_0) = 0$, i.e., no particles are produced before oscillations start, exactly what we expect.

Similarly to the mode equation, this general expression for $n_k(\eta)$ will also be valid for any scalar field potential. The form for $\Omega_k(\eta)$ and $E(\eta)$ will also remain exactly the same, with the only difference being the way the scalar oscillations $\varphi(\eta)$ are described. The n_k expressions in the literature ([5,24,26]) look different from our expression, but they represent the same number density under the choice of initial conditions of the mode equation, which we show in Appendix D.

C. Number density of fermions in $\lambda\phi^4$ inflation

Now, we shall use the general framework above and apply it to the case of $\lambda\phi^4$ inflation with an inflaton quartic coupling λ and Yukawa coupling h , as in Eq. (2.3) and Eq. (2.11) respectively. All conformal variables can first be transformed into dimensionless variables using these couplings and the initial amplitude of the inflaton φ_0 , as follows:

$$|\alpha_j|^2 + |\beta_j|^2 = 1, \quad j = 1, 2. \quad (2.24)$$

We can assume that the operators with $s = +$ and comoving momentum $\mathbf{k}$ are equivalent to other operators with $s = -$ and comoving momentum $-\mathbf{k}$, which implies $\hat{c}_{k,\pm} \equiv \hat{c}_{-k,\mp}, \hat{d}_{k,\pm} \equiv \hat{d}_{-k,\mp}$. This is a consequence of a similar correspondence between the $R_\pm$ and $\bar{R}_\pm$ eigenvectors [see Eq. (A2)]. We use the properties of the Bogoliubov coefficients and the nondiagonal coefficients to first see: $\alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta$. Using these, we rewrite Eq. (2.18) in terms of the time-dependent Bogoliubov coefficients and the new operators:

$$\begin{aligned} \kappa^2 &= \frac{k^2}{\lambda\varphi_0^2}, & q &= \frac{h^2}{\lambda}, & \tau &= (\sqrt{\lambda}\varphi_0)\eta, \\ X_\kappa &= (\sqrt{\lambda}\varphi_0)X_k, & f(\tau) &= \frac{\varphi(\tau)}{\varphi_0}, & \Omega_\kappa(\tau) &= \frac{\Omega_k(\eta)}{\sqrt{\lambda}\varphi_0}. \end{aligned} \quad (2.27)$$

We take $f(\tau)$ to be the Jacobi elliptic cosine from Eq. (2.6), thereby neglecting backreaction [16]. We show that this assumption is justified in Appendix F. We consider fermions to be light compared to the amplitude of inflaton oscillations and therefore can assume the bare mass of the fermion $m_\psi = 0$. Then, using $\dot{}$ to represent derivatives with respect to τ , the mode equation can be written using dimensionless variables:

$$\ddot{X}_\kappa(\tau) + (\kappa^2 + qf^2(\tau) - i\sqrt{q}\dot{f}(\tau))X_\kappa(\tau) = 0. \quad (2.28)$$

And we can write,

$$\Omega_\kappa(\tau) = \sqrt{\kappa^2 + qf^2(\tau)}. \quad (2.29)$$

The parameter q is called the ‘coupling parameter’ for this mode equation, while the parameter κ characterizes the various fermion momentum modes. We choose the same initial conditions of the mode equation, which were described earlier in Eq. (2.22), and they can be represented in these new variables (with $\tau = 0$ representing the beginning of oscillations):

$$\begin{aligned} X_\kappa(0) &= [2\Omega_\kappa(0)[\Omega_\kappa(0) + \sqrt{q}f(0)]^{-1/2}, \\ \dot{X}_\kappa(0) &= -i\Omega_\kappa(0)X_\kappa(0). \end{aligned} \quad (2.30)$$

These initial conditions ensure $C = 1$ again and $E_\kappa(0) = \Omega_\kappa(0)$; hence the relevant number density expression can be written in the dimensionless variables,

$$n_\kappa(\tau) = \frac{\Omega_\kappa(\tau) - E_\kappa(\tau)}{2\Omega_\kappa(\tau)}, \quad (2.31)$$

where

$$E_\kappa(\tau) = \sqrt{q}f(\tau)(|\dot{X}_\kappa|^2 + \Omega_\kappa^2(\tau)|X_\kappa|^2) + 2\Omega_\kappa^2(\tau)\text{Im}(X_\kappa\dot{X}_\kappa^*). \quad (2.32)$$

For small values of the coupling parameter q , the number density as a function of dimensionless conformal time τ follows a smooth pattern. However, for larger values of q , the variation of the number density with time has very rapid changes as a consequence of nonadiabaticity, with jumps in the pattern whenever $f(\tau) = 0$. These patterns are shown in Fig. 1.

The spiky pattern for larger q values can be approximated by a smooth function $\bar{n}_\kappa(\tau)$ that matches the value of $n_\kappa(\tau)$ when τ is an integer multiple of the period T . This $\bar{n}_\kappa(\tau)$ is given by [10,24]

$$\bar{n}_\kappa(\tau) = F_\kappa \sin^2(\nu_\kappa \tau), \quad (2.33)$$

where

$$\nu_\kappa = \frac{1}{T} \cos^{-1}(|\text{Re}(X_\kappa^1(T))|),$$

$$F_\kappa = \frac{1}{\sin^2(\nu_\kappa T)} \cdot \frac{\kappa^2}{\Omega_\kappa^2} (\text{Im}(X_\kappa^1(T)))^2. \quad (2.34)$$

$X_\kappa^1(\tau)$ used here is the numerical solution of the same differential Eq. (2.28), but with the initial conditions:

$$X_\kappa^1(0) = 1, \quad \dot{X}_\kappa^1(0) = 0. \quad (2.35)$$

The details of why $n_\kappa(\tau)$ matches $\bar{n}_\kappa(\tau)$ at certain times is discussed in [10], and we describe the procedure in our notation in Appendix E. As an example, we plot $n_\kappa(\tau)$ from Eq. (2.31) and $\bar{n}_\kappa(\tau)$ from Eq. (2.33) for $q = 1$ in Fig. 2. We have chosen a particular κ that we explain in Sec. III.

Instead of $n_\kappa(\tau)$ and $\bar{n}_\kappa(\tau)$ as functions of time, if we plot them as functions of κ for a fixed value of time that is an integral multiple of T , $n_\kappa(nT)$, and $\bar{n}_\kappa(nT)$ as functions of κ will match for any given q . We use $\bar{n}_\kappa(\tau)$ to approximate the long-period behavior of $n_\kappa(\tau)$ because $\bar{n}_\kappa(\tau)$ is numerically much simpler to calculate. As they have the same pattern as functions of κ at points of time where they match, we can use the $\bar{n}_\kappa(\tau)$ approximation to study the pattern of filling of the κ modes for the number density of produced fermions.

D. Late-time filling of fermion modes

Now, we can study the filling of the κ modes by seeing the variation of $\bar{n}_\kappa(\tau)$ with κ for fixed values of q and at fixed instants of time. We do so for four values of the coupling parameter: $q = 0.001$, $q = 0.01$, $q = 1$, and $q = 100$, and at two particular instants of time; $\tau = 5T$ and $20T$. The plots for these cases are shown in Fig. 3. We also plot the F_κ envelopes [see Eq. (2.34)] for the q values on their respective graphs.

From the plots in Fig. 3, we can see that the lower κ modes of the F_κ envelopes are well populated. We call this low κ regime ‘bulk region’. The higher κ modes exhibit resonance peaks where F_κ is only large near discrete values of κ , and these resonances are a consequence of constructive interference between the inflaton oscillations and the fermion modes. We describe how κ modes corresponding to resonance peaks can be predicted using an analytic or semianalytic relation, without having to solve the mode equation in Eq. (2.28) in Sec. III. In Fig. 1, we choose

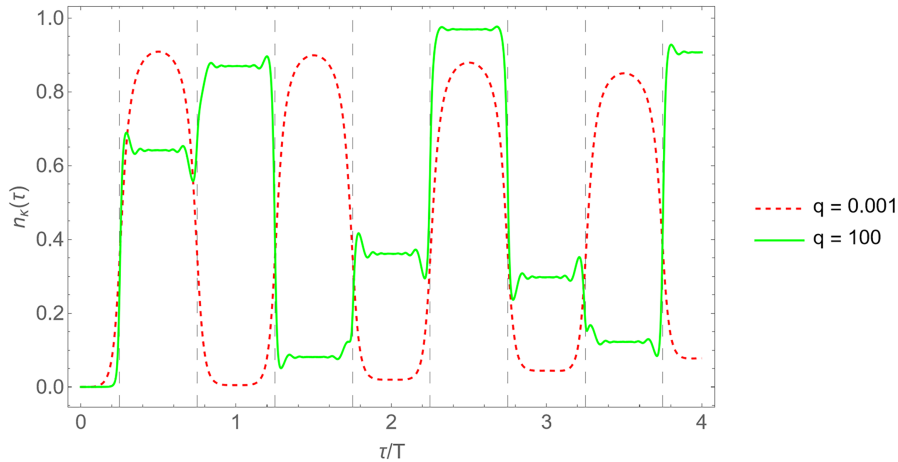


FIG. 1. $n_\kappa(\tau)$ from Eq. (2.31) as a function of time, using the period T of $f(\tau)$ for two different q values. The dashed vertical lines correspond to $\tau = (2n + 1)T/4$, $n = 0, 1, 2, \dots$, the instants of time when $f(\tau) = 0$. We choose $\kappa = 0.01$ for $q = 0.001$ and $\kappa = 1$ for $q = 100$, which we explain in Sec. IID.

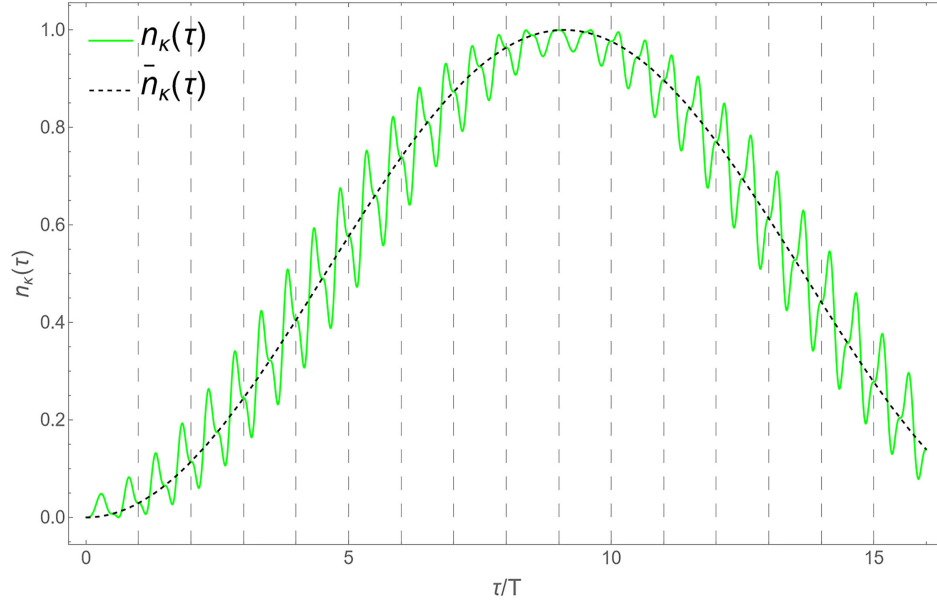


FIG. 2. Numerically evaluated functions $\bar{n}_\kappa(\tau)$ and $n_\kappa(\tau)$ as a function to τ/T , where T is the period of the inflaton solution $f(\tau)$. We choose $q = 1$ and $\kappa = 1.05236$, which corresponds to one of the resonance peaks as shown in Fig. 3. $\bar{n}_\kappa(\tau)$ and $n_\kappa(\tau)$ are guaranteed to be identical at integer multiples of T (and also match at some other points). Dashed vertical lines represent when τ is an integer multiple of T .

$\kappa = 0.01$ and $\kappa = 1$ for $q = 0.001$ and $q = 100$, respectively, because those values of κ are in the bulk region as can be seen in Fig. 3.

From Fig. 3, we see that the occupancy of each mode oscillates with κ within the envelope F_κ , and that the frequency of the oscillations in κ space increase with time. This means that at late time, modes that are nearby one another in κ will decohere. Hence, if the inflaton oscillations and subsequent particle production occur for a long time $\tau \gg T$, we can average the $\sin^2(\nu_\kappa \tau)$ term in Eq. (2.33) to $1/2$, and we can approximate $\bar{n}_\kappa$ simply from the F_κ envelope, as

$$\bar{n}_\kappa(\tau) \xrightarrow{\tau \gg T} \langle F_\kappa \sin^2(\nu_\kappa \tau) \rangle = F_\kappa/2. \quad (2.36)$$

We conclude from this that we can approximate the filling of the κ modes of the produced fermions by only the F_κ envelope. As such, we study the κ modes of the F_κ envelope in terms of its bulk region and resonance peaks, which are a consequence of the behavior of the oscillatorlike equation from Eq. (2.28). The exact patterns of how momentum modes are filled vary with q , depending on whether q is less than $\mathcal{O}(0.1)$ or greater. Although F_κ can be numerically evaluated, it becomes difficult to numerically resolve resonance peaks at higher κ values, in particular for smaller values of q , as the peak widths are very narrow. Thus, it becomes important to use analytical approximations to estimate the contributions of different κ regions to the number density.

To obtain an expression for the total number density of fermions, we first see that we can use the conformal variables and the density of states in $\mathbf{k}$ space to write the total comoving number density of fermions by integrating over the number density per mode as

$$n_{\psi+\bar{\psi}} = \frac{2 \times 2 \times 4\pi}{(2\pi)^3} \int dk k^2 n_k = \frac{2}{\pi^2} \int dk k^2 n_k. \quad (2.37)$$

The additional factors of 2 come from counting particles and antiparticles, and taking into account spin-up and spin-down fermions. The 4π comes from considering a sphere in $\mathbf{k}$ space and $(2\pi)^3$ comes from the expression for the density of states. We approximate n_k by $\bar{n}_\kappa$ from Eq. (2.33), and use $\bar{n}_\kappa(\tau) \xrightarrow{\tau \gg T} F_\kappa/2$. Hence, if we convert the integral above into dimensionless variables using $k = \kappa \sqrt{\lambda} \varphi_0$ [from Eq. (2.27)], we can write the total comoving number density as

$$n_{\psi+\bar{\psi}} = \frac{2\lambda^{3/2}\varphi_0^3}{\pi^2} \int d\kappa \kappa^2 \bar{n}_\kappa \approx \frac{\lambda^{3/2}\varphi_0^3}{\pi^2} \int d\kappa \kappa^2 F_\kappa. \quad (2.38)$$

The total physical number density is obtained from the total comoving number density by dividing by a^3 (where a is the scale factor) and hence,

$$n = \frac{n_{\psi+\bar{\psi}}}{a^3} = \frac{2\lambda^{3/2}\varphi_0^3}{\pi^2 a^3} \int d\kappa \kappa^2 \bar{n}_\kappa \approx \frac{\lambda^{3/2}\varphi_0^3}{\pi^2 a^3} \int d\kappa \kappa^2 F_\kappa. \quad (2.39)$$

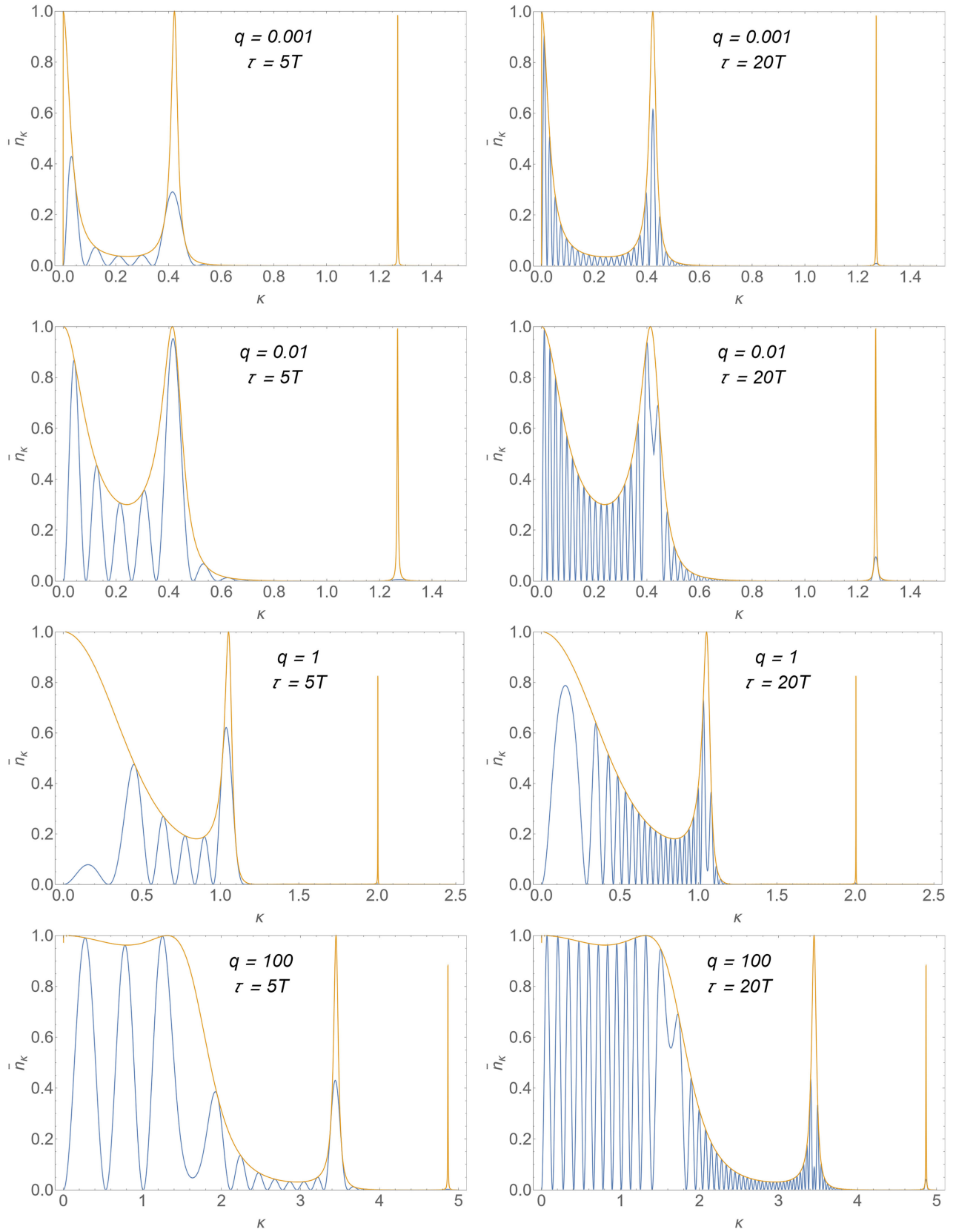


FIG. 3. $\bar{n}_k(\tau)$ (in blue) for $q = 0.001$ (top row), $q = 0.01$ (second row), $q = 1$ (third row) and $q = 100$ (bottom row) with its F_k envelope (in orange) at $\tau = 5T$ on the left and $\tau = 20T$ on the right.

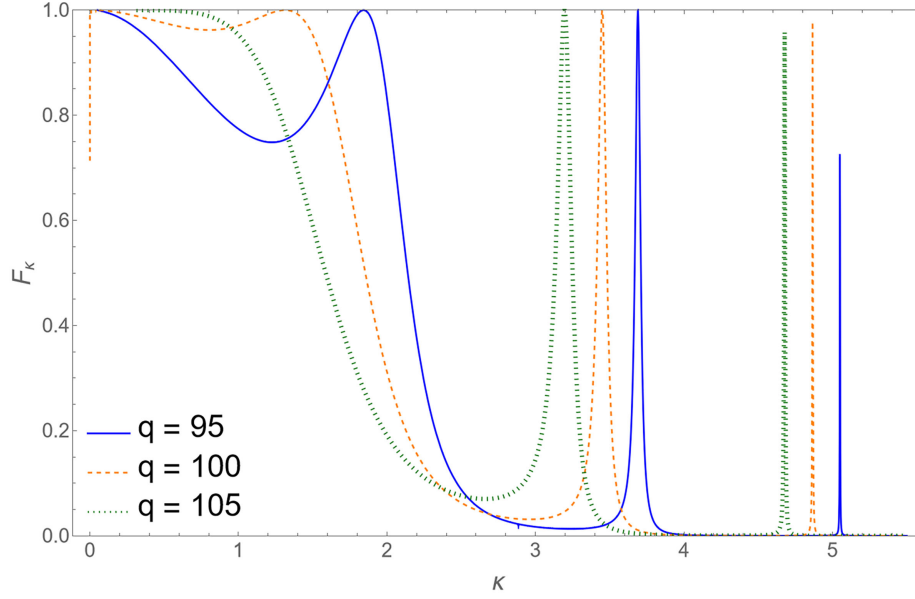


FIG. 4. F_κ as a function of κ for three different values of q , showing how the envelope evolves as q increases. The κ value of each resonance peak decreases with increasing q .

III. PREDICTING THE LOCATIONS OF RESONANCE PEAKS

In this section, we describe how the κ values of the resonance peaks of the momentum spectrum can be found analytically or semianalytically, without the need to solve a differential equation. We also provide physical explanations for the relations that help predict the κ values of the resonance peaks.

The envelope F_κ is defined in Eq. (2.34) and computed by numerically solving the mode equation of Eq. (2.28) with the initial conditions of Eq. (2.35). For small values of the coupling parameter q ($\lesssim 0.01$), the bulk region located at small κ is narrow, while the resonance peaks (the spikes at larger values of κ) are independent of q . The bulk region and the resonance peaks are well separated for small q values. For large q ($\gtrsim 1$), the resonance peak positions decrease in κ with increasing q and we can see this evolution in Fig. 4. For larger q values, the bulk region of the F_κ envelope can merge with a resonance peak for some values of q , and we can also see this effect in Fig. 4.

In [5], it was found that for $q \ll 1$, the resonance peaks are located at odd multiples of π/T , which can be written as

$$\kappa = \frac{\pi}{T}(2l+1), \text{ where } l=0,1,2,3,\dots(\text{small } q). \quad (3.1)$$

We numerically verify this relation for $l=0$ in Fig. 5. For small values of q , it is difficult to verify Eq. (3.1) explicitly for peaks at higher values of l (and thus larger κ) as those peaks become very narrow and difficult to resolve numerically.

We now give a physics argument for why Eq. (3.1) holds, based on a semiperturbative description of the process $n\varphi \rightarrow \Psi\bar{\Psi}$. Effectively, the relation is a consequence of

energy conservation in this process. To show this, we define the time average of $\Omega_\kappa(\tau)$ from Eq. (2.29) over a single period of inflaton oscillations as follows:

$$\bar{\Omega}_\kappa(q) \equiv \frac{1}{T} \int_0^T d\tau \Omega_\kappa(\tau), \quad (3.2)$$

where $\Omega_\kappa(\tau)$ and $\bar{\Omega}_\kappa(q)$ are functions of both q and κ . $\bar{\Omega}_\kappa(q)T$ represents the phase accumulated in the Ψ and $\bar{\Psi}$ wave functions over one time period of the inflaton oscillations (refer to Eq. (3.16) of [24]). Then, the integrals over internal spacetime points that appear in the amplitude for the process $n\varphi \rightarrow \Psi\bar{\Psi}$ yield a term of the form:

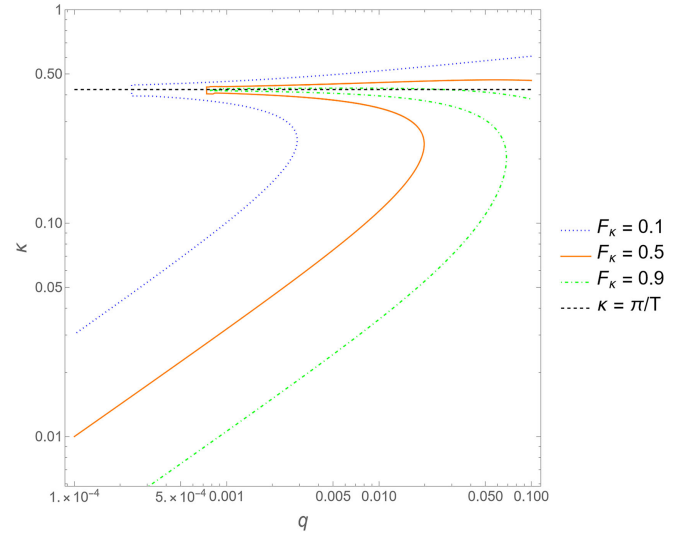


FIG. 5. F_κ contours in the (q, κ) plane, with both axes scaled logarithmically. F_κ gets large at $\kappa \approx \pi/T$ (black dashed line), the resonance peak corresponding to $l=0$ from Eq. (3.1).

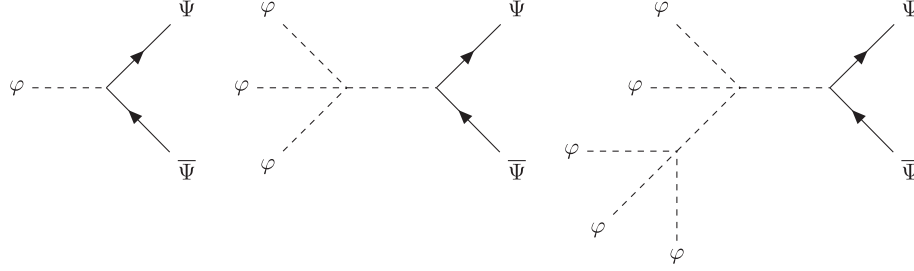


FIG. 6. Feynman diagrams describing the perturbative process $n\phi \rightarrow \Psi\bar{\Psi}$, for small q . Shown are the diagrams for $n = 1$ (left), $n = 3$ (middle), and a representative diagram for $n = 5$ (right).

$$\int d^4x f(\tau)^n e^{i(\kappa_\Psi^\mu + \kappa_{\bar{\Psi}}^\mu)x_\mu} \sim \delta^{(3)}(\kappa_\Psi + \kappa_{\bar{\Psi}}) \int d\tau e^{-in\omega_\phi\tau} e^{2i\Omega_\kappa(\tau)\tau}, \quad (3.3)$$

where we keep the term involving the fundamental frequency of $f(\tau)$ [see Eq. (2.7)], and $\omega_\phi = 2\pi/T$, with T being the period of inflaton oscillations.⁴ The ϕ field is a classical background field with trivial spatial dependence which implies that it does not carry any 3-momentum. The $\delta^{(3)}$ function then ensures momentum conservation, so that the fermion 3-momentum is equal in magnitude and opposite in direction to that of the antifermion, i.e., $\kappa_\Psi = -\kappa_{\bar{\Psi}}$, and $\kappa_\Psi = \kappa_{\bar{\Psi}} = \kappa$.

If $\Omega_\kappa(\tau)$ were a constant, the τ integral in Eq. (3.3) would yield the usual energy-conserving δ function. Instead, the periodic variation of $\Omega_\kappa(\tau)$ causes the phase to oscillate more and more rapidly as τ becomes large, while the accumulated phase for the fermion-antifermion pair approaches $2\bar{\Omega}_\kappa(q)\tau$ to higher and higher fractional precision. Replacing $\Omega_\kappa(\tau)$ with its time-average then gives the energy-conservation condition,

$$n\omega_\phi = 2\bar{\Omega}_\kappa(q) \Rightarrow \bar{\Omega}_\kappa(q) = \frac{\pi}{T}n. \quad (3.4)$$

At small q , $\bar{\Omega}_\kappa(q) \approx \kappa$ so this reduces to Eq. (3.1) if n is odd.

We can qualitatively explain the resonance structure of Eq. (3.4) from the perturbative amplitude of $n\phi \rightarrow \Psi\bar{\Psi}$ using Feynman diagrams. For small values of $q = h^2/\lambda$ [from Eq. (2.27)], the inflaton quartic coupling λ is much larger than the square of the Yukawa coupling h . Then the process is dominated by Feynman diagrams of the form shown in Fig. 6. Clearly, for each subsequent addition of a ϕ^4 vertex to the $n = 1$ case (left diagram of Fig. 6), the number of external ϕ increases by 2. This implies that n must be odd, i.e., $n = 2l + 1$, where $l = 0, 1, 2, \dots$, and we

⁴Higher harmonics in $f(\tau)$ from Eq. (2.7) contribute only odd multiples of ω_ϕ ; this holds for the solution of any scalar potential symmetric under $\phi \rightarrow -\phi$ and will be important to preserve our conclusion below that only odd values of n contribute.

have proved Eq. (3.1) for small q . Note that this argument will hold for any scalar potential symmetric under $\phi \rightarrow -\phi$.

For large values of q , the square of the Yukawa coupling is much larger than λ and we can neglect the ϕ^4 vertices. Then $n\phi \rightarrow \Psi\bar{\Psi}$ is described by Feynman diagrams having the general form shown in Fig. 7.

For any even n in Fig. 7, i.e., an even number of external ϕ vertices, the diagram will have an odd number of fermion propagators, in which case the amplitude involves only terms containing an odd number of γ -matrices in between the external fermion and antifermion spinors. These can be reduced using anticommutation relations down to a single gamma matrix, which must be dotted into an external fermion momentum because the initial-state scalars are effectively at rest. For massless fermions, all such terms are then zero by the Dirac equation. Hence, the amplitude of diagrams of the form of Fig. 7 is nonzero only for odd values of n . We thus obtain the same criterion for n as for small q , i.e., $n = 2l + 1$, where $l = 0, 1, 2, \dots$.

Thus, as a consequence of energy conservation, we can write a general semianalytic relation that enables us to calculate the κ values corresponding to resonance peaks for any q :

$$\bar{\Omega}_\kappa(q) = \frac{\pi}{T}(2l + 1), \quad \text{where } l = 0, 1, 2, 3, \dots \quad (3.5)$$

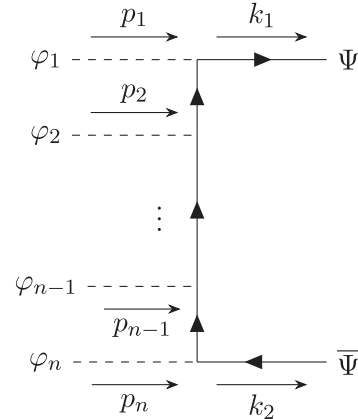


FIG. 7. General form of Feynman diagram describing the perturbative process $n\phi \rightarrow \Psi\bar{\Psi}$, for large q .

We can numerically verify that Eq. (3.5) holds by plotting the contours of F_κ in the $(q, \bar{\Omega}_\kappa \cdot T/\pi)$ plane instead of the (q, κ) plane of Fig. 5. We expect resonance peaks whenever $\bar{\Omega}_\kappa \cdot T/\pi$ is equal to an odd positive integer and this is exactly what we observe in Fig. 8. We provide further justification for our energy conservation arguments in Appendix G by redoing our analysis with a $\phi\phi\bar{\psi}\psi$ interaction rather than a Yukawa coupling.

We can thus identify each resonance peak for all values of q by its corresponding l , according to Eq. (3.5). For a fixed q , $\bar{\Omega}_\kappa$ cannot be less than its value when $\kappa = 0$. This $\bar{\Omega}_0$ (smallest value of $\bar{\Omega}_\kappa$ for a fixed q) is given by

$$\bar{\Omega}_0(q) = \langle \Omega_0(\tau) \rangle = q^{1/2} \langle |f(\tau)| \rangle = \frac{\pi}{T} \sqrt{2} q^{1/2}, \quad (3.6)$$

where $\langle \dots \rangle$ denotes a time average over a complete period of inflaton oscillations i.e., $\langle |f| \rangle = (1/T) \int_0^T d\tau \sqrt{f(\tau)^2} = \sqrt{2}\pi/T$. This lower bound $\bar{\Omega}_0(q)$ is indicated by the lowermost smooth green curve in Fig. 8 and indicates which l values are possible for a fixed q .

In order to find the positions of the peaks in κ for a fixed value of q , we first construct an interpolating function for $\bar{\Omega}_\kappa(q)$ for a wide range of values of q and κ . Then, we can invert Eq. (3.5) for a fixed value of l to get the value of κ corresponding to that value of l . However, before applying this method, we need to take into account the lowest l value accessible for a particular q due to the lower bound $\bar{\Omega}_0(q)$,

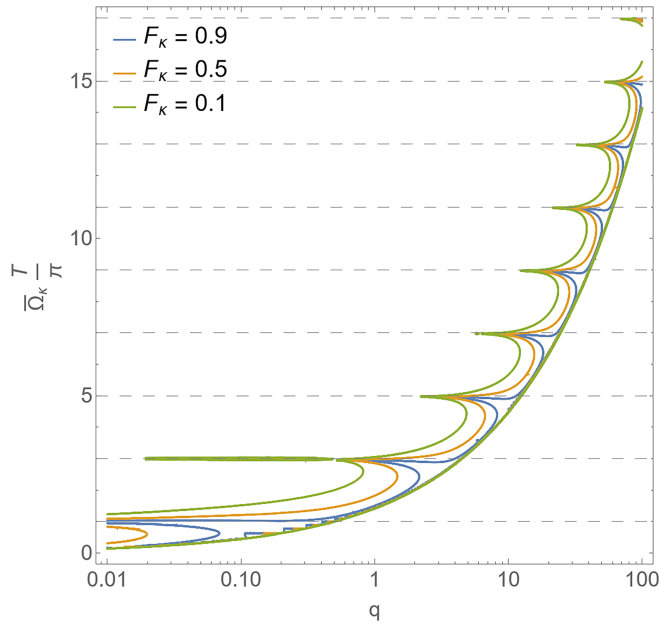


FIG. 8. Contours corresponding to $F_\kappa = 0.9$ (blue), $F_\kappa = 0.5$ (orange), $F_\kappa = 0.1$ (green) in the q vs $\bar{\Omega}_\kappa T/\pi$ plane. The horizontal dashed lines are at odd integers. In addition to the spiky pattern of the resonance peaks, there is a smooth curve at the base of the spikes that corresponds to contours for $\kappa = 0$ and determines which l peaks are accessible for a particular q .

which can be obtained as follows:

$$l_0(q) = \left\lceil \frac{1}{2} \left(\bar{\Omega}_0(q) \frac{T}{\pi} - 1 \right) \right\rceil = \left\lceil \frac{1}{2} (\sqrt{2} q^{1/2} - 1) \right\rceil, \quad (3.7)$$

where $\lceil x \rceil$ represents the ceiling function that gives the nearest integer greater than or equal to x . The peak in F_κ corresponding to this l_0 is sometimes merged into the bulk region at small κ , which we can see happening from the evolution of the F_κ envelope of Fig. 4 with increasing q . In general, for any fixed q , we can first use Eq. (3.7) to find l_0 and then invert Eq. (3.5) to find the value of κ corresponding to the peak l_0 and all higher peaks. We used this procedure to find the approximate κ value used in Fig. 2 corresponding to the $l_0 = 1$ peak for $q = 1$. Finally, using Eq. (3.6), we are also able to determine the values of q at which the value of l_0 changes,

$$q_0(l) = \left(\frac{\pi(2l+1)}{T \langle |f| \rangle} \right)^2 = \frac{(2l+1)^2}{2}, \quad \text{where} \\ l = 0, 1, 2, 3, \dots \quad (3.8)$$

These values will correspond to the discontinuities in the curves in Figs. 12 and 15.

IV. CONTRIBUTIONS TO THE TOTAL NUMBER DENSITY

In this section, we will compute how the bulk region around $\kappa = 0$ and the resonance peaks at higher κ contribute to the total number density by numerically evaluating integrals of the form in Eqs. (2.38) and (2.39). We will divide up the integral into bulk and resonance peak contributions, show how they behave separately and then find analytical approximations for all integrals. We numerically evaluate the different contributions to the integral,

$$I \equiv \int_0^\infty d\kappa \kappa^2 F_\kappa. \quad (4.1)$$

We find approximations (as functions of q) for the different contributions and for the total integral itself. In particular, we find good approximations in two regimes; small q ($q \lesssim 0.01$) and large q ($q \gtrsim 10$). We also give numerical results for the intermediate regime, but do not find a good analytical approximation there.

A. Small q

We first focus on evaluating the integral I over a range of small q values; $10^{-5} \leq q \leq 10^{-2}$. We expect these results to be valid for smaller values of q as well. For all values of q less than $q_0(0) = 0.5$, $l_0 = 0$ and, therefore, resonance peaks corresponding to all l values are present for $q < 0.5$. We observe, however, that the resonance peaks corresponding to $l \geq 2$ are very narrow and, as such, the numerical

estimations of their contributions to I become unreliable. Below we give evidence that the contribution of the $l = 2$ peak is small compared to the total integral.

To estimate the total integral for small q , we integrate the bulk region and the first two resonance peaks corresponding to $l = 0$ and $l = 1$,

$$I_S(q) = \int_0^{(\kappa_1+\kappa_2)/2} d\kappa \kappa^2 F_\kappa. \quad (4.2)$$

The upper limit of the this integral is chosen because it represents the κ -point midway between the $l = 1$ and $l = 2$ resonance peaks. Because the peaks are narrow compared to their spacing (see the top row of Fig. 3), this upper limit of the integral ensures that we are taking into account only the contributions from the bulk and the first two peaks $l = 0$ and $l = 1$. The total integral I_S can be broken up into three parts, with each part describing the contributions from the bulk region and each of the first two resonance peaks, as follows:

$$I_S(q) = \underbrace{\int_0^{\kappa_0/2} d\kappa \kappa^2 F_\kappa}_{\text{Bulk, } I_b} + \underbrace{\int_{\kappa_0/2}^{(\kappa_0+\kappa_1)/2} d\kappa \kappa^2 F_\kappa}_{l=0 \text{ Peak, } I_0} + \underbrace{\int_{(\kappa_0+\kappa_1)/2}^{(\kappa_1+\kappa_2)/2} d\kappa \kappa^2 F_\kappa}_{l=1 \text{ Peak, } I_1}. \quad (4.3)$$

Figure 9 shows the value of the three component integrals normalized to the total integral, as a function of q . We observe in Fig. 9 that most of the contribution to the fermion density for small q values comes from the resonance peaks $l = 0$ and $l = 1$. The peak $l = 0$ contributes 65–70% of the total integral, the peak $l = 1$ contributes

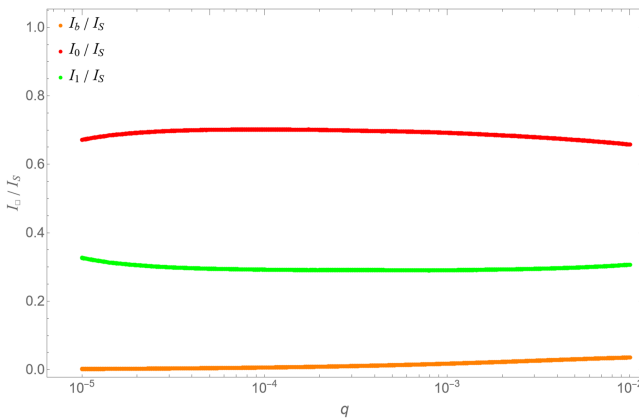


FIG. 9. Contributions of different components as fractions of the total integral $I_S(q)$ from Eq. (4.2) for $10^{-5} \leq q \leq 10^{-2}$. The contributions defined in Eq. (4.3) are given by the bulk (orange, bottom), the $l = 0$ peak (red, top), and the $l = 1$ peak (green, middle).

30–35%, while the bulk region contributes less than 5% of the total. As such, we focus on finding approximations to the contributions of the resonance peaks.

Plotting I_0 vs q with both axes logarithmically scaled, we find that the values follow a $q^{1/2}$ power law. Fitting a straight line to the log-log data using least-squares regression gives

$$I_0(q) = \int_{\kappa_0/2}^{(\kappa_0+\kappa_1)/2} d\kappa \kappa^2 F_\kappa \approx 0.26 \times q^{1/2}. \quad (4.4)$$

The numerical estimate as well as the fit line are shown in Fig. 10. Repeating the procedure for I_1 , we similarly find

$$I_1(q) = \int_{(\kappa_0+\kappa_1)/2}^{(\kappa_1+\kappa_2)/2} d\kappa \kappa^2 F_\kappa \approx 0.11 \times q^{1/2}. \quad (4.5)$$

This fit is also shown in Fig. 10.

Repeating the same procedure for the bulk region, $I_b = \int_0^{\kappa_0/2} d\kappa \kappa^2 F_\kappa$, we find that it follows a power law $\sim q^{0.9}$. Because the bulk is only a small fraction of the total contribution (see Fig. 9), the power law dependence of the total integral is still well-described by $q^{1/2}$, as shown in Fig. 10.

We are unable to find a good approximation for the resonance peaks $l = 2$ and higher by this method directly, as the peaks are too narrow. The numerical evaluation becomes unreliable due to round-off error as $\text{Re}(X_\kappa^1(T))$ [from Eq. (2.34)] is very close to -1 . As such, we try to estimate the contribution from the peak $l = 2$ by using $Y_\kappa = X_\kappa + 1$ and substituting it for X_κ in Eqs. (2.28), (2.34), and (2.35). We find that this trick improves numerical calculations for a small range of q values between $q = 0.005$ and $q = 0.01$, since Wolfram Mathematica handles very small values better than small differences between $\mathcal{O}(1)$ values. We then evaluate the integral using the F_κ from Eq. (2.34) (with Y_κ now) over a small range of κ around $\kappa = 5\pi/T$ [from Eq. (3.1), corresponding to the peak $l = 2$ for small q] for three values $q = 0.005, 0.008, 0.01$. We find that the log-log data for the three integral values approximately follows a $q^{1/2}$ power law (similar to I_0 and I_1) and we find the fit to these data points using least-squares regression as follows:

$$I_2(q) = \int_{5\pi/T-0.01}^{5\pi/T+0.01} d\kappa \kappa^2 F_\kappa \approx 0.012 \times q^{1/2}. \quad (4.6)$$

The numerical values of the integral and their corresponding fit are also shown in Fig. 10. We thus see that I_2 is about ten times smaller than I_1 , justifying that it can be ignored. We note that the three points computed for I_2 are approximately the same size as I_b , but we expect that this is a coincidence and that at smaller q , I_2 will be larger than I_b as shown by the extrapolation in Fig. 10.

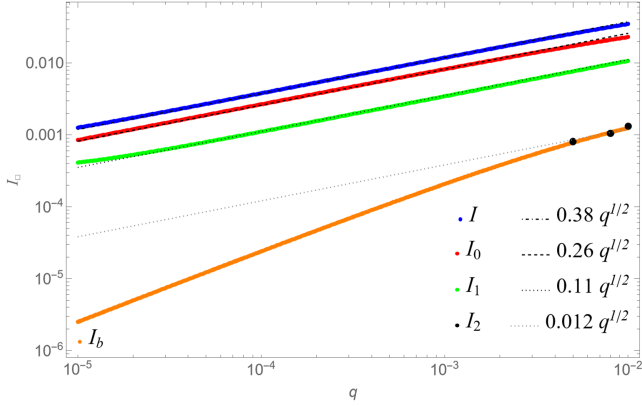


FIG. 10. Contributions of different components and the total integral for $10^{-5} \leq q \leq 10^{-2}$. Values of the total integral are given by I_S from Eq. (4.2) (blue, top) with the approximation from Eq. (4.7) (black, dot-dashed). Also shown are contributions of the peak $l = 0$ (red, second from top) given by I_0 from Eq. (4.3) with its corresponding approximation from Eq. (4.4) (black, dashed), the peak $l = 1$ (green, third from top) given by I_1 from Eq. (4.3) with its corresponding approximation from Eq. (4.5) (black, dotted), and contributions of the bulk given by I_b (orange, bottom) from Eq. (4.3). We also show contributions of the $l = 2$ peak for $q = 0.005, 0.008, 0.01$ given by I_2 (black circles, bottom right) and its approximation from Eq. (4.6) (gray, dotted) which we extrapolate down to $q = 10^{-5}$. The three points for I_2 are approximately the same size as I_b , but we expect this to be a coincidence and expect I_2 to be larger than I_b for smaller q , as shown by the extrapolation.

We now fit the total integral,

$$I_S(q) = \int_0^{(\kappa_1 + \kappa_2)/2} d\kappa \kappa^2 F_\kappa \approx 0.38 \times q^{1/2}. \quad (4.7)$$

which is shown as the top lines in Fig. 10. This approximation in Eq. (4.7) for I can now be used in the number density formulas from Eqs. (2.38) and (2.39) to approximate the number density of produced fermions as a function of the coupling parameter q for small values of q ($\lesssim 0.01$). As we have only considered the peaks $l = 0$ and $l = 1$ for the approximation in Eq. (4.7), we can estimate the error in the approximation by comparing I_2 from Eq. (4.6) and I_S from Eq. (4.7). We find $I_2/I_S \approx 0.03$, which implies that dropping the $l = 2$ peak gives an error of around 3%. We observe from the approximations in Eqs. (4.4), (4.5), and (4.6) that the proportionality constant decreases significantly with increasing l from 0 to 2. We expect that this pattern continues so that higher $l(> 2)$ peaks contribute even less to the total number density; we therefore neglect them in our analysis.

B. Large q

For larger values of the coupling parameter ($q \gtrsim 10$), we first discuss the boundary of the bulk region and how it could be used to approximate I from Eq. (4.1). In [7], it was

shown that the boundary of the bulk region of F_κ is proportional to $q^{1/4}$, which was shown to be a consequence of the nonadiabaticity condition for particle production; $|\dot{\Omega}_\kappa| \gtrsim \Omega_\kappa^2$. The nonadiabaticity condition arises whenever Ω_κ changes very rapidly and is needed for nonresonant particle production, resulting in the large F_κ in the bulk region of the momentum distribution. For our case, we can first write the nonadiabaticity condition for Ω_κ from Eq. (2.29) as follows:

$$\left| \frac{d\Omega_\kappa}{d\tau} \right| \gtrsim \Omega_\kappa^2 \Rightarrow \kappa \lesssim \sqrt{(qf\dot{f})^{2/3} - qf^2}, \quad (4.8)$$

where we have chosen a time when f and $\dot{f}$ are both positive. Particle production for large q occurs whenever the effective fermion mass $\sqrt{q}f(\tau)$ crosses zero.⁵ We can see this in Fig. 1, where for large q there is a sudden change in $n_\kappa(\tau)$ in the vicinity of times where $f(\tau) = 0$. This indicates that the nonadiabaticity condition will be valid in a narrow region of τ around the points where $f(\tau) = 0$. Similarly to the procedure in [7], we can find the maximum κ that results in particle production by first setting κ equal to the right-hand side of the inequality on the right of Eq. (4.8). Then, if f_* is the value of f that corresponds to the maximum value of κ , it can be found by maximizing κ with respect to f as follows:

$$\begin{aligned} \left. \frac{d\kappa}{df} \right|_{f=f_*} &= \left. \frac{d((qf\dot{f})^{2/3} - qf^2)^{1/2}}{df} \right|_{f=f_*} = 0 \\ \Rightarrow f_* &= \frac{q^{-1/4} \dot{f}_*^{1/2}}{3^{3/4}}, \end{aligned} \quad (4.9)$$

where $\dot{f}_*$ is the slope when $f(\tau) = 0$ (and at a time when it is positive). As the τ range for which the nonadiabaticity condition is satisfied is very narrow, we approximate $\dot{f}_*$ as a constant equal to the slope of f when it is zero. For $f(\tau)$ from Eq. (2.6), $\dot{f}_* = 1/\sqrt{2}$. We use f_* from Eq. (4.9) for the maximum κ given by the inequality on the right of Eq. (4.8) and see for the corresponding value κ_* ,

$$\begin{aligned} qf_*\dot{f}_* &= (\kappa_*^2 + qf_*^2)^{3/2} \\ \Rightarrow \kappa_* &= \left(\frac{2q^{1/2}\dot{f}_*}{3\sqrt{3}} \right)^{1/2} = \left(\frac{2}{27} \right)^{1/4} q^{1/4} \approx 0.52q^{1/4}. \end{aligned} \quad (4.10)$$

This $\kappa_* \approx 0.52q^{1/4}$ implies that the boundary of the bulk is $\propto q^{1/4}$. If the major contribution to the fermion number density comes from the bulk region, we thus expect $I(q) \propto q^{3/4}$; we will show numerically that this is indeed the case. A similar approximation was used in [5,26]. For larger κ values, particle production occurs via resonant

⁵As before, we are assuming the bare mass of the fermions is negligible.

production as in the small- q case. Estimating the boundary of the bulk is difficult for large q because for many q values, the lowest peaks are partially merged into the bulk. The points in the contours of F_κ in the (q, κ) plane where the lowest resonance peaks are separated from the bulk correspond to the dips in Fig. 11 because those are where the bulk is narrowest. Fitting the points at these dips indeed yields a $q^{1/4}$ power law as expected from the nonadiabaticity condition, as we show for the $F_\kappa = 0.5$ contour in Fig. 11.

For many larger values q , the bulk region merges with the lowest resonance peak denoted l_0 , given by the function of q in Eq. (3.7). This merging can be seen in Fig. 4 as the lowest peak moves down in κ with increasing q . The merging can also be observed in slices of fixed q in Fig. 11 as for a fixed q , we observe that the resonance peak gets wider as it gets closer to the bulk region and ultimately assimilates into the bulk. Hence, rather than trying to find approximations for the bulk and the first peak of the integral I from Eq. (4.1) (as done for small q), we treat the bulk and first peak as a single component.

For large q all resonance peaks with $l \geq l_0$ exist, but our numerical analysis indicates that the cumulative sum is well-approximated by the bulk and the first six resonance peaks. Therefore, we truncate our computation of the total integral after the sixth peak. As before, the positions of the resonances are computed using Eq. (3.5). Then, similar to Eq. (4.2), the total integral for the large q values is evaluated as

$$I_L(q) = \int_0^{(\kappa_{l_0+5} + \kappa_{l_0+6})/2} d\kappa \kappa^2 F_\kappa. \quad (4.11)$$

We find numerical estimations for different contributions to I_L by first assuming that the bulk region and the l_0 peak form one component and then take into account the other resonance peaks (which are reasonably well-separated from the bulk). We break up I_L into six components with the different components representing contributions from different parts as follows:

$$\begin{aligned} I_L(q) = & \underbrace{\int_0^{(\kappa_{l_0} + \kappa_{l_0+1})/2} d\kappa \kappa^2 F_\kappa}_{\text{Bulk and } l_0 \text{ Peak, } I_{b+l_0}} + \underbrace{\int_{(\kappa_{l_0} + \kappa_{l_0+1})/2}^{(\kappa_{l_0+1} + \kappa_{l_0+2})/2} d\kappa \kappa^2 F_\kappa}_{l_0+1 \text{ Peak, } I_{l_0+1}} \\ & + \underbrace{\int_{(\kappa_{l_0+1} + \kappa_{l_0+2})/2}^{(\kappa_{l_0+2} + \kappa_{l_0+3})/2} d\kappa \kappa^2 F_\kappa}_{l_0+2 \text{ Peak, } I_{l_0+2}} + \underbrace{\int_{(\kappa_{l_0+2} + \kappa_{l_0+3})/2}^{(\kappa_{l_0+3} + \kappa_{l_0+4})/2} d\kappa \kappa^2 F_\kappa}_{l_0+3 \text{ Peak, } I_{l_0+3}} \\ & + \underbrace{\int_{(\kappa_{l_0+3} + \kappa_{l_0+4})/2}^{(\kappa_{l_0+4} + \kappa_{l_0+5})/2} d\kappa \kappa^2 F_\kappa}_{l_0+4 \text{ Peak, } I_{l_0+4}} + \underbrace{\int_{(\kappa_{l_0+4} + \kappa_{l_0+5})/2}^{(\kappa_{l_0+5} + \kappa_{l_0+6})/2} d\kappa \kappa^2 F_\kappa}_{l_0+5 \text{ Peak, } I_{l_0+5}}. \end{aligned} \quad (4.12)$$

These integrals, normalized to the total, are shown in Fig. 12 for various q values in the range $10 < q < 3 \times 10^4$.

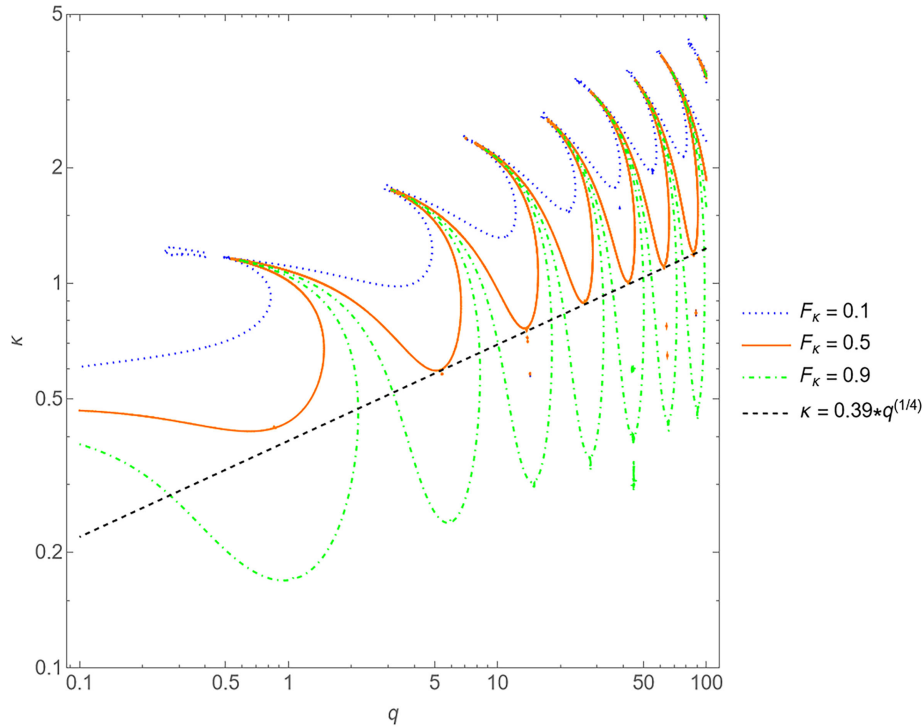


FIG. 11. F_κ contours in the (q, κ) plane, with both axes scaled logarithmically. $F_\kappa = 0.5$ contour follows a pattern which matches the line representing $\kappa = 0.39 \times q^{1/4}$ for $q \gtrsim 1$.

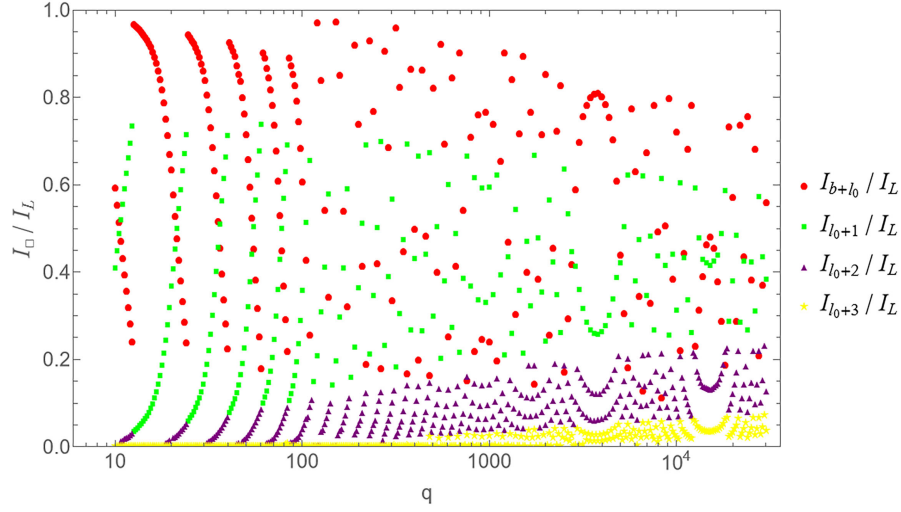


FIG. 12. Contributions of different components as fractions of the total integral I_L defined in Eq. (4.11). Shown are contributions of a component with the bulk region and peak l_0 taken together I_{b+l_0} (red circles), as well as I_{l_0+1} (green squares), I_{l_0+2} (purple triangles), I_{l_0+3} (yellow stars), with the integrals defined in Eq. (4.12).

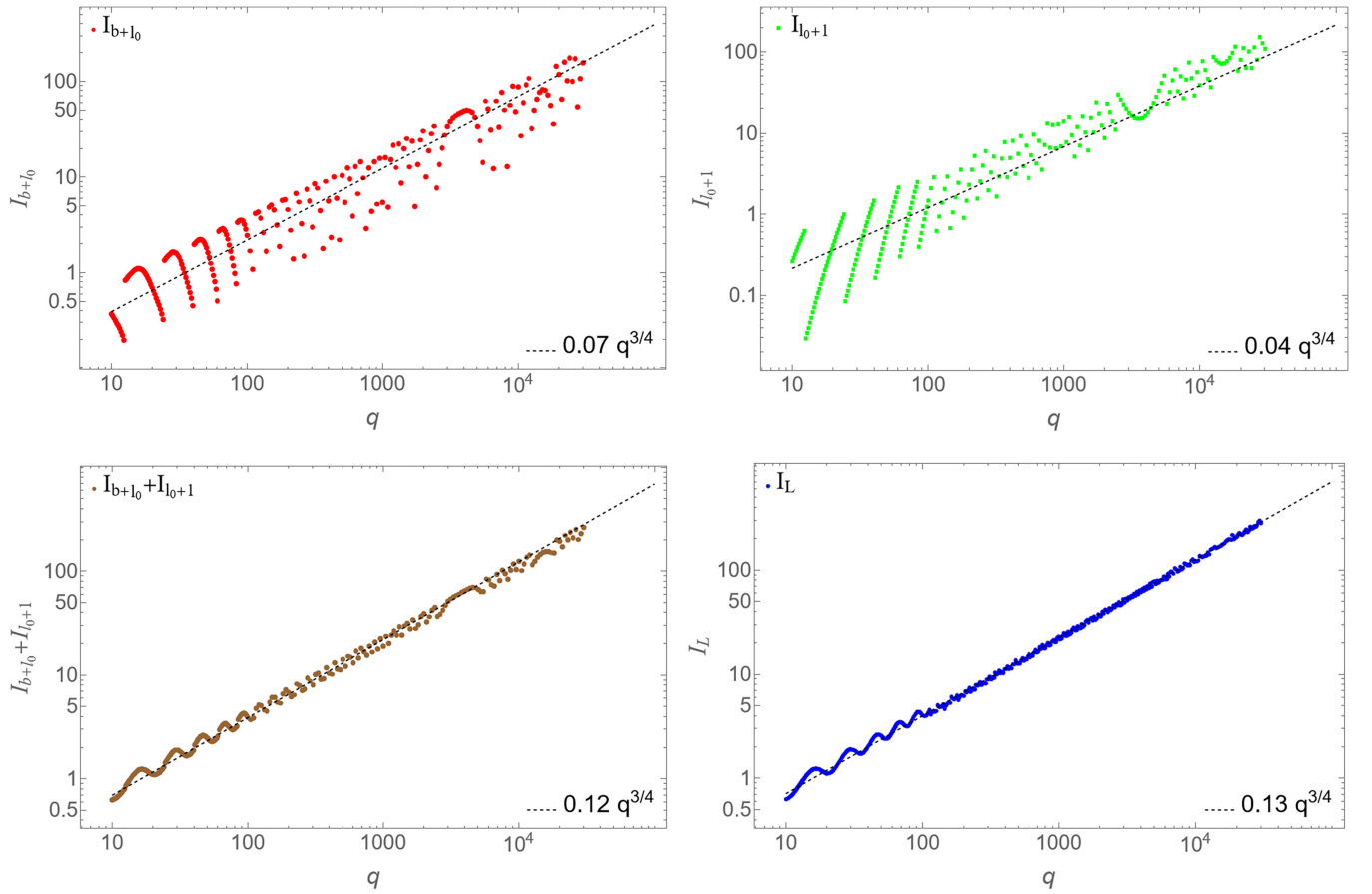


FIG. 13. Different integrals as a function of q for $10 < q < 3 \times 10^4$. *Top left*: The integral I_{b+l_0} [see Eq. (4.12)] over the bulk region and peak l_0 as well as the approximation from Eq. (4.13). *Top right*: I_{l_0+1} over the second peak and the approximation from Eq. (4.14). *Bottom left*: $I_{b+l_0} + I_{l_0+1}$ over the bulk, first peak, and second peak, as well as the fit from Eq. (4.15). *Bottom right*: Total integral I_L [see Eq. (4.11)] with the approximation from Eq. (4.16).

We observe that although the bulk region and the l_0 peak together (red circles) account for the highest contribution for many values of q , the contribution from the $l_0 + 1$ peak (green squares) surpasses that of the bulk + l_0 component for some values of q , implying that it is not only the bulk region that has a relevant contribution. Although it is subdominant, the $l_0 + 2$ peak (purple triangles) also has a significant contribution, as much as 20% of the total integral, while the $l_0 + 3$ and higher peaks make up a smaller contribution.

Then, similar to the small q case, we fit the numerical values of the bulk + l_0 and $l_0 + 1$ integrals as a function of q to a power law. The points are shown in the top two plots of Fig. 13. Although the values of the integrals oscillate around the fit as q varies, both contributions are reasonably approximated by a $q^{3/4}$ power law, where the fit is also shown in the top two plots of Fig. 13. The fits are given by

$$I_{b+l_0}(q) = \int_0^{(\kappa_{l_0}+\kappa_{l_0+1})/2} d\kappa \kappa^2 F_\kappa \approx 0.069 \times q^{3/4}, \quad (4.13)$$

$$I_{l_0+1}(q) = \int_{(\kappa_{l_0}+\kappa_{l_0+1})/2}^{(\kappa_{l_0+1}+\kappa_{l_0+2})/2} d\kappa \kappa^2 F_\kappa \approx 0.038 \times q^{3/4}. \quad (4.14)$$

When we add up the contributions from the bulk + l_0 component and the $l_0 + 1$ peak, we find that the scatter diminishes significantly and the data points more clearly follow a $q^{3/4}$ power law, which can be seen in the bottom left plot of Fig. 13. Hence, an overall approximation for the total contribution from the bulk region, l_0 peak and the $l_0 + 1$ peak is found from the best fit to the data points as

$$I_{b+l_0} + I_{l_0+1} = \int_0^{(\kappa_{l_0+1}+\kappa_{l_0+2})/2} d\kappa \kappa^2 F_\kappa \approx 0.12 \times q^{3/4}. \quad (4.15)$$

When we add all the contributions in I_L from Eq. (4.11), we find that the scatter in the calculations reduces even more and the total integral clearly follows a $q^{3/4}$ power law, which is represented in the bottom right plot of Fig. 13. The proportionality constant is found from the best-fit to the data points and hence, the relevant approximation can be written as

$$I_L(q) = \int_0^{(\kappa_{l_0+5}+\kappa_{l_0+6})/2} d\kappa \kappa^2 F_\kappa \approx 0.13 \times q^{3/4}. \quad (4.16)$$

We see that including peaks with $l > l_0 + 1$ changes the total integral by less than 10% which shows that the contributions to I from increasing l are rapidly decreasing, implying I_L is a good approximation for I . This fit in Eq. (4.16) for I is can be used in the number density formulae from Eqs. (2.38) and (2.39), to approximate the total number density of produced fermions as a function of the coupling parameter q for large values of $q \gtrsim 10$.

C. Intermediate q

For the intermediate range of q values between 10^{-2} and 10, we use the same integral limits to evaluate the total integral and the individual contributions as in Eqs. (4.11) and (4.12). Neither the $q^{1/2}$ approximation from the small q case nor the $q^{3/4}$ approximation from the large q case gives a good approximation for the total integral in this intermediate range of q , as can be seen in Fig. 14. Although there is a major deviation from the $q^{1/2}$ power law for $q \gtrsim 0.01$, we find that for values of q greater than

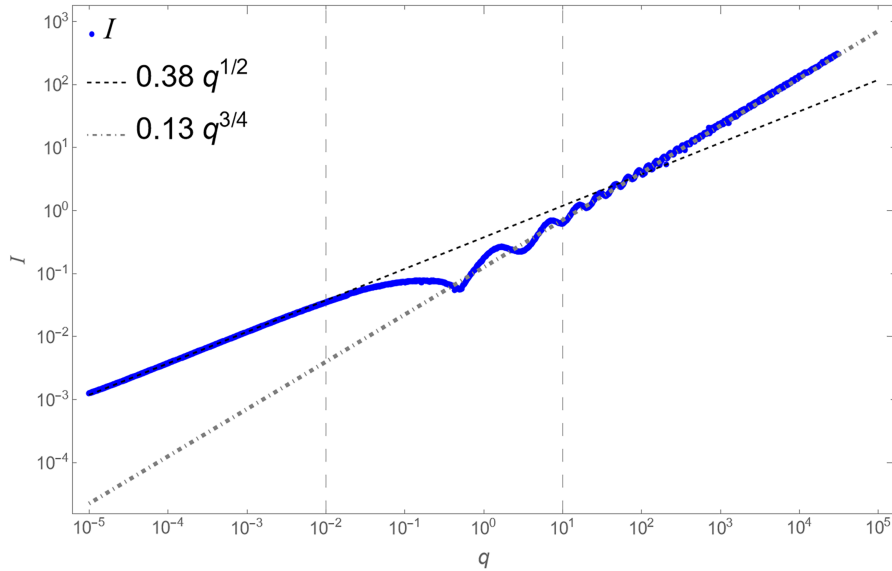


FIG. 14. Total integral $I(q)$ for $10^{-5} \leq q \leq 3.0 \times 10^4$, with the approximations for the small q and large q cases described by Eqs. (4.7) and (4.16). The vertical dashed lines divide the small q , intermediate q and large q regions.

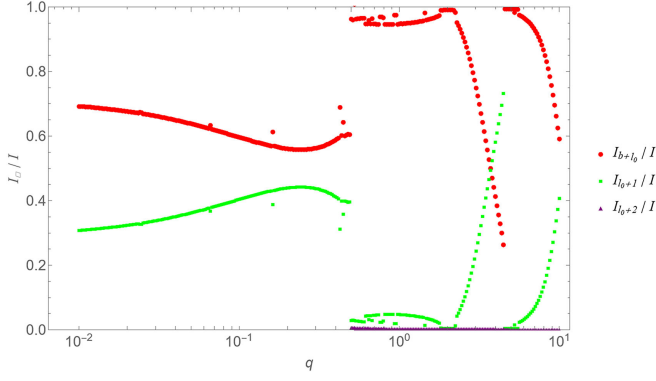


FIG. 15. Contributions of different components for $0.01 \leq q \leq 10$ as fractions of the total integral. For $q < q_0(0) = 0.5$, the normalization is I_S from Eq. (4.2), while for $q \geq q_0(0)$, the normalization is I_L from Eq. (4.11). Shown are contributions of the bulk and l_0 peak, I_{b+l_0} (red circles), the $l_0 + 1$ peak I_{l_0+1} (green squares), and the $l_0 + 2$ peak I_{l_0+2} (purple triangles, for $q > q_0(0)$ only).

$q_0(0) = 0.5$, an oscillating pattern starts to form around the $q^{3/4}$ power-law fit which ultimately converges into the expression from Eq. (4.16) at larger q . The $q^{3/4}$ power-law fit seems to be decent for $1 < q < 10$ but with a maximum relative error $\sim 50\%$.

We also plot fractions of the total integral contributed by the first three components, and this is shown in Fig. 15. Similar to the large q case, the major contributions come from the bulk + l_0 component and the $l_0 + 1$ peak, which can be seen in a more pronounced manner for this intermediate q range as the contribution of the $l_0 + 2$ peak remains very small. For q less than $q_0(0) = 0.5$, the bulk + l_0 component has the dominant contribution over that of the $l_0 + 1$ peak. Then for values of q just above each transition point $q_0(l)$ [from Eq. (3.8), with $l = 1, 2, 3, \dots$], the largest contribution again comes from the bulk + l_0 component. However, this fraction decreases with increasing q and the contribution from the $l_0 + 1$ peak dominates at values of q just below the next transition point $q_0(l + 1)$. The jumps in Fig. 15 at $q = 0.5$ and $q = 4.5$ [corresponding to $q_0(0)$ and $q_0(1)$ from Eq. (3.8), respectively] are physical and are a consequence of the l value of the lowest resonance peak changing from 0 to 1 and from 1 to 2, respectively. We expect such jumps over the whole range of q whenever the l value of the lowest resonance peak changes and we observe this pattern in Fig. 12 as well.

V. DISCUSSION AND CONCLUSIONS

We considered nonperturbative production of fermions that are produced out of thermal equilibrium during coherent oscillations of a scalar field. We study the production of such particles through the mechanism of fermionic preheating for $\lambda\phi^4$ inflation, under the assumption that the bare mass of the fermions is negligible. Production is described by a differential equation with a time-dependent natural frequency

$\Omega_\kappa(\tau) \equiv \sqrt{\kappa^2 + qf^2(\tau)}$, which represents an effective relativistic energy, where κ is the dimensionless comoving momentum of the fermions, $f(\tau)$ describes inflaton oscillations, and $q = h^2/\lambda$ is a coupling parameter with h being the Yukawa coupling between inflaton and fermions, and λ being the inflaton quartic coupling. The momentum distribution of the fermions produced is described by an envelope function F_κ , which contains a highly occupied ‘bulk region’ around $\kappa = 0$ and resonance peaks for higher κ .

It was observed in [5] that for $q \ll 1$, resonance peaks of F_κ are present when κ is an odd multiple of π/T (T being the time period of inflaton oscillations). We provide a physics explanation for this relation and generalize the idea to find the momentum value of resonance peaks for any q . We use the time average of Ω_κ to find a simple semianalytic relation that predicts the κ values corresponding to resonance peaks in the momentum distribution of the fermions produced for any q without the need to numerically solve a differential equation. This is described as: $\bar{\Omega}_\kappa(q) = \frac{\pi}{T}(2l + 1)$, $l = 0, 1, 2, 3, \dots$, which we use to find and label κ corresponding to the resonance peak l as κ_l for various q . However, $l = 0$ does not correspond to the first resonance peak for all q values. The l value representing the first resonance peak is called l_0 and is given by Eq. (3.7) as a function of q . Because of the physics behind it, we expect this semianalytic relation describing the positions of resonance peaks in the momentum spectrum to be true for any symmetric inflaton potential.

As shown in Eq. (2.36), we use F_κ to describe the number density per mode of the particles produced ($\bar{n}_\kappa$), which implies that for the total number density (n):

$$n \propto N_f \int_0^\infty d\kappa \kappa^2 \langle \bar{n}_\kappa \rangle = \frac{N_f}{2} \int_0^\infty d\kappa \kappa^2 F_\kappa \equiv \frac{N_f}{2} I, \quad (5.1)$$

where N_f is the number of chiral fermion species. Hence, we use F_κ to estimate the total number density of fermions through the approximations we find for the integral I .

We observe that the momentum spectrum of the fermions produced is not completely degenerate. We find that for small values of the coupling parameter $q \lesssim 0.01$, the main contributions ($\sim 95\%$) to I are actually from momentum shells at κ corresponding to the resonance peaks labeled $l = 0$ and $l = 1$ (at $\kappa_0 = \pi/T$ and $\kappa_1 = 3\pi/T$), instead of the bulk region (an approximately filled sphere around $\kappa = 0$). We can see this from the plots of the contributions of different components of F_κ in Figs. 9 and 10.

For larger q , we evaluate and plot the contributions to I of different components of F_κ in Figs. 12 and 13 using the values of κ_l for different q , and find that $\sim 80\text{--}90\%$ of I is taken into account with κ between 0 and the peak $l_0 + 1$ [where Eq. (3.7) gives l_0 for any q]. Overall, we observe contributions from an approximately half-filled sphere with subdominant contributions from momentum shells at resonant values of momentum.

Thereafter, we extract novel analytic approximations for the contributions of different components of F_κ to the number density through I . For $q \lesssim 0.01$, we find that the major contributions, which are from the resonance peaks, follow $q^{1/2}$ power-law approximations such as $I_0(q) \approx 0.26 \times q^{1/2}$ for the $l = 0$ peak, as shown in Fig. 10. We therefore find a $q^{1/2}$ power-law approximation for I for small q , given by $I_S(q) \approx 0.38 \times q^{1/2}$ from Eq. (4.7). For larger q ($\gtrsim 10$), we find that the contributions of different components follow $q^{3/4}$ power-law approximations, as shown in Fig. 13. We therefore can approximate the total integral at large q by $I_L(q) \approx 0.13 \times q^{3/4}$ from Eq. (4.16).

The fermions produced by the mechanism described in this work could be a promising dark matter candidate. In Ref. [19], it was found that for a single flavor of fully-degenerate Majorana fermions ψ to make up all dark matter, their mass m_ψ must be greater than about 2 keV as a consequence of the effects on structure formation when their Fermi momentum p_F remains relativistic, i.e., $p_F \gtrsim m_\psi$, to a sufficiently late redshift. We can adapt this limit on m_ψ for our case of Dirac fermions (equivalent to $N_f = 2$ Majorana flavors) produced with the nonthermal momentum distributions predicted by our analysis, assuming that these fermions make up all the dark matter.

We first consider small $q \lesssim 0.01$, for which the fermion momentum spectrum is dominated by the first two resonance peaks. To demonstrate the method, we start by considering only the $l = 0$ resonance peak and assuming that all the fermions produced are in a single narrow Fermi shell with momentum $\kappa_0 = \pi/T$. These fermions become relativistic when $m_\psi \simeq \kappa_0$. The bound on m_ψ can then be found by rescaling the fermion number density in Ref. [19] by the ratio,

$$\frac{(N_f/2) \times I_0(q)}{\int_0^{p_F=\kappa_0} p^2 dp} = \frac{3N_f I_0(q)}{2\kappa_0^3}, \quad (5.2)$$

where $I_0(q)$ is the integral of the $l = 0$ resonance momentum shell from Eq. (4.4) (note that this first-pass approximation ignores 30–35% of the fermion number density). Using the fact that the energy density of dark matter $\rho_\psi = nm_\psi$ remains fixed, we can then rescale the limit on m_ψ from Ref. [19] as follows:

$$m_\psi > 2 \text{ keV} \left(\frac{2\kappa_0^3}{3N_f I_0(q)} \right)^{1/4} \approx 2 \text{ keV} \left(\frac{0.56}{q^{1/8}} \right). \quad (5.3)$$

For $q = 10^{-5}$, we find that the limit is strengthened to $m_\psi \gtrsim 5 \text{ keV}$, but for $q = 0.01$, we still find $m_\psi \gtrsim 2 \text{ keV}$. The constraint on m_ψ is not improved for $q = 0.01$, which we attribute to the fact that the peak $l = 0$ for $q = 0.01$ is quite broad, as can be seen in Fig. 3, so that it again approximates a (half)-filled Fermi sphere.

For a better approximation at small q , we can instead consider the contributions from the first two momentum shells, at $\kappa_0 = \pi/T$ and $\kappa_1 = 3\pi/T$, containing $\sim 65\%$ and $\sim 35\%$ of the total fermion number density, respectively. Assuming that having only $\sim 35\%$ of the dark matter be relativistic is sufficient to trigger the constraints from structure formation, we can follow the same procedure, now taking into account the full number density and setting $m_\psi \simeq \kappa_1$. In this case we find

$$m_\psi > 2 \text{ keV} \left(\frac{2\kappa_1^3}{3N_f I_S(q)} \right)^{1/4} \approx 2 \text{ keV} \left(\frac{1.2}{q^{1/8}} \right). \quad (5.4)$$

The limit on m_ψ for the particles to make up all dark matter is then strengthened for $q = 10^{-5}$ ($q = 0.01$) to $m_\psi \gtrsim 10 \text{ keV}$ ($m_\psi \gtrsim 4 \text{ keV}$). We expect the actual bound to be between the ones from (5.3) and (5.4).

For larger q values, the momentum spectrum can be reasonably approximated as a half-filled Fermi sphere. Taking into account the extra factor of $N_f = 2$ for Dirac fermions and assuming that these fermions make up all the dark matter, the bound $m_\psi > 2 \text{ keV}$ from Ref. [19] remains unchanged.

Although our results are for $\lambda\phi^4$ inflation, they are easily generalizable to $m^2\phi^2$ inflation [22], hybrid inflation [37], or coherent axion oscillations [38,39]. We generically expect the total number density of produced fermions for any coherently oscillating scalar field with a symmetric potential to follow the same power laws that we found for $\lambda\phi^4$ inflation; proportional to $q^{1/2}$ for small values of q , and proportional to $q^{3/4}$ for large values of q . We check $I_S(q)$ from Eq. (4.2) for $m^2\phi^2$ inflation and obtain a nearly identical fit as the one for $\lambda\phi^4$ inflation; $I_S(q) \approx 0.39 \times q^{1/2}$.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: DETAILS OF THE CALCULATION OF H_D IN TERMS OF $\hat{a}_{k,s}$, $\hat{b}_{k,s}$ OPERATORS

Here, we write the steps involved in representing the Hamiltonian $H_D(\eta)$ in terms of the creation ($\hat{a}_{k,s}^\dagger$, $\hat{b}_{k,s}^\dagger$) and annihilation ($\hat{a}_{k,s}$, $\hat{b}_{k,s}$) operators. $H_D(\eta_0)$ is diagonal in these operators and the operators annihilate the ground state of the Hamiltonian. However, $H_D(\eta)$ at $\eta > \eta_0$ is not diagonal in these operators and that is the expression we are looking for. First, we write the equations that describe all the relevant terms from Eqs. (2.12) and (2.16). The orthonormal basis spinors $R_\pm$ and $\bar{R}_\pm$ are defined through the eigenequations,

$$\begin{aligned} \gamma^0 R_\pm(\mathbf{k}) &= +R_\pm(\mathbf{k}), & \gamma^0 \bar{R}_\pm(\mathbf{k}) &= -\bar{R}_\pm(\mathbf{k}), \\ (\hat{\mathbf{k}} \cdot \boldsymbol{\Sigma}) R_\pm(\mathbf{k}) &= \pm R_\pm(\mathbf{k}), & (\hat{\mathbf{k}} \cdot \boldsymbol{\Sigma}) \bar{R}_\pm(\mathbf{k}) &= \pm \bar{R}_\pm(\mathbf{k}). \end{aligned} \quad (\text{A1})$$

For simplicity, the comoving momentum ($\mathbf{k}$) of the particles can be assumed along the z -direction i.e., $\mathbf{k} = (0, 0, k)$, and we can use $\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$ [40], where $\boldsymbol{\sigma}$ are the Pauli matrices. Using this assumption and the eigenequations above, we can show the following relations:

$$R_\pm(\pm\mathbf{k}) = R_\mp(\mp\mathbf{k}), \quad \bar{R}_\pm(\pm\mathbf{k}) = \bar{R}_\mp(\mp\mathbf{k}). \quad (\text{A2})$$

The ansatz used for Ψ in Eq. (2.13) can then be used to define $u_{k,s}(\eta)$, with ' here indicating derivatives with respect to η :

$$\begin{aligned} u_{k,\pm}(\eta) e^{+i\mathbf{k}\cdot\mathbf{x}} &= [i\gamma^0 \partial_\eta + i\gamma^j \partial_j + (h\varphi(\eta) + m_\psi a(\eta))] X_k(\eta) R_\pm(\mathbf{k}) e^{+i\mathbf{k}\cdot\mathbf{x}}, \\ \Rightarrow u_{k,\pm}(\eta) &= [iX'_k(\eta) - (\gamma \cdot \mathbf{k} - (h\varphi(\eta) + m_\psi a(\eta)) X_k(\eta))] R_\pm, \end{aligned} \quad (\text{A3})$$

where $a(\eta)$ is the scale factor. The charge conjugate $v_{k,s}(\eta)$ is defined through the charge conjugation operator $C = i\gamma_0\gamma_2$ and $\bar{u}_{k,\pm} = u_{k,\pm}^\dagger \gamma_0$ as

$$\begin{aligned} v_{k,\pm}(\eta) &= C \bar{u}_{k,\pm}^T(\eta), \\ \Rightarrow v_{k,\pm}(\eta) &= [-iX'_k(\eta) + (\gamma \cdot \mathbf{k} + (h\varphi(\eta) + m_\psi a(\eta)) X_k(\eta))] X_k^*(\eta) \bar{R}_\pm. \end{aligned} \quad (\text{A4})$$

To evaluate the Hamiltonian H_D in terms of the fermion creation and annihilation operators, we first use Eqs. (A3) and (A4) in Eq. (2.16) and use the resultant expression in Eq. (2.17). The definition of the δ -function can then be used to first remove the d^3x -integral,

$$\int e^{i(\mathbf{k}-\mathbf{K})\cdot\mathbf{x}} d^3x = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{K}). \quad (\text{A5})$$

Thereafter, eigenequations from Eq. (A1) can be used with Eq. (A2) to substitute values for terms with $R_\pm$ and $\bar{R}_\pm$ and this ultimately yields Eq. (2.18).

APPENDIX B: DERIVING RELATION BETWEEN E , F AND Ω_k

We will now show how the relation between E , F , and Ω_k from Eq. (2.21) comes about. To do so, we first assume $f(\eta) = g(\eta) \equiv X_k(\eta)$ for Eq. (2.20), and we can then write:

$$|g'(\eta)|^2 + \Omega_k^2(\eta) |g(\eta)|^2 + 2(m_\psi a(\eta) + h\varphi(\eta)) \text{Im}(g(\eta) g'^*(\eta)) = C. \quad (\text{B1})$$

Equations (2.14) and (2.19) can also be written as

$$\begin{aligned} g''(\eta) + [k^2 + (h\varphi(\eta) + m_\psi a(\eta))^2 - ih\varphi'(\eta)] g(\eta) &= 0, \\ E(\eta) &= [h\varphi(\eta) + m_\psi a(\eta)] [|g'(\eta)|^2 + \Omega_k^2(\eta) |g(\eta)|^2] + 2\Omega_k^2(\eta) \text{Im}(g(\eta) g'^*(\eta)), \\ F(\eta) &= k(g'^*(\eta))^2 + k\Omega_k^2(\eta) (g^*(\eta))^2. \end{aligned} \quad (\text{B2})$$

Using these, we can show that

$$\begin{aligned}
E(\eta)^2 &= [h\varphi(\eta) + m_\psi a(\eta)]^2 [|g'(\eta)|^2 + \Omega_k^2(\eta)|g(\eta)|^2]^2 + 4\Omega_k^4(\eta) \text{Im}(g(\eta)g'^*(\eta))^2 \\
&\quad + 4\Omega_k^2(\eta) \text{Im}(g(\eta)g'^*(\eta)) [h\varphi(\eta) + m_\psi a(\eta)] [|g'(\eta)|^2 + \Omega_k^2(\eta)|g(\eta)|^2], \\
|F(\eta)|^2 &= k^2 [|g'(\eta)|^4 + \Omega_k^4(\eta)|g(\eta)|^4] + k^2 \Omega_k^2 [2|g(\eta)|^2 |g'(\eta)|^2 - 4 \text{Im}(g(\eta)g'^*(\eta))^2].
\end{aligned} \tag{B3}$$

Then, we can add these up and use Eq. (B1) to show

$$E(\eta)^2 + |F(\eta)|^2 = \Omega_k^2 [|f'(\eta)|^2 + \Omega_k^2(\eta)|f(\eta)|^2] + 2(m_\psi a + h\varphi) \text{Im}(f(\eta)f'^*(\eta))^2 = \Omega_k^2 C^2. \tag{B4}$$

APPENDIX C: DERIVATION OF $|\beta|^2$

We will show how we can calculate the fermion (or antifermion) number density per mode $n_k(\eta) = |\beta|^2$. There are two equivalent ways to show that this relation holds (in either the Schrödinger or the Heisenberg picture). One is to consider time-dependent vacuum and use $|0_\eta\rangle$ to represent the ground state of the Hamiltonian at time η , which is annihilated by new operators $\hat{c}_{k,s}, \hat{d}_{k,s}$ defined in Eq. (2.24). Then $n_k(\eta)$ is given by the expectation value of the time-independent number density operator $\hat{a}_{k,s}^\dagger \hat{a}_{k,s}$ (or $\hat{b}_{k,s}^\dagger \hat{b}_{k,s}$) as $n_k(\eta) = \langle 0_\eta | \hat{a}_{k,s}^\dagger \hat{a}_{k,s} | 0_\eta \rangle = \langle 0_\eta | \hat{b}_{k,s}^\dagger \hat{b}_{k,s} | 0_\eta \rangle$. Alternatively, we consider the time-independent vacuum $|0\rangle$ that is annihilated by $\hat{a}_{k,s}, \hat{b}_{k,s}$, to represent the ground state of the Hamiltonian at time η_0 . Then $n_k(\eta)$ is given by the expectation value of the number density operator at time η , $\hat{c}_{k,s}^\dagger \hat{c}_{k,s}$ (or $\hat{d}_{k,s}^\dagger \hat{d}_{k,s}$) as $n_k(\eta) = \langle 0 | \hat{c}_{k,s}^\dagger \hat{c}_{k,s} | 0 \rangle = \langle 0 | \hat{d}_{k,s}^\dagger \hat{d}_{k,s} | 0 \rangle$, where the time dependence is carried in the transformation from the $\hat{a}, \hat{b}$ operators to the $\hat{c}, \hat{d}$ operators. We can use the first option, and use the Bogoliubov transformations described by Eqs. (2.23) and (2.24), to see that

$$\begin{aligned}
n_k(\eta) &= \langle 0_\eta | \hat{a}_{k,s}^\dagger \hat{a}_{k,s} | 0_\eta \rangle \\
&= \langle 0_\eta | (\alpha \hat{c}_{k,s}^\dagger - \beta^* \hat{d}_{k,s}) (\alpha^* \hat{c}_{k,s} - \beta \hat{d}_{k,s}^\dagger) | 0_\eta \rangle = |\beta|^2.
\end{aligned} \tag{C1}$$

As the Bogoliubov transformation diagonalizes the Hamiltonian $H_D(\eta)$, we can set the nondiagonal terms of Eq. (2.25) to 0 and this implies

$$F\alpha^2 - F^*\beta^2 - 2E\alpha\beta = 0. \tag{C2}$$

Due to their property in Eq. (2.24), α and β can then be written as

$$\alpha = \cos \theta e^{i\phi_\alpha}, \quad \beta = \sin \theta e^{i\phi_\beta}. \tag{C3}$$

Substituting these into Eq. (C2) gives (using $\phi = \phi_\alpha - \phi_\beta$)

$$F \cos^2 \theta e^{i\phi} - F^* \sin^2 \theta e^{-i\phi} = 2E \cos \theta \sin \theta. \tag{C4}$$

As the right-hand side of this equation is purely real, the left-hand side of the equation is its own conjugate and we can write

$$F \cos^2 \theta e^{i\phi} - F^* \sin^2 \theta e^{-i\phi} = F^* \cos^2 \theta e^{-i\phi} - F \sin^2 \theta e^{i\phi} \Rightarrow e^{2i\phi} = F^*/F. \tag{C5}$$

Equation (C4) can then be solved as a quadratic equation in $\sin^2 \theta$, and after making the substitution from Eq. (C5), the following equation is obtained:

$$\sin^2 \theta = |\beta|^2 = \frac{(F^*/F)(4(|F|^2 + E^2) \pm 4E\sqrt{E^2 + |F|^2})}{8(F^*/F)(E^2 + |F|^2)} = \frac{1}{2} \pm \frac{E}{2\sqrt{E^2 + |F|^2}}. \tag{C6}$$

We need to choose the negative solution to ensure that $n_k(\eta_0) = |\beta(\eta_0)|^2 = 0$.

APPENDIX D: COMPARING DIFFERENT $n_k(\eta)$ EXPRESSIONS

We will show here that the expression for $n_k(\eta)$ that we have derived is equivalent to the number density equations present in [5,24,26]. The relevant equations from existing literature also assume the same initial conditions as in Eq. (2.22) that imply $C = 1$. Together with the additional assumption of light fermions i.e., $m_\psi \approx 0$, the number density from Eq. (2.26) can then be written as

$$n_k(\eta) = \frac{\Omega_k(\eta) - h\varphi(\eta)[|X'_k(\eta)|^2 + \Omega_k^2(\eta)|X_k(\eta)|^2] - 2\Omega_k^2(\eta) \text{Im}(X_k(\eta)X'^*_k(\eta))}{2\Omega_k(\eta)}. \tag{D1}$$

Then, Eq. (B1) can be used with $C = 1$ and $m_\psi = 0$ to write

$$\begin{aligned} |X'_k(\eta)|^2 + \Omega_k^2(\eta)|X_k(\eta)|^2 + 2h\varphi(\eta)\text{Im}(X_k(\eta)X'^*_k(\eta)) &= 1 \\ \Rightarrow |X'_k(\eta)|^2 + \Omega_k^2(\eta)|X_k(\eta)|^2 &= 1 - 2h\varphi(\eta)\text{Im}(X_k(\eta)X'^*_k(\eta)). \end{aligned} \quad (\text{D2})$$

Substituting this in Eq. (D1) gives

$$n_k(\eta) = \frac{1}{2} - \frac{k^2}{\Omega_k(\eta)}\text{Im}(X_k(\eta)X'^*_k(\eta)) - \frac{h\varphi(\eta)}{2\Omega_k(\eta)}. \quad (\text{D3})$$

This is of the same form as the number density equation written in [5,26].

Alternatively, Eq. (D2) can be used to write

$$\Omega_k(\eta) = \Omega_k(\eta) \times 1 = \Omega_k(\eta)[|X'_k(\eta)|^2 + \Omega_k^2(\eta)|X_k(\eta)|^2 + 2h\varphi(\eta)\text{Im}(X_k(\eta)X'^*_k(\eta))]. \quad (\text{D4})$$

Substituting this in Eq. (D1) gives

$$n_k(\eta) = \left(\frac{\Omega_k(\eta) - h\varphi(\eta)}{2\Omega_k(\eta)} \right) [|X'_k(\eta)|^2 + \Omega_k^2(\eta)|X_k(\eta)|^2 - 2\Omega_k\text{Im}(X_k(\eta)X'^*_k(\eta))]. \quad (\text{D5})$$

This is of the same form as the number density equation written in [24].

APPENDIX E: DERIVATION OF $\bar{n}_\kappa(\tau)$

We will now show the derivation for $\bar{n}_\kappa(\tau)$ from Eq. (2.33), following the procedure of [10], using the example and notation we have used for $\lambda\phi^4$ inflation. Although all equations are written for the particular case of $\lambda\phi^4$ inflation, we can write the relevant equations for any other model (such as $m^2\phi^2$ inflation) in this form as well, by using different transformations to write all the equations in dimensionless variables, and the same ideas will be valid.

In [10], expressions are derived which describe the production of particles in vacuum by a spatially uniform periodic electric field. These were then applied to a particular case of particle production from optical laser light. The general ideas of the derivation are also applicable for particle production from the inflaton, and we make the approximation that the inflaton oscillations start at some point of time and occur for N periods, during which particle production takes place and then stops.

To start the derivation, we first know that $f(\tau) = cn(\tau, 1/2)$ from Eq. (2.6) is periodic with a period T i.e., $f(\tau + T) = f(\tau)$. As mentioned below Eq. (2.6), we set $\tau_1 = 0$ so that $f(\tau)$ is at its maximum at $\tau = 0$. This implies $f(-\tau) = f(\tau) \Rightarrow \dot{f}(-\tau) = -\dot{f}(\tau)$. Thus, applying the transformation $[f(\tau) \rightarrow f^*(-\tau), X_\kappa(\tau) \rightarrow X^*_\kappa(-\tau)]$ to Eq. (2.28) yields the exact same differential equation, using the fact that $f(\tau)$ is real. Then, let us introduce two solutions, X^1_κ and X^2_κ , of the differential equation [Eq. (2.28)] with the solutions having the following initial conditions:

$$X^1_\kappa(0) = 1, \quad \dot{X}^1_\kappa(0) = 0, \quad X^2_\kappa(0) = 0, \quad \dot{X}^2_\kappa(0) = 1, \quad (\text{E1})$$

where $\dot{}$ implies derivatives with respect to τ . Then we can write the Wronskian for X^1_κ and X^2_κ , and use Abel's identity for Eq. (2.28) with the above initial conditions to get the following relation for all τ :

$$X^1_\kappa(\tau)\dot{X}^2_\kappa(\tau) - \dot{X}^1_\kappa(\tau)X^2_\kappa(\tau) = 1. \quad (\text{E2})$$

Due to the symmetry of the differential equation under a transformation involving a complex conjugate and $\tau \rightarrow -\tau$, we can say that one of the solutions X^1_κ or X^2_κ is even under the transformation, while the other is odd under the transformation. Then, let us assume the following:

$$\begin{aligned} X^{1*}_\kappa(-\tau) &= X^1_\kappa(\tau), & X^{2*}_\kappa(-\tau) &= -X^2_\kappa(\tau). \\ \therefore \dot{X}^{1*}_\kappa(-\tau) &= -\dot{X}^1_\kappa(\tau), & \dot{X}^{2*}_\kappa(-\tau) &= \dot{X}^2_\kappa(\tau). \end{aligned} \quad (\text{E3})$$

Due to the periodicity of $f(\tau)$, we can write solutions $X^1_\kappa(\tau + T)$ and $X^2_\kappa(\tau + T)$ as linear combinations of $X^1_\kappa(\tau)$ and $X^2_\kappa(\tau)$. This can be done in the following manner:

$$\begin{aligned} X^1_\kappa(\tau + T) &= a_1X^1_\kappa(\tau) + b_1X^2_\kappa(\tau), \\ X^2_\kappa(\tau + T) &= a_2X^1_\kappa(\tau) + b_2X^2_\kappa(\tau). \end{aligned} \quad (\text{E4})$$

The coefficients a_i, b_i ($i = 1, 2$) can be found by looking at the above equations at $\tau = 0$ and the time derivative of the equations at $\tau = 0$, and using Eq. (E1). After finding the coefficients, we can write

$$\begin{aligned} X^1_\kappa(\tau + T) &= X^1_\kappa(T)X^1_\kappa(\tau) + \dot{X}^1_\kappa(T)X^2_\kappa(\tau), \\ X^2_\kappa(\tau + T) &= X^2_\kappa(T)X^1_\kappa(\tau) + \dot{X}^2_\kappa(T)X^2_\kappa(\tau). \end{aligned} \quad (\text{E5})$$

Now, let us define a matrix $\mathbf{M}(\tau)$ using $X^1_\kappa(\tau)$ and $X^2_\kappa(\tau)$:

$$\mathbf{M}(\tau) = \begin{pmatrix} X_\kappa^1(\tau) & \dot{X}_\kappa^1(\tau) \\ X_\kappa^2(\tau) & \dot{X}_\kappa^2(\tau) \end{pmatrix}. \quad (\text{E6})$$

We have $\mathbf{M}(0) = \mathbf{I}_{2 \times 2}$, from Eq. (E1). Using the relation from Eq. (E2), we have $|\mathbf{M}(\tau)| = 1$ for all τ . Then, by Eq. (E5), we find

$$\begin{aligned} \mathbf{M}(\tau + T) &= \begin{pmatrix} X_\kappa^1(T) & \dot{X}_\kappa^1(T) \\ X_\kappa^2(T) & \dot{X}_\kappa^2(T) \end{pmatrix} \begin{pmatrix} X_\kappa^1(\tau) & \dot{X}_\kappa^1(\tau) \\ X_\kappa^2(\tau) & \dot{X}_\kappa^2(\tau) \end{pmatrix} \\ &= \mathbf{M}(T)\mathbf{M}(\tau). \end{aligned} \quad (\text{E7})$$

If we take $\tau = T$, Eq. (E7) implies $\mathbf{M}(2T) = (\mathbf{M}(T))^2$; taking $\tau = 2T$ implies $\mathbf{M}(3T) = (\mathbf{M}(T))^3$, and so on. In general, we can write

$$\mathbf{M}(nT) = (\mathbf{M}(T))^n, \quad (\text{E8})$$

where $n = 1, 2, 3, \dots$. Now, let us introduce an matrix operation (represented by $^\Delta$) which gives another 2×2 matrix with the complex conjugate of all terms and with switched diagonal terms. For example, applying it on $\mathbf{M}(\tau)$ implies

$$(\mathbf{M}(\tau))^\Delta = \begin{pmatrix} X_\kappa^1(\tau) & \dot{X}_\kappa^1(\tau) \\ X_\kappa^2(\tau) & \dot{X}_\kappa^2(\tau) \end{pmatrix}^\Delta = \begin{pmatrix} \dot{X}_\kappa^{2*}(\tau) & \dot{X}_\kappa^{1*}(\tau) \\ X_\kappa^{2*}(\tau) & X_\kappa^{1*}(\tau) \end{pmatrix}. \quad (\text{E9})$$

Similarly to taking complex conjugates, if we apply this operation on two arbitrary matrices W_1 and W_2 , then it can be proven that

$$(W_1 W_2)^\Delta = W_2^\Delta W_1^\Delta. \quad (\text{E10})$$

Applying $^\Delta$ twice on a matrix will yield the same matrix. Now, we apply $^\Delta$ to $\mathbf{M}(-\tau)$ and multiply the result with $\mathbf{M}(\tau)$. Then using Eqs. (E3) and (E2), we find

$$\mathbf{M}^\Delta(-\tau)\mathbf{M}(\tau) = \mathbf{I}_{2 \times 2} \Rightarrow \mathbf{M}^\Delta(-\tau) = \mathbf{M}^{-1}(\tau). \quad (\text{E11})$$

Then let us apply Eq. (E7) with $\tau = -T/2$ and use Eq. (E11) to get

$$\mathbf{M}(T) = \mathbf{M}\left(\frac{T}{2}\right)\mathbf{M}^{-1}\left(-\frac{T}{2}\right) = \mathbf{M}\left(\frac{T}{2}\right)\mathbf{M}^\Delta\left(\frac{T}{2}\right). \quad (\text{E12})$$

Applying $^\Delta$ on both sides of this equation implies

$$\begin{aligned} \mathbf{M}^\Delta(T) &= \left(\mathbf{M}^\Delta\left(\frac{T}{2}\right)\right)^\Delta \mathbf{M}^\Delta\left(\frac{T}{2}\right) = \mathbf{M}\left(\frac{T}{2}\right)\mathbf{M}^\Delta\left(\frac{T}{2}\right) \\ &= \mathbf{M}(T). \end{aligned} \quad (\text{E13})$$

We can now use this with Eq. (E8) and find

$$\mathbf{M}^\Delta(nT) = ((\mathbf{M}(T))^n)^\Delta = (\mathbf{M}^\Delta(T))^n = (\mathbf{M}(T))^n = \mathbf{M}(nT). \quad (\text{E14})$$

If we compare the terms of the matrices on both sides of the above equation, we find that the following relations are valid:

$$\begin{aligned} \dot{X}_\kappa^{1*}(nT) &= \dot{X}_\kappa^1(nT), & X_\kappa^{2*}(nT) &= X_\kappa^2(nT), \\ X_\kappa^{1*}(nT) &= \dot{X}_\kappa^2(nT). \end{aligned} \quad (\text{E15})$$

Then, for an arbitrary unimodular 2×2 matrix $\mathbf{C}$ (i.e., $|\mathbf{C}| = 1$), it can be proven that the following relation holds for $n = 1, 2, 3, \dots$:

$$\mathbf{C}^n = \left(\frac{\sinh(nD)}{\sinh(D)}\right)\mathbf{C} - \left(\frac{\sinh((n-1)D)}{\sinh(D)}\right)\mathbf{I}, \quad (\text{E16})$$

where D is obtained from the trace of $\mathbf{C}$ as

$$\cosh(D) = \text{Tr}(\mathbf{C})/2. \quad (\text{E17})$$

We can see that Eq. (E16) is trivially true for $n = 1$. For $n = 2$, it can be simplified as follows:

$$\mathbf{C}^2 = (\text{Tr}(\mathbf{C}))\mathbf{C} - \mathbf{I}. \quad (\text{E18})$$

This equation will be true if $|\mathbf{C}| = 1$. Thereafter, we can prove by induction that Eq. (E16) will be true for all $n = 1, 2, 3, \dots$. We can then use Eq. (E16) for the matrix $\mathbf{M}(\tau)$, defined in Eq. (E6), at $\tau = nT$. Then, using Eq. (E8), this implies

$$\mathbf{M}(nT) = \left(\frac{\sinh(nD)}{\sinh(D)}\right)\mathbf{M}(T) - \left(\frac{\sinh((n-1)D)}{\sinh(D)}\right)\mathbf{I}, \quad (\text{E19})$$

where D is obtained from the trace of $\mathbf{M}(T)$ as [using Eq. (E15)]

$$\cosh(D) = \frac{\text{Tr}(\mathbf{M}(T))}{2} = \frac{X_\kappa^1(T) + \dot{X}_\kappa^2(T)}{2} = \text{Re}(X_\kappa^1(T)), \quad (\text{E20})$$

where the last equality comes from Eq. (E15). This implies that $\cosh(D)$ is purely real, which means that D needs to be purely real or purely imaginary. Now, we can rewrite the Eq. (D2) identity in dimensionless variables as

$$|\dot{X}_\kappa(\tau)|^2 + \Omega_\kappa^2(\tau)|X_\kappa(\tau)|^2 = 1 - 2\sqrt{q}f(\tau)\text{Im}(X_\kappa(\tau)\dot{X}_\kappa^*(\tau)). \quad (\text{E21})$$

Substituting this into Eq. (2.31) and simplifying gives the equation,

$$n_\kappa(\tau) = \frac{1}{2} - \frac{\sqrt{q}f(\tau)}{2\Omega_\kappa(\tau)} - \frac{\kappa^2}{\Omega_\kappa(\tau)} \text{Im}(X_\kappa(\tau)\dot{X}_\kappa^*(\tau)), \quad (\text{E22})$$

which is the same as Eq. (D3), written in dimensionless variables. We can also use the identity of Eq. (2.20) with X_κ^1 and X_κ^2 , at $\tau = 0$, and the initial conditions from Eq. (E1) to get

$$C = -i\sqrt{q}f(0). \quad (\text{E23})$$

Now we use Eq. (2.20) again with $X_\kappa^1(NT)$ and $X_\kappa^2(NT)$ and Eq. (E15), where N is the number of oscillation periods that we are considering (starting from $\tau = 0$), and plug in the C above [with $f(0) = f(NT)$] to get

$$X_\kappa^1(nT)(\Omega_\kappa^2(NT)X_\kappa^2(NT) + \dot{X}_\kappa^1(NT) + 2\sqrt{q}f(NT) \text{Im}(X_\kappa^1(NT))) = 0. \quad (\text{E24})$$

Then we have that the oscillations $f(\tau)$ for Eq. (2.28), which drive the mode equation, are ‘turned off’ at times $\tau > NT$ or $\tau < 0$. This would imply that f will be constant for these times and $\dot{f}$ will be 0. Hence, Eq. (2.28) takes the form of a simple harmonic oscillator equation for these times as

$$\ddot{X}_\kappa(\tau) + \Omega_{\kappa\pm}^2 X_\kappa(\tau) = 0, \quad (\text{E25})$$

where: $\Omega_{\kappa+} = \Omega_\kappa(NT) = \sqrt{\kappa^2 + qf_+^2}$ and $\Omega_{\kappa-} = \Omega_\kappa(0) = \sqrt{\kappa^2 + qf_-^2}$ represent the fixed values of $\Omega_\kappa(\tau)$ for $\tau \geq NT$ and $\tau \leq 0$, respectively. We need N to be large for the $\tau \geq NT$ approximation to be valid for our mode equation. The general solution for Eq. (E25) for the $\tau \geq NT$ case is given by

$$X_\kappa^+(\tau) = Ae^{i\Omega_{\kappa+}\tau} + Be^{-i\Omega_{\kappa+}\tau}, \quad (\text{E26})$$

where A and B are some complex coefficients. Then, using this general solution with the initial conditions from Eq. (2.30), for a time $\tau \leq 0$, we can write the solution as follows:

$$X_\kappa^-(\tau) = X_\kappa(0)e^{i\Omega_{\kappa-}\tau}. \quad (\text{E27})$$

For $0 \leq \tau \leq NT$, we can use the initial conditions from Eq. (2.30) and the solutions X_κ^1 and X_κ^2 defined by the initial conditions from Eq. (E1) to construct the following approximate solution for Eq. (2.28):

$$X_\kappa(\tau) = X_\kappa(0)(X_\kappa^1(\tau) + i\Omega_\kappa(\tau)X_\kappa^2(\tau)). \quad (\text{E28})$$

Clearly, this $X_\kappa(\tau)$ smoothly joins to $X_\kappa^-(\tau)$ at $\tau = 0$ [as $X_\kappa(0) = X_\kappa^-(0)$ and $\dot{X}_\kappa(0) = \dot{X}_\kappa^-(0)$ for the $X_\kappa(\tau)$ above]. We also expect it to smoothly join $X_\kappa^+(\tau)$ at $\tau = NT$, which implies the boundary conditions at $\tau = NT$. This is

possible if the following relations are satisfied:

$$X_\kappa(0)(X_\kappa^1(NT) + i\Omega_\kappa(NT)X_\kappa^2(NT)) = Ae^{i\Omega_\kappa(NT)NT} + Be^{-i\Omega_\kappa(NT)NT}, \quad (\text{E29})$$

$$\frac{X_\kappa(0)}{i\Omega_\kappa(NT)}(\dot{X}_\kappa^1(NT) + i\Omega_\kappa(NT)\dot{X}_\kappa^2(NT)) = Ae^{i\Omega_\kappa(NT)NT} - Be^{-i\Omega_\kappa(NT)NT}. \quad (\text{E30})$$

Then, let us consider Eq. (E26) at $\tau = NT$ and plug it into Eq. (E22) and this gives

$$\bar{n}_\kappa(NT) = \frac{1}{2} - \frac{\sqrt{q}f(NT)}{2\Omega_\kappa(NT)} - \kappa^2(|B|^2 - |A|^2). \quad (\text{E31})$$

We can use Eqs. (E29) and (E30) to get an expression for $(|B|^2 - |A|^2)$ in terms of $X_\kappa^1(NT)$ and $X_\kappa^2(NT)$. We can then substitute the expression into Eq. (E31). Finally, we use Eqs. (2.30), (E15), (E24), and the periodicity of $f(\tau)$ to simplify the expression for $\bar{n}_\kappa(NT)$ and we get

$$\bar{n}_\kappa(NT) = \frac{\kappa^2}{\Omega_\kappa^2} (\text{Im}(X_\kappa^1(NT)))^2. \quad (\text{E32})$$

Now, we can compare the $(1, 1)$ entries of the matrices on both sides of Eq. (E19) and use Eq. (E6) to get the following equation for $X_\kappa^1(NT)$:

$$X_\kappa^1(NT) = \left(\frac{\sinh(ND)}{\sinh(D)} \right) X_\kappa^1(T) - \left(\frac{\sinh((N-1)D)}{\sinh(D)} \right), \quad (\text{E33})$$

where D is given by Eq. (E20). For the above equation, we find that the terms of form $[\sinh(ND)/\sinh(D)]$ will be purely real. This is a consequence of the fact that D is purely real or purely imaginary. Hence, we can plug the value for $X_\kappa^1(NT)$ into Eq. (E32) and we get the following:

$$\bar{n}_\kappa(NT) = \frac{\kappa^2}{\Omega_\kappa^2} \frac{\sinh^2(ND)}{\sinh^2(D)} (\text{Im}(X_\kappa^1(T)))^2. \quad (\text{E34})$$

Equation (E20) implies that D will be purely real or purely imaginary. However, as we consider fermions, we also need to ensure that $\bar{n}_\kappa(NT)$ respects the Pauli principle by ensuring that $\bar{n}_\kappa(NT) \leq 1$. If D were purely real, $\sinh^2(ND)/\sinh^2(D)$ grows exponentially with N and will not be bounded.⁶ Hence, the only possibility for us is that D is purely imaginary and $-1 < \text{Re}(X_\kappa^1(T)) < 1$. Hence, we can write $D = id$, where $0 \leq d \leq \pi$. This leads to the following relations:

⁶This case corresponds to considering bosons.

$$\cosh(D) = \cos(d), \quad \sinh(ND) = i \sin(Nd). \quad (\text{E35})$$

We use these relations in Eq. (E34) and get the expression for $\bar{n}_\kappa(nT)$ as

$$\bar{n}_\kappa(NT) = \frac{\kappa^2 \sin^2(Nd)}{\Omega_\kappa^2 \sin^2(d)} (\text{Im}(X_\kappa^1(T)))^2, \quad (\text{E36})$$

$$\cos(d) = \text{Re}(X_\kappa^1(T)). \quad (\text{E37})$$

This is of the same form as Eq. (30) of [10]. This expression for $\bar{n}_\kappa(NT)$ matches $n_\kappa(\tau)$ from Eq. (2.31) at integer multiples of T . As shown in [24], we want to convert $\bar{n}_\kappa(NT)$ to the continuous function $\bar{n}_\kappa(\tau)$ in Eq. (2.33). This is because we want to focus only on the long period of $n_\kappa(\tau)$ to study the filling of κ modes, and $\bar{n}_\kappa$ provides a numerically stable approximation for $n_\kappa(\tau)$, without any of the ‘spiky’ short period behavior of $n_\kappa(\tau)$. We can then do this by considering $\tau = NT$ and comparing $\sin^2(Nd)$ with $\sin^2(\nu_\kappa \tau)$. There are infinitely many solutions which will satisfy this, but we shall choose the following [instead of $\cos(\nu_\kappa T) = -\text{Re}(X_\kappa^1(T))$, which was the case in [24]]:

$$\cos(\nu_\kappa T) = |\text{Re}(X_\kappa^1(T))|. \quad (\text{E38})$$

This is done to ensure that ν_κ is always the lowest frequency, in order to match the low-frequency pattern followed by $n_\kappa(\tau)$, irrespective of the sign of $\text{Re}(X_\kappa^1(T))$. This gives us the formula for ν_κ from Eq. (2.34).

APPENDIX F: BACKREACTION

In Sec. II C, we mentioned that we neglect backreaction for our analysis. Here, we show that for the range of q values that we consider, the energy transferred into the fermions is a very small fraction of the energy of the inflaton field, justifying neglecting backreaction. We can write an integral similar to the one in Eq. (2.38) to describe the total comoving energy density accumulated in the fermions produced [24]. We approximate the energy density per mode of the fermions by $\bar{\Omega}_\kappa$ from Eq. (3.2) and approximate the occupancy of the κ modes by $F_\kappa/2$, with F_κ described in Eq. (2.34). Then, the ratio of the total comoving energy density in the fermions to that of the inflaton field is given by

$$\frac{\rho_\psi}{\rho_\phi} = \frac{4\lambda}{\pi^2} \int d\kappa \kappa^2 \bar{\Omega}_\kappa F_\kappa, \quad (\text{F1})$$

where $\rho_\phi = \lambda \phi_0^4/4$. We have used the time average of Ω_κ and $\bar{n}_\kappa$ and hence, the ratio in Eq. (F1) is also independent of time. Guided by cosmic microwave background observations [24,41] of the amplitude of the power spectrum of scalar density perturbations A_S , we set $\lambda = 10^{-13}$. We then calculate this ratio for three different q values and obtain

TABLE I. Three q values and the ρ_ψ/ρ_ϕ ratios they correspond to, for $\lambda = 10^{-13}$.

q	ρ_ψ/ρ_ϕ
10^2	1.2×10^{-12}
10^3	1.8×10^{-11}
10^5	5.9×10^{-9}

the corresponding fractions for the comoving energy density of the inflaton field transferred into the produced fermions, as shown in Table I.

We numerically fit these values and obtain an approximate parametric estimate for the ratio. The estimate will be different for different values of λ , and we also account for this in the following equation:

$$\frac{\rho_\psi}{\rho_\phi} \approx (4.0 \times 10^{-15}) \times \left(\frac{\lambda}{10^{-13}} \right) \times q^{1.2}. \quad (\text{F2})$$

Based on this, we find that less than 1% of the energy of the inflaton is transferred into the fermions for $q \lesssim 10^{10}$, assuming $\lambda = 10^{-13}$. We focus on much smaller q values in our analysis, justifying neglecting backreaction.

In [16], backreaction was analyzed for fermionic preheating with a $m^2 \phi^2$ inflaton potential and it was found, from the fraction of the energy density in the produced fermions, that backreaction effects are not relevant for $q \lesssim 10^8$.

APPENDIX G: RESONANCE PEAKS FOR FERMIONS WITH A $\phi\phi\bar{\psi}\psi$ COUPLING

As further justification for our energy conservation arguments from Sec. III and the relation Eq. (3.5), we apply our analysis to the case of production of light fermions ($m_\psi \approx 0$) with the same $\lambda\phi^4$ inflaton potential but with a $h_1\phi\phi\bar{\psi}\psi$ coupling between the inflaton and fermions, where h_1 is now a dimensionful coupling. The description of fermion production for this case and the equations representing the number density will be exactly the same as the ones in Sec. II, with the only differences being how the mode equation and the coupling parameter are defined. Similar to Eq. (2.14), the mode equation for the case with a $h_1\phi\phi\bar{\psi}\psi$ coupling can first be written in conformal variables as

$$X_k''(\eta) + [k^2 + (h_1\phi(\eta)\varphi(\eta))^2 - i(2h_1\phi(\eta)\varphi'(\eta))]X_k(\eta) = 0. \quad (\text{G1})$$

We can write this mode equation in dimensionless variables by using the transformations from Eq. (2.27) for most of the variables, with the exception of the coupling parameter. Instead of q from Eq. (2.27), we define q_1 as

$$q_1 = \frac{h_1^2 \varphi_0^2}{\lambda}. \quad (\text{G2})$$

We can then write the mode equation for this scenario in dimensionless variables as

$$\ddot{X}_\kappa(\tau) + (\kappa^2 + q_1 f^4(\tau) - 2i\sqrt{q_1} f(\tau) \dot{f}(\tau)) X_\kappa(\tau) = 0. \quad (\text{G3})$$

And we can write its ‘frequency’ as

$$\Omega_\kappa(\tau) = \sqrt{\kappa^2 + q_1 f^4(\tau)}. \quad (\text{G4})$$

Here $f(\tau)$ is the same Jacobi elliptic function from Eq. (2.6). F_κ is defined the same way as in Eq. (2.34) but it is now computed by numerically solving the mode equation of Eq. (G3).

We can apply a similar energy conservation argument based on the perturbative description of the process $n\varphi \rightarrow \Psi\bar{\Psi}$ as the one described in Sec. III. We can then find a semianalytic relation that can predict the locations of resonance peaks for this case, using the time average of $\Omega_\kappa(\tau)$ from Eq. (G4), represented by $\bar{\Omega}_\kappa(q_1)$. As described in Eqs. (3.3) and (3.4), the integrals over internal spacetime points that appear in the amplitude for the process $n\varphi \rightarrow \Psi\bar{\Psi}$ yield a term of the form:

$$\delta^{(3)}(\mathbf{\kappa}_\Psi + \mathbf{\kappa}_{\bar{\Psi}}) \int d\tau f(\tau)^n e^{2i\bar{\Omega}_\kappa(q_1)\tau}, \quad (\text{G5})$$

where we have used the large τ approximation for the accumulated phase of the fermion-antifermion pair. We can provide a simple qualitative explanation of the relevant physics by looking at the lowest order term in the expansion of $f(\tau)$ from Eq. (2.7):

$$\frac{4\pi\sqrt{2}}{T} \cos(\omega_\varphi \tau) = \frac{2\pi\sqrt{2}}{T} (e^{i\omega_\varphi \tau} + e^{-i\omega_\varphi \tau}), \quad (\text{G6})$$

where $\omega_\varphi = 2\pi/T$. Plugging this term into the amplitude of Eq. (G5) yields a term that gives the relation from Eq. (3.4).

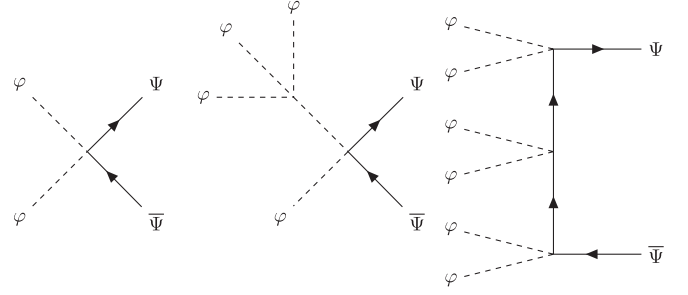


FIG. 16. Feynman diagrams describing the perturbative process $n\varphi \rightarrow \Psi\bar{\Psi}$. Shown first is the diagram for $n = 2$ (left), which is the same for both small and large q_1 values. For small q_1 , shown then is a representative diagram for $n = 4$ (middle). For large q_1 , shown is the diagram for $n = 6$ (right).

As done before, we can then qualitatively explain the resonance structure for this case using Feynman diagrams.

For small values of q_1 , $\bar{\Omega}_\kappa(q_1) \approx \kappa$ and relevant diagrams are of the form of the left and middle ones in Fig. 16. Clearly, for each subsequent addition of a φ^4 vertex to the $n = 2$ case (left diagram of Fig. 16), the number of external φ increases by 2. This implies that n must be even, i.e., $n = 2l$, where $l = 1, 2, 3, \dots$. Hence, for small values of q_1 , resonance peaks of F_κ should be present whenever κ is an even multiple of π/T . This is exactly what we observe for $q_1 = 0.01$ in the left plot of Fig. 17, where we show the locations of the resonance peaks corresponding to $l = 1, 2, 3$.

For large values of q_1 , the relevant diagrams are of the form of the left and right ones in Fig. 16. Diagrams with an odd number of fermion propagators will have zero amplitude based on the arguments in Sec. III. Thus, at large q_1 naively we would expect resonance peaks only for $\bar{\Omega}_\kappa(q_1) = (4l + 2)\pi/T$. However, peaks at all even multiples of π/T again appear; e.g., for $\bar{\Omega}_\kappa(q_1) = 4\pi/T$ the diagram corresponding to $n = 6$ is relevant, as if we plug Eq. (G6) into Eq. (G5) for $n = 6$, we still obtain three energy conservation relations corresponding to resonance peaks:

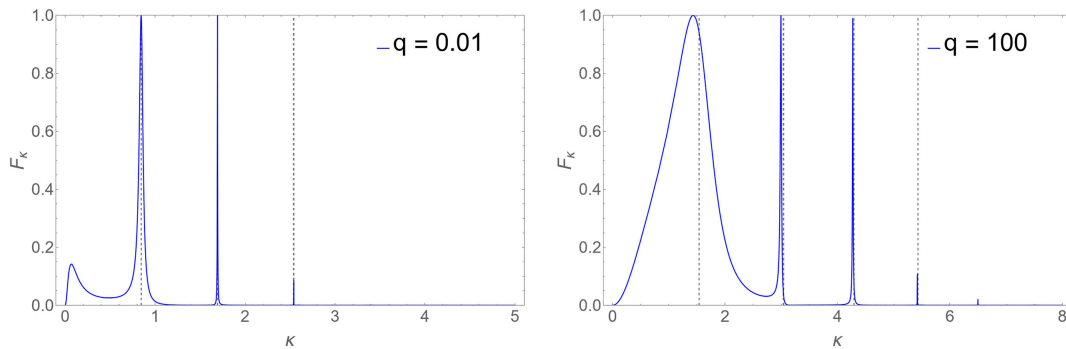


FIG. 17. F_κ (in blue) for $q = 0.01$ (left) and $q = 100$ (right). Dashed vertical lines represent the locations of first few resonance peaks, found using Eq. (G8).

$$\bar{\Omega}_\kappa(q_1) = \frac{2\pi}{T}, \quad \bar{\Omega}_\kappa(q_1) = \frac{4\pi}{T}, \quad \bar{\Omega}_\kappa(q_1) = \frac{6\pi}{T}. \quad (\text{G7})$$

The first two of these relations correspond respectively to moving either one or two of the inflatons in the right diagram of Fig. 16 from the initial to the final state. For the case involving a $\lambda\phi^4$ potential, an additional contribution to these peaks comes from the higher-order terms in the series expansion of the Jacobi elliptic function described in Eq. (2.7).

Hence, resonance peaks are present whenever $\bar{\Omega}_\kappa(q_1)$ is an even multiple of π/T for all q_1 in this model and we arrive at a semianalytic relation that predicts the locations of resonance peaks for all q_1 values:

$$\bar{\Omega}_\kappa(q_1) = \frac{\pi}{T}(2l), \quad \text{where } l = 1, 2, 3, \dots \quad (\text{G8})$$

Here, $\bar{\Omega}_0(q_1)$ sets a bound for the smallest l value that is possible for a particular q_1 . Similar to Eq. (3.7), this

smallest accessible l value can be obtained as follows:

$$l_0(q_1) = \left\lceil \frac{1}{2} \left(\bar{\Omega}_0(q_1) \frac{T}{\pi} \right) \right\rceil. \quad (\text{G9})$$

Using these, we find the locations of the first four resonance peaks with $q_1 = 100$ and they are approximately consistent with what we observe for the numerically evaluated F_κ , as shown in the right plot of Fig. 17. In Fig. 17, it also appears that the bulk region vanishes for large q , but exploring why this happens is beyond the scope of this work.

Finally, we also analyze the scenario with a $h_1\phi\phi\bar{\psi}\psi$ coupling for a $m^2\phi^2$ inflaton potential and find that the relation in Eq. (G8) works for that case as well. For small q_1 values however, the higher resonance peaks (with $l \geq 2$) vanish as only the left diagram of Fig. 16 contributes. The higher resonance peaks appear again for large q_1 values since the same arguments we gave above for the case with a $\lambda\phi^4$ potential are still applicable for the $m^2\phi^2$ scenario.

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- [1] L. Kofman, A. D. Linde, and A. A. Starobinsky, Reheating after inflation, *Phys. Rev. Lett.* **73**, 3195 (1994).
 - [2] A. D. Dolgov and D. P. Kirilova, On particle creation by a time dependent scalar field, *Sov. J. Nucl. Phys.* **51**, 172 (1990).
 - [3] J. H. Traschen and R. H. Brandenberger, Particle production during out-of-equilibrium phase transitions, *Phys. Rev. D* **42**, 2491 (1990).
 - [4] Y. Shtanov, J. H. Traschen, and R. H. Brandenberger, Universe reheating after inflation, *Phys. Rev. D* **51**, 5438 (1995).
 - [5] P. B. Greene and L. Kofman, Preheating of fermions, *Phys. Lett. B* **448**, 6 (1999).
 - [6] A. D. Dolgov and A. D. Linde, Baryon asymmetry in inflationary universe, *Phys. Lett.* **116B**, 329 (1982).
 - [7] L. Kofman, A. D. Linde, and A. A. Starobinsky, Towards the theory of reheating after inflation, *Phys. Rev. D* **56**, 3258 (1997).
 - [8] N. Bhusal, M. E. Chávez M., M. A. G. Garcia, A. G. Menkara, and M. Pierre, Fermion reheating with a quartic inflaton potential, *J. Cosmol. Astropart. Phys.* **06** (2026) 064.
 - [9] N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, A new method in the theory of superconductivity, *Fortschr. Phys.* **6**, 605 (1958).
 - [10] V. M. Mostepanenko and V. M. Frolov, Particle creation from vacuum by homogeneous electric field with a periodic time dependence, *Yad. Fiz.* **19**, 885 (1974).
 - [11] E. W. Kolb and A. J. Long, Cosmological gravitational particle production and its implications for cosmological relics, *Rev. Mod. Phys.* **96**, 045005 (2024).
 - [12] P. Adshead and E. I. Sfakianakis, Fermion production during and after axion inflation, *J. Cosmol. Astropart. Phys.* **11** (2015) 021.
 - [13] P. Adshead, L. Pearce, M. Peloso, M. A. Roberts, and L. Sorbo, Phenomenology of fermion production during axion inflation, *J. Cosmol. Astropart. Phys.* **06** (2018) 020.
 - [14] B. A. Bassett, S. Tsujikawa, and D. Wands, Inflation dynamics and reheating, *Rev. Mod. Phys.* **78**, 537 (2006).
 - [15] E. W. Kolb, A. D. Linde, and A. Riotto, GUT baryogenesis after preheating, *Phys. Rev. Lett.* **77**, 4290 (1996).
 - [16] G. F. Giudice, M. Peloso, A. Riotto, and I. Tkachev, Production of massive fermions at preheating and leptogenesis, *J. High Energy Phys.* **08** (1999) 014.
 - [17] C. Caprini and D. G. Figueroa, Cosmological backgrounds of gravitational waves, *Classical Quantum Gravity* **35**, 163001 (2018).
 - [18] D. J. H. Chung, E. W. Kolb, and A. Riotto, Nonthermal supermassive dark matter, *Phys. Rev. Lett.* **81**, 4048 (1998).
 - [19] M. Carena, N. M. Coyle, Y.-Y. Li, S. D. McDermott, and Y. Tsai, Cosmologically degenerate fermions, *Phys. Rev. D* **106**, 083016 (2022).
 - [20] R. Allahverdi, R. Brandenberger, F.-Y. Cyr-Racine, and A. Mazumdar, Reheating in inflationary cosmology: Theory and applications, *Annu. Rev. Nucl. Part. Sci.* **60**, 27 (2010).
 - [21] M. A. Amin, M. P. Hertzberg, D. I. Kaiser, and J. Karouby, Nonperturbative dynamics of reheating after inflation: A review, *Int. J. Mod. Phys. D* **24**, 1530003 (2014).
 - [22] A. D. Linde, Chaotic inflation, *Phys. Lett.* **129B**, 177 (1983).
 - [23] P. B. Greene and L. Kofman, On the theory of fermionic preheating, *Phys. Rev. D* **62**, 123516 (2000).

- [24] P. B. Greene, Aspects of preheating after inflation, Ph.D. thesis, Toronto University, 2002, URL <http://hdl.handle.net/1807/121431>.
- [25] Y. Akrami *et al.* (Planck Collaboration), Planck 2018 results. X. Constraints on inflation, *Astron. Astrophys.* **641**, A10 (2020).
- [26] J. Garcia-Bellido, S. Mollerach, and E. Roulet, Fermion production during preheating after hybrid inflation, *J. High Energy Phys.* **02** (2000) 034.
- [27] F. Bezrukov, D. Gorbunov, and M. Shaposhnikov, On initial conditions for the hot big bang, *J. Cosmol. Astropart. Phys.* **06** (2009) 029.
- [28] M. A. G. Garcia, K. Kaneta, Y. Mambrini, K. A. Olive, and S. Verner, Freeze-in from preheating, *J. Cosmol. Astropart. Phys.* **03** (2022) 016.
- [29] J. Klaric, A. Shkerin, and G. Vacalis, Non-perturbative production of fermionic dark matter from fast preheating, *J. Cosmol. Astropart. Phys.* **02** (2023) 034.
- [30] E. W. Kolb, D. J. H. Chung, and A. Riotto, WIMPzillas!, *AIP Conf. Proc.* **484**, 91 (1999).
- [31] O. E. Bjaelde and S. Das, Dark matter decaying into a Fermi sea of neutrinos, *Phys. Rev. D* **82**, 043504 (2010).
- [32] G. Choi, M. Suzuki, and T. T. Yanagida, Degenerate fermion dark matter from a broken $U(1)_{B-L}$ gauge symmetry, *Phys. Rev. D* **102**, 035022 (2020).
- [33] G. Choi, M. Suzuki, and T. T. Yanagida, Degenerate Sub-keV fermion dark matter from a solution to the Hubble tension, *Phys. Rev. D* **101**, 075031 (2020).
- [34] E. W. Kolb and M. S. Turner, *The Early Universe* (Taylor and Francis, London, 2019), Vol. 69, ISBN 978-0-429-49286-0, 978-0-201-62674-2.
- [35] Wolfram Research, JacobiCN, <https://reference.wolfram.com/language/ref/JacobiCN.html> (1988).
- [36] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York City, 1964), Ninth Dover Printing, Tenth GPO Printing ed.
- [37] A. D. Linde, Hybrid inflation, *Phys. Rev. D* **49**, 748 (1994).
- [38] J. Preskill, M. B. Wise, and F. Wilczek, Cosmology of the invisible axion, *Phys. Lett.* **120B**, 127 (1983).
- [39] L. F. Abbott and P. Sikivie, A cosmological bound on the invisible axion, *Phys. Lett.* **120B**, 133 (1983).
- [40] M. Thomson, *Modern particle physics* (Cambridge University Press, New York, 2013), ISBN 978-1-107-03426-6, 978-1-139-52536-7.
- [41] N. Aghanim *et al.* (Planck Collaboration), Planck 2018 results. VI. Cosmological parameters, *Astron. Astrophys.* **641**, A6 (2020); *Astron. Astrophys.* **652**, C4(E) (2021).