

Resonant locking between binary systems induced by gravitational waves

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The interaction of gravitational waves (GWs) with matter is thought to be typically negligible in the Universe. We identify an exception in the case of resonant interactions, where GWs emitted by a background binary system, such as an inspiraling supermassive black hole (SMBH) binary, cause a resonant response in a stellar-mass foreground binary and the frequencies of the two systems become, and remain, synchronized. We point out that this previously unexplored dynamical phenomenon is not only possible, but can lead to $\mathcal{O}(30)$ binary systems becoming resonantly locked in the host galaxy of merging SMBHs of mass $10^{8.5-11} M_\odot$, each of which has a significantly reduced merger time. We predict $\mathcal{O}(10^{10})$ binary systems have been locked in the Universe's history. Resonant locking could be detected through anomalous inspiral of binary systems.

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I. INTRODUCTION

The interaction of gravitational waves (GWs) with matter has been of interest since its prediction by Einstein [1]. Bondi [2,3] showed that GWs carry energy, and some of this energy can be dissipated in matter that the GWs pass through. Such dissipation can occur in an expanding Universe [4,5], a viscous fluid [6], radiation in the early Universe [7,8], neutrinos and dark matter [9,10], galaxies and clusters [11–13], accretion disks [14,15], oscillating stars [15–18], binary systems [19–29], planets [30–32], spacecraft [33–41], or crystals in laboratories [42]. Gravitational waves act as periodic perturbations on astrophysical systems and, while the associated dissipation is ordinarily negligible away from resonance [13,19,20], the cumulative effect in resonance can dramatically enhance the response [17,22,26,27,43,44].

Since it is known that Newtonian resonant perturbations commonly produce resonantly locked configurations for multiplanetary systems or moons in mean motion resonance, we are motivated to investigate if a similar capture of astrophysical systems into resonance is possible at a much larger distance from the source—even in the wave zone, defined as the region outside of one reduced GW wavelength from the GW source. In particular, inspiraling

supermassive black hole binaries (SMBHBs) are expected to form in most galaxies following galaxy collisions [45] whose GWs sweep through periods from thousands of years to hours, or even minutes, until they merge. Almost all binaries in the host galaxy will inevitably be in resonance with these GWs at some point during the SMBHB inspiral.

In this paper, we investigate the phenomenon of a stellar-mass astrophysical binary system (“foreground, FG, binary”) undergoing resonant locking (or frequency synchronization) with the GWs emitted by an inspiraling SMBHB (“background, BG, binary”). See Fig. 5 in the Appendix for an illustration of the setup. We adopt geometrized units: $G = c = 1$.

II. EQUATION OF MOTION

Table I defines the binary parameters used in this work. Since the BG binary is much more massive than the FG binary, we neglect feedback effects. Subscripts “f” and “b” denote quantities of the FG and BG binaries, respectively. To leading order in v_f/c , the equation for geodesic deviation gives the relative force F^i between the two FG binary components [[46], Eq. (8.43)],

$$\frac{F^i}{\mu_f} = \ddot{r}_f^i + \frac{M_f}{r_f^3} r_f^i = R^{0i0j} r_{f,j}, \quad (1)$$

where dots ($\dot{}$) represent time derivatives, commas do *not* refer to partial differentiation, and R is the Riemann tensor

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TABLE I. Definitions of binary parameters and variables used in this paper.

ϕ	True anomaly.
n	Mean motion frequency.
v	Orbital velocity.
T	Orbital period.
$\mathcal{M}$	Chirp mass.
M	Total mass.
μ	Reduced mass.
η	Symmetric mass ratio.
r^i	Separation vector.
r	Separation vector length.
a	Semi-major axis.
e	Eccentricity.
i	Inclination.
Ω	Longitude of the ascending node.
ω	Argument of periapsis.
ϖ	Longitude of periapsis.
u	Mean anomaly.
ψ	Phase of the GW.
Q^{ij}	Quadrupole moment.

due to the BG binary. The lowest-order (quadrupole) contribution to this force¹ is [29]

$$\begin{aligned} \frac{F^h}{\mu_f} = & \frac{1}{D} r_{f,i} A_1^{ih} - \frac{1}{2D^3} r_{f,i} (D^i D_k A_2^{kh} + D^h D_k A_2^{ki}) \\ & + \frac{1}{6D^3} r_f^h D_j D_k (2A_2^{jk} - A_3^{jk}) + \frac{1}{2D^5} r_{f,i} D^i D^h D_k D_l A_3^{kl}, \end{aligned} \quad (2)$$

where repeated indices are summed over, D^i is the vector pointing to the FG binary from the BG binary, $D = \sqrt{D^i D_i}$, and

$$A_1^{ab} = \ddot{Q}_b^{ab} + \frac{2}{D} \ddot{Q}_b^{ab} + \frac{3}{D^2} \ddot{Q}_b^{ab} + \frac{3}{D^3} \dot{Q}_b^{ab} + \frac{3}{D^4} Q_b^{ab}, \quad (3a)$$

$$A_2^{ab} = 2\ddot{Q}_b^{ab} + \frac{8}{D} \ddot{Q}_b^{ab} + \frac{18}{D^2} \ddot{Q}_b^{ab} + \frac{30}{D^3} \dot{Q}_b^{ab} + \frac{30}{D^4} Q_b^{ab}, \quad (3b)$$

$$A_3^{ab} = \ddot{Q}_b^{ab} + \frac{10}{D} \ddot{Q}_b^{ab} + \frac{45}{D^2} \ddot{Q}_b^{ab} + \frac{105}{D^3} \dot{Q}_b^{ab} + \frac{105}{D^4} Q_b^{ab}. \quad (3c)$$

III. TOY MODEL

For general D^i the problem is analytically intractable. To illustrate resonant locking, we briefly present a toy model in the far-wave zone ($D \gg \lambda$, with λ being the reduced wavelength of the BG binary's GWs), where the force reduces to [19,20,26,27]

¹Higher moments at fixed frequency are suppressed by powers of v_b/c .

$$\frac{F^i}{\mu_f} = \ddot{r}_f^i + \frac{M_f}{r_f^3} r_f^i = \frac{1}{2} \ddot{h}_{ij}^i r_f^j, \quad (4)$$

where $h_{ij} = 2\ddot{Q}_{ij}/D$ is the GW strain of the BG binary. In the quadrupole approximation, $\psi = 2\phi$. For simplicity, we assume both binaries are circular.

IV. EVOLUTION OF THE FG BINARY

The rate of work done due to GW stimulated emission or absorption is

$$\frac{dE}{dt} = W_{\text{GW}} = \frac{\mu_f}{2} \ddot{r}_f^i \ddot{h}_{ij}^j r_f^i. \quad (5)$$

In Cartesian coordinates the GW perturbation is [47]

$$h_{ij} = \begin{pmatrix} A_+ \cos(\psi_b) & A_\times \sin(\psi_b) & 0 \\ A_\times \sin(\psi_b) & -A_+ \cos(\psi_b) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (6)$$

where $A_{+,\times}$ are the GW polarization amplitudes. For a circular BG source [48]

$$A_+ = -A_b(1 + \cos^2 \chi), \quad A_\times = -2A_b \cos \chi, \quad (7)$$

where $A_b = 2\mathcal{M}_b^{5/3} n_b^{2/3}/D$ and χ is the angle between the BG binary's orbital axis and D^i .

We take D^i to be along the positive $\hat{z}$ axis, and the FG binary's orbit to be in an arbitrary plane

$$r_f^i = \mathbf{O}_z(\omega_f) \mathbf{O}_x(t_f) (r_f \cos(\phi_f), r_f \sin(\phi_f), 0), \quad (8)$$

where $\mathbf{O}_\alpha(\beta)$ are rotation matrices about the $\hat{\alpha}$ axis by an angle β . Averaging Eq. (5) over one FG orbit and keeping secular resonant terms yields

$$\frac{W_{\text{GW}}}{4\mathcal{M}_f^{5/3} n_f^{-1/3} n_b^2 A_b} \rightarrow \mathcal{A} \sin(2\phi_f - 2\phi_b + \vartheta), \quad (9)$$

where $\mathcal{A}(t_f, \omega_f, \chi)$ and $\vartheta(t_f, \omega_f, \chi)$ are given in Eqs. (A1) and (A2). The energy of the FG binary system is, to leading order in the center-of-mass frame,

$$E_f = \frac{1}{2} \mu_f \dot{r}_f^2 - \frac{M_f \mu_f}{r_f} = -\frac{\mathcal{M}_f^{5/3} n_f^{2/3}}{2}, \quad (10)$$

from Kepler's third law. Thus, $E_f \propto n_f^{2/3}$ and

$$\frac{\dot{n}_f}{n_f} = \frac{3}{2} \frac{\dot{E}_f}{E_f} = -\frac{3}{\mathcal{M}_f^{5/3} n_f^{2/3}} \dot{E}_f. \quad (11)$$

Let us decompose $\dot{E}_f \equiv W_{\text{GW}} + L_f$, where [49]

$$L_f = -\frac{32}{5} \frac{\mathcal{M}_f^5}{r_f^5 \eta_f} = -\frac{32}{5} (n_f \mathcal{M}_f)^{10/3} \quad (12)$$

is the FG binary's own GW loss. This gives

$$\dot{n}_f = \frac{96 \mathcal{M}_f^{5/3}}{5} n_f^{11/3} - 24 \mathcal{A} \frac{\mathcal{M}_b^{5/3}}{D} n_b^{8/3} \sin(2\phi_f - 2\phi_b + \vartheta), \quad (13)$$

where the BG binary's phase evolves as predicted by general relativity (GR) [50]

$$\phi_b = \frac{n_{0,b}^{-5/3}}{32 \mathcal{M}_b^{5/3}} \left[1 - \left(1 - \frac{256}{5} \mathcal{M}_b^{5/3} n_{0,b}^{8/3} t \right)^{5/8} \right] + \phi_{0,b}, \quad (14)$$

to leading post-Newtonian order, with $\phi_{0,b}$ and $n_{0,b}$ being the initial phases and angular velocities, respectively. Equations (13) and (14) give a closed description of the FG binary's evolution over time.

V. FORCED PENDULUM

Defining the relative phase $\theta = 2\phi_f - 2\phi_b + \vartheta$, neglecting GW emission from the FG binary (since $\mathcal{M}_f \ll \mathcal{M}_b$) and taking t_f , ω_f , and χ fixed, Eq. (13) reduces to

$$\ddot{\theta} = -C_A \sin(\theta) - C_F, \quad (15)$$

where

$$C_A(t) = 48 \mathcal{A} \frac{n_b^{8/3}(t) \mathcal{M}_b^{5/3}}{D} = 24 \mathcal{A} \frac{\eta_b v_b^5(t)}{D \tilde{\lambda}(t)} > 0, \quad (16a)$$

$$C_F(t) = \frac{192}{5} n_b^{11/3}(t) \mathcal{M}_b^{5/3} = \frac{48 \eta_b v_b^5(t)}{5 \tilde{\lambda}^2(t)} > 0. \quad (16b)$$

Equation (15) is coincidentally the equation of motion of a forced pendulum with amplitude C_A and forcing C_F .

A pendulum admits two types of motion: rotation (2π cycles) and libration (oscillations about the stable equilibrium). Resonant locking corresponds to libration, where $\dot{\theta}$ remains bounded as n_b and n_f increase synchronously.

VI. RESONANT LOCKING IN THE TOY MODEL

Locking requires a transition from rotation to libration. This process is governed by the phase-space structure of the forced pendulum (Fig. 1). From Eq. (16), C_A , C_F , and $C_F/C_A \propto n_b$ grow with time causing the separatrix to stretch vertically. Trajectories near it can be captured, moving the pendulum into libration and hence leading to locking. The initial phase θ_0 dictates proximity to the separatrix at resonance, and a uniform $\theta_0 \in [0, 2\pi)$ yields an (approximately) uniform distribution between the inner and

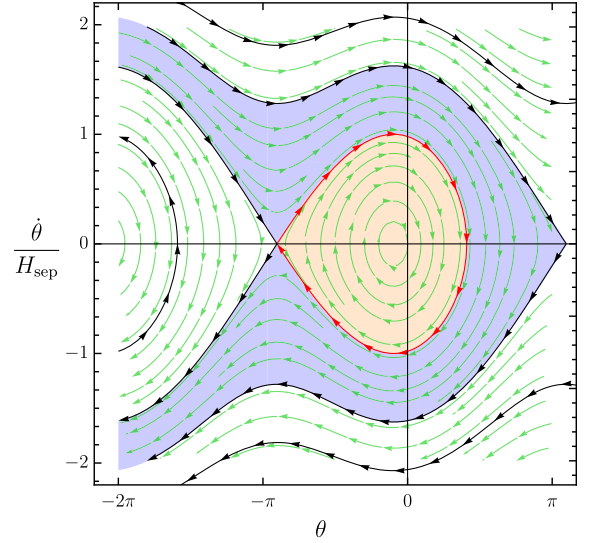


FIG. 1. Phase-space of the forced pendulum [Eq. (15)] with $C_F/C_A = 0.3$. Closed librating contours are orange, open rotating contours are blue, the red separatrix separates these two regions and the outer separatrices are black. We have normalized $\dot{\theta}$ to the separatrix height, H_{sep} .

outer separatrices at $\theta = -\pi$. This is illustrated in two videos: one showing capture, linked here [51], and one showing no capture, linked here [52].

Suppose the system enters libration at frequency $n_{b,\text{lock}}$, and reduced wavelength $\tilde{\lambda}_{\text{lock}} = 1/(2n_{b,\text{lock}})$, in the wave zone. The FG binary will remain in the wave zone provided D is fixed, and n_f will oscillate about n_b provided $C_F < C_A$. Since $C_F/C_A \propto n_b$, libration will inevitably end when $C_F > C_A$, after which $\ddot{\theta} < 0$ permanently and the system resumes rotation. Thus, during locking,

$$\frac{C_F}{C_A} = \frac{4}{5 \mathcal{A}} D n_b = \frac{2 D \tilde{\lambda}_{\text{lock}}}{5 \mathcal{A}} \frac{n_b}{n_{\text{lock}}} \leq 1, \quad (17)$$

where $D_{\tilde{\lambda}} = D/\tilde{\lambda}$. Since $\max(\mathcal{A}) = 1$ for $(t_f, \chi) \in [0, \pi)$, we require $D_{\tilde{\lambda}} < 5/2$. This contradicts the $D \gg \tilde{\lambda}$ limit, confirming that a full EoM treatment in the $D \sim \tilde{\lambda}$ regime is necessary. Nonetheless, this toy model is valuable for qualitatively illustrating resonant locking.

VII. IMPLICATION: PRODUCING MERGING SYSTEMS

Resonant locking increases n_f , reducing the FG binary's merger time. For the sake of the argument, consider $\mathcal{A} = 1$. The FG binary's time to coalescence is [50]

$$t_{\text{merge},f}(n_f) = \frac{5}{256} \mathcal{M}_f^{-5/3} n_f^{-8/3}. \quad (18)$$

If locking starts at n_{start} and terminates at frequency n_{end} (where $C_F = C_A$), then

$$\frac{t_{\text{merge},f}(n_{\text{start}})}{t_{\text{merge},f}(n_{\text{end}})} = \left(\frac{n_{\text{end}}}{n_{\text{start}}}\right)^{8/3} = \left(\frac{5}{2D_\lambda}\right)^{8/3}. \quad (19)$$

To illustrate how resonant locking can drive binaries toward merger, consider a system with $M_f = 40M_\odot$, $r_f = 0.5$ AU, and $D_\lambda = 1$, which gives $D = 100$ AU. During locking, the orbital period goes from 7.24 to 2.90 days, and the merger time is brought down from $5.76T_H$ to $0.50T_H$, where T_H is a Hubble time. Thus, binaries that would not merge within T_H via GW emission alone can do so after resonant locking with a faster-inspiralling BG binary. This may have important implications for GW population-synthesis models.

We now turn to the full treatment using the general force in Eq. (2).

VIII. GENERAL EOM

When $D \sim \lambda$, each term in Eq. (3) contributes approximately equally, so the general expression is required. We also include the FG binary's orbit about the BG binary (the “outer” orbit), giving a total of three intrinsic Euler angle triplets $(\Omega_i, \iota_i, \omega_i)$ with $i \in \{f, b, \text{outer}\}$. Orbital elements are allowed to vary self-consistently via the Lagrange planetary equations [LPEs; Eq. (A3) in Appendix and surrounding text] under the influence of the full perturbing force—Eq. (2). We focus on low-eccentricity systems ($e \lesssim 0.1$), leaving the general eccentric case for future work.

This general treatment raises two questions: (1) Is resonant locking still possible? (2) If so, how does capture occur?

Figure 2 demonstrates locking is indeed possible for a noncircular FG binary (initially $e_f = 0.1$ and remains

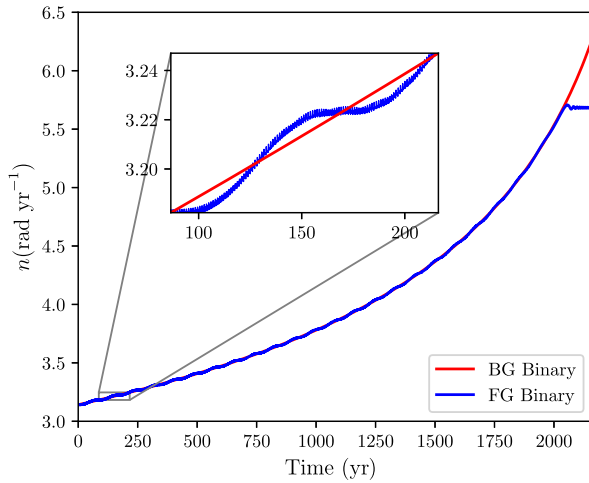


FIG. 2. We numerically integrate Eq. (A3) with the general force [Eq. (2)] using initial conditions $\mathcal{M}_b = 8.71 \times 10^8 M_\odot$, $D_\lambda = 1$, $e_{b,0} = 0$, $\phi_{b,0} = 0$, $T_{b,0} = T_{f,0} = 2$ yrs, $\phi_{f,0} = 0.65\pi$, $e_{f,0} = 0.1$, $\iota_f = 0$, $\iota_{\text{outer}} = 0$ and all other orbital elements are initially set to zero. This gives $T_{\text{outer}} \approx 22.6$ yr. The FG binary's mean motion n_f remains trapped in resonance until decoupling at $T_f \approx 1.11$ yr.

approximately constant). Starting at resonance, n_f oscillates slowly about n_b due to secular terms, while faster oscillations arise from nonsecular terms.

In fact, we have found that it is possible for locking to begin in the near zone and persist as the FG binary moves into the wave zone. This is shown in Fig. 6 and is discussed further in the Appendix. However, since our primary interest lies in GW-induced resonances, we confine our study to the wave zone and defer an analysis of near-zone resonances to future work.

We discuss below the conditions needed for generic FG/BG/outer binary configurations to produce resonant locking like that shown in Fig. 2.

During locking, the GW must increase the FG binary's mean-motion frequency at the same rate that the BG binary's frequency changes, i.e. $\dot{n}_f \geq \dot{n}_b$. If this does not occur, the GW will outrun the binary and resonance cannot be sustained. Since the GW interaction strength varies along the outer orbit, and the BG binary's inspiral timescale is much longer than T_{outer} , we must outer-orbit average. We arrive at the condition

$$\bar{n}_f = -\frac{3n_f}{2a_f} \bar{a}_f = -\frac{3n_f}{2a_f} \frac{1}{T_{\text{outer}}} \int_0^{T_{\text{outer}}} \dot{a}_f dt \geq \dot{n}_b, \quad (20)$$

where $\dot{a}_f$ comes from Eq. (A3a). Since the relative phase between the FG binary and BG binary controls whether energy is given or taken from the FG binary, we maximize $\bar{n}_f$ over the initial relative phase between the FG and BG binaries. In other words, if, at resonance, there exists an initial relative phase that satisfies Eq. (20), then locking is possible.

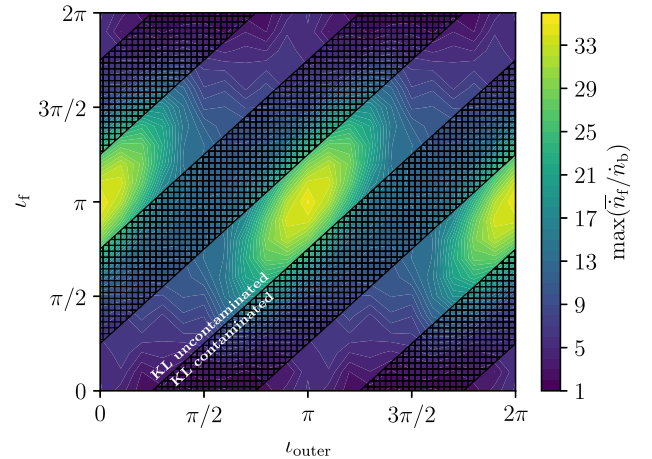


FIG. 3. Maximum value of $\bar{n}_f/\dot{n}_b$ when varying the relative phase of the FG and BG binaries as a function of the inclinations ι_f and ι_{outer} relative to the BG binary for $D = \lambda$. The hatched regions represent the parameter space removed due to von Zeipel-Lidov-Kozai (ZLK) oscillations. The condition for resonance capture, $\bar{n}_f/\dot{n}_b > 1$, is satisfied for all ι_f and ι_{outer} .

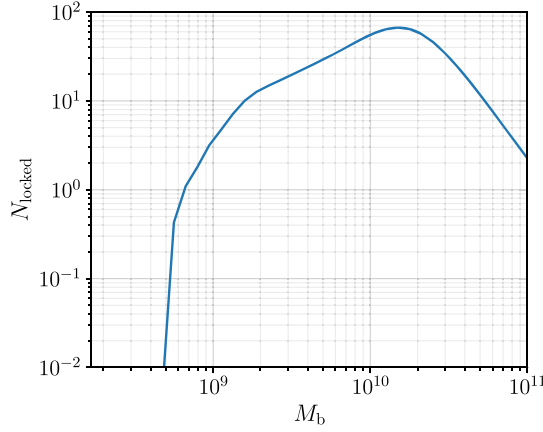


FIG. 4. Number of locked binary systems as a function of M_b , using Eq. (26).

Note that while $\dot{n}_f$ can fall below $\dot{n}_b$ during parts of the outer orbit, the slow drift of the GW frequency ensures the system remains within the resonance bandwidth over one outer orbital period and so it is the averaged rate of change that governs locking. This condition is both sufficient and necessary for resonant locking, provided the binary has a suitable initial phase relative to the GW. To verify this numerically, we randomly sampled system parameters and found that all cases satisfying $\bar{n}_f \geq \dot{n}_b$ indeed achieved resonant locking, including extremal values where $\bar{n}_f/\dot{n}_b$ is close to unity and $\bar{n}_f/\dot{n}_b$ is much larger than unity.

Our system has 12 initial degrees of freedom (nine Euler angles and three phases). To identify which are important for low-eccentricity orbits, it is instructive to consider first the circular case ($e = 0$), since our results remain essentially unchanged for small e .² In this limit, only three nondegenerate parameters affect the evolution: ι_f and ι_{outer} relative to the BG binary’s orbital plane, and the relative phase $\Delta\phi_0 = \phi_{f,0} - \phi_{b,0}$. Note that we must extend the domain of inclinations from $[0, \pi)$ to $[0, 2\pi)$ in order to absorb the degeneracies with longitudes Ω_{outer} and Ω_f . The inclinations also obey a $(\iota_f, \iota_{\text{outer}}) \rightarrow (2\pi - \iota_f, 2\pi - \iota_{\text{outer}})$ symmetry.

Resonant locking is possible if, for given ι_f and ι_{outer} , there exists a $\Delta\phi_0$ such that $\bar{n}_f \geq \dot{n}_b$. This will of course depend on D_λ .

In the toy model discussed above, only a single resonance appeared, at $n_f = n_b$. Now that we allow the FG binary to orbit the BG binary, we get a total of five distinct resonances at $n_f = n_b + kn_{\text{outer}}$ for $k \in \{-2, -1, 0, 1, 2\}$.³ These resonances do not overlap and their strengths depend on the Euler angles. Figure 3 shows the maximum of $\bar{n}_f/\dot{n}_b$ across all five resonances, when $\Delta\phi_0$ is varied, as a function

of ι_{outer} and ι_f for $D_\lambda = 1$. We find values between 1.7 and 35, implying that locking can occur for any orbital configuration at this distance. It is around $D_\lambda \gtrsim 2$, however, that $\bar{n}_f/\dot{n}_b < 1$ everywhere (the exact distance depends on ι_f and ι_{outer} —see Appendix Fig. 7).

The remaining challenge is resonance capture: acquiring the phase needed for locking. In our toy circular-restricted model [Eq. (15)], a FG binary with a suitable initial relative phase could lock with the GWs by crossing the separatrix. In the general treatment, however, variations in the FG binary’s longitude of periapsis, $\varpi = \omega + \Omega$, inhibit separatrix crossing and thus resonance capture.

We identify two mechanisms that can lead to locking, both confirmed numerically. One is separatrix crossing with the additional requirement that the outer orbit’s separation distance, a_{outer} , must be decreasing. Specifically we require $\dot{a}_{\text{outer}}/a_{\text{outer}} \gtrsim \dot{n}_b/n_b$.⁴ Once captured, locking can persist even if a_{outer} stops decreasing. A dynamical process that could cause such behavior of a_{outer} is the “tidal” disruption of a wide triple near the periapsis of its eccentric orbit around the BG, thus depositing the inner binary of the triple, the FG binary, on a more tightly bound orbit about the BG binary than it originally was. We leave this mechanism for future work.

The second mechanism is getting kicked into resonance via dynamical scattering encounters—the primary process we consider next. The nature of these encounters of course depends on the binary’s astrophysical environment.

IX. POSTMERGER GALAXIES

Since capture is probabilistic, resonant locking is most effective in dense environments containing many stellar-mass binary objects (smBOs) around a strong GW source. A natural setting is in the bulge of a postgalaxy merger, where the SMBHB forms our BG binary and the surrounding smBOs act as FG binaries (see Appendix for details of the galaxy/bulge model). As the SMBHB inspirals, it sweeps through almost all relevant frequencies, giving each smBO an opportunity to be captured. We model the kicks from scattering encounters and estimate the resulting number of resonantly locked systems.

X. KICKED INTO RESONANCE BY DYNAMICAL ENCOUNTERS

Dynamical encounters, which represent the dominant noise source in galactic bulges, can knock the FG binary either *into* resonance or knock it *out* of resonance. Although the aforementioned separatrix approach is not strictly valid in our full treatment, we will apply it here as it provides a transparent and intuitive framework for gaining

²We confirm using numerical simulations as in Fig. 2.

³Allowing the FG binary to orbit the BG binary introduces new terms proportional to $\cos(2(\phi_f - \phi_b + k\phi_{\text{outer}}))$, where $k \in \{-2, -1, 0, 1, 2\}$, when expanding Eq. (2).

⁴This was verified numerically, and is identical to the condition that the separatrix increases in both height *and* width in the $D \gg \lambda$ regime.

an order-of-magnitude estimate of this stochastic capture rate. We note that the resonances in this general treatment are of similar strength to, and are sometimes stronger than, the resonance in the toy model and hence our estimate is a conservative underestimate.

Consider an encounter between the FG binary and a perturber of mass M_p , impact parameter b , and velocity v , taken as the bulge velocity dispersion from the $M_{\text{SMBH}}-\sigma$ relation [[53], Eq. (7)],

$$v = \sigma = 200 \text{ km/s} \left(\frac{M_{\text{SMBH}}}{3.08 \times 10^8 M_\odot} \right)^{1/4.38}. \quad (21)$$

There are two regimes to consider: slow encounters, where $(v/b)T_f < \gamma$, and fast encounters, where $(v/b)T_f > \gamma$, with $\gamma = \mathcal{O}(1)$ (the exact value depends on one's definition of “fast” and “slow” and the results are insensitive to it). During a slow encounter, the binary rotates rapidly, and the work done by the perturbing star averages to zero. Such encounters hence cannot push the binary into/out of resonance. In the fast regime the phase is effectively frozen during the interaction and the perturber delivers a net kick, changing the FG binary's energy by [[54], Eq. (8.42)]

$$\Delta E \sim \frac{G^2 M_p^2 M_f r_f^2}{2v^2 b^4} = \frac{G^{8/3} M_p^2 M_f^{5/3}}{2n_f^{4/3} v^2 b^4}, \quad (22)$$

where G is the gravitational constant. This can be negative or positive depending on the phase. These fast encounters are the ones relevant for kicks into resonance.

Further suppose that the FG binary is not in resonance during a fast encounter and that $\Gamma_{\text{into}}(T_f, D, t)$ is the rate of kicks that are just the right strength to knock the binary system inside the separatrix. The probability density that such a kick first occurs at time t follows the exponential density distribution of a Poisson process

$$\mathcal{P}_{\text{into}}(T_f, D, t_1, t) = \Gamma_{\text{into}} \exp \left[- \int_{t_1}^t \Gamma_{\text{into}} dt' \right], \quad (23)$$

where $t_1(M_b)$ marks the time the FG binary first enters the wave-zone; $\lambda(t_1) = D$. The full expression for Γ_{into} is given in Eq. (A12).

Resonance must then persist until natural decoupling, i.e. until $\bar{n}_f/\bar{n}_b$ drops below unity. We take this to be at $D = 2\lambda$ (the exact distance depends on t_f and t_{outer} and typically lies between 1.5λ and 2.3λ). If $\Gamma_{\text{out}}(t, D)$ is the rate of kicks large enough to remove the binary from resonance (i.e. a kick with $2\Delta n_f > H_{\text{sep}}$), the survival probability of a system that was captured at time t is

$$P_{\text{stable}} = \exp \left[- \int_t^{t_2} \Gamma_{\text{out}} dt' \right], \quad (24)$$

where t_2 marks the time the FG binary naturally decouples, defined implicitly through $\lambda(t_2) = D/2$. The full expression for Γ_{out} is given in Eq. (A14).

Other dynamical processes, such as the ZLK effect [55,56], can interfere with resonant locking and we must thus constrain our parameter space. In particular, ZLK is important for $40^\circ \lesssim t_f - t_{\text{outer}} \pmod{180^\circ} \lesssim 140^\circ$ [57] as it induces eccentricity oscillations. We therefore exclude this inclination range. Note, however, that ZLK oscillations do not change the FG binary's orbital frequency and will thus not necessarily disrupt locking even if oscillations are large—however, we leave this to future work. Assuming inclinations are uniform in $\cos i$, this removes 77.5% of parameter space, so we multiply P_{locked} by 0.225.

Further, recall that in the general treatment, we found nonoverlapping resonances at $n_f = n_b + kn_{\text{outer}}$ for $k \in \{-2, -1, 0, 1, 2\}$. We show in the Appendix in Fig. 8, that all five resonances are of similar strength. Consequently, locking can occur in identical fashion at each resonance. For concreteness, we also demonstrate that resonant locking is possible at each resonance in Fig. 9. Some (in fact, most) regions of (t_f, t_{outer}) space are home to multiple of these resonances [cf. Fig. 8(f)]. On average, each configuration with $t_f - t_{\text{outer}} \pmod{180^\circ} \lesssim 40^\circ$, or $t_f - t_{\text{outer}} \pmod{180^\circ} \gtrsim 140^\circ$ hosts 2.59 resonances.⁵ We hence multiply P_{locked} by 2.59.

The probability of being locked is therefore

$$P_{\text{locked}}(T_f, D) = 2.59 \times 0.225 \int_{t_1}^{t_2} \mathcal{P}_{\text{into}} P_{\text{stable}} dt, \quad (25)$$

which depends on the FG binary's initial period and distance from the SMBHB, the BG binary's mass, and the perturber mass, which we take to be the average star mass $\langle m_{\text{star}} \rangle = 0.3 M_\odot$ [58].

XI. THE NUMBER OF GW LOCKED SYSTEMS

The cumulative number of binaries that lock and remain locked until natural decoupling during a galaxy merger is

$$\frac{N_{\text{locked}}}{N_{\text{bin}}} = \int_0^\infty \int_{T_{\text{min}}}^{T_{\text{max}}} P_{\text{locked}}(D, \lambda) f_D(D) f_T(T) dT dD, \quad (26)$$

where $\lambda(T) = cT/(4\pi)$, $f_D(D)$ is the radial distribution, $f_T(T)$ is the period distribution, and $N_{\text{bin}} = 0.25 N_{\text{stars}}$ is the number of binaries in the bulge [59,60] where $N_{\text{stars}} = M_{\text{bulge}}/\langle m_{\text{star}} \rangle$ and M_{bulge} comes from the $M_{\text{bulge}}-M_{\text{SMBH}}$ relation [Eq. (A7)]. See Appendix for details, including the integration bounds.

⁵This is calculated by averaging the number of resonances shown in Fig. 8(f) across the non-ZLK contaminated parameter space.

Figure 4 shows N_{locked} for varying M_b . It peaks at $M_b \sim 10^{10} M_\odot$ due to competing effects. At small M_b , higher central densities shorten evaporation times [Eq. (A15)], reducing T_{max} and hence bottlenecking N_{locked} for $M_b \lesssim 10^{8.5} M_\odot$. At larger M_b , the BG binary's inspiral accelerates, so FG binaries spend less time in the $1 < D_\lambda < 2$ zone, increasing P_{stable} . At very large $M_b > 10^{10} M_\odot$, the inspiral is too rapid for binaries to be knocked into resonance in the first place, lowering P_{into} . The result is a peak near $10^{10} M_\odot$ where these effects balance.

XII. DISCUSSION

We find that up to $\mathcal{O}(30)$ binaries can become resonantly locked with a SMBH binary during a high-mass galaxy merger (Fig. 4). Further, using the $M_{\text{SMBH}}-M_{\text{DM}}$ relation [[53], Eq. (18)] and integrating over the dark matter (DM) halo mass function [[53], Eq. (10)], we estimate that $\mathcal{O}(10^{10})$ binaries have been locked at some point during the cosmic history of the observable Universe (see Appendix).

A direct signature of locking could be anomalous inspiral detected either by EM observations of binary pulsars or GW observations of compact objects. Future detectors could observe a resonantly locked binary inspiraling more quickly than what is predicted by GR. This would be a smoking-gun signature of locking.

There are limitations to the work discussed here. We have only considered perturbations to the binary from passing stars when there could be other local effects (such as the presence of planets/gas) that influence the dynamics. We have not studied binaries with large eccentricities. However, given that an eccentric orbit may be decomposed in Fourier modes [23], we would expect similar behavior except with additional harmonics. The combined effect of a resonant GW interaction in the presence of ZLK oscillations with inclinations $40^\circ < i_f - i_{\text{outer}} \pmod{180^\circ} < 140^\circ$ has been ignored. We ignore near-zone resonant locking. We wish to explore these directions in future work, as well as the possibility of entering the resonance via a ‘‘tidal’’ disruption of a hierarchical triple, on an initially eccentric outer orbit, as alluded to above.

XIII. CONCLUSION

We have studied a previously unrecognized dynamical mechanism: the resonant locking of stellar-mass binaries with gravitational waves generated by supermassive black hole mergers. We expect this phenomenon to take place in high mass postmerger galaxies, with billions of BH binaries being resonantly locked in the Universe's history.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX

1. Expressions for $\mathcal{A}$ and ϑ

The expressions for $\mathcal{A}$ and ϑ in Eq. (9) are

$$\mathcal{A} = \frac{1}{8} \sqrt{4(C_{\text{tf}} + C_\chi)^2(1 + C_{\text{tf}} C_\chi)^2 + S_\chi^4 S_{\text{tf}}^4 \cos(2\omega_{\text{f}})}, \quad (\text{A1})$$

$$\vartheta = \pi + \arctan \left(\frac{1 - t_\chi^4 t_{\text{tf}}^4}{1 + t_\chi^4 t_{\text{tf}}^4} \tan(2\omega_{\text{f}}) \right), \quad (\text{A2})$$

respectively, with $t_x = \tan(x/2)$, $S_x = \sin(x)$ and $C_x = \cos(x)$ for $x \in \{\chi, i_f\}$.

2. The lagrange planetary equations

The LPEs are a set of six coupled first-order differential equations that describe how the six orbital elements (a , e , i , u , ω , and Ω) of a binary system evolve in the presence of an external perturbation ([61] Sec. 1.1), ([62] Sec. 11.9), ([63] Sec. 12.2)

$$\dot{a} = \sqrt{\frac{4a}{GM}} \frac{\partial \mathcal{L}_p}{\partial M}, \quad (\text{A3a})$$

$$\dot{e} = -\sqrt{\frac{1-e^2}{GMae^2}} \frac{\partial \mathcal{L}_p}{\partial \omega} + \frac{1-e^2}{\sqrt{GMae^2}} \frac{\partial \tilde{\mathcal{L}}_p}{\partial M}, \quad (\text{A3b})$$

$$\frac{di}{dt} = -\frac{\csc i}{\sqrt{GMa(1-e^2)}} \frac{\partial \mathcal{L}_p}{\partial \Omega} + \frac{\tan i}{\sqrt{GMa(1-e^2)}} \frac{\partial \tilde{\mathcal{L}}_p}{\partial \omega}, \quad (\text{A3c})$$

$$\dot{u} = \sqrt{\frac{GM}{a^3}} - \sqrt{\frac{4a}{GM}} \frac{\partial \tilde{\mathcal{L}}_p}{\partial a} - \frac{1-e^2}{\sqrt{GMae^2}} \frac{\partial \mathcal{L}_p}{\partial e}, \quad (\text{A3d})$$

$$\dot{\omega} = \sqrt{\frac{1-e^2}{GMae^2}} \frac{\partial \mathcal{L}_p}{\partial e} - \frac{\cos \iota}{\sqrt{GMa(1-e^2)} \sin \iota} \frac{\partial \mathcal{L}_p}{\partial \iota}, \quad (\text{A3e})$$

$$\dot{\Omega} = \frac{1}{\sqrt{GMa(1-e^2)} \sin \iota} \frac{\partial \mathcal{L}_p}{\partial \iota}, \quad (\text{A3f})$$

where $\mathcal{L}_p$ is the Lagrangian corresponding to the perturbing force. The general force given in Eq. (2) implies a perturbing Lagrangian given by [29]

$$\mathcal{L}_p = \frac{Q_f^{ij}}{2D} A_{1,ij} - Q_f^{ij} \frac{D_i D_j}{2D^3} A_{2,kj} + Q_f^{ij} \frac{D_i D_j D^k D^l}{4D^5} A_{3,kl}, \quad (\text{A4})$$

where $A_{n,ij}$ are defined in Eq. (2).

Note that post-Newtonian (PN) effects are not contained in the LPEs by default, but are typically incorporated by adding the corresponding PN terms to the perturbing Lagrangian, i.e., treating them as additional perturbative forces on the Newtonian orbit [64,65]. While we accounted for PN effects in the near-field terms of the BG binary in Eq. (2), but we do not add additional PN corrections here to the FG binary, which are generally subdominant,⁶ but we wish to explore this in future work. In addition, we neglect the center-of-mass momentum kick imparted to the FG binary by the passing gravitational wave [20], Eq. (32)], as the resulting velocity change is entirely negligible and has neither theoretical nor observational consequences.

3. Maintaining locking across zone boundary

Here, we demonstrate that the FG binary can remain locked as it crosses the near-zone/wave-zone boundary, i.e. a FG binary system initially in resonance in the near zone (where the dominant effect comes from the Newtonian quadrupole-quadrupole interaction) can remain locked as it moves into the wave zone. This is shown in Fig. 6, where locking begins at $D_\lambda = 0.5$, and ends at $D_\lambda \approx 1.84$. However, if we are not careful, the strong D^{-5} scaling of the Newtonian term can cause our $5n_f = n_b + kn_{\text{outer}}$ resonances to overlap for small D , invalidating our treatment. Due to this, as well as our motivation to study GW-induced resonances, we will restrict ourselves to the wave zone and leave near-zone resonances to future work.

4. Decoupling during resonant locking

A binary system with a given ι_f and ι_{outer} will decouple from resonance when there no longer exists a relative phase, $\Delta\phi_0$, such that Eq. (20) is satisfied. We saw in Fig. 3 that such a $\Delta\phi_0$ exists in all $(\iota_f, \iota_{\text{outer}})$ space when $D = \lambda$. Figure 7 shows $\max(\bar{n}_f/\bar{n}_b)$ for $D = 2\lambda$. There are small

⁶Note that in the early inspiral phase of the BG binary, the resonance condition together with the assumption of $M_b \gg M_f$ implies that $v_f/c = (M_f/M_b)^{1/3} v_b/c \ll v_b/c \ll 1$.

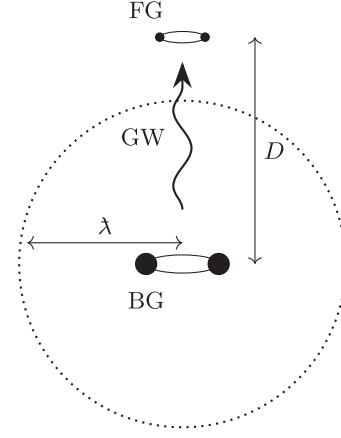


FIG. 5. Illustration of the setup. The dotted line represents the wave-zone/near-zone boundary. The BG binary is emitting GWs in all directions. Note that, despite the depiction, these GWs actually form near the wave-zone/near-zone boundary rather than inside the near zone.

regions of parameter space left that can sustain locking, but these also drop below unity by $D = 2.3\lambda$.

5. Galaxy model

We select the galaxy's SMBH mass, which is the BG binary's mass M_b , and model our galaxy accordingly. Specifically, we focus on FG binaries in the galactic bulge. We assume the distribution of FG binaries is spherically symmetric. We assume that their distance and period distribution functions, given by $f_D(D)$ and $f_T(T)$, respectively, are independent. We assume the radial matter

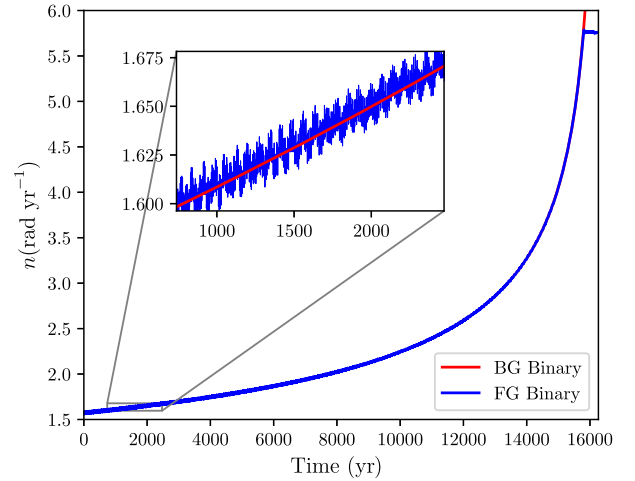


FIG. 6. We demonstrate sustained resonant locking as the FG binary crosses the near-zone/wave-zone boundary. Specifically, we numerically integrate Eq. (A3) with the general force [Eq. (2)] using initial conditions $\mathcal{M}_b = 8.71 \times 10^8 M_\odot$, $D_\lambda = 0.5$, $e_{b,0} = 0$, $\phi_{b,0} = 0$, $T_{b,0} = T_{f,0} = 4$ yrs, $\phi_{f,0} = 0.65\pi$, $e_{f,0} = 0.1$, $\iota_f = 0$, $\iota_{\text{outer}} = 0$ and all other orbital elements are initially set to zero. This gives $T_{\text{outer}} \approx 22.6$ yr. The FG binary's mean motion n_f remains trapped in resonance until decoupling at $T_f \approx 1.09$ yr.

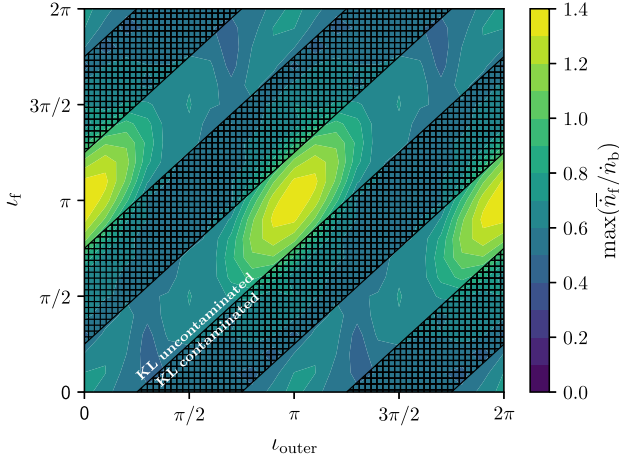


FIG. 7. Maximum value of $\tilde{n}_f/\dot{n}_b$ due to the BG binary when marginalizing over all relative phases as a function of the FG orbit and outer orbit inclinations for $D = 2\lambda$. The hatched regions represent the parameter space removed due to ZLK oscillations. At this distance, $\tilde{n}_f/\dot{n}_b$ does not exceed 1.4.

distribution in the bulge follows a Hernquist profile [66]

$$f_D(D) \propto 4\pi D^2 \frac{1}{\left(\frac{D}{D_0}\right)\left(1 + \frac{D}{D_0}\right)^3} \propto 4\pi D^2 \rho_D(D), \quad (\text{A5})$$

where $\rho_D(D)$ is the radial mass density distribution and $D_0 = R_e/1.8153$ is the bulge scale length with R_e being the half-light radius. When combining the R_e - M_{bulge} relation [67] with the M_{bulge} - M_{SMBH} relation [[53], Eq. (10)], one finds

$$D_0 = 3.1312 \times \left(\frac{M_b}{10^9 M_\odot}\right)^{0.748} \text{ kpc}. \quad (\text{A6})$$

Further, we normalize our distributions such that the total number of stars is $N_{\text{stars}} = M_{\text{bulge}}/\langle m_{\text{star}} \rangle$ where $\langle m_{\text{star}} \rangle = 0.3 M_\odot$ is the average star mass [58] and M_{bulge} comes from the M_{bulge} - M_{SMBH} relation [[53], Eq. (10)]

$$M_{\text{bulge}} = 1.8 \times 10^{11} M_\odot \left(\frac{M_b}{10^9 M_\odot}\right)^{0.85 \pm 0.06}. \quad (\text{A7})$$

For the period distribution, we use Öpik's law [68] which states that $f_T(T) = 1/(T \ln [T_{\text{max}}/T_{\text{min}}])$. T_{max} is given in Eq. (A17), while we choose $T_{\text{min}} = 10^{-5}$ days. Our choice for T_{min} is somewhat arbitrary, however the very weak logarithmic dependence ensures that its precise value has minimal impact on our results, even if changed by a few orders of magnitude.

6. Thrown into resonance by encounters

Here, we consider an encounter between the FG binary and a perturber of mass M_p , impact parameter b , and velocity v , taken as the bulge velocity dispersion from the M_{SMBH} - σ relation [Eq. (21)]. The interaction rate of such an encounter is

$$\Gamma = \pi b^2 n_{\text{stars}} v = \pi b^2 n_{\text{stars}} \sigma \quad (\text{A8})$$

where n_{stars} is the local stellar number density. We use the separatrix treatment presented in our simplified toy model to give an estimate of the chance for the binary system to get kicked into the separatrix by a random scattering encounter by a third object. This puts upper and lower limits on the size of the kick.

The separatrix height is [cf. Fig. 1; this follows from the equation of motion, Eq. (15)]

$$H_{\text{sep}} = \sqrt{4\sqrt{C_A^2 - C_F^2} + 4C_F \sin^{-1}\left(\frac{C_F}{C_A}\right) - 2\pi C_F}. \quad (\text{A9})$$

We consider two cases: (1) the pendulum is close to exact resonance, namely on its final oscillation before reaching $\dot{\theta} = 0$, in which case we take the size of the kick to be such that $H_{\text{sep}}/2 < \Delta n_f < 3H_{\text{sep}}/2$, and (2) the pendulum is far from resonance, in which case $|n_f - n_b| - H_{\text{sep}}/2 < |\Delta n_f| < |n_f - n_b| + H_{\text{sep}}/2$.

We label the lower and upper bounds as Δn_{lower} and Δn_{upper} , respectively. From Kepler's law $3\Delta E_f/E_f = 2\Delta n_f/n_f$ and hence, from Eq. (22), there exist impact parameters b_{min} and b_{max} such that Δn_f sits within the desired range. However, there lies the simultaneous condition that $(v/b)T < \gamma$ for work to be done [see text below Eq. (21)], and thus for the binary's frequency to change. Thus,

$$b_{\text{max}}(t) = \min \left[\left(\frac{6G^2 M_p^2}{\sigma^2 n_f^2 \left(\frac{|\Delta n_f|}{n_f}\right)_{\text{lower}}} \right)^{1/4}, \sigma T_f / \gamma \right], \quad (\text{A10})$$

$$b_{\text{min}}(t) = \min \left[\left(\frac{6G^2 M_p^2}{\sigma^2 n_f^2 \left(\frac{|\Delta n_f|}{n_f}\right)_{\text{upper}}} \right)^{1/4}, \sigma T_f / \gamma \right]. \quad (\text{A11})$$

From Eq. (A8), the interaction rate of such encounters is

$$\Gamma_{\text{into}} = \pi n_{\text{stars}} \sigma (b_{\text{max}}^2 - b_{\text{min}}^2). \quad (\text{A12})$$

7. Thrown out of resonance by encounters

Again, continuing the toy separatrix model, getting kicked out of resonance would require an encounter where $|\Delta n_f| < H_{\text{sep}}/2$. From this, we get a lower bound on the impact parameter of encounters that do not disrupt resonance

$$b_{\text{out}} = \min \left[\left(\frac{6G^2 M_p^2}{\sigma^2 n_f^2 \left(\frac{H_{\text{sep}}}{2n_f}\right)} \right)^{1/4}, \sigma T_f / \gamma \right], \quad (\text{A13})$$

where, from Eq. (A8), the corresponding interaction rate is

$$\Gamma_{\text{out}} = \pi n_{\text{stars}} \sigma b_{\text{out}}^2. \quad (\text{A14})$$

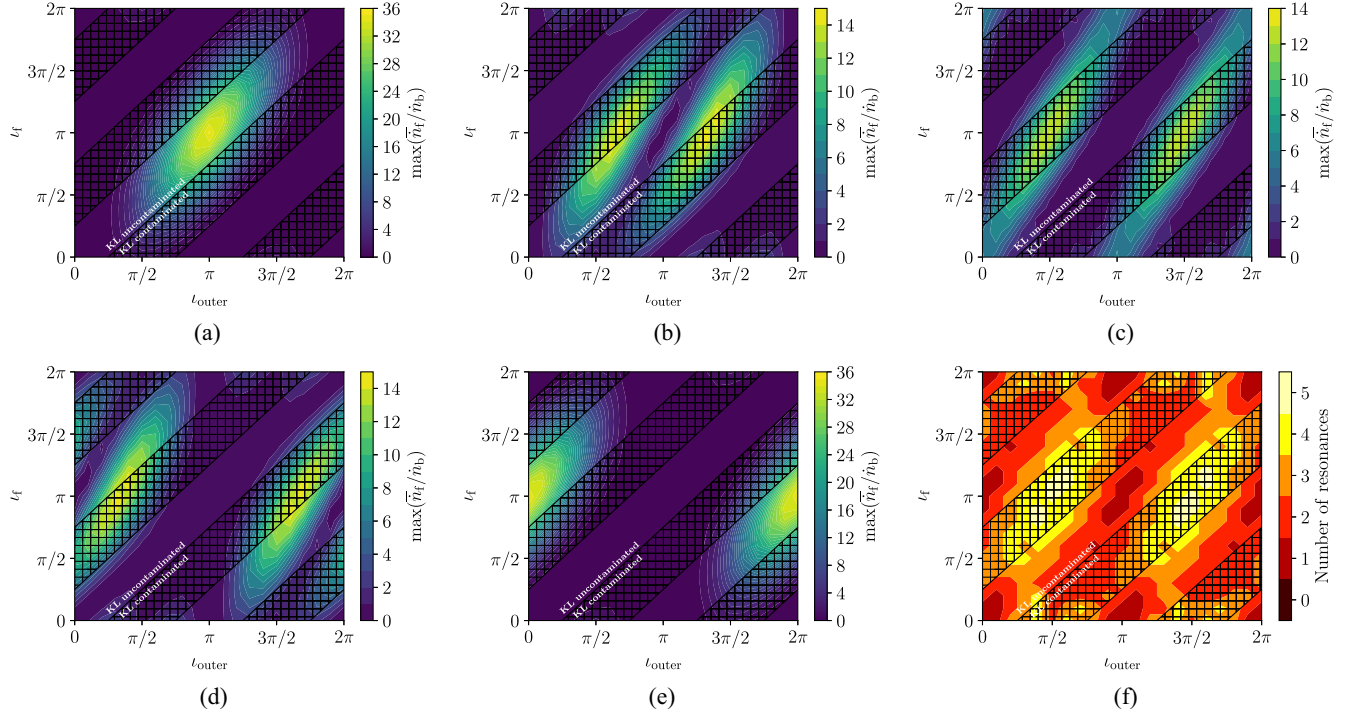


FIG. 8. Maximum value of $\tilde{n}_f/\dot{n}_b$ are shown in panels (a)–(e) for all five resonances, while panel (f) shows the total number of resonances where $\max(\tilde{n}_f/\dot{n}_b) > 1$. The hatched regions represent the parameter space removed due to ZLK oscillations. (a) $n_f = n_b + 2n_{\text{outer}}$. (b) $n_f = n_b + n_{\text{outer}}$. (c) $n_f = n_b$. (d) $n_f = n_b - n_{\text{outer}}$. (e) $n_f = n_b - 2n_{\text{outer}}$. (f) The number of resonances for which $\max(\tilde{n}_f/\dot{n}_b) > 1$.

8. Resonant locking for different resonances

Figure 8 shows $\max(\tilde{n}_f/\dot{n}_b)$ as a function of t_f and t_{outer} for all five resonances found in the general treatment, $n_f = n_b + kn_{\text{outer}}$ for $k \in \{\pm 2, \pm 1\}$. Figure 8(f) shows the number of these resonances that have $\max(\tilde{n}_f/\dot{n}_b) > 1$ as a function of t_f and t_{outer} .

As a means of demonstration, Fig. 9 shows resonant locking occurring for each of the four new resonances found in the general treatment, $n_f = n_b + kn_{\text{outer}}$ for $k \in \{\pm 2, \pm 1\}$ ($k = 0$ is shown in Fig. 2). In all cases, the initial conditions are set such that the system starts within the separatrix analogous to the orange region in Fig. 1.

9. Integration limits

As discussed above, we choose $T_{\text{min}} = 10^{-5}$ days. T_{max} comes from the condition that the FG binary's separation distance must be less than each component's Hill radius and that it must not be so soft that it would evaporate within a Hubble time, T_H , before the galaxy merger. The evaporation time of a binary with separation distance a in an environment with velocity dispersion σ and local mass density ρ_D is [[54], Eq. (7.173)]

$$t_{\text{evap}} \sim \frac{0.061\sigma}{G\rho_D a}. \quad (\text{A15})$$

We must have $t_{\text{evap}} > T_H$ for such a binary to still exist in our postmerger galaxy. Thus,

$$T_{\text{max}}(D) = \min \left[\frac{4\pi}{c} \sqrt{\frac{D^3}{6r_{b,s}}}, 2\pi \sqrt{\frac{1}{0.6GM_\odot} \left(\frac{0.061\sigma}{G\rho_D(D)T_H \ln \Lambda} \right)^3} \right]. \quad (\text{A16})$$

10. Number of locked binaries over the history of all mergers

This may be expressed as

$$N_{\text{ever}} = N_g \int_{M_{\text{min}}}^{M_{\text{max}}} N_{\text{locked}} \langle N_{\text{mergers}} \rangle f_{\text{SMBH}} dM_{\text{SMBH}} \quad (\text{A17})$$

where N_g is the number of galaxies in the observable Universe, $\langle N_{\text{mergers}} \rangle \equiv \langle N_{\text{mergers}}(M_{\text{SMBH}}) \rangle$ is the average number of mergers a galaxy with SMBH mass M_{SMBH} experiences in its lifetime, and f_{SMBH} is the SMBH mass distribution function. We conservatively take $\langle N_{\text{mergers}} \rangle = \mathcal{O}(1)$ to be unity.

The SMBH mass distribution function is found from the DM halo mass function. Assuming there is one DM halo per

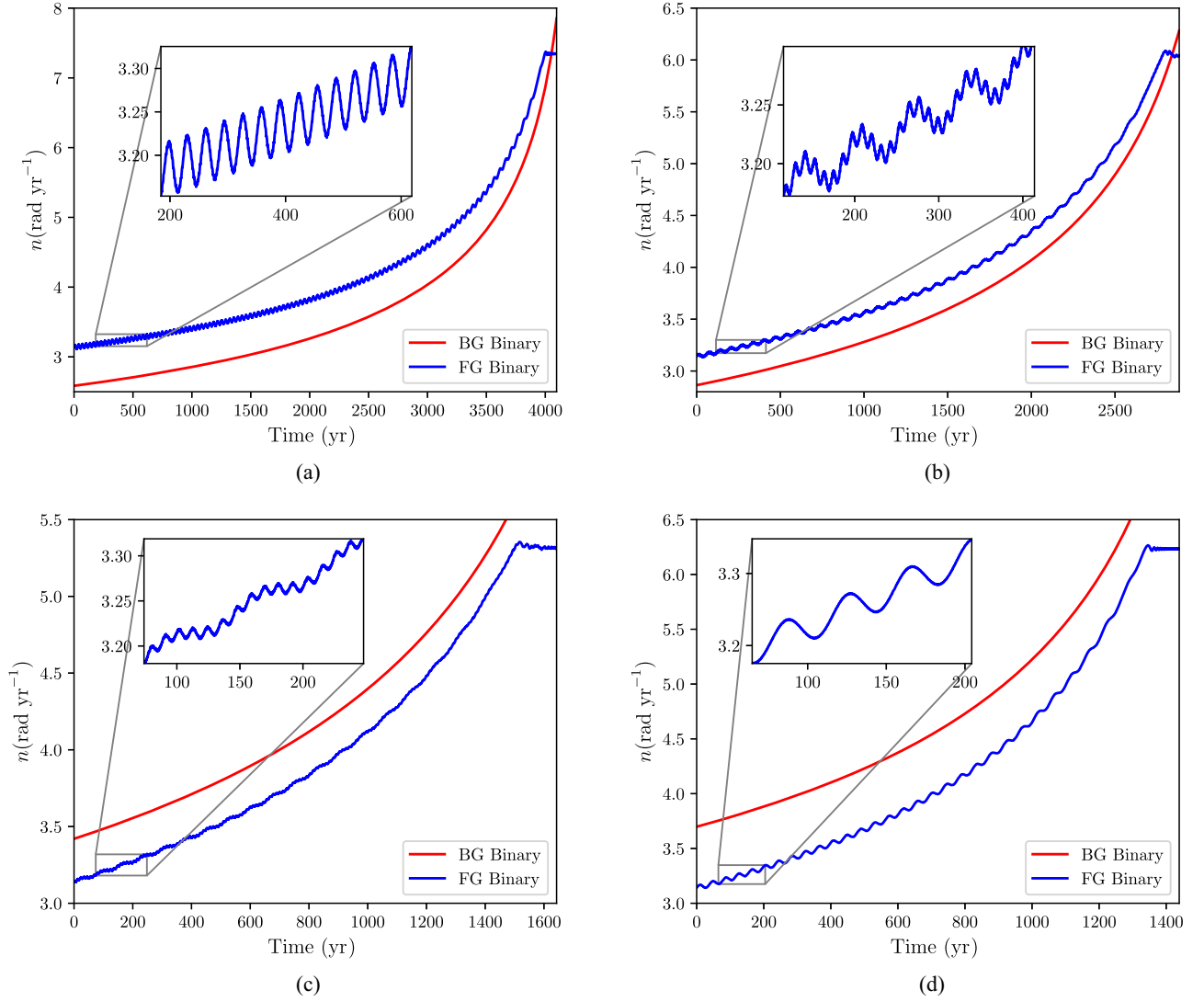


FIG. 9. Numerically integrating Eq. (A3) for resonances $n_f = n_b + kn_{\text{outer}}$ where $k = \{2, 1, -1, -2\}$ in panels (a)–(d), respectively. We have set $\mathcal{M}_b = 8.71 \times 10^8 M_\odot$, $D_\lambda = 1$, $e_{b,0} = 0$, $\phi_{b,0} = 0$, $T_{b,0} = T_{f,0} = 2$ yrs, $e_{f,0} = 0.1$ initially, while all other unspecified orbital elements were initially set to zero. (a) $n_f = n_b + 2n_{\text{outer}}$, $\iota_f = \pi$, $\iota_{\text{outer}} = \pi$. (b) $n_f = n_b + n_{\text{outer}}$, $\iota_f = \pi/2$, $\iota_{\text{outer}} = \pi/2$. (c) $n_f = n_b - n_{\text{outer}}$, $\iota_f = \pi/2$, $\iota_{\text{outer}} = 3\pi/2$. (d) $n_f = n_b - 2n_{\text{outer}}$, $\iota_f = \pi$, $\iota_{\text{outer}} = 0$.

SMBH, then using $f_{\text{SMBH}} = f_{\text{DM}} dM_{\text{DM}}/dM_{\text{SMBH}}$, with $f_{\text{DM}} \propto M_{\text{DM}}^{-1.806}$ [[53] Fig. 25(a)] (we never reach the exponential truncation of the mass function) and $dM_{\text{DM}}/dM_{\text{SMBH}}$ from Ref. [53], Eq. (18), we get

$$f_{\text{SMBH}} = f_{\text{M},0} \begin{cases} M_{\text{SMBH}}^{-1.298}, & \text{if } M_{\text{SMBH}} < M_{\text{kink}} \\ \frac{M_{\text{kink}}^{2.072}}{M_{\text{SMBH}}^{3.370}}, & \text{if } M_{\text{SMBH}} > M_{\text{kink}} \end{cases} \quad (\text{A18})$$

where $M_{\text{kink}} = 1.2 \times 10^8 M_\odot$ and $f_{\text{M},0}$ is the normalization constant. Assuming the smallest SMBH mass is $10^6 M_\odot$ and the largest is $10^{11} M_\odot$, we normalize the distribution and integrate Eq. (A17). Assuming $N_g = 2 \times 10^{12}$ [69], we find that

$$N_{\text{ever}} \approx 7.2 \times 10^9. \quad (\text{A19})$$

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