

Global determination of $|V_{us}|$ Matthew Kirk¹ and Danny van Dyk*Institute for Particle Physics Phenomenology and Department of Physics,
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In light of ongoing issues with the first-row unitarity test of the Cabibbo-Kobayashi-Maskawa matrix, we showcase a global fit to measurements of $K \rightarrow \ell\nu$, $K \rightarrow \pi\ell\nu$, $\tau \rightarrow K\nu$, and $\tau \rightarrow K\pi\nu$ decays for the first time. Fitting the semileptonic and three-body τ decay data simultaneously becomes computationally feasible because we employ a simple form factor parametrization for the $K\pi$ form factor that manifestly connects the semileptonic and pair-production regions. We find good agreement with the data and use our analyses to infer $|V_{us}|$ and the parameters for the $K\pi$ form factors. Our result for $|V_{us}|$ sits close to the results extracted from exclusive $K_{\ell 2}$ and somewhat above the $K_{\ell 3}$ decay and inclusive τ decay result. Moreover, our results for the $K \rightarrow \pi$ vector form factor at zero momentum transfer are compatible with lattice QCD determinations, despite not using lattice QCD inputs for this quantity in our fit of the hadronic matrix elements. We are further able to determine some of the pole parameters of the individual scalar and vector $K\pi$ resonances with masses below the τ mass. We caution that our results account only for short-distance electromagnetic corrections but not long-distance contributions; this is due to the lack of a consistent description of long-distance corrections to the $K\pi$ matrix elements both above and below threshold for an arbitrary model of the form factors.

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I. INTRODUCTION

Since the discovery of the Higgs boson, the complete field content of the Standard Model (SM) has now been discovered, and so far there has been no direct signal of any new particles at the LHC. As such, we increasingly turn to precision tests of the SM in the hope of discovering a significant deviation. A clear prediction of the SM is the unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, manifesting as a wealth of predicted relationships between elements of that matrix which can be tested against experimental determinations. Among others, we have the “first-row unitarity” prediction

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1. \quad (1.1)$$

For a long time, this relationship held within experimental and theoretical uncertainties, but increasing precision on both fronts means we can test this relation at unprecedented level. Taking for example the Particle Data Group (PDG) averages for V_{ud} ¹ and V_{us} [5,6] and neglecting the contribution due to

¹We note several recent new calculations of various corrections to β decay [1–4] that could shift the value of V_{ud} . We continue to use the PDG value for comparison and look forward to a comprehensive update including the latest calculations.

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V_{ub} on account of its size, one obtains

$$|V_{ud}|^2 + |V_{us}|^2 = 0.9983 \pm 0.0006(V_{ud}) \pm 0.0004(V_{us}), \quad (1.2)$$

showing a discrepancy with respect to unitarity of more than 2σ . With uncertainties at the level of less than one part per mille, this is currently the most stringent test of unitarity in the CKM matrix. Using a single value for $|V_{us}|$, as we have done here, hides another discrepancy, whereby $|V_{us}|$ values extracted from leptonic kaon decays ($K \rightarrow \mu\nu$, commonly known as $K_{\mu 2}$ ²) disagree with those from semileptonic kaon decays ($K \rightarrow \pi\ell\nu$, commonly known as $K_{l 3}$), leading to the PDG result above receiving an inflated error. Additional determinations of $|V_{us}|$ from inclusive τ decays are less precise than either of these methods, but appear to give yet another discrepant result.

The tension in this first-row unitarity test has been interpreted in terms of a variety of beyond the Standard Model (BSM) physics. Of particular interest for BSM model building is the observed discrepancy between the $|V_{us}|$ extractions from two- and three-body kaon decays, as this

²In fact there is yet another layer, in that the value of V_{us} from $K_{\mu 2}$ is typically obtained from the experimental ratio $\Gamma(K \rightarrow \mu\nu)/\Gamma(\pi \rightarrow \mu\nu)$, a lattice determination of f_K/f_π , and then either using CKM unitarity to eliminate V_{ud} or using a V_{ud} from, e.g., superallowed β decays.

could indicate a nonzero right-handed current—this thread has been followed in Refs. [7–14].

In this work we conduct a pilot study that simultaneously analyses data from K_{l2} , K_{l3} , $\tau \rightarrow K\pi\nu$ and $\tau \rightarrow K\nu$ decays, which are all $s \rightarrow u\ell\nu$ transitions (and in the SM all governed by a single effective operator). At leading order in α_e , the three-body kaon and τ decays are jointly described by two hadronic form factors, evaluated in different regions of phase space. We are able to conduct a global analysis at leading order in α_e by making use of a new novel parametrization of these form factors, which explicitly takes into account the resonances present in the $\tau \rightarrow K\pi\nu$ decay spectrum (shown in Fig. 2), as first developed in Ref. [15]. Beyond leading order in α_e , long-distance process-specific corrections arise. These have been calculated for K_{l3} decays in Ref. [16], while for two hadron τ decays only partial results are available: corrections assuming pointlike mesons are provided in Refs. [17,18]; meson structure-dependent corrections with a real emitted photon have been recently calculated in Ref. [19]; there has not yet been a calculation of the structure-dependent virtual photon corrections, however. Furthermore, these corrections are not consistently available for a common form factor parametrization that covers both the semileptonic and pair-production phase space regimes. Hence, they are not included in our analysis. To go beyond this pilot study, both the process-specific electromagnetic corrections and isospin-breaking effects must be accounted for.

II. THEORETICAL FRAMEWORK

Weak decay processes are very commonly described within the framework of the weak effective theory (WET), which is constructed from SM fields with masses below the scale of electroweak symmetry breaking. Within this theory, one identifies sectors corresponding to sets of operators that do not mix under the renormalization group equations at leading order in the Fermi constant $G_F \sim g^2/M_W^2$ [20]. For the kaon and τ decays at hand, the relevant sector is described by the Lagrangian

$$\mathcal{L}^{us\ell\nu} = \frac{4G_F}{\sqrt{2}} V_{us} \sum_i C_i^\ell(\mu) \mathcal{O}_i^\ell + \text{H.c.} \quad (2.1)$$

Here $\mathcal{O}_i^\ell$ are a set of local operators of mass dimension six involving an ℓ -flavored charged lepton field, and $C_i^\ell(\mu)$ are their WET Wilson coefficients. In the SM, only a single operator with $V - A$ structure contributes, i.e., only a single operator features a nonvanishing Wilson coefficient. The operator reads

$$\mathcal{O}_{V,L}^\ell = [\bar{u}\gamma^\mu P_L s][\bar{\ell}\gamma_\mu P_L \nu], \quad (2.2)$$

where $2P_{L/R} \equiv 1 \mp \gamma_5$. To order α_e , the SM value of its Wilson coefficient is lepton-flavor universal and reads

$$C_{V,L}^\ell(\mu) = 1 + \frac{\alpha_e}{\pi} \ln\left(\frac{M_Z}{\mu}\right) \simeq 1.01. \quad (2.3)$$

The numerical approximation above holds for $\mu \simeq 1$ GeV, which is chosen close to the geometric mean of M_K and M_τ and therefore minimizes potential logarithmic enhancements in the matrix elements of the operators. We assume $\mu = 1$ GeV whenever we discuss values of WET Wilson coefficients throughout this work.

Allowing for BSM effects, but still assuming the presence of only left-handed neutrinos, there exist four more operators that can contribute to the decays under consideration at mass dimension six. For the purposes of this study we briefly consider only contributions by a right-handed $\bar{u}s$ current, corresponding to the operator

$$\mathcal{O}_{V,R}^\ell = [\bar{u}\gamma^\mu P_R s][\bar{\ell}\gamma_\mu P_L \nu]. \quad (2.4)$$

III. HADRONIC MATRIX ELEMENTS

In order to calculate amplitudes and (subsequently) decay rates and other observables based on the Lagrangian Eq. (2.1), one needs access to the (hadronic) matrix element of the $us\ell\nu$ operators for the appropriate external hadronic states. In this section, we discuss the hadronic matrix elements relevant to this study.

A. Preliminaries

To determine $|V_{us}|$ from individual processes, information about process-specific hadronic matrix elements are required. In the description of leptonic decays of a kaon $K \rightarrow \ell\nu$ (dubbed K_{l2}), to order α_e^0 , only a single decay constant f_K emerges. The latter is commonly defined as [21]

$$\langle 0 | \bar{u}\gamma^\mu \gamma_5 s | K^-(p_K) \rangle \equiv -i p_K^\mu f_K. \quad (3.1)$$

In the description of the semileptonic decay of a kaon $K \rightarrow \pi\ell\nu$ (dubbed K_{l3}), two hadronic form factors f_+ and f_0 emerge at order α_e^0 . These form factors are scalar-valued functions of the dilepton mass square q^2 . They are commonly defined as³

$$\begin{aligned} & \langle \pi^+(p_\pi) | \bar{u}\gamma_\mu s | \bar{K}^0(p_K) \rangle \\ & \equiv f_+(q^2) \left[(p_K + p_\pi)_\mu - \frac{M_K^2 - M_\pi^2}{q^2} q_\mu \right] \\ & + f_0(q^2) \left[\frac{M_K^2 - M_\pi^2}{q^2} q_\mu \right], \end{aligned} \quad (3.2)$$

³We define the form factors in terms of the $\bar{K}^0$ initial state, since experimental data for the semileptonic rate of a neutral kaon are much more precise than for the charged kaon. Assuming isospin symmetry, the form factors for $K_{L,S}^0 \rightarrow \pi^+$ and $K^- \rightarrow \pi^0$ transitions are related to the $\bar{K}^0 \rightarrow \pi^+$ one in the following way: $\langle \pi^+ | j_{\bar{u}s} | \bar{K}^0 \rangle = \pm \sqrt{2} \langle \pi^+ | j_{\bar{u}s} | K_{L,S}^0 \rangle = -\sqrt{2} \langle \pi^0 | j_{\bar{u}s} | K^- \rangle$.

and the relevant phase space is $m_\ell^2 \leq q^2 \leq (M_K - M_\pi)^2 \equiv t_-$. To avoid an unphysical pole in the hadronic matrix element, the form factors must coincide at $q^2 = 0$, i.e., they must fulfil $f_+(0) = f_0(0)$. Since the phase space interval is numerically small, the form factors are commonly expanded in q^2 around the point $q^2 = 0$ to linear or quadratic order, however, more sophisticated parametrizations are discussed in the literature [5,22]. The world average of lattice computations provides a value for $f_+(0)$, i.e., the normalization at zero momentum transfer [21]. In the definition (3.2), f_+ parametrizes a P -wave $K\pi$ state and f_0 parametrizes an S -wave $K\pi$ state. The form factor f_0 also emerges in hadronic matrix elements of the scalar $\bar{u}s$ current,

$$\langle \pi^+(p_\pi) | \bar{u}s | \bar{K}^0(p_K) \rangle \equiv f_0(q^2) \frac{M_K^2 - M_\pi^2}{m_s - m_u}. \quad (3.3)$$

In the description of the hadronic decay of a τ to a two-body strange final state $\tau \rightarrow K\pi\nu$, both form factors emerge once more: the relevant hadronic matrix elements are related by crossing symmetry to the one for the K_{l3} decay. The phase space for the τ decay is $t_+ \equiv (M_K + M_\pi)^2 \leq q^2 \leq m_\tau^2$. However, the point $q^2 = (M_K + M_\pi)^2$ is a branch point of the form factors corresponding to the onset of the $K\pi$ production threshold. As a consequence, both form factors become multivalued and develop an imaginary part. In experimental studies, they are commonly parametrized using a sum of Breit-Wigner terms [23]. More sophisticated analyses (involving unitarity and analyticity arguments and, e.g., based on an Omnès representation) are available [24–28].

When inferring $|V_{us}|$ from a single process only, relations between the various hadronic matrix elements are not relevant. However, for a global determination of $|V_{us}|$, these relations are essential to ensure consistency of the determination.

The first of these relations emerges when using K_{l3} and τ decays in a joint analysis. Such an analysis needs to ensure that the two form factors are consistently used across phase space boundaries. To this end, we adapt a recently proposed parametrization for the pion form factor [15] to both of the $K\pi$ form factors; its discussion is relegated to Sec. III D.

The second of these relations emerges when using K_{l2} and K_{l3} or τ decays in a joint analysis. It connects the f_0 form factor with the kaon decay constant f_K at two q^2 points, by means of the so-called Callan-Treiman relations [29–31]. These relations emerge from chiral symmetry arguments and the exploitation of equal-time commutation relations at the zero-recoil point $q^2 = (m_K^2 - m_\pi^2)$ using the soft-pion theorem [29,30] and the point $q^2 = (m_\pi^2 - m_K^2)$ using the soft-kaon theorem [31]. These (generalized) Callan-Treiman relations read

$$\begin{aligned} f_0(m_K^2 - m_\pi^2) &= \frac{f_K}{f_\pi} + \Delta_{\text{CT}}, \\ f_0(m_\pi^2 - m_K^2) &= \frac{f_\pi}{f_K} + \Delta_{\text{CT}}^\sim. \end{aligned} \quad (3.4)$$

The corrections were calculated to next-to-leading order in chiral perturbation theory [32], giving

$$\Delta_{\text{CT}} = (-3.5 \pm 8.0) \times 10^{-3}, \quad \Delta_{\text{CT}}^\sim = 0.03 \pm 0.03. \quad (3.5)$$

The error on Δ_{CT} is directly taken from Ref. [33],⁴ while the error on $\Delta_{\text{CT}}^\sim$ is based on the range $-0.035 < \Delta_{\text{CT}}^\sim < 0.11$ provided in Ref. [33].

B. Properties

The form factors $f_+(q^2)$ and $f_0(q^2)$ exhibit useful properties that can be exploited in their parametrization.

P1 Both form factors parametrize $K\pi$ states with definite spin-parity quantum numbers J^P . In the f_+ case, these are $J^P = 1^-$, while in the f_0 case these are $J^P = 0^+$. Since both the kaon and the pion are spin $S^P = 0^-$ states, the total angular momentum coincides with the orbital angular momentum of the two mesons. There are no known states with these quantum numbers below the $K\pi$ threshold. Hence, the form factors are free of bound-state poles, which would manifest as poles on the first Riemann sheet.

P2 At the $K\pi$ pair-production threshold t_+ , the form factors are finite.

P3 Starting at t_+ , the two form factors each have a right-handed branch cut and therefore develop an imaginary part. For f_+ with $L = 1$, this discontinuity develops as $(t_+ - q^2)^{3/2}$, while for f_0 with $L = 0$, it is proportional to $(t_+ - q^2)^{1/2}$.

P4 Further right-handed branch cuts start at higher pair-production thresholds, e.g., $(M_K + 2M_\pi)^2$ and $(M_K + M_\eta)^2$.

P5 Using a suitable phase convention, both form factors are real-valued functions on the real q^2 axis for $q^2 < t_+$. As a consequence, when expanding the form factors around any point on the real q^2 axis to the left of t_+ , the expansion coefficients must be real valued.

P6 The combination of individual branch cuts shows an intricate pattern of resonances, which reflects the presence of a series of poles off the real axis on the second and higher indexed Riemann sheets. These poles are associated with resonances in the $K\pi$, $K\pi\pi$, $K\eta$ production channels. We summarize these resonances and their properties in Table I.

⁴Reference [33] in turn cites a private communication with H. Leutwyler.

TABLE I. Summary of the properties of the known resonances emerging in the description of the form factors f_+ and f_0 , as recorded in the PDG review [5].

Name	S^P	Mass (GeV)	Approx.	
			Width (GeV)	Channels
$K^*(896)$	1^-	0.890	0.052	$K\pi$
$K^*(1410)$	1^-	1.368	0.212	$K\pi, K\pi\pi$ (via $K^*\pi, K\rho$)
$K^*(1680)$	1^-	1.718	0.320	$K\pi, K\pi\pi$ (via $K^*\pi, K\rho$), $K\eta$
$\kappa = K_0^*(700)$	0^+	0.680	0.300	$K\pi$
$K_0^*(1430)$	0^+	1.425	0.270	$K\pi, K\eta$
$K_0^*(1950)$	0^+	1.957	0.170	$K\pi$

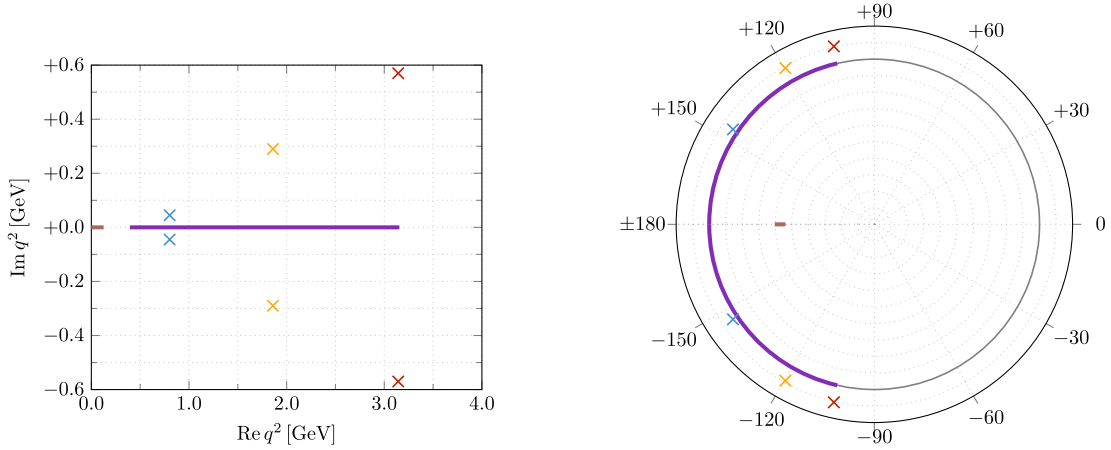


FIG. 1. Illustration of the relevant phase space regions and locations of the resonance poles in the complex q^2 (left) and z (right) planes for the f_+ form factor. The phase space of $\tau \rightarrow K\pi\nu$ is illustrated as purple lines and curves. The phase space of $K \rightarrow \pi\ell\bar{\nu}$ is illustrated as brown lines. The resonance poles are marked with colored crosses [$K^*(896)$, light blue; $K^*(1410)$, yellow; $K^*(1680)$, red].

P7 In the limit $|q^2| \rightarrow \infty$, the form factor $f_+(q^2) \sim 1/q^2 \rightarrow 0$, on account of the exchange of hard gluon interactions dominating the transition [34]. The same argument implies $f_0(q^2) \sim 1/q^2 \rightarrow 0$ as $q^2 \rightarrow \infty$.

A visual summary of the relevant phase space and pole properties is depicted in Fig. 1.

C. Dispersive bounds

Dispersive bounds for hadronic form factors, developed first in the context of semileptonic kaon decay [35,36], have now been firmly established for more than half a century in the context of heavy-to-heavy meson form factors [37,38]. The bounds on a form factor F take the schematic form

$$\int_{q^2=t_+}^{\infty} \omega_F(q^2) |F|^2 dq^2 \leq 1 \quad (3.6)$$

where ω is a positive-definite weight function determined in the derivation of the bound. It is closely related to the outer functions to be defined below.

In the modern formulation, the standard parametrization of dispersively bounded form factors for evaluation in the

semileptonic region is

$$f_+(q^2) = \frac{1}{\phi_+(q^2)} \sum_{k=0}^{N_+} a_k^{(+)} \left[z(q^2; t_0) \right]^k, \\ f_0(q^2) = \frac{1}{\phi_0(q^2)} \sum_{k=0}^{N_0} a_k^{(0)} \left[z(q^2; t_0) \right]^k. \quad (3.7)$$

In the above expressions, we have made use of the conformal mapping

$$q^2 \mapsto z(q^2; t_s) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_s - q^2}}{\sqrt{t_+ - q^2} + \sqrt{t_s - q^2}}, \quad (3.8)$$

with t_s an arbitrary parameter that fulfils $t_s < t_+$. The parameter t_0 used in Eq. (3.7) can be chosen freely. Here and throughout this work, we choose $t_0 = -4 \text{ GeV}^2$. The above change of variable maps the form factors' first Riemann sheet onto the unit open disk, their second Riemann sheet onto the complement of the unit disk, and the branch cut connecting both sheets onto the unit

circle. The expansion coefficients $a_k^{(+)}$ and $a_k^{(0)}$ are not independent by virtue of the identity $f_+(0) = f_0(0)$. The prefactors in Eq. (3.7) involve the outer functions ϕ_+ and ϕ_0 . For $\bar{u}s$ currents these are given by

$$\begin{aligned}\phi_+(t) &= \left| \frac{dz}{dt} \right|^{-1/2} \sqrt{\frac{1}{32\pi\chi_{1^-}(-Q^2)}} \left[\frac{t-t_+}{-z(t;t_+)} \frac{t-t_-}{-z(t;t_-)} \right]^{3/4} \times \frac{-z(t;0)}{t} \left(\frac{-z(t;-Q^2)}{t+Q^2} \right)^{3/2} \\ \phi_0(t) &= \left| \frac{dz}{dt} \right|^{-1/2} \sqrt{\frac{3}{32\pi\chi_{0^+}(-Q^2)}} \sqrt{t_+t_-} \left[\frac{t-t_+}{-z(t;t_+)} \frac{t-t_-}{-z(t;t_-)} \right]^{1/4} \times \frac{-z(t;0)-z(t;-Q^2)}{t} \frac{1}{t+Q^2},\end{aligned}\quad (3.9)$$

within which the χ_{1^-} and χ_{0^+} are normalizations. The normalizations are obtained from the local operator product expansion of a two-point correlation function with momentum transfer $-Q^2 < 0$. In this work, we use $Q^2 = 4 \text{ GeV}^2$, evaluating the expressions provided in Ref. [39] with $\alpha_s(2 \text{ GeV}) = 0.307$, $m_s = (93.5 \pm 0.8) \text{ MeV}$ [5], and condensate values from Ref. [40]. We obtain

$$\begin{aligned}\chi_{1^-}(Q^2 = 4 \text{ GeV}^2) &= (3.446 \pm 0.009) \times 10^{-3} \text{ GeV}^{-2} \\ \chi_{0^+}(Q^2 = 4 \text{ GeV}^2) &= (2.07 \pm 0.41) \times 10^{-4}.\end{aligned}\quad (3.10)$$

The perturbative convergence of χ_{0^+} is poor, and we have therefore assigned an uncertainty of 20%, larger than what one obtains as a nominal parametric uncertainty.

The dispersive bound on the form factors as given in Eq. (3.6) then implies a simple bound on the expansion parameters a_k

$$\sum_{k=0}^{\infty} |a_k^{(+)}|^2 \leq 1, \quad \sum_{k=0}^{\infty} |a_k^{(0)}|^2 \leq 1. \quad (3.11)$$

D. Improved parametrization

The dispersively bounded parametrization for the $K \rightarrow \pi$ form factors given in Eq. (3.7) respects properties P1 and P5. However, it does not satisfy the remaining properties. The specific issues are:

- (1) Properties P2 and P3 are violated, since the outer functions leads to a divergence in both f_+ and f_0 at threshold, while the imaginary part of f_+ develops as $(t_+ - q^2)^{3/2}$ above threshold. The effect of violating this property has been discussed in the context of the pion form factor [41] and of the $B \rightarrow \pi$ vector form factor [42]. In the context of the latter, it inspired a new parametrization thereof [43]. The concerns of Refs. [41–43] apply here as well: since the end point of the K_{l3} decay $t_- \simeq 0.13 \text{ GeV}^2$ and the branch point $t_+ \simeq 0.40 \text{ GeV}^2$ are very close to each other, mismodeling of the form factor in the vicinity of the branch point can have undue effect on

the shape of the form factors in the K_{l3} phase space and/or hinder the description of the τ decay.

- (2) Property P4 is violated, since the mapping from q^2 to z introduces only a single, square-root-like branch cut. We do not attempt to improve on this issue here and refer to Ref. [44] for a prospective solution.
- (3) Property P6 is violated, insofar that a pattern of multiple resonances cannot be described by the truncated series expansion in z . Even for a single resonance, this has not been shown to be possible; see the discussion in Ref. [41] where on the order of 60 parameters are required to approximately fit the ρ peak while producing unphysical features near the $\pi\pi$ threshold.
- (4) Property P7 is violated, since one finds for the asymptotic behavior $f_+(q^2) \sim (q^2)^{1/4}$ and $f_0(q^2) \sim (q^2)^{3/4}$ as $q^2 \rightarrow \infty$.

Three of these issues (nos. 1, 3, and 4) have been previously discussed and addressed by means of a new and simple parametrization in the context of the pion form factor [15]. Here, we propose to apply those modifications to the $K \rightarrow \pi$ form factors, generalizing that work to account for the existence of multiple above threshold poles. We summarize the changes to the standard parametrization made in that work as follows:

- (i) We include an overall weight function that removes the nominal pole at threshold and generates the correct asymptotic behavior for $q^2 \rightarrow \infty$, to ensure properties P2 and P7. The weight functions are simplest to describe in z space, taking the forms $W_+(z) = (1+z)^2(1-z)^{5/2}$ and $W_0(z) = (1+z)^1(1-z)^{7/2}$.
- (ii) We explicitly account for above threshold resonances, which are located outside the unit circle (corresponding to the second Riemann sheet as required by property P6) at

$$z_r = z((M_r - i\Gamma_r/2)^2; t_0)^{-1}$$

for a resonance with mass M_r and width Γ_r . Each resonance is included as a conjugate pair through a factor $1/(z - z_r)(z - z_r^*)$, so as to maintain property P5.

The new parametrization then reads

$$\begin{aligned} f_+(q^2) &= \frac{W_+(q^2)}{\phi_+(q^2)} \prod_{r=1}^{M_+} \frac{1}{z - z_{r,+}} \frac{1}{z - z_{r,+}^*} \sum_{k=0}^{N_+} b_k^{(+)} \left[z(q^2; t_0) \right]^k, \\ f_0(q^2) &= \frac{W_0(q^2)}{\phi_0(q^2)} \prod_{r=1}^{M_0} \frac{1}{z - z_{r,0}} \frac{1}{z - z_{r,0}^*} \sum_{k=0}^{N_0} b_k^{(0)} \left[z(q^2; t_0) \right]^k. \end{aligned} \quad (3.12)$$

Two linear relations exist between the expansion coefficients introduced in Eq. (3.12).

The first relation reads

$$b_0^{(+)} = -\frac{1}{X'} \left(\sum_{n=1}^{N_+} (-1)^n b_n^{(+)} (X' - nX) \right), \quad (3.13)$$

where we use

$$X(z) = \frac{W_+(z)}{\phi_+(z)} \prod_{r=1}^{M_+} \frac{1}{z - z_{r,+}} \frac{1}{z - z_{r,+}^*}, \quad X' = \frac{dX}{dz}. \quad (3.14)$$

By construction, the contributions to the imaginary part arise from odd powers in $(z + 1) \sim (q^2 - t_+)^{1/2}$. Equation (3.13) ensures that the leading contribution to the imaginary part of f_+ , which behaves as $\propto (q^2 - t_+)^{1/2}$, vanishes. As a consequence, $\Im f_+(q^2 = t_+) \propto (q^2 - t_+)^{3/2}$ as required by property P3.

The second relation reads

$$b_0^{(0)} = \frac{\tilde{\phi}_0(z_0)}{\tilde{\phi}_+(z_0)} \frac{\prod_{r=1}^{M_0} |z_0 - z_{r,0}|^2}{\prod_{r=1}^{M_+} |z_0 - z_{r,+}|^2} \sum_{n=0}^N b_n^{(+)} z_0^n - \sum_{n=1}^N b_n^{(0)} z_0^n, \quad (3.15)$$

where we abbreviate $z_0 = z(0; t_0)$. This relation ensures that $f_+(0) = f_0(0)$.

With this parametrization, we find ourselves unable to express the dispersive bound in a closed form, i.e., similar to Eq. (3.11) involving the expansion coefficients. However, we can still determine the saturation of the dispersive bound due to the $K\pi$ form factors by numerically evaluating the integral

$$\text{saturation}_{+,0} \equiv \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\vartheta \left| \phi_{+,0}(q^2(z)) f_{+,0}(q^2(z)) \right|^2 \Big|_{z=e^{i\vartheta}}. \quad (3.16)$$

The quantities χ_{1^-} and χ_{0^+} (and therefore the normalization of the bound saturations) feature sizable uncertainties as indicated in Eq. (3.10).

IV. ANALYSIS SETUP

We perform a Bayesian analysis of the available data on $K_{l2}, K_{l3}, \tau \rightarrow K\pi\nu$ and $\tau \rightarrow K\nu$ decays. The central quantity for this analysis is the so-called posterior probability density function (PDF),

$$P(\vec{x}|D, M) \propto P(D|\vec{x}, M) P_0(\vec{x}|M). \quad (4.1)$$

In the course of our analysis, we either optimize this posterior PDF with respect to the parameters $\vec{x}$ or we draw random samples $\vec{x} \sim P(\vec{x}|D, M)$. The former provides for a best-fit point while the latter provides insight into the extent of the allowed parameter space and is crucial input to producing posterior predictions. The set of varied parameters $\vec{x}$ is tied to the underlying fit model M . The fit models used in this analysis are discussed in Sec. IV B. The posterior PDF [Eq. (4.1)] is expressed in terms of the likelihood function $P(D|\vec{x}, M)$ and the prior PDF $P_0(\vec{x}|M)$. The former accounts for the experimental and theoretical constraints following from any of the labeled datasets D , which are discussed in Sec. IV A. The latter accounts for our *a priori* knowledge of the model parameters; see Sec. IV B for a discussion of the models, their parameters, and their prior PDFs.

Our analysis is carried out by using the EOS software [45] in version 1.0.19 [46]. To draw the posterior samples, EOS uses dynamical nested sampling [47] as implemented in the open-source Dynesty software [48]. The use of dynamical nested sampling ensures simultaneously high accuracy for the posterior predictions and the estimate of the evidence. The contents of the remainder of this section provide a detailed explanation of the analysis file used in the course of this study. The analysis file is available at Zenodo [49].

A. Experimental data

Our analysis uses the following experimental results for exclusive processes mediated by the $us\ell\nu$ sector of the WET:

$K^- \rightarrow \mu^- \bar{\nu}$: We use the PDG world average [5] of branching ratio measurements of the muon mode, which reads

$$\mathcal{B}(K^- \rightarrow \mu^- \bar{\nu})_{\text{PDG 2024}} = (63.60 \pm 0.16)\%. \quad (4.2)$$

This average includes measurements by the KLOE and OSPK experiments [50,51] and provides a total of one observation to our analysis.

The corresponding electron mode has, to the best of our knowledge, only been measured by means of the ratio $\Gamma(K^- \rightarrow e^- \bar{\nu})/\Gamma(K^- \rightarrow \mu^- \bar{\nu})$. Including it in the fit would therefore induce (potentially large) correlations between the measurement of this ratio and the muon mode measurement. Hence, we do not include the electron mode measurement in our analysis.

$K^- \rightarrow \pi^0 \{e^-, \mu^-\} \bar{\nu}$: Two sets of measurements of the branching ratios of these decays enter the world

average [5]: a joint measurement of both decay modes by KLOE [52] and a measurement of each mode without known correlations [51]. The average is clearly dominated by the KLOE measurement

$$\begin{aligned}\mathcal{B}(K^- \rightarrow \pi^0 e^- \bar{\nu})_{\text{KLOE}} &= (4.965 \pm 0.038(\text{stat}) \pm 0.037(\text{sys}))\% \\ \mathcal{B}(K^- \rightarrow \pi^0 \mu^- \bar{\nu})_{\text{KLOE}} &= (3.233 \pm 0.029(\text{stat}) \pm 0.026(\text{sys}))\%\end{aligned}\quad (4.3)$$

which features a linear correlation coefficient of $\rho = 0.627$. To avoid issues with ignoring the unknown correlations for the measurement presented in Ref. [51], we use only the KLOE measurement, which contributes two observations to our analysis.

$K_S^0 \rightarrow \pi^+ \{e^-, \mu^-\} \bar{\nu}$: The world average includes a measurement of the branching ratio for $K_S \rightarrow \pi^+ e^- \bar{\nu}$ by KLOE [53]

$$\mathcal{B}(K_S^0 \rightarrow \pi^+ e^- \bar{\nu})_{\text{KLOE}} = (3.455 \pm 0.17(\text{stat}) \pm 0.075(\text{sys})) \times 10^{-2}\% \quad (4.4)$$

and a determination of the same branching ratio extracted from the ratio $\Gamma(K_S^0 \rightarrow \pi^+ e^- \bar{\nu})/\Gamma(K_L^0 \rightarrow \pi^+ e^- \bar{\nu})$. To the accuracy of our analysis, the latter ratio is fully determined. Hence, we use only the KLOE measurement [53], which contributes one observation to our analysis.

The world average does not yet include a measurement of the branching ratio for the muon mode $K_S \rightarrow \pi^+ \mu^- \bar{\nu}$ by the KLOE-2 experiment [54]

$$\mathcal{B}(K_S^0 \rightarrow \pi^+ \mu^- \bar{\nu})_{\text{KLOE-2}} = (2.280 \pm 0.055(\text{stat}) \pm 0.085(\text{sys})) \times 10^{-2}\%. \quad (4.5)$$

We use this measurement, which contributes one observation to our analysis.

$K_L^0 \rightarrow \pi^+ \{e^-, \mu^-\} \bar{\nu}$: The world average [5] is determined from two individual correlated measurements of the branching ratios for $K_L \rightarrow \pi^+ \ell^- \bar{\nu}$, with $\ell = e, \mu$. The measurement by the KLOE experiment [55] reads

$$\begin{aligned}\mathcal{B}(K_L^0 \rightarrow \pi^+ e^- \bar{\nu})_{\text{KLOE}} &= (20.035 \pm 0.025(\text{stat}) \pm 0.075(\text{sys}))\% \\ \mathcal{B}(K_L^0 \rightarrow \pi^+ \mu^- \bar{\nu})_{\text{KLOE}} &= (13.490 \pm 0.025(\text{stat}) \pm 0.075(\text{sys}))\%\end{aligned}\quad (4.6)$$

with a correlation coefficient $\rho = -0.25$. When using the above, we combine the given statistical and systematic uncertainties in quadrature. The measurement by the KTeV experiment [56] reads

$$\begin{aligned}\mathcal{B}(K_L^0 \rightarrow \pi^+ e^- \bar{\nu})_{\text{KTeV}} &= (20.335 \pm 0.055)\% \\ \mathcal{B}(K_L^0 \rightarrow \pi^+ \mu^- \bar{\nu})_{\text{KTeV}} &= (13.505 \pm 0.045)\%\end{aligned}\quad (4.7)$$

with a correlation coefficient $\rho = +0.15$. These two joint determinations of the branching ratios are mutually compatible only at the 2.7σ level, with the incompatibility primarily in the electron mode. In the reported PDG world averages, the electron mode and the muon mode are combined individually, applying the standard PDG averaging procedure to each mode. This requires inflating the reported error on the electron average by a scale factor $S = 3.1$, while the error on the muon average is unchanged [5]. While there is no widely agreed on procedure for combining discrepant multivariate measurements, we choose to apply a PDG-like scale factor to the joint measurements from both experiments, inflating both electron and muon uncertainties by the common scale factor $S = 2.1$.

$\tau^- \rightarrow K^- \nu$: We use the PDG world average [5] for the branching ratio

$$\mathcal{B}(\tau^- \rightarrow K^- \nu)_{\text{PDG 2024}} = (0.685 \pm 0.023)\%, \quad (4.8)$$

which includes measurements from the ALEPH [57], CLEO [58], DELPHI [59], and OPAL [60] experiments. It contributes a total of one observation to our analysis.

$\tau^- \rightarrow K_S^0 \pi^- \nu$: We use the PDG world average [5] for the branching ratio

$$\mathcal{B}(\tau^- \rightarrow K_S^0 \pi^- \nu)_{\text{PDG 2024}} = (0.4195 \pm 0.0110)\%, \quad (4.9)$$

which includes measurements by the ALEPH [57], Belle [61], L3 [62], and OPAL [63] experiments. This average includes a scale factor of 1.5 and contributes one observation to our analysis.

In addition, we include a measurement of the normalized decay rate

$$\begin{aligned}P(\tau^- \rightarrow K_S^0 \pi^- \nu) \\ \equiv \frac{1}{\Gamma(\tau^- \rightarrow K_S^0 \pi^- \nu)} \frac{d\Gamma(\tau^- \rightarrow K_S^0 \pi^- \nu)}{dq^2}\end{aligned}\quad (4.10)$$

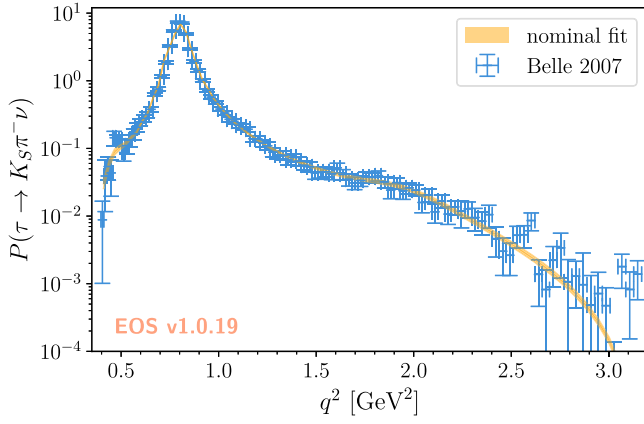


FIG. 2. Normalized differential τ decay data from Belle [23] (blue error bars), along with the posterior prediction and 68% probability envelope of our nominal model fit (orange band).

by the Belle experiment [23] (these data can be seen in Fig. 2). It contributes 100 observations to our analysis.

$\tau^- \rightarrow K^- \pi^0 \nu$: We use the PDG world average [5] for the branching ratio

$$\mathcal{B}(\tau^- \rightarrow K^- \pi^0 \nu) = (0.426 \pm 0.016)\%, \quad (4.11)$$

which includes measurements by the ALEPH [57], BABAR [64], CLEO [65], and OPAL [66] experiments. This average contributes one observation to our analysis.

In the following, we will refer to combinations of the above as these named datasets. These are defined as:

3body: All three-body decay modes, i.e., the branching ratios for all $K \rightarrow \pi \ell \nu$ decays, as well as the branching ratios of both $\tau \rightarrow K \pi \nu$ modes and the differential measurement from Belle. It accounts for a total of 110 observations.

all: The combination of the datasets **3body** with the branching ratios for the $\tau \rightarrow K \nu$ and $K \rightarrow \mu \nu$ decays. It accounts for a total of 112 observations.

B. Fit models, parameters of interest, nuisance parameters, and their priors

We use the following fit models:

FF: In this model, the parameters of interest are those of our new $K \rightarrow \pi$ form factors (FFs) given in Eq. (3.12). Here we fix $|V_{us}| = 0.22515$ which is the value obtained using $K_{\mu 2}/\pi_{\mu 2} + \text{unitarity}$ [5,6]. This fit model is exclusively used in conjunction with the **3body** experimental dataset. For imposing the Callan-Treiman relations, we vary the parameters f_K , f_π , Δ_{CT} , and $\tilde{\Delta}_{CT}$ within informative Gaussian priors, which are discussed below. For the description of the $K\pi$ form factors, it features additionally $2(M_+ + M_0) + (N_+ + N_0 - 2)$ real-valued parameters. Unless otherwise stated, we choose uniform

priors for each of these parameters. The intervals are chosen wide enough to avoid cutting into the peak due to the likelihood. For details on the prior ranges and choices of the model meta parameters M_+ , M_0 , and $N_+ = N_0$, we refer to the analysis file `analysis_FF.yaml` that is part of the Zenodo data [49].

CKM: This model is based on the FF model, except that we additionally float $|V_{us}|$ in the range $[0.223, 0.227]$ as the new parameter of interest and utilize the lattice QCD result for the semileptonic kaon form factors at $q^2 = 0$ as an additional theoretical input. This fit model is exclusively used with the dataset **all**. For further details, we refer to the analysis file `analysis_CKM.yaml` in the Zenodo data [49].

BSM: This model is based on the FF model. We use it to study the potential size of BSM effects in the data by introducing two new degrees of freedom in the form of the WET Wilson coefficients $C_{V,L}$ and $C_{V,R}$. For simplicity we assume both to be real-valued quantities that are varied in the ranges

$$1.000 \leq C_{V,L} \leq 1.025, \quad -0.01 \leq C_{V,R} \leq 0.02. \quad (4.12)$$

The new parameters are varied alongside the same hadronic FF parameters as in the FF model. Further hadronic matrix elements only arise if BSM operators with $\bar{u}\sigma^{\mu\nu}s$ currents are included in the analysis. Since we only consider vector-type currents in this fit model, we are able to consistently fit the form factor from data while simultaneously quantifying potential BSM effects. To provide an absolute scale for the magnitude of these Wilson coefficients, we fix $|V_{us}| = 0.22515$, as in the FF fit model. We assume lepton-flavor universality of the Wilson coefficients. As in the CKM model, we additionally constrain the semileptonic kaon form factors based on the lattice QCD world average. For further details, we refer to the analysis file `analysis_BSM.yaml` in the Zenodo data [49]. This fit model is exclusively used with the dataset **all**.

The informative priors referred to in the description of the FF model are chosen as follows:

$K^- \rightarrow \mu^- \bar{\nu}$: The hadronic input to leptonic decays of a pseudoscalar meson is expressed in the SM by means of a single scalar parameter, the kaon decay constant

$$f_K = (155.7 \pm 0.3) \text{ MeV} \quad (4.13)$$

from the 2024 FLAG report [21], based on lattice QCD results provided in Refs. [67–70].

$\bar{K}^0 \rightarrow \pi^+ \ell^- \nu$: The hadronic input to the semileptonic decays of a kaon includes the normalization of the f_+ form factor at $q^2 = 0$. The 2024 FLAG report [21], based on results by the ETMC [71] and FNAL/MILC [72] Collaborations provides the average

$$f_+(0) = 0.9698 \pm 0.0017. \quad (4.14)$$

Callan-Treiman: The Callan-Treiman theorem depends on the ratio of the kaon and pion decay constants. In the latest FLAG report [21] this ratio is given to 0.16% precision, while the pion decay constant is only known to 0.6% (and as given above, the kaon decay constant is known to 0.19%). We therefore choose the more conservative option by treating the pion and kaon decay constants as separate nuisance parameters, using the above kaon value and

$$f_\pi = (130.2 \pm 0.8) \text{ MeV} \quad (4.15)$$

for the pion decay constant likelihood (based on lattice results provided in Refs. [65,73–75]). The corrections Δ_{CT} and $\Delta_{\text{CT}}^\sim$ to the Callan-Treiman relations are included via two uncorrelated Gaussian likelihoods using the numerics in Eq. (3.5).

For all fit models we impose the Callan-Treiman relations and the dispersive bounds. The latter are imposed by numerically computing the saturation and penalizing parameter points for which the saturation exceeds 100%. To so, we use Eq. (A3) of Ref. [76] to impose a smooth penalty function that accounts for the relative uncertainties of χ_{1^-} and χ_{0^+} .

V. RESULTS

A. Fits extracting the hadronic matrix elements

We have conducted a variety of fits using the FF model, either fixing individual resonance parameters according to their PDG values or letting them float. We successfully fit the available data 3body, which excludes any theoretical information on the $K \rightarrow \pi$ form factors in the semileptonic region. We accept fits that feature a p value in excess of 3%, which is our *a priori* threshold. Among these accepted fits, the worst performing is $M_+ = M_0 = 1$ and $N_+ = N_0 = 3$ at $p \simeq 6\%$, i.e., accounting for the $K^*(892)$ and the $K_0^*(700)$ only. With respect to the resonances appearing in the τ decay, our finding can be summarized as follows:

- (1) We are able to fit the resonance parameters (mass and width) of the $K^*(892)$ from the available data. To do this, we fix the parameters relating to the remaining resonances in the f_+ and f_0 form factors. We obtain pole parameters for the $K^*(892)$,

$$\begin{aligned} M_{892} &= (891.87 \pm 0.18(\text{stat}) \pm 0.05(\text{model})) \text{ MeV}, \\ \Gamma_{892} &= (45.5 \pm 0.4(\text{stat}) \pm 0.5(\text{model})) \text{ MeV}, \end{aligned} \quad (5.1)$$

that are in a good agreement with the PDG values. We discuss how we assess the model uncertainty below.

- (2) We are unable to fit the resonance parameters of the $K_0^*(700)$ from the available data, regardless of our choice of treatment of the remaining resonances and their parameters.

- (3) Fixing the $K_0^*(700)$ resonance parameters to those of the PDG, we are able to fit *either* the $K^*(1410)$ or the $K_0^*(1430)$ resonance parameters, but not both at the same time. Given the closeness of these two resonances, we speculate that separating them from each other is not possible given the available data. Since they differ in their total angular momentum, we hold that additional information in form of the angular observable arising from the $K\pi$ helicity angle has the potential to change this outcome, i.e., to cleanly separate the two resonances from each other and fit their parameter simultaneously.
- (4) We are unable to fit the resonance parameters of the $K^*(1680)$ from the available data. This is likely related to the fact that it is situated very close to the end point of the τ decay phase space $\sqrt{q^2} \leq m_\tau \simeq 1.777 \text{ GeV}$, which causes a small decay rate in the region close to $K^*(1680)$.

In light of our findings, we choose to use as our nominal result a model containing $K^*(892)$, $K^*(1410)$, $K_0^*(700)$, and $K_0^*(1430)$ [i.e., $M_+ = M_0 = 2$ in Eq. (3.12)], as such a model contains the four resonances we know to be present and to which the available data are sensitive. We find that going to at least order z^3 in the expansion is needed to produce an acceptable p-value, and therefore we go one order higher to ensure the systematic truncation uncertainties are under control (i.e., we use $N_+ = N_0 = 4$). At higher orders in the expansion, we see degeneracies appearing, indicating that the parameters are poorly constrained. This model fit has a p-value of 72%, and the posterior prediction for the differential τ decay rate is shown in Fig. 2. We additionally use a model that also contains the $K^*(1680)$ (i.e., $M_+ = 3$, $M_0 = 2$). Here we find that going to z^4 in the expansion is required for an acceptable p-value, and again use the next higher order to study truncation errors. The difference between (pseudo) observables in these four representative models

$$\begin{aligned} &\{(M_+, M_0, N_+, N_0) \\ &: (2, 2, 3, 3), (2, 2, 4, 4), (3, 2, 4, 4), (3, 2, 5, 5)\} \end{aligned} \quad (5.2)$$

is used as our estimate of the model uncertainty.

1. $K \rightarrow \pi$ form factor

Our fits produce posterior-predictive results for the $K \rightarrow \pi$ form factor. We first compare the value at zero momentum transfer $q^2 = 0$. From our representative fit models as given in Eq. (5.2), we obtain

$$f_+(0) = 0.9713 \pm 0.0019(\text{stat}) \pm 0.0030(\text{model}). \quad (5.3)$$

This result should be compared to the world average of the $K \rightarrow \pi$ form factor from lattice QCD determinations [21], which is provided in Eq. (4.14). We find excellent agreement

between our result and the world average of lattice QCD predictions.

Experimental studies of the K_{l3} decay have access to the shape of the decay distribution and therefore access to the shape of the form factors. This shape is probed through a model that Taylor expands the form factors about $q^2 = 0$,

$$f_{+,0}(t) = f_{+,0}(0) \left(1 + \frac{\lambda'_{+,0} t}{m_\pi^2} + \frac{\lambda''_{+,0} t^2}{2m_\pi^4} \right). \quad (5.4)$$

Our model is not limited to the first two terms in the expansion. However, restricting the degrees of freedom in the Taylor expansion above is liable to introduce bias in the results.⁵ Here, we compare the results for the linear coefficient from a joined analysis of the electron and muon mode λ'_+ [77],

$$\begin{aligned} \lambda'_{+|\text{NA48/2}}^{\text{quadratic model}} &= 0.02424 \pm 0.00075(\text{stat}) \pm 0.00130(\text{sys}), \\ \lambda'_{+|\text{NA48/2}}^{\text{linear model}} &= 0.01447 \pm 0.00063(\text{stat}) \pm 0.00117(\text{sys}), \end{aligned} \quad (5.5)$$

with our posterior prediction

$$\begin{aligned} \lambda'_+ &= 0.0238 \pm 0.0007(\text{stat}) \pm 0.0020(\text{model}), \\ \lambda'_0 &= 0.0119 \pm 0.0011(\text{stat}) \pm 0.0010(\text{model}). \end{aligned} \quad (5.6)$$

We find excellent agreement at the levels of 0.17σ and 1.29σ , respectively.

2. Predictions for D decays

Another use case for these form factor results comes from hadronic D decay studies, where the topological amplitudes of the annihilation type can (up to corrections of order $1/N_c^2$) be written in terms of the D decay constant and the scalar $K \rightarrow \pi$ form factor evaluated at $q^2 = m_D^2$ or $q^2 = m_{D_s}^2$ [78].

In Ref. [78], the authors assume $f_0(m_{D_s}^2) = f_0(m_D^2)$ and take an estimate based on extrapolations in Ref. [23], which reads

$$1 \lesssim |f_0(m_{D_s}^2)| \lesssim 4.5, \quad -\pi \lesssim \arg f_0(m_{D_s}^2) \lesssim \pi \quad (5.7)$$

as a suitable range to describe the available τ decay data. Using our results, we are able to predict the absolute value more precisely

$$|f_0(m_D^2)| \approx |f_0(m_{D_s}^2)| = 1.7 \pm 0.7(\text{stat}) \pm 0.2(\text{model}), \quad (5.8)$$

while for the phase our fit still provides no sensitivity. Both quantities are strongly correlated, with the correlation coefficient close to unity. For the ratio between both quantities

⁵As evidenced by the substantial differences in the results for λ'_+ from a linear and a quadratic fit in the PDG world average [5].

we find

$$|f_0(m_D^2)|/|f_0(m_{D_s}^2)| \simeq 1 \pm 0.2. \quad (5.9)$$

B. Fits extracting $|V_{us}|$

Using the representative models given in Eq. (5.2), we perform a series of new fits using the CKM fit model and the all dataset to determine our CKM posterior. Having found good agreement with the world average of lattice QCD predictions for $f_+(0)$ in our FF fit, we now include this input as a prior going forward. In this way, we determine $|V_{us}|$ within a global analysis. We find that all fits continue to have acceptable p values $\geq 3\%$ and therefore provide the first global result for $|V_{us}|$ of

$$|V_{us}| = 0.2249 \pm 0.0004(\text{stat}) \pm 0.0003(\text{model}). \quad (5.10)$$

We show this value in Fig. 3, alongside other determinations of V_{us} , and as can be seen our result is in good agreement with previous results. However, there is still a tension with CKM unitarity. If we combine our result for $|V_{us}|$ with the $|V_{ud}|$ result obtained from β decay, we find

$$|V_{ud}|^2 + |V_{us}|^2 = 0.9986 \pm 0.0006(V_{ud}) \pm 0.0002(V_{us}), \quad (5.11)$$

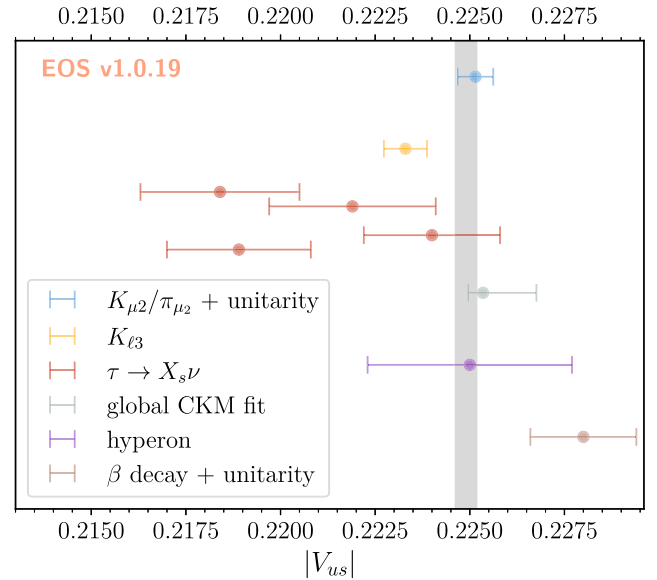


FIG. 3. Our result for $|V_{us}|$ shown as the shaded vertical band, in comparison to results in the literature. We show four independent results obtained from inclusive τ decays, corresponding to results from Refs. [79,80] (topmost data point), Refs. [81,82] (second from top), Refs. [82,83] (second from bottom), and Ref. [84] (bottom). We refer the interested reader to discussions in the HFLAV report [85,86] for details of the differences between these determinations. The K_{l3} , hyperon, and β decay + unitarity results are taken from Refs. [5,6], while the global CKM fit is obtained by the CKMfitter Collaboration [87,88].

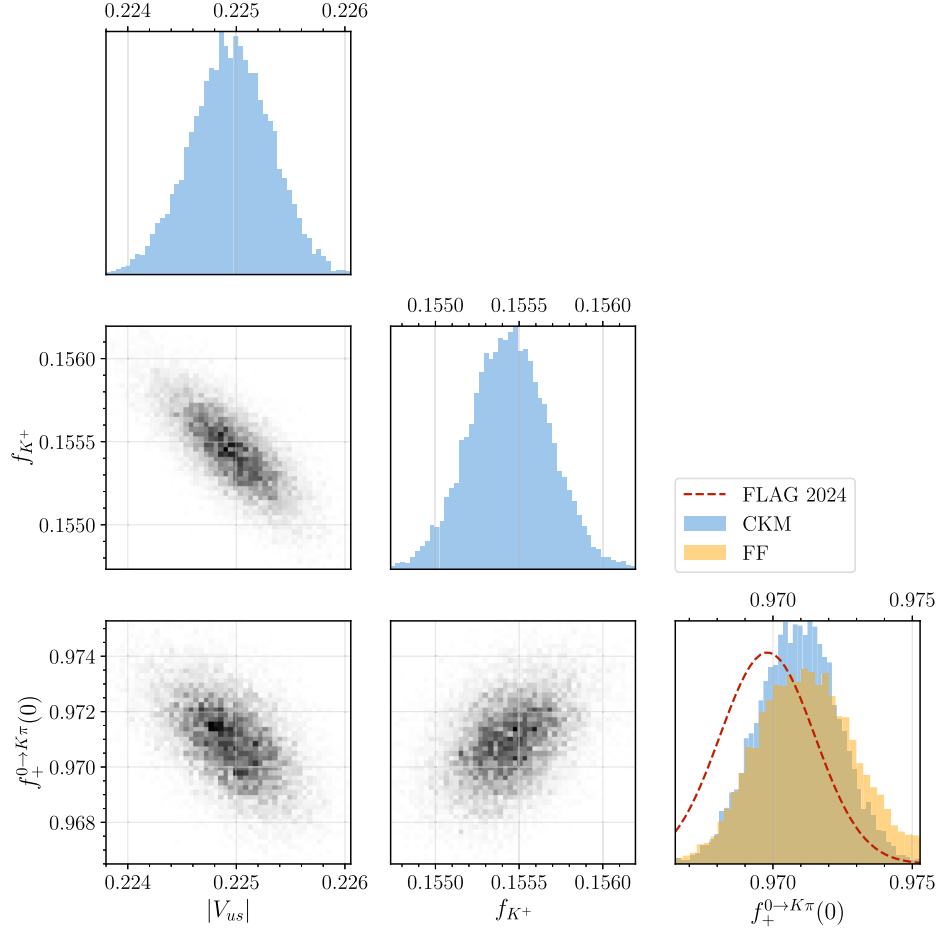


FIG. 4. Joint distribution of $|V_{us}|$, f_K , and $f_+(0)$ in our CKM fit model (blue), along with a comparison to the $f_+(0)$ posterior in our FF fit model (orange) and the FLAG 2024 average (red dotted).

showing a continued tension with unitarity of more than 2σ . However, we caution that the small error quoted in Eq. (5.10) does not yet account for long-distance electromagnetic effects or isospin-breaking effects that can reach the percent level. To improve upon this first global $|V_{us}|$ analysis, it is therefore paramount to develop a framework to include these effects consistently across all of the decays used here.

Despite now including lattice data on $f_+(0)$ in this fit, our posterior prediction for this quantity remains very similar to the results obtained for the FF posterior discussed in Sec. VA. To illustrate this, we show the joint distributions of $|V_{us}|$, f_K , and $f_+(0)$ in Fig. 4. We see a strong negative correlation between $|V_{us}|$ and both f_K and $f_+(0)$. The correlation between $|V_{us}|$ and f_K is expected since the extremely precise $K \rightarrow \mu\nu$ branching ratio measurement depends on the product. The correlation between $|V_{us}|$ and $f_+(0)$ can also be understood, since the K_{l3} decays effectively depend on this normalization along with some mild q^2 dependence. The value obtained for $f_+(0)$ in the CKM posterior remains in good agreement with the FLAG world average [21].

C. Fits searching for BSM effects

Given the long-standing discrepancy between V_{us} values extracted from two- and three-body kaon decays, we use our final fit model to study whether the fit quality is improved by BSM effects. The BSM model allows for the presence of a right-handed current in the form of a nonzero contribution by the WET Wilson coefficient $C_{V,R}$ [corresponding to the operator defined in Eq. (2.4)]. As discussed in Sec. IV B, no further hadronic matrix elements beyond those described in Sec. III A arise in this setting. Our fit gives

$$C_{V,R} = 0.0027 \pm 0.0017(\text{stat}) \pm 0.0020(\text{model}), \quad (5.12)$$

showing a very mild preference for BSM physics (albeit only at the 1σ level due to the model variation we observe), in general agreement with other previous results [7, 11–13, 89]. However, again we caveat this statement with the fact that, given our neglect of long-distance EM and isospin-breaking corrections, our results should not be directly compared with the more sophisticated analyses available in the literature.

VI. CONCLUSIONS

In this paper, we have presented a pilot study showing that data on $K \rightarrow \ell \nu$, $K \rightarrow \pi \ell \nu$, and $\tau \rightarrow K \pi \nu$ decays can be simultaneously described within a single framework, using a new simple form factor parametrization recently developed for the pion vector form factor. For this, we make use of theoretical constraints from unitarity in the form of dispersive bounds and chiral perturbation theory calculations for the Callan-Treiman relations, and lattice QCD calculations for the pion and kaon decay constants. This parametrization transparently takes into account the resonances present in the τ decay spectrum, obeys the required properties P1–P3 and P5–P7, and does not require input on the hadronic phase shift as needed in parametrizations based on the Omnès representation.

As discussed in Sec. VA, we find that we are consistently able to fit the $K^*(892)$ resonance pole parameters within our setup and with the current experimental datasets on τ decays. We find that our results for the pole parameters are in good agreement with the PDG world average. On the other hand, higher resonances in the f_+ spectrum, along with all resonances in the f_0 spectrum, are resistant to being fitted. We attribute these issues to different effects. For the $K^*(1680)$, we suspect insufficient statistical power due to limited precision of the dataset close to the end point of the phase space. For the $K^*(1410)$ and $K_0^*(1430)$, the lack of data on the angular distribution in the $K\pi$ helicity angle prohibits us from separating both resonances. We arrive at a nominal fit model that contains the $K^*(892)$, $K^*(1410)$, $K_0^*(700)$, and $K_0^*(1430)$, with the model uncertainty estimated by comparison with a fit model that additionally includes the $K^*(1680)$, a fit model that increases the z truncation order by one and a fit model that does both.

In our initial fits, when studying only the hadronic parameters, we further produce posterior predictions for the form factors and their derivatives at $q^2 = 0$. Since we do not make use of any lattice data on the $K\pi$ form factors, we are able to use these predictions to test the consistency of our setup. We do so by comparing them to lattice QCD results of the form factor and experimental results for the derivatives of the form factors. In both cases, we find excellent agreement. We additionally provide a new estimate for the absolute value of the f_0 form factor at $q^2 = m_D^2$ [Eq. (5.8)], which is an important input to nonleptonic D decay calculations, at

approximately twice the precision obtained previously in the literature.

With our form factor fits in hand, we then include $\tau \rightarrow K \nu$ and $K \rightarrow \mu \nu$ experimental data together with the lattice QCD world average of $f_+(0)$ to perform the first global $|V_{us}|$ fit, finding a result [Eq. (5.10)] that is in good agreement with previous individual $|V_{us}|$ determinations while still in tension with first-row unitarity at the level of around 2σ .

As a final investigation, we study a model that allows for a beyond the Standard Model contribution to a right-handed current, corresponding to the operator defined in Eq. (2.4). Such an effect has been studied in the literature as a possible explanation of the discrepancy seen between individual $|V_{us}|$ determinations from two- and three-body decays. Our fit favors a nonzero central value for the BSM Wilson coefficient $C_{V,R} \simeq 0.0027$, which aligns well with BSM studies in the literature. However, given our model uncertainties, our results for a right-handed BSM coupling are compatible with zero at the 1σ level.

We identify two key issues that limit further progress. First, we lack a consistent description of the long-distance electromagnetic corrections to the $K\pi$ matrix elements both in the semileptonic and in the pair-production region that is independent of the choice of form factor parametrization. Second, we lack measurements on the angular distribution in the $K\pi$ helicity angle for the τ decay data. Such measurements would make it possible to better separate contributions to the f_+ and f_0 form factors. We look forward to updating our analysis to provide a more accurate global $|V_{us}|$ determination once these issues are overcome.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [49].

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