

Lensing in matched exterior and interior Kottler solutions

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We analyze gravitational lensing in a matched Kottler spacetime with cosmological constant Λ , where a uniform-density fluid sphere described by the interior Kottler solution is smoothly joined to the exterior Kottler region. We derive an analytic expression for the deflection angle of light rays traversing both regions, isolating the corrections induced by the interior mass distribution relative to the vacuum Kottler case. The interior terms systematically reduce the bending angle and tangential magnification while enhancing the radial one, leading to a scale-dependent effect on the total magnification. Numerical evaluation shows a mild amplification for galaxy lenses and a demagnification for clusters, reflecting the contrast between compact and diffuse configurations. The induced Λ contribution from the interior corrections further reduces the bending, but only slightly, consistent with its role in the vacuum Kottler case. These results demonstrate that lensing observables retain a measurable imprint of interior structure in Schwarzschild–de Sitter-like lenses, crucial for refining realistic strong-lensing models.

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I. INTRODUCTION

Gravitational lensing is a sensitive probe of both local matter distributions and the geometry of the Universe. While the Schwarzschild solution describes deflection by point-like masses, realistic astrophysical lenses such as galaxies and clusters possess extended interiors and are embedded in a cosmological background. The Kottler or Schwarzschild–de Sitter (SdS) solution incorporates the cosmological constant Λ , whose impact on lensing has been actively debated since Rindler and Ishak [1]. In the exterior SdS spacetime, Λ affects measurable angles only through geometry, with negligible effect compared to mass. In interior SdS extensions, however, Λ explicitly enters the geodesic equations due to the nonvacuum matter profile, raising the question of how both the cosmological constant and finite-size structure alter standard predictions [2].

Cluster-lensed quasars provide a natural setting for this problem. Some images form very close to or even inside the lens mass distribution, corresponding to photon trajectories that traverse a fluid or cored interior. Such images, typically

faint and demagnified, motivate extending lens models beyond the exterior SdS case [3–5].

In this work we adopt a spherically symmetric interior with constant energy density as a minimal analytic model that allows an exact matching to the exterior Kottler spacetime and a transparent identification of interior contributions to lensing observables. While such a constant-density interior is not intended as a realistic halo model, cored or noncuspy inner mass distributions are physically motivated and observationally supported. In particular, studies of galaxy rotation curves indicate that the inner regions of many spiral, dwarf, and low-surface-brightness galaxies are better described by cored, roughly constant-density profiles than by steep cusps [6–12]. Similar evidence for softened inner mass distributions has also been reported for relaxed galaxy clusters from strong-lensing, stellar-dynamical, and x-ray analyses [13–15]. Within this context, the constant-density interior serves as a controlled analytic benchmark for isolating the effects of extended mass distributions before considering more realistic profiles [16–18].

Despite progress on the roles of Λ and interior matter in lensing, a unified treatment of lensing in matched SdS spacetime—where an extended interior is smoothly joined to the exterior Kottler region—remains of interest for bridging the gap between idealized point-mass lens models and realistic extended astrophysical systems and for assessing the extent to which lensing observables encode information about both cosmological curvature and interior structure.

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The paper is organized as follows. In Sec. II we present the matched interior-exterior Kottler spacetime and summarize the basic properties of null geodesics in both regions. Section III is devoted to the analytic derivation of the light-deflection angle for trajectories crossing the interior and exterior domains. In Sec. IV we analyze the resulting lensing observables, with particular emphasis on magnification properties and the role of the cosmological constant. Numerical illustrations for galaxy- and cluster-scale lenses are discussed in Sec. V. Finally, Sec. VI summarizes our main results and outlines possible extensions of this work.

II. MATCHED KOTTLER METRIC

We model the lens as a uniform-density fluid sphere, of total mass M and radius r_B , matched smoothly to the exterior SdS solution using the exact global metric

$$ds^2 = f(r)dt^2 - g(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where $f(r) = g(r) = 1 - \delta(r) - \lambda(r)^2$, for $r \geq r_B$, and

$$f(r) = \frac{1}{4} \left[\frac{3\delta_B}{\delta_B + \lambda_B^2} \sqrt{g(r_B)} - \left(\frac{3\delta_B}{\delta_B + \lambda_B^2} - 2 \right) \sqrt{g(r)} \right]^2, \\ g(r) = 1 - (\delta_B + \lambda_B^2) \frac{r^2}{r_B^2}, \quad (2)$$

for $r \leq r_B$, with $\delta(r) = 2GM/r$, $\lambda(r) = \sqrt{\Lambda/3}r$, $\delta_B = \delta(r_B)$, and $\lambda_B = \lambda(r_B)$.

Assuming a uniform fluid density, satisfying $4\pi r_B^3 \rho = 3M$, the pressure profile within the mass distribution is isotropic and can be expressed as

$$p(r) = \rho \frac{\sqrt{g(r)} - \sqrt{g(r_B)}}{3(1 - \lambda_B^2/\delta_B)^{-1} \sqrt{g(r_B)} - \sqrt{g(r)}} \quad (r \leq r_B), \quad (3)$$

which vanishes at the boundary surface $r = r_B$, consistent with the matching to the exterior vacuum solution, but increases monotonically inward and diverges at the center $r = 0$ if the compactness parameter δ_B attains the critical value

$$\delta_B = \frac{4}{9} \left(1 + \sqrt{1 - \frac{9}{4} \lambda_B^2} \right). \quad (4)$$

Such a singular situation is clearly prevented under physically relevant configurations characterized by extremely low compactness,

$$\delta_B \ll \frac{8}{9} - \frac{1}{2} \lambda_B^2 \quad (\lambda_B^2 \ll 1), \quad (5)$$

ensuring finite pressure throughout the fluid sphere. This condition holds for stellar or fluid structures with radii

much larger than their Schwarzschild radii, which are ubiquitous in the Universe, except for compact bodies such as neutron stars and black holes. Note that the bound in (5) generalizes the Buchdahl limit [19], $\delta_B \leq 8/9$, for a stable Schwarzschild perfect fluid, to include the repulsive effect of Λ . However, the assumption of uniform density implies that sound waves propagate instantaneously throughout the mass distribution, thereby violating causality. Although the inclusion of Λ modifies the pressure profile, it does not resolve this issue unless a more realistic nonuniform density profile were employed [17].

III. LIGHT DEFLECTION ANGLE

The photon trajectory lies in the equatorial plane $\theta = \pi/2$, with azimuthal variation ruled by the equation

$$d\varphi(r) = \pm \frac{dr}{u(r)}, \quad u(r) = r\sqrt{g(r)} \left(\frac{f(r_0)r^2}{r_0^2 f(r)} - 1 \right)^{\frac{1}{2}}, \quad (6)$$

where r_0 is the pericenter inside the SdS fluid ($r_0 \leq r_B$) [Eq. (1)], related to the conserved angular momentum per unit mass, associated with the φ component $J(=r^2\dot{\varphi})$, by $J = \pm r_0/\sqrt{f(r_0)}$. The second constant of motion associated with the t component $E(=if(r))$ is absorbed into a redefinition of the affine parameter ($E = 1$). For configurations where φ increases along the path, it is natural to take $J > 0$, while preserving full generality by $J \rightarrow -J$. The plus and minus signs in Eq. (6) reflect outgoing and ingoing geodesics relative to the lens (Fig. 1). The reception angle α at Earth (r_E, π) ($r_E \gg r_B$) and the emission angle σ at the source (r_S, φ_S) ($r_S \gg r_B$) relate to r_0 by the tetrad projection,

$$J(r_0) = \frac{r_E \sin \alpha}{\sqrt{f(r_E)}} = \frac{r_S \sin \sigma}{\sqrt{f(r_S)}}. \quad (7)$$

The deflection angle is

$$\Delta\varphi = \alpha + \sigma - \varphi_S, \quad (8)$$

where φ_S is obtained by integrating Eq. (6) across both regions,

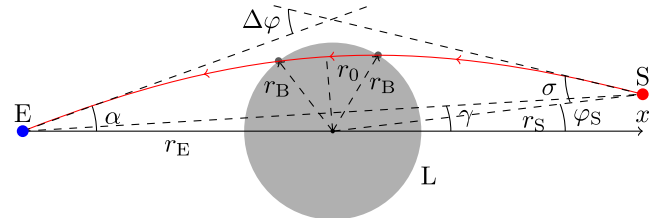


FIG. 1. Light ray emitted by a source S, bent outside and inside the SdS fluid lens L, and received at the Earth E.

$$\varphi_S = \pi - \left(\int_{r_B}^{r_E} + \int_{r_B}^{r_S} \right) \frac{dr}{u_{\text{ex}}(r)} - 2 \int_{r_0}^{r_B} \frac{dr}{u_{\text{in}}(r)}, \quad (9)$$

where $u_{\text{ex}}(r)$ and $u_{\text{in}}(r)$ are the relevant expressions of $u(r)$ [Eq. (6)] for the exterior and interior SdS metrics. These integrals are denoted by I_E , I_S , and I_B , respectively, and evaluated to first order in $\delta_B = 2GM/r_B \ll 1$, as appropriate for noncompact bodies whose sizes greatly exceed their Schwarzschild radii. We extend this for photons crossing into the interior SdS region, in the sense that approximating in terms of $\delta_0 = \delta(r_0) = 2GM/r_0$ amounts to the same thing, provided $r_{\text{Schw}} \ll r_0 \leq r_B$. The explicit evaluation of the integrals I_E , I_S , and I_B is deferred to the Appendix. The results are

$$I_E \simeq \arcsin \frac{r_0}{r_B} - \arcsin \frac{r_0}{r_E} + \frac{\delta_0}{2} \{ C(r_E) - C(r_B) + (1 - h_0)[\varepsilon(r_E) - \varepsilon(r_B)] \}, \quad (10)$$

$$I_B \simeq \frac{\pi}{2} - \arcsin \frac{r_0}{r_B} + \frac{\delta_0}{2} \left[C(r_B) + (1 - h_0)\varepsilon(r_B) - 2 \left(1 - \frac{r_0^2}{r_B^2} \right)^{\frac{3}{2}} \right], \quad (11)$$

where $C(r_E)$, $C(r_B)$, $\varepsilon(r_E)$, and $\varepsilon(r_B)$ are defined by

$$C(r) = \left(1 - \frac{r_0^2}{r^2} \right)^{\frac{1}{2}} \left[1 + \left(1 + \frac{r_0}{r} \right)^{-1} \right], \quad (12)$$

$$\varepsilon(r) = \frac{r_0}{r} \left(1 - \frac{r_0^2}{r^2} \right)^{-\frac{1}{2}}. \quad (13)$$

Here, $h_0 = h(r_0)$ ($r \leq r_B$) is given by

$$h(r) = \frac{r^3}{r_B^3} \left(3 \frac{1 - \sqrt{1 - \lambda_B^2} \sqrt{1 - \lambda(r)^2}}{\lambda(r)^2} - 2 \right), \quad (14)$$

which characterizes the first-order correction to $f(r)$,

$$f(r) \simeq 1 - \delta(r)h(r) - \lambda(r)^2 \quad (r \leq r_B), \quad (15)$$

with $0 < h(r) \leq h(r_B) = 1$, ensuring continuity at the boundary. A similar formula to (10) holds for I_S , replacing r_E with r_S . Inserting I_E , I_B , and I_S back into (9), the term $\delta_0[C(r_B) + (1 - h_0)\varepsilon(r_B)]$ drops out, yielding

$$\Delta\varphi \simeq \Delta\varphi_{\text{ex}} - 2\delta_0 \left[\left(1 - \frac{r_0^2}{r_B^2} \right)^{\frac{3}{2}} - (1 - h_0) \frac{\varepsilon(r_E, r_S)}{4} \right], \quad (16)$$

where $\varepsilon(r_E, r_S) = \varepsilon(r_E) + \varepsilon(r_S)$. The δ_0 term represents a subtractive correction induced by the SdS fluid, which

vanishes for a grazing trajectory, $r_B = r_0$ ($h_0 = 1$), recovering the standard deflection in the vacuum SdS,

$$\Delta\varphi_{\text{ex}} \simeq \alpha + \sigma - \arcsin \frac{r_0}{r_E} - \arcsin \frac{r_0}{r_S} + 2\delta_0[C(r_E) + C(r_S)]. \quad (17)$$

The light deflection is explicitly offset by $2\delta_0(1 - r_0^2/r_B^2)^{3/2}$ as long as $r_0 < r_B$, whereas it is slightly enhanced by the term $\delta_0(1 - h_0)\varepsilon(r_E, r_S)/2$, which depends on small ratios r_0/r_E and r_0/r_S [Eq. (13)], with the Earth and source being very far from the lens for typical situations. Physically, the mass enclosed up to the radius, growing as r^3 for a uniform density, reflects a weaker curvature inside the fluid sphere than outside, where the mass is fully located within that radius. Specifically, this additive term includes, through h_0 , a Λ contribution that reduces the light bending, consistent with its effect outside the mass distribution [1]. This reduction is very small and can be made explicit by expanding h_0 to first order in λ_B^2 ,

$$h_0 \simeq \frac{1}{2} \frac{r_0}{r_B} \left[3 - \frac{r_0^2}{r_B^2} + \left(1 - \frac{r_0^2}{r_B^2} \right)^2 \frac{\Lambda}{4} r_B^2 \right], \quad (18)$$

and then substituting into (16),

$$\Delta\varphi \simeq \Delta\varphi_{\text{ex}} - 2\delta_0 \left[\left(1 - \frac{r_0^2}{r_B^2} \right)^{\frac{3}{2}} - \frac{\varepsilon(r_E, r_S)}{4} \left(1 + \frac{1}{2} \frac{r_0}{r_B} \right) \times \left(1 - \frac{r_0}{r_B} \right)^2 - \delta_B \frac{\varepsilon(r_E, r_S)}{4} \left(1 - \frac{r_0^2}{r_B^2} \right)^2 \frac{\Lambda}{4} r_B^2 \right]. \quad (19)$$

This remains true if one considers only the bending within the SdS fluid. Substituting (18) into (11), one gets

$$2I_B - \pi \simeq -2 \arcsin \frac{r_0}{r_B} + 3 \frac{GM}{r_B} \frac{r_0}{r_B} \left(1 - \frac{r_0^2}{r_B^2} \right)^{\frac{1}{2}} - \frac{GM}{r_0} \left(1 - \frac{r_0^2}{r_B^2} \right)^{\frac{3}{2}} \frac{\Lambda}{4} r_0^2. \quad (20)$$

In Ref. [2], Schücker confirmed this direct dependence of light bending on Λ for a light ray confined inside a global SdS fluid—unlike the exterior case—but found that the Λ effect appears reversed compared with the usual expectation. As stated earlier, the subtractive Λ term in (20) is, however, suppressed by an additive one arising from I_E and I_S [Eq. (10)], leaving only residual dependencies encoded in r_0 via $2\delta_0(1 - r_0^2/r_B^2)^{3/2}$ [Eq. (11)], subsequently added, along with a remaining Λ term (19), to the total bending. Furthermore, the bending inside the fluid increases linearly with the compactness parameter δ_B , consistent with physical intuition that a more compact configuration induces stronger deflection. Although both corrections stem from the same central mass, the subtractive δ_0 term (16), $2\delta_0(1 - r_0^2/r_B^2)^{3/2}$,

in the total bending accounts, as noted, for the reduced interior curvature relative to the vacuum case, while the additive δ_B term (20) in the interior bending reflects the cumulative effect of the distributed mass along the light path. Using (15) with (18) in Eq. (7) leads to, after a redefinition of the impact parameter, a quadratic,

$$a \frac{r_0^4}{r_B^4} + \left(\frac{r_B^2}{J_\Lambda^2} - b \right) \frac{r_0^2}{r_B^2} + c = 1, \quad \frac{1}{J_\Lambda^2} = \frac{1}{J^2} + \frac{\Lambda}{3}, \quad (21)$$

solved by

$$r_0 = \frac{r_B}{\sqrt{2a}} \left[\sqrt{\left(\frac{r_B^2}{J_\Lambda^2} - b \right)^2 + 4a(1-c) - \frac{r_B^2}{J_\Lambda^2} + b} \right]^{\frac{1}{2}}, \quad (22)$$

where a , b , and c are small coefficients defined by

$$a = \frac{3}{4} \frac{\delta_B}{2} \lambda_B^2, \quad b = \frac{\delta_B}{2} + 2a, \quad c = 3 \frac{\delta_B}{2} + a. \quad (23)$$

This generalizes the pericenter expression to account for a trajectory that penetrates the SdS fluid. The grazing case ($r_B = r_0$) reduces Eq. (21) to a depressed cubic, whose real root determines r_0 in the vacuum SdS [20,21]. Linearizing (22) to leading order in δ_B and λ_B^2 yields

$$r_0 \simeq J_\Lambda \left[1 - \frac{1}{2} \frac{\delta_B}{2} \left(3 - \frac{J_\Lambda^2}{r_B^2} \right) - \frac{3}{2} \frac{\delta_B^2}{4} \frac{J_\Lambda^2}{r_B^2} - \frac{3}{8} \frac{\delta_B}{2} \left(1 - 2 \frac{J_\Lambda^2}{r_B^2} \right) \lambda_B^2 \right] \quad (24)$$

$$\simeq J \left[1 - \frac{1}{2} \frac{GM}{r_B} \left(3 - \frac{J^2}{r_B^2} \right) - \frac{\Lambda}{6} J^2 \right], \quad (25)$$

which shows that r_0 decreases linearly with both δ_B and Λ . The first reflects the focusing effect of increasing interior mass, while the latter, though counterintuitive, originates from the geometric dilution of spacetime curvature by Λ , which weakens the gravitational attraction. Although the light trajectory approaches more closely to the center, the overall bending is reduced—consistent with the effect observed in the vacuum SdS [1,22], where the Λ term remains identical even for penetrating trajectories. Sereno [22] confirmed this effect by showing that Λ reduces the Einstein-ring radius and leads to a slight demagnification of lensing images, consistent with a reduced deflection relative to the pure Schwarzschild.

Given that the Earth and the source are located at large but finite distances from the lens, $2GM/r_E \ll \lambda_E^2 (= \Lambda r_E^2/3)$ and $2GM/r_S \ll \lambda_S^2 (= \Lambda r_S^2/3)$, the angles are small enough to justify these approximations of (7):

$$r_0 \simeq J \simeq \frac{\alpha r_E}{\sqrt{1 - \lambda_E^2}} \simeq \frac{\sigma r_S}{\sqrt{1 - \lambda_S^2}}. \quad (26)$$

This allows the bending angle to be expressed in terms of measurable quantities as

$$\Delta\varphi \simeq \Delta\varphi_{\text{ex}} - \frac{4GM}{\alpha r_E} \sqrt{1 - \lambda_E^2} \left(1 - \frac{\alpha^2}{\beta^2} \right)^{\frac{3}{2}} \quad (27)$$

$$\simeq \frac{4GM}{\alpha r_E} \sqrt{1 - \lambda_E^2} \left[1 - \left(1 - \frac{\alpha^2}{\beta^2} \right)^{\frac{3}{2}} \right] - \alpha\eta(\Lambda), \quad (28)$$

where the angle $\beta = (r_B/r_E) \sqrt{1 - \lambda_E^2}$ is introduced by analogy with (26), and

$$\Delta\varphi_{\text{ex}} \simeq \frac{4GM}{\alpha r_E} \sqrt{1 - \lambda_E^2} - \alpha\eta(\Lambda), \quad (29)$$

where $\eta(\Lambda)$ is a correction term arising from the presence of Λ , defined by

$$\eta(\Lambda) = \frac{1 - \sqrt{1 - \lambda_E^2}}{\sqrt{1 - \lambda_E^2}} + \frac{r_E}{r_S} \frac{1 - \sqrt{1 - \lambda_S^2}}{\sqrt{1 - \lambda_E^2}}. \quad (30)$$

For $\Lambda = 0$ [$\eta = 0$], (28) recovers the matched exterior-interior Schwarzschild bending angle, where (29) reduces to the standard Einstein (or Schwarzschild) bending angle.

IV. LENS EQUATION AND MAGNIFICATION

The interior fluid sphere structure modifies the lens equation $\gamma = \alpha - (d_{LS}/d_S)\Delta\varphi$ [23], relating the bending angle and the unlensed angular position of the source γ ,

$$\gamma = \alpha \left[1 + \frac{d_{LS}}{d_S} \eta(\Lambda) \right] - \frac{4GM}{\alpha d_L} \frac{d_{LS}}{d_S} \left[1 - \left(1 - \frac{\alpha^2}{\beta^2} \right)^{\frac{3}{2}} \right], \quad (31)$$

by using angular diameter distances d_L (Earth-to-lens), d_S (Earth-to-source), and d_{LS} (lens-to-source), derived in the de Sitter geometry *à la* Sereno [22],

$$d_L = \frac{r_E}{\sqrt{1 - \lambda_E}}, \quad d_S = \frac{r_E + r_S}{\sqrt{1 - \lambda_E}}, \quad d_{LS} = r_S. \quad (32)$$

The magnification of the image is factored into tangential $\mu_t = (\gamma/\alpha)^{-1}$ and radial $\mu_r = (d\gamma/d\alpha)^{-1}$ components, $\mu = \mu_t \mu_r$,

$$\mu_t = \left(1 - \frac{d_{LS}}{d_S} \frac{\Delta\varphi}{\alpha} \right)^{-1}, \quad \mu_r = \left(1 - \frac{d_{LS}}{d_S} \frac{d\Delta\varphi}{d\alpha} \right)^{-1}, \quad (33)$$

and expressed relative to corresponding exterior SdS ones, μ_{tex} and μ_{rex} , as

$$\mu_t^{-1} = \mu_{\text{tex}}^{-1} + \frac{e^2}{\alpha^2} \left(1 - \frac{\alpha^2}{\beta^2} \right)^{\frac{3}{2}}, \quad (34)$$

$$\mu_r^{-1} = \mu_{\text{rex}}^{-1} - \frac{\epsilon^2}{\alpha^2} \left(1 + 2 \frac{\alpha^2}{\beta^2}\right) \left(1 - \frac{\alpha^2}{\beta^2}\right)^{\frac{1}{2}}, \quad (35)$$

$$\mu_{\text{rex}}^{-1} = 1 + \tilde{\eta}(\Lambda) - \frac{\epsilon^2}{\alpha^2}, \quad \mu_{\text{rex}}^{-1} = 1 + \tilde{\eta}(\Lambda) + \frac{\epsilon^2}{\alpha^2}, \quad (36)$$

where, for the sake of simplicity, we have set $\tilde{\eta}(\Lambda) = (d_{\text{LS}}/d_S)\eta(\Lambda)$ and introduced the Einstein ring angle $\epsilon = \sqrt{4GMd_{\text{LS}}/d_L d_S}$. The interior fluid adds a positive term $\propto (1 - \alpha^2/\beta^2)^{3/2}$ to μ_{rex}^{-1} , so μ_t is systematically reduced (demagnified) relative to μ_{rex} , consistent with weaker lensing from a distributed interior mass. Conversely, the interior subtracts a positive term $\propto (1 - \alpha^2/\beta^2)^{1/2}$ from μ_{rex}^{-1} , so μ_r tends to be enhanced (magnified) compared with the exterior solution due to cumulative lensing effects inside the fluid sphere. *A priori*, the total magnification may be either diminished or enhanced. We express it in terms of the total exterior magnification μ_{ex} as

$$\mu^{-1} = \mu_{\text{ex}}^{-1} - 3 \frac{\epsilon^2}{\beta^2} \left[1 + \tilde{\eta}(\Lambda) - \frac{\epsilon^2 \zeta(\alpha)}{\beta^2} \frac{1}{3}\right] \left(1 - \frac{\alpha^2}{\beta^2}\right)^{\frac{1}{2}}, \quad (37)$$

$$\mu_{\text{ex}}^{-1} = (\mu_{\text{tex}} \mu_{\text{rex}})^{-1} = [1 + \tilde{\eta}(\Lambda)]^2 - \frac{\epsilon^4}{\alpha^4}, \quad (38)$$

where we have defined a positive function

$$\zeta(\alpha) = \frac{\beta^4}{\alpha^4} \left\{ 1 - \frac{\alpha^2}{\beta^2} + \left(1 + 2 \frac{\alpha^2}{\beta^2}\right) \left[1 - \left(1 - \frac{\alpha^2}{\beta^2}\right)^{\frac{3}{2}}\right] \right\}. \quad (39)$$

Notably, increasing Λ systematically raises $\tilde{\eta}(\Lambda)$ [Eq. (30)], leading to demagnification of tangential, radial [Eq. (36)], and total exterior magnifications [Eq. (38)], consistent with previous analyses [22,24]. The interior correction enters with a minus sign in front of the Λ -dependent term (37), yielding a slight magnifying effect that is suppressed by the small factor $(\epsilon/\beta)^2$ and thus negligible compared with the dominant exterior demagnification induced by Λ .

V. NUMERICAL RESULTS AND DISCUSSION

We now evaluate our results as a function of the image angle α over the range $[0.9\beta, \beta]$ for two illustrative setups, adopting $r_E = r_S = 3$ Gpc and $\Lambda = 10^{-52} \text{m}^{-2}$:

(i) Galaxy scale: $M = 5 \times 10^{11} M_\odot$ and $r_B = 25$ kpc.

(ii) Cluster scale: $M = 10^{14} M_\odot$ and $r_B = 250$ kpc.

The resulting $\Delta\varphi(\alpha)$, $\mu_t(\alpha)$, $\mu_r(\alpha)$, and $\mu(\alpha)$ curves are displayed in Figs. 2–9, then compared against the pure exterior SdS predictions to isolate the impact of the interior structure. The numerical trends confirm the analytic structure of Eqs. (27), (34), and (35) for both galaxy and cluster scales: the interior terms systematically drive $\Delta\varphi \leq \Delta\varphi_{\text{ex}}$ (Figs. 2 and 3), reflecting the reduced curvature inside the

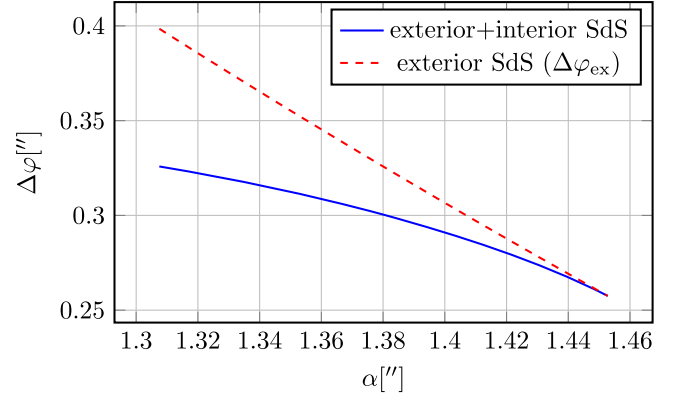


FIG. 2. Evolution of bending angle versus reception angle for a galaxy lens in matched exterior and interior SdS.

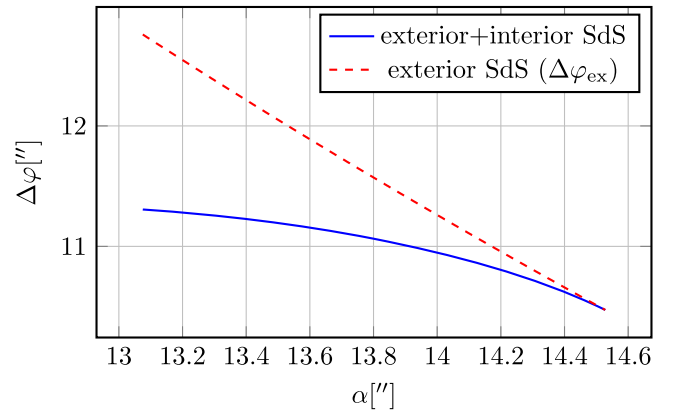


FIG. 3. Evolution of bending angle versus reception angle for a cluster lens in matched exterior and interior SdS.

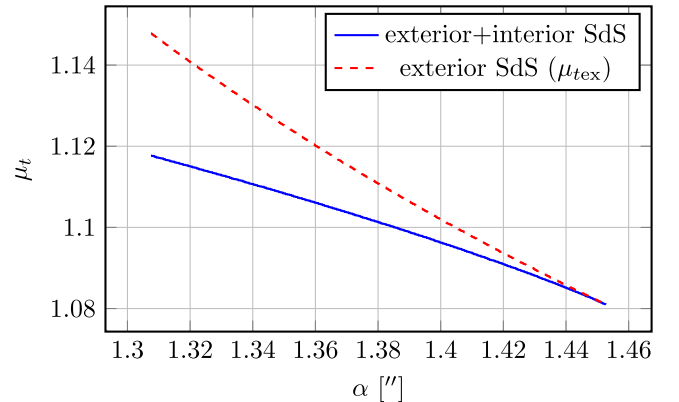


FIG. 4. Evolution of tangential magnification versus reception angle for a galaxy lens in matched exterior and interior SdS.

fluid sphere relative to the vacuum exterior; $\mu_t \leq \mu_{\text{tex}}$ (Figs. 4 and 7), owing to the weaker angular focusing of a distributed mass compared to a point-like lens; and $\mu_r \geq \mu_{\text{rex}}$ (Figs. 5 and 8), arising from the cumulative

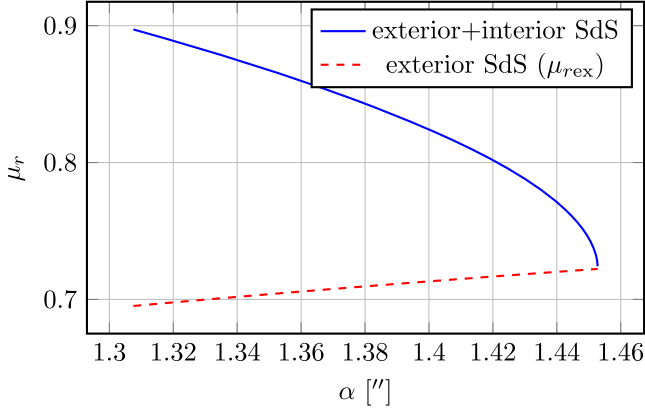


FIG. 5. Evolution of radial magnification versus reception angle for a galaxy lens in matched exterior and interior SdS.

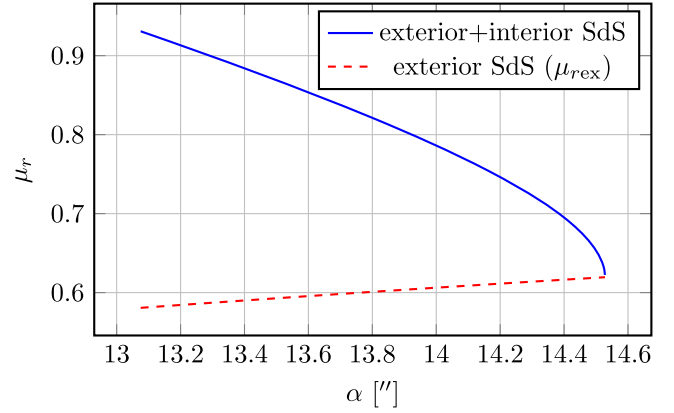


FIG. 8. Evolution of radial magnification versus reception angle for a cluster lens in matched exterior and interior SdS.

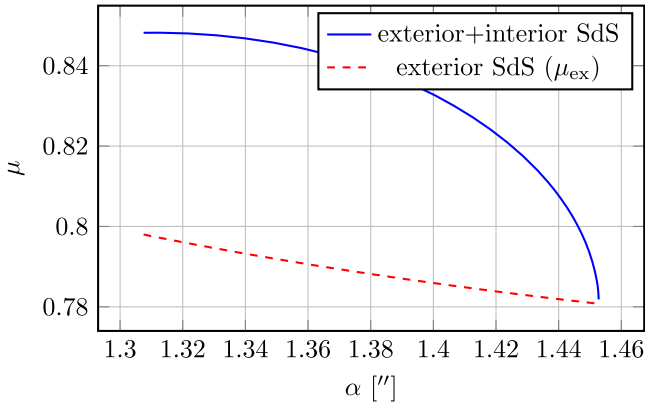


FIG. 6. Evolution of total magnification versus reception angle for a galaxy lens in matched exterior and interior SdS.

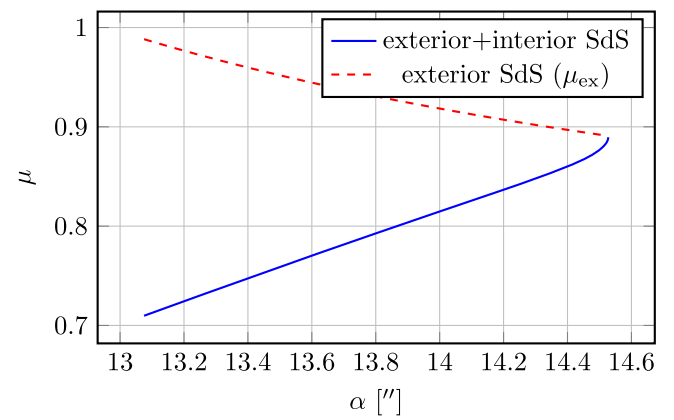


FIG. 9. Evolution of total magnification versus reception angle for a cluster lens in matched exterior and interior SdS.

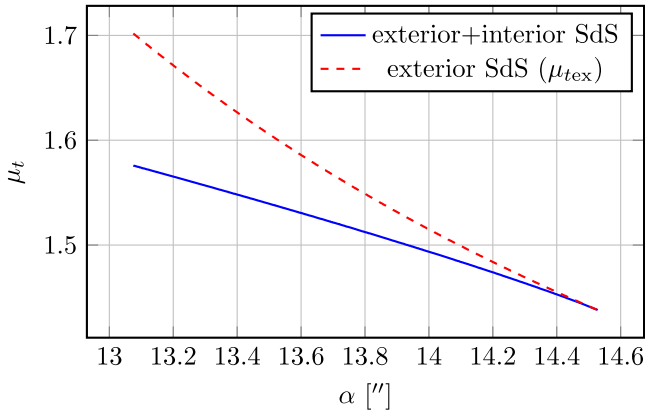


FIG. 7. Evolution of tangential magnification versus reception angle for a cluster lens in matched exterior and interior SdS.

deflection accumulated along the interior path. The sign of the net correction to μ in Eq. (37) is controlled by the compactness parameter $\epsilon^2/\beta^2 (\propto \delta_B)$, leading to a net but slight magnification for the galaxy scale (Fig. 6), where the

radial enhancement dominates, and a demagnification for the cluster case (Fig. 9), where tangential suppression prevails: for small ϵ^2/β^2 (galaxy parameters), the leading $\mathcal{O}(\epsilon^2/\beta^2)$ term dominates, yielding $\mu \geq \mu_{\text{ex}}$, while for larger ϵ^2/β^2 (cluster parameters), the higher-order contribution flips the sign, producing $\mu \leq \mu_{\text{ex}}$.

VI. SUMMARY

We have explored gravitational lensing within a matched Kottler spacetime, modeling the lens as a uniform-density fluid sphere smoothly joined to the exterior Kottler solution. Although the constant-density interior adopted here is not meant to provide a realistic description of dark-matter halos, it provides a controlled analytic framework that permits exact matching and a transparent interpretation of lensing observables associated with extended mass distributions. While quantitative predictions depend on the specific mass profile, the qualitative lensing effects identified here are expected to persist for more general cored matter distributions, including models with an energy density falling off radially as $\rho \propto r^{-2}$.

This expectation is supported by studies of realistic galaxy and cluster halos, showing that different cored and isothermal profiles yield similar observable trends despite variations in the detailed inner structure [7,8,10,13–15].

Analytic expressions for the bending angle of light rays crossing both interior and exterior regions were derived, isolating distinctive corrections induced by the interior mass distribution compared with the vacuum SdS case. We showed that the inclusion of interior structure systematically weakens the bending angle and tangential magnification, while enhancing the radial magnification, yielding scale-dependent signatures in total magnification. These analytic insights were corroborated by numerical examples, demonstrating that galaxy-scale lenses exhibit a slight net magnification, whereas cluster-scale lenses yield a net demagnification. This difference arises from lens compactness: more compact galaxy lenses produce radial magnification boosts, whereas diffuse clusters have dominant tangential demagnification. Additionally, the interior correction carries a small negative Λ -dependent contribution to the bending, consistent with its role in the vacuum SdS case. At the level of magnifications, Λ induces a dominant exterior demagnification, while the interior term provides only a negligible magnifying effect suppressed by $(\epsilon/\beta)^2$.

These results highlight the critical influence of interior structure on lensing predictions and motivate extensions to more realistic density profiles [16–18] and dynamical backgrounds [25–29], which would clarify how robust these corrections to the deflection angle and magnification signatures are and whether they can serve as discriminants in forthcoming strong-lensing surveys. In Ref. [30], a companion study addressed the time delay in the same framework, offering a complementary probe of interior structure directly relevant to strong-lensing cosmography.

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DATA AVAILABILITY

All data supporting the results of the study are reported within this article.

APPENDIX: EVALUATION OF I_E , I_S , AND I_B

In this appendix, we evaluate the integrals

$$\begin{aligned} I_E &= \int_{r_B}^{r_E} \frac{dr}{u_{\text{ex}}(r)}, & I_S &= \int_{r_B}^{r_S} \frac{dr}{u_{\text{ex}}(r)}, \\ I_B &= \int_{r_0}^{r_B} \frac{dr}{u_{\text{in}}(r)}, \end{aligned} \quad (\text{A1})$$

which enter the analytic expression for the light deflection angle (8) through the inclination angle (9), $\varphi_S = \pi - I_E - I_S - 2I_B$.

We expand the integrands $u_{\text{ex}}(r)$ and $u_{\text{in}}(r)$ (6) consistently in the small parameter $\delta_0 = 2GM/r_0$. Introducing the dimensionless variable $x = r_0/r$, we get

$$u_{\text{ex}}(x)^{-1} \simeq \frac{x^2}{r_0 \sqrt{1-x^2}} \left(1 + \frac{\delta_0 h_0 - x^3}{2(1-x^2)} \right), \quad (\text{A2})$$

$$\begin{aligned} u_{\text{in}}(x)^{-1} &\simeq \frac{x^2}{r_0 \sqrt{1-x^2}} \left[1 + 3 \frac{\delta_0 r_0^3}{2 r_B^3} \frac{\sqrt{1-\lambda_B^2} \sqrt{1-\lambda_0^2}}{\lambda_0^2 (1-x^2)} \right. \\ &\quad \times \left. \left(\frac{\sqrt{1-\lambda_0^2} x}{\sqrt{x^2-\lambda_0^2}} - 1 \right) \right], \end{aligned} \quad (\text{A3})$$

where h_0 is given by (14),

$$h_0 = \frac{r_0^3}{r_B^3} \left(3 \frac{1 - \sqrt{1-\lambda_B^2} \sqrt{1-\lambda_0^2}}{\lambda_0^2} - 2 \right). \quad (\text{A4})$$

The identity

$$\frac{\sqrt{1-\lambda_0^2} x - \sqrt{x^2-\lambda_0^2}}{\lambda_0^2 (1-x^2)} = \frac{1}{\sqrt{1-\lambda_0^2} x + \sqrt{x^2-\lambda_0^2}} \quad (\text{A5})$$

readily shows that the correction term in u_{in} is $\mathcal{O}(\delta_0)$. At $x_B = r_0/r_B$, one has $u_{\text{ex}} = u_{\text{in}}$, ensuring continuity across the matching surface.

The approximate expressions for u_{ex} (A2) and u_{in} (A3) require the use of the following standard integrals:

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x, \quad (\text{A6})$$

$$\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{1-x^2}}, \quad (\text{A7})$$

$$\int \frac{x^3 dx}{(1-x^2)^{\frac{3}{2}}} = \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}, \quad (\text{A8})$$

$$\int \frac{x dx}{(1-x^2)^{\frac{3}{2}} \sqrt{x^2-\lambda_0^2}} = \frac{1}{1-\lambda_0^2} \frac{\sqrt{x^2-\lambda_0^2}}{\sqrt{1-x^2}}. \quad (\text{A9})$$

The evaluation of I_E [Eq. (A1)] then yields

$$\begin{aligned} I_E &\simeq \arcsin x_B - \arcsin x_E + \frac{\delta_0}{2} \left(\sqrt{1-x_E^2} + \frac{1-h_0 x_E}{\sqrt{1-x_E^2}} \right. \\ &\quad \left. - \sqrt{1-x_B^2} - \frac{1-h_0 x_B}{\sqrt{1-x_B^2}} \right), \end{aligned} \quad (\text{A10})$$

which is identical to (10), with $x_E = x(r_E) = r_0/r_E$. The integral I_S [Eq. (A1)] follows analogously from I_E upon replacing r_E with r_S . For I_B in (A1), we obtain

$$I_B \simeq \frac{\pi}{2} - \arcsin x_B + 3 \frac{\delta_0}{2} x_B^2 \frac{\sqrt{1-\lambda_0^2} - \sqrt{1-\lambda_B^2}}{\lambda_B^2} \times \frac{\sqrt{1-\lambda_B^2}}{\sqrt{1-x_B^2}}, \quad (\text{A11})$$

which can be rewritten in terms of h_0 as

$$I_B \simeq \frac{\pi}{2} - \arcsin x_B + \frac{\delta_0}{2} \left[\frac{1-h_0 x_B}{\sqrt{1-x_B^2}} - (1-2x_B^2) \sqrt{1-x_B^2} \right]. \quad (\text{A12})$$

This last expression is identical to (11).

Although the integrals (A7) and (A9) individually exhibit apparent divergences at $r = r_0$ in the evaluation of I_B , these singular terms cancel exactly in their combined contribution, leaving a regular result. Alternatively, the identity (A5) may be used from the outset to recast the integrand into a manifestly finite form at $r = r_0$.

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- [1] W. Rindler and M. Ishak, Contribution of the cosmological constant to the relativistic bending of light revisited, *Phys. Rev. D* **76**, 043006 (2007).
 - [2] T. Schücker, Lensing in an interior Kottler solution, *Gen. Relativ. Gravit.* **42**, 1991 (2010).
 - [3] J. N. Winn, D. Rusin, and C. S. Kochanek, The central image of a gravitationally lensed quasar, *Nature (London)* **427**, 613 (2004).
 - [4] N. Inada *et al.*, Discovery of a fifth image of the large separation gravitationally lensed quasar SDSS J1004 + 4112, *Publ. Astron. Soc. Jpn.* **57**, L7 (2005).
 - [5] S. Ertl, S. Schuldt, S. H. Suyu, P. L. Schechter, A. Halkola, and J. Wagner, The missing quasar image in the gravitationally lensed quasar HE0230–2130: Implications for the cored lens mass distribution and dark satellites, *Astron. Astrophys.* **685**, A15 (2024).
 - [6] P. Salucci, The constant-density region of the dark haloes of spiral galaxies, *Mon. Not. R. Astron. Soc.* **320**, L1 (2001).
 - [7] F. C. van den Bosch, and R. A. Swaters, Dwarf galaxy rotation curves and the core problem of dark matter haloes, *Mon. Not. R. Astron. Soc.* **325**, 1017 (2001).
 - [8] W. J. G. de Blok, S. S. McGaugh, A. Bosma, and V. C. Rubin, Mass density profiles of low surface brightness galaxies, *Astrophys. J.* **552**, L23 (2001).
 - [9] G. Gentile, P. Salucci, U. Klein, D. Vergani, and P. Kalberla, The cored distribution of dark matter in spiral galaxies, *Mon. Not. R. Astron. Soc.* **351**, 903 (2004).
 - [10] P. Salucci, The mass distribution in spiral galaxies, in *Dark Galaxies and Lost Baryons*, Proceedings of the International Astronomical Union Vol. 244, edited by J. I. Davies and M. J. Disney (Cambridge University Press, Cambridge, England, 2008), pp. 53–62.
 - [11] F. Donato, G. Gentile, P. Salucci, C. Frigerio Martins, M. I. Wilkinson, G. Gilmore, E. K. Grebel, A. Koch, and R. Wyse, A constant dark matter halo surface density in galaxies, *Mon. Not. R. Astron. Soc.* **397**, 1169 (2009).
 - [12] K. A. Oman *et al.*, The unexpected diversity of dwarf galaxy rotation curves, *Mon. Not. R. Astron. Soc.* **452**, 3650 (2015).
 - [13] D. J. Sand, T. Treu, G. P. Smith, and R. S. Ellis, The dark matter distribution in the central regions of galaxy clusters: Implications for cold dark matter, *Astrophys. J.* **604**, 88 (2004).
 - [14] A. B. Newman, T. Treu, R. S. Ellis, D. J. Sand, C. Nipoti, J. Richard, and E. Jullo, The density profiles of massive, relaxed galaxy clusters. I. The total density over three decades in radius, *Astrophys. J.* **765**, 24 (2013).
 - [15] Newman, T. Treu, R. S. Ellis, and D. J. Sand, The density profiles of massive, relaxed galaxy clusters. II. Separating luminous and dark matter in cluster cores, *Astrophys. J.* **765**, 25 (2013).
 - [16] J. F. Navarro, C. S. Frenk, and S. D. M. White, A universal density profile from hierarchical clustering, *Astrophys. J.* **490**, 493 (1997).
 - [17] L. Gabbanelli, J. Ovalle, A. Sotomayor, Z. Stuchlik, and R. Casadio, A causal Schwarzschild-de Sitter interior solution by gravitational decoupling, *Eur. Phys. J. C* **79**, 486 (2019).
 - [18] R. Dehghani, P. Salucci, and H. Ghaffarnejad, Navarro-Frenk-White dark matter profile and the dark halos around disk systems, *Astron. Astrophys.* **643**, A161 (2020).
 - [19] H. A. Buchdahl, General relativistic fluid spheres, *Phys. Rev.* **116**, 1027 (1959).
 - [20] F. Finelli, M. Galaverni, and A. Gruppiso, Light bending as a probe of the nature of dark energy, *Phys. Rev. D* **75**, 043003 (2007).
 - [21] M. Ishak, W. Rindler, and J. Dossett, More on lensing by a cosmological constant, *Mon. Not. R. Astron. Soc.* **403**, 2152 (2010).
 - [22] M. Sereno, Influence of the cosmological constant on gravitational lensing in small systems, *Phys. Rev. D* **77**, 043004 (2009).
 - [23] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer-Verlag, Berlin, 1992).
 - [24] F. Zhao and J. Tang, Gravitational lensing effects of Schwarzschild-de Sitter black hole, *Phys. Rev. D* **92**, 083011 (2015).
 - [25] T. Schücker, Strong lensing in the Einstein-Straus solution, *Gen. Relativ. Gravit.* **41**, 1595 (2009).
 - [26] K.-E. Boudjemaa, M. Guenouche, and S. R. Zouzou, Time delay in the Einstein-Straus solution, *Gen. Relativ. Gravit.* **43**, 1707 (2011).

-
- [27] M. Guenouche and S. R. Zouzou, Deflection of light and time delay in closed Einstein-Straus solution, [Phys. Rev. D **98**, 123508 \(2018\)](#).
- [28] M. Guenouche, Effect of the spatial curvature on light bending and time delay in a curved Einstein-Straus–de Sitter spacetime, [Phys. Rev. D **110**, 063508 \(2024\)](#).
- [29] M. Guenouche, Deflection of light and time delay in a hyperbolic Einstein-Straus–de Sitter solution, [Phys. Rev. D **112**, 063550 \(2025\)](#).
- [30] M. Guenouche, Time delay in matched exterior and interior Kottler solutions, [arXiv:2509.11049](#).