

Edge physics and the Casimir interaction in Maxwell-Chern-Simons theory on a strip

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We study Maxwell-Chern-Simons theories on a strip $\mathcal{M} = \mathbb{R}^{1,1} \times [0, h]$ within the Symanzik framework for quantum field theory with boundaries. We add the most general local boundary functional within the quadratic one-tangential-derivative truncation adopted throughout the paper and derive the admissible boundary conditions from locality and the variational principle, with gauge symmetry implemented through boundary Ward identities. The strip supports two boundary $U(1)$ current algebras, with opposite levels fixed by the Chern-Simons coupling κ and by boundary orientation, while edge velocities depend on local boundary couplings. They are equal and opposite on the flip-symmetric branch, which is the natural choice for a spatially symmetric strip. The same boundary data determine the reflection coefficients for bulk modes and yield a closed determinant (scattering) representation of the finite interaction energy. In the Maxwell limit ($\kappa = 0$) the interedge interaction is long ranged and produces a power-law Casimir force, whereas in Maxwell-Chern-Simons theory the topological mass $m = \kappa g^2$ generates Yukawa suppression at large separations $mh \gg 1$. The determinant form also gives analytic control of the asymptotic regimes and separates topological boundary data (levels and anomaly inflow) from dynamical bulk effects (dispersion, boundary mixing, and Casimir interaction).

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I. INTRODUCTION

Three-dimensional (3D) gauge theories provide a clean setting in which topological and dynamical mechanisms coexist and can be sharply separated. Pure Chern-Simons (CS) theory has no propagating bulk degrees of freedom and is insensitive to the metric, yet in the presence of a boundary it supports physical edge excitations whose current algebra level is fixed by the CS coupling κ and by boundary orientation [1–3]. Adding a Maxwell term produces Maxwell-Chern-Simons (MCS) theory, where a massive propagating bulk mode coexists with the topological CS sector [4–6]. On a strip geometry $\mathcal{M} = \mathbb{R}^{1,1} \times [0, h]$, the width h becomes a genuine physical parameter: bulk fluctuations mediate an interaction between the two boundaries and, in a “wire” interpretation, this interaction is measured by the Casimir force [7,8]. The strip is therefore a natural setting in which to track how boundary data and bulk propagation combine in a finite-width observable.

Here we analyze Maxwell and Maxwell-Chern-Simons theory on a strip within the Symanzik framework, focusing on the interplay between boundary data, propagating bulk modes, and finite-width vacuum effects. In pure CS, the absence of local bulk dynamics implies that the strip width h introduces no new scale: observables are controlled only by κ and by boundary orientation, while the separation itself is invisible. Each boundary nevertheless supports a chiral current algebra with equal and opposite levels, with an equivalent description in terms of chiral bosons propagating in opposite directions, as in quantum-Hall edge physics [9–12]. On the flip-symmetric branch—the natural choice when the two sides of the strip are related by spatial reflection—the two edge velocities are equal in magnitude and opposite in sign, reproducing the standard counter-propagating configuration. In MCS theory this rigidity is lost: Maxwell dynamics makes the bulk nontopological and turns h into a genuine control parameter.

Our starting point is the Symanzik formulation of local quantum field theory with boundaries [13–15]. In this approach boundary interactions are encoded by the most general local boundary functional compatible with power counting and the symmetries of the problem, and admissible boundary conditions are derived from separability and locality rather than imposed *a priori*. In the standard approach to QFT on bounded domains (and in spectral/heat-kernel analyses) one instead fixes a well-posed,

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self-adjoint problem by *choosing* boundary conditions, equivalently selecting a self-adjoint extension of the relevant differential operator (e.g., [14,16,17]). Here we apply the Symanzik logic to Maxwell and MCS gauge theories and compute the *finite* interedge vacuum interaction on the strip, starting from the most general *local* boundary functional within the quadratic, one-tangential-derivative truncation compatible with power counting. The result is a closed determinant (scattering) representation for the interaction energy in terms of the boundary reflection data, with a transparent crossover between the long-ranged Maxwell regime and the Yukawa-suppressed topologically massive theory. Although much of the analysis is classical, the Casimir interaction is extracted from the quadratic fluctuation determinant. Locality ensures that h -independent contributions can be subtracted consistently without affecting the physical h dependence [7,8]. From the boundary point of view, the relevant data are local operators supported at $x^2 = 0$ and $x^2 = h$, constrained by dimensional analysis and the global symmetries of the problem. From the bulk point of view, the same data appear as boundary conditions on the gauge field and on its normal derivatives. The Symanzik prescription demands that these two viewpoints be strictly equivalent. This bulk–boundary correspondence is our organizing principle throughout [13,15,18,19].

Concretely, our analysis proceeds in three steps:

- (i) We construct the most general local quadratic boundary functional compatible with power counting and derive from it the boundary conditions on the gauge field. These conditions relate tangential components of A_μ to the normal field strengths at the two boundaries and organize the admissible dynamics into branches in the boundary-parameter space, conveniently parametrized by two sets of coefficients $a_n^{(\alpha)}$ on each boundary component $\partial\mathcal{M}_\alpha$ ($\alpha = 0, h$). In particular, we identify a flip-symmetric branch exchanging the two edges, which plays a distinguished role in the resulting boundary dynamics.
- (ii) We implement residual gauge symmetry in the presence of a boundary through a broken Ward identity. The CS term is the only source of breaking, and its boundary variation fixes the Kac-Moody central terms (levels) of the edge current algebras purely in terms of κ and boundary orientation, independent of the Maxwell dynamics [1,2], in agreement with the general CS/RCFT correspondence [20]. The corresponding boundary degrees of freedom are described by scalar edge fields living on $\partial\mathcal{M}_0$ and $\partial\mathcal{M}_h$. While the levels are “topological” and hence robust, the edge velocities are dynamical quantities controlled by the local boundary couplings. Equal and opposite velocities arise only on a flip-symmetric branch.

- (iii) Following [21], we interpret the derived boundary conditions as scattering data for bulk fluctuations. After Fourier transform along $\mathbb{R}^{1,1}$ (and Wick rotation when appropriate), bulk profiles depend on a momentum scale Q determined by the bulk dispersion. In MCS this includes the topological mass $m = \kappa g^2$ [4,5]. The boundary conditions determine the reflection matrices at $x^2 = 0$ and $x^2 = h$, and the corresponding multiple-reflection series is resummed by a determinant representation for the h -dependent vacuum energy and the Casimir force [22–24].

For pure Chern-Simons theory on a strip, the two boundaries support chiral edge sectors with opposite Kac-Moody levels, while the absence of propagating bulk modes prevents any finite-width Casimir interaction [25]. By contrast, existing analyses of Maxwell-Chern-Simons theory with a *single* boundary derive the boundary current algebra and chiral edge dynamics in different gauges, but do not address the genuinely two-boundary problem or the associated interedge vacuum interaction [19,26]. More generally, in systems with two boundaries global data can couple the two boundary sectors and are not captured by the local boundary conditions at each edge separately [27].

In the present strip setup we stay within a local Symanzik framework on $\mathbb{R}^{1,1} \times [0, h]$. Starting from the most general local quadratic boundary functional within the truncation defined in Sec. III, we classify the admissible boundary branches, show that the Kac-Moody central term depends only on κ and boundary orientation, and reinterpret the resulting boundary conditions as reflection data for the propagating bulk mode. This leads to a closed determinant formula for the finite Casimir interaction and keeps distinct the universal algebraic data from the branch-dependent dynamical data.

The paper is organized as follows. Section II introduces the geometry, bulk actions, and conventions. Section III classifies the local boundary terms and derives the corresponding boundary conditions from the Symanzik requirement. Section IV contains the broken Ward identity and the derivation of the boundary current algebra, while Sec. V discusses the associated boundary degrees of freedom and the role of the flip symmetry exchanging the two boundaries. Maxwell and MCS theory on the strip are then analyzed in Sec. VI, and the Casimir energy and force are computed in Sec. VII using the determinant/scattering representation. Asymptotic regimes, in particular $mh \gg 1$, are discussed in Sec. VIII before the concluding Section. Technical details are collected in the Appendixes.

II. GEOMETRY, CONVENTIONS, AND BULK ACTIONS

We work in flat Minkowski $2 + 1$ (3D) spacetime, with coordinates $x^\mu = (x^0, x^1, x^2)$, $\mu = 0, 1, 2$, and metric

$$\eta_{\mu\nu} = \text{diag}(-, +, +). \quad (2.1)$$

The Levi-Civita symbol is $\epsilon^{\mu\nu\rho}$, with $\epsilon^{012} = +1$. Boundary indices are denoted by $i, j = 0, 1$, and we use $\epsilon^{2ij} \equiv \epsilon^{ij}$ with $\epsilon^{01} = +1$. The strip region is

$$\mathcal{M} = \mathbb{R}^{1,1} \times [0, h], \quad \partial\mathcal{M} = \Sigma_0 \cup \Sigma_h, \quad (2.2)$$

with boundaries Σ_0 at $x^2 = 0$ and Σ_h at $x^2 = h$. The outward-pointing unit normals are

$$n_\mu|_{\Sigma_0} = -\delta_\mu^2, \quad n_\mu|_{\Sigma_h} = +\delta_\mu^2. \quad (2.3)$$

The opposite orientations of Σ_0 and Σ_h will be relevant for boundary contributions induced by the CS term. On each boundary component Σ_α we use the induced 2D Minkowski metric $\eta_{ij} = \text{diag}(-, +)$ to raise and lower tangential indices, $V^i = \eta^{ij}V_j$. When convenient we denote points on the two boundary components by the embeddings $X^\mu(x^i) = (x^0, x^1, 0)$ and $\bar{X}^\mu(x^i) = (x^0, x^1, h)$. Accordingly, for any bulk field $\mathcal{O}$ we write $\mathcal{O} \equiv \mathcal{O}|_{x^2=0}$ and $\mathcal{O}(\bar{X}) \equiv \mathcal{O}|_{x^2=h}$. We consider an Abelian gauge field A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and bulk action

$$S_{\text{bulk}}[A] = \int_{\mathcal{M}} d^3x \left[-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right]. \quad (2.4)$$

The mass dimension of the gauge field is $[A_\mu] = 1/2$, the Maxwell coupling g^2 is dimensionless and the CS coupling κ has dimension of mass. The topological mass scale is [4,5]

$$m \equiv \kappa g^2. \quad (2.5)$$

The relevant limiting regimes are:

- (i) Maxwell limit: $\kappa \rightarrow 0$ at fixed g^2 ($m = 0$);
- (ii) CS limit: $g^2 \rightarrow \infty$ at fixed κ (pure CS);
- (iii) MCS regime: $g^2, \kappa \neq 0$ (one massive bulk mode of mass $|m|$).

We use a covariant gauge-fixing term

$$S_{\text{gf}}[A] = -\frac{1}{2\xi} \int_{\mathcal{M}} d^3x (\partial_\mu A^\mu)^2, \quad (2.6)$$

where ξ is the dimensionless gauge parameter. With this convention the gauge-fixing normalization is independent of g^2 . In the Abelian theory Faddeev-Popov ghosts decouple and will be omitted. To generate boundary correlators we couple external sources J_α^i just on the boundary components Σ_0 and Σ_h : since our focus is the boundary current algebra, bulk sources are not introduced (they can be added straightforwardly, but are not needed in what follows),

$$S_J = \int_{\Sigma_0} d^2x J_{(0)}^i A_i + \int_{\Sigma_h} d^2x J_{(h)}^i A_i, \quad (2.7)$$

and define $Z[J]$ and $W = \ln Z$ as the functionals of Green's and connected Green's functions, respectively. Boundary correlators of the gauge field are generated by

$$\langle A_{i_1}(x_1) \dots A_{i_n}(x_n) \rangle \equiv \frac{\delta W}{\delta J_{i_1}^{(\alpha)}(x_1) \dots \delta J_{i_n}^{(\alpha)}(x_n)} \Big|_{J=0}, \quad (2.8)$$

$x \in \Sigma_\alpha, \alpha \in \{0, h\}.$

For the Casimir problem it is convenient to Wick rotate to Euclidean time and Fourier transform along the directions parallel to the boundaries. Denoting by (ζ, k) the Euclidean frequency and spatial momentum along x^1 , bulk profiles along x^2 behave as $e^{\pm Qx^2}$ with

$$Q(\zeta, k) \equiv \sqrt{\zeta^2 + k^2 + m^2}. \quad (2.9)$$

We take $Q(\zeta, k) \geq 0$, so that $Q(\zeta, k)$ acts as an inverse decay length along the transverse direction x^2 . In the Maxwell limit one sets $m = 0$. In the next Section we construct the most general local quadratic boundary functional compatible with power counting and derive the corresponding boundary conditions from the variational principle and separability.

III. LOCAL BOUNDARY ACTION AND DERIVED BOUNDARY CONDITIONS

Following Symanzik [13], boundary interactions are encoded by local terms supported at $x^2 = 0$ and $x^2 = h$,

$$S_{\text{bd}} = \int_{\mathcal{M}} d^3x [\delta(x^2) \mathcal{L}_0 + \delta(x^2 - h) \mathcal{L}_h], \quad (3.1)$$

where $\mathcal{L}_0$ and $\mathcal{L}_h$ are local functionals of boundary fields and tangential derivatives. The guiding principles are:

- (i) Locality: $\mathcal{L}_{0,h}$ contain only local operators on each boundary.
- (ii) Separability: in the absence of explicit nonlocal couplings, Σ_0 and Σ_h are independent. This is implemented by independent sources in (2.7) and by excluding bilocal boundary terms.

We work at quadratic order in the gauge field and keep a finite set of local operators sufficient to generate the most general *linear* boundary conditions relating A_i and the normal field strength F_{2i} . On the 2D boundary the density $\mathcal{L}_\alpha$ has mass dimension two, so by power counting we allow local terms with at most one tangential derivative. A convenient basis is

TABLE I. Local boundary branches implied by the δF_{2i} boundary variation (Appendix A).

Branch on Σ_α	Extra condition	Edge currents?	Comment
$a_4^{(\alpha)} = 0$	None beyond (3.5)	Yes	Used in Secs. IV–V
$a_4^{(\alpha)} \neq 0$	$A_i _{\Sigma_\alpha} = 0$	No	Dirichlet-like tangential sector

$$\begin{aligned} \mathcal{L}_\alpha &= \frac{a_1^{(\alpha)}}{2} A_i A^i + \frac{a_2^{(\alpha)}}{2} \epsilon^{ij} A_i \partial_0 A_j \\ &\quad + \frac{a_3^{(\alpha)}}{2} \epsilon^{ij} A_i \partial_1 A_j + a_4^{(\alpha)} A_i F_2^i, \\ \alpha &\in \{0, h\}, \end{aligned} \quad (3.2)$$

where the coefficients $a_n^{(\alpha)}$ parametrize the most general local boundary physics within this truncation, and

$$F_{2i} = \partial_2 A_i - \partial_i A_2 \quad (3.3)$$

is the component of the Abelian strength tensor that mixes tangential and normal fields. Boundary conditions are *derived* from the variational principle and will be further constrained by residual gauge symmetry and separability in the next Section. Accordingly, “most general” refers here to local, quadratic boundary terms built from the tangential pullback A_i and at most one tangential derivative, modulo tangential integrations by parts and linear redefinitions that leave the induced boundary conditions unchanged. Within this truncation, terms containing $\partial_i A^i$ are redundant up to integrations by parts, while dependence on the normal component is kept only through F_{2i} so that the resulting boundary equations remain linear in the tangential data and in their normal derivative. Equation (3.2) therefore provides a convenient basis for the linear local boundary conditions studied below. Let

$$S_{\text{tot}} = S_{\text{bulk}} + S_{\text{gf}} + S_{\text{bd}} + S_J. \quad (3.4)$$

Unless otherwise stated, by “boundary conditions” we mean the conditions induced by the local boundary action on the tangential components A_i ($i = 0, 1$) and on the normal field-strength components F_{2i} evaluated at Σ_0 and Σ_h . Varying S_{tot} yields bulk equations of motion and boundary contributions at $x^2 = 0, h$. Requiring $\delta S_{\text{tot}} = 0$ for admissible variations gives linear boundary conditions on each boundary component, which can be written compactly as

$$(\mathcal{B}_{ij}^{(\alpha)} A^j + \mathcal{C}_{ij}^{(\alpha)} F_2^j)|_{x^2=\alpha} = 0, \quad \alpha \in \{0, h\}. \quad (3.5)$$

The matrices $\mathcal{B}^{(\alpha)}$ and $\mathcal{C}^{(\alpha)}$ depend on (g^2, κ) and on the boundary coefficients in (3.2). Their explicit form for the quadratic truncation (3.2) is collected in Appendix A.

A. Boundary branches and the edge-active sector

A key point of the local boundary functional (3.2) is the presence of the term $a_4^{(\alpha)} A_i F_2^i$, which depends explicitly on the normal field-strength components F_{2i} . As shown in Appendix A, the boundary variation contains a term proportional to δF_{2i} and yields the extra boundary condition

$$a_4^{(\alpha)} A^i|_{\Sigma_\alpha} = 0, \quad \alpha \in \{0, h\}, \quad (3.6)$$

which is Eq. (A13) in the Appendix.

Equation (3.6) splits the space of local boundary data into two qualitatively different *branches* on each boundary component, summarized in Table I:

- (i) Edge-active branch ($a_4^{(\alpha)} = 0$): the admissible boundary conditions are entirely encoded in the tangential equations (3.5) [equivalently, (A10)–(A11) in Appendix A]. This is the branch that supports nontrivial boundary currents and edge dynamics discussed in Secs. IV–V.
- (ii) Dirichlet-like branch ($a_4^{(\alpha)} \neq 0$): the extra condition forces $A_i|_{\Sigma_\alpha} = 0$, which suppresses the boundary current sector. This branch is still admissible as a local boundary condition for the bulk problem, but it does not lead to the edge current algebra analyzed below.

In the remainder of the paper, whenever we discuss the boundary current algebra and the associated chiral edge modes, we work on the edge-active branch $a_4^{(\alpha)} = 0$.

IV. BROKEN WARD IDENTITY AND BOUNDARY CURRENT ALGEBRA

Consider an infinitesimal Abelian gauge transformation

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda. \quad (4.1)$$

The Maxwell term is gauge invariant also in the presence of a boundary. In covariant Lorenz gauge the gauge-fixing term S_{gf} is not separately gauge invariant: using Eq. (A8), its variation is proportional to $(\partial \cdot A)$ and generates a bulk term plus a boundary-supported term. In the Ward identity these contributions are local in the boundary coordinates and therefore produce only contact terms, or equivalently local improvements of the boundary currents, exactly as for the local Symanzik functional S_{bd} . They do not affect the universal coefficient of the CS breaking and

hence do not modify the Kac-Moody level. The CS term instead produces the nontrivial boundary breaking

$$\delta_\Lambda S_{\text{CS}} = \frac{\kappa}{2} \int_{\partial\mathcal{M}} d^2x \Lambda \epsilon^{ij} \partial_i A_j = \sum_{\alpha \in \{0, h\}} \sigma_\alpha \frac{\kappa}{2} \int_{\Sigma_\alpha} d^2x \Lambda \epsilon^{ij} \partial_i A_j, \quad (4.2)$$

where the sign σ_α encodes the opposite boundary orientations,

$$\sigma_0 = -1, \quad \sigma_h = +1. \quad (4.3)$$

Performing the gauge transformation (4.1), with Λ arbitrary on each Σ_α and vanishing at infinity along the boundary directions, yields a broken Ward identity. The source term varies as

$$\delta_\Lambda S_J = \sum_{\alpha \in \{0, h\}} \int_{\Sigma_\alpha} d^2x J_{(\alpha)}^i \partial_i \Lambda = - \sum_{\alpha \in \{0, h\}} \int_{\Sigma_\alpha} d^2x \Lambda \partial_i J_{(\alpha)}^i. \quad (4.4)$$

The Ward-Takahashi identity obtained from the change of variables is, in its primary form, an *integrated* functional identity [28,29]. Combining the CS boundary variation (4.2) with the source variation (4.4), and including the local contributions coming from the gauge variations of S_{bd} and S_{gf} , one may write it as

$$\sum_{\alpha \in \{0, h\}} \int_{\Sigma_\alpha} d^2x \Lambda \left[\partial_i J_{(\alpha)}^i - \sigma_\alpha \frac{\kappa}{2} \epsilon^{ij} \partial_i A_j \right] \Big|_{\Sigma_\alpha} - (\text{local terms from } S_{\text{bd}} \text{ and } S_{\text{gf}}) = 0. \quad (4.5)$$

Because Λ can be chosen as an arbitrary test function independently on each boundary component, Eq. (4.5) is equivalent to a local relation on each Σ_α , namely

$$\partial_i J_{(\alpha)}^i = \sigma_\alpha \frac{\kappa}{2} \epsilon^{ij} \partial_i \langle A_j \rangle_J \Big|_{\Sigma_\alpha} + (\text{local terms from } S_{\text{bd}} \text{ and } S_{\text{gf}}), \quad x \in \Sigma_\alpha, \quad \alpha \in \{0, h\}. \quad (4.6)$$

This step is where separability is crucial: in the absence of explicit interboundary couplings, the two boundary components admit independent gauge parameters and independent sources, so the Ward identity splits into two local identities, one on Σ_0 and one on Σ_h . Here “local terms from S_{bd} and S_{gf} ” denotes the boundary-supported contributions produced by the gauge variation of the Symanzik action together with the covariant gauge-fixing term. They are local differential polynomials in the boundary fields and in $(\partial \cdot A)$, and depend on the coefficients $a_n^{(\alpha)}$ as well as on the gauge-fixing

choice. In the Ward identity they contribute only contact terms and possible local improvements of the boundary currents. By contrast, the CS breaking term controls the Kac-Moody central extension [1,2] and depends only on the CS coupling κ and on boundary orientation, not on Maxwell dynamics. To derive Ward identities among correlators and, in particular, the Kac-Moody central term, it is convenient to rewrite (4.6) as a functional equation for $W[J]$. Using Eq. (2.8), Eq. (4.6) can be written as

$$\partial_i J_{(\alpha)}^i = \sigma_\alpha \frac{\kappa}{2} \epsilon^{ij} \partial_i \frac{\delta W}{\delta J_{(\alpha)}^j} + (\text{local terms from } S_{\text{bd}} \text{ and } S_{\text{gf}}), \quad x \in \Sigma_\alpha. \quad (4.7)$$

The boundary current on each component is obtained from the generating functional as

$$j_{(\alpha)}^i \equiv \frac{\delta W}{\delta J_{(\alpha)}^i}, \quad x \in \Sigma_\alpha, \quad \alpha \in \{0, h\}. \quad (4.8)$$

The CS breaking selects the gauge-invariant boundary combination

$$\mathcal{J}_{(\alpha)} \equiv \epsilon^{ij} \partial_i A_j \Big|_{\Sigma_\alpha}, \quad \alpha \in \{0, h\}. \quad (4.9)$$

This combination is gauge invariant since $\epsilon^{ij} \partial_i \partial_j \Lambda \equiv 0$, and it is precisely the object that appears in the boundary variation (4.2). At vanishing external sources $J_{(\alpha)}^i = 0$, the local resolution of the Ward identity implies

$$\mathcal{J}_{(\alpha)} = 0 \Rightarrow, \quad A_i \Big|_{\Sigma_0} = \partial_i \phi, \quad A_i \Big|_{\Sigma_h} = \partial_i \bar{\phi}, \quad (4.10)$$

locally on each boundary. This introduces two scalar edge fields ϕ and $\bar{\phi}$ living on Σ_0 and Σ_h , respectively. Differentiating (4.7) with respect to the sources and then setting the sources to zero yields identities for boundary-current correlators. Focusing on a fixed boundary component Σ_α , we start from the broken Ward identity for $W = \ln Z$, Eq. (4.7), and take one functional derivative with respect to $J_0^{(\alpha)}(y)$. Setting the sources to zero and using $j_{(\alpha)}^i \equiv \delta W / \delta J_{(\alpha)}^i$, this yields a relation for connected (two-point) correlators,

$$\partial_{x^0} \delta^{(2)}(x - y) = \sigma_\alpha \frac{\kappa}{2} \epsilon^{ij} \partial_i^x \langle j_{(\alpha)}^j j_{(\alpha)}^0(y) \rangle + (\text{local contact terms from } S_{\text{bd}}). \quad (4.11)$$

Expanding the ϵ -tensor ($\epsilon^{01} = +1$) gives

$$\epsilon^{ij} \partial_i^x \langle j_{(\alpha)}^j j_{(\alpha)}^0(y) \rangle = \partial_{x^0} \langle j_{(\alpha)}^1 j_{(\alpha)}^0(y) \rangle - \partial_{x^1} \langle j_{(\alpha)}^0 j_{(\alpha)}^0(y) \rangle. \quad (4.12)$$

In the subsequent Bjorken-Johnson (BJL) high-frequency limit, the ∂_{x^0} term contributes only local contact terms (from the time-ordering and from S_{bd}), so one may equivalently write the Ward identity in the form

$$\partial_i^x \langle j_{(\alpha)}^i j_{(\alpha)}^0(y) \rangle = -\sigma_\alpha 2\kappa \partial_{x^1} \delta^{(2)}(x-y) + (\text{contact terms from } S_{\text{bd}}). \quad (4.13)$$

The right-hand side is a pure contact term: it is fixed by the CS breaking, is independent of Maxwell dynamics, and can be reabsorbed into local improvements of the boundary currents without modifying the coefficient of $\partial_{x^1} \delta(x^1 - y^1)$ that determines the level.

To extract the equal-time commutator we use the standard BJL high-frequency limit of the *time-ordered* two-point function [30,31]: taking the Fourier transform with respect to $(x^0 - y^0)$ and multiplying by the frequency p_0 , the $p_0 \rightarrow \infty$ limit isolates the equal-time commutator of the charge densities,

$$\lim_{p_0 \rightarrow \infty} p_0 \int d(x^0 - y^0) e^{ip_0(x^0 - y^0)} \langle j_{(\alpha)}^0 j_{(\alpha)}^0(y) \rangle = i \langle [j_{(\alpha)}^0, j_{(\alpha)}^0(y)] \rangle_{x^0=y^0}. \quad (4.14)$$

Using (4.13) in this limit gives the $U(1)$ Kac-Moody central term on each boundary component [1,2]. Denoting by $\rho_{(\alpha)}(x^1) \equiv j_{(\alpha)}^0(x^0, x^1)$, at equal time x^0 one obtains

$$[\rho_{(\alpha)}(x^1), \rho_{(\alpha)}(y^1)]_{x^0=y^0} = -i\sigma_\alpha 2\kappa \partial_{x^1} \delta(x^1 - y^1). \quad (4.15)$$

This commutator defines a $U(1)$ Kac-Moody algebra with level $k_\alpha = -2\sigma_\alpha \kappa$. Locality of the boundary action (no interboundary couplings) implies that currents living on different boundary components commute,

$$[\rho_{(0)}(x^1), \rho_{(h)}(y^1)]_{x^0=y^0} = 0. \quad (4.16)$$

The central term in (4.15) is topological: it depends only on the CS coupling (and on boundary orientation). By contrast, the dynamical data of the edge sector—in particular the velocities of the boundary modes—depend on the boundary couplings $a_n^{(\alpha)}$ and on Maxwell dynamics. These velocities will be kept explicit and derived in the next Section.

This conclusion is consistent with earlier analyses of Maxwell-Chern-Simons theories with boundary, where the boundary Kac-Moody level is fixed by the topological CS term, while the Maxwell sector does not affect the central extension [19,26].

V. EDGE SCALAR FIELDS AND CHIRAL DYNAMICS

In this Section we make explicit the emergence of the two 2D scalar degrees of freedom ϕ and $\bar{\phi}$ (4.10) living on the boundary components Σ_0 and Σ_h . Throughout we use the boundary embeddings introduced in Sec. II ($X^i \equiv (x^0, x^1, 0)$ and $\bar{X}^i \equiv (x^0, x^1, h)$ with $i = 0, 1$), so that boundary quantities are evaluated as $A_i \equiv A_i(x^0, x^1, 0)$ and $A_i(\bar{X}) \equiv A_i(x^0, x^1, h)$. We will also introduce the two edge velocities, which are fixed by the local boundary data and control the chiral propagation of boundary modes.

A. Boundary currents and scalar potentials at vanishing sources

The two edge scalar fields ϕ and $\bar{\phi}$ were defined in Eq. (4.10) from the local form of the Ward identity at vanishing sources. In this Section we relate them to the conserved boundary currents and derive their chiral dynamics. At vanishing external sources, the boundary currents $j_{(\alpha)}^i$ defined in Eq. (4.8) are conserved on each boundary component,

$$\partial_i j_{(\alpha)}^i = 0, \quad x \in \Sigma_\alpha, \quad \alpha \in \{0, h\}. \quad (5.1)$$

We parametrize any conserved 2D current locally in terms of a scalar potential as

$$j_{(0)}^i = \epsilon^{ij} \partial_j \phi, \quad j_{(h)}^i(\bar{X}) = \epsilon^{ij} \partial_j \bar{\phi}(\bar{X}). \quad (5.2)$$

This fixes the normalization convention for ϕ and $\bar{\phi}$, which, from (4.10), are defined up to constant shifts. As we show below, following standard practice [32], the Kac-Moody algebra of Sec. IV translates into canonical equal-time commutation relations for the scalar edge fields. We follow the normalization/conventions of Ref. [33]. In particular, the equal-time Kac-Moody commutator (4.15) on each boundary component fixes the universal central extension and depends only on the CS coupling κ .

B. Chirality constraints and edge velocities

The boundary dynamics is further constrained by the boundary conditions (3.5) originating from the Symanzik boundary action S_{bd} (3.1). In the regime relevant for the boundary current algebra (long wavelength along the boundary), the tangential boundary condition on Σ_α reduces to a local linear relation among the boundary values of the tangential gauge field [see Appendix A, Eq. (A12)],

$$\left(a_1^{(\alpha)} \eta_{ij} + \sigma_\alpha \frac{\kappa}{2} \epsilon_{ij} \right) A^j \Big|_{\Sigma_\alpha} = 0, \quad (5.3)$$

where σ_α is the orientation factor (4.3). At vanishing sources, the Ward identity implies $\mathcal{J}_{(\alpha)} = \epsilon^{ij} \partial_i A_j = 0$, and hence

$A_i|_{\Sigma_0} = \partial_i \phi$ and $A_i|_{\Sigma_h} = \partial_i \bar{\phi}$ locally, as in (4.10). Writing the tangential gauge field $A_i|_{\Sigma_\alpha}$ in terms of edge scalars as in (4.10), Eq. (5.3) reads, for a generic boundary component Σ_α ,

$$\left(a_1^{(\alpha)} \eta_{ij} + \sigma_\alpha \frac{\kappa}{2} \epsilon_{ij} \right) \partial^j \Phi_{(\alpha)} = 0, \quad \Phi_{(0)} \equiv \phi, \quad \Phi_{(h)} \equiv \bar{\phi}, \quad (5.4)$$

which, in components, is equivalent to

$$(\partial_0 + v_{(\alpha)} \partial_1) \Phi_{(\alpha)} = 0, \quad (5.5)$$

describing chiral bosons moving on the boundaries with constant velocities [10,12,32]

$$v_{(\alpha)} = \sigma_\alpha \frac{2a_1^{(\alpha)}}{\kappa}. \quad (5.6)$$

Equation (5.6) is the long-wavelength consequence of the tangential low-energy boundary conditions (A12): at this order the derivative structures multiplied by $a_2^{(\alpha)}$ and $a_3^{(\alpha)}$ are subleading, since they enter as $a_2^{(\alpha)} \partial_0 \sim a_2^{(\alpha)} p_0$ and $a_3^{(\alpha)} \partial_1 \sim a_3^{(\alpha)} p_1$ and therefore vanish relative to the $a_1^{(\alpha)}$ mass term as the tangential momenta $p_i \rightarrow 0$. They do not enter the chirality constraint. The leading edge velocity is thus controlled only by $a_1^{(\alpha)}$ and κ , even though $a_2^{(\alpha)}$ and $a_3^{(\alpha)}$ reappear in the full reflection data through ϱ_α in Sec. VI. Likewise, within the present classical/quadratic analysis there is no explicit g^2 -dependence in (5.6). Any g^2 -dependence of the edge velocity can arise only indirectly through loop renormalization of the local boundary couplings $a_n^{(\alpha)}$, a problem that lies beyond the scope of the present work. Therefore, on the two boundary components, we have

$$(\partial_0 + v \partial_1) \phi = 0, \quad (\partial_0 + \bar{v} \partial_1) \bar{\phi}(\bar{X}) = 0, \quad (5.7)$$

where

$$v \equiv v_{(0)} = -\frac{2a_1^{(0)}}{\kappa}, \quad \bar{v} \equiv v_{(h)} = +\frac{2a_1^{(h)}}{\kappa}. \quad (5.8)$$

In general v and $\bar{v}$ are independent because the two sets of local boundary couplings $a_n^{(0)}$ and $a_n^{(h)}$ are independent.

Flip symmetry. A useful discrete symmetry of the strip is the reflection $x^2 \mapsto h - x^2$, which exchanges the two boundary components $\Sigma_0 \leftrightarrow \Sigma_h$ while reversing the orientation factor σ_α . Imposing flip symmetry at the level of the local Symanzik data amounts to

$$a_n^{(h)} = a_n^{(0)}, \quad n = 1, 2, 3, 4, \quad (5.9)$$

while σ_α (4.3) remains fixed by geometry. Under (5.9) one finds

$$\bar{v} = -v. \quad (5.10)$$

Equation (5.10) has a direct physical interpretation: when the strip is symmetric under the exchange of its two boundary components, the two chiral edge velocities are equal in magnitude and opposite in sign. This reproduces the hallmark situation of Hall-type edge physics, where counterpropagating chiral modes live on opposite edges [10,12]. The same structure persists in the present MCS setting, even though Maxwell dynamics renders the bulk theory nontopological. In other words, the chirality constraint and the relative sign of the edge velocities are controlled by the CS sector only (and thus by κ) [1,2,20], while the Maxwell term affects bulk propagation and, in particular, the Casimir interaction studied later (and may influence the effective values of the boundary couplings through renormalization, without altering the CS-controlled central term). This is the basic separation used throughout the rest of the paper: within the present local framework the Kac-Moody level is universal, while the edge velocities, reflection data, and finite-width interaction are branch dependent.

VI. MAXWELL-CHERN-SIMONS THEORY ON A STRIP: SPECTRAL SETUP

In this Section we set up the strip boundary-value problem for the MCS theory defined by the bulk action (2.4) with covariant gauge fixing (2.6). The admissible boundary conditions are those derived from the Symanzik boundary action S_{bd} (3.1). We will use them only through their compact form (3.5) and through the explicit identification of $\mathcal{B}_{ij}^{(\alpha)}$ and $\mathcal{C}_{ij}^{(\alpha)}$ reported in Appendix A. The purpose of this section is to close, in a self-contained way, the chain

$$\begin{aligned} \text{boundary conditions} &\Rightarrow R_0(\zeta, k), R_h(\zeta, k) \\ &\Rightarrow \text{strip secular equation}, \end{aligned} \quad (6.1)$$

where $R_\alpha(\zeta, k)$, with $\alpha \in \{0, h\}$, are the reflection matrices [8,23] defined below. The chain (6.1) provides the natural starting point for the finite Casimir prescription implemented in Sec. VII. The goal is spectral: for fixed tangential momentum we express the strip compatibility condition in terms of the reflection matrices R_0 and R_h and isolate the unique propagation factor that carries all the h dependence. Throughout we keep explicit the dependence on the bulk couplings (g^2, κ) , since κ controls the boundary current algebra in Sec. IV while g^2 controls bulk propagation.

A. Fourier decomposition and transverse decay constant

We Fourier transform along the boundary directions (x^0, x^1) ,

$$A_i = \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} e^{-i\omega x^0 + ikx^1} \tilde{A}_i(\omega, k; x^2), \quad (6.2)$$

where the dispersion relation is

$$\omega^2 = k^2 + p^2 + m^2, \quad (6.3)$$

m is the topological mass (2.5), and p is the transverse momentum. For Casimir purposes we Wick rotate $\omega \rightarrow i\zeta$ and set

$$p_E^2 \equiv \zeta^2 + k^2, \quad Q(\zeta, k) \equiv \sqrt{p_E^2 + m^2}. \quad (6.4)$$

In Euclidean variables the bulk dependence along x^2 is therefore a linear combination of $e^{\pm Qx^2}$. Since we work in a covariant gauge, the quadratic fluctuation operator splits into a physical (transverse) sector and a gauge (longitudinal) sector. The longitudinal/ghost sector produces only h -independent contributions, as discussed in Sec. VII B. Therefore, for the h -dependent interaction we can restrict to the physical tangential channel and encode the boundary conditions through the corresponding reflection matrices. In practice, we will extract the strip spectrum and reflection data by projecting the boundary conditions onto the physical tangential sector. This produces a closed boundary-value problem for the physical tangential amplitudes and their normal derivatives. It is worth emphasizing that this 2×2 structure does *not* mean that Abelian MCS theory carries two independent propagating bulk particles: the bulk still contains a single topologically massive mode. The two-component space is instead the space of tangential boundary data (A_0, A_1) at fixed (ζ, k) . Local boundary operators can mix these two tangential components, so the reflection problem is naturally matrix-valued even though the propagating bulk content is a single mode.

B. Boundary conditions in momentum space and reflection matrices

The most general local boundary conditions derived in Sec. III can be written compactly as

$$(\mathcal{B}_{ij}^{(\alpha)} A_j + \mathcal{C}_{ij}^{(\alpha)} F_{2j})|_{x^2=\alpha} = 0, \quad (6.5)$$

where $\mathcal{B}^{(\alpha)}$ is a local operator along the boundary and $\mathcal{C}^{(\alpha)}$ is an algebraic matrix. Their explicit expressions in terms of the Symanzik boundary parameters $a_n^{(\alpha)}$ are given in Appendix A.

We focus on the tangential sector and denote the tangential boundary vector by $\tilde{A} = (\tilde{A}_0, \tilde{A}_1)^T$. For fixed (ζ, k) , bulk solutions can be decomposed as

$$\tilde{A}(\zeta, k; x^2) = e^{-Qx^2} \tilde{A}^{\text{in}}(\zeta, k) + e^{+Qx^2} \tilde{A}^{\text{out}}(\zeta, k), \quad (6.6)$$

so that

$$\partial_2 \tilde{A} = -Q \tilde{A}^{\text{in}} + Q \tilde{A}^{\text{out}}. \quad (6.7)$$

Inserting (6.6)–(6.7) into the projected (physical tangential) form of (3.5) yields

$$(\mathcal{B}^{(0)} - Q\mathcal{C}^{(0)}) \tilde{A}^{\text{in}} + (\mathcal{B}^{(0)} + Q\mathcal{C}^{(0)}) \tilde{A}^{\text{out}} = 0, \quad (6.8)$$

and, at the upper boundary after shifting the transverse coordinate, we have

$$(\mathcal{B}^{(h)} - Q\mathcal{C}^{(h)}) \tilde{A}_{(h)}^{\text{in}} + (\mathcal{B}^{(h)} + Q\mathcal{C}^{(h)}) \tilde{A}_{(h)}^{\text{out}} = 0. \quad (6.9)$$

Here $\tilde{A}_{(h)}^{\text{in/out}}$ are defined by shifting the transverse coordinate $y = h - x^2$ so that $\tilde{A} = e^{-Qy} \tilde{A}_{(h)}^{\text{in}} + e^{+Qy} \tilde{A}_{(h)}^{\text{out}}$; hence $\tilde{A}_{(h)}^{\text{in}} = e^{+Qh} \tilde{A}^{\text{out}}$ and $\tilde{A}_{(h)}^{\text{out}} = e^{-Qh} \tilde{A}^{\text{in}}$. Assuming the non-degeneracy condition

$$\det(\mathcal{B}^{(\alpha)} + Q\mathcal{C}^{(\alpha)}) \neq 0, \quad (6.10)$$

which is generic for local elliptic boundary conditions, the above equations define the reflection matrices. Degenerate loci, where the determinant vanishes, can support additional boundary zero modes or threshold states and require a separate treatment. In what follows we therefore work on the open nondegenerate region of boundary-parameter space where the reflection problem is well defined. These loci remain physically interesting, since they may signal critical boundary sectors with extra low-energy degrees of freedom.

$$\tilde{A}^{\text{out}} = R_0(\zeta, k) \tilde{A}^{\text{in}},$$

$$R_0(\zeta, k) = -(\mathcal{B}^{(0)} + Q\mathcal{C}^{(0)})^{-1}(\mathcal{B}^{(0)} - Q\mathcal{C}^{(0)}), \quad (6.11)$$

and

$$\tilde{A}_{(h)}^{\text{out}} = R_h(\zeta, k) \tilde{A}_{(h)}^{\text{in}},$$

$$R_h(\zeta, k) = -(\mathcal{B}^{(h)} + Q\mathcal{C}^{(h)})^{-1}(\mathcal{B}^{(h)} - Q\mathcal{C}^{(h)}). \quad (6.12)$$

For the quadratic Symanzik boundary action S_{bd} (3.1), the operator $\mathcal{B}^{(\alpha)}$ is a linear combination of η_{ij} and ϵ_{ij} , with momentum dependence through tangential derivatives. Before Wick rotation, in momentum space one may write (see Appendix A)

$$\begin{aligned} \mathcal{B}_{ij}^{(\alpha)}(\omega, k) &= a_1^{(\alpha)} \eta_{ij} + q_\alpha(\omega, k) \epsilon_{ij}, \\ q_\alpha(\omega, k) &\equiv a_2^{(\alpha)}(-i\omega) + a_3^{(\alpha)}(ik) + \sigma_\alpha \frac{\kappa}{2}, \end{aligned} \quad (6.13)$$

while from (A18) one has, in the same truncation,

$$\mathcal{C}_{ij}^{(\alpha)} = c_\alpha \eta_{ij}, \quad c_\alpha \equiv a_4^{(\alpha)} + \sigma_\alpha \frac{1}{g^2}. \quad (6.14)$$

Any 2×2 matrix of the form $u\eta_{ij} + v\epsilon_{ij}$ is inverted as

$$(u\eta + v\epsilon)^{-1} = \frac{u}{u^2 + v^2}\eta - \frac{v}{u^2 + v^2}\epsilon. \quad (6.15)$$

Applying (6.15) to (6.11)–(6.12) with (6.13)–(6.14), and using the multiplication rule $(u\eta + v\epsilon)(u'\eta + v'\epsilon) = (uu' - vv')\eta + (uv' + vu')\epsilon$, one finds that on each boundary component

$$R_\alpha(\zeta, k) = r_\alpha^{(\eta)}(\zeta, k)\eta + r_\alpha^{(\epsilon)}(\zeta, k)\epsilon, \quad (6.16)$$

with

$$\begin{aligned} r_\alpha^{(\eta)}(\zeta, k) &= -\frac{(a_1^{(\alpha)})^2 + q_\alpha(i\zeta, k)^2 - Q(\zeta, k)^2 c_\alpha^2}{(a_1^{(\alpha)} + Q(\zeta, k)c_\alpha)^2 + q_\alpha(i\zeta, k)^2}, \\ r_\alpha^{(\epsilon)}(\zeta, k) &= -\frac{2Q(\zeta, k)c_\alpha q_\alpha(i\zeta, k)}{(a_1^{(\alpha)} + Q(\zeta, k)c_\alpha)^2 + q_\alpha(i\zeta, k)^2}. \end{aligned} \quad (6.17)$$

Equivalently, after complexifying the tangential boundary space one may diagonalize ϵ and write

$$S^{-1}R_\alpha S = \begin{pmatrix} r_\alpha^{(+)} & 0 \\ 0 & r_\alpha^{(-)} \end{pmatrix}, \quad r_\alpha^{(\pm)} \equiv r_\alpha^{(\eta)} \pm ir_\alpha^{(\epsilon)}. \quad (6.18)$$

The two scalar quantities $r_\alpha^{(\pm)}$ are the two boundary-response channels of the same bulk excitation; they are not additional propagating bulk degrees of freedom.

C. Strip secular equation and relevant limits

To obtain the strip compatibility condition one relates the boundary data at Σ_0 and Σ_h through bulk propagation across the strip. In Euclidean momentum space this introduces a factor e^{-Qh} for each traversal, so that a complete round trip between the two boundaries yields the matrix product $R_0(\zeta, k)R_h(\zeta, k)e^{-2Qh}$. This is precisely the building block of the standard scattering (multiple-reflection) formulation of Casimir interactions, see, e.g., [21–23]. The compatibility (or secular) condition on the strip follows directly from the consistency of the bulk profile with the boundary relations at $x_2 = 0, h$. Starting from the general Euclidean solution (6.6) and applying the local Symanzik boundary conditions on both sides, one finds that an incoming amplitude reproduces itself after two reflections only if

$$\det [\mathbb{I} - R_0(\zeta, k)R_h(\zeta, k)e^{-2Q(\zeta, k)h}] = 0. \quad (6.19)$$

This represents the strip secular equation [8,23] in the multiple-reflection form. The detailed step-by-step derivation is given in Appendix B. Equation (6.19) is the starting point for the Casimir interaction analysis in Sec. VII.

Two limiting regimes will be central in the Casimir analysis:

- (i) Maxwell limit $\kappa \rightarrow 0$ at fixed g^2 : then $m = \kappa g^2 \rightarrow 0$ and $Q \rightarrow \sqrt{\zeta^2 + k^2}$, while the CS contribution in (6.13) disappears. On the boundary-algebra side, the CS breaking term in the broken Ward identity (4.6) vanishes with κ , hence the Kac-Moody central extension found in Sec. IV is absent in the Maxwell theory.
- (ii) Switching off Maxwell dynamics at fixed κ (formally $g^2 \rightarrow \infty$): then $m = \kappa g^2 \rightarrow \infty$ and $Q \sim m$, so propagation across the strip is exponentially suppressed, $e^{-2Qh} \sim e^{-2mh}$. Accordingly any separation-dependent vacuum interaction mediated by bulk propagation becomes short ranged. The detailed consequences of these limits for the Casimir interaction and force will be worked out in Secs. VII and VIII.

VII. FINITE CASIMIR ENERGY ON A STRIP

In this Section we compute the finite Casimir interaction for Maxwell and MCS theory on the strip $\mathcal{M} = \mathbb{R}^{1,1} \times [0, h]$, using the local Symanzik boundary construction developed in the previous Sections. The physically meaningful quantities are the interaction energy $E_{\text{int}}(h)$ and the corresponding Casimir force

$$F(h) \equiv -\frac{dE_{\text{int}}(h)}{dh}. \quad (7.1)$$

We compute the vacuum energy per unit length along x^1 . After Wick rotation, the one-loop interaction admits a representation as an integral over the Euclidean momenta (ζ, k) parallel to the boundaries, and is entirely controlled by the reflection data derived in Sec. VI.

A. Finite Casimir logic: Renormalized observable

Local counterterms generated by renormalization are integrals of local densities: bulk terms supported on $\mathcal{M}$ and boundary terms supported on each connected component Σ_0 and Σ_h . Locality implies that boundary counterterms split additively between the two components and are independent of the separation h . Therefore, any separation-dependent vacuum interaction is finite and scheme independent. Let $E_{\text{vac}}^{\text{ren}}(h)$ denote the renormalized vacuum energy per unit length along x^1 . A convenient and fully equivalent definition of the interaction part is the infinite-separation subtraction

$$E_{\text{int}}(h) \equiv E_{\text{vac}}^{\text{ren}}(h) - \lim_{h \rightarrow \infty} E_{\text{vac}}^{\text{ren}}(h), \quad (7.2)$$

which removes all bulk contributions and all boundary-local pieces supported on a single boundary component, independently of the detailed renormalization scheme.

Equation (7.2) implements the finite Casimir prescription in the present gauge-theory setting with local Symanzik boundary conditions [13,21]. The renormalized interaction is therefore entirely encoded in the physical tangential sector through the multiple-reflection factor derived in Sec. VI; see Sec. VII B for the explicit gauge-fixed decomposition.

B. Scattering/log-determinant representation

For fixed Euclidean frequency ζ and tangential momentum k , the physical bulk excitation propagates across the strip with inverse decay length (along x^2) $Q(\zeta, k)$ (6.4). The logic of this Subsection is the following: isolate the physical tangential sector, encode each boundary into a reflection operator, and then resum the repeated round trips across the strip. In this way the dependence on the chosen boundary branch is carried entirely by R_0 and R_h , whereas the dependence on the separation is carried entirely by the propagation factor e^{-2Qh} . The local boundary conditions derived from the Symanzik boundary action S_{bd} (3.1) define, for each fixed (ζ, k) , linear reflection maps between incoming and outgoing Euclidean bulk amplitudes in the physical tangential sector. In Sec. VI we encoded these boundary conditions at Σ_0 and Σ_h into reflection matrices $R_0(\zeta, k)$ and $R_h(\zeta, k)$ acting on the physical boundary data space. Bulk propagation from one boundary to the other produces a factor e^{-Qh} , so that a complete round trip yields the multiple-reflection factor

$$M(\zeta, k; h) \equiv R_0(\zeta, k)R_h(\zeta, k)e^{-2Q(\zeta, k)h}. \quad (7.3)$$

The strip secular equation (6.19) derived in Sec. VI can be written as $\det(\mathbb{I} - M(\zeta, k; h)) = 0$. Here and in the following, $\mathbb{I}$ denotes the identity endomorphism on the physical boundary data space and must not be confused with the tangential metric η_{ij} . In the present truncation this space is two-dimensional, so the determinant and trace below are taken over 2×2 matrices in the physical tangential channel. At one loop the vacuum functional is a ratio of functional determinants of the quadratic fluctuation operators, including gauge fixing and ghosts. In the present Abelian theory the Faddeev-Popov ghosts are free and do not couple to the boundary parameters $a_n^{(\alpha)}$. Their contribution, together with that of the longitudinal gauge sector selected by the covariant gauge fixing (2.6), can only modify bulk terms or single-boundary self-energies. In particular, it does not generate any boundary-parameter-dependent round-trip factor across the strip, and therefore drops out of the interaction energy after the infinite-separation subtraction (7.2). After projection onto the physical tangential sector, the only object that carries the nonlocal strip information is the round-trip factor $R_0R_h e^{-2Qh}$; no analogous longitudinal/ghost factor survives the infinite-separation subtraction. This yields the

standard scattering/log-determinant formula for the interaction energy per unit length [7,23,34–36]

$$E_{\text{int}}(h) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \ln \det(\mathbb{I} - M(\zeta, k; h)). \quad (7.4)$$

The determinant formula (7.4) is the compact way to resum the multiple-reflection series. For *passive* boundaries, the physically admissible parameter domain is characterized by the convergence of the round-trip expansion, i.e., by the condition that the spectral radius of the multiple-reflection operator satisfies

$$\rho(M(\zeta, k; h)) < 1 \quad \text{for all } (\zeta, k) \text{ contributing to the integral.} \quad (7.5)$$

In practice, for any given set of boundary parameters one can check passivity by verifying that all eigenvalues λ of $M(\zeta, k; h)$ satisfy $|\lambda| < 1$ for all Euclidean momenta $(\zeta, k) \in \mathbb{R}^2$. In the present context “passive” is meant in this minimal scattering sense: the quadratic boundary response does not amplify a Euclidean round trip, the logarithm in (7.4) stays on its principal real branch, and the resulting interaction defines a stable Casimir observable. The condition can be made completely explicit on a simple flip-symmetric analytic branch. Consider the edge-active subclass

$$\begin{aligned} a_2^{(0)} &= a_2^{(h)} = a_3^{(0)} = a_3^{(h)} = a_4^{(0)} = a_4^{(h)} = 0, \\ a_1^{(0)} &= a_1^{(h)} \equiv a_1. \end{aligned} \quad (7.6)$$

Then $c_0 = -1/g^2$, $c_h = +1/g^2$, $q_0 = -\kappa/2$, $q_h = +\kappa/2$, and the eigenvalues (6.18) satisfy

$$|r_0^{(\pm)}(\zeta, k)r_h^{(\pm)}(\zeta, k)| = 1. \quad (7.7)$$

Therefore the round-trip eigenvalues take the form

$$\lambda_{\pm}(M(\zeta, k; h)) = e^{-2Q(\zeta, k)h} e^{\pm i\vartheta(\zeta, k)}, \quad (7.8)$$

with a real phase $\vartheta(\zeta, k)$, and one obtains the exact passivity statement

$$\rho(M(\zeta, k; h)) = e^{-2Q(\zeta, k)h} < 1 \quad (h > 0). \quad (7.9)$$

Thus this entire one-parameter flip-symmetric family is passive for every real a_1 and every Euclidean momentum. More generally, passivity defines an open region in boundary-parameter space: because the eigenvalues of M depend continuously on $a_n^{(\alpha)}$, any sufficiently small deformation of the analytic branch (7.6)—including mild asymmetry between the two boundaries and small nonzero

$a_2^{(\alpha)}, a_3^{(\alpha)-}$ —remains passive. At large Euclidean momentum this is further reinforced by the universal suppression e^{-2Qh} , while possible violations can only arise in a bounded low-momentum region and are readily diagnosed numerically from the eigenvalues of M . Beyond such special subclasses, the passive domain is branch dependent and is best determined numerically from the eigenvalues of M .

It is useful to state precisely the range of validity of the log-determinant formula (7.4). There are two distinct requirements. First, the single-boundary reflection matrices must be well defined, which is the nondegeneracy condition (6.10). Second, the strip round-trip operator must define a passive Euclidean scattering problem: in the massive case $\det[\mathbb{I} - M(\zeta, k; h)]$ should have no zeros on the Euclidean integration domain, while in a strictly massless limit the standard threshold point $Q = 0$ is treated by the usual infrared limiting prescription. Away from such an integrable threshold, the round-trip eigenvalues should stay inside the unit disk, so that the logarithm remains on its principal real branch; equivalently, $\rho(M) < 1$ pointwise. A simple sufficient condition is $|r_0^{(\pm)}(\zeta, k)r_h^{(\pm)}(\zeta, k)| \leq 1$ on the whole domain, since the propagation factor $e^{-2Qh} < 1$ then guarantees $\rho(M) < 1$; the flip-symmetric family (7.6) saturates this bound with $|r_0^{(\pm)}r_h^{(\pm)}| = 1$ and is therefore passive for every $h > 0$, and the same holds in an open neighborhood of it. If the reflection matrices remain well defined but the passive bound is not satisfied, (7.4) should not be interpreted as a convergent round-trip logarithm. In that case the appropriate prescription is the secular/zeta representation of section VII C, or equivalently the force formula (7.10) integrated with $E_{\text{int}}(h \rightarrow \infty) = 0$, provided the chosen contour does not cross zeros of $D(\zeta, k; h) = \det[\mathbb{I} - M]$. Zeros of D on the integration contour, away from the massless threshold just mentioned, signal strip bound states, zero modes, or instabilities and require a separate treatment. They are distinct from the single-boundary degeneracies of (6.10), although both types of singular behavior lie outside the generic elliptic passive sector considered here. In particular, the flip-symmetric configurations used in the numerical analysis below, and their finite neighborhood, are entirely within the regime where (7.4) applies.

In the passive domain one may equivalently work with the Casimir force, obtained by differentiating (7.4) with respect to h and using $\partial_h M = -2QM$,

$$F(h) = -\frac{dE_{\text{int}}}{dh} = -\int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \times \int_{-\infty}^{+\infty} \frac{dk}{2\pi} Q(\zeta, k) \text{Tr} \left[\frac{M(\zeta, k; h)}{\mathbb{I} - M(\zeta, k; h)} \right]. \quad (7.10)$$

Equation (7.10) is the operational definition used in practice: it is manifestly real in the passive domain (7.5), and the interaction energy is then obtained by integrating

the force with the boundary condition $E_{\text{int}}(h \rightarrow \infty) = 0$. Outside (7.5) the round-trip series ceases to converge; if the secular function has no contour zeros, one may still use the secular/zeta or force prescription just described, but this nonpassive sector is not used in the numerical analysis below.

1. A simple check: Diagonal reflection data (no channel mixing)

A useful consistency check is obtained when the boundary conditions do not mix polarization channels, so that the reflection operators are diagonal in a suitable basis and the round-trip operator reduces to a scalar factor $M(\zeta, k; h) = r_0(\zeta, k)r_h(\zeta, k)e^{-2Q(\zeta, k)h}$. In this case the interaction energy (7.4) reduces to the standard scattering expression

$$E_{\text{int}}(h) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \times \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \ln \left(1 - r_0(\zeta, k)r_h(\zeta, k)e^{-2Q(\zeta, k)h} \right), \quad (7.11)$$

which is manifestly real in the passive domain $|r_0 r_h e^{-2Qh}| < 1$ (in particular, for $|r_{0,h}| \leq 1$). This illustrates explicitly how the general determinant formula reproduces the usual scattering result in a nonmixing limit and clarifies the admissible parameter domain discussed in Eqs. (7.5)–(7.10).

2. How to evaluate the interaction numerically

Equations (7.4) and (7.10) are well suited for numerical evaluation because all separation dependence enters through the factor e^{-2Qh} . It is convenient to work with dimensionless variables.

MCS regime ($m \neq 0$). Introduce

$$a \equiv mh, \quad \hat{\zeta} \equiv \frac{\zeta}{m}, \quad \hat{k} \equiv \frac{k}{m}, \quad \hat{Q}(\hat{\zeta}, \hat{k}) = \sqrt{1 + \hat{\zeta}^2 + \hat{k}^2}. \quad (7.12)$$

Then

$$E_{\text{int}}(h) = \frac{m^2}{2} \int \frac{d\hat{\zeta}}{2\pi} \int \frac{d\hat{k}}{2\pi} \ln \det \left[\mathbb{I} - R_0(\hat{\zeta}, \hat{k})R_h(\hat{\zeta}, \hat{k})e^{-2a\hat{Q}} \right], \quad (7.13)$$

and

$$F(h) = -m^3 \int \frac{d\hat{\zeta}}{2\pi} \int \frac{d\hat{k}}{2\pi} \hat{Q} \text{Tr}[(\mathbb{I} - \hat{M})^{-1} \hat{M}],$$

$$\hat{M} \equiv R_0(\hat{\zeta}, \hat{k}) R_h(\hat{\zeta}, \hat{k}) e^{-2a\hat{Q}}. \quad (7.14)$$

The integrands are exponentially suppressed for large $(\hat{\zeta}, \hat{k})$ at fixed $a > 0$. A robust implementation proceeds as follows. (i) For each $(\hat{\zeta}, \hat{k})$ compute the reflection matrices $R_0(\hat{\zeta}, \hat{k})$ and $R_h(\hat{\zeta}, \hat{k})$ from the explicit expressions in Sec. VI. (ii) Form the product $P(\hat{\zeta}, \hat{k}) \equiv R_0(\hat{\zeta}, \hat{k}) R_h(\hat{\zeta}, \hat{k})$ and compute either (a) the eigenvalues of P and thus the scalar factors in the determinant, or (b) directly $\ln \det(\mathbb{I} - \hat{M})$ and $\text{Tr}[(\mathbb{I} - \hat{M})^{-1} \hat{M}]$ by standard 2×2 linear algebra. (iii) Integrate over $(\hat{\zeta}, \hat{k})$ with a quadrature adapted to the exponential tail. Polar coordinates $\hat{\zeta} = \rho \cos \theta$, $\hat{k} = \rho \sin \theta$ are often efficient,

$$\int \frac{d\hat{\zeta}}{2\pi} \int \frac{d\hat{k}}{2\pi} (\dots) = \int_0^\infty \frac{d\rho}{2\pi} \rho \int_0^{2\pi} \frac{d\theta}{2\pi} (\dots), \quad (7.15)$$

with $\hat{Q} = \sqrt{1 + \rho^2}$. Time-domain approaches provide an alternative numerical route in more general geometries (see Ref. [37] for a representative formulation).

Maxwell regime ($m \rightarrow 0$). It is natural to scale momenta with h instead of m : set

$$\rho \equiv h \sqrt{\zeta^2 + k^2}, \quad Qh = \rho, \quad (7.16)$$

so that

$$E_{\text{int}}(h) = \frac{1}{2h^2} \int_0^\infty \frac{d\rho}{2\pi} \rho \int_0^{2\pi} \frac{d\theta}{2\pi} \ln \times \det[\mathbb{I} - R_0(\rho, \theta) R_h(\rho, \theta) e^{-2\rho}]. \quad (7.17)$$

If the reflection data are effectively scale independent in this limit, (7.17) makes the expected h^{-2} scaling of the energy per unit length manifest, and the force scales as h^{-3} .

3. Numerical consistency checks and controlled limits

The following checks validate numerical implementations and connect with analytic limits.

(i) *Large-separation/large-mass regime* ($a = mh \gg 1$).

For $a \gg 1$ the factor $e^{-2a\hat{Q}}$ is uniformly small and one may expand $\ln \det(\mathbb{I} - \hat{M}) = -\text{Tr}(\hat{M}) + \mathcal{O}(\hat{M}^2)$. Keeping the leading term gives

$$E_{\text{int}}(h) \simeq -\frac{1}{2} \int \frac{d\zeta}{2\pi} \int \frac{dk}{2\pi} \text{Tr}[R_0(\zeta, k) R_h(\zeta, k)] e^{-2Qh}, \quad (7.18)$$

and the corresponding leading force follows from (7.10) as

$$F(h) \simeq - \int \frac{d\zeta}{2\pi} \int \frac{dk}{2\pi} Q(\zeta, k) \times \text{Tr}[R_0(\zeta, k) R_h(\zeta, k)] e^{-2Qh}. \quad (7.19)$$

- (ii) *Subtraction stability and ultraviolet tails.* Because the interaction is defined by the subtraction (7.2), any purely local contribution must cancel. Numerically, stability can be checked by verifying that $E_{\text{int}}(h)$ and $F(h)$ are insensitive to the precise truncation of the (ζ, k) integration domain once the integrand is implemented in the form (7.4)–(7.10). In practice, the exponential tail driven by e^{-2Qh} should dominate the truncation error for sufficiently large cutoffs.
- (iii) *Symmetry checks and benchmark limits.* If the boundary data are chosen such that $R_0(\zeta, k) = R_h(\zeta, k)$, the integrands in (7.4) and (7.10) are symmetric under exchange of the two boundaries. If, in addition, boundary parameters enforce parity invariance (no ϵ -type mixing), then $R_\alpha(\zeta, k)$ becomes proportional to η and the determinant reduces to a product of identical scalar-channel factors. This provides a controlled benchmark against known scalar/Robin limits [7,34]; the corresponding scalar reflection coefficients are identified explicitly in the Maxwell limit in Sec. VIII C, Eq. (8.4), where they are shown to be of Robin type with Dirichlet and Neumann as limiting cases. In particular, when the ultraviolet round-trip matrix approaches the identity in both channels, Eq. (8.9) gives $E_{\text{int}}(h) \sim -\zeta(3)/(8\pi h^2)$ and $F(h) \sim -\zeta(3)/(4\pi h^3)$, namely the standard two-channel Dirichlet benchmark in $2+1$ dimensions. As a further consistency check, in the weak-reflection regime where $\|R_0(\zeta, k) R_h(\zeta, k)\|$ is small, the leading approximation (7.18)–(7.19) must be reproduced.

C. Zeta-function viewpoint

We connect the scattering formula (7.4) with a spectral zeta-function formulation in a way that is fully consistent with the Euclidean strip setup of Sec. VI and with the finite Casimir subtraction (7.2) [38,39]. For fixed Euclidean tangential momentum (ζ, k) , the strip problem reduces to an ordinary differential problem in the transverse coordinate $x^2 \in [0, h]$ in the physical channel. All spectral information relevant to the interaction is encoded in the separation-dependent secular function

$$D(\zeta, k; h) \equiv \det(\mathbb{I} - M(\zeta, k; h))$$

$$= \det(\mathbb{I} - R_0(\zeta, k) R_h(\zeta, k) e^{-2Q(\zeta, k)h}), \quad (7.20)$$

whose zeros correspond to strip eigenmodes in the physical channel. A standard way to encode this spectral information is provided by the argument principle [8,38,39]. Let $D(\omega, k; h)$ denote the (Minkowski) secular function whose zeros in ω at fixed k determine the strip eigenfrequencies $\{\omega_n(k; h)\}$ in the physical channel. Then a mode sum can be rewritten as a contour integral,

$$\sum_n f(\omega_n(k; h)) = \frac{1}{2\pi i} \oint_{\mathcal{C}} d\omega f(\omega) \partial_\omega \ln D(\omega, k; h), \quad (7.21)$$

where $\mathcal{C}$ encloses the zeros of D and $f(\omega)$ is an appropriate test function. Choosing $f(\omega)$ to match the vacuum-energy weight and deforming the contour in the standard way to the imaginary axis, $\omega = i\zeta$, one obtains the Euclidean representation in terms of $Q(\zeta, k)$ and the multiple-reflection denominator (7.20). In this way a mode sum, or equivalently a reduced spectral determinant, turns into the scattering/log-determinant integral (7.4) [34,36,38,39]. In zeta regularization one introduces an arbitrary mass scale μ and defines the regularized vacuum energy per unit length as a regularized logarithm of a reduced physical determinant. Ultraviolet divergences are bulk local and boundary local. Implementing the finite Casimir prescription through the subtraction (7.2) removes all local pieces and isolates the unique separation-dependent part [13]. Equivalently, the interaction energy can be expressed directly in terms of the secular function (7.20) as

$$E_{\text{int}}(h) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \ln D(\zeta, k; h), \quad (7.22)$$

which is equivalent to (7.4). This makes explicit that the zeta-function and scattering viewpoints are two equivalent encodings of the same spectral data: the separation dependence is carried by the multiple-reflection factor, while ultraviolet structure is purely local and is absorbed into independent renormalizations of the boundary couplings on Σ_0 and Σ_h . The general force formula (7.10) provides the natural starting point for asymptotic expansions and for numerical evaluation once a specific choice of Symanzik boundary parameters is made.

VIII. ASYMPTOTIC REGIMES AND CONSISTENCY CHECKS

We now analyze the main asymptotic regimes of the finite Casimir interaction encoded by the scattering/log-determinant representation (7.4): the pure Chern-Simons limit, the pure Maxwell limit, the short-distance regime, and the large-separation regime.

A. Worked flip-symmetric example and numerics

Practical checks of the above analytical limits rely on the force formula (7.10), which is numerically more robust

than differentiating a discretized $E_{\text{int}}(h)$. An equivalent single-channel form can be obtained by diagonalizing the 2×2 physical sector through (6.18). To make the formalism concrete, we evaluate (7.4)–(7.10) on the same minimal flip-symmetric, edge-active branch (7.6). Because in this case the round-trip eigenvalues have the form (7.8), the two-dimensional Euclidean integral reduces to a radial one once the phase $\vartheta(Q)$ is extracted. For the explicit plots below we set

$$g^2 = 1, \quad \kappa = 1, \quad m = \kappa g^2 = 1, \quad (8.1)$$

so that the dimensionless separation is simply $a \equiv mh = h$. Figure 1 shows two representative choices, $a_1/\kappa = 1/2$ and $a_1/\kappa = 2$, for which the passivity condition is automatically satisfied by (7.9). At zero Euclidean momentum one finds

$$\begin{aligned} & \frac{1}{2} \text{Tr}[R_0(0, 0)R_h(0, 0)] \\ &= \frac{(4a_1^2 - 7\kappa^2)(4a_1^2 + \kappa^2)}{(4a_1^2 - 8a_1\kappa + 5\kappa^2)(4a_1^2 + 8a_1\kappa + 5\kappa^2)}. \end{aligned} \quad (8.2)$$

Combined with the large-distance asymptotic formula (8.13), this shows that the Yukawa tail is repulsive for $2|a_1| < \sqrt{7}|\kappa|$ and attractive for $2|a_1| > \sqrt{7}|\kappa|$. This analytic criterion is visible in the right panel of Fig. 1: the $a_1/\kappa = 1/2$ curve crosses to a repulsive tail, whereas the $a_1/\kappa = 2$ curve remains attractive at all separations. The same numerical data also verify the ultraviolet laws $E_{\text{int}}(h) \sim C_{\text{UV}}/h^2$ and $F(h) \sim 2C_{\text{UV}}/h^3$, since the rescaled quantities $a^2 E_{\text{int}}$ and $a^3 F$ approach the universal plateaux $C_{\text{UV}} = -\zeta(3)/(8\pi)$ and $2C_{\text{UV}}$ as $a \rightarrow 0$.

The worked example already displays the main numerical benchmarks used throughout this Section: the ultraviolet plateaux $a^2 E_{\text{int}} \rightarrow C_{\text{UV}}$ and $a^3 F \rightarrow 2C_{\text{UV}}$ as $a \rightarrow 0$, the Yukawa falloff for $mh \gg 1$, and invariance under exchange of the two boundaries on the flip-symmetric branch.

To display the role of the topological mass directly, Fig. 2 shows the unrescaled interaction energy $E_{\text{int}}(h)$ on the same flip-symmetric branch ($g^2 = 1$, $a_1 = 1/2$) for several values of κ , including the Maxwell case $\kappa = 0$. At short distances all curves collapse onto the universal Maxwell behavior $E_{\text{int}} \simeq -\zeta(3)/(8\pi h^2)$, since the topological mass is negligible for $h \ll m^{-1}$. At large separations the difference is qualitative: the Maxwell interaction remains long ranged and attractive, decaying as $1/h^2$, whereas for $\kappa \neq 0$ the interaction is Yukawa screened on the scale $h \sim m^{-1}$ and, on this branch, develops a small repulsive tail before vanishing exponentially (here $2|a_1| < \sqrt{7}|\kappa|$ for all the displayed $\kappa > 0$, cf. Eq. (8.2)). The semilog panel makes the contrast between the power-law and exponential decay explicit.

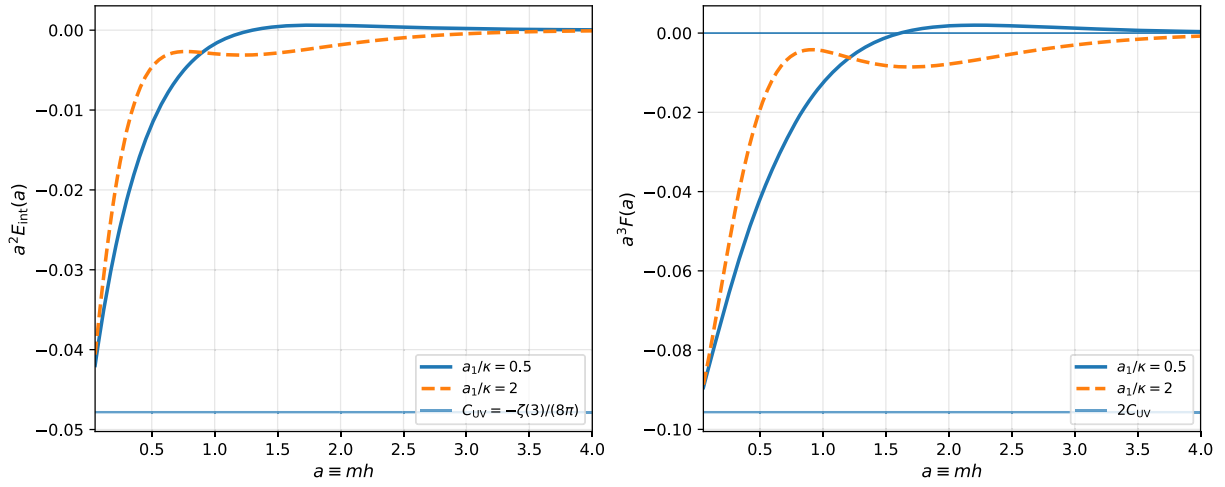


FIG. 1. Numerical evaluation of the minimal flip-symmetric, edge-active branch (7.6) with units (8.1). Left: the rescaled interaction energy $a^2 E_{\text{int}}(a)$. Right: the rescaled force $a^3 F(a)$. Solid and dashed curves correspond respectively to $a_1/\kappa = 1/2$ and $a_1/\kappa = 2$; the horizontal reference lines show the universal ultraviolet limits $C_{\text{UV}} = -\zeta(3)/(8\pi)$ and $2C_{\text{UV}}$. The $a_1/\kappa = 1/2$ branch develops a repulsive large-distance Yukawa tail, whereas the $a_1/\kappa = 2$ branch remains attractive. In both cases the unscaled interaction is exponentially suppressed as e^{-2mh}/h for $mh \gg 1$.

B. Pure Chern-Simons limit $g^2 \rightarrow \infty$ at fixed κ

At fixed κ , the limit $g^2 \rightarrow \infty$ drives $m \rightarrow \infty$ and exponentially suppresses the physical massive propagation that enters the tangential scattering channel. Because $Q(\zeta, k) \geq m$, one has $e^{-2Qh} \leq e^{-2mh} \rightarrow 0$ for all finite h , hence

$$E_{\text{int}}(h) \rightarrow 0, \quad F(h) \rightarrow 0, \quad (g^2 \rightarrow \infty, \kappa \text{ fixed}). \quad (8.3)$$

This matches the expected physical picture: a pure CS theory supports no propagating bulk degrees of freedom

and therefore cannot mediate a separation-dependent vacuum force. Consequently, edge current algebras and anomaly inflow persist, controlled only by κ [1,40], whereas the *distance-dependent* part of the vacuum energy, being dynamical, vanishes. This cleanly separates topological from dynamical physics on the strip.

C. Pure Maxwell limit $\kappa = 0$

For $\kappa = 0$, one has $m = 0$ and thus $Q(\zeta, k) = \sqrt{\zeta^2 + k^2}$ (2.9). Under these conditions Eq. (7.4) reduces to the standard massless-strip form, with local reflection data

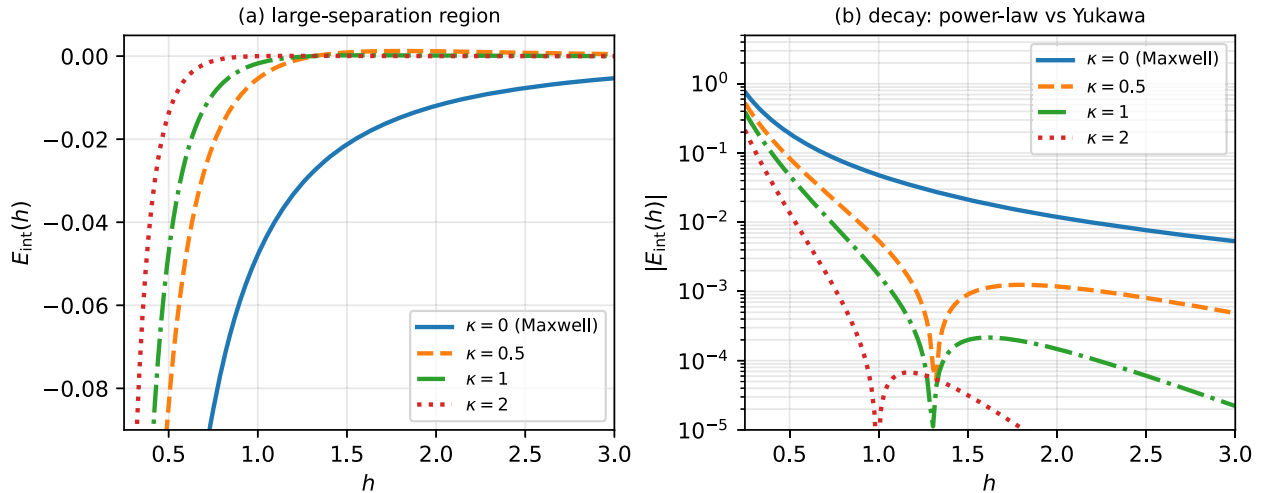


FIG. 2. Unrescaled interaction energy $E_{\text{int}}(h)$ on the flip-symmetric edge-active branch (7.6) with $g^2 = 1$ and $a_1 = 1/2$, for $\kappa = 0$ (Maxwell), 0.5, 1 and 2. Left: large-separation region; the Maxwell curve ($\kappa = 0$) stays attractive and long-ranged, while the $\kappa \neq 0$ curves are Yukawa-screened and turn slightly repulsive before vanishing. Right: $|E_{\text{int}}(h)|$ on a logarithmic scale, contrasting the power-law decay of the Maxwell limit with the exponential decay $\sim e^{-2mh}$ for $\kappa \neq 0$; the downward cusps mark the separation at which E_{int} changes sign. All curves share the same ultraviolet behavior $E_{\text{int}} \rightarrow -\zeta(3)/(8\pi h^2)$ as $h \rightarrow 0$.

determined by the boundary functional [13,23] only. This reduction can be made fully explicit. In the Maxwell limit the parity-odd coefficient loses its Chern-Simons contribution, $q_\alpha \rightarrow a_2^{(\alpha)}(-i\omega) + a_3^{(\alpha)}(ik)$. Hence, on the edge-active branch ($a_2 = a_3 = a_4 = 0$), one has $q_\alpha = 0$: the ϵ part of each reflection matrix drops out and R_α becomes proportional to the tangential metric, $R_\alpha = r_\alpha(Q)\eta$, with the scalar reflection coefficient

$$r_\alpha(Q) = \frac{Qc_\alpha - a_1^{(\alpha)}}{Qc_\alpha + a_1^{(\alpha)}} = \frac{\sigma_\alpha Q - a_1^{(\alpha)}g^2}{\sigma_\alpha Q + a_1^{(\alpha)}g^2}, \quad c_\alpha = \frac{\sigma_\alpha}{g^2}. \quad (8.4)$$

Equation (8.4) is exactly the reflection coefficient of a massless scalar obeying the oriented Robin boundary condition $(n_\alpha \cdot \partial + \lambda_\alpha)\phi = 0$, with $\lambda_\alpha = g^2 a_1^{(\alpha)}$ and effective Robin length $\ell_\alpha = 1/\lambda_\alpha$; the orientation signs are those encoded in σ_α . The limit $a_1^{(\alpha)} \rightarrow 0$ gives a Neumann condition ($r_\alpha \rightarrow +1$), while $a_1^{(\alpha)} \rightarrow \infty$ gives a Dirichlet condition ($r_\alpha \rightarrow -1$), with generic $a_1^{(\alpha)}$ interpolating between the two. This identifies precisely the effective scalar boundary conditions induced by the Symanzik functional in the Maxwell limit, and clarifies the “scalar/Robin” benchmark invoked in Sec. VII B 3: the parity-even subclass is the Robin family (8.4), with Dirichlet and Neumann as endpoints. On the flip-symmetric branch (7.6) the two scalar coefficients are reciprocal, $r_0(Q)r_h(Q) = 1$ for every Q , so the round-trip factor reduces to e^{-2Qh} and

$$E_{\text{int}}(h) = \int \frac{d\zeta dk}{2\pi 2\pi} \ln(1 - e^{-2Qh}) = -\frac{\zeta(3)}{8\pi h^2}, \quad (\kappa = 0), \quad (8.5)$$

exactly for all h and independently of a_1 . Thus the flip-symmetric Maxwell limit reproduces the standard massless strip round-trip factor $r_0 r_h = 1$; with the two identical tangential response channels already accounted for in the determinant normalization of (7.4), this gives (8.5). A nonzero topological mass then turns this power law into the Yukawa-screened interaction of the MCS theory. Dimensional analysis for an energy per unit length along the x^1 direction implies the scaling

$$E_{\text{int}}(h) \propto \frac{1}{h^2}, \quad F(h) \propto \frac{1}{h^3}, \quad (m = 0), \quad (8.6)$$

with coefficients that depend on the ultraviolet behavior of the product $R_0(\zeta, k)R_h(\zeta, k)$. Physically, the interaction remains long-ranged because the bulk is gapless: field modes with arbitrarily small $Q(\zeta, k)$ propagate across the strip. The scaling (8.6) therefore arises directly from dimensional reasoning and the extensivity of the energy along the boundary. The overall coefficient measures the effective reflectivity of each boundary at high Euclidean

momentum, i.e., how strongly the local Symanzik coupling departs from a perfectly transparent interface.

D. Short-distance regime $h \rightarrow 0$

As $h \rightarrow 0$, the integrals in (7.4) and (7.10) are dominated by high values of $|\zeta|$ and $|k|$. In this limit $Q(\zeta, k) \simeq \sqrt{\zeta^2 + k^2}$ and the mass m becomes negligible, so the leading short-distance behavior is universally Maxwell-like,

$$E_{\text{int}}(h) = \frac{C_{\text{UV}}}{h^2} + O(h^0),$$

$$F(h) = \frac{2C_{\text{UV}}}{h^3} + O(h^{-1}), \quad (h \rightarrow 0). \quad (8.7)$$

The coefficient C_{UV} depends only on the ultraviolet limit of the round-trip matrix. If this limit exists as a finite constant matrix,

$$\mathcal{R}_{\text{UV}} \equiv \lim_{|\zeta|, |k| \rightarrow \infty} R_0(\zeta, k)R_h(\zeta, k), \quad (8.8)$$

let $\{\lambda_+, \lambda_-\}$ denote its eigenvalues in the 2×2 physical tangential sector. A closed analytic expression then follows from the polar expansion of (7.4):

$$C_{\text{UV}} = -\frac{1}{16\pi} [\text{Li}_3(\lambda_+) + \text{Li}_3(\lambda_-)]. \quad (8.9)$$

Here $\text{Li}_s(z)$ denotes the polylogarithm, $\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ for $|z| < 1$, and extended to other values of z by analytic continuation. In particular Li_3 is the trilogarithm. This arises by expanding the logarithm, $\ln(1-x) = -\sum_{n \geq 1} x^n/n$, which turns the energy into a sum over n round trips. In 3D the momentum integral yields the characteristic $1/n^3$ weight, hence Li_3 [23]. For the flip-symmetric branch (7.6) used in Fig. 1, the ultraviolet limit entering Eq. (8.9) is completely explicit: from Eq. (6.17) one finds, as $Q \rightarrow \infty$, $r_\alpha^{(\eta)} \rightarrow -1$ and $r_\alpha^{(\epsilon)} \rightarrow 0$ on both boundaries, so Eq. (6.16) gives $R_\alpha \rightarrow -\eta$. Therefore

$$\mathcal{R}_{\text{UV}} = \lim_{Q \rightarrow \infty} R_0 R_h = (-\eta)^2 = \mathbb{1}, \quad (8.10)$$

hence $\lambda_+ = \lambda_- = 1$ and Eq. (8.9) reduces to

$$C_{\text{UV}} = -\frac{\text{Li}_3(1)}{8\pi} = -\frac{\zeta(3)}{8\pi}, \quad (8.11)$$

which is precisely the two-channel Dirichlet benchmark quoted in Sec. VII B 3. Equations (8.7)–(8.9) provide a stringent combined test: for sufficiently small h , the numerical values of $h^2 E_{\text{int}}(h)$ and $h^3 F(h)$ must approach C_{UV} and $2C_{\text{UV}}$, respectively. The universality of (8.7) reflects the locality of the boundary action. For $h \ll m^{-1}$, the relevant fluctuations have $|\zeta|, |k| \sim 1/h$,

for which $Q \gg m$. These modes probe only a thin layer near each boundary, so the mass and topological terms are almost invisible at leading order. Accordingly, the interaction probes the ultraviolet limit of the boundary reflection data, with topological corrections suppressed as $O(mh)$. The resulting asymptotics coincides with that of a massless field confined by local linear boundary operators. When $\mathcal{R}_{UV}$ is proportional to the identity, $\lambda_+ = \lambda_-$ and Eq. (8.9) reduces to the familiar scalar-channel expression. The sign and magnitude of $\text{Li}_3(\lambda_{\pm})$ specify whether the leading short-distance force is attractive or repulsive, linking boundary reflectivity directly to the macroscopic Casimir effect. The special case $\lambda_+ = \lambda_- = 1$ gives $\text{Li}_3(1) = \zeta(3)$ and provides a convenient numerical check [7,8].

E. Large-separation regime $h \rightarrow \infty$

When $m \neq 0$ and $mh \gg 1$, the bulk mass gap renders the interaction short ranged. Expanding the logarithm in (7.4) for $\|R_0 R_h\| e^{-2Qh} \ll 1$ yields the dominant “weak round-trip” form [23]

$$E_{\text{int}}(h) \simeq -\frac{1}{2} \int \frac{d\zeta}{2\pi} \int \frac{dk}{2\pi} \text{Tr}[R_0(\zeta, k) R_h(\zeta, k)] e^{-2Q(\zeta, k)h}, \quad (mh \gg 1). \quad (8.12)$$

A Laplace analysis around $(\zeta, k) = (0, 0)$, using $Q = \sqrt{m^2 + p^2} = m + \frac{p^2}{2m} + \dots$, gives

$$E_{\text{int}}(h) \sim -\frac{m}{8\pi h} \text{Tr}[R_0(0, 0) R_h(0, 0)] e^{-2mh}, \quad (mh \gg 1), \quad (8.13)$$

up to subleading powers of $(mh)^{-1}$. In the regime $mh \gg 1$ only the lowest-lying, slowest-decaying modes with $Q \simeq m$ contribute appreciably, while higher momenta are exponentially damped. The associated force, obtained from (7.10), exhibits the same exponential suppression $F(h) \sim e^{-2mh}$, identifying $\xi_{\text{topo}} = m^{-1}$ as the correlation length of the topological mass. Equations (8.12)–(8.13) therefore make explicit that the finite-width Casimir interaction quantifies the infrared transparency of the two boundaries through the quantity $\text{Tr}[R_0(0, 0) R_h(0, 0)]$. In the strict limit $h \rightarrow \infty$, the two boundaries decouple and each supports its own chiral edge dynamics, fully consistent with the single-boundary reduction of Ref. [19]. While E_{int} vanishes exponentially, local boundary observables remain finite and the $U(1)$ Kac-Moody algebras on Σ_0 and Σ_h retain opposite levels $\pm\kappa$, matching the one-boundary formulation [1,19,40].

IX. DISCUSSION AND OUTLOOK

The asymptotic analysis of Sec. VIII shows that the short-distance interaction is controlled by ultraviolet locality, while at large separation it is screened by the

topological mass. In this sense the strip geometry turns local boundary response and bulk propagation into a single finite observable, namely the interaction energy between the two boundary components. The scattering representation (7.4) makes this structure explicit by factorizing the problem into reflection data at each boundary and evanescent propagation across the strip [23].

The scattering formulation also provides a practical criterion for the admissible (passive) boundary-parameter domain: the multiple-reflection operator must satisfy $\rho(M) < 1$ over the relevant Euclidean momenta, Eq. (7.5). When this condition fails, the quadratic boundary response no longer defines a passive scattering problem. Degenerate loci where (6.10) fails have a different status: they lie outside the generic elliptic sector analyzed here and may correspond to additional boundary zero modes or critical boundary phases.

A further structural result is the locality of the ultraviolet sector. Ultraviolet divergences split into bulk-local and boundary-local contributions. Bulk counterterms renormalize bulk couplings, while boundary-local divergences renormalize the Symanzik functional on each boundary component independently: counterterms supported on Σ_0 depend only on the coefficients $a_n^{(0)}$, and those supported on Σ_h depend only on the coefficients $a_n^{(h)}$, where the $a_n^{(\alpha)}$ appear in the Symanzik boundary action S_{bd} (3.1). After this renormalization, the interaction becomes finite and its nonlocal content is fully encoded by the scattering combination $R_0^{\text{ren}}(\zeta, k) R_h^{\text{ren}}(\zeta, k) e^{-2Q(\zeta, k)h}$, with no further non-local subtraction.

The analysis also keeps distinct the algebraic and dynamical sectors. The CS term controls anomaly inflow and fixes the boundary current algebra in terms of κ and boundary orientation, independently of the Maxwell coupling [1,40]. Dispersion data, edge velocities, and finite-width interedge mixing instead depend on the local boundary functional and on bulk propagation across the strip. The interaction range is therefore set by the topological mass scale, while in the Maxwell limit $\kappa = 0$ the parity-even scattering coefficients reduce to those of a massless scalar with oriented Robin boundary conditions, with Neumann and Dirichlet as limiting cases. Within this picture the “flip” operation exchanging Σ_0 and Σ_h plays a simple organizing role: imposing flip symmetry relates the two boundary functionals, hence the two reflection operators, and selects branches in which the two edges carry opposite chiralities while sharing the same algebraic data fixed by κ .

Several extensions are natural. Finite temperature and real-time response should be accessible within the same reflection framework. One may also relax planar symmetry and study curved or inhomogeneous boundaries, or couple the strip to boundary matter and to non-Abelian sectors. Interedge observables beyond the vacuum energy, such as

boundary current correlators across the strip, should likewise be governed by the same reflection/propagation data.

Recent developments on rank-2 and higher-rank gauge theories provide a complementary direction: CS-like mechanisms can also endow tensor gauge fields with a topological mass, giving rise to quasitopological and fractonic phases [41–44]. These models share structural parallels with the present MCS framework—most notably the coexistence of gauge invariance, topological mass generation, and chiral-like boundary responses—and provide a natural setting in which to test an extension of the scattering/finite-Casimir logic to higher-rank tensor fields.

Finally, general sign theorems for Casimir forces rely on passivity or reflection positivity, whereas the Symanzik functionals considered here allow for a broader class of boundary responses. The repulsive regime visible in Fig. 1 is therefore not in conflict with the Kenneth-Klich theorem [45]: that proof assumes a reflection-positive (in scattering language, passive) response, whereas the local Symanzik matrices encoded by the coefficients $a_n^{(a)}$ need not be positive semidefinite on the full parameter space. It would be useful to characterize more sharply which choices lead to an always-attractive interaction. Related boundary algebras arise in BF-type theories with boundary [46] and on the boundary of 3D Maxwell theory [47]; in higher-dimensional BF reductions, boundary scalar–tensor dualities can even be fermionized into 3 + 1D massless Weyl spinors [48], while higher-rank gauge theories can also exhibit boundary-sensitive spectra and Casimir-like mechanisms [49,50], and Chern–Simons–like terms can generate a quasitopological mass in 3D linearized gravity [51]. It would be worthwhile to explore whether a scattering/finite-Casimir formulation similar to the one developed here can be adapted to those contexts.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

APPENDIX A: BOUNDARY VARIATION AND BOUNDARY CONDITIONS

In this Appendix we collect the boundary terms arising from the variation of the bulk Maxwell and CS actions and of the local quadratic boundary functional (3.2). Requiring the total boundary variation to vanish yields the explicit boundary conditions.

For

$$S_{\text{Max}}[A] = -\frac{1}{4g^2} \int_{\mathcal{M}} d^3x F_{\mu\nu} F^{\mu\nu}, \quad (\text{A1})$$

one finds

$$\begin{aligned} \delta S_{\text{Max}} &= -\frac{1}{2g^2} \int_{\mathcal{M}} d^3x F^{\mu\nu} \delta F_{\mu\nu} \\ &= \frac{1}{g^2} \int_{\mathcal{M}} d^3x (\partial_\mu F^{\mu\nu}) \delta A_\nu + \frac{1}{g^2} \int_{\partial\mathcal{M}} d^2x n_\mu F^{\mu\nu} \delta A_\nu. \end{aligned} \quad (\text{A2})$$

Restricting to tangential components $\nu = i$ gives the boundary contribution

$$\delta S_{\text{Max}} \Big|_{\partial\mathcal{M}} = \frac{1}{g^2} \int_{\Sigma_h} d^2x F^{2i} \delta A_i - \frac{1}{g^2} \int_{\Sigma_0} d^2x F^{2i} \delta A_i, \quad (\text{A3})$$

where the outward-pointing unit normals $n_\mu|_{\Sigma_a}$ are defined in (2.3).

For

$$S_{\text{CS}}[A] = \frac{\kappa}{2} \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \quad (\text{A4})$$

the variation yields, up to bulk terms,

$$\delta S_{\text{CS}} = \kappa \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\rho} \delta A_\mu \partial_\nu A_\rho + \frac{\kappa}{2} \int_{\partial\mathcal{M}} d^2x n_\nu \epsilon^{\mu\nu\rho} A_\mu \delta A_\rho. \quad (\text{A5})$$

Since on the strip $n_\nu = \pm\delta_\nu^2$, only tangential components contribute and we obtain

$$\delta S_{\text{CS}} \Big|_{\partial\mathcal{M}} = \frac{\kappa}{2} \int_{\Sigma_h} d^2x \epsilon^{ij} A_i \delta A_j - \frac{\kappa}{2} \int_{\Sigma_0} d^2x \epsilon^{ij} A_i \delta A_j. \quad (\text{A6})$$

The opposite sign on Σ_0 and Σ_h is entirely due to the opposite outward normals.

For completeness we note that

$$S_{\text{gf}}[A] = -\frac{1}{2\xi} \int_{\mathcal{M}} d^3x (\partial_\mu A^\mu)^2 \quad (\text{A7})$$

varies as

$$\delta S_{\text{gf}} = \frac{1}{\xi} \int_{\mathcal{M}} d^3x \partial_\mu (\partial \cdot A) \delta A^\mu - \frac{1}{\xi} \int_{\partial\mathcal{M}} d^2x n_\mu (\partial \cdot A) \delta A^\mu. \quad (\text{A8})$$

In Sec. III we derive tangential boundary conditions for A_i . The last term in (A8) does not affect those conditions provided the normal variations are fixed (equivalently, one

restricts to admissible variations such that $n_\mu \delta A^\mu = 0$ at the boundary). We therefore omit (A8) from the tangential boundary conditions below.

Varying and integrating by parts S_{bd} (3.1) along the boundary directions (x^0, x^1) gives

$$\begin{aligned} \delta S_{\text{bd}} = & \sum_{\alpha \in \{0, h\}} \int_{\Sigma_\alpha} d^2x (a_1^{(\alpha)} A^i + a_2^{(\alpha)} \epsilon^{ij} \partial_0 A_j \\ & + a_3^{(\alpha)} \epsilon^{ij} \partial_1 A_j + a_4^{(\alpha)} F_{2i}) \delta A_i \\ & + \sum_{\alpha \in \{0, h\}} \int_{\Sigma_\alpha} d^2x a_4^{(\alpha)} A^i \delta F_{2i}. \end{aligned} \quad (\text{A9})$$

The total tangential boundary variation is the sum of (A3), (A6), and the δA_i part of (A9). Requiring it to vanish for arbitrary δA_i yields the tangential boundary conditions on the two boundary components Σ_0 and Σ_h , namely

$$\begin{aligned} -\frac{1}{g^2} F^{2i} + a_1^{(0)} A^i + a_2^{(0)} \epsilon^{ij} \partial_0 A_j + a_3^{(0)} \epsilon^{ij} \partial_1 A_j \\ + a_4^{(0)} F_{2i} - \frac{\kappa}{2} \epsilon^{ij} A_j \Big|_{\Sigma_0} = 0, \end{aligned} \quad (\text{A10})$$

and

$$\begin{aligned} \frac{1}{g^2} F^{2i} + a_1^{(h)} A^i + a_2^{(h)} \epsilon^{ij} \partial_0 A_j + a_3^{(h)} \epsilon^{ij} \partial_1 A_j \\ + a_4^{(h)} F_{2i} + \frac{\kappa}{2} \epsilon^{ij} A_j \Big|_{\Sigma_h} = 0. \end{aligned} \quad (\text{A11})$$

For later use in Sec. V, it is convenient to isolate from Eqs. (A10)–(A11) the leading tangential constraint for modes slowly varying along the boundary directions, $p_i = (p_0, p_1) \rightarrow 0$. In this regime one neglects the tangential-derivative terms proportional to $a_2^{(\alpha)}$ and $a_3^{(\alpha)}$ and isolates the leading *algebraic* tangential sector. Moreover, in the sector relevant for the boundary current algebra we work on the branch $a_4^{(\alpha)} = 0$ [cf. Eq. (A13)] and drop terms involving normal data such as F_{2i} . One then obtains, on each boundary component Σ_α ,

$$\left(a_1^{(\alpha)} \eta_{ij} + \sigma_\alpha \frac{\kappa}{2} \epsilon_{ij} \right) A^j \Big|_{\Sigma_\alpha} = 0, \quad \alpha \in \{0, h\}, \quad (\text{A12})$$

where σ_α are defined in (4.3). Equation (A12) is the “tangential low-energy” condition used to derive the chirality constraints and the edge velocities. The δF_{2i} part of (A9) gives, on each boundary component,

$$a_4^{(\alpha)} A^i \Big|_{\Sigma_\alpha} = 0, \quad \alpha \in \{0, h\}. \quad (\text{A13})$$

Equation (A13) is the extra condition associated with the explicit dependence of $\mathcal{L}_\alpha$ on F_{2i} . In particular, on branches where $a_4^{(\alpha)} = 0$ the boundary conditions reduce to the single tangential equations (A10)–(A11) with $a_4^{(\alpha)}$ removed. The strip spectral analysis of Sec. VI and the explicit Casimir examples of Secs. VII–VIII are restricted to this edge-active branch; the Dirichlet-like alternative $a_4^{(\alpha)} \neq 0$ is admissible but is not pursued further here.

For the purposes of the main text it is useful to recast (A10)–(A11) in the compact operator form

$$(\mathcal{B}_{ij}^{(\alpha)} A^j + \mathcal{C}_{ij}^{(\alpha)} F_{2j}) \Big|_{x^2=\alpha} = 0, \quad \alpha \in \{0, h\}, \quad (\text{A14})$$

where $\mathcal{B}_{ij}^{(\alpha)}$ is a local differential operator along the boundary and $\mathcal{C}_{ij}^{(\alpha)}$ is an algebraic matrix. To make the identification explicit, introduce the tangential differential operator

$$\mathcal{D}_{ij}^{(\alpha)} \equiv a_1^{(\alpha)} \eta_{ij} + a_2^{(\alpha)} \epsilon_{ij} \partial_0 + a_3^{(\alpha)} \epsilon_{ij} \partial_1 + \sigma_\alpha \frac{\kappa}{2} \epsilon_{ij}. \quad (\text{A15})$$

Then (A10)–(A11) are equivalently written as

$$\mathcal{D}_{ij}^{(\alpha)} A^j + \left(a_4^{(\alpha)} \eta_{ij} + \sigma_\alpha \frac{1}{g^2} \eta_{ij} \right) F_{2j} \Big|_{\Sigma_\alpha} = 0, \quad (\text{A16})$$

where the sign σ_α reflects the opposite orientations of Σ_0 and Σ_h , so that the Maxwell boundary contribution gives $n_\mu F^{\mu i} = \pm F^{2i}$. Comparing (A14) and (A16) one therefore reads off

$$\mathcal{B}_{ij}^{(\alpha)} = \mathcal{D}_{ij}^{(\alpha)}, \quad (\text{A17})$$

$$\mathcal{C}_{ij}^{(\alpha)} = \left(a_4^{(\alpha)} + \sigma_\alpha \frac{1}{g^2} \right) \eta_{ij}. \quad (\text{A18})$$

APPENDIX B: DERIVATION OF THE SECULAR EQUATION ON THE STRIP

In this Appendix we derive the strip secular equation (6.19) starting from the bulk decomposition and the local Symanzik boundary conditions.

For each fixed Euclidean frequency ζ and tangential momentum k , the bulk equations in the physical (tangential) sector reduce to a second order ordinary differential equation along the transverse direction $x^2 \in [0, h]$, whose solutions decay or grow as $e^{\mp Q x^2}$, with

$$Q(\zeta, k) = \sqrt{\zeta^2 + k^2 + m^2}. \quad (\text{B1})$$

As introduced in Eq. (6.6), the general bulk profile of the tangential components in momentum space can be written as

$$\tilde{A}(\zeta, k; x^2) = e^{-Qx^2} \tilde{A}^{\text{in}}(\zeta, k) + e^{+Qx^2} \tilde{A}^{\text{out}}(\zeta, k), \quad (\text{B2}) \quad \tilde{A}_h^{\text{in}}(\zeta, k) = e^{+Qh} \tilde{A}^{\text{out}}(\zeta, k), \quad \tilde{A}_h^{\text{out}}(\zeta, k) = e^{-Qh} \tilde{A}^{\text{in}}(\zeta, k). \quad (\text{B9})$$

where $\tilde{A}^{\text{in}}$ and $\tilde{A}^{\text{out}}$ are two-component vectors collecting the tangential amplitudes [on the boundary data space spanned by (A_0, A_1)].

The local Symanzik boundary conditions derived in Sec. III can be written in the compact form

$$[B^{(\alpha)} \tilde{A}(\zeta, k; x^2) + C^{(\alpha)} \partial_2 \tilde{A}(\zeta, k; x^2)]|_{x^2=\alpha} = 0, \quad \alpha=0, \quad h, \quad (\text{B3})$$

where the 2×2 matrices $B^{(\alpha)}$ and $C^{(\alpha)}$ are given in Eqs. (A17)–(A18) in terms of the boundary coefficients $a_n^{(\alpha)}$.

1. Lower boundary

Evaluating (B3) at $x^2 = 0$ and using $\partial_2 \tilde{A} = -Qe^{-Qx^2} \tilde{A}^{\text{in}} + Qe^{+Qx^2} \tilde{A}^{\text{out}}$, one obtains

$$(B^{(0)} - QC^{(0)}) \tilde{A}^{\text{in}}(\zeta, k) + (B^{(0)} + QC^{(0)}) \tilde{A}^{\text{out}}(\zeta, k) = 0. \quad (\text{B4})$$

On a generic (elliptic) branch we assume $\det(B^{(0)} + QC^{(0)}) \neq 0$, so that (B4) can be solved for the outgoing amplitude as

$$\tilde{A}^{\text{out}}(\zeta, k) = R_0(\zeta, k) \tilde{A}^{\text{in}}(\zeta, k), \quad R_0(\zeta, k) = -(B^{(0)} + QC^{(0)})^{-1} (B^{(0)} - QC^{(0)}), \quad (\text{B5})$$

which defines the reflection matrix at the lower boundary Σ_0 .

2. Propagation to the upper boundary

The profile evaluated at $x^2 = h$ is

$$\tilde{A}(\zeta, k; h) = e^{-Qh} \tilde{A}^{\text{in}}(\zeta, k) + e^{+Qh} \tilde{A}^{\text{out}}(\zeta, k), \quad (\text{B6})$$

and its transverse derivative reads

$$\partial_2 \tilde{A}(\zeta, k; h) = -Qe^{-Qh} \tilde{A}^{\text{in}}(\zeta, k) + Qe^{+Qh} \tilde{A}^{\text{out}}(\zeta, k). \quad (\text{B7})$$

It is convenient to introduce at the upper boundary its own incoming and outgoing amplitudes, $\tilde{A}_h^{\text{in}}$ and $\tilde{A}_h^{\text{out}}$, defined as in (6.6) with respect to the shifted transverse coordinate $y = h - x^2$, i.e., the bulk profile in a neighborhood of the upper boundary is decomposed as

$$\tilde{A}(\zeta, k; x^2) = e^{-Qy} \tilde{A}_h^{\text{in}}(\zeta, k) + e^{+Qy} \tilde{A}_h^{\text{out}}(\zeta, k), \quad y = h - x^2. \quad (\text{B8})$$

Comparing (B8) with (6.6) immediately yields

3. Upper boundary

Applying (B3) at $x^2 = h$ and using the decomposition in terms of $\tilde{A}_h^{\text{in/out}}$ one finds a relation of the same form as (B4),

$$(B^{(h)} - QC^{(h)}) \tilde{A}_h^{\text{in}}(\zeta, k) + (B^{(h)} + QC^{(h)}) \tilde{A}_h^{\text{out}}(\zeta, k) = 0, \quad (\text{B10})$$

which, under the generic nondegeneracy condition $\det(B^{(h)} + QC^{(h)}) \neq 0$, can be solved as

$$\tilde{A}_h^{\text{out}}(\zeta, k) = R_h(\zeta, k) \tilde{A}_h^{\text{in}}(\zeta, k), \quad R_h(\zeta, k) = -(B^{(h)} + QC^{(h)})^{-1} (B^{(h)} - QC^{(h)}), \quad (\text{B11})$$

defining the reflection matrix at the upper boundary Σ_h .

4. Compatibility condition and secular equation

A nontrivial mode confined between the two boundaries must be self-consistent under a complete round trip: starting from an incoming amplitude at Σ_0 , propagating to Σ_h , reflecting, and propagating back to Σ_0 must reproduce the original amplitude.

Using (B5), (B11) and the propagation relations (B9), one finds that a complete round trip acts on the incoming amplitude at Σ_0 as

$$\tilde{A}^{\text{in}}(\zeta, k) \longrightarrow R_0(\zeta, k) R_h(\zeta, k) e^{-2Q(\zeta, k)h} \tilde{A}^{\text{in}}(\zeta, k). \quad (\text{B12})$$

The condition for self-consistency is that this transformation has eigenvalue 1, i.e.,

$$\det[\mathbb{I} - R_0(\zeta, k) R_h(\zeta, k) e^{-2Q(\zeta, k)h}] = 0. \quad (\text{B13})$$

This is precisely the secular (compatibility) condition quoted in Eq. (6.19) in the main text. Since $\det(AB) = \det(BA)$, the order of R_0 and R_h inside the determinant is immaterial for the spectrum.

Equation (B13) admits the standard multiple-reflection interpretation familiar from scattering approaches to the Casimir effect, see, e.g., [22,23]: the factor e^{-2Qh} encodes propagation between the two boundaries, while R_0 and R_h describe local boundary response. Its zeros determine the allowed strip modes in the physical channel and provide the natural input for the logarithmic determinant representation of the Casimir interaction in Sec. VII.

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