

Equation of state and surface tension of QCD in the first order phase transition region

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We build up a complete description of QCD phase structure by applying the parametrization of the chiral and deconfinement order parameters upon the calculations from functional QCD approaches. In particular in the first order phase transition region at high chemical potential, both the phase transition line using Maxwell construction and the coexistence boundary lines from the spinodal decomposition are determined. We compute the thermodynamic quantities including the number density, the energy density, the pressure, and also the free energy for both stable and unstable phases of QCD. Additionally, after applying a phenomenological description of the inhomogeneity of the QCD free energy, we obtain the surface tension of the first order phase transition of QCD.

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I. INTRODUCTION

The phase structure and thermodynamics of strong interaction matter at finite temperature T and baryon chemical potential μ_B is of great importance for understanding quantum chromodynamics (QCD). From the theoretical side, it has been extensively studied via effective models and extrapolation methods [1–12], and first principle QCD approaches like lattice QCD simulation [13–15], functional QCD (fQCD) methods including the Dyson-Schwinger equations [16–22] and functional Renormalization Group (fRG) approach [23–25]. One of the main goals of these theoretical studies is to explain the experimental data from the Beam Energy Scan at the Relativistic Heavy-Ion Collider [26–32]. This will also need the assistance of nonequilibrium physics like hydrodynamics simulations and dynamical evolution equations, where the equation of state of QCD serves as an essential input [33–39]. Moreover, the description of QCD phase transition, as well as the phase transition in other QCD-like theories, is also important in cosmological studies as the cosmic first

order phase transition can be related to the gravitational wave (GW) production through the thermodynamic quantities, particularly the trace anomaly and the surface tension of the system are required [40–46]. The quantitative description is required for further constraining the potential role of such phase transitions in the GW production.

A direct precision calculation of the QCD thermodynamic quantities has been extended to the chemical potential region up to $\mu_B/T \lesssim 4$ [47]. This region remains the crossover region as the same calculation also suggests the critical end point is located at $\mu_B/T \sim 5$ –6. However, for thermodynamic quantities in the first order phase transition region, there is still lack of precision calculations from first principle QCD approaches. Alternatively, one may utilize the Ising parametrization to study the QCD thermodynamics by mapping QCD system to Ising model [48–51]. One may apply the Ising mapping to map the order parameters of QCD instead of the thermodynamic quantities, and after that, one can incorporate the parametrized order parameters into the quark propagator to calculate the QCD thermodynamic quantities [50]. This method has provided a quantitatively accurate description of the QCD thermodynamics for low chemical potential, and has also enabled the description of Maxwell construction in the first order phase transition region. Here in this article, we extend this method to provide a complete description of the first order phase transition, incorporating the application of unstable phase description from Ising model as introduced in Ref. [51]. The obtained trace anomaly and surface tension of QCD can open access

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for the precision phenomenological studies related to QCD first order phase transition both in hydrodynamics simulations and the cosmological studies.

The article is organized as follows: In Sec. II, we present the framework of the parametrization of order parameters and the construction of thermodynamic quantities. In Sec. III, we show the numerical results of the constructed QCD phase diagram including the phase transition line and the coexistence boundary line, and then the respective QCD thermodynamic quantities, in particular, the surface tension in the first order phase transition region. In Sec. IV, we summarize the main results and make a few discussions.

II. THE GENERAL METHOD

In functional QCD approaches, the quark propagator is the central element in determining the thermodynamic quantities. The chiral and deconfinement order parameters are directly linked to the properties of the propagator. As a result, parametrizing the order parameters and integrating them into the quark propagator offers a clear and detailed microscopic description of the system. Additionally, this parametrization preserves higher-order information, even when a mean-field ansatz is applied, as it effectively encapsulates the nonperturbative properties of QCD's correlation functions.

A. From order parameters to QCD thermodynamics

Generally speaking, the QCD phase transitions are characterized by two order parameters, the chiral and the deconfinement phase transitions correspond to the chiral condensate $\langle \bar{q}q \rangle$ and the Polyakov loop Φ , respectively. Microscopically, the two order parameters are embedded in the Green functions, i.e., the full quark propagator (inverse) in the language of functional approaches,

$$S_q^{-1}(p) = i(\tilde{\omega}_n + gA_4)\gamma_4 Z_q^E(\mathbf{p}, \tilde{\omega}_n) + i\boldsymbol{\gamma} \cdot \mathbf{p} Z_q^M(\mathbf{p}, \tilde{\omega}_n) + Z_q^E(\mathbf{p}, \tilde{\omega}_n) M_q(\mathbf{p}, \tilde{\omega}_n), \quad (1)$$

where $\tilde{\omega}_n = \omega_n + i\mu_q$ with ω_n the Matsubara frequencies, μ_q the quark chemical potential, and $\mathbf{p}$ the spacial momentum, along with Z_q^E , Z_q^M , and M_q the dressing functions and A_4 the gluonic background field condensate, which is related to the Polyakov loop L as [52,53],

$$L[A_4] = \frac{1}{N_c} \text{tr} \mathcal{P} e^{i \int_0^\beta dx_4 A_4}, \quad (2)$$

with $\beta = 1/T$. The quantities in the quark propagator, M_q and A_4 , can be considered as the order parameters of chiral and deconfinement phase transition, respectively.

We then applied a further approximation on the dressing functions at each (T, μ_q) in Eq. (1),

$$Z_q^{E,M}(\mathbf{p}, \tilde{\omega}_n; T, \mu_q) \rightarrow 1, \\ M_q(\mathbf{p}, \tilde{\omega}_n; T, \mu_q) \rightarrow M_q(0; T, \mu_q). \quad (3)$$

This then leads to a simple expression of quark propagator as,

$$S_q^{-1}(p) = i[\tilde{\omega}_n + gA_4(T, \mu_q)]\gamma_4 + i\boldsymbol{\gamma} \cdot \mathbf{p} + M_q(T, \mu_q), \quad (4)$$

in the expression we have left out the momentum index in M_q and labeled the T and μ_q dependence in A_4 and M_q explicitly. Note that this approximation is just to take the quark mass in the infrared limit, which generally contains the main information of QCD in the nonperturbative regime. We emphasize that for a direct calculation from functional QCD methods, one will need to acquire the correct momentum dependence in the quark mass function and wave function, and only after this, one can describe the QCD phase transition. However, here we do not aim at solving the quark mass and wave function, instead, we only need the correct T and μ_B dependence of the QCD thermodynamic quantities. Taking the infrared limit will not change the T and μ_B dependence of the order parameters, and consequently, it can still give a good description of QCD thermodynamics. This has been previously verified as the obtained thermodynamic quantities via this approximation are in good agreement with the results from lattice QCD simulations [50].

Such an approximation allows one to give an analytic expression of number density n_q in terms of the two order parameters of QCD as,

$$n_q(T, \mu_q) = -T \sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3} \text{tr}_{C,D} [\gamma_4 S_q(\mathbf{p}, \omega_n)] \\ = 2N_c \int \frac{d^3\mathbf{k}}{(2\pi)^3} [f_q^+(\mathbf{k}; T, \mu_q) - f_q^-(\mathbf{k}; T, \mu_q)], \quad (5)$$

with

$$f_q^\pm = \frac{L(T, \mu_q)x_\pm^2 + 2L(T, \mu_q)x_\pm + 1}{x_\pm^3 + 3L(T, \mu_q)x_\pm^2 + 3L(T, \mu_q)x_\pm + 1}, \\ x_\pm(\mathbf{k}; T, \mu_q) = \exp[(E_q(\mathbf{k}; T, \mu_q) \mp \mu_q)/T], \\ E_q(\mathbf{k}; T, \mu_q) = \sqrt{\mathbf{k}^2 + M_q^2(T, \mu_q)}. \quad (6)$$

With the knowledge of the two order parameters M_q and L at each temperature and chemical potential, the number density can be obtained. The pressure is then calculated together with $P(T, 0)$ from lattice QCD simulation via the relation [40,54,55],

$$P(T, \mu_B) = P(T, 0) + \sum_q \int_0^{\mu_q} d\mu_q n_q(T, \mu). \quad (7)$$

The other thermodynamic quantities can be further calculated.

B. The parametrization of the chiral and deconfinement order parameters

As mentioned above, a direct computation with precision for M_q and L has been conducted within the functional QCD method in the chemical potential region up to $\mu_B/T \lesssim 4$ [47]. Together with the results from the lattice QCD simulation, there exists some instructions for the parametrization of M_q and L . First of all, one expects the QCD chiral phase transition line for a wide range of chemical potential can be parametrized as,

$$\frac{T_c(\mu_B = 3\mu_q)}{T_c(0)} = 1 - \kappa \left(\frac{3\mu_q}{T_c(0)} \right)^2 - \kappa_2 \left(\frac{3\mu_q}{T_c(0)} \right)^4 + \dots \quad (8)$$

For the $(2+1)$ -flavor case, the consistent results (central averages) are $T_c(0) = 155$ MeV, $\kappa = 0.016$, and $\kappa_2 = 3.5 \times 10^{-4}$ [13–15, 21, 23]. Note that the validity of the parametrization beyond $\mu_B \approx 700$ MeV is lack of evidence from direct calculations [21–23, 56]. Nevertheless, there are several hints that the phase transition line should follow this parametrization, for instance, at zero temperature, the liquid gas transition at $\mu_B \approx 930$ MeV gives a constraint on the end point of the line [57]. The study of the thermodynamic quantities also suggests that the first order phase transition line should follow a quadratic behavior of μ_B [58]. Here we will then extend this parametrization to the whole temperature region, and adjust the parameters to yield the phase transition at zero temperature at $\mu_B = 930$ MeV.

Moreover, the functional QCD approaches also give the estimation of the location of the critical end point (CEP) at about $\mu_B^{\text{CEP}} \approx 600$ to 650 MeV [21–23, 47, 56]. Here we apply the CEP at

$$(T^{\text{CEP}}, \mu_B^{\text{CEP}}) = (106, 600) \text{ MeV}.$$

This gives a general constraint on the order parameters of QCD phase transitions.

Now for the order parameter of chiral phase transition, M_q , it is found that the functional QCD data [21, 23] fits well with the feature of the 3D Ising parametrization [48, 49]. Note that in principle, any parametrization works equivalently if it matches with the data from direct computation, here for simplicity, we choose such a 3D Ising parametrization. In the Ising parametrization, one can parametrize M_q as,

$$M_q(T, \mu_q) = \frac{M_0}{2} [1 - M_{\text{Ising}}(T, \mu_q)]. \quad (9)$$

$M_0 = 350$ MeV is chosen as the typical mass scale for the light-flavor quarks suggested by the lattice [59, 60] and functional QCD [23, 61]. M_{Ising} is parametrized with the order parameter potential as,

$$F(M_{\text{Ising}}, r, h) = -hM_{\text{Ising}} + \frac{1}{2}rM_{\text{Ising}}^2 + \frac{\lambda}{6}M_{\text{Ising}}^6, \quad (10)$$

where $r = (T - T_c)/T_c$ is the reduced temperature and h is the external magnetic field. Note that the potential is different from the Ising mean field potential, as the fQCD computation suggests that for the effective potential of order parameter ϕ (here is simply the quark mass $\phi = M_q$), the ϕ^6 term is dominant near the QCD phase transition region [62]. This ϕ^6 dominance is natural in a 3D field theory, as the unit of the field in 3d is the square root of the energy scale that is determined by the kinetic term, and hence, ϕ^6 becomes relevant following the instruction of Wilsonian's renormalization group. Such a potential could describe both the second order and first order phase transition including the spinodal line that gives the coexistence boundary as proposed in Ref. [51], and as will be illustrated below, the parameter λ is the control parameter for the range of the coexistence region.

The order parameter $M_{\text{Ising}}(r, h)$ can be obtained from the potential via,

$$\frac{\partial F}{\partial M_{\text{Ising}}} = -h + rM_{\text{Ising}} + \lambda M_{\text{Ising}}^5 = 0. \quad (11)$$

In order to match the phase transition line Eq. (8) precisely, one can take a nonlinear mapping from r, h to T, μ_B as,

$$\begin{aligned} \frac{3\mu_q - \mu_B^{\text{CEP}}}{\mu_B^{\text{CEP}}} &= -r\omega\rho \cos \alpha_1 - h\omega \cos \alpha_2, \\ \frac{T - T^{\text{CEP}}}{T^{\text{CEP}}} &= f_{PT}(r) + h\omega \sin \alpha_2, \end{aligned} \quad (12)$$

where

$$\begin{aligned} \alpha_1 &= \arctan \left[2\kappa \left(\frac{\mu_B^{\text{CEP}}}{T_c(0)} \right) + 4\kappa_2 \left(\frac{\mu_B^{\text{CEP}}}{T_c(0)} \right)^3 \right], \\ \alpha_2 &= \alpha_1 + \frac{\pi}{2}, \quad \omega = 1, \quad \rho = 2. \end{aligned} \quad (13)$$

The function $f_{PT}(r)$ is chosen such that the mapping match the chiral phase transition line. Hence,

$$\begin{aligned} f_{PT}(r) &= \frac{T_c(0)}{T^{\text{CEP}}} \left[1 - \kappa \left(\frac{\mu_B^{\text{CEP}}}{T_c(0)} \right)^2 (1 - r\omega\rho \cos \alpha_1)^2 \right. \\ &\quad \left. - \kappa_2 \left(\frac{\mu_B^{\text{CEP}}}{T_c(0)} \right)^4 (1 - r\omega\rho \cos \alpha_1)^4 \right] - 1. \end{aligned} \quad (14)$$

For the Polyakov loop expectation value, we take the fRG result at zero chemical potential from Ref. [63], which

is denoted as $\Phi(T, 0) = L(t)$ with $t = T/T_c(0)$. The following fit function is taken here:

$$L(t) = 2 \left[1 + \exp \left(\frac{1 + l_1 t^3}{m_1 t + m_2 t^6} \right) \right]^{-1}, \quad (15)$$

with $l_1 = 2.732$, $m_1 = 0.5495$ and $m_2 = 1.831$. For the Polyakov loop at finite chemical potential, we consider it along with the same phase transition line as the chiral phase transition and thus we have,

$$\Phi(T, \mu_q) = L(t_\Phi),$$

$$t_\Phi = \frac{T}{T_c(0)} + \kappa \left(\frac{\mu_B}{T_c(0)} \right)^2 + \kappa_2 \left(\frac{\mu_B}{T_c(0)} \right)^4. \quad (16)$$

To sum up, the above parametrization of the T and μ_B dependence of the M_q and L opens an access for analyzing the QCD phase structure and thermodynamic quantities, including the first order phase transition region. Note that here the parametrization of the Polyakov loop is relatively crude, however, based on the fQCD computation [47], the Polyakov loop contributes less to the lower order of the QCD thermodynamic quantities, such as pressure, number density, and energy density. A self-consistent calculation is required to investigate higher order fluctuations and also other detailed information, which will be pursued in future studies.

III. NUMERICAL RESULTS

In this Section, we present the numerical results based on the above parametrization of $M_q(T, \mu_B)$ and $L(T, \mu_B)$. We give the chiral phase transition of QCD including the phase transition line and the coexistence boundary. We also compute the thermodynamic quantities of QCD. Specifically, we calculate the pressure and the free energy in the first order phase transition, which then enables one to perform the Maxwell construction, and analyze the supercooling/superheating phase. Furthermore, by incorporating a phenomenological description of the inhomogeneity of QCD, we determine the surface tension during the QCD first order phase transition.

A. QCD phase diagram

Here we focus on the chiral phase transition, which dominates the thermodynamic properties of QCD for temperatures below approximately 300 MeV. In the parametrization of the order parameter, the phase transition line has already been incorporated. Therefore, the phase transition line can be directly read off from the chiral order parameter M_q , which its temperature and chemical potential dependence is depicted in Fig. 1. At low chemical potential, M_q behaves as a monotonous function, while as the chemical potential increases, the slope of the M_q

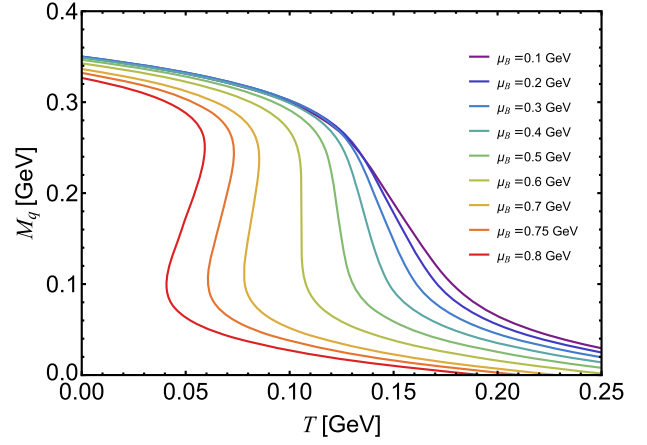


FIG. 1. Temperature dependence of the order parameter of chiral phase transition at different chemical potential with the parameter in Ising potential being taken as $\lambda = 1$.

function becomes steeper. At the CEP location, the slope becomes divergent.

For larger chemical potential that the system steps into the first order phase transition region, the function M_q becomes multivalued, leading to the bend over behavior in Fig. 1. For each temperature, the multivalued function, in specific, has three values in the first order phase transition, which corresponds to three phases: the stable and the metastable phase, i.e., the chiral symmetry breaking phase (Nambu phase) and the chiral symmetry preserved phase (Wigner phase), and the unstable phase located in between the stable and metastable phase. The upper and lower bounds of the chemical potential give the boundaries of the spinodal decomposition, thereby defining the coexistence region where the stable and metastable phases can coexist. The respective phase diagram is depicted in Fig. 2.

As mentioned above, the range of coexistence region depends on the parameter λ . As the value of λ decreases, the coexistence region becomes broader. A further analysis for

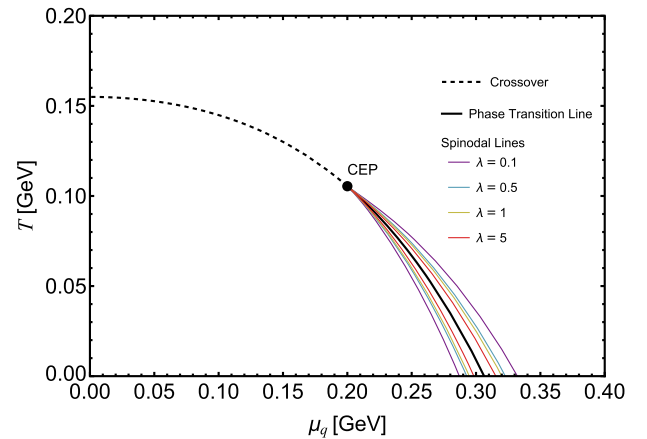


FIG. 2. Phase diagram of chiral phase transition including the phase transition line and the spinodal lines.

determining the parameter requires additional information from the calculation of functional QCD approaches in first order phase transition region, here we will adopt $\lambda = 1$ to calculate the thermodynamic quantities. Note that the phase transition line in the first order phase transition is directly from the parametrization of Eq. (8), instead of being determined from the thermodynamic quantities, which requires the Maxwell construction from the pressure. This will be illustrated below, which is in accordance with the direct parametrization. Such a phase transition line is the ideal case of phase transition, with the pressures of the stable phase and the metastable phase being equal. If considering the system evolves from high temperature to low temperature, the phase transition can happen in the coexistence region at a lower temperature after the evolution crosses the phase transition line, this is the so-called supercooling phenomenon. Similarly for the opposite evolution direction, the superheating phenomenon can take place. As the spinodal lines define the boundary of the coexistence region, they are also the lower and upper bound of the supercooling/heating phenomenon, respectively.

B. QCD thermodynamic quantities

After employing Eq. (5) together with the parametrization of two order parameters in Eqs. (9) and (16), one can directly calculate the quark number density in the temperature-chemical potential plane as depicted in Fig. 3. For large temperatures that correspond to the crossover region, the number density increases monotonously as the chemical potential becomes larger. When the temperature is below the CEP temperature, a first order phase transition occurs at a specific chemical potential. Similar to the behavior observed in M_q , here the complete parametrization that includes the metastable and unstable phases results in a bend-over behavior in the number density. This multivalued region defines the coexistence region of the first order phase transition and the boundaries represent the spinodal lines.

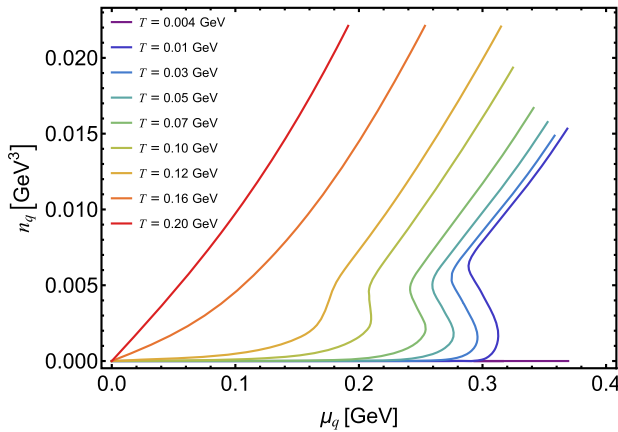


FIG. 3. Number density as a function of μ_B at several temperatures, calculated from the quark propagator as in Eq. (5).

To determine the phase transition line, one can compute the pressure upon the integration of number density, and apply the Maxwell construction, which yields the pressure of Nambu and Wigner phases to be equal at a certain chemical potential, i.e., $P_N(\mu_q) = P_W(\mu_q)$. Note that since the number density is multivalued in μ_q , the integral respective to μ_q is ill defined, one can apply the intrinsic coordinate on the trajectory of the number density. The number density and μ_q can be then defined with the intrinsic coordinate as $n_q(T, s)$ and $\mu_q(s)$. The difference of the pressure P_q between finite μ_q and $\mu_q = 0$, which is the integral that follows the trajectory of function n_q can be defined along with s as,

$$P_q(T, \mu) = \sum_q \int_0^{\mu_q} n_q(T, s) \frac{\partial \mu_q(s)}{\partial s} ds. \quad (17)$$

The pressure of the Nambu and Wigner phase, as well as the unstable phase, can be calculated in our parametrization. The pressure as a function of chemical potential at different temperatures has been depicted in Fig. 4. In the first order phase transition, the pressure is also multivalued. In general, the pressure of the stable phase is larger than that of the metastable phase, and the Nambu and Wigner phase intersect at one specific chemical potential, which precisely corresponds to the phase transition point under the Maxwell construction.

In Fig. 5, we illustrate the details of the Maxwell construction. When the Nambu and Wigner phase exceed the phase transition chemical potential, they become metastable phases, representing the superheating/cooling phenomenon. The line that connects the two phases is the pressure of unstable phase. The instability of this phase can be read off from its slope that corresponds to a negative susceptibility. This is further illustrated clearly in the plot of number density in the right panel of Fig. 5. Besides, the dotted line is the phase transition line under the Maxwell

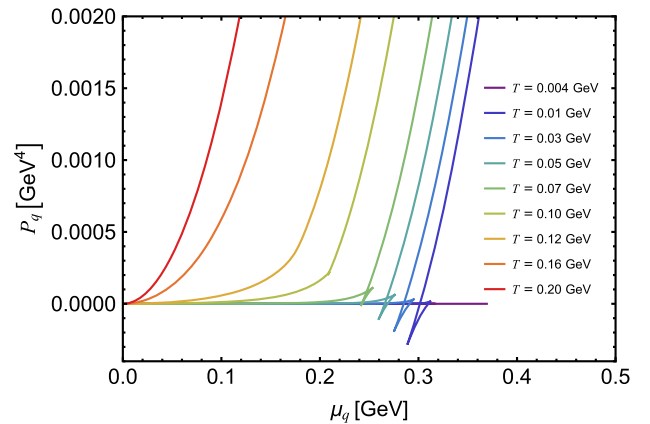


FIG. 4. Pressure under the consideration of the metastable and unstable phase, calculated from Eq. (19).

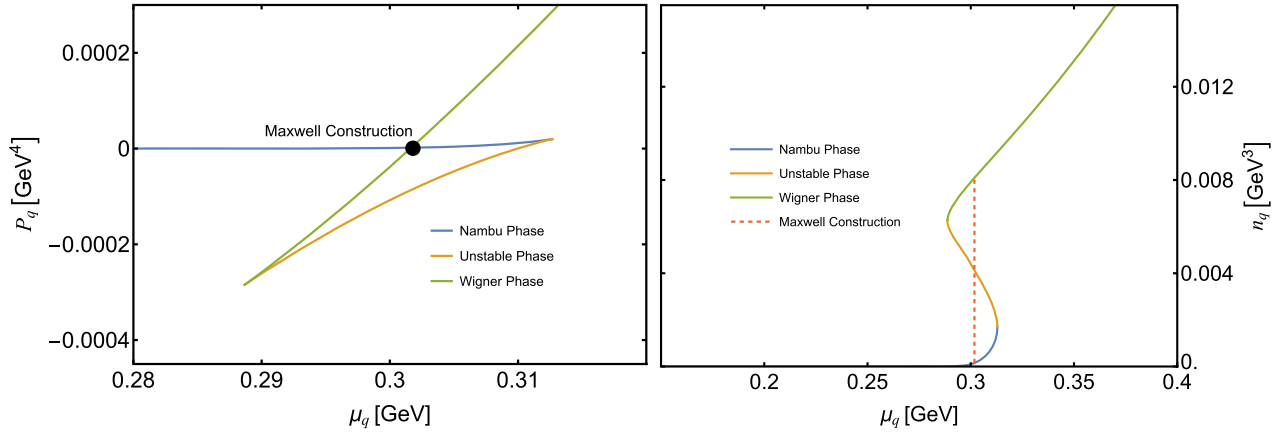


FIG. 5. Maxwell construction of the pressure at $T = 0.01$ GeV (left panel) and the respective number densities and the phase transition chemical potential (right panel).

construction with the ending point value being the corresponding number density for the Nambu and Wigner phase.

After having the pressure P_q , one can also calculate the trace anomaly difference between two phases in the coexistence region. The trace anomaly difference denoting as $\Delta\theta$ can be defined as,

$$\begin{aligned}\Delta\theta &= \Delta P_q - \frac{1}{4}T\Delta s_q - \frac{1}{4}\mu_q\Delta n_q \\ &= -\frac{1}{4}T^5 \frac{\partial \Delta P_q / T^4}{\partial T} - \frac{1}{4}\mu_q \frac{\partial \Delta P_q}{\partial \mu_q},\end{aligned}\quad (18)$$

with Δs_q the quark entropy density difference, and Δn_q the number density difference. The results of trace anomaly are depicted in Fig. 6. As the temperature decreases, the trace anomaly difference becomes larger, which is due to the increasing chemical potential and the number density.

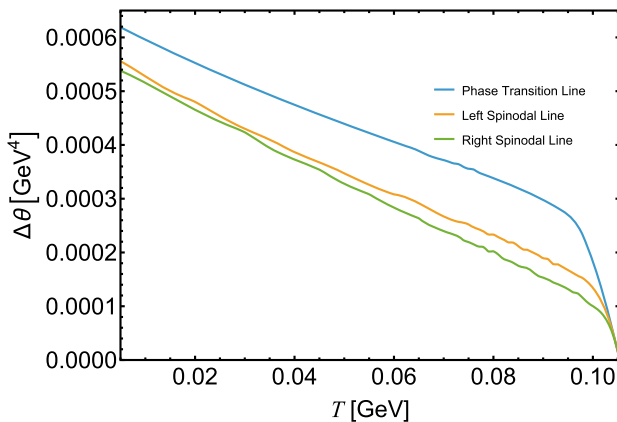


FIG. 6. Trace anomaly for one flavor quark along the phase transition line and also the spinodal lines.

C. Free energy and surface tension

One can also calculate the free energy of the system using

$$F_T(T, n_q) = F_T(T, 0) + \int_0^{n_q} \mu_q(T, n_q) dn_q. \quad (19)$$

It is more important to see the difference between the free energy of the system and the free energy under the Maxwell construction, which can be defined as,

$$\Delta F_T(T, n_q) = F_T(T, n_q) - F_T^M(n_q), \quad (20)$$

where the Maxwell construction $F_T^M(n_q)$ is the linear construction of free energy as,

$$F_T^M(n_q) = F_T(n_N) + \frac{F_T(n_N) - F_T(n_W)}{n_N - n_W} (n_q - n_N). \quad (21)$$

The slope of the above Maxwell construction is precisely the chemical potential at phase transition. The corresponding results for difference free energy at several temperatures are depicted in Fig. 7.

The results of the difference free energy can be further incorporated to calculate the surface tension of the system. A phenomenological inhomogeneous free energy density for the first order phase transition system has been proposed [64], which includes a gradient term to characterize the space distribution of the number density. The gradient term serves as the kinetic term of the system and determines the distribution of number density together with the free energy as the potential, and we have

$$F_T(\mathbf{r}) = n_q \mu_q + \frac{1}{2} C (\nabla n_q)^2, \quad (22)$$

where $C = \frac{a^2}{n_B^2} E_g$ with $n_B = \frac{1}{3} n_q^{\text{CEP}} = 0.19 \text{ fm}^{-3}$ and $E_g = 0.572 \text{ GeV} \cdot \text{fm}^{-3}$ to be the baryon number density and the

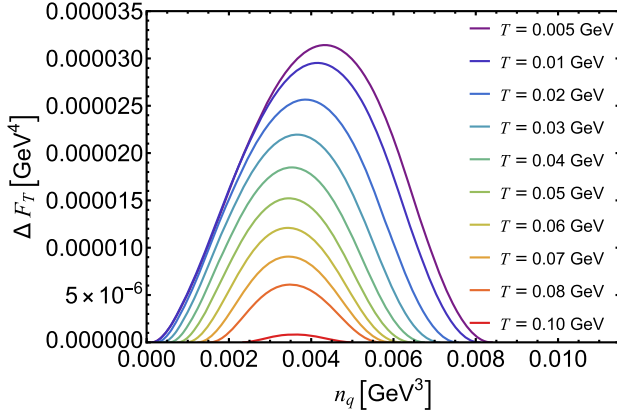


FIG. 7. Deviation of free energy for one flavor quark from the Maxwell construction at the phase transition points.

energy density at the CEP. For the parameter α , the measure of the interface's thickness, we choose it to be 0.33 fm as that in Refs. [65,66].

By comparison with Eq. (20), one immediately gets,

$$\Delta F_T - \frac{1}{2} C \left(\frac{\partial n_q}{\partial r} \right)^2 = 0, \quad (23)$$

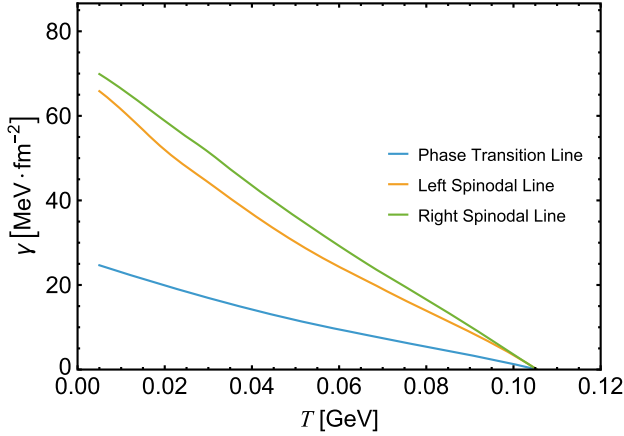


FIG. 8. Surface tension for one flavor quark along the phase transition line and also the spinodal lines.

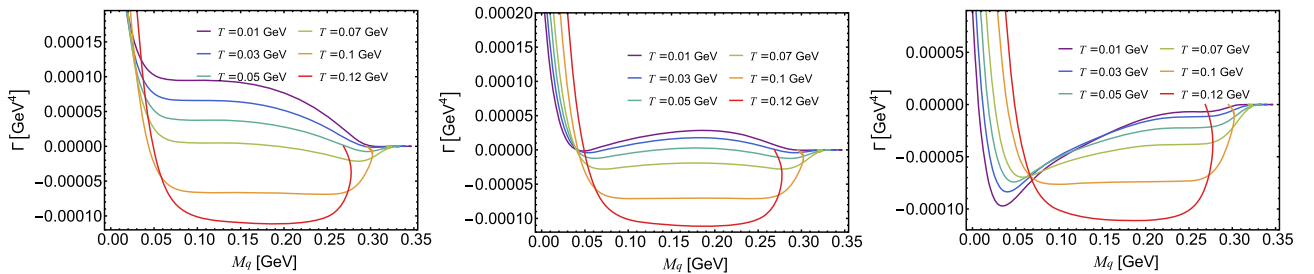


FIG. 9. Effective potential of Chiral phase transition as the function of the order parameter $\phi = M_q$ at the phase transition point and also the spinodal boundaries. At each temperature, the corresponding chemical potential in three figures increases from left to right.

with the spherical symmetry being adopted. The radial distribution of the number density and also the free energy can be calculated. The interface tension is then expressed as,

$$\gamma(T) = \int_0^\infty \Delta F_T(r) dr = \int_{n_N}^{n_W} \sqrt{\frac{C}{2} \Delta F_T(n_q)} dn_q, \quad (24)$$

where n_N and n_W are number densities of the coexisting Nambu and Wigner phase. The corresponding results of surface tension are depicted in Fig. 8. Note that such a calculation can be done in the whole coexistence region. In detail, the surface tension on the phase transition line gives a lower bound, while at the spinodal line, the surface tension is larger. A larger surface tension leads to a quicker phase transition, which is compatible with the super-cooling/heating phenomenon.

Moreover, the effective potential of QCD as a function of the order parameter of chiral phase transition $\phi = M_q$ can be defined by utilizing the relation between n_q and M_q . The effective potential $\Gamma(\phi)$ is expressed as,

$$\Gamma(\phi = M_q; T, \mu_q) = F_T(T, n_q(M_q)) - \mu_q n_q(T, M_q). \quad (25)$$

The results of the effective potential are depicted in Fig. 9. Below the CEP temperature, the effective potential gives the typical “double-well” structure as a signal of the first order phase transition. At the phase transition point under the Maxwell construction, the two minima become equal. On the spinodal boundaries, the minimum point for the metastable phase becomes a saddle point. The effective potential of the order parameter can be further applied for studying the dynamical evolution of the nucleation and also the spinodal decomposition process during the QCD phase transition.

It is worth mentioning that the current calculations can be directly applied to calculate the GW spectral. In particular, the parameter α can be directly calculated from the trace anomaly between two phases, as $\alpha \sim \theta$. The trace anomaly depends on the flavor, for example, a factor two should be involved as in the conventional QCD phase transition, the up and down quarks experience the QCD phase transition simultaneously. The parameter β is related

to the dynamics of phase transition, which is in principle determined by the order parameter potential in Fig. 9. Such a determination is beyond the scope of this work, however, the parameter β can be estimated via the surface tension as in Ref. [46] with the relation between β and the surface tension as $\beta \sim \gamma^{-3/2}$.

We would also like to emphasize that the GW signals obtained from the QCD phase transition in the standard model, are very unlikely to be detected [43,44,46]. In specific, the surface tension of QCD is too small, which then gives a large value of β and cannot generate the strong enough GW signals. However, if incorporating the scenario beyond the standard model, either by a dark QCD sector, or in some scenarios, the quark mass and the coupling strength can be altered, the situation will be different. Our frame here can be easily adopted to study these QCD-like scenarios, by adjusting the phase transition line and CEP location, as well as the parametrization of order parameters, and hence, we offer a platform for calculating the GW signals from the cosmic first order phase transition of QCD-like theories.

IV. SUMMARY

We develop a unified framework to describe the QCD thermodynamics upon the numerical results from lattice QCD simulations and the functional QCD approaches. The key is to retreat the quark mass and the gluonic background condensate as the order parameters of chiral and deconfinement order parameters, respectively. On one hand, these two elements can be calculated directly through the functional QCD approaches and easily parametrized. On the other hand, QCD thermodynamic quantities are microscopically encoded in these elements through the quark

propagator, providing a bridge between first principle calculations and broader phenomenology studies.

Using this framework, we calculate the QCD thermodynamic quantities encompassing stable, metastable phase and also the unstable phase. This first completes the phase diagram of QCD in particular in the first order phase transition. The phase transition line can be determined through the Maxwell construction, and the coexistence region with the spinodal lines are also determined.

A more interesting extension is the calculation of the free energy of the system and its deviation from that of the Maxwell construction. This provides insights into the dynamics of the first order phase transition. Moreover, with the assist of the phenomenological description of the inhomogeneity, one can calculate the space distribution of the free energy and the number density. The surface tension is then determined along both the phase transition line and the spinodal lines, offering valuable insights into phenomena observed in heavy-ion collision experiments and cosmological studies related to QCD phase transitions.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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