

Integrable systems for generalized toric polygons and Higgsed 5D $\mathcal{N} = 1$ theories

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The interplay between toric Calabi-Yau 3-folds, cluster integrable systems, and 5-dimensional quantum field theories has proved fruitful. We extend this framework to generalized toric polygons (GTPs) and show that their integrable systems arise from refined birational transformations of known cluster integrable systems acting on the Casimirs and Hamiltonians as well as the Poisson structure and spectral curves. We argue that these transformations are realized as Hanany-Witten transitions producing (p, q) 5-brane webs dual to GTPs. We show that the resulting 5D $\mathcal{N} = 1$ theory is obtained by Higgsing a higher-rank theory whose associated toric Calabi-Yau has a toric diagram of the same shape as the GTP.

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Introduction. The study of integrable systems has led to a broad range of insights into 4D $\mathcal{N} = 2$ supersymmetric gauge theories [1]. An important feature of 4D $\mathcal{N} = 2$ theories is that they admit an uplift to 5D $\mathcal{N} = 1$ theories [2–5], which can be realized as type IIB 5-brane webs [6–8]. When such a 5D $\mathcal{N} = 1$ theory is geometrically engineered in M-theory via a toric Calabi-Yau 3-fold [9], the 5-brane webs—also referred to as (p, q) -web diagrams—are dual to the Newton polygon of the associated toric Calabi-Yau 3-fold. In this way, the story comes full circle: the toric Calabi-Yau 3-fold underlying the (p, q) -web also corresponds to a cluster integrable system [10–20].

The family of cluster integrable systems associated with toric Calabi-Yau 3-folds is realized by brane tilings [21–23]. These brane tilings describe 4D $\mathcal{N} = 1$ supersymmetric gauge theories [24–27] that arise as world volume theories on D3-branes probing the toric Calabi-Yau 3-folds associated with the cluster integrable systems. Under T-duality, the probe D3-branes map to D5-branes suspended from an NS5-brane wrapping a holomorphic curve Σ , which yields the type IIB brane configuration corresponding to the brane tiling. A brane tiling consists of a bipartite periodic graph on a 2-torus and is also referred to as a dimer model [28]. Directed paths along the edges of this bipartite graph generate a path algebra whose generators provide canonical coordinates for an integrable system, which we refer to here as a cluster integrable system.

It was recently argued in [29,30] that if two toric Calabi-Yau 3-folds with reflexive polygons [31,32] as their toric diagrams are related by a birational transformation, then the associated cluster integrable systems are birationally equivalent [33–36]. A birational transformation φ_A acts on the Newton polynomial $P(x, y)$ of a toric diagram Δ . Its zero locus, $\Sigma: P(x, y) = 0$ defines the holomorphic curve, also known as the Calabi-Yau mirror curve [37,38], wrapped by the NS5-brane in the type IIB brane configuration

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represented by a brane tiling. Under a suitable identification of coefficients, the same curve Σ is the spectral curve of the cluster integrable system, which is written in terms of Hamiltonians associated with internal points of Δ together with Casimirs associated with external points.

The birational equivalence is realized by a refined birational map that matches Casimirs and Hamiltonians, identifies the spectral curve, and preserves the Poisson structure of cluster variables. Accordingly, birational equivalence between cluster integrable systems was first introduced in [29] for toric diagrams with the same number of internal points, so that the same number of commuting Hamiltonians can be identified on both sides. In this paper, we addresses the complementary situation: what is the correct notion of birational equivalence between integrable systems when a birational transformation changes the number of internal points of the toric diagram?

Our work is motivated by the dual (p, q) -web description, which exists for every toric diagram of a toric Calabi-Yau 3-fold. Starting from a (p, q) -web in which each external 5-brane ends on a separate 7-brane, a Hanany-Witten transition [39] generically produces brane configurations in which several external 5-brane legs terminate on the same 7-brane. The object dual to such a web is no longer an ordinary toric diagram. Instead, it is described by a generalized toric polygon (GTP) [40–46], whose boundary edges contain white vertices indicating external 5-brane legs ending on common 7-branes. In the Newton polynomial and associated spectral curve this leads to natural combinations of boundary coefficients, while in the integrable system it leads to a controlled reduction, or freezing, of degrees of freedom. We show that birational maps relating ordinary toric diagrams with different numbers of internal points are naturally interpreted as birational equivalences between an ordinary cluster integrable system and a reduced integrable system associated with a GTP of the same shape.

In what follows we briefly recall the construction of cluster integrable systems in the form needed here, define the reduction mechanism for a integrable system associated with a GTP from the (p, q) -web viewpoint, and demonstrate the mechanism explicitly for a birational equivalence between corresponding integrable systems associated to the dP_1 [27,32,47,48] and $L^{2,5,1}$ [23,49–51] geometries.

Cluster integrable systems and birational transformations. A bipartite periodic graph on T^2 , also referred to as a dimer model [28] or brane tiling [21–23] determines, upon choosing a fundamental domain in T^2 , a Kasteleyn matrix $K(x, y)$ [52] whose entries are weighted sums of edges e_{ij} in the brane tiling with monomials in $(x, y) \in (\mathbb{C}^*)^2$ that encode the lattice translations $w(e_{ij}) \in \mathbb{Z}^2$ of the end points of the edges from white node to black node in the universal cover of T^2 . We use here the determinant of the Kasteleyn matrix $K(x, y)$ in order to obtain the Newton polynomial of the form

$$P(x, y) = \sum_{\substack{P(n_x, n_y) \\ (n_x, n_y) \in \Delta}} \bar{p}_{(n_x, n_y)} x^{n_x} y^{n_y}, \quad (1)$$

where the sum runs over perfect matchings $p_{(n_x, n_y)}$ of the brane tiling, which are associated to vertices with coordinates $(n_x, n_y) \in \mathbb{Z}^2$ in the toric diagram Δ . Each monomial coefficient $\bar{p}_{(n_x, n_y)}$ in $P(x, y)$ is a product of e_{ij} that form a perfect matching $p_{(n_x, n_y)}$ in the brane tiling. The spectral curve of the associated cluster integrable system is the zero locus, $\Sigma: P(x, y) = 0$.

A convenient way to organize coefficients of $P(x, y)$ is to factor out a monomial coefficient $\bar{p}_0$ associated with a reference perfect matching p_0 [29,30] in the brane tiling, so that each monomial coefficient in the reduced polynomial $P_0(x, y) = \bar{p}_0^{-1} \cdot P(x, y)$ becomes a loop variable in the cluster integrable system. These loop variables are expressed as ratios of directed edge variables $e_{ij}^\pm$ [29,30] corresponding to products of directed paths in the brane tiling. Terms in $P_0(x, y)$ corresponding to boundary lattice points of Δ have monomial coefficients that are Casimirs $\delta_{(n_x, n_y)} = (\bar{p}_0)^{-1} \cdot \bar{p}_{(n_x, n_y)}$, while terms corresponding to internal points have coefficients that are Hamiltonians of the form $H = \sum_u \gamma_u$, where γ_u are 1-loops. More precisely, we note here that ratios between $\delta_{(n_x, n_y)}$ associated with consecutive boundary lattice points in Δ are more elementary Casimirs [10–12] that are independent of choice of reference perfect matching. For the boundary reference perfect matching used here, the $\delta_{(n_x, n_y)}$ are products of these elementary Casimirs and hence have vanishing Poisson brackets with all other variables. We therefore refer to the $\delta_{(n_x, n_y)}$ as Casimirs throughout this work.

Let f_i be face variables, also known as cluster variables, associated with faces of the brane tiling, or equivalently nodes of the corresponding quiver. The face variables are expressed in terms of cyclic products of directed edge variables $e_{ij}^\pm$ that form a directed path counterclockwise along the boundary edges of a face. Within the path algebra defined by $(e_{ij}^\pm)^{-1} = e_{ij}^\mp$ and $e_{ij}^+ \cdot e_{ij}^- = 1$, the face variables f_i satisfy the Poisson brackets $\{f_i, f_j\} = I_{i,j} f_i f_j$, where $I_{i,j}$ is the adjacency matrix of the quiver. As for the face variables f_i , one can also identify a finite set of directed zigzag paths z_r [28,53–56] that are likewise expressed as cyclic products of $e_{ij}^\pm$. Casimirs and the 1-loops in the Hamiltonians can be expressed as products of f_i and z_r .

We can expand the Newton polynomial for Δ as follows:

$$P(x, y) = \sum_{m=-a}^b C_m(x, y) y^m, \quad (2)$$

where $a, b > 0$, and $C_m(y)$ are subpolynomials of $P(x, y)$ for $-a \leq m \leq b$. Using this expansion, we can introduce a birational transformation of the form,

$$\varphi_A: (x, y) \mapsto (x, A(x)y), \quad (3)$$

where the Laurent polynomial $A(x)$ is chosen such that $A(x)^{-m}$ is a polynomial divisor of $C_m(x)$ for all m . Calling the resulting Newton polynomial $P'(x, y)$ and the corresponding toric diagram Δ' , [29,30] provided evidence that, when Δ and Δ' have the same number of internal lattice points, the corresponding cluster integrable systems are birationally equivalent. The subtlety addressed in this work is that φ_A may change the number of internal points of the toric diagram. With this in mind, we explain why the appropriate target of the birational transformation in this case is a reduced integrable system associated with a generalized toric polygon of the same shape as Δ' .

Generalized toric polygons and (p, q) -webs. In a (p, q) -web dual to an ordinary toric diagram, each of the external 5-brane legs ends on a distinct 7-brane. These external 5-brane legs are also in one-to-one correspondence with zigzag paths of the associated brane tiling, where the $\mathbb{Z}^2$ direction of the external legs matches the winding number of the corresponding zigzag path on T^2 . Under a Hanany-Witten transition [39], multiple external 5-branes can end up terminating on the same 7-brane. The dual polygon is then no longer a toric diagram in the conventional sense and is instead a generalized toric polygon (GTP) [40,41,43,43–46], where white vertices on the boundary of the GTP indicate grouped 5-branes ending on the same 7-brane. Let us denote the GTP by $\tilde{\Delta}$, whereas Δ refers to an ordinary toric diagram of the same shape as $\tilde{\Delta}$.

A central question is how the dynamics of the $5D \mathcal{N} = 1$ theory corresponding to a GTP $\tilde{\Delta}$ is described by an integrable system. We first consider the dual (p, q) -web corresponding to Δ , where each external 5-brane ends on a distinct 7-brane. The multiple external 5-branes that are later identified with the same 7-brane initially correspond to distinct zigzag paths in the brane tiling for Δ . These zigzag paths parametrize the Casimirs and loop variables of the corresponding cluster integrable system. Upon identifying the 5-branes to the same 7-brane, the corresponding zigzag paths are set equal, which imposes constraints on the cluster variables of the original cluster integrable system. This process is known as freezing, and it effectively reduces the dimension of the phase space while fixing some of the Hamiltonians as constants. In the $5D \mathcal{N} = 1$ theory, this corresponds to Higgsing. The reduced system remains integrable: it inherits a Poisson structure and a set of commuting Hamiltonians from the parent cluster integrable system.

When a birational transformation φ_A relates two toric diagrams Δ and Δ' that have different numbers of internal points and hence correspond to cluster integrable systems with different numbers of Hamiltonians, we freeze canonical variables in the cluster integrable system for Δ' so that it reduces to an integrable system corresponding to a GTP $\tilde{\Delta}'$

of the same shape as Δ' . We show in an explicit example that the (p, q) -webs corresponding to Δ and $\tilde{\Delta}'$ are related by a Hanany-Witten transition and that the cluster integrable system for Δ is birationally equivalent to the reduced integrable system corresponding to $\tilde{\Delta}'$. In this paper, we demonstrate this mechanism explicitly for the birational transformation relating the dP_1 and $L^{2,5,1}$ models.

The integrable systems for dP_1 and $L^{2,5,1}$. The dP_1 brane tiling and toric diagram Δ with one internal point are shown in the Supplemental Material [57]. The spectral curve of the associated cluster integrable system takes the following form:

$$\Sigma: \frac{z_1}{y} + \frac{x}{y} + \frac{y}{z_2 z_3 x} + \frac{y^2}{z_3 x} - H = 0, \quad (4)$$

where the monomial coefficients correspond to Casimirs $\delta_{(n_x, n_y)}$ expressed in terms of the zigzag paths $z_{1, \dots, 4}$. The single internal point of the reflexive toric diagram for dP_1 corresponds to the Hamiltonian $H = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, which is in terms of the following 1-loops:

$$\begin{aligned} \gamma_1 &= \frac{z_1 z_2 f_3}{f_4}, & \gamma_2 &= z_1 z_2 f_3, \\ \gamma_3 &= z_1 z_2 f_2 f_3, & \gamma_4 &= \frac{z_1 z_2}{f_4}. \end{aligned} \quad (5)$$

The 1-loops $\gamma_{1, \dots, 4}$ are expressed in terms of the zigzag paths $z_{1, \dots, 4}$ and face loops $f_{1, \dots, 4}$, whose expressions in terms of edge variables of the dP_1 brane tiling are given in the Supplemental Material [57].

The $L^{2,5,1}$ brane tiling and toric diagram Δ' with two internal points are summarized in the Supplemental Material [57]. The spectral curve of the cluster integrable system can be written in the following factorized form:

$$\Sigma': \frac{1}{y} - H_1 + H_2 y - \frac{y}{w_2 x} - (x + w_3)(x + w_4) \frac{w_1 y^2}{x} = 0, \quad (6)$$

where the monomial coefficients correspond to the Casimirs $\delta'_{(n_x, n_y)}$ written in terms of the zigzag paths $w_{1, \dots, 5}$ of the $L^{2,5,1}$ brane tiling, whose expressions in terms of edge variables are given in the Supplemental Material [57]. We note here that (6) coincides with the Seiberg-Witten curve of the associated $5D$ theory $SU(3)_2 + 1F$ up to a standard reparametrization. The two Hamiltonians corresponding to the two internal points of the $L^{2,5,1}$ toric diagram are written in terms of 1-loops $\alpha_{1,2,3}$, $\beta_{1,2,3}$ and ν as follows:

$$\begin{aligned}
H_1 &= \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 + \nu, \\
H_2 &= \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2 \\
&\quad + \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 + \alpha_2\nu + \beta_2\nu.
\end{aligned} \tag{7}$$

The 1-loops can be expressed in terms of the zigzag paths $w_{1,\dots,5}$ and face loops $h_{1,\dots,7}$ of the $L^{2,5,1}$ brane tiling as shown in the Supplemental Material [57]. Here, we introduce a parametrization of face loops $h_{1,\dots,6}$ in terms of a new auxiliary set of variables $\eta_{1,\dots,6}$, as follows:

$$h_i = \left(\frac{w_4}{w_3}\right)^{\frac{(-1)^i}{3}} \frac{\eta_{i+2}}{\eta_i}, \quad \eta_{i+6} = \eta_i, \quad \prod_{i=1}^6 \eta_i = 1, \tag{8}$$

with periodicity $\eta_{i+6} = \eta_i$. Using this parametrization, we can write the 1-loops as follows:

$$\alpha_m = \frac{(w_1 w_4)^{\frac{1}{3}}}{\eta_{2m} \eta_{2m+1}}, \quad \beta_m = \frac{(w_1 w_3)^{\frac{1}{3}}}{\eta_{2m-1} \eta_{2m}}, \quad \nu = \frac{(w_1^2 w_3 w_4)^{\frac{1}{3}} w_5}{(\eta_1)^2 \eta_2 \eta_6}, \tag{9}$$

where $m = 1, 2, 3$.

We can now introduce the following refined birational transformation:

$$\varphi_A: (x, y) \mapsto (x, (x + z_1)y), \tag{10}$$

where z_1 corresponds to a zigzag path in the dP_1 brane tiling. Under the above birational map, the spectral curve Σ of the dP_1 model in (4) becomes

$$\begin{aligned}
\Sigma^\vee = \varphi_A \Sigma: & \frac{1}{y} - H + \frac{1}{z_2 z_3} (x + z_1) \frac{y}{x} \\
& + \frac{1}{z_3} \left(x + 2z_1 + \frac{z_1^2}{x} \right) y^2 = 0,
\end{aligned} \tag{11}$$

whose Newton polygon $\Delta^\vee$ has the same shape as the toric diagram Δ' for the $L^{2,5,1}$ model. The question we want to answer in this work is what the corresponding integrable system for $\Sigma^\vee$ is and how it is related to the cluster integrable system with spectral curve Σ' in (6) corresponding to the $L^{2,5,1}$ model.

The GTP reduction and birational equivalence. We now show how the cluster integrable system for $L^{2,5,1}$ with spectral curve Σ' in (6) can be reduced to the integrable system corresponding to $\Sigma^\vee$ in (11). To do so, we note that the (p, q) -web for the $L^{2,5,1}$ model must have the 5-branes corresponding to zigzag paths w_3 and w_4 end on the same 7-brane, imposing the constraint $w_3 = w_4$. Under this identification, the resulting (p, q) -web is related to the (p, q) -web for the dP_1 model by a Hanany-Witten transition, as illustrated in Fig. 1.

Under this constraint, the original Hamiltonians for the $L^{2,5,1}$ model in (7) can be expressed as

$$\begin{aligned}
H_1|_{w_3=w_4} &= \frac{(w_1 w_3)^{\frac{2}{3}} w_5}{(\eta_1)^2 \eta_2 \eta_6} + \sum_{m=1}^6 \frac{(w_1 w_3)^{\frac{1}{3}}}{\eta_m \eta_{m+1}}, \\
H_2|_{w_3=w_4} &= (w_1 w_3)^{\frac{2}{3}} \left(\sum_{i=1}^3 \eta_{2i} \right) \left(\sum_{j=1}^3 \eta_{2j-1} \right) \\
&\quad + \frac{w_1 w_3 w_5 (\eta_3 + \eta_5)}{\eta_1}.
\end{aligned} \tag{12}$$

To obtain an integrable system with the correct number of Hamiltonians, we impose the additional freezing constraint $\eta_1 + \eta_3 + \eta_5 = 0$, which yields

$$\begin{aligned}
\tilde{H}_1 &= \frac{(w_1 w_3)^{\frac{2}{3}} w_5}{\eta_1^2 \eta_2 \eta_6} - (w_1 w_3)^{\frac{1}{3}} \left(\frac{\eta_5}{\eta_1 \eta_2 \eta_3} + \frac{\eta_1}{\eta_3 \eta_4 \eta_5} + \frac{\eta_3}{\eta_5 \eta_6 \eta_1} \right), \\
\tilde{H}_2 &= -w_1 w_3 w_5,
\end{aligned} \tag{13}$$

where we can see that $\tilde{H}_2$ is now a constant, leaving only a single dynamical Hamiltonian $\tilde{H}_1$ after freezing. Internal white lattice points of a GTP, which encode s -rule consistency in the corresponding brane configuration, are reflected in additional freezing conditions in the corresponding integrable system.

The resulting spectral curve of the reduced integrable system corresponds to the GTP with the same shape as the toric diagram for $L^{2,5,1}$. It takes the following form:

$$\begin{aligned}
\tilde{\Sigma}': & \frac{1}{y} - \tilde{H}_1 - w_1 w_3 w_5 (x + w_3) \frac{y}{x} \\
& - w_1 \left(x + 2w_3 + \frac{w_3^2}{x} \right) y^2 = 0.
\end{aligned} \tag{14}$$

By comparing the spectral curve $\tilde{\Sigma}'$ above with $\Sigma^\vee$ in (11) obtained from birationally transforming the spectral curve Σ for the dP_1 model, we obtain the following identifications between zigzag variables:

$$z_1 = w_3, \quad z_2 = \frac{1}{w_3 w_5}, \quad z_3 = -\frac{1}{w_1}, \quad z_4 = -w_1 w_5, \tag{15}$$

together with the map between the 1-loops $\gamma_{1,\dots,4}$ from the dP_1 model and the 1-loops in the reduced $L^{2,5,1}$ model, expressed in terms of $w_{1,\dots,5}$ and $\eta_{1,\dots,6}$, as follows:

$$\begin{aligned}
\gamma_1 &= \frac{(w_1 w_3)^{\frac{2}{3}} w_5}{(\eta_1)^2 \eta_2 \eta_6}, & \gamma_2 &= -\frac{(w_1 w_3)^{\frac{1}{3}} \eta_3}{\eta_5 \eta_6 \eta_1}, \\
\gamma_3 &= -\frac{(w_1 w_3)^{\frac{1}{3}} \eta_1}{\eta_3 \eta_4 \eta_5}, & \gamma_4 &= -\frac{(w_1 w_3)^{\frac{1}{3}} \eta_5}{\eta_1 \eta_2 \eta_3}.
\end{aligned} \tag{16}$$

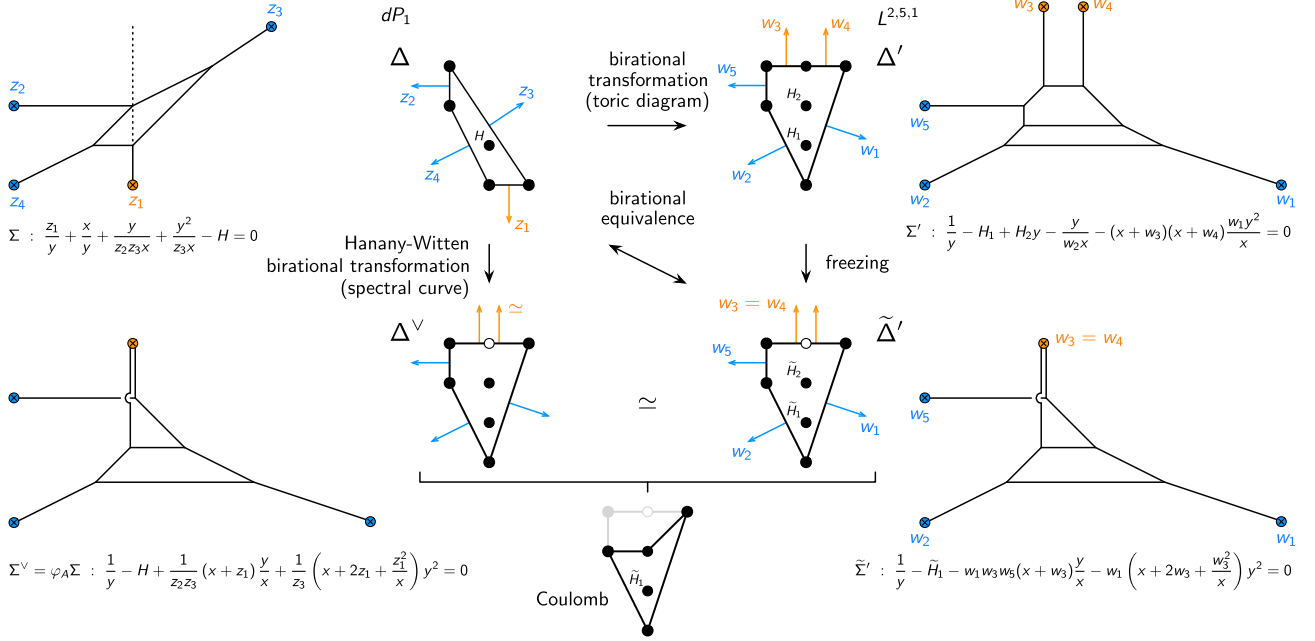


FIG. 1. The toric diagrams Δ of the dP_1 model and Δ' of the $L^{2,5,1}$ model are related by a birational transformation. Under a Hanany-Witten transition, the (p, q) -web dual to Δ becomes a (p, q) -web where two external 5-branes end on the 7-brane, making the dual polygon $\Delta^\vee$ a GTP. Because $\Delta^\vee$ is of the same shape as Δ' , we can identify the two 5-branes terminating on the same 7-brane with zigzag paths w_3 and w_4 in the brane tiling of $L^{2,5,1}$. In the corresponding cluster integrable system, we can impose $w_3 = w_4$ resulting in a reduced (frozen) integrable system that describes the dynamics of the $5D \mathcal{N} = 1$ theory corresponding to the GTP $\Delta^\vee$. We also note that the reduced integrable system is also birationally equivalent to the cluster integrable system corresponding to Δ .

Using $\prod_{m=1}^6 \eta_m = 1$ and (8), the 1-loops $\gamma_{1,\dots,4}$ above can be expressed as products of the original face variables h_i of the $L^{2,5,1}$ model.

We note here that the Poisson brackets between $\gamma_{1,\dots,4}$ in (16) satisfy the original Poisson brackets from the dP_1 model. This implies that the cluster integrable system for the dP_1 model with spectral curve Σ in (4) is birationally equivalent to the reduced integrable system with spectral curve $\tilde{\Sigma}'$ in (14), obtained by freezing the original cluster integrable system for the $L^{2,5,1}$ model with spectral curve Σ' in (6).

When we look at the exponents of $x, y \in \mathbb{C}^*$ in the terms in $\tilde{\Sigma}'$ in (14) we see that they correspond to vertices of the GTP with the same shape as the toric diagram for the $L^{2,5,1}$ model. Consider the associated $5D \mathcal{N} = 1$ pure $SU(2)_{3\pi}$ theory [58]. Here, $\theta = 3\pi$ labels the (p, q) -web diagram and the nonconvex polygon embedded in the GTP that encodes the theory's Coulomb branch. In this polygon, the vertex associated with $\tilde{H}_2$, which is internal in the original toric diagram for $L^{2,5,1}$, becomes a boundary vertex, whereas the sole remaining internal vertex corresponds to $\tilde{H}_1$, which is the only dynamical Hamiltonian of the reduced integrable system. As such, we can see that the reduced integrable system correctly describes the dynamics of the $5D \mathcal{N} = 1$ pure $SU(2)_{3\pi}$ theory corresponding to the GTP and the embedded nonconvex polygon shown in Fig. 1.

From the viewpoint of the integrable system, our work provides a concrete construction of an integrable system associated with a GTP, and equivalently with an embedded nonconvex polygon that describes the $5D \mathcal{N} = 1$ theory with Coulomb moduli. Our work thus extends the construction of these integrable systems beyond the original scope of cluster integrable systems first introduced in [10–12]. Moreover, from the viewpoint of birational transformations, our example shows that when the birational map changes the number of internal points from one in the dP_1 model to two in the $L^{2,5,1}$ model, the reduced integrable system corresponding to the spectral curve $\tilde{\Sigma}'$ in (14) and the GTP is nevertheless birationally equivalent to the original cluster integrable system corresponding to the dP_1 model. Our work therefore shows that birational transformations that change the number of internal points in toric diagrams still define birational equivalences between corresponding integrable systems.

Summary and outlook. In our work, we have identified a natural extension of birational equivalence between cluster integrable systems to the case in which the birational map changes the number of internal points of the toric diagram. The key observation is that the dual (p, q) -web admits brane configurations in which multiple external 5-brane legs end on a common 7-brane after a Hanany-Witten move. This leads to GTPs whose additional boundary

information allows us to introduce freezing of the integrable system corresponding to the original toric diagram. The resulting reduced integrable system is birationally equivalent to the original cluster integrable system prior to the birational transformation. Moreover, the reduced integrable system corresponds naturally to the GTP and describes the dynamics of the Higgsed $5D \mathcal{N} = 1$ theory. We expect our observations to extend beyond the example based on the dP_1 and $L^{2,5,1}$ models. In future work, we plan to construct new integrable systems corresponding to a variety of GTPs and nonconvex polygons, while reporting on a new family of birational equivalences.

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Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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