

Complete classification of domain wall solutions in the $\mathbb{Z}_2$ -symmetric 2HDM

Richard A. Battye^{*,†}, Steven J. Cotterill[†], and Adam K. Thomasson[‡]

Department of Physics and Astronomy, *The University of Manchester*, Manchester, United Kingdom



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We present a complete classification of domain wall solutions in the two-Higgs-doublet model (2HDM) with a global $\mathbb{Z}_2$ symmetry, categorized as superconducting, CP -violating, or neither, depending on the scalar particle masses and the ratio of the two Higgs doublets' vacuum expectation values. We demonstrate that any domain wall solution can be reduced to depend on only six of the eight general field components, with further field reductions possible within different regions of the parameter space. Furthermore, we show that the superconducting solutions can be used to construct stable, current-carrying domain walls in two spatial dimensions. Similarly, the CP -violating solutions allow for two-dimensional configurations where CP symmetry is locally broken on the $\mathbb{Z}_2$ -symmetric wall, which could provide an out-of-equilibrium environment for CP -violating processes to occur.

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I. INTRODUCTION

Domain walls can form when a discrete symmetry is spontaneously broken at a phase transition in the early Universe [1,2]. These topological defects are generally viewed as problematic because, if absolutely stable, even a small population of domain walls would quickly dominate the Universe's energy density unless the symmetry-breaking transition occurs at a sufficiently low energy scale [3]. However, if the discrete symmetry is only approximate—that is, if it is explicitly or softly broken—then any domain walls produced can decay on cosmologically acceptable timescales, rendering them compatible with observations [4]. Decaying domain walls can act as a potential source of gravitational waves (GWs), producing a stochastic background that may be observable with current or near-future experiments. As these metastable walls collapse under their own tension, they emit GWs with a characteristic spectral shape, typically peaking at frequencies determined by the wall energy scale and decay time [5,6].

Domain wall solutions in a simple $\mathbb{Z}_2$ -symmetric model are well known [2]. If that model is coupled to an additional field with an unbroken $U(1)$ symmetry, one finds superconducting wall solutions [7,8], which can form stable

ring-like configurations in two dimensions known as *kinky vortons* [8].

The two-Higgs-doublet model (2HDM) [9] extends the Standard Model (SM) scalar sector by adding a second complex Higgs doublet (for a review see, for example, Ref. [10]). Such models are well motivated and have been extensively studied—for example, they can introduce new sources of CP violation [11,12], provide mechanisms for baryogenesis [13–16], and even supply dark matter candidates [15,17]. The 2HDM predicts the emergence of five physical scalar particles: h and H (CP -even neutral), A (CP -odd neutral), and $H^\pm$ (charged). In the so-called *alignment limit* (which we consider here), h is identified with the observed SM Higgs boson. Phenomenological studies of the 2HDM are often performed in the $\mathbb{Z}_2$ -symmetric variant, as it can eliminate flavor-changing neutral currents (FCNCs); this variant naturally leads to the formation of domain walls upon spontaneous symmetry breaking. Other discrete symmetries, $CP1$ and $CP2$, can also lead to the formation of domain walls [18–20]. In all cases, the discrete symmetries would need to be softly broken to avoid a domain wall overclosure problem [21].

Field-theoretic simulations of domain wall formation and evolution in the 2HDM have revealed some intriguing features. Notably, some domain wall solutions induce a nonzero photon mass at the center of the wall [22,23]. This local breaking of $U(1)_{\text{EM}}$ has many potential implications [24,25] that could be relevant if these effects last for a sufficient time—that is, if the soft-breaking scale is sufficiently low. More recent work [26] has shown that ring-like wall configurations akin to the *kinky vortons* found in the simpler $\mathbb{Z}_2 \times U(1)$ model [8,27,28] might also form in the 2HDM. These objects were not stable, but they lasted much longer than would have been expected based

*Contact author: richard.battye@manchester.ac.uk

†Contact author: steven.cotterill@manchester.ac.uk

‡Contact author: adam.thomasson@manchester.ac.uk

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on standard arguments of domain wall decay, with currents being observed on the walls. Kinky vortons are $(2+1)$ -dimensional analogues to vortons [29–34], where the superconducting string is replaced by a superconducting domain wall, allowing for studies of the stability within a lower-dimensional setting where greater dynamical range is available. The ultimate significance of such solutions for our $(3+1)$ -dimensional Universe, however, remains to be determined.

Inspired by the developments presented in Refs. [18,22,23,25,26], in this paper we fully categorize the parameter space of the $\mathbb{Z}_2$ -symmetric 2HDM, identifying four possible subclasses of $\mathbb{Z}_2$ wall solutions. These have been categorized into standard solutions, superconducting solutions,¹ CP -violating solutions,² and superconducting and CP -violating solutions.³ We identify distinct regions of the parameter space where each subclass is the energy-minimizing field configuration. We go on to demonstrate the existence of stable, current-carrying domain walls, formed from superconducting solutions, from which we suggest it might be possible to construct kinky vortons within this model. We also identify a novel two-dimensional solution in which a $CP1$ domain wall forms longitudinally along the core of a $\mathbb{Z}_2$ domain wall. This stable composite structure could provide a natural mechanism for out-of-equilibrium CP violation. Our findings serve to successfully explain the observations of previously performed full dynamical simulations of this model.

II. 2HDM WITH $\mathbb{Z}_2$ SYMMETRY

A. Formalism

The Lagrangian density of the 2HDM (neglecting gauge fields and fermion couplings) can be written as

$$\mathcal{L} = (\partial^\mu \Phi_1)^\dagger (\partial_\mu \Phi_1) + (\partial^\mu \Phi_2)^\dagger (\partial_\mu \Phi_2) - V(\Phi_1, \Phi_2), \quad (1)$$

where Φ_1 and Φ_2 are the two complex doublets. When the potential is restricted to be symmetric under the $\mathbb{Z}_2$ transformation $\Phi_1 \rightarrow \Phi_1, \Phi_2 \rightarrow -\Phi_2$ it takes the most general form of

$$\begin{aligned} V = & -\mu_1^2 (\Phi_1^\dagger \Phi_1) - \mu_2^2 (\Phi_2^\dagger \Phi_2) + \lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \lambda_2 (\Phi_2^\dagger \Phi_2)^2 \\ & + \lambda_3 (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + (\lambda_4 - |\lambda_5|) [\text{Re}(\Phi_1^\dagger \Phi_2)]^2 \\ & + (\lambda_4 + |\lambda_5|) [\text{Im}(\Phi_1^\dagger \Phi_2)]^2, \end{aligned} \quad (2)$$

where the requirement of Hermiticity demands that $\mu_1^2, \mu_2^2, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$. In general, $\lambda_5 \in \mathbb{C}$, but it can

always be made real by a choice of basis [18], which we choose to represent in the potential using $|\lambda_5|$. It is often convenient to rewrite the potential in the so-called *bilinear field space* formalism [35–37],

$$V = -\frac{1}{2} M_\mu R^\mu + \frac{1}{4} L_{\mu\nu} R^\mu R^\nu, \quad (3)$$

where M_μ and $L_{\mu\nu}$ are constant coefficient matrices (their explicit forms can be found in, for example, Ref. [18]) and R^μ , which is invariant under a global electroweak (EW) transformation, is given by

$$R^\mu = \Phi^\dagger (\sigma^\mu \otimes \sigma^0) \Phi = \begin{pmatrix} \Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 \\ \Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1 \\ -i[\Phi_1^\dagger \Phi_2 - \Phi_2^\dagger \Phi_1] \\ \Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2 \end{pmatrix}, \quad (4)$$

where $\Phi = (\Phi_1)$. By introducing the $SU(2)_L$ -invariant object, $\Phi_1^\dagger i\sigma^2 \Phi_2$, R^μ can be promoted to a null 6-vector, R^A for $A = 0, \dots, 5$, that incorporates charged degrees of freedom [18]. In particular, defining two additional components

$$\begin{aligned} R^4 &= \Phi_1^\dagger i\sigma^2 \Phi_2 - \Phi_2^\dagger i\sigma^2 \Phi_1^*, \\ R^5 &= -i(\Phi_1^\dagger i\sigma^2 \Phi_2 + \Phi_2^\dagger i\sigma^2 \Phi_1^*), \end{aligned} \quad (5)$$

one finds the useful identity $R^\mu R_\mu = R_4^2 + R_5^2$. The remaining degrees of freedom can be packaged into an $SU(2)_L$ -vector $n^a = -\Phi^\dagger (\sigma^0 \otimes \sigma^a) \Phi$ (with $a = 1, 2, 3$), as discussed in Ref. [38].

The two Higgs doublets of the model can be represented in various ways. In the *linear representation*, we expand the fields in terms of eight real scalar components ϕ_i :

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \\ \phi_5 + i\phi_6 \\ \phi_7 + i\phi_8 \end{pmatrix}. \quad (6)$$

Alternatively, one can describe the most general field configuration by applying an arbitrary EW rotation to a generic state containing only the degrees of freedom which affect the potential. In this *general representation*, we write the field as an EW rotated vacuum state,

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{v_{\text{SM}}}{\sqrt{2}} (\sigma^0 \otimes U) \begin{pmatrix} 0 \\ f_1 \\ f_+ \\ f_2 e^{i\xi} \end{pmatrix}, \quad (7)$$

¹The $U(1)$ symmetry of electromagnetism is broken locally on the wall.

² CP symmetry is locally broken on the wall, with the two vacua being CP -preserving.

³The solution is simultaneously superconducting and CP -violating.

where $v_{\text{SM}} = 246$ GeV is the SM Higgs vacuum expectation value, and $U \in U(1)_Y \times SU(2)_L$ is a constant group element given, using the representation of Ref. [22], by

$$U = e^{\frac{i}{2}\chi} \begin{pmatrix} \cos \frac{1}{2}\gamma_1 e^{\frac{i}{2}\gamma_2} & \sin \frac{1}{2}\gamma_1 e^{\frac{i}{2}\gamma_3} \\ -\sin \frac{1}{2}\gamma_1 e^{-\frac{i}{2}\gamma_3} & \cos \frac{1}{2}\gamma_1 e^{-\frac{i}{2}\gamma_2} \end{pmatrix}. \quad (8)$$

This generates an eight-component field configuration, described by the *vacuum manifold parameters* f_1, f_+, f_2, ξ and the *EW group parameters* $\chi, \gamma_1, \gamma_2, \gamma_3$. This configuration is *charge breaking* in general, since a nonzero f_+ means that the $U(1)$ symmetry of electromagnetism is broken. To enforce an electrically neutral vacuum, far from any defects, one can impose the condition $f_+ = 0$, which is equivalent to $R^\mu R_\mu = 0$ and is often referred to as the *neutral vacuum condition* [36]. As we and others have shown, this condition does not hold in the core of certain defect solutions, meaning that those solutions carry a nonzero electromagnetic charge condensate.

There are useful relations connecting the bilinear formalism to the representations above. For example, one can express the 6-vector R^A in terms of the general representation fields $(f_1, f_+, f_2, \xi, \chi)$ as

$$R^A = \frac{v_{\text{SM}}^2}{2} \begin{pmatrix} f_1^2 + f_+^2 + f_2^2 \\ 2f_1 f_2 \cos \xi \\ 2f_1 f_2 \sin \xi \\ f_1^2 - f_+^2 - f_2^2 \\ -2f_1 f_+ \cos \chi \\ -2f_1 f_+ \sin \chi \end{pmatrix}, \quad (9)$$

and the EW group parameters can be related to the linear basis fields via

$$\begin{aligned} \gamma_1 &= 2 \arctan \left(\sqrt{\frac{\phi_1^2 + \phi_2^2}{\phi_3^2 + \phi_4^2}} \right), \\ \gamma_2 &= \chi - 2 \arctan \left(\frac{\phi_4}{\phi_3} \right), \\ \gamma_3 &= 2 \arctan \left(\frac{\phi_2}{\phi_1} \right) - \chi. \end{aligned} \quad (10)$$

For our discussions in Sec. IV, it is useful to note a simplified form of the potential that emerges thanks to its $SU(2)_L \times U(1)_Y$ invariance,

$$\begin{aligned} V &= -\frac{\mu_1^2}{2} f_1^2 - \frac{\mu_2^2}{2} (f_+^2 + f_2^2) + \frac{\lambda_1}{4} f_1^4 + \frac{\lambda_2}{4} (f_+^2 + f_2^2)^2 \\ &\quad + \frac{\lambda_3}{4} f_1^2 (f_+^2 + f_2^2) + \frac{1}{4} (\lambda_4 - |\lambda_5| \cos 2\xi) f_1^2 f_2^2, \end{aligned} \quad (11)$$

when we consider the general vacuum element of Eq. (7). We have set $v_{\text{SM}} = 1$ in this expression for brevity, which we continue to do in all expressions related to the potential and energy densities from here on. We may consistently do this, as it corresponds only to setting the energy and length scales, which we do in all numerical work presented: this is detailed in Sec. III of Appendix A.

B. Parameters

The potential parameters of Eq. (2) can be exchanged for a more physical set of quantities: the masses of the five scalar particles, M_h, M_H, M_A , and $M_{H^\pm}$; the vacuum expectation value of the standard model, v_{SM} ; and the mixing angles between the CP -even and CP -odd fields of the model, α and β ; with the ratio of the vacuum expectation values of the two field doublets given by $\tan \beta$, details of which can be found, for example, in Ref. [22]. To ensure that h corresponds to the observed Higgs boson, we always work in the *alignment limit* of $\cos(\alpha - \beta) = 1$ [21,39]. In this limit, the potential parameters can be written in simple closed forms in terms of the physical masses. For completeness, we list these relations here:

$$\begin{aligned} \mu_1^2 &= \frac{1}{2} M_h^2, & \mu_2^2 &= \frac{1}{2} M_h^2, \\ \lambda_1 &= \frac{M_h^2 + M_H^2 \tan^2 \beta}{2v_{\text{SM}}^2}, & \lambda_2 &= \frac{M_h^2 + M_H^2 \cot^2 \beta}{2v_{\text{SM}}^2}, \\ \lambda_3 &= \frac{(M_h^2 - M_H^2) + 2M_{H^\pm}^2}{v_{\text{SM}}^2}, \\ \lambda_4 &= \frac{M_A^2 - 2M_{H^\pm}^2}{v_{\text{SM}}^2}, & |\lambda_5| &= \frac{M_A^2}{v_{\text{SM}}^2}. \end{aligned} \quad (12)$$

Throughout this work, we refer to several specific parameter sets which have been chosen to illustrate different qualitative regimes of the model and to facilitate comparison with earlier work: they do not necessarily represent phenomenologically viable models. These parameter choices are summarized in Table I. In each case,

TABLE I. Table of mass parametrizations used in this work, with masses given in GeV. All parametrizations used assume $M_h = 125$ GeV, $v_{\text{SM}} = 246$ GeV, and $\cos(\alpha - \beta) = 1$. Parameter set A was that used in Ref. [22].

Parameter set	M_H	M_A	$M_{H^\pm}$	$\tan \beta$
A	200	200	200	0.85
B	600	200	300	0.85
C	600	300	300	0.85
D	600	300	200	0.85
E	600	400	300	0.85
F	600	300	400	0.85
G	750	900	125	0.35
H	500	125	250	0.85

we list the heavy scalar masses M_H , M_A , $M_{H^\pm}$, and $\tan\beta$ (with alignment limit assumed). The values of $M_h = 125$ GeV and $v_{\text{SM}} = 246$ GeV are fixed by experiment [40]; however, as detailed in Sec. III of Appendix A, we set the energy and length scales in our numerical work by rescaling the parameters such that these are both set to unity. For most investigations in this work—for example, the parameter scans of Sec. IV—we maintain that $\tan\beta = 0.85$ unless otherwise stated, but we show that the conclusions drawn are not limited only to this case.

C. Field configuration reduction

While the general field *Ansatz* of Eq. (7) involves eight continuous field components, finite-energy arguments can reduce the number of independent degrees of freedom needed to describe a domain wall. We introduce a convenient representation of the EW rotation matrix of Eq. (8) that leads to a symmetric form for this simplification.

However, the parameter reduction we discuss is not unique to a particular representation and can be understood by looking at the internal structure of $SU(2)$. The group admits the decomposition $U = U_1 U_2 = e^{i|u|\hat{u}^a \sigma^a} e^{i|v|\hat{v}^b \sigma^b}$, with $\hat{u}^a$ being a chosen constant direction, and v^b restricted to lie in the plane spanned by $\hat{u}^a$ and one of its two perpendicular directions, where $U_1 \in U(1)_{\hat{u}^a} \subset SU(2)$ and $U_2 \in SU(2)/U(1)_{\hat{u}^a}$. The group parameters only enter into the energy density through the gradients; therefore, we need only consider $U^\dagger \partial_x U = U_2^\dagger U_1^\dagger (\partial_x U_1) U_2 + U_2^\dagger \partial_x U_2$, where $U_1^\dagger \partial_x U_1 = i e^{-i|u|\hat{u}^a \sigma^a} (\partial_x |u|) \hat{u}^b \sigma^b e^{i|u|\hat{u}^a \sigma^a} = i(\partial_x |u|) \hat{u}^a \sigma^a$, thanks to the fixed $\hat{u}^a$, such that only the derivatives of $|u|$ enter the energy density. Similarly, the $U(1)_Y$ gauge parameter enters only through its derivatives, in complete analogy with the internal $U(1)$ subgroup of $SU(2)$,

whereas there is an explicit dependence on the two components of v^b , as well as on their derivatives. As such, only two of the group parameters appear explicitly in the energy density.

Our parametrization introduces three new angles η_1 , η_2 , η_3 defined by

$$\eta_1 = \gamma_2 - \gamma_3, \quad \eta_2 = \chi + \gamma_3, \quad \eta_3 = \chi - \gamma_3, \quad (13)$$

in terms of which we can rewrite the group element as a product of two matrices:

$$U = \begin{pmatrix} e^{\frac{i}{2}\eta_2} & 0 \\ 0 & e^{\frac{i}{2}\eta_3} \end{pmatrix} \begin{pmatrix} \cos \frac{1}{2}\gamma_1 e^{\frac{i}{2}\eta_1} & \sin \frac{1}{2}\gamma_1 \\ -\sin \frac{1}{2}\gamma_1 & \cos \frac{1}{2}\gamma_1 e^{-\frac{i}{2}\eta_1} \end{pmatrix}. \quad (14)$$

This representation is advantageous when considering static, finite-energy domain wall solutions. In particular, by demanding that the energy density of the wall remain localized, one finds that η_2 and η_3 can be expressed in terms of the remaining six components in the wall configuration and as such explicitly eliminated; we will show later that they in fact converge to zero in all numerical solutions we have found. As a result, a general domain wall solution can be described using only six independent functions of x : namely, f_1 , f_+ , f_2 , ξ , γ_1 , and η_1 . In specific regions of parameter space, further simplifications occur—for example, some of these functions vanish, yielding special subclasses of solutions, as we will explore in Sec. IV.

To illustrate our proposed field reduction, we can write the one-dimensional energy density for the general field configuration and examine its structure. The one-dimensional energy density for this configuration is given by

$$\begin{aligned} \mathcal{E} = & \frac{1}{2}(\partial_x f_1)^2 + \frac{1}{2}(\partial_x f_+)^2 + \frac{1}{2}(\partial_x f_2)^2 + \frac{1}{2}f_2^2(\partial_x \xi)^2 + \frac{1}{8}(f_1^2 + f_+^2 + f_2^2)[(\partial_x \gamma_1)^2 + c_1^2(\partial_x \eta_1)^2] \\ & + \frac{1}{2}c_\xi[(f_2 \partial_x f_+ - f_+ \partial_x f_2) \partial_x \gamma_1 + f_+ f_2 s_1 c_1 \partial_x \xi \partial_x \eta_1] + \frac{1}{2}s_\xi[(f_+ \partial_x f_2 - f_2 \partial_x f_+) s_1 c_1 \partial_x \eta_1 + f_+ f_2 \partial_x \xi \partial_x \gamma_1] \\ & - \frac{1}{2}f_2^2 c_1^2 \partial_x \xi \partial_x \eta_1 + \mathcal{E}_{\eta_2} + \mathcal{E}_{\eta_3} + V, \end{aligned}$$

where

$$\begin{aligned} \mathcal{E}_{\eta_2} = & \frac{\partial_x \eta_2}{2} \left[f_2^2 s_1^2 \partial_x \xi + \frac{1}{2} f_+^2 c_1^2 \partial_x \eta_1 + \frac{1}{2} f_+ f_2 s_\xi \partial_x \gamma_1 + s_1 c_1 \left\{ f_+ f_2 c_\xi \left(\partial_x \xi + \frac{\partial_x \eta_1}{2} \right) + (f_+ \partial_x f_2 - f_2 \partial_x f_+) s_\xi \right\} \right] \\ & + \frac{(\partial_x \eta_2)^2}{8} [(f_1^2 + f_2^2) s_1^2 + f_+^2 c_1^2 + 2 f_+ f_2 s_1 c_1 c_\xi], \\ \mathcal{E}_{\eta_3} = & \frac{\partial_x \eta_3}{2} \left[f_2^2 c_1^2 \partial_x \xi - \frac{1}{2} (f_1^2 + f_2^2) c_1^2 \partial_x \eta_1 + \frac{1}{2} f_+ f_2 s_\xi \partial_x \gamma_1 - s_1 c_1 \left\{ f_+ f_2 c_\xi \left(\partial_x \xi - \frac{\partial_x \eta_1}{2} \right) + (f_+ \partial_x f_2 - f_2 \partial_x f_+) s_\xi \right\} \right] \\ & + \frac{(\partial_x \eta_3)^2}{8} [(f_1^2 + f_2^2) c_1^2 + s_1^2 f_+^2 - 2 f_+ f_2 s_1 c_1 c_\xi], \end{aligned} \quad (15)$$

where we have defined $s_1 = \sin \frac{1}{2}\gamma_1$, $c_1 = \cos \frac{1}{2}\gamma_1$, $s_\xi = \sin(\xi - \frac{1}{2}\eta_1)$, and $c_\xi = \cos(\xi - \frac{1}{2}\eta_1)$ for brevity. The two EW group parameters η_2 and η_3 do not appear explicitly, but only implicitly through their first derivatives. Taking the resulting equation of motion for η_2 ,

$$\partial_x \left\{ \frac{1}{2} \partial_x \eta_2 [s_1^2(f_1^2 + f_2^2) + c_1^2 f_+^2 + 2s_1 c_1 f_+ f_2 c_\xi] + \left[s_1^2 f_2^2 \partial_x \xi + \frac{1}{2} c_1^2 f_+^2 \partial_x \eta_1 + \frac{1}{2} f_+ f_2 s_\xi \partial_x \gamma_1 \right. \right. \\ \left. \left. + s_1 c_1 \left\{ f_+ f_2 c_\xi \left(\partial_x \xi + \frac{1}{2} \partial_x \eta_1 \right) + (f_+ \partial_x f_2 - f_2 \partial_x f_+) s_\xi \right\} \right] \right\} = 0, \quad (16)$$

we see that the contents of the outermost brackets must therefore equal a constant. For finite-energy solutions, the derivatives of all functions must go to zero far from the wall, and as such, the constant must equal zero. We may also use the same argument for η_3 , such that we obtain the two expressions

$$\partial_x \eta_2 = - \frac{s_1^2 f_2^2 \partial_x \xi + \frac{1}{2} c_1^2 f_+^2 \partial_x \eta_1 + \frac{1}{2} f_+ f_2 s_\xi \partial_x \gamma_1 + s_1 c_1 \{ f_+ f_2 c_\xi (\partial_x \xi + \frac{1}{2} \partial_x \eta_1) + (f_+ \partial_x f_2 - f_2 \partial_x f_+) s_\xi \}}{\frac{1}{2} s_1^2 (f_1^2 + f_2^2) + \frac{1}{2} c_1^2 f_+^2 + s_1 c_1 f_+ f_2 c_\xi}, \quad (17)$$

$$\partial_x \eta_3 = - \frac{c_1^2 f_2^2 \partial_x \xi - \frac{1}{2} c_1^2 (f_1^2 + f_2^2) \partial_x \eta_1 + \frac{1}{2} f_+ f_2 s_\xi \partial_x \gamma_1 - s_1 c_1 \{ f_+ f_2 c_\xi (\partial_x \xi - \frac{1}{2} \partial_x \eta_1) + (f_+ \partial_x f_2 - f_2 \partial_x f_+) s_\xi \}}{\frac{1}{2} c_1^2 (f_1^2 + f_2^2) + \frac{1}{2} s_1^2 f_+^2 - s_1 c_1 f_+ f_2 c_\xi}, \quad (18)$$

which may be substituted into the remaining six equations of motion, or back into the energy density to eliminate the dependence on η_2 and η_3 explicitly. This results in any domain wall solution within the model depending on only six field components, $f_1, f_+, f_2, \xi, \gamma_1$, and η_1 , from which six consistent equations of motion may be derived.⁴

In Sec. IV, we use this reduced field description to classify all possible domain wall solutions and identify the parameter conditions under which each occurs.

III. MOTIVATION

A. Full dynamical simulations

Numerical simulations of the 2HDM from random initial conditions (RIC) indicate [22] that the domain wall solution of Ref. [18] does not naturally form in simulations of the same parameters (parameter set A). Instead, the walls in the simulation develop a nonzero $R^\mu R_\mu$ (i.e., the walls become superconducting). Moreover, although the bilinear

components R^0, R^1 , and R^3 behaved as in the known solution (R^1 changes sign across the wall, and R^0, R^3 have the expected extrema), the simulations showed a notable discrepancy: R^2 is nonzero on the domain walls, whereas in the Ref. [18] solution R^2 vanishes.⁵ In other words, the domain walls that form dynamically tend to excite fields that are found to be zero in the solution of Ref. [18].

Figure 1 illustrates these findings by showing the spatial profiles of R^1, R^2 , and $R^\mu R_\mu$ in a set of $(2+1)$ -dimensional field simulations from RIC, for four different parameter sets. Fig. 1(a) reproduces the result for parameter set A (used in Ref. [22]), showing that both R^2 and $R^\mu R_\mu$ are mildly localized on the domain walls. We also show corresponding simulations for three alternative parameter sets (B, C, and D) to highlight how the wall structure varies with the mass hierarchy; these specific choices will be analyzed in detail in Sec. IV, but here we note the key differences. In all three additional parameter sets, either R^2 and/or $R^\mu R_\mu$ become large on the walls. When $M_A = M_{H^\pm}$ (parameter set C), both quantities are moderately elevated on the walls. However, when $M_{H^\pm} < M_A$ (set D), the simulations show a significantly larger $R^\mu R_\mu$ condensate and a much smaller R^2 on the walls. This trend reverses for $M_{H^\pm} > M_A$ (set B), where a prominent R^2 accompanies a suppressed $R^\mu R_\mu$. Note that where $R^\mu R_\mu$ is found to be nonzero on domain walls, a “winding” is observed in both the extended components of R^4 and R^5 , although it is not depicted here. These enhanced features correlate directly

⁴In the language of classical mechanics, η_2 and η_3 are *cyclic* (or *ignorable*) generalized coordinates: the one-dimensional Lagrangian density (or static energy density) depends on them only through their first derivatives, so their conjugate momenta are conserved. The finite energy conditions described above require these conserved momenta to vanish identically. In this special case, the cyclic coordinates could also be eliminated directly by substitution into the Lagrangian also, without altering the equations of motion for the remaining fields. However, in the generic case, where the associated conjugate momenta are nonzero, direct substitution into the Lagrangian changes the variational problem and yields incorrect reduced equations of motion; the consistent procedure is to perform a partial Legendre transform in the cyclic coordinates (Routhian reduction) or to work in the Hamiltonian formalism [41–43], as we have done here by using the one-dimensional energy density.

⁵Reference [22] did not comment on the nonzero R^2 observed on the walls; it was first drawn to attention in our recent work of Ref. [26]; the present work is the first to analyze this feature in the 2HDM.

with the expected minimum-energy solutions for each parameter set, as we will demonstrate in Sec. IV. In contrast, the original (parameter set A) solution from Ref. [18] (which we confirm to be of minimum energy) exhibits neither feature found to occur in dynamical simulations.

Some definitions used throughout this work which are important to highlight are the following:

- (1) *Minimum-energy solution*: The lowest possible energy solution for a given set of parameters ($M_H, M_A, M_{H^\pm}, \tan\beta$), where all general field components are allowed to vary, and homogeneous Neumann boundary conditions are imposed.
- (2) *Naturally bounded solution*: Solutions where homogenous Neumann boundary conditions are imposed on all field variables. These boundary conditions are *natural* for domain wall solutions, such that there is no variation of the energy of the fields in the vacuum. These solutions are not

necessarily of minimum energy—for example, when a reduced field *Ansatz* is used.

- (3) *Standard solution*: Domain wall solution found in Ref. [18], where $f_+ = \xi = \gamma_1 = \eta_1 = \eta_2 = \eta_3 \equiv 0$, described only by the general field variables f_1 and f_2 .

We will show in Sec. IV that the parameter space of the model divides into distinct regions, characterized by whether the minimum-energy wall solution supports nonzero $R^\mu R_\mu$, nonzero R^2 , both, or neither. These solutions can be described by reduced *Ansätze*, obtained by restricting the general field configuration of Sec. II C, thereby greatly simplifying the analysis of each region. However, as we have detailed above with Fig. 1, simulations reveal that the fields do not fully relax to these minimum-energy configurations: the walls exhibit small but nonzero values of $R^\mu R_\mu$ and/or R^2 even when the corresponding minimum-energy state does not. These apparent “excitations” of the walls can be traced back to the fact that the fields emerge

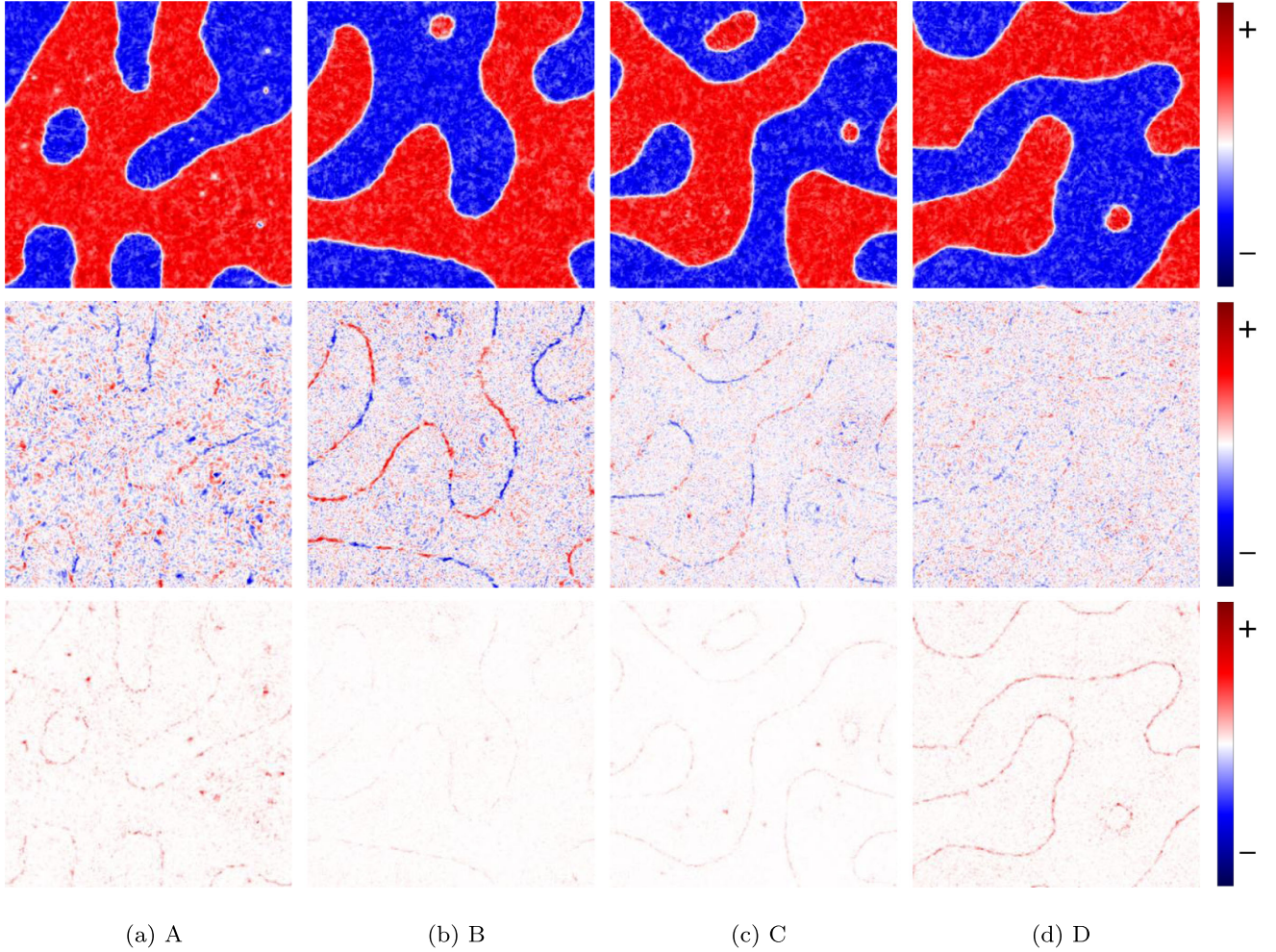


FIG. 1. $(2 + 1)$ -dimensional simulations of the 2HDM with a $\mathbb{Z}_2$ -symmetric potential for different mass parametrizations (A, B, C, and D). Shown are the bilinear components R^1 (top), R^2 (middle), and $R^\mu R_\mu$ (bottom). Color mappings of each individual component are normalized across all parameter sets, with each component type having a separate scale.

from random initial conditions, as expected in a phase transition, leading to relative EW transformations between neighboring vacua. These field configurations have been previously discussed in Refs. [22,23,25], but their role has not yet been systematically characterized. We first aim to address these relative EW transformations to explain the departures from the minimum-energy state observed in full dynamical simulations.

It is important to note here that a nonzero value of R^2 on a domain wall requires an interpolating profile of the phase ξ between 0 and π across the wall, and thus a nonzero value of f_2 at its center. This follows directly from the structure of R^2 in Eq. (9). The field component $f_2 e^{i\xi}$ can be written in two equivalent ways depending on the choice of domain, $f_2 \in \mathbb{R}_{\geq 0}$, $\xi \in (-\pi, \pi]$ or $f_2 \in \mathbb{R}$, $\xi \in (-\pi/2, \pi/2]$. For solutions with $f_2(0) = 0$, the wall may be described either as an interpolation from negative to positive values of f_2 with $\xi \equiv 0$ or, equivalently, as a discontinuity in ξ between 0 and π and $f_2 \geq 0$ —both descriptions coincide smoothly in the limit $f_2 \rightarrow 0$. For solutions with $f_2(0) \neq 0$, the wall can be considered to interpolate smoothly in ξ with $f_2 > 0$, producing a nonzero R^2 , or as an equivalent description that involves discontinuities in both ξ and f_2 at the center. In both cases, we choose the description without discontinuities, as this is both simpler and more convenient for numerical work, although this does mean that we have chosen different domains for the two scenarios. We will often refer to the former type of solution by stating that “the wall is in f_2 ,” and to the latter by stating that “the wall is in ξ ”. Previous studies have only considered walls in f_2 .

B. Relative electroweak transformations of the vacua

As mentioned above, it has been explained [22] that one can generally perform a relative EW transformation of the vacuum upon either side of the domain wall. In practice, this means that the vacuum expectation values on either side of the wall need not be in the same electroweak gauge—they can differ by a constant $U(1)_Y \times SU(2)_L$ transformation. This situation was first explored in Ref. [22], as it would be a natural occurrence in full dynamical simulations from RIC.

Such a field configuration takes the form

$$\begin{aligned} \Phi(-\infty) &= \frac{v_{\text{SM}}}{\sqrt{2}} \begin{pmatrix} 0 \\ \bar{f}_1 \\ 0 \\ -\bar{f}_2 \end{pmatrix}, \\ \Phi(+\infty) &= \frac{v_{\text{SM}}}{\sqrt{2}} (\sigma^0 \otimes U) \begin{pmatrix} 0 \\ \bar{f}_1 \\ 0 \\ \bar{f}_2 \end{pmatrix}, \end{aligned} \quad (19)$$

where $U \equiv U(\chi, \gamma_1, \gamma_2, \gamma_3)$ is a constant $U(1)_Y \times SU(2)_L$ rotation matrix as given in Eq. (8) and $(\bar{f}_1, \bar{f}_2) = (\cos \beta, \sin \beta)$ are the vacuum expectation values for the two fields, with the sign of the lower component changing as in a $\mathbb{Z}_2$ domain wall. Reference [22] only studied a small number of different relative rotations, whereas Ref. [23] considered specific lines within the parameter space using a different but equivalent EW representation. Here we report a systematic study of the solutions as functions of χ , γ_1 , γ_2 , and γ_3 , completing the delineation of this issue.

First, we note that we can express

$$(\sigma^0 \otimes U) \begin{pmatrix} 0 \\ \bar{f}_1 \\ 0 \\ \bar{f}_2 \end{pmatrix} = \begin{pmatrix} \bar{f}_1 \sin \frac{1}{2} \gamma_1 e^{\frac{i}{2} \theta_2} \\ \bar{f}_1 \cos \frac{1}{2} \gamma_1 e^{\frac{i}{2} \theta_1} \\ \bar{f}_2 \sin \frac{1}{2} \gamma_1 e^{\frac{i}{2} \theta_2} \\ \bar{f}_2 \cos \frac{1}{2} \gamma_1 e^{\frac{i}{2} \theta_1} \end{pmatrix}, \quad (20)$$

where $\theta_1 = \chi - \gamma_2 = \eta_3 - \eta_1$ and $\theta_2 = \chi + \gamma_3 = \eta_2$. This has one less EW degree of freedom than the general element in Eq. (8), due to the demand of the unbroken $U(1)_{\text{EM}}$ symmetry in the vacuum. Importantly, some of the remaining continuous parameters are physically redundant, as an additional global EW rotation can be applied that does not alter $\Phi(-\infty)$ but does alter $\Phi(+\infty)$. This transformation is the unbroken $U(1)_{\text{EM}}$ symmetry at negative infinity, but not at positive infinity, and it takes the form $\sigma^0 \otimes \bar{U}$, where

$$\bar{U} = \begin{pmatrix} e^{-\frac{i}{2} \theta_2} & 0 \\ 0 & 1 \end{pmatrix}, \quad (21)$$

which eliminates the θ_2 dependence of $\Phi(+\infty)$,

$$\Phi(+\infty) = \frac{v_{\text{SM}}}{\sqrt{2}} \begin{pmatrix} \bar{f}_1 \sin \frac{1}{2} \gamma_1 \\ \bar{f}_1 \cos \frac{1}{2} \gamma_1 e^{\frac{i}{2} \theta_1} \\ \bar{f}_2 \sin \frac{1}{2} \gamma_1 \\ \bar{f}_2 \cos \frac{1}{2} \gamma_1 e^{\frac{i}{2} \theta_1} \end{pmatrix}. \quad (22)$$

Choosing parameter set A for this study, we solved for kink solutions by identifying the far vacuum values given by Eq. (22) with those in the linear representation and evolving the equations of motion within the latter. We used the *approximate treatment* detailed in Sec. I of Appendix A for this investigation, with numerical settings of $n_x = 300$, $\Delta x = 0.1$, $\delta = 10^{-5}$ for computational ease, as such a treatment and resolution is acceptable when working in the linear representation: the field equations are only softly coupled. As is the nature of this type of solution, we impose fixed boundary conditions on the fields, varying the two angles γ_1 and θ_1 between 0 and 4π for full coverage of the field configuration. The values of R^2 and $R^\mu R_\mu$ at the center

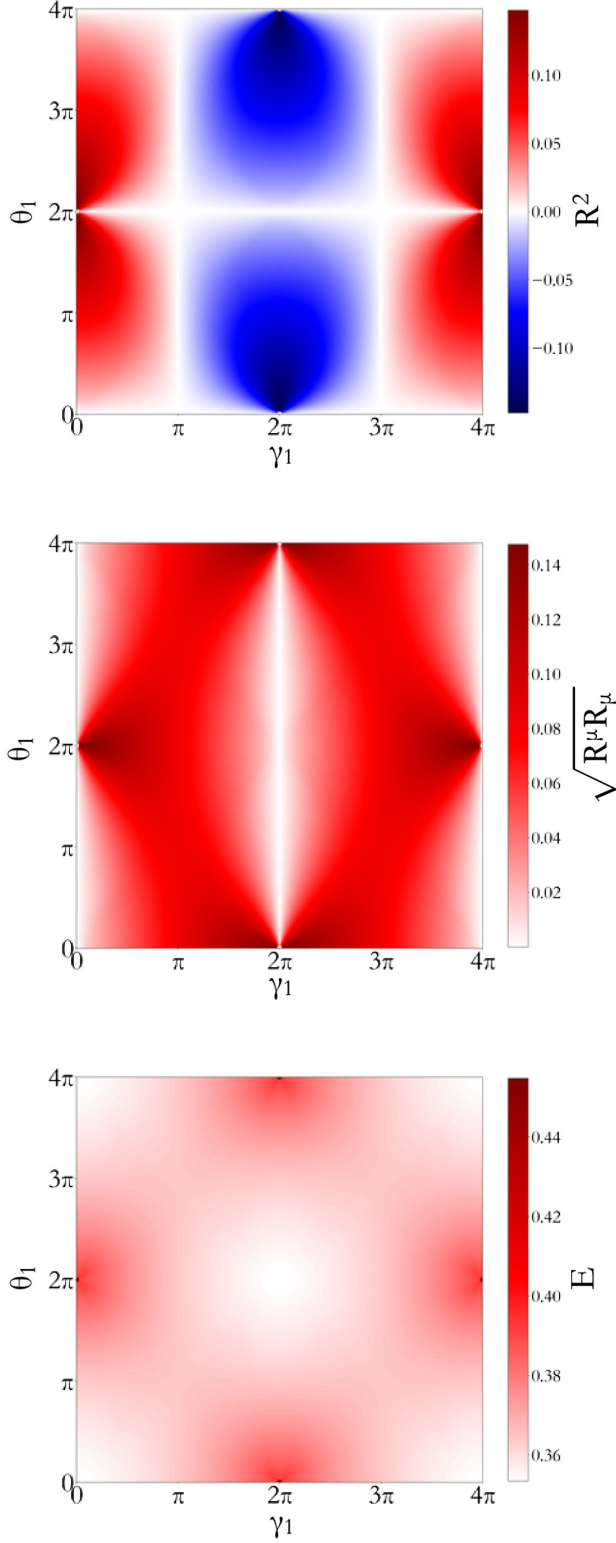


FIG. 2. Values of R^2 and $\sqrt{R^\mu R_\mu}$ at the center of fixed boundary domain wall solutions, and the total energy, E , as functions of γ_1 and θ_1 in parameter set A, for relatively EW rotated vacua. The rotations were varied in increments of $\pi/128$ in the range $[0, 4\pi]$, such that 512^2 different solutions were computed.

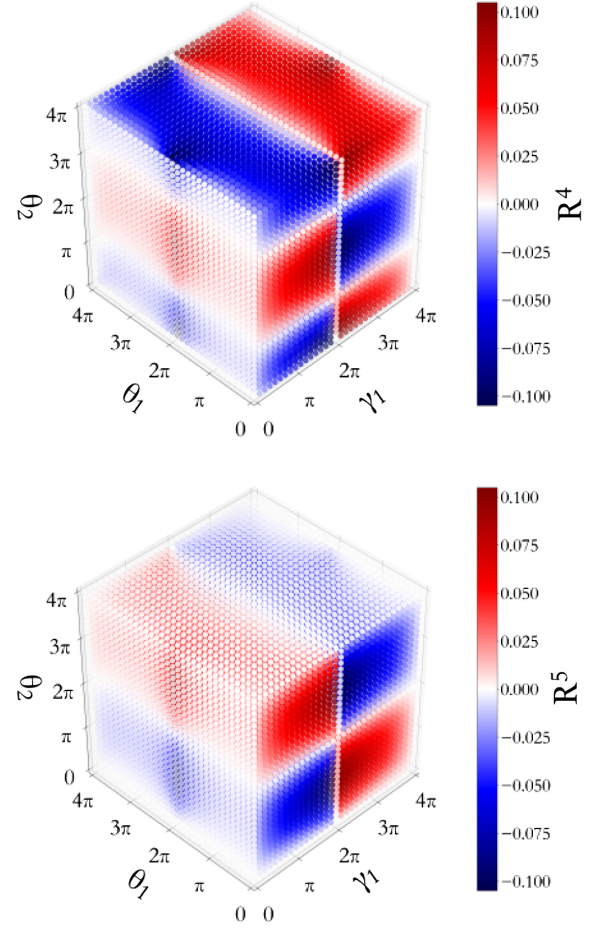


FIG. 3. Values of R^4 and R^5 at the center of fixed boundary domain wall solutions, as functions of γ_1 and θ_1 in parameter set A for relatively EW rotated vacua. The rotations were varied in increments of $\pi/8$ in the range $[0, 4\pi]$, such that 32^3 different solutions were computed.

of the kink solutions are shown in Fig. 2 across this relative transformation space, with the energy of the solutions also included. This analysis shows that only under zero or an exact $\gamma_1 = \theta_1 = 2\pi$ rotation does the field configuration correspond to that of the minimum-energy solution for this parameter set, and as such any small relative rotation between the two vacua in simulations will result in altered wall properties—i.e., $R^2 \neq 0$ and/or $R^\mu R_\mu \neq 0$. This fully explains the features of the simulation depicted in Fig. 1(a).

It is clear that there are symmetries in the (γ_1, θ_1) space. In particular, $\gamma_1 \rightarrow 4\pi - \gamma_1$ and $\theta_1 \rightarrow 4\pi - \theta_1$, that effectively reduce the space to $0 \leq \gamma_1, \theta_1 < 2\pi$, which can be easily understood by observing the structure of the trigonometric functions in Eq. (22). The symmetry in γ_1 is equivalent to the global EW transformation $(\sigma^0 \otimes \sigma^3)$, and the symmetry in θ_1 simply corresponds to complex conjugation, both of which affect $\Phi(+\infty)$ in general, but not $\Phi(-\infty)$, similarly to the transformation of Eq. (21).

We also present in Fig. 3 the components of R^4 and R^5 . While these extended components are invariant under an $SU(2)_L$ transformation, they are manifestly variant under a simple $U(1)$ phase change, and as such θ_2 also contributes to their variation. The variation of the energy under the three rotation parameters is not included, as the energy was found to be invariant under θ_2 .

We can map the solutions that we have produced in the linear representation to the general using Eqs. (9), (10), allowing us to highlight the following features. Given our observation of nonzero values in the bilinear components of R^2 and $R_\mu R^\mu$ (including the winding of R^4) within full dynamical simulations, we present in Fig. 4 a selection of solutions designed to illustrate how these features manifest in the general representation. These solutions are restricted to those where either R^2 is nonzero at the center of the wall, for which the domain wall forms in ξ , and those where

$R^\mu R_\mu$ is nonzero, for which the domain wall forms in f_2 . We can immediately see from these solutions that for both a nonzero R^2 and $R^\mu R_\mu$, the two EW parameters of χ and γ_3 are identically zero and can be removed from the problem, which is unsurprising given the elimination of $\theta_2 = \chi + \gamma_3$. We observe that in the case where $R^\mu R_\mu \neq 0$, then $\text{sgn}(f_+) = \text{sgn}(\partial_x \gamma_1)$, which was also found to be the case in a set of solutions presented in Ref. [25], and in the case of $R^2 \neq 0$, that $\text{sgn}(\partial_x \xi) = \text{sgn}(\partial_x \gamma_2)$, a previously unexplored observation of this type of solution. We note also that in each case, one could apply a global rotation to shift either γ_1 or γ_2 by a constant such that they are zero at the center of the solution, which highlights that the relevant structure of each type of solution is the nonzero value of $\partial_x \gamma_i$ at the center of the kink—i.e., the interpolation of γ_i .

The effects of γ_1 and γ_2 upon f_+ and ξ , respectively, were the motivation for the form of our analytic reduction of the

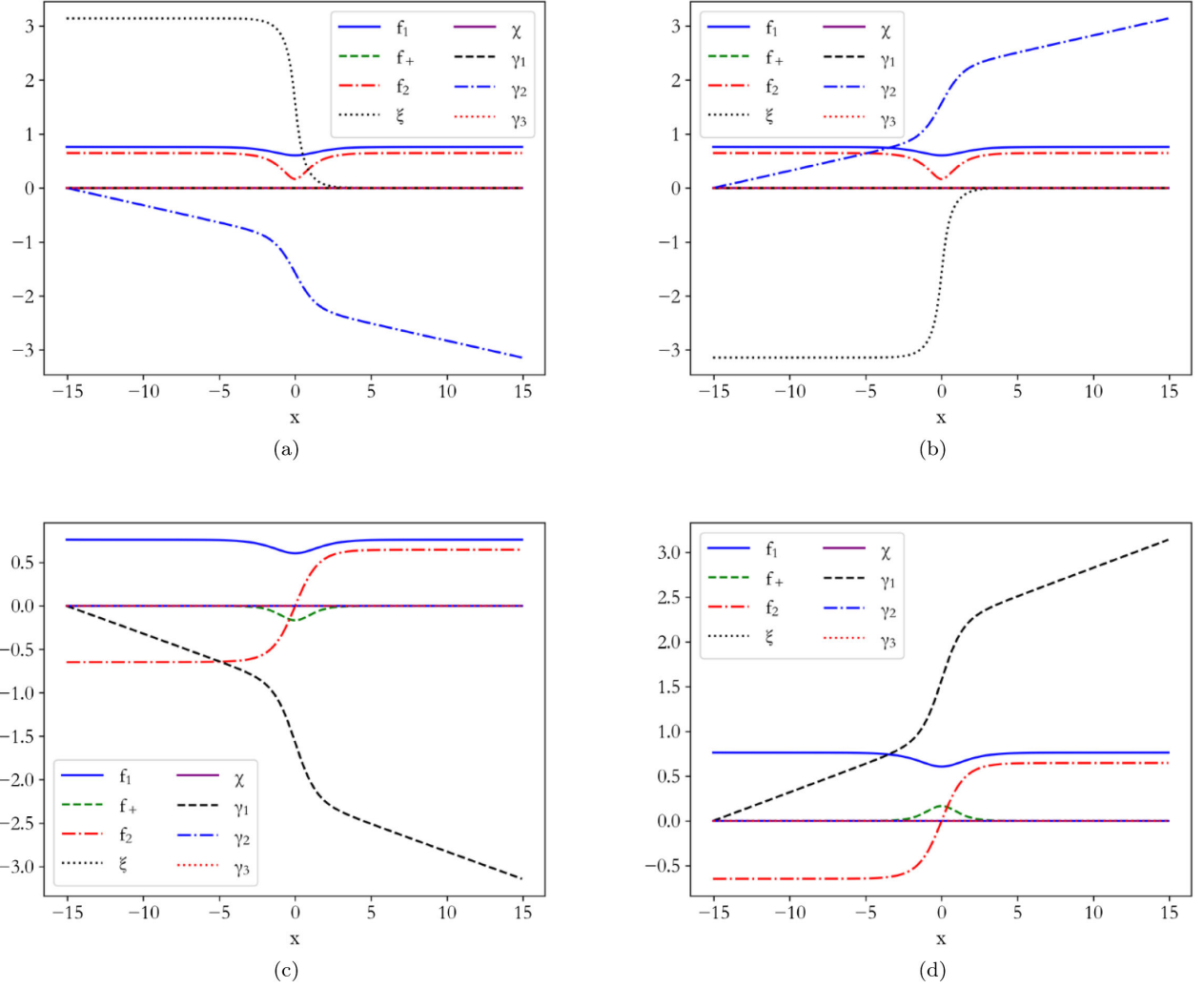


FIG. 4. Relatively EW rotated domain wall solutions where the solution exhibits either nonzero R^2 (top) or R^4 (bottom) in parameter set A. Solutions where the value at the center of the wall is positive are depicted on the left, and those of equal magnitude but negative are shown on the right. Within the space of $(\gamma_1, \theta_1, \theta_2)$ presented are solutions (a) $(0, \pi, 0)$, (b) $(2\pi, \pi, 0)$, (c) $(3\pi, 0, 0)$, and (d) $(\pi, 0, 0)$. In (a) and (b), the field components f_+ , χ , γ_1 , γ_3 are identically zero for all x ; in (c) and (d), ξ , χ , γ_2 , γ_3 are identically zero for all x .

general field configuration in Sec. II C, as the structure of the field configuration is preserved in γ_1 and η_1 , respectively.

IV. GENERAL DOMAIN WALL SOLUTIONS WITH $\mathbb{Z}_2$ SYMMETRY

Before turning to the general minimum-energy domain wall solutions, it is instructive to first consider several special subclasses that arise within the reduced field configuration of Sec. II C. These subclasses correspond to distinct regions of parameter space where certain field components consistently vanish, leading to simplified *Ansätze* with clear and characteristic structure. Indeed, some of these solutions coincide directly with those already identified in Refs. [18,22,25]. Presenting these subclasses first is not merely pedagogical: our subsequent analysis reveals that the fully general solutions always reduce to one of these subclasses, depending on the region of parameter space under consideration. In this sense, the subclasses serve as natural building blocks, and their study both clarifies the organization of the parameter space and is essential for interpreting the general solution landscape. All solutions presented in this section are naturally bounded. Only when we analyze the unrestricted reduced general field configuration do we obtain the minimum-energy solutions.

All solutions in this section were obtained using the *full treatment* of Sec. I in Appendix A. For broad parameter scans, we used numerical settings of $n_x = 600$, $\Delta x = 0.05$, $\delta = 10^{-5}$ to efficiently cover the parameter range. For individual example solutions, we used $n_x = 3000$, $\Delta x = 0.01$, $\delta = 10^{-7}$ to achieve higher accuracy in the profiles.

A. $\gamma_1 = \eta_1 = \eta_2 = \eta_3 = 0$

We first consider the case where all EW group parameters in Eq. (14) are set to zero. We find that this restricted *Ansatz* does not generically yield what we have defined as the *standard solution*, originally presented in Ref. [18]. This earlier work found the field configuration to enforce $f_+ \equiv \xi \equiv 0$, but we find that this behavior only arises for specific choices of model parameters. In fact, we find that specific regions of parameter space admit either superconducting or *CP*-violating behavior, with $f_+ \neq 0$ or $\xi \neq 0$, respectively, subject to the mass hierarchy of M_A and $M_{H^\pm}$. The dependence on the M_A and $M_{H^\pm}$ mass ordering is explained in the semianalytic categorization of the general case in Sec. IV D.

When $M_A > M_{H^\pm}$, there is a region of the parameter space, shown in the dark-shaded area of Fig. 5(a), in which the wall develops a stable condensate; an explicit example is provided in Fig. 5(b). In this case $\xi \equiv 0$, so the potential no longer depends on the combination $\lambda_4 + |\lambda_5|$, and the solutions become independent of M_A .

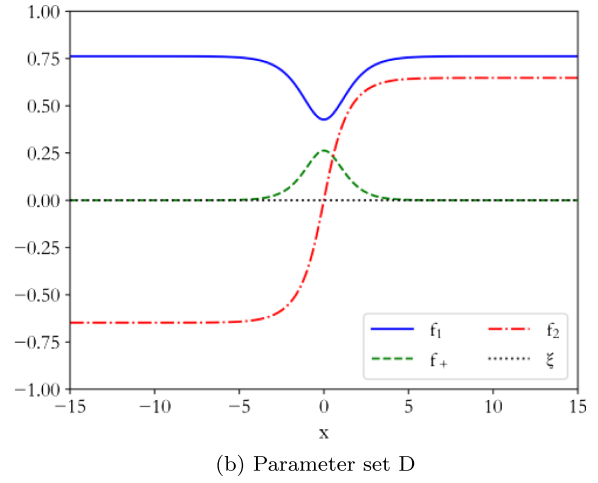
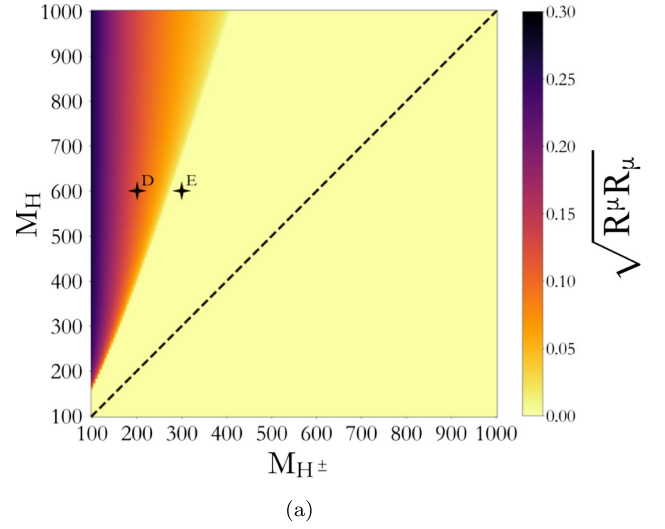


FIG. 5. (a) $\sqrt{R^\mu R_\mu}$ at the center of the domain wall for naturally bounded solutions of the field configuration [Eq. (7)] with the field restriction $\gamma_1 = \eta_1 = \eta_2 = \eta_3 = 0$, and parameter restriction $M_A > M_{H^\pm}$. Scalar masses are given in GeV, and we set M_A to be 100 GeV greater than $M_{H^\pm}$; however, the magnitude of this mass difference does not affect the solutions. There is a clear dark-shaded region where there exist solutions with a stable condensate. The dashed line represents $M_H = M_{H^\pm}$. (b) Example solution for parameter set D, where a condensate forms in f_+ .

Conversely, when $M_A < M_{H^\pm}$, a region exists where the wall exists smoothly in ξ rather than f_2 . This generates a nonzero R^2 at the center of the wall, giving rise to a qualitatively new type of solution. This parameter region is illustrated in Fig. 6(a), and an example solution form is given in Fig. 6(b). Such ξ -wall configurations have been overlooked in previous studies, which have focused exclusively on walls in f_2 . For this type of solution, $f_+ \equiv 0$, and the potential then depends directly on the combination of $\lambda_3 + \lambda_4$, eliminating the dependence on $M_{H^\pm}$.

Outside of these two regions, the solutions take the form shown in Fig. 7, which we identify as the *standard solution*.

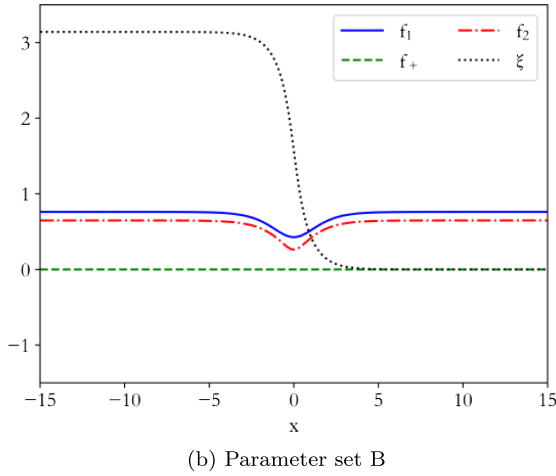
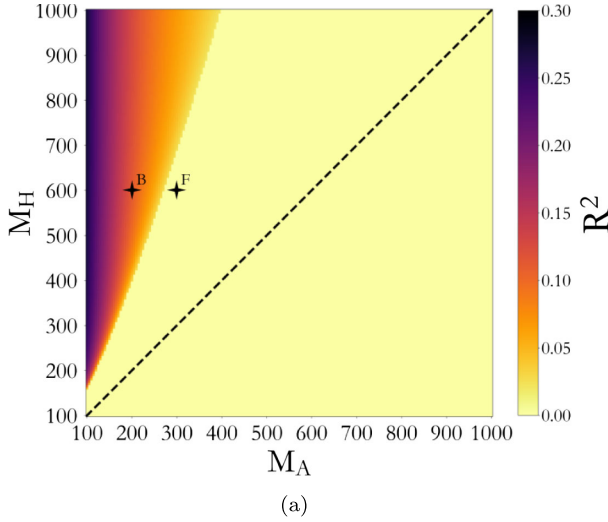


FIG. 6. (a) R^2 at the center of the domain wall for naturally bounded solutions of the field configuration [Eq. (7)] with the field restriction $\gamma_1 = \eta_1 = \eta_2 = \eta_3 = 0$, and parameter restriction $M_A < M_H^\pm$. Scalar masses are given in GeV, and we set $M_{H^\pm}$ to be 100 GeV greater than M_A ; however, the magnitude of this mass difference does not affect the solutions. There is a clear dark-shaded region where there exist solutions with domain wall in ξ . The dashed line represents $M_H = M_A$. (b) Example solution for parameter set B, where the wall exists smoothly in ξ .

This subclass is described entirely by f_1 and f_2 , with $f_+ \equiv \xi \equiv 0$, which consequently makes the solutions independent of both M_A and $M_{H^\pm}$. The appearance of the condensates or ξ -walls in this restricted configuration can be understood as reductions to the potential energy when $M_{H^\pm}$ or M_A , respectively, are sufficiently small compared to M_H , with the precise boundary depending on $\tan\beta$. The general conditions are developed in the semianalytic discussion of Sec. IV D. Note that the regions of $R^\mu R_\mu \neq 0$ and $R^2 \neq 0$ in the two different mass orderings of this restricted *Ansatz* coincide exactly in extent.

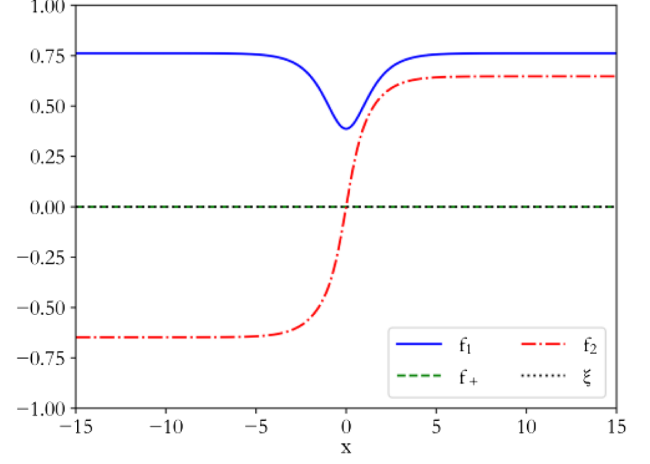


FIG. 7. Naturally bounded domain wall solution in parameter sets E and F for the configuration [Eq. (7)] with the field restriction $\gamma_1 = \eta_1 = \eta_2 = \eta_3 = 0$. This type of solution is fully described by the field components f_1 and f_2 only and is identified as the *standard solution* form.

We do not distinguish the superconducting ($f_+ \neq 0$) or *CP*-violating ($\xi \neq 0$) solutions within this *Ansatz* as distinct subclasses of solution. As shown in Secs. IV B and IV C, whenever the configurations shown in Figs. 5(b) and 6(b) are possible under this restricted *Ansatz*, lower-energy solutions arise once γ_1 or η_1 , respectively, are allowed to vary. In other words, these configurations are never minimum-energy solutions when γ_1 or η_1 are unrestricted. We have included the above discussion of condensates and ξ -walls in this restricted *Ansatz* as a precursor to the following subclasses and to complete the delineation of this *Ansatz*.

The central point is to distinguish the *standard solution* as a distinct simple subclass of solution, characterized by the form in Fig. 7 and dependent solely on the general representation fields f_1 and f_2 as previously defined, with $f_+ \equiv \xi \equiv 0$.

B. $\gamma_1 \neq 0$, $\xi = \eta_1 = \eta_2 = \eta_3 = 0$

In Ref. [25], it was shown that allowing only the parameter γ_1 of Eq. (8) to vary yields a naturally bounded solution with a stable condensate at the center of the wall. Their analysis, performed for a single parameter set ($M_H = 800$ GeV, $M_A = 500$ GeV, $M_{H^\pm} = 400$ GeV, $\tan\beta = 0.85$), identified a solution similar to the type we display in Fig. 8(b), which is for parameter set E.

Examining the energy functional confirms that a nonzero f_+ at the center of the wall lowers the energy further should γ_1 interpolate across the wall, with $\text{sgn}(f_+) = \text{sgn}(\partial_x f_2 \partial_x \gamma_1)$. This follows directly from the energy density term for this subclass,

$$-\frac{1}{2} f_+ \partial_x f_2 \partial_x \gamma_1, \quad (23)$$

in agreement with the analysis of Ref. [25]. Furthermore, as also noted (but not explicitly demonstrated) in Ref. [25], the values that γ_1 takes at the boundaries, and thus the amplitude of the condensate f_+ , are directly related to the masses of the physical scalars, a result which we confirm.

We restrict $\xi \equiv 0$ here, such that the domain wall is explicitly in f_2 , the solutions can admit a condensate and are again independent of M_A . This allows us to perform a parameter scan of M_H and $M_{H^\pm}$ which is shown in Fig. 8(a). We observe an enhanced region of the parameter space where a stable condensate exists upon the wall, accompanied by an interpolating profile for γ_1 . Outside of this region, we find the field configuration to relax to the previously determined standard solution, with f_+ and γ_1 vanishing. We observe that the effect of allowing γ_1 to vary can be dramatic; the solutions in Figs. 7 and 8(b) share identical scalar masses, yet only the latter supports a condensate. We also note that equivalent solutions exist with the opposite condensate sign, realized when γ_1 interpolates in the opposite direction, as already pointed out in Ref. [25]. These equivalent solutions are simply related by a global rotation.

The mass dependence of $\sqrt{R^\mu R_\mu}$ at the center of the wall closely mirrors the behavior seen in Ref. [44], where condensation onto the core of the string occurs when $M_H \gtrsim M_{H^\pm}$. We shall return to and quantify this correspondence later in Sec. IV D, where we provide a semi-analytic explanation of the parameter dependence of the solution subclasses.

We now propose the form of the solution in Fig. 8(b) as our second subclass of solution, the *superconducting solution*, characterized by a stable condensate on the wall and an interpolating γ_1 profile with local breaking of the electromagnetic $U(1)$ symmetry.

C. $\eta_1 \neq 0, f_+ = \gamma_1 = \eta_2 = \eta_3 = 0$

The findings of Sec. III B suggest that an interpolating profile in η_1 should accompany a wall in ξ . From the energy functional, it clearly follows that interpolating ξ and η_1 profiles lower the energy further whenever $\text{sgn}(\partial_x \xi) = \text{sgn}(\partial_x \eta_1)$, due to the term in the energy density,

$$-\frac{1}{2} f_2^2 \partial_x \xi \partial_x \eta_1. \quad (24)$$

As suggested in Sec. IV A and to be shown explicitly in Sec. IV D, the presence or absence of a condensate, and likewise the existence of a wall in ξ , is controlled by the hierarchy of M_A and $M_{H^\pm}$. Here, we restrict $f_+ \equiv 0$ such that we are explicitly in the ξ -wall regime and the solutions are independent of $M_{H^\pm}$.

In Fig. 9(a), we show a mass parameter scan for this *Ansatz*, with an example solution in Fig. 9(b). This, akin to the variation of γ_1 , reveals an enhanced region where ξ -wall solutions exist. Outside of this region, the lowest-energy

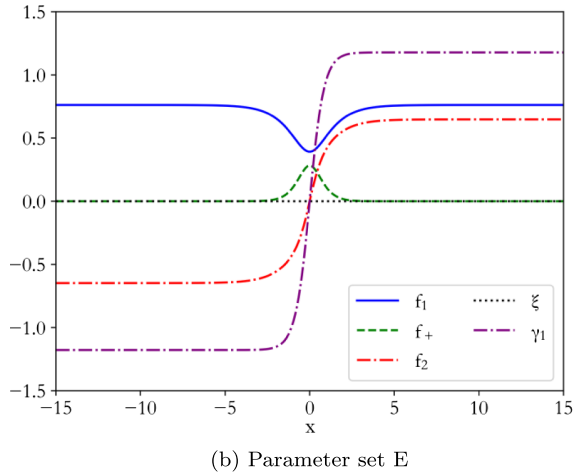
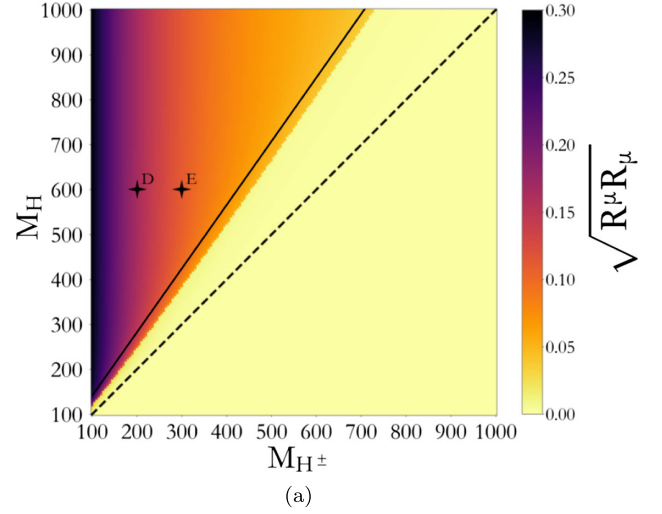


FIG. 8. (a) $\sqrt{R^\mu R_\mu}$ at the center of the domain wall for naturally bounded solutions of the field configuration [Eq. (7)] with the field restriction $\eta_1 = \eta_2 = \eta_3 = 0$, as functions of the two dependent mass parameters M_H and $M_{H^\pm}$ in GeV. Note that there is a clear region where there exist solutions with a stable superconducting condensate and in all cases $\xi \equiv 0$. The solid line represents a semianalytic prediction of the boundary using Eqs. (29) and (32), while the dashed line represents $M_H = M_{H^\pm}$. (b) Example solution for parameter set E, which is identified as the *superconducting solution* form.

configuration again relaxes to the standard solution. We likewise see that the effect of allowing η_1 to vary can be dramatic; the solutions in Figs. 7 and 9(b) share identical scalar masses, yet in the former the domain wall is in f_2 , and in the latter it is in ξ . We also note that equivalent solutions exist with the opposite sign of R^2 , realized when η_1 interpolates in the opposite direction and ξ interpolates from $-\pi$ to zero, passing through $-\pi/2$ at the center of the wall. These equivalent solutions are simply related by a $CP1$ transformation of $\Phi \rightarrow \Phi^*$.

We therefore propose the type of solution seen in Fig. 9(b) as our third subclass, the *CP-violating solution*,

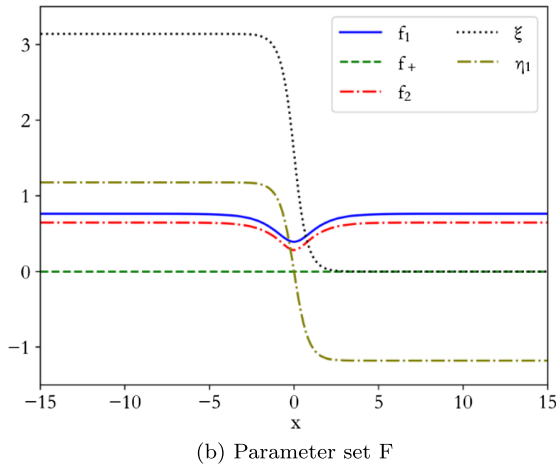
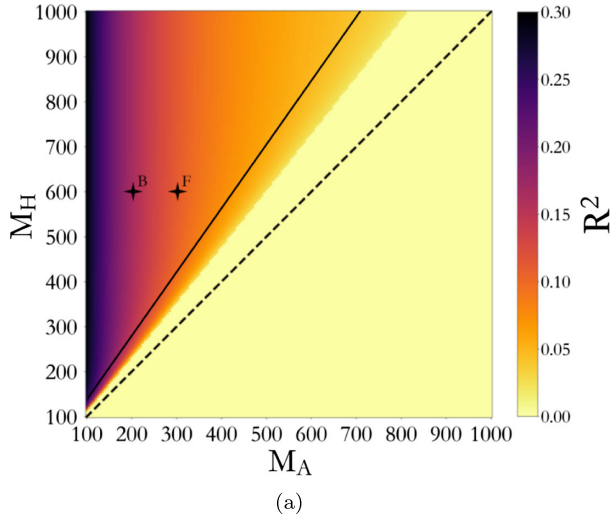


FIG. 9. (a) R^2 at the center of the domain wall for naturally bounded solutions of the field configuration [Eq. (7)] with the restriction $f_+ = \gamma_1 = \eta_2 = \eta_3 = 0$, as functions of the two dependent mass parameters M_H and M_A in GeV. Note that there is a clear region, where there exist solutions with a domain wall in ξ as opposed to f_2 . The solid line represents a semianalytic prediction of the boundary using Eqs. (29) and (32), while the dashed line represents $M_H = M_A$. (b) Example solution for parameter set F, which is identified as the *CP*-violating solution form.

characterized by the domain wall residing in ξ with an interpolating η_1 profile, causing a local breaking of *CP* symmetry, while the vacua remain *CP*-symmetric. As with the superconducting case of Sec. IV B, we will delineate the precise boundary of this solution space in Sec. IV D.

D. General solutions

We now consider the minimum-energy domain wall solutions obtained from the unrestricted field configuration, consistently reduced to six independent fields, as outlined in Sec. II C. Solving the full equations of motion allows us

to partition the parameter space into four distinct subclasses: standard (Sec. IV A), superconducting (Sec. IV B), *CP*-violating (Sec. IV C), and simultaneously superconducting and *CP*-violating solutions.

The delineation of the parameter space can be understood first by categorizing the space into two distinct regions: those that contain a wall in ξ , and those that contain a wall in f_2 . Neglecting gradient energy and considering only the potential (11), we first consider under what conditions there will be a reduction to the energy for a ξ profile interpolating between 0 and π , naturally passing through $\pi/2$ at the center of the domain wall. Let us consider the term proportional to

$$\lambda_4 - |\lambda_5| \cos 2\xi = \frac{2}{v_{\text{SM}}} [M_A^2 \sin^2 \xi - M_{H^\pm}^2], \quad (25)$$

which is the only part of the potential depending on M_A and ξ . At the center of the domain wall, where $\xi = \pi/2$, this takes a form $\propto (M_A^2 - M_{H^\pm}^2)$, suggesting that for a *CP*-violating wall to reduce the energy requires $M_A < M_{H^\pm}$.

We have confirmed this condition to hold by performing a parameter scan of M_H , M_A , $M_{H^\pm}$ for three fixed values of $\tan \beta$, for the unrestricted configuration. This is shown in Fig. 10, where we see that, if the solution is nonstandard, when $M_A \leq M_{H^\pm}$, a domain wall forms in ξ , whereas if $M_A \geq M_{H^\pm}$, a stable condensate exists on the wall, with $M_A = M_{H^\pm}$ yielding solutions where both occur simultaneously.

Second, it is clear that the boundary between standard and nonstandard (superconducting and/or *CP*-violating) solutions is dependent on $\tan \beta$ and the ratio of M_H/M_X , where $X = A, H^\pm$. This can be understood by considering the effective masses of f_+ and f_2 about the standard solution at the center of the wall, where $f_+ = f_2 = \xi = 0$. Neglecting gradient energies, the sign of these effective masses determines whether it is energetically favorable for the standard solution to develop a nonzero value in either f_+ or f_2 .

In the mass regime $M_{H^\pm} \leq M_A$, the relevant fluctuation is in f_+ . Expanding the potential about the standard solution at the center of the wall gives the effective mass

$$\begin{aligned} M_+^2(0) &= -\frac{1}{2}\mu_2^2 + \frac{1}{4}\lambda_3 f_1^2(0) \\ &= \frac{1}{4}M_h^2[(1 - \hat{M}_H^2 + 2\hat{M}_{H^\pm}^2)f_1^2(0) - 1], \end{aligned} \quad (26)$$

where $\hat{M}_H = M_H/M_h$ and $\hat{M}_{H^\pm} = M_{H^\pm}/M_h$. A negative $M_+^2(0)$ signifies an instability in the f_+ direction, leading to condensation and the superconducting solutions discussed in Sec. IV B.

Conversely, in the opposite mass regime, $M_{H^\pm} \geq M_A$, the relevant fluctuation is in f_2 . Expanding about the standard solution at the center of the wall yields

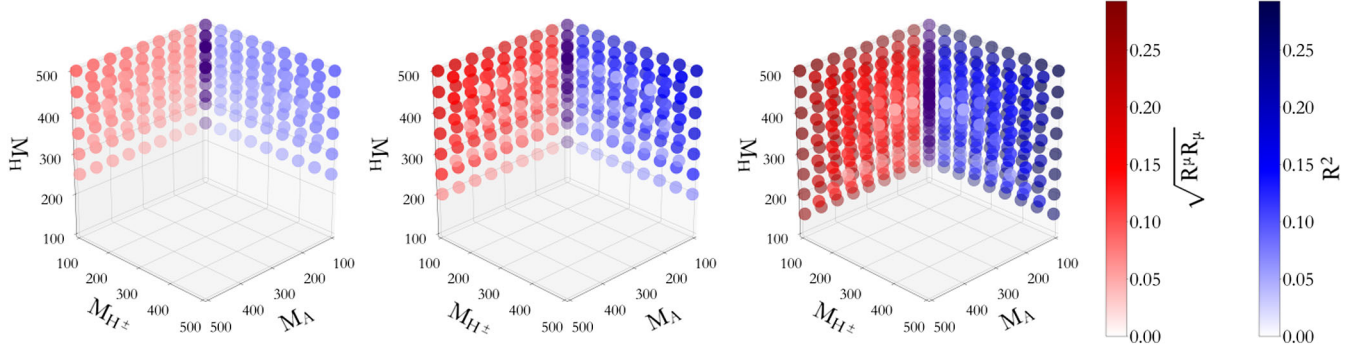


FIG. 10. $\sqrt{R^\mu R_\mu}$ and R^2 at the center of minimum-energy domain wall solutions, for varying scalar masses, M_H , M_A and $M_{H^\pm}$, where $\tan \beta = 0.10$ (left), $\tan \beta = 0.25$ (center), and $\tan \beta = 0.85$ (right). Masses with no data point represent standard solutions. Note the distinct regions where a superconducting and/or CP -violating solution exist. For an increasing $\tan \beta$, we observe an increasing region of nonstandard solutions, in addition to increasing amplitudes of $\sqrt{R^\mu R_\mu}$ and R^2 at the center of the wall.

$$\begin{aligned} M_2^2(0) &= -\frac{1}{2}\mu_2^2 + \frac{1}{4}(\lambda_3 + \lambda_4 + |\lambda_5|)f_1^2(0) \\ &= \frac{1}{4}M_h^2[(1 - \hat{M}_H^2 + 2\hat{M}_A^2)f_1^2(0) - 1], \end{aligned} \quad (27)$$

where, as previously, $\hat{M}_A = M_A/M_h$. In making this identification, we implicitly assume that any nonzero f_2 is accompanied by $\xi = \pi/2$ at the wall center. Recalling our distinction in Sec. III, it is clear that this assumption is true, such that the $\mathbb{Z}_2$ symmetry is respected in each coordinate basis of the wall. This is precisely the behavior realized in the CP -violating configurations of Sec. IV C. A negative $M_2^2(0)$ signifies an instability of the standard solution in the f_2 direction, leading to CP -violating configurations.

We see that these two effective mass expressions $M_+^2(0)$ and $M_2^2(0)$ exhibit an exact symmetry under the exchange of $M_{H^\pm} \leftrightarrow M_A$. This symmetry naturally unifies the classification of the superconducting and CP -violating regimes, allowing them to be treated simultaneously within the same framework, given our approximation of neglecting gradient energies. To fully categorize the regions of mass space, we must therefore evaluate where either M_+^2 or M_2^2 will be zero in terms of M_H , M_X , and $\tan \beta$.

We can evaluate the mass dependence of $f_1(0)$ using the reduced standard solution *Ansatz* of

$$\Phi = \frac{v_{\text{SM}}}{\sqrt{2}} \begin{pmatrix} 0 \\ f_1 \\ 0 \\ f_2 \end{pmatrix}, \quad (28)$$

and rearranging Eqs. (26), (27), we find that one would therefore expect a standard solution when

$$\hat{M}_X^2 \gtrsim \frac{1}{2}(\hat{M}_H^2 + f_1^{-2}(0) - 1), \quad (29)$$

where as before, $X = A, H^\pm$ for the appropriate regime.

We can obtain a rough estimate of the critical masses by assuming $\partial_x^2 f_1 = 0$ at the center, in which case a quick analysis of the equation of motion for f_1 reveals that

$$\tilde{f}_1(0) = \frac{1}{1 + \hat{M}_H^2 \tan^2 \beta}, \quad (30)$$

where we define $\tilde{f}_1(0)$ to be an approximate expression for $f_1(0)$. This approximation holds exactly if $\hat{M}_H = 1$, where f_1 is found to be a constant, and as such $\partial_x^2 f_1 = 0$. Away from $\hat{M}_H = 1$, however, f_1 is found to develop a local extremum at the wall, as can be seen in all our example solutions. We evaluated $f_1(0)$ across the parameter intervals $\hat{M}_H = [1, 8]$, $\tan \beta = [0.25, 8]$ and found this simple approximation to differ by up to approximately 70% on this interval, with it expected to worsen for higher values of $\hat{M}_H$. However, the benefit of this simple analytic estimate is the ability to make the first approximation that we would expect a standard solution if

$$\hat{M}_{H^\pm, A}^2 \gtrsim \frac{1}{2}\hat{M}_H^2(1 + \tan^2 \beta), \quad (31)$$

and a superconducting and/or CP -violating solution otherwise, given the appropriate mass-ordering regime.

While this simple analytic evaluation provides a good first approximation, we have found that a set of minimal correctional terms may be introduced to significantly improve the evaluation of $f_1(0)$ on the interval we evaluated. Introducing these terms provides us with the more robust expression to be substituted into Eq. (29),

$$\begin{aligned} \tilde{f}_1(0) &= \frac{1}{1 + \hat{M}_H^2 \tan^2 \beta} [1 + (\hat{M}_H - 1)(c_1 \hat{M}_H \\ &\quad + c_2 \hat{M}_H^2 + c_3 \tan \beta + c_4 \tan^2 \beta)], \end{aligned} \quad (32)$$

with

$$\begin{aligned} c_1 &= 0.1158, & c_2 &= -0.0090, \\ c_3 &= 0.0824, & c_4 &= 0.0078, \end{aligned} \quad (33)$$

where the correction terms were obtained from a minimax (Chebyshev) fit to the numerical data across the analyzed intervals, anchored to the exact expression at $\hat{M}_H = 1$. These numerical corrections improve the prediction of $f_1(0)$ such that it only differs by up to 7% over our tested intervals, which, given the neglecting of gradient energy in our approximation, provides an adequate prediction. This can be seen in Figs. 8(a) and 9(a), where the solid lines represent the prediction using Eqs. (29) and (32). This may be a less concise prediction compared to our first approximation, but it can be used for a more robust prediction if required, away from $\hat{M}_H = 1$. It must be stressed that this is still only an approximate predictive tool and will not be valid for any given combination of $\hat{M}_H$ and $\tan\beta$, specifically those outside our fitted intervals. Alternatively, one could simply evaluate the domain wall solution using the restricted standard solution *Ansatz* for any given parameter set and identify using (29) the approximate form of the minimum-energy field configuration. This would be the most accurate approach but requires some numerical effort for each combination of $\hat{M}_H$ and $\tan\beta$.

In Appendix B, we have included a brief discussion on the nature of the field components present in the general field configuration which identically vanish in each of the solution regimes. It transpires that in each solution regime, the fields restricted to be zero in each of the subclasses identically vanish, greatly simplifying the parameter space. Furthermore, in Appendix C, we have included a selection of minimum-energy solutions for varying parameter sets to further demonstrate our findings.

Given our classification of the minimum-energy solutions, we now refer back to our RIC simulations, where it can be seen that in Fig. 1(b) the walls are dominated by a nonzero R^2 , reflecting the CP -violating minimum-energy solution of parameter set B; in Fig. 1(d) the walls are dominated by nonzero $R^\mu R_\mu$, reflecting the corresponding superconducting solution of parameter set D; while the simulation of parameter set C in Fig. 1(c) presents both nonstandard features in line with its minimum-energy solution, found in Fig. 18(b). This shows the success of our classification, with full dynamical simulations agreeing completely with our minimum-energy solutions while also showing evidence of relative EW rotations, as previously discussed.

V. CURRENT-CARRYING SOLUTIONS

We now turn our attention to the case of current-carrying solutions. The earliest realization that topological defects such as cosmic strings can stably support currents goes back to Witten's seminal work on superconducting strings [29]. Subsequently, Davis and Shellard [30] provided a

detailed analysis of vortex superconductivity and characterized stable loop configurations (vortons) supported by these currents. Here, we use the term *superconducting* to indicate that the defect supports a condensate, as occurs in our superconducting subclass of domain walls where the $U(1)_{\text{EM}}$ symmetry is broken locally on the wall. By contrast, we use *current-carrying* to refer specifically to solutions in which this condensate sustains a propagating current along the defect.

Building on our findings of Sec. IV D, and in agreement with Refs. [44,45], we propose the following superconducting domain wall *Ansatz*:

$$\Phi = \frac{v_{\text{SM}}}{\sqrt{2}} \left[\sigma^0 \otimes \begin{pmatrix} \cos \frac{1}{2} \gamma_1 & \sin \frac{1}{2} \gamma_1 \\ -\sin \frac{1}{2} \gamma_1 & \cos \frac{1}{2} \gamma_1 \end{pmatrix} \right] \begin{pmatrix} 0 \\ f_1 \\ f_+ \\ f_2 \end{pmatrix}, \quad (34)$$

subject to the parameter restrictions of $\hat{M}_{H^\pm} < \hat{M}_A$ and $\hat{M}_{H^\pm}^2 \lesssim \frac{1}{2}(\hat{M}_H^2 + f_1^{-2}(0) - 1)$, whereby we expect a superconducting condensate to form with $f_+ \neq 0$ at the center of the wall.

To produce a current-carrying field configuration, we perform a space-time-dependent transformation on the superconducting *Ansatz*, using the degree of freedom which is unbroken in the vacuum, acting on Eq. (34) with

$$e^{\frac{i}{2}\psi_\mu x^\mu} [\sigma^0 \otimes e^{\frac{i}{2}\psi_\nu x^\nu \sigma^3}], \quad (35)$$

where $\psi_\mu \psi^\mu = \omega^2 - k^2 \equiv \kappa$, such that we obtain, for a domain wall lying along the y axis, the current-carrying *Ansatz*

$$\begin{aligned} \Phi &= \frac{v_{\text{SM}}}{\sqrt{2}} \begin{pmatrix} f_1 \sin \frac{1}{2} \gamma_1 e^{i(\omega t + ky)} \\ f_1 \cos \frac{1}{2} \gamma_1 \\ (f_+ \cos \frac{1}{2} \gamma_1 + f_2 \sin \frac{1}{2} \gamma_1) e^{i(\omega t + ky)} \\ -f_+ \sin \frac{1}{2} \gamma_1 + f_2 \cos \frac{1}{2} \gamma_1 \end{pmatrix} \\ &= \frac{v_{\text{SM}}}{\sqrt{2}} \begin{pmatrix} g_1 e^{i(\omega t + ky)} \\ g_2 \\ g_3 e^{i(\omega t + ky)} \\ g_4 \end{pmatrix}. \end{aligned} \quad (36)$$

Here, we have introduced g_i functions to replace f_1, f_+, f_2, γ_1 in order to simplify numeric calculations.

This *Ansatz* yields the following Lagrangian and energy densities:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}(\partial_x g_1)^2 - \frac{1}{2}(\partial_x g_2)^2 - \frac{1}{2}(\partial_x g_3)^2 - \frac{1}{2}(\partial_x g_4)^2 \\ &\quad + \frac{1}{2}\kappa(g_1^2 + g_3^2) - V, \end{aligned} \quad (37)$$

$$\mathcal{E} = \frac{1}{2}(\partial_x g_1)^2 + \frac{1}{2}(\partial_x g_2)^2 + \frac{1}{2}(\partial_x g_3)^2 + \frac{1}{2}(\partial_x g_4)^2 + \frac{1}{2}(\omega^2 + k^2)(g_1^2 + g_3^2) + V, \quad (38)$$

where we have that $g_i \equiv g_i(x)$. This construction is directly analogous to the method of Ref. [45] for superconducting strings, adapted here to the wall case. This introduces the further parameter of κ to fix in the equations of motion (κ is often called χ in the literature on current-carrying strings; however, we use κ to distinguish from the EW group parameter χ), which plays the role of an effective mass term for the current-carrying components g_1, g_3 , controlling the magnitude and regime of the solution. In the literature on current-carrying strings, solutions are classified into three regimes: chiral ($\kappa = 0$), magnetic ($\kappa < 0$), or electric ($\kappa > 0$).

A subtlety arises for this *Ansatz* because the transformation (35) used to generate the current-carrying *Ansatz* is not limited to only adding a phase to the condensate components, but also couples to the EW component of γ_1 . Whether γ_1 relaxes to zero in the vacuum therefore depends on the sign of κ . In the magnetic regime, the effective mass term $\frac{1}{2}\kappa(g_1^2 + g_3^2)$ has been found to suppress the current-carrying components away from the wall, which acts to drive $\gamma_1 \rightarrow 0$ in the vacuum, yielding consistent solutions. We have found, however, that in the chiral and electric regimes, this suppression of the current-carrying components does not occur, and γ_1 remains nonzero in the vacuum. Thus, within this *Ansatz*, we have found that only magnetic-current-carrying configurations can be constructed without inducing nonzero vacuum energy. We cannot, however, exclude the possibility that there exist parameter sets in which chiral or electric solutions are consistent; we have simply not found any examples in our explorations.

We suggest that the limitation of this *Ansatz* could be resolved in the gauged version of the theory, as one may choose to work in a gauge where $\gamma_1 = 0$ globally without

loss of generality; however, as this work is concerned only with the global theory, we limit our discussion now to the magnetic regime.

In Fig. 11, we present kink solutions for this *Ansatz* for varying values of negative κ . As is usual for superconducting defects, the condensate amplitude decreases with more negative κ , and we also see the direct influence of κ on the convergence of γ_1 .

It is important to mention that current-carrying defect solutions may sometimes be unstable. We therefore analyze when the solutions are predicted to be unstable to simple longitudinal (pinching) and transverse perturbations. These will be apparent in simulations of a perturbed wall, which provides a more robust test of stability.

A semianalytic analysis based on those detailed in Refs. [46–48] for current-carrying strings allows us to predict where such instabilities will occur for a given parameter set. Using the g_i representation of Eq. (36), we may write the total energy of the kink solution as

$$E_{\text{kink}} = \int \mathcal{E} dx = \int (\mathcal{E}_{13} + \mathcal{E}_{24}) dx, \quad (39)$$

where $\mathcal{E}_{13}$ is the part of the energy density containing the contributions of the fields associated with the current, g_1 and g_3 , while $\mathcal{E}_{24}$ is the remaining energy density, consequently containing terms which only involve the unassociated fields g_2 and g_4 :

$$\begin{aligned} \mathcal{E}_{13} = & \frac{1}{2}(\partial_x g_1)^2 + \frac{1}{2}(\partial_x g_3)^2 + \frac{1}{2}(\omega^2 + k^2)(g_1^2 + g_3^2) \\ & - \frac{1}{2}\mu_1^2 g_1^2 - \frac{1}{2}\mu_2^2 g_3^2 \\ & + \frac{1}{4}\lambda_1 g_1^2 (g_1^2 + 2g_2^2) + \frac{1}{4}\lambda_2 g_3^2 (g_3^2 + 2g_4^2) \\ & + \frac{1}{4}\lambda_3 (g_1^2 g_3^2 + g_1^2 g_4^2 + g_2^2 g_3^2) \\ & + \frac{1}{4}(\lambda_4 - |\lambda_5|)(g_1^2 g_3^2 + 2g_1 g_2 g_3 g_4), \end{aligned} \quad (40)$$

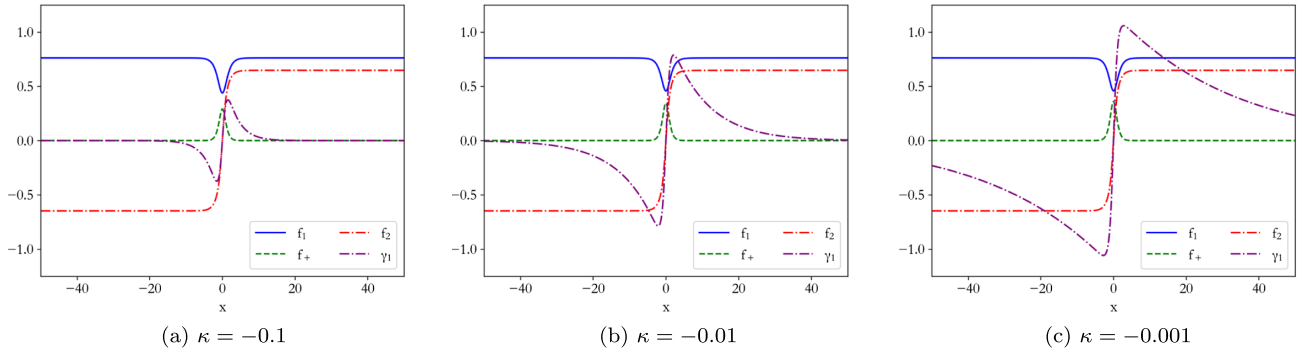


FIG. 11. Example minimum-energy kink solutions for the current-carrying *Ansatz* of Eq. (36) for varying values of κ in parameter set D. Numerical settings of $n_x = 20000$, $\Delta x = 0.01$, $\delta = 10^{-7}$. Note the standard reduction in the amplitude of the condensate for more negative values of κ , and the convergence rate of $\gamma_1 \rightarrow 0$.

$$\begin{aligned} \mathcal{E}_{24} = & \frac{1}{2}(\partial_x g_2)^2 + \frac{1}{2}(\partial_x g_4)^2 - \frac{1}{2}\mu_1^2 g_2^2 - \frac{1}{2}\mu_2^2 g_4^2 \\ & + \frac{1}{4}\lambda_1 g_2^4 + \frac{1}{4}\lambda_2 g_4^4 + \frac{1}{4}(\lambda_3 + \lambda_4 - |\lambda_5|)g_2^2 g_4^2. \end{aligned} \quad (41)$$

Taking the equations of motion for g_1 and g_3 , multiplying by $\frac{1}{2}g_1$ and $\frac{1}{2}g_3$, respectively, integrating each over the cross section of the solution, and simplifying the derivatives using integration by parts shows that

$$\begin{aligned} \int \left[\frac{1}{2}(\partial_x g_1)^2 - \frac{1}{2}\kappa g_1^2 - \frac{1}{2}\mu_1^2 g_1^2 \right. \\ \left. + \frac{1}{2}\lambda_1 g_1^2(g_1^2 + g_2^2) + \frac{1}{4}\lambda_3 g_1^2(g_3^2 + g_4^2) \right. \\ \left. + \frac{1}{4}(\lambda_4 - |\lambda_5|)(g_1^2 g_3^2 + g_1 g_2 g_3 g_4) \right] dx = 0, \end{aligned} \quad (42)$$

$$\begin{aligned} \int \left[\frac{1}{2}(\partial_x g_3)^2 - \frac{1}{2}\kappa g_3^2 - \frac{1}{2}\mu_2^2 g_3^2 \right. \\ \left. + \frac{1}{2}\lambda_2 g_3^2(g_3^2 + g_4^2) + \frac{1}{4}\lambda_3 g_3^2(g_1^2 + g_2^2) \right. \\ \left. + \frac{1}{4}(\lambda_4 - |\lambda_5|)(g_1^2 g_3^2 + g_1 g_2 g_3 g_4) \right] dx = 0. \end{aligned} \quad (43)$$

These can be substituted into the energy [Eq. (39)], such that

$$E_{\text{kink}} = \tau + \omega^2 \Sigma_2, \quad (44)$$

where $\Sigma_2 = \int (g_1^2 + g_3^2) dx$, and

$$\begin{aligned} \tau = \int \left[\mathcal{E}_{24} - \frac{1}{4}\lambda_1 g_1^4 - \frac{1}{4}\lambda_2 g_3^4 \right. \\ \left. - \frac{1}{4}(\lambda_3 + \lambda_4 - |\lambda_5|)g_1^2 g_3^2 \right] dx. \end{aligned} \quad (45)$$

The Lagrangian may be rewritten using the same technique to yield

$$L_{\text{kink}} = \int \mathcal{L} dx = -\tau. \quad (46)$$

A calculation and diagonalization of the energy momentum tensor, details of which can be found in Refs. [28,48], shows the propagation speeds of longitudinal and transverse perturbations to be

$$c_L^2 = 1 + \frac{2\kappa \Sigma_2'}{\Sigma_2}, \quad c_T^2 = 1 + \frac{\kappa \Sigma_2}{\tau} \quad (47)$$

in the magnetic regime, where $\Sigma_2' = \partial \Sigma_2 / \partial \kappa$. Both are required to be greater than zero in order for a solution to be

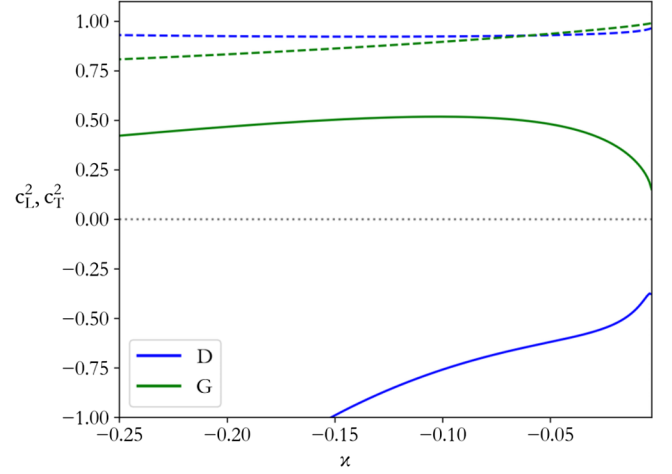


FIG. 12. Perturbation propagation speeds, c_L^2 and c_T^2 for current-carrying solutions on the interval $\kappa = [-0.250, -0.001]$, with $\Delta\kappa = 0.001$, for two parameter sets using kink solutions with numerical settings of $n_x = 8000$, $\Delta x = 0.05$, $\delta = 10^{-6}$. Solid lines represent c_L^2 and dashed c_T^2 . Parameter set D is predicted to be unstable for the full range of κ , whereas parameter set G is found to be stable. We find this prediction of stability to be correct.

stable, in addition to being less than unity to respect causality.

The variation of these speeds is shown in Fig. 12 across a range of κ for two different parameter sets, with Σ_2 and τ having been evaluated for each individual solution on the interval and Σ_2' computed using a second-order finite difference scheme. We find that parameter set G is predicted to be stable to such perturbations, whereas parameter set D is not. This analysis demonstrates further that our *Ansatz* is only valid in the magnetic regime for the global theory, as from Eq. (47) it is clear that both speeds should tend to unity as they approach the chiral limit; however, from Fig. 12 we see that this is clearly not the case. This is a direct consequence of the behavior of γ_1 in the vacuum and its influence on Σ_2 ; as we approach the chiral limit, γ_1 tends to a finite value, and as such, Σ_2 diverges at $\kappa = 0$. Noteworthy also is that in agreement with Ref. [48], we find that $c_L^2 < c_T^2$ for all tested parameter sets.

To test the stability of our solutions, we constructed y -directed domain walls using kink solutions for parameter set G, we set the values of the fields in x by identifying them with those of the relevant kink solution and assigned the phase of the current associated with the coordinates of y and t . These solutions were then evolved in full two-dimensional dynamics in the linear representation. We performed our simulations on rectangular grids of $P_x = 4096$, $P_y = 640$, $\Delta x = 0.05$, $\Delta t = 0.01$ (such that the length of the wall is $L = 32$) using periodic boundaries in y and homogeneous Neumann boundaries in x . The dimensions of the simulation were chosen to best accommodate the width of the wall and allow for the solution's stability to be tested over a large time period.

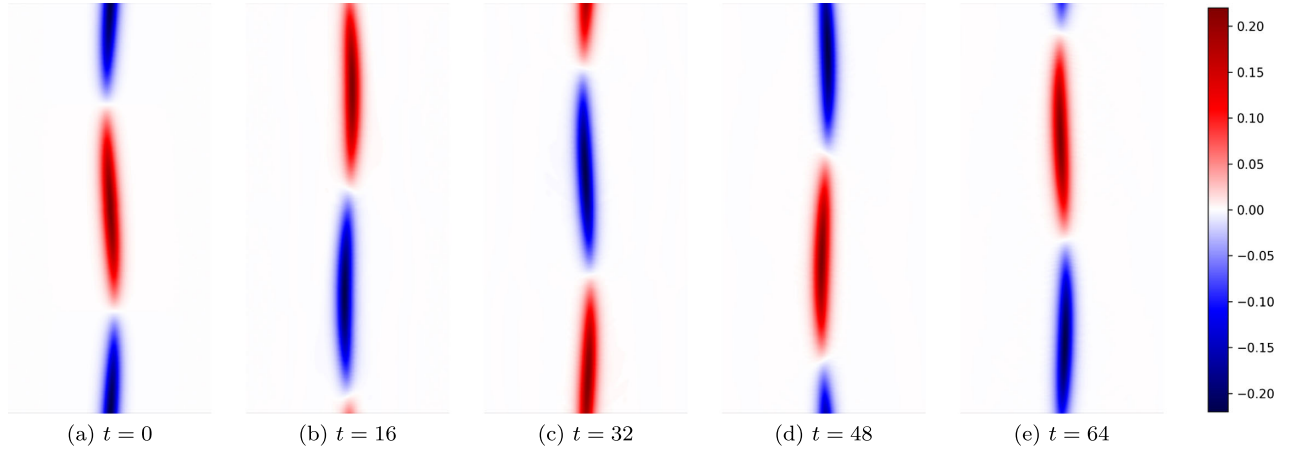


FIG. 13. Snapshots of the evolution of the bilinear component R^4 of a magnetic infinite wall solution, within a full dynamical simulation, for parameter set G and $\kappa = -0.025$; the wall has winding number $N = 1$ and phase frequency $\omega = 0.116$. The full simulation runs until $t = 16000$, with no signs of instability observed.

In Fig. 13, we show snapshots of the early evolution of one of these walls for a value of $\kappa = -0.025$; the wall is initialized with a winding of $N = 1$ ($k = 2\pi N/L$), a phase frequency of $\omega = 0.116$, and a sinusoidal perturbation of unitary amplitude along the y direction for a more robust test of stability. We observe the current along the wall length and a coherent oscillation of the entire object induced by the initial perturbation. This current-carrying wall was tested up to $t = 16000$, after which the current had performed approximately 300 full revolutions, and no signs of any instability were observed. We found similar stability for all other walls with values of κ where parameter set G is predicted to be stable on the basis of $c_L^2, c_T^2 > 0$. We also tested current-carrying walls for parameter set D, and found them all to decay due to a growing pinching instability akin to Ref. [49].

One technical point of interest is that in order to allow for a reliable representation of the dynamics, we found that a spatial resolution of $\Delta x \leq 0.05$ was required in order for the condensate to remain localized on the wall over longer periods of time. This is likely due to the associated length scales of the solution.

Of course, such simulations do not absolutely confirm that these current-carrying solutions are stable; however, given the temporal extent to which we have tested them, it is highly suggestive of stability. Given the natural formation of circular current-carrying objects observed in full dynamical simulations [26], we propose that these solutions provide the foundation for constructing kinky vortons, a topic currently under investigation which will be presented in future work.

VI. CP-VIOLATING DOMAIN WALLS

In Sec. IV, we demonstrated the existence of a new subclass of domain wall solution, of the type shown in Fig. 9(b), which can be described by the reduced *Ansatz* of

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{v_{\text{SM}}}{\sqrt{2}} \begin{pmatrix} 0 \\ f_1 e^{-\frac{1}{2}i\eta_1} \\ 0 \\ f_2 e^{i(\xi - \frac{1}{2}\eta_1)} \end{pmatrix}, \quad (48)$$

subject, of course, to the previous parameter restrictions of $M_{H^\pm} > M_A$ and $\hat{M}_A^2 \lesssim \frac{1}{2}(\hat{M}_H^2 + f_1^{-2}(0) - 1)$.

This reduced *Ansatz* admits two domain wall solutions, related by a $CP1$ transformation, $\Phi \rightarrow \Phi^*$, shown in Fig. 14. They differ only by the sign of R^2 across the wall, while their vacuum R^μ configurations remain identical. We have used the further parameter set H, as it resides well within the mass regime of these solutions. Motivated by this, and by observations from full dynamical simulations in this mass regime, we propose that a $CP1$ domain wall can form longitudinally upon the underlying $\mathbb{Z}_2$ wall.

To test this hypothesis, we constructed a two-dimensional grid with the $\mathbb{Z}_2$ wall aligned along the y axis. Opposite sides of the grid were initialized with the two CP -violating solutions of Fig. 14(b), with a thin interpolating section (of tanh form) at the center. This served as the initial condition for solving the two-dimensional equations of motion with homogeneous Neumann boundary conditions in both x and y , ensuring relaxation to a minimum-energy configuration.

We present in Fig. 15 a fully converged (to a tolerance of 10^{-7}) two-dimensional domain wall solution in the bilinear components R^1 and R^2 , along with the full bilinear component profiles along the domain wall. The converged solution clearly shows a superposition, where a $CP1$ domain wall exists upon a $\mathbb{Z}_2$ domain wall. The longitudinal profiles of R^μ directly correspond to the forms of the $CP1$ kink solution found in Ref. [22]. Explicitly, this solution realizes local maximal CP violation confined to the $\mathbb{Z}_2$ domain wall.

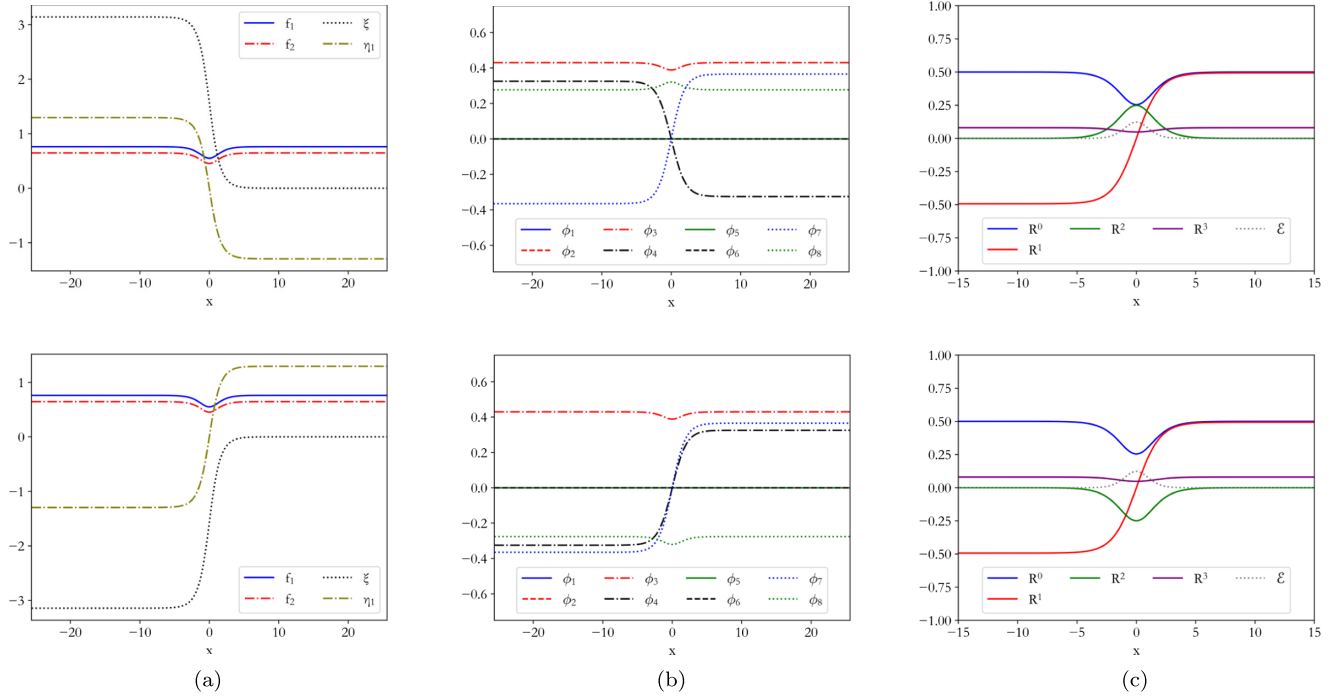


FIG. 14. Minimum-energy kink solutions for parameter set H, depicting fields in (a) the general representation, (b) the linear representation (middle), and (c) the bilinear representation (bottom) for two equivalent CP -violating solutions. In the linear representation, ϕ_i for $i = 1, 2, 5, 6$ are globally zero. Numerical settings are $n_x = 5000$, $\Delta x = 0.01$, $\delta = 10^{-7}$.

To test the stability of this composite domain wall structure, we took the solution which we have presented in Fig. 15 and evolved it within a full dynamical simulation from a sinusoidally perturbed initial state, using numerical settings of $P_x = 1024$, $\Delta x = 0.05$, $\Delta t = 0.01$ and Neumann boundaries in both spatial dimensions. We show in Fig. 16 the evolution of this simulation, where we see an oscillation of the entire object due to the initial perturbation, with the $CP1$ wall persisting upon the $\mathbb{Z}_2$ wall under the full dynamics and exhibiting no signs of instability. While this simulation is only a short test of stability, it demonstrates

that the composite structure does indeed persist under full dynamical evolution, suggesting stability of the solution.

The existence of this stable field configuration completes the explanation of the features found to exist within full dynamical $(2+1)$ -dimensional simulations of this model. That is to say that the previously observed “winding” in R^2 is in fact the formation of $CP1$ domain walls upon those which naturally occur in the $\mathbb{Z}_2$ -symmetric model.

Although speculative, the existence of such solutions has far-reaching cosmological implications. Since domain walls must ultimately decay to avoid overclosing the

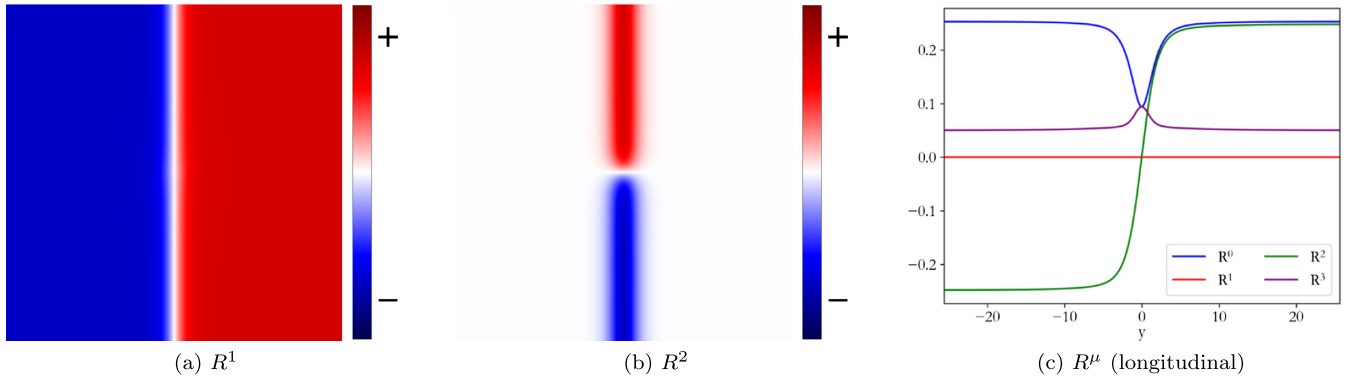


FIG. 15. Stable two-dimensional domain wall solution for parameter set H, where initial conditions were constructed such that a $CP1$ domain wall existed upon the $\mathbb{Z}_2$ domain wall. We see this two-dimensional configuration to be stable, with maximal CP violation occurring longitudinally upon the domain wall.

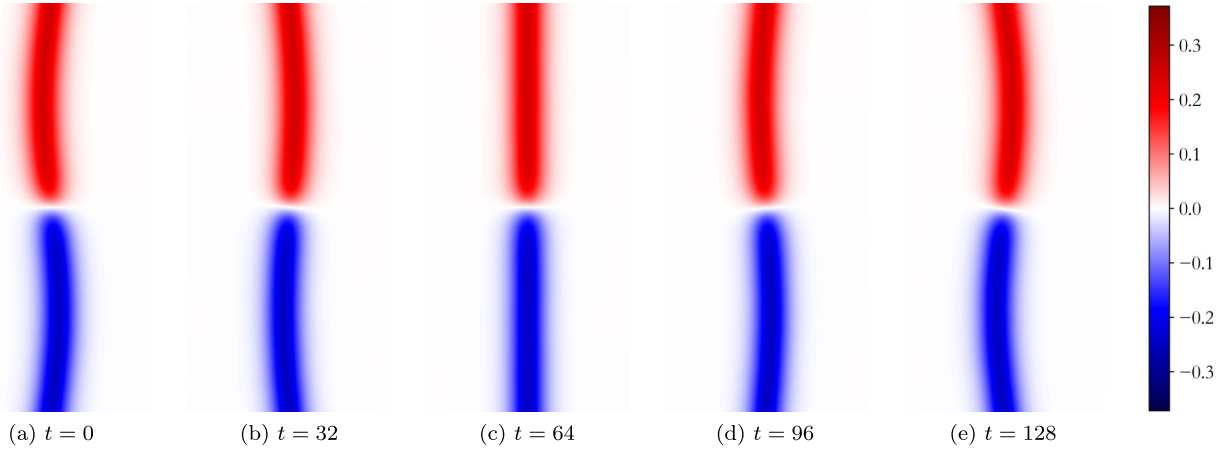


FIG. 16. Snapshots of the evolution, within a full dynamical simulation, of the bilinear component R^2 of a composite domain wall structure, where a $CP1$ wall exists longitudinally upon a Z_2 wall, initialized in a sinusoidally perturbed state. No signs of instability were observed.

Universe, the presence of local CP violation along them would naturally create an out-of-equilibrium environment, potentially conducive to baryogenesis. Unlike the “winding” of a condensate on a two-dimensional wall (a proxy for the winding of a three-dimensional string), this superposition of domain walls can be straightforwardly extended to three dimensions, strengthening its cosmological relevance.

VII. CONCLUSIONS

The underlying objective of this work was to explain the phenomena observed in full dynamical, two-dimensional simulations of the Z_2 -symmetric 2HDM. To this end, we have presented a comprehensive study of domain wall solutions in this model, identifying two broad classes of solutions: (i) those connecting vacua related by an electroweak rotation, and (ii) those corresponding to distinct energy-minimizing configurations. Together, these two classes provide a comprehensive explanation of the simulation dynamics.

We have categorized the models parameter space into four distinct regions, in which different subclasses of solution are the energy-minimizing field configuration. We summarize these solutions below in terms of the scaled mass parameters $\hat{M}_H$, $\hat{M}_{H^\pm}$, $\hat{M}_A$, and $\tan\beta$,

- (1) *Standard solutions* (cf. Sec. IV A) occur when

$$\hat{M}_{\min}^2 \gtrsim \frac{1}{2}(\hat{M}_H^2 + f_1^{-2}(0) - 1),$$

where $\hat{M}_{\min} = \min(\hat{M}_{H^\pm}, \hat{M}_A)$. These solutions are fully described by the field components f_1 and f_2 .

- (2) *Superconducting solutions* (cf. Sec. IV B) arise when

$$\hat{M}_{H^\pm} < \hat{M}_A \quad \text{and} \quad \hat{M}_{H^\pm}^2 \lesssim \frac{1}{2}(\hat{M}_H^2 + f_1^{-2}(0) - 1).$$

These are characterized by the field components f_1 , f_+ , f_2 , and the $SU(2)$ parameter γ_1 .

- (3) *CP -violating solutions* (cf. Sec. IV C) appear when

$$\hat{M}_A < \hat{M}_{H^\pm} \quad \text{and} \quad \hat{M}_A^2 \lesssim \frac{1}{2}(\hat{M}_H^2 + f_1^{-2}(0) - 1).$$

These are described by the field components f_1 , f_2 , ξ , and the $SU(2)$ parameter η_1 .

- (4) *Simultaneously superconducting and CP -violating solutions* occur when

$$\hat{M}_A = \hat{M}_{H^\pm} \quad \text{and} \quad \hat{M}_A^2 \lesssim \frac{1}{2}(\hat{M}_H^2 + f_1^{-2}(0) - 1).$$

These involve the full set of field components f_1 , f_+ , f_2 , ξ , and the $SU(2)$ parameters γ_1 , η_1 .

Together with the effects of relatively EW transformed vacua, the above solutions provide a complete explanation of the structures observed in $(2+1)$ -dimensional simulations of the model from random initial conditions. The excellent agreement between this theoretical classification and full dynamical simulations gives us confidence that the relevant minimum-energy configurations have been identified. A concise delineation of the parameter space as we have outlined can be found in Fig. 17, a flowchart mapping any given parameter set combinations to the approximate solution subclass.

As in our previous work on superconducting strings in the 2HDM [45], we have demonstrated that the superconducting subclass of domain wall solutions can support persistent currents, resulting in apparently stable current-carrying walls in two dimensions. These walls could serve as the basis for constructing closed-loop configurations, so-called kinky vortons, in the 2HDM, which would be the $(2+1)$ -dimensional analogues of vortons in the $U(1)$ -symmetric variant of the theory. If such objects can be

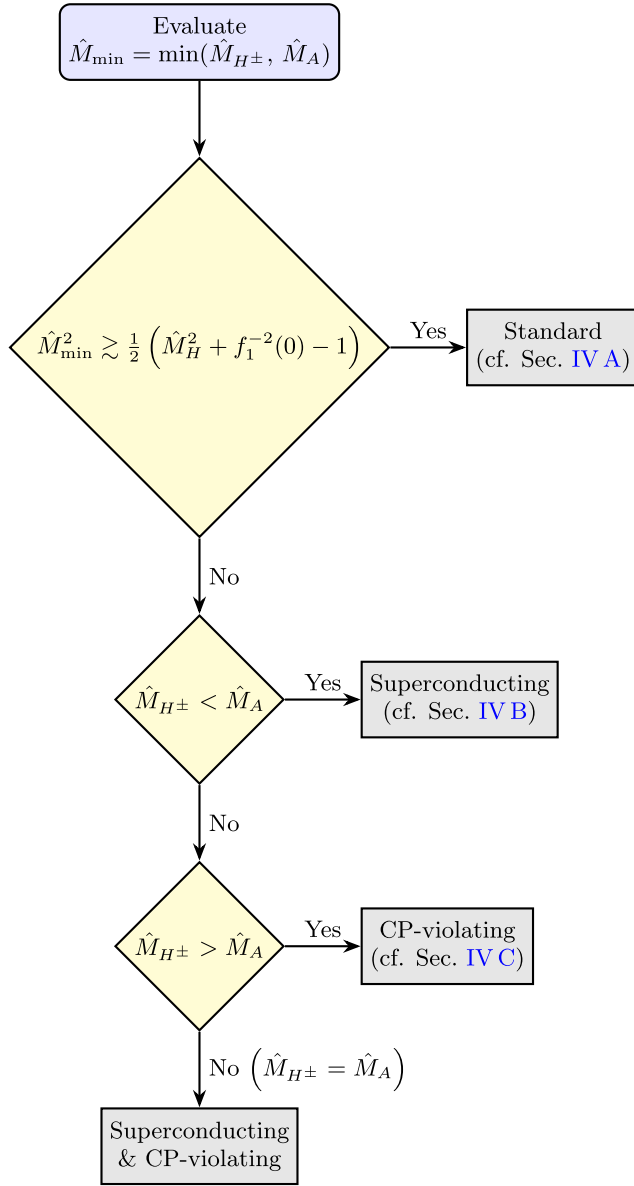


FIG. 17. Flowchart which, given a parameter set, will yield the approximate domain wall solution type in the $\mathbb{Z}_2$ -symmetric 2HDM. The quantity $f_1(0)$ may be evaluated directly using the reduced *Ansatz* of the standard solution for the best prediction, using Eq. (29) for a robust approximate prediction, or Eq. (30) close to $\hat{M}_H^2 = 1$. This classification remains an approximation due to the neglecting of gradient energies.

successfully constructed, they would provide a valuable lower-dimensional test of vorton properties within this model, offering insight into their stability in a far more computationally tractable setting.

Finally, we have discovered a new type of domain wall solution in the 2HDM: a CP -violating wall configuration in which CP symmetry is maximally locally broken along the $\mathbb{Z}_2$ -symmetric domain wall. Unlike the current-carrying walls (which primarily serve as proxies for current-carrying strings in three dimensions), this CP -violating wall solution

should have an analogue in higher dimensions as well. In realistic scenarios, domain walls must ultimately decay to avoid cosmological problems; notably, the presence of CP violation along such unstable walls means they naturally fulfil two of Sakharov's three conditions (CP violation and departure from equilibrium) for generating a matter-antimatter asymmetry [50]. Whether these walls can actually produce a net baryon asymmetry depends on the details of how they interact with Standard Model fermions, and further work will be necessary to determine if these CP -violating walls can efficiently fuel baryogenesis.

In summary, this study has elucidated the rich spectrum of domain wall solutions in the 2HDM and linked our theoretical classification of these solutions with their observed dynamics in simulations. We have provided a complete categorization of the minimum-energy field configurations and identified the conditions under which each subclass occurs, thereby offering a predictive roadmap for domain wall phenomena in this model. These results have significant cosmological implications: by understanding which domain wall solutions occur—and whether they are stable or metastable—we gain insight into the possible relics of the electroweak phase transition and their roles in the early Universe. For example, the existence of current-carrying walls and the prospect of kinky vortons offer a computationally tractable way to explore vorton dynamics in the 2HDM. Meanwhile, the CP -violating walls introduce a novel mechanism that could be relevant to baryogenesis. We conclude that the 2HDM, a well-motivated extension of the Standard Model, supports a far more varied set of topological phenomena than previously recognized. Future work will focus on exploring the viability of kinky vortons, examining how these domain wall solutions interact with particles, and investigating their three-dimensional extensions in realistic cosmological settings. Our findings broaden the theoretical groundwork for such studies and open several new avenues at the intersection of particle physics and cosmology.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: NUMERICAL TECHNIQUES

1. One-dimensional kink solutions

One-dimensional, static kink solutions, appropriate for the study of domain walls, can be obtained by using variational techniques to solve the set of second-order differential equations of motion,

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial (dq/dx)} \right) - \frac{\partial \mathcal{L}}{\partial q} = 0, \quad (\text{A1})$$

where q is the set of field functions of a given representation—e.g., $q = \{\phi_1, \dots, \phi_8\}$. The general form of the Lagrangian density for the 2HDM is given by, in one spatial dimension,

$$\mathcal{L} = -\frac{d\Phi_1^\dagger}{dx} \frac{d\Phi_1}{dx} - \frac{d\Phi_2^\dagger}{dx} \frac{d\Phi_2}{dx} - V(\Phi_1, \Phi_2). \quad (\text{A2})$$

These solutions correspond to static solutions to the equations of motion, subject to the boundary conditions imposed. The solutions are solved for on one-dimensional arrays of size n_x , grid spacing of Δx subject to a tolerance of δ .

a. Approximate treatment

Here, we implement a method similar to the variational technique of successive overrelaxation (SOR) where functions of each field variable, q^i , are introduced:

$$\begin{aligned} F_1(q^i) &= \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial (dq^i/dx)} \right) - \frac{\partial \mathcal{L}}{\partial q^i}, \\ F_2(q^i) &= \frac{\partial F_1(q^i)}{\partial q^i}. \end{aligned} \quad (\text{A3})$$

We start with an initial trial field configuration (often a linear interpolation between the two boundary values). Then, at each iteration n , we simultaneously update all field variables q^i according to

$$q_{n+1}^i = q_n^i - w \frac{F_1(q_n^i)}{F_2(q_n^i)} \quad (\text{A4})$$

until all changes are below a tolerance δ . When implementing this technique, we use $w = 1.5$. We applied either Neumann or Dirichlet boundary conditions, as appropriate for the solution type. Spatial derivatives were evaluated with a fourth-order finite-difference stencil.

b. Full treatment

Where the equations of motion were highly coupled—for example, in the cases detailed in Sec. IV—a more robust treatment for solving the one-dimensional equations of motion was required for stable convergence. This was done by implementing a modified Newton's method approach with iterative updates:

$$\mathbf{q}_{n+1} = \mathbf{q}_n - w \mathbf{J}_n^{-1} \mathbf{F}_n, \quad (\text{A5})$$

where

$$\begin{aligned} \mathbf{q} &= \begin{pmatrix} q^0 \\ \vdots \\ q^i \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_1(q^0) \\ \vdots \\ F_1(q^i) \end{pmatrix}, \\ \mathbf{J} &= \begin{pmatrix} \frac{\partial F_1(q^0)}{\partial q^0} & \cdot & \cdot & \frac{\partial F_1(q^0)}{\partial q^i} \\ \vdots & \cdot & \cdot & \vdots \\ \frac{\partial F_1(q^i)}{\partial q^0} & \cdot & \cdot & \frac{\partial F_1(q^i)}{\partial q^i} \end{pmatrix}. \end{aligned} \quad (\text{A6})$$

Generally, where this method was implemented, the equations were very stiff and thus required the use of a lower relaxation coefficient of $w < 1$ in order for stable convergence. In addition, derivatives were approximated to sixth order for improved accuracy.

2. Full dynamical simulations

Here, $(2 + 1)$ -dimensional simulations were performed on regular square grids of P^2 data points with grid spacing Δx and time-step Δt , or in the case of our infinite-domain wall solutions, rectangular grids of $P_x \times P_y$ grid points. The equations of motion were discretized using central finite-difference methods with temporal derivatives calculated to second order and spatial derivatives to fourth order. Neumann or periodic boundary conditions were used at the boundaries of the simulation grid. These full dynamical simulations are performed in the linear representation of Eq. (6).

3. Dimensionless rescaling

Within numerical studies, it is convenient to rescale the potential for dimensionless length and energy. Given that the values of M_h and v_{SM} are fixed by experiment, these quantities are used for the rescaling as follows:

$$\Phi \rightarrow v_{\text{SM}} \Phi, \quad x \rightarrow M_h^{-1} x, \quad (\text{A7})$$

which results in

$$\lambda_i \rightarrow \frac{v_{\text{SM}}^2}{M_h^2} \lambda_i, \quad \mu_j^2 \rightarrow \frac{1}{M_h^2} \mu_j^2, \quad (\text{A8})$$

such that the model parameters only depend on the mass ratios. This rescaling is such that lengths are expressed in units of M_h^{-1} and energy densities in units of $v_{\text{SM}}^2 M_h^2$. These two parameters can then subsequently be set to unity to achieve a simple numerical scale.

APPENDIX B: VANISHING FIELDS

In the regions of parameter space corresponding to each solution subclass, any field that is not part of that subclass's *Ansatz* indeed vanishes identically in the minimum-energy solution. In Ref. [25], it was proposed that under variation of all eight general field components in their parameter set, which under our classification would have a superconducting solution, there remained a small but nonzero *CP*-violating phase ξ at the wall, in addition to minimal nonzero profiles in the other fields of their chosen representation: we suspect these were numerical artifacts.

Further to this point, we show in Fig. 18 minimum-energy solutions for three different parameter sets with contrasting solutions. We see that all of the solutions are

correctly identified by our classification. Included in the inset plots are the quantities $\partial_x \eta_2$ and $\partial_x \eta_3$, which we used to reduce the field configuration so that it is described by only six functions. As can be seen, in all solutions $\partial_x \eta_3$ is effectively zero in all cases, as is $\partial_x \eta_2$ if the solution has a condensate. We show in Fig. 18(d) that this minimal profile in both of these quantities is a numerical artifact of the resolution of a solution.

For *CP*-violating solutions, $\partial_x \eta_2$ does not fully vanish even at high resolution. However, this seems to be entirely a numerical effect: if we consider only the nonvanishing fields (f_1, f_2, ξ, η_1) in a *CP*-violating solution, one can show that the equations of motion are satisfied regardless of $\partial_x \eta_2$'s behavior. We have verified this by testing many parameter sets beyond those shown here. This confirms that

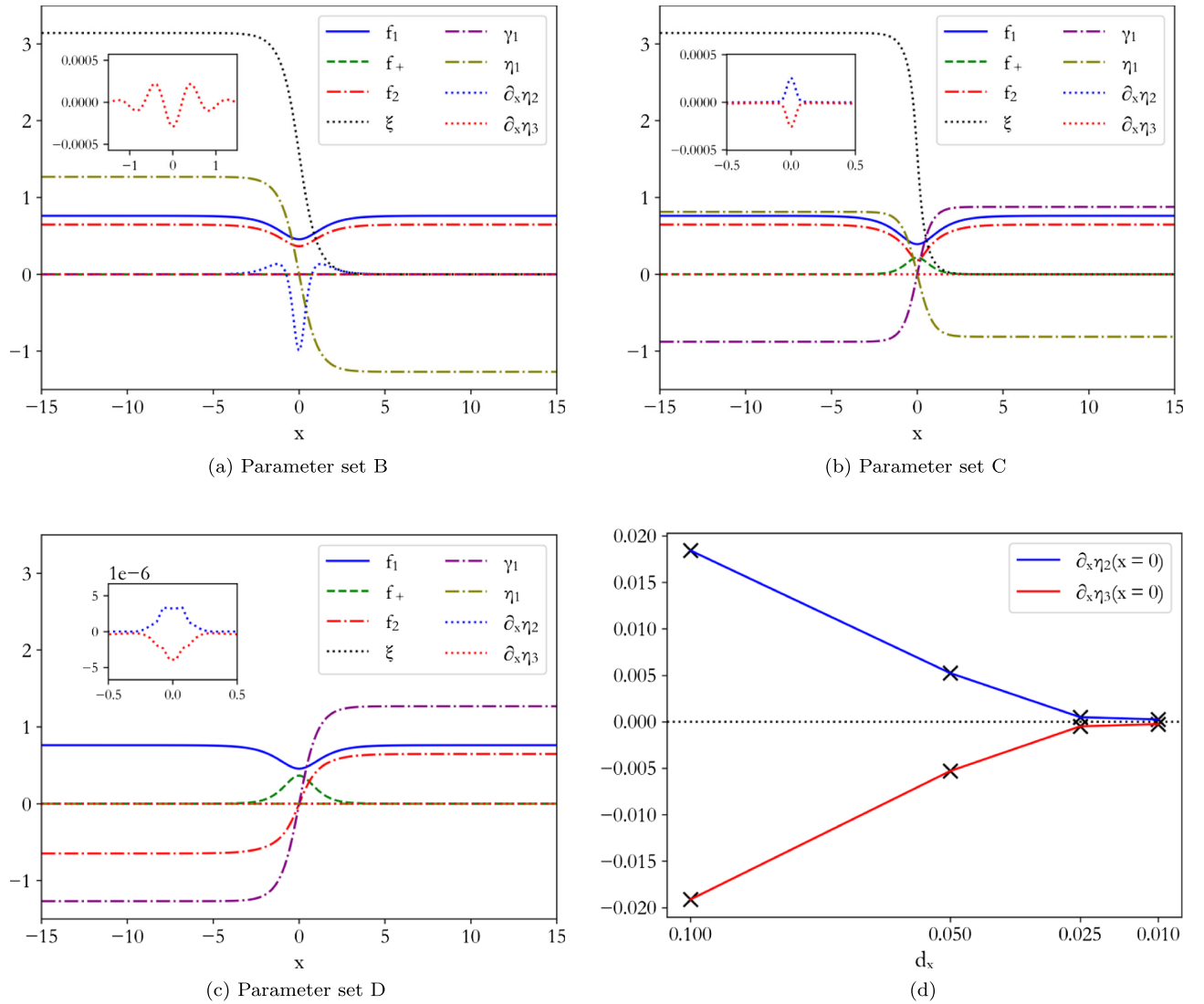


FIG. 18. (a),(b),(c) Minimum-energy domain wall solutions in the $\mathbb{Z}_2$ -symmetric 2HDM, for three different parameter sets. (d) Convergence toward zero of the general solution quantities $\partial_x \eta_2$ and $\partial_x \eta_3$ for minimum-energy solutions where $M_{H^\pm} \leq M_A$. Their value at the center of the kink solution is shown as a function of the spatial resolution d_x of the numerical solution. Note that d_x decreases to the right.

our subclasses of solutions successfully describe the full parameter space of the model, meaning that in each solution regime, the minimum-energy configuration involves only the subset of fields we identified for that regime (all other fields are exactly zero). Furthermore, while we do see a tiny nonzero CP -violating phase within the superconducting solution regime, this was found to follow the same behavior

as $\partial_x \eta_2$ and $\partial_x \eta_3$, in that it reduces for smaller grid spacings and smaller solution tolerance.

APPENDIX C: FURTHER MINIMUM-ENERGY SOLUTION EXAMPLES

Here, we present a selection of additional minimum-energy solutions for various parameter sets in Fig. 19.

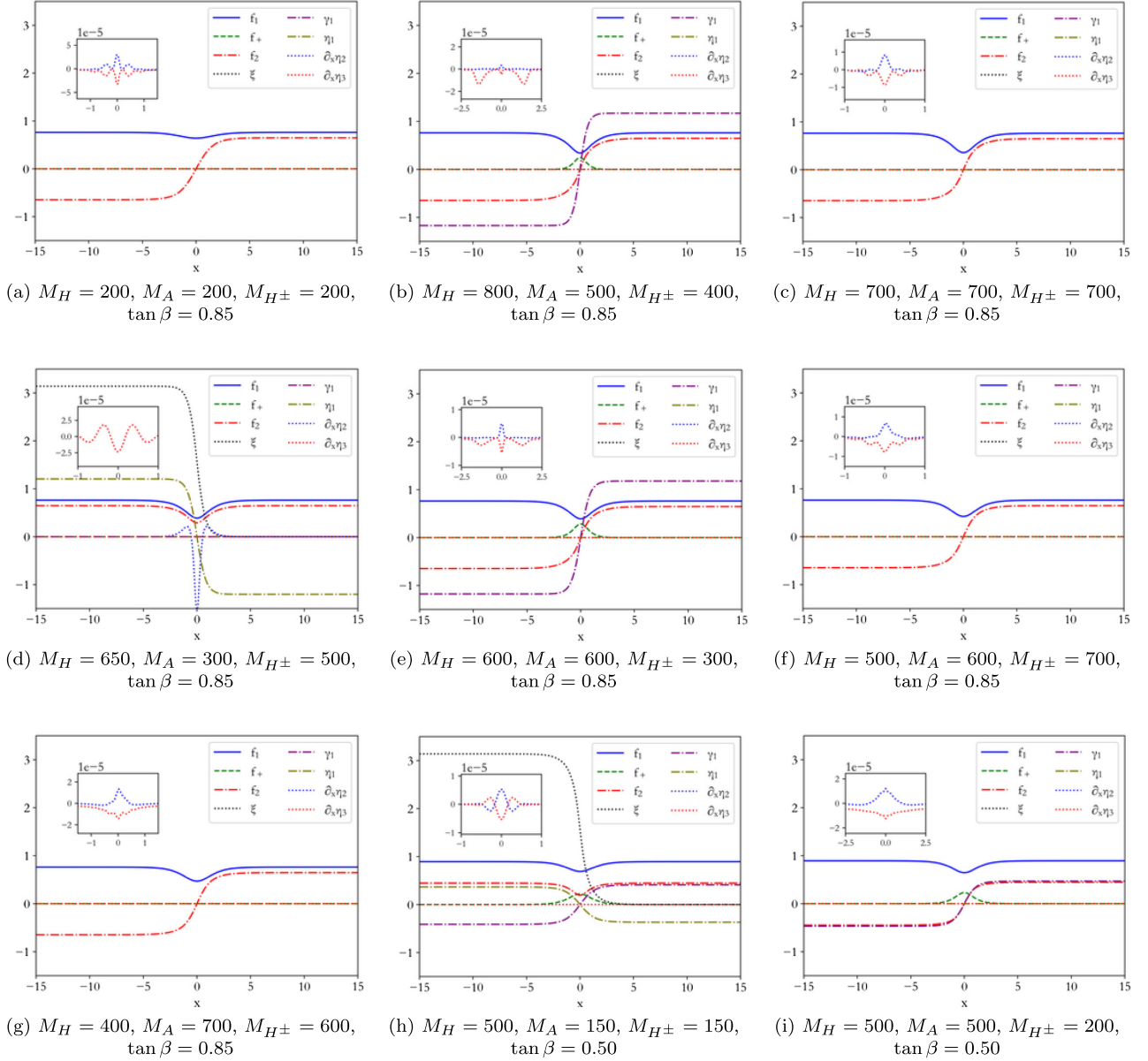


FIG. 19. Example minimum-energy kink solutions for the vacuum configuration of Eqs. (7), (14), for a selection of scalar mass (in GeV) and $\tan \beta$ values. Inset plots show the quantities $\partial_x \eta_2$ and $\partial_x \eta_3$, which converge to zero or an otherwise equivalent profile for each solution.

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