

$Z\alpha^2$ correction to superallowed beta decays in effective field theory and implications for $|V_{ud}|$

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Superallowed ($0^+ \rightarrow 0^+$) beta decays currently provide the most precise extraction of quark mixing in the Standard Model. Their interpretation as a measurement of $|V_{ud}|$ relies on a reliable first-principles computation of QED radiative corrections expressed as a series in $Z\alpha$ and α . In this work, we provide the first model-independent result for two-loop, $O(Z\alpha^2)$, long-distance radiative corrections where the nuclei are treated as heavy pointlike particles. We use renormalization group analysis to obtain new results at $O(Z\alpha^3)$ for the coefficient of double logarithms in the ratio of the maximal beta energy to the inverse nuclear size, E_m/R^{-1} . We use the Kinoshita-Lee-Nauenberg theorem to obtain new results at $O(Z^2\alpha^3)$ for the coefficient of logarithms in the ratio of maximal beta energy to the electron mass, $\log(2E_m/m)$. We identify a structure-dependent, and, therefore, short-distance, contribution to the traditional $Z\alpha^2$ correction that should be revisited. We provide the first comprehensive update to the long-distance corrections in almost 40 years and comment on the impact of our findings for extractions of $|V_{ud}|$. We find that shifts in the long-distance corrections are $2.5\times$ larger than past estimates of their uncertainty, $1.5\times$ larger than the statistical uncertainty from the combined fit of superallowed decays, and about $1/2$ the size of estimated systematic error, which stems dominantly from nuclear structure effects.

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I. INTRODUCTION

The lifetimes of superallowed ($0^+ \rightarrow 0^+$) beta emitters have an immediate connection to fundamental constants of nature and, in particular, $|V_{ud}|$ [1,2]. Because of a number of approximate symmetries and small expansion parameters [1], $|V_{ud}|$ can be extracted with a high level of precision (currently estimated at $\sim 3 \times 10^{-4}$) [2]. An important set of inputs for this program are long-distance QED radiative corrections, which are required at high perturbative order. Recently, two of us have found an error in the $O(Z^2\alpha^3)$ correction [3,4], which results in sizable shifts to $|V_{ud}|$ (e.g., an estimated shift of 3.2×10^{-4} in the $\mathcal{F}t$ value for ^{26}Al [3]).

It has recently been realized that all long-distance electromagnetic effects can be computed using a heavy particle effective field theory (EFT), including the

Fermi function and other radiative corrections [3–8]. This provides a model-independent definition of the corrections (supplanting previous estimates based on the independent particle model [9–11]) and allows the full machinery of renormalization-group improved perturbation theory to be employed. For superallowed decays, only one composite operator appears at leading power:

$$\mathcal{L}_{\text{eff}} = -\mathcal{C}(\mu)(\phi_v^{[A,Z]})^* \phi_v^{[A,Z+1]} \bar{\nu}_e \not{e} (1 - \gamma_5) e + \text{H.c.} \quad (1)$$

For given final states, the matrix element

$$\mathcal{M}(\mu) = \langle (\phi_v^{[A,Z]})^* \phi_v^{[A,Z+1]} \bar{\nu}_e \not{e} (1 - \gamma_5) e \rangle \quad (2)$$

can be computed perturbatively using heavy particle Feynman rules [3,4,6] (see [7,8,12–16] for related work on neutron beta decay using heavy-baryon theories). Physical, and, therefore, μ -independent, amplitudes are given by the product of $\mathcal{C}(\mu)$ and $\mathcal{M}(\mu)$.

A complementary program based on both chiral effective theory [15–19] and dispersion relations [20–26] has been launched examining short-distance modes and allows for control over the resummation of electroweak logarithms

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[27,28] and nuclear structure-dependent effects; lattice QCD can also play an important role at the hadronic scale [23,29,30]. When matching onto Eq. (1), all such short-distance effects (including those stemming from nuclear structure) are subsumed into $\mathcal{C}(\mu)$ at leading power. At subleading power, other Wilson coefficients multiplying higher-dimensional operators appear but are numerically subleading.

To compute the lifetime of a beta emitter through $O(Z\alpha^2)$, we require corrections to the rate for $A \rightarrow B\nu e$ through $O(Z\alpha^2)$ and corrections to the rate for $A \rightarrow B\nu e\gamma$ through relative $O(Z\alpha)$. In this work, we present two new theoretical results. First, we compute relative $O(Z\alpha)$ virtual corrections to the real photon spectrum, finding that the result factorizes into the product of the first-order Fermi function and the tree-level real photon spectrum. Second, we obtain the two-loop virtual corrections at $O(Z\alpha^2)$ in the limit of a massless electron by making use of recently obtained results for neutron decay [7]. These two results are combined with existing analyses of the operator's anomalous dimension to supply a new numerical evaluation of long-distance QED corrections to nuclear beta decay. We include terms enhanced by logarithms in the ratio of the maximal beta energy to the inverse nuclear size, $L = \log(\Lambda/E_m)$ with $\Lambda \sim 1/R^{-1}$, up to $O(Z^2\alpha^4 L)$ [4]. Using our $O(Z\alpha^2)$ result, we apply the Kinoshita-Lee-Nauenberg (KLN) theorem [31,32] to further control terms at $O(Z^2\alpha^3, Z\alpha^3, \alpha^2)$ that are enhanced by logarithms in the ratio of maximal beta energy to the electron mass, $\log(E_m/m)$.

The rest of the paper is organized as follows: In Sec. II, we explain why there are no $O(Z\alpha^2)$ corrections from real emission beyond those captured by the Fermi function. Section III presents the extraction of the $O(Z\alpha^2)$ double-virtual correction to $A \rightarrow Be^+\nu_e$. In Sec. IV, we combine these results and integrate over phase space to provide the $O(Z\alpha^2)$ radiative correction to the lifetime of a super-allowed emitter A . We provide an update for the long-distance corrections computed using Eq. (1), including an exploration of the numerical impact of some known higher-order corrections. We conclude with a summary and discussion in Sec. V.

II. $O(Z\alpha^2)$ CORRECTION TO THE PHOTON SPECTRUM

In this section, we demonstrate that real-virtual corrections at $O(Z\alpha^2)$ are given by the product of the Fermi function and the Sirlin g function (more precisely, its EFT analog).¹ This factorization stems not from a separation of scales but because most of the virtual corrections are out of phase (i.e., they are “pure imaginary”) with the leading-

¹Real photons were not explicitly discussed in past calculations of the $Z\alpha^2$ correction [9–11].

order real emission amplitude and, therefore, do not contribute to the rate at this order; at higher orders, we do not expect this factorization to hold.

Real photon emission contributes to the lifetime of any β emitter beginning at $O(\alpha)$. The relative $O(Z\alpha)$ correction to real photon emission then influences the decay rate at $O(Z\alpha^2)$. Let us consider

$$A \rightarrow B\nu_e e^+(p)\gamma(q) \quad (3)$$

and write the matrix element as a Dirac structure to be contracted as $\bar{u}_{\nu_e} \not{\epsilon}(1 - \gamma_5) \mathcal{M} v_e$. The matrix element is then expanded as

$$\mathcal{M} = \mathcal{M}^{(0)} + \mathcal{M}^{(1)} + \dots \quad (4)$$

The tree-level matrix element for single-photon emission is given by the sum of two terms:

$$\begin{aligned} \mathcal{M}_a^{(0)} &= e \frac{-\not{p} - \not{q} + m}{2p \cdot q} \not{\epsilon}^*, \\ \mathcal{M}_b^{(0)} &= -e \frac{v \cdot \epsilon^*}{-v \cdot q}, \end{aligned} \quad (5)$$

where $v_\mu = (1, 0, 0, 0)$ is the four-velocity of the nucleus [$\mathcal{M}_a^{(0)}$ and $\mathcal{M}_b^{(0)}$ correspond to diagrams (a) and (b) in Fig. 1, without the additional background field attachment]. At one-loop order, three diagrams contribute (cf. Fig. 1):

$$\begin{aligned} \mathcal{M}_a^{(1)} &= Ze^2 \int \frac{d^D L}{(2\pi)^D} \frac{(-1)}{\mathbf{L}^2} \frac{\not{p} + \not{q} + \mathbf{L} - m}{2(\mathbf{p} + \mathbf{q}) \cdot \mathbf{L} + \mathbf{L}^2 - 2p \cdot q - i0} \\ &\quad \times \not{p} \mathcal{M}_a^{(0)}, \\ \mathcal{M}_b^{(1)} &= Ze^2 \int \frac{d^D L}{(2\pi)^D} \frac{(-1)}{\mathbf{L}^2} \frac{\not{p} + \mathbf{L} - m}{2\mathbf{p} \cdot \mathbf{L} + \mathbf{L}^2 - i0} \not{p} \mathcal{M}_b^{(0)}, \\ \mathcal{M}_c^{(1)} &= Ze^3 \int \frac{d^D L}{(2\pi)^D} \frac{(-1)}{\mathbf{L}^2} \frac{\not{p} + \not{q} + \mathbf{L} - m}{2(\mathbf{p} + \mathbf{q}) \cdot \mathbf{L} + \mathbf{L}^2 - 2p \cdot q - i0} \\ &\quad \times \not{\epsilon}^* \frac{\not{p} + \mathbf{L} - m}{2\mathbf{p} \cdot \mathbf{L} + \mathbf{L}^2 - i0} \not{p}. \end{aligned} \quad (6)$$

The loop integrations are Euclidean, performed in $D = 3 - 2\epsilon$ dimensions, and the background field Feynman rule enforces $L^0 = v \cdot L = 0$.

At $O(Z\alpha^2)$, we need consider only the real part of the interference between $\mathcal{M}^{(0)}$ and $\mathcal{M}^{(1)}$. It turns out that, due to the structure of the one-loop integrals, almost all contributions to the matrix element are imaginary. This simple observation stems from the fact that three-dimensional Euclidean integrals yield a prefactor of $1/(-E^2 - i0)^{n/2+\epsilon}$ with n an odd integer and E^2 a positive number. For $\epsilon \rightarrow 0$, this gives $i(-1)^{(n-1)/2}/E^n$ such that $\mathcal{M}^{(1)}$ is out of phase with $\mathcal{M}^{(0)}$. Real contributions arise only in the presence of divergences, where

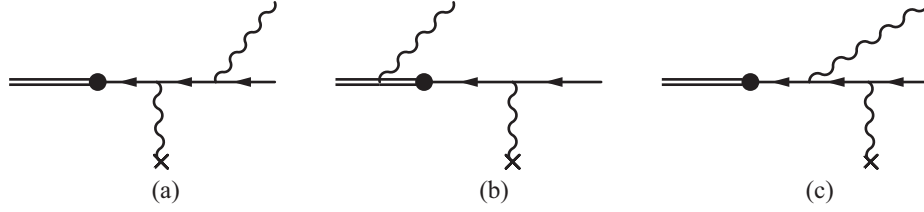


FIG. 1. Feynman diagrams that correct the real-photon amplitude at $O(Z\alpha)$. The circle denotes the weak vertex, and we use the Feynman rules discussed in [4,6] that make Z counting manifest. The diagrams (a), (b), and (c) correspond to the amplitudes in Eq. (6).

$$\left(\frac{1}{-E^2 - i0}\right)^{n/2+\epsilon} \times \frac{1}{\epsilon} = (-1)^{(n-1)/2} E^{-n} \left(\frac{i}{\epsilon} - 2i \log E - \pi + \mathcal{O}(\epsilon)\right). \quad (7)$$

The product $(i) \times \log(-1 - i0)$ supplies a real part proportional to π . Indeed, this is the source of the $O(Z\alpha)$ term in the expansion of the Fermi function (for a final-state positron):

$$\begin{aligned} \text{Diagram} &= Ze^2 \int \frac{d^D L}{(2\pi)^D} \frac{1}{\mathbf{L}^2} \frac{-2E_e - \not{L}\not{\psi}}{2\mathbf{p} \cdot \mathbf{L} + \mathbf{L}^2 - i0} \\ &= \frac{iZ\bar{\alpha}}{\beta} \left(\frac{\mu^2}{-4\mathbf{p}^2 - i0}\right)^\epsilon \left[\frac{1}{2\epsilon} + \left(1 + \frac{m}{E}\right) \frac{1}{2(1-2\epsilon)}\right] \\ &\rightarrow -\frac{\pi Z\bar{\alpha}}{2\beta}. \end{aligned} \quad (8)$$

where in the last line we have kept only the real piece. Notice that this logarithmic contribution is tied to the infrared divergence as $|\mathbf{L}| \rightarrow 0$. The same result is obtained by retaining only the IR-divergent contribution to the amplitude (“ $\sim$ ” denotes equality up to finite, pure imaginary, parts):

$$\text{Diagram} \sim Ze^2 \int \frac{d^D L}{(2\pi)^D} \frac{1}{\mathbf{L}^2} \frac{-2E_e}{2\mathbf{p} \cdot \mathbf{L} + \mathbf{L}^2 - i0} = \frac{iZ\bar{\alpha}}{\beta} \left(\frac{\mu^2}{-4\mathbf{p}^2 - i0}\right)^\epsilon \left[\frac{1}{2\epsilon}\right] \rightarrow -\frac{\pi Z\bar{\alpha}}{2\beta}. \quad (9)$$

Real contributions to $\mathcal{M}^{(1)}$ occur only when a $1/\epsilon$ divergence is present. Each of the integrals is UV finite, and their IR limits are given by²

$$\begin{aligned} \mathcal{M}_a^{(1)} &\sim 0, \\ \mathcal{M}_b^{(1)} &\sim \left[Ze^2 \int \frac{d^D L}{(2\pi)^D} \frac{1}{\mathbf{L}^2} \frac{-2E_e}{2\mathbf{p} \cdot \mathbf{L} + \mathbf{L}^2}\right] \mathcal{M}_b^{(0)}, \\ \mathcal{M}_c^{(1)} &\sim \left[Ze^2 \int \frac{d^D L}{(2\pi)^D} \frac{1}{\mathbf{L}^2} \frac{-2E_e}{2\mathbf{p} \cdot \mathbf{L} + \mathbf{L}^2}\right] \mathcal{M}_a^{(0)}. \end{aligned} \quad (10)$$

Using the integral (9), we see that

$$\mathcal{M}^{(1)} \sim \left(-\frac{\pi Z\bar{\alpha}}{2\beta}\right) (\mathcal{M}_a^{(0)} + \mathcal{M}_b^{(0)}) = \left(-\frac{\pi Z\bar{\alpha}}{2\beta}\right) \mathcal{M}^{(0)}, \quad (11)$$

and, hence, after performing spin or polarization sums and phase-space averages,

²The amplitudes represent hard matching coefficients, and the IR divergences are canceled by corresponding UV divergences in the soft function.

$$\begin{aligned} \frac{\langle |\mathcal{M}_{1\gamma}|^2 \rangle}{\langle |\mathcal{M}_{1\gamma}^{(0)}|^2 \rangle} &= 1 - \frac{\pi Z\alpha}{\beta} + \mathcal{O}(Z^2\alpha^2) \\ &= F(Z\alpha, E) + \mathcal{O}(Z^2\alpha^2). \end{aligned} \quad (12)$$

We expect this simple factorization to break at relative $O(Z^2\alpha^2)$. The result (12) holds for massive electrons (i.e., for arbitrary values of m/E). We have checked this result by explicitly computing the relevant master integrals.

III. $\mathcal{O}(Z\alpha^2)$ CORRECTIONS WITHOUT PHOTON EMISSION

Next, consider the double-virtual $O(Z\alpha^2)$ correction to the decay rate. We can extract the result in the limit of a massless electron from our recent work on neutron beta decay [7,8].³ In what follows, we neglect terms of $O(m^2/E_e^2)$.

³We have recently been informed of an explicit calculation by Crosas and Mereghetti for finite lepton mass [33]. Their result agrees with our extraction in the small electron-mass limit.

Consider the matrix element for $[Z+1] \rightarrow [Z]e^+\nu_e$, as a Dirac structure to be contracted as $\bar{u}_{\nu_e} \not{\epsilon}(1-\gamma_5) \mathcal{M}_{[Z+1] \rightarrow [Z]e^+\nu_e} v_e$. This can be related to the amplitude for the related process $[-Z-1] \rightarrow [-Z]e^-\bar{\nu}_e$ contracted as $\bar{u}_e \mathcal{M}_{[-Z-1] \rightarrow [-Z]e^-\bar{\nu}_e} \not{\epsilon}(1+\gamma_5) v_{\nu_e}$. The relation, to all orders in QED, is

$$\mathcal{M}_{[Z+1] \rightarrow [Z]e^+\nu_e} = \mathcal{C} \overline{\mathcal{M}}_{[-Z-1] \rightarrow [-Z]e^-\bar{\nu}_e}^* \mathcal{C}^{-1}, \quad (13)$$

where $\bar{\mathcal{M}} = \gamma^0 \mathcal{M}^\dagger \gamma^0$ and $\mathcal{C}$ satisfies $\mathcal{C} = \mathcal{C}^{-1} = \mathcal{C}^T = \mathcal{C}^*$, $\mathcal{C} \gamma^\mu \mathcal{C}^{-1} = -(\gamma^\mu)^*$. Using a “blob” for all diagrams not involving the background field (which is denoted by $\times$), the amplitude for $[-Z-1] \rightarrow [-Z]e^-\bar{\nu}_e$ can be expanded as

$$\begin{aligned} \mathcal{M}_{[-Z-1] \rightarrow [-Z]e^-\bar{\nu}_e} = & \text{Diagram 1} \\ & + (-Z) \times \text{Diagram 2} + (-Z)^2 \times \text{Diagram 3} \\ & + (-Z) \times \left(\text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} \right. \\ & \left. + \text{Diagram 7} + \text{Diagram 8} \right) + \dots, \end{aligned} \quad (14)$$

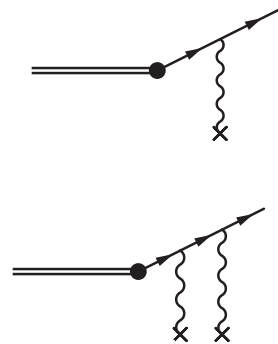
where the double line denotes a heavy particle with charge -1 , the solid line with arrow denotes an electron, and the cross denotes insertion of a background field with charge $+1$. In particular, at $Z = -1$, this expression reduces to the neutron decay amplitude that we have computed in Refs. [7,8]. Solving for the contributions of interest, we have, diagrammatically,

$$\begin{aligned} & \text{Diagram 9} + \text{Diagram 10} + \dots \\ & = \mathcal{M}_H - \mathcal{M}_H|_{v \rightarrow -v} - \left(\text{Diagram 11} + \text{Diagram 12} + \dots \right). \end{aligned} \quad (15)$$

In Eq. (15), the left-hand side (i.e., the first line) denotes all contributions containing at least one background field photon and one nonbackground field photon. The quantity $\mathcal{M}_H$ is the hard function describing neutron beta decay [7]. The final term in parentheses contains terms with only background field photons. Diagrams representing wave function renormalization have not been displayed but are included when computing physical amplitudes. At one-loop order, the left-hand side of Eq. (15) vanishes by definition, and the right-hand side is verified to vanish using Eq. (8) of Ref. [7] for $\mathcal{M}_H - \mathcal{M}_H|_{v \rightarrow -v}$ and Eq. (37) of Ref. [5] for the one-loop background field diagram. At two-loop order, we compute, in the small- m limit,

$$\begin{aligned} \mathcal{M}_H - \mathcal{M}_H|_{v \rightarrow -v} = & \frac{\bar{\alpha}}{4\pi} (F_H^{(1)} - \tilde{F}_H^{(1)}) + \left(\frac{\bar{\alpha}}{4\pi} \right)^2 [(F_R^{(2)} - \tilde{F}_R^{(2)}) \\ & + (F_H^{(2)} - \tilde{F}_H^{(2)}) + F_J^{(1)} (F_H^{(1)} - \tilde{F}_H^{(1)})], \end{aligned} \quad (16)$$

where $\tilde{F} = F|_{v \rightarrow -v}$ and F_H is the hard function, F_J the jet function, and F_R the remainder function for neutron decay defined in Eq. (16) of Ref. [7]. The background field diagrams in this limit are



$$\begin{aligned} & = \bar{\alpha} \left[\frac{i}{2} (2L_E - i\pi - 1) \right], \\ & = \bar{\alpha}^2 \left[-\frac{1}{8} (2L_E - i\pi)^2 + \frac{5}{4} - \frac{\pi^2}{12} \right], \end{aligned} \quad (17)$$

where $L_E = \log(2E/\mu)$. Using Eqs. (16) and (17) in Eq. (15), thus, determines the $\mathcal{O}(Z\alpha^2)$ contributions to the nuclear amplitude in Eqs. (13) and (14). Together with the $Z\alpha^2$ contribution including real photon emission from Eq. (12) and the known $\mathcal{O}(\alpha)$ correction, the total electron spectrum for $[Z+1] \rightarrow [Z]e^+\nu_e$ is

$$\begin{aligned} \frac{d\Gamma}{dE} = & \left(\frac{d\Gamma}{dE} \right)_{\text{tree}} \times \tilde{F}(Z\bar{\alpha}_1, E, \mu) \times \left(1 + \frac{\bar{\alpha}_1}{2\pi} \left(\hat{g}(E, \mu) + \frac{11}{4} \right) \right. \\ & \left. + Z\bar{\alpha}_1^2 \left[-\left(\frac{2n_e}{3} + 1 \right) L_E + \frac{8n_e}{9} + \frac{11}{4} - \zeta_2 \right] + \dots \right), \end{aligned} \quad (18)$$

where $\tilde{F}(Z\bar{\alpha}_1, E)$ is the all-orders expression containing the leading Z dependence [cf. Eq. (22) below],⁴ and $\hat{g}(E, \mu)$ is

⁴The definition (22) includes the additional factor $4\eta/(1+\eta)^2 = \mathcal{O}((Z\alpha)^4)$ appearing in the complete $\overline{\text{MS}}$ hard function [3].

the Sirlin g function but with m_p replaced by the $\overline{\text{MS}}$ subtraction point μ [cf. Eq. (C1) below; the term $11/4$ converts between $\overline{\text{MS}}$ and Sirlin's scheme [3]]. Here, n_e is the number of massless fermions (i.e., electrons) in the theory; in practice, we will always take $n_e = 1$. We have expressed the result in terms of the $\overline{\text{MS}}$ coupling with $n_e = 1$ electron flavor. This is related to on-shell α (or, equivalently, $\bar{\alpha}_0$) by $\alpha = \bar{\alpha}_1 [1 - \frac{n_e}{3\pi} \bar{\alpha}_1 L_m + (\frac{n_e}{3\pi} \bar{\alpha}_1 L_m)^2 + \dots]$, where $L_m = \log(m^2/\mu^2)$. We remark that the $Z\bar{\alpha}_1^2$ correction remains finite as $m \rightarrow 0$. This finite $m \rightarrow 0$ limit is guaranteed by the KLN theorem [31,32] (see Appendix B and related discussion in [9]) and provides a nontrivial check of our result.

Conventionally, the $\mathcal{O}(Z\alpha^2)$ correction is defined via a multiplicative factor⁵

$$\begin{aligned} \frac{d\Gamma}{dE} = & \left(\frac{d\Gamma}{dE} \right)_{\text{tree}} \times \tilde{F}(Z\alpha, E, \mu) \\ & \times (1 + \delta^{(1,0)}(E) + \delta^{(2,1)}(E) + \dots), \end{aligned} \quad (19)$$

with the series defined in terms of on-shell α . Translating from Eq. (18), we have

$$\delta^{(1,0)}(E) = \frac{\alpha}{2\pi} \left(\hat{g}(E, \mu) + \frac{11}{4} \right), \quad (20)$$

$$\begin{aligned} \delta^{(2,1)}(E) = & Z\alpha^2 \left[\frac{n_e}{3} L_m - \left(\frac{2n_e}{3} + 1 \right) L_E \right. \\ & \left. + \frac{8n_e}{9} + \frac{11}{4} - \zeta_2 + \mathcal{O}\left(\frac{m^2}{E^2} \right) \right]. \end{aligned} \quad (21)$$

We note that the energy dependence and the dependence on the electron mass match the existing calculation by Sirlin and Zucchini [9]. The finite contribution differs, as expected, since our calculation is performed in a (model-independent) pointlike EFT as opposed to the independent particle model used in [9].

Using our observation that real-virtual corrections factorize at $\mathcal{O}(Z\alpha^2)$, the relation (15) for the $\mathcal{O}(Z\alpha^2)$ virtual corrections, and the result from Ref. [7] for the two-loop correction at $Z = -1$, we thus obtain the new result (21) for $\delta^{(2,1)}(E)$, representing the complete result for the long-distance QED corrections through $\mathcal{O}(Z\alpha^2)$ in the limit of a massless electron. Our recent work on neutron beta decay also supplies the virtual corrections at $\mathcal{O}(\alpha^2)$ in the limit of a massless electron, but $\mathcal{O}(\alpha^2)$ real-virtual and real-real corrections are currently unknown [as are the finite parts of the $\mathcal{O}(Z^2\alpha^3)$ corrections].

⁵Our notation tracks powers of Z and α separately. The notation used by Sirlin and Zucchini is related by $\delta_2(E) = \delta^{(2,1)}(E)$ and $\delta_3 = \delta^{(3,2)}(E)$.

A. Comment on short-distance terms

The correction derived in Eq. (21) is computed in the low-energy effective theory where nuclear structure is not explicitly present. In past literature, it has been emphasized that nuclear structure modifies the resulting correction [9–11], and so before proceeding we comment on how these effects appear in the EFT formalism.

First, logarithms of the form $\log(\Lambda/2E_m)$ (where Λ is a nuclear scale) are captured by renormalization group (RG) evolution and, specifically, the $O(Z\alpha^2)$ coefficient of the anomalous dimension; these terms are included in our results. Second, there can be finite corrections induced at the nuclear scale. These are absorbed as $O(Z\alpha^2)$ corrections to the Wilson coefficient $\mathcal{C}(\mu)$ and depend on nuclear structure.

In the context of the present paper, it is sufficient to show that these corrections factorize from the long-distance matrix element. This is guaranteed because they are absorbed inside the Wilson coefficient $\mathcal{C}(\mu)$ which appears as the tree-level vertex at the heart of every diagram (at arbitrary order in perturbation theory). They are, therefore, not a part of the long-distance corrections that we discuss below but instead arise from short-distance regions that probe the scale of nuclear structure. They should be properly booked as a $O(Z\alpha^2)$ contribution to the nuclear-structure-dependent corrections (sometimes called δ_{NS} in the literature).

For the present work, we have taken an ansatz for the short-distance correction from the work of Sirlin and Zucchini as implemented by Towner and Hardy [2,34]. After subtracting the $\overline{\text{MS}}$ low-energy matrix element, this may be interpreted as their estimate for the structure-dependent short-distance region. We write this as $\mathcal{C}(\Lambda) \rightarrow \mathcal{C}(\Lambda)(1 + \frac{1}{2}\delta_{\text{NS,TH}}^{(2,1)})$, where $\delta_{\text{NS,TH}}^{(2,1)}$ is defined in Appendix C.

IV. CORRECTION TO THE LIFETIME

Let us now combine the above results and obtain a prediction for the decay rate. We are interested in long-distance QED corrections that are typically defined after extracting the leading result containing powers of $Z\alpha$. In the EFT approach we follow here, this is given by [3,5] (all expressions are for positron emitters)

$$\tilde{F}(Z\alpha, E, \mu) = \frac{4\eta}{(1+\eta)^2} \frac{2(1+\eta)}{[\Gamma(2\eta+1)]^2} \times |\Gamma(\eta+i\xi)|^2 e^{\pi\xi} (2pr)^{2(\eta-1)}, \quad (22)$$

where $r = 1/(e^{\gamma_E}\mu)$ (with γ_E the Euler-Mascheroni constant), $\eta = \sqrt{1 - Z^2\alpha^2}$, and $\xi = -Z\alpha E/p$. We define

$$1 + \delta'_R(\mu_H) \equiv R(\mu_H, \mu_L)M(\mu_L, E_m, m) + O\left(\frac{q_{\text{ext}}}{\Lambda}\right), \quad (23)$$

with [notice that $1 + \delta'_R(\mu_H)$ is independent of μ_L]

$$R(\mu_H, \mu_L) = \left[\frac{\mathcal{C}(\mu_L)/\mathcal{C}(\mu_H)}{\exp[(1-\eta)\log(\mu_H/\mu_L)]} \right]^2, \quad (24)$$

$$M(\mu_L, E, m) = \frac{\int d\Pi |\mathcal{M}(\mu_L)|^2}{\int d\Pi \tilde{F}(Z\alpha, E, \mu_L)}, \quad (25)$$

where $E_m \sim 10$ MeV is set by the nuclear mass splitting, $\Lambda \sim 100$ MeV represents a short-distance scale, and $\int d\Pi$ denotes integration over electron, neutrino, and photon phase space (when considering real emission). We do not consider terms suppressed by q_{ext}/Λ , where q_{ext} denotes the external lepton momentum, which arise from higher-dimensional operators in the pointlike effective theory.

We have split the long-distance correction $(1 + \delta'_R)$ into a piece induced by RG flow, $R(\mu_H, \mu_L)$, and the low-energy matrix element contribution $M(\mu_L, E_m, m)$. The denominators are the full amplitude in the limit that $\alpha \rightarrow 0$ with $Z\alpha$ held fixed. The RG term is governed by the anomalous dimension and QED beta function, contains all large logarithms, and does not depend on energy. The matrix-element term is evaluated at a low-scale $\mu_L \sim q_{\text{ext}}$ and does not contain large logarithms. With the above definition, we may write the decay rate as

$$\Gamma = \left(\frac{1}{\pi^3} \int_m^{E_m} dE p E (E_m - E)^2 \tilde{F}(Z, E, \mu_H) \right) \times |\mathcal{C}(\mu_H)|^2 \times (1 + \delta'_R(\mu_H)) + O\left(\frac{q_{\text{ext}}}{\Lambda}\right), \quad (26)$$

where $E_m = M_A - M_B = Q_{\text{EC}} - m$ is the energy emitted in the beta decay (we neglect nuclear recoil). Comparing to the typical parametrization, written as $(1 + \Delta_R) \times (1 + \delta_{\text{NS}} - \delta_C)(1 + \delta'_R)$, we see that $\mathcal{C}(\mu_H)$ has subsumed all short distance corrections (including nuclear structure dependence) as well as fundamental constants G_F and $|V_{ud}|$.

The factor $R(\mu_H, \mu_L)$ accounts for RG evolution from $\mu_H \sim \Lambda$ down to the dynamical scale of the beta decay $\mu_L \sim E_m$. It can be computed to high accuracy and controls contributions that are enhanced by logarithms of the form $\log(\mu_H/\mu_L) \sim \log(E_m R)$ (see Appendix A for details). It is conventional in the beta decay literature to include the Fermi function in the phase-space measure. We implement this as

$$\langle \dots \rangle_F = \frac{\int_m^{E_m} dE p E (E_m - E)^2 \tilde{F}(Z, E, \mu) \times (\dots)}{\int_m^{E_m} dE p E (E_m - E)^2 \tilde{F}(Z, E, \mu)}. \quad (27)$$

The μ dependence in $\tilde{F}(Z, E, \mu)$ cancels to all orders in $Z\alpha$ between numerator and denominator in Eq. (27).

A. Comment on one-loop corrections

In this work, we restrict ourself to an analysis at leading power, which is equivalent to taking $ER \rightarrow 0$ (and by extension $mR \rightarrow 0$ and $|\mathbf{p}|R \rightarrow 0$). At subleading power, it is well known that one encounters structure-dependent terms that can depend on lepton energy. For example, as we discuss in Appendix D, if one includes certain structure-dependent terms in the shape factor or definition of the Fermi function, this can slightly alter the phase-space average of the one-loop radiative corrections.

The appropriate quantity to use at leading power is the pointlike Fermi function or, equivalently (for the purposes of phase-space averaging), the resummation of all leading-in- $Z\alpha$ diagrams in the EFT. Any structure dependence that depends on E or $\mathbf{p}$ will correspond to higher-dimensional operators in the pointlike effective theory, beyond the single operator listed in Eq. (1). The one-loop renormalization of these operators, and their contribution to the real photon spectrum, is not guaranteed to be captured by the ansatz of $d\Phi F(Z, E) \rightarrow d\Phi F(Z, E) \times (1 + \delta^{(1,0)}(E))$.

In Appendix D, we find that, in order to reproduce Towner and Hardy's phase-space averages of the one-loop corrections through three digits, structure-dependent corrections must be included. We account for these effects using the analytic expressions taken from Ref. [35] (see Appendix D for a discussion). The numerical size of

$$\frac{\alpha}{2\pi} \langle g(E, E_m, m) \rangle_F - \frac{\alpha}{2\pi} \bar{g}, \quad (28)$$

with $\bar{g}$ taken from Table V of [34], gives a sense for the error induced by the factorization ansatz $d\Phi F(Z, E) \times (1 + \delta^{(1,0)}(E))$ at subleading power. We estimate the uncertainty due to this effect in Appendix D.

B. New results for outer corrections

Here, we present our results for the pointlike long-distance corrections, $(1 + \delta'_R(\mu_H))$ defined in this work. We present results for $R(\mu_H, \mu_L)$ and $M(\mu_L)$ separately and use the residual dependence on μ_L as an estimate of neglected higher-order perturbative corrections. We fix $\mu_H = \Lambda = \sqrt{6}/\langle r^2 \rangle^{1/2}$, where $\langle r^2 \rangle^{1/2}$ is the root-mean-square radius of the daughter nucleus, and vary $\mu_L = E_m \dots 4E_m$. We use the form of the expressions in terms of $\bar{\alpha}_L$ given in Eqs. (A8) and (B9), with central value $\mu_L = 2E_m$.

1. Finite-mass corrections

The scale variation used in Table II is an estimate of missing finite terms at $O(Z^2\alpha^3)$, i.e., those without a logarithmic enhancement, and other higher-order effects. It does not account for any error incurred from the small-mass expansion of $\delta^{(2,1)}(E)$, nor does it account for the pointlike approximation used herein.

To estimate these missing uncertainties, we simply assign an error of $Z\alpha^2 m^2/E_m^2$ for finite-mass corrections to $\delta^{(2,1)}(E)$. The size of m^2/E_m^2 is largest for the lightest nuclei ^{10}C , ^{14}O , and ^{26m}Al ; however, the modest values of Z for these nuclei help keep this error under control:

$$\begin{aligned} ^{10}\text{C}: Z\alpha^2 \left(\frac{m}{E_m}\right)^2 &= 0.36 \times 10^{-4}, \\ ^{14}\text{O}: Z\alpha^2 \left(\frac{m}{E_m}\right)^2 &= 0.18 \times 10^{-4}, \\ ^{26m}\text{Al}: Z\alpha^2 \left(\frac{m}{E_m}\right)^2 &= 0.12 \times 10^{-4}. \end{aligned} \quad (29)$$

This quantity is smaller than 1×10^{-5} for all other nuclei. The estimated error is, thus, smaller than the perturbative error estimated in Table I, except for the three lightest beta emitters where the two uncertainties are comparable.

2. Mixing of structure-dependent and radiative corrections

Structure-dependent terms that depend on the electron's energy at tree level can be identified with higher-dimensional operators in the effective theory used here. These operators receive different radiative corrections than the leading-order operator in Eq. (1). To the best of our knowledge, these corrections have not been calculated. We estimate, on the basis of the results in Appendix D, that these terms are $\sim 10^{-5}$ for light nuclei but can grow to become $\sim 10^{-4}$ for larger- Z beta emitters. These are smaller than the perturbative error estimated in Table I.

C. Comparison to Towner and Hardy

In order to take the above analysis and infer the associated shift in $|V_{ud}|$, we reinterpret the analysis of Towner and Hardy [2]. The details are discussed in Appendix C, and we simply quote the main results here. Towner and Hardy's calculation of $\delta_2^{(\text{TH})}$ contains the sum of the short- and long-distance contributions. We do not adjust their value of $\delta_2^{(\text{TH})}$ and, rather, interpret it as a long-distance piece that precisely matches our formulas and a short-distance piece that we then infer from their calculation. This short-distance piece is interpreted as a $Z\alpha^2$ shift to $\mathcal{C}(\mu)$ or, equivalently, to an $O(Z\alpha^2)$ contribution to the nuclear-structure-dependent correction which we denote as $\delta_{\text{NS}}^{(2,1)}$.

We first reorganize the existing ingredients of Towner and Hardy's calculation according to the factorization theorem that can be derived from Eq. (1) (see [3] for details). This leads to the shift denoted "convention shift" in Table II. Next, we compare the results of the present paper with the long-distance corrections of Towner and Hardy defined in this new convention. This leads to the

TABLE I. Radiative corrections computed using the conventions described in the main text for the ten nuclei with the lowest $\mathcal{F}t$ errors in [2]. We fix $\mu_H = \Lambda = \sqrt{6}/\langle r^2 \rangle^{1/2}$ with the root-mean-square radius of the daughter, $\langle r^2 \rangle^{1/2}$, taken from [36] (except for ^{34}Cl , for which we use $\langle r^2 \rangle^{1/2} = 3.39$ fm [37]) to match [2], while scale variation errors show dependence on different choices of μ_L . We take a central value of $\mu_L = 2E_m$ and vary $\mu_L \rightarrow 2\mu_L$ and $\mu_L \rightarrow \frac{1}{2}\mu_L$ (for ^{26m}Al , there is an accidental cancellation and we inflate the error estimate by hand to 10^{-5}). The RG factor is computed using Eq. (A8), while the matrix element factor is computed using the contributions under the integral in Eq. (B9).

Beta emitter	Z_{daughter}	Λ [MeV]	$2E_m$ [MeV]	$R(\mu_H, \mu_L)$	$M(\mu_L)$	$1 + \delta'_R(\mu_H)$
^{10}C	5	199.1	2.794	1.01634	0.99793	1.01424(5)
^{14}O	7	188.9	4.641	1.01464	0.99805	1.01266(3)
^{26m}Al	12	159.3	7.443	1.01309	0.99826	1.01133(1)
^{34}Cl	16	147.2	9.961	1.01226	0.99842	1.01066(2)
^{34}Ar	17	142.6	11.102	1.01181	0.99844	1.01023(2)
^{38m}K	18	142.0	11.066	1.01199	0.99851	1.01048(3)
^{42}Sc	20	137.8	11.831	1.01191	0.99860	1.01050(4)
^{46}V	22	134.0	13.083	1.01166	0.99869	1.01033(6)
^{50}Mn	24	132.1	14.247	1.01152	0.99878	1.01029(7)
^{54}Co	26	130.9	15.467	1.01141	0.99888	1.01027(9)

TABLE II. Update to the long-distance radiative corrections from Towner and Hardy using the procedure described in Appendix C for the ten nuclei considered in this work. Additional digits are shown when they affect the final digit displayed in the “total shift” column. The convention shift stems from a reorganization of Towner and Hardy’s calculation of $(1 + \delta'_R)_{\text{TH}}$. The total shift includes both the effects of this rearrangement and the incorporation of the higher-order terms calculated in this work. The long-distance shift does not alter the $Z\alpha^2$ short-distance correction $\delta_{\text{NS,TH}}^{(2,1)}$, whose size we estimate using Towner and Hardy’s calculation. This quantity is sizable and should be revisited (in addition to uncertainties on $\delta_{\text{NS}} - \delta_C$) in a consistent microscopic theory of the nucleus.

Beta emitter	Z_{daughter}	Convention shift	Long-distance shift	Total shift	$\delta_{\text{NS,TH}}^{(2,1)} [10^{-4}]$
^{10}C	5	1.000105	0.999967	1.00007	2.1
^{14}O	7	1.00012	1.00000	1.00012	2.9
^{26m}Al	12	1.00013	1.00006	1.00019	4.3
^{34}Cl	16	1.00013	1.00014	1.00027	5.5
^{34}Ar	17	1.00013	1.00016	1.00029	5.8
^{38m}K	18	1.00015	1.00017	1.00032	6.2
^{42}Sc	20	1.000154	1.000214	1.00037	6.8
^{46}V	22	1.00015	1.00027	1.00042	7.2
^{50}Mn	24	1.00015	1.00033	1.00048	7.6
^{54}Co	26	1.00016	1.00039	1.00055	8.0

quantity denoted “long-distance shift.” The product of convention shift and long-distance shift gives the total shift.

In Fig. 2, we plot the total shifts from Table II as applied to the $\mathcal{F}t$ values from Fig. 3 of [2]; we do not adjust the error bars. The shifts are sizable. Our analysis incorporates many terms that are not present in Towner and Hardy’s analysis. Let us list these for illustration: (i) subleading logarithms at $O(\bar{\alpha}^2 L)$ from the RG evolution [+0.4], (ii) the coefficient at $Z^2 \bar{\alpha}^3 L$ [−0.7] and $Z^2 \bar{\alpha}^3 L L_m$ [+1.9], (iii) the coefficient at $O(Z \bar{\alpha}^3 L^2)$ [+0.3], (iv) the coefficient at $O(Z^2 \bar{\alpha}^4 L^3)$ [+0.02], and (v) the coefficient of $\log(m)$ singularity at $O(Z^2 \bar{\alpha}^3)$ [−1.0]. The numbers in square brackets give the shift to the total radiative correction,

$1 + \delta'_R(\mu_H = \Lambda)$, for ^{26m}Al (in units of 10^{-4}) relative to a “baseline” where this term is set to zero by hand.

If we perform a weighted average of the $\mathcal{F}t$ values of the ten nuclei⁶ considered here, we find $\langle \mathcal{F}t \rangle = (3072.1 \pm 0.6)$ s. If the updated radiative corrections are applied using the shift discussed above, we instead obtain $\langle \mathcal{F}t \rangle = (3073.0 \pm 0.6)$ s. This shift is $2.5\times$ larger than the estimated uncertainty from δ'_R in [2], is $1.5\times$ larger

⁶Towner and Hardy take the 15 most precise superallowed emitters and obtain $\langle \mathcal{F}t \rangle = (3072.24 \pm 0.57)$ s [2]. The 0.57 error is statistical and should be compared to the estimated theory errors from $\delta_{\text{NS}} [\pm 1.72]$ and $\delta'_R [\pm 0.36]$.

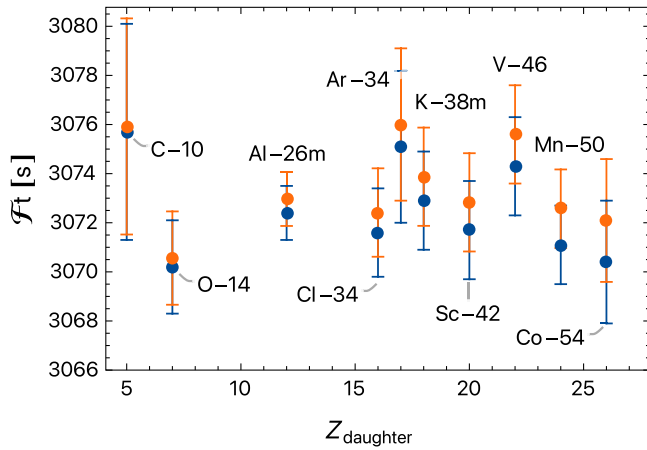


FIG. 2. A reproduction of Fig. 3 of Towner and Hardy [2] for the $\mathcal{F}t$ values of the ten nuclei considered in this work. Error bars show purely experimental errors and do not account for theoretical uncertainties related to radiative corrections. The data points of Towner and Hardy are shown in blue (dark gray). The same data points multiplied by the “total shift” column in Table II are shown in orange (light gray).

than the statistical error, and is roughly half the estimated error from δ_{NS} ; the shift exacerbates the first-row Cabibbo-Kobayashi-Maskawa unitarity deficit (however, see, e.g., [38] for nuclear structure effects which do the opposite).

V. SUMMARY AND DISCUSSION

In this work, we have provided a comprehensive update to the long-distance radiative corrections for superallowed beta decays. Our results are derived using a rigorous factorization of the decay amplitude at leading power [i.e., neglecting the interference of structure-dependent higher-dimensional operators and $O(\alpha)$ radiative corrections]. This factorization of the amplitude has allowed us to quantify errors incurred by Towner and Hardy’s factorization ansatz, as well as to control higher-order radiative corrections proportional to logarithms of the form $\log(\Lambda/(2E_m))$ and $\log(2E_m/m)$. We replace Sirin’s heuristic estimate for the $Z^2\alpha^3$ correction with a systematic determination of logarithmic enhancements using renormalization group methods.

Our three main theoretical results are as follows.

- (i) The amplitude factorizes at leading power, and this factorization theorem constrains the possible form of higher-order radiative corrections.
- (ii) The virtual corrections to real-photon radiation are captured by the Fermi function at $O(Z\alpha)$.
- (iii) The $O(Z\alpha^2)$ matrix element of the pointlike effective theory has been presented. Combined with the results from [3–5], this allows us to control all effects of $O(Z\alpha^2)$ including logarithmic enhancements which we count numerically as $\log(\Lambda/2E_m) \sim \log(2E_m/m) \sim Z$.

The relevant formulas for these results are given by Eqs. (23)–(25) for the factorization theorem, Eq. (12) for real-photon corrections, and Eqs. (18) and (A8) [or Eqs. (21) and (A7) in terms of on-shell α] for the analytic formulas for long-distance radiative corrections. Our scale variation uncertainty suggests that currently unknown $Z^2\alpha^3$ corrections and finite-mass effects are $\sim 3 \times 10^{-5}$ in size. This is much smaller than $\delta_{\text{NS,TH}}^{(2,1)}$, and this quantity should be revisited in light of its sizable impact on $|V_{ud}|$.

Future work will involve a careful merger of the long-distance EFT calculations presented here with short-distance results. Ideally, both theories will be formulated using minimal subtraction or another closely related scheme. The results presented here are sufficient for $\sim 10^{-4}$ precision, which we expect to be sufficient for the near-term and medium-term future of precision beta decay experiments.

Motivated by this goal, and given that the long-distance region is perturbatively calculable, we list some open questions in the low-energy theory whose answers would reduce theoretical uncertainty. These include finite electron-mass corrections to the long-distance $Z\alpha^2$ results presented here; the proper one-loop renormalization of higher-dimensional operators; a calculation of the $Z^2\alpha^3$ matrix element in the small electron-mass limit; and an analysis of $O(\alpha^2)$ corrections including real-virtual corrections and double-real emission. Most pressing, however, is the control of the $Z\alpha^2$ contribution to the short-distance matching, $\delta_{\text{NS}}^{(2,1)}$, and the control of one-loop radiative corrections sensitive to nuclear structure. We look forward to future progress on these open questions and toward a consistent implementation of the results presented above with a short-distance matching calculation.

Note added. Recently, the authors of Ref. [33] informed us of their own independent calculation of the $Z\alpha^2$ correction. Their analysis includes full electron-mass dependence (going beyond the result derived herein) and, furthermore, includes corrections to correlation coefficients. Our work provides an independent extraction of the small- m limit of the $Z\alpha^2$ correction, a novel analysis of real-virtual corrections which explains their factorization, higher-order renormalization group evolution, and supplies KLN constraints on the amplitude at three-loop order. A more detailed account has been added in Appendix E.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: RESUMMATION OF LOGARITHMS

For convenience, we provide here a discussion of the renormalization group flow (see also Ref. [3]). The renormalization group equation governing the flow of $\mathcal{C}(\mu)$ is

$$\frac{d \log \mathcal{C}}{d \log \mu} = \gamma_{\mathcal{O}}. \quad (\text{A1})$$

Since $\bar{\alpha}(\mu)$ flows with μ , it is convenient to express the solution to this equation as

$$\log \left(\frac{\mathcal{C}(\mu_H)}{\mathcal{C}(\mu_L)} \right) = \int_{\bar{\alpha}_L}^{\bar{\alpha}_H} \frac{\gamma_{\mathcal{O}}(\bar{\alpha})}{\beta(\bar{\alpha})} d\bar{\alpha}, \quad (\text{A2})$$

where the β function is defined via

$$\beta(\bar{\alpha}) \equiv \frac{d}{d \log \mu} \bar{\alpha}(\mu) = -2\alpha \left[\frac{\bar{\alpha}}{4\pi} \beta_0 + \left(\frac{\bar{\alpha}}{4\pi} \right)^2 \beta_1 + \dots \right]. \quad (\text{A3})$$

For QED,⁷ the relevant values are $\beta_0 = -(4/3)n_e$ and $\beta_1 = -4n_e$. One expands the anomalous dimension as

$$\gamma_{\mathcal{O}} = \frac{\bar{\alpha}}{4\pi} [\gamma_0^{(1)}] + \left(\frac{\bar{\alpha}}{4\pi} \right)^2 [Z^2 \gamma_1^{(0)} + Z \gamma_0^{(1)} + \gamma_0^{(2)}] + \left(\frac{\bar{\alpha}}{4\pi} \right)^3 [Z^2 \gamma_2^{(1)} + \dots] + \left(\frac{\bar{\alpha}}{4\pi} \right)^4 [Z^4 \gamma_3^{(0)} + \dots] + \dots, \quad (\text{A4})$$

where we have retained only the terms needed for the present analysis, which can be found in Ref. [4].

Next, introduce the parameter λ and count $\alpha \sim \lambda^2$ and $Z \sim 1/\lambda$. The RG equations may be expanded and integrated order by order in λ . This yields an expression for the ratio $\mathcal{C}(\mu_H)/\mathcal{C}(\mu_L)$ as a function of $\bar{\alpha}_H \equiv \bar{\alpha}(\mu_H)$ and $\bar{\alpha}_L \equiv \bar{\alpha}(\mu_L)$ as given explicitly in [3]. Integrating Eq. (A3), we obtain

$$\frac{\bar{\alpha}_H}{\bar{\alpha}_L} = 1 - \frac{\bar{\alpha}_L}{2\pi} \beta_0 L + \left(\frac{\bar{\alpha}_L}{2\pi} \right)^2 \left(\beta_0^2 L^2 - \frac{\beta_1}{2} L \right) + \mathcal{O}(\lambda^4), \quad (\text{A5})$$

where we assign the scaling $L = \log(\mu_H/\mu_L) \sim 1/\lambda$. The $\overline{\text{MS}}$ and on-shell couplings are given by⁸

$$\bar{\alpha}(\mu_L) = \alpha \left[1 - \frac{\alpha}{2\pi} \beta_0 \log \left(\frac{\mu_L}{m} \right) + \left(\frac{\alpha}{2\pi} \right)^2 \beta_0^2 \log^2 \left(\frac{\mu_L}{m} \right) + \mathcal{O}(\lambda^3) \right], \quad (\text{A6})$$

where we take $\log(\mu_L/m) \lesssim 1/\lambda$.⁹ Introducing $L_m = \log(m^2/\mu_L^2)$, we obtain the renormalization factor in Eq. (24), expressed in terms of on-shell α ,

⁷In principle, we could also include a dynamical muon for $\mu \geq m_\mu$ and dynamical pions for $\mu \geq m_\pi$. For simplicity, we opt instead to work exclusively in $n_e = 1$ QED. The price of this choice is that matching from a microscopic theory at the scale $\mu_H = \Lambda$ will involve small logarithms at $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(Z\alpha^3)$ of the form $\log(\Lambda/m_\mu) \approx 0.5$, $\log(m_\pi/m_\mu) \approx 0.25$, and $\log(\Lambda/m_\pi) \approx 0.35$ with numerical values obtained using $\Lambda = 190$ MeV; these count as $\mathcal{O}(\lambda^4)$ [see discussion above Eq. (A5)] in our counting and are absorbed into the matching calculation that defines the short-distance correction.

⁸The largest corrections in our counting are all $\mathcal{O}(\lambda)$, e.g., $\mathcal{O}(Z\alpha)$ and $Z^2\alpha^2 L$. Therefore, when relating α and $\bar{\alpha}$, one need work through $\mathcal{O}(\lambda^2)$ only when a global error of $\mathcal{O}(\lambda^3)$ is desired.

⁹Higher-order terms can be obtained from Eqs. (29)–(32) of [39].

$$\begin{aligned}
R(\mu_H, \mu_L) = & 1 + \alpha \left[\frac{3}{2\pi} L \right] + \alpha^2 \left[ZL + \frac{13}{8\pi^2} L^2 + \left(\frac{1}{3} - \frac{35}{48\pi^2} \right) L - \frac{1}{2\pi^2} LL_m \right] \\
& + \alpha^3 \left[\frac{2}{3\pi} Z^2 L^2 + \left(\frac{\pi}{6} - \frac{3}{\pi} \right) Z^2 L + \frac{13}{6\pi} ZL^2 + \frac{221}{144\pi^3} L^3 - \left(\frac{2Z(Z+1)}{3\pi} L + \frac{13L^2}{12\pi^3} \right) L_m + \frac{L}{6\pi^3} L_m^2 \right] \\
& + \alpha^4 \left[\frac{13}{9\pi^2} Z^2 L^3 - \frac{5Z^2}{3\pi^2} L^2 L_m + \frac{Z^2}{3\pi^2} LL_m^2 \right] + \mathcal{O}(\lambda^4),
\end{aligned} \tag{A7}$$

where we have allowed $L_m \sim 1/\lambda$ (given that $2E_m/m$ is often numerically comparable to $\Lambda/2E_m$ for the nuclei of interest) and truncated our expansion at $\mathcal{O}(\lambda^3)$. Equation (A7) generalizes the equivalent expression given in Supplemental Material of Ref. [3] for $L_m \neq 0$. We also supply the result for $R(\mu_H, \mu_L)$ in terms of $\bar{\alpha}_L$:

$$\begin{aligned}
R(\mu_H, \mu_L) = & 1 + \bar{\alpha}_L \left[\frac{3}{2\pi} L \right] + \bar{\alpha}_L^2 \left[ZL + \frac{13}{8\pi^2} L^2 + \left(\frac{1}{3} - \frac{35}{48\pi^2} \right) L \right] \\
& + \bar{\alpha}_L^3 \left[\frac{2}{3\pi} Z^2 L^2 + \left(\frac{\pi}{6} - \frac{3}{\pi} \right) Z^2 L + \frac{13}{6\pi} ZL^2 + \frac{221}{144\pi^3} L^3 - \frac{2}{3\pi} Z^2 LL_m \right] \\
& + \bar{\alpha}_L^4 \left[\frac{13}{9\pi^2} Z^2 L^3 - \frac{Z^2}{\pi^2} L^2 L_m - \frac{Z^2}{3\pi^2} LL_m^2 \right] + \mathcal{O}(\lambda^4).
\end{aligned} \tag{A8}$$

Note that the explicit dependence on L_m arises from the conventional definition of R involving on-shell α .

The analysis written above can be readily extended to $\mathcal{O}(\lambda^4)$ with currently available ingredients; this pushes well beyond the $\mathcal{O}(10^{-4})$ precision target of superallowed beta decays. Therefore, logarithmic enhancements are completely controlled for the purposes of $|V_{ud}|$ extractions.

APPENDIX B: KINOSHITA-LEE-NAUENBERG CONSTRAINTS

The KLN theorem [31,32] guarantees that the inclusive decay rate for a superallowed beta decay has a finite limit as $m \rightarrow 0$. This implies that corrections of the form $\bar{\alpha}_1^2(\mu) \log(\mu/m)$ (for example) are forbidden for some fixed μ as $m \rightarrow 0$. As a result, we can guarantee that our perturbative expressions are free of any electron-mass logarithms to all orders in perturbation theory if the result is expressed in terms of $\bar{\alpha}_L$ rather than α . This proves to be a powerful constraint on the amplitude as was already appreciated by Sirlin and Zucchini in their work at $\mathcal{O}(Z\alpha^2)$ [9].

Let us consider the full amplitude

$$\Gamma = |\mathcal{C}(\mu_H)|^2 \left| \frac{\mathcal{C}(\mu_L)}{\mathcal{C}(\mu_H)} \right|^2 \int d\Phi_e |\mathcal{M}_{0\gamma}(\mu_L)|^2 \left[1 + \int d\Phi_{1\gamma} \frac{|\mathcal{M}_{1\gamma}(\mu_L)|^2}{|\mathcal{M}_{0\gamma}(\mu_L)|^2} + \dots \right], \tag{B1}$$

where $d\Phi_e = pE(E_m - E)^2 dE$ while $d\Phi_{1\gamma}$ is the additional phase space of a single photon defined at fixed electron energy.

We know the full leading-in- $Z\alpha$ expression for $|\mathcal{M}_{0\gamma}(\mu_L)|^2$ [5] and can readily express the result in terms of $\bar{\alpha}_L$. We may then define corrections as a series in $\bar{\alpha}_L$:

$$\begin{aligned}
\Gamma = & |\mathcal{C}(\mu_H)|^2 \left| \frac{\mathcal{C}(\mu_L)}{\mathcal{C}(\mu_H)} \right|^2 \int d\Phi_e \tilde{F}(Z\bar{\alpha}_L, E, \mu_L) \left[1 + \frac{\bar{\alpha}_L}{2\pi} \left(\hat{g}(E, \mu_L) + \frac{11}{4} \right) + Z\bar{\alpha}_L^2 \left(-\frac{5}{3} L_E + \frac{8}{9} + \frac{11}{4} - \zeta_2 \right) \right. \\
& \left. + Z^2 \bar{\alpha}_L^3 f^{(3,2)} + \bar{\alpha}_L^2 f^{(2,0)} + \dots \right],
\end{aligned} \tag{B2}$$

where the functions $f^{(2,3)}$ are currently undetermined but whose $\log(m)$ terms will be constrained by the KLN theorem. The function $\hat{g}(E, \mu)$ contains electron-mass singularities (see, e.g., [40]):

$$\begin{aligned}
\hat{g}(E, \mu) + \frac{11}{4} = & \log\left(\frac{2E_m}{m}\right) \left[\frac{4(1-x)}{3x} - 3 + \frac{(1-x)^2}{6x^2} + 4 \log\left(\frac{1-x}{x}\right) \right] \\
& + 3 \log\left(\frac{\mu}{2E_m}\right) + 2 - \frac{2\pi^2}{3} + 4(\log x - 1) \left[\frac{1-x}{3x} - \frac{3}{2} + \log\left(\frac{1-x}{x}\right) \right] \\
& + \frac{(1-x)^2}{6x^2} \log x + O\left(\frac{m^2}{E^2} \log \frac{m}{E}\right),
\end{aligned} \tag{B3}$$

where $x = E/E_m$. The electron-mass singularity cancels (as is guaranteed by the KLN theorem) when integrated over tree-level phase space:

$$\int_0^1 dx x^2 (1-x)^2 \left[\frac{4(1-x)}{3x} - 3 + \frac{(1-x)^2}{6x^2} + 4 \log\left(\frac{1-x}{x}\right) \right] = 0. \tag{B4}$$

This observation, and the fact that the Fermi function becomes energy independent at $O(Z\alpha)$ for a massless electron, fixes the $\log(m)$ dependence of $\delta^{(2,1)}$ as was first noticed by Sirlin and Zucchini [9].

We may now play the same game to fix the $\log(m)$ dependence of the $O(\alpha^2)$ and $O(Z^2\alpha^3)$ terms. This is most easily phrased in terms of $\bar{\alpha}_L$, because this quantity is fixed as $m \rightarrow 0$. We demand that Γ be given by

$$\Gamma = \Gamma^{(0)} + \frac{\bar{\alpha}_L}{2\pi} \Gamma^{(1)} + \left(\frac{\bar{\alpha}_L}{2\pi}\right)^2 \Gamma^{(2)} + \left(\frac{\bar{\alpha}_L}{2\pi}\right)^3 \Gamma^{(3)} + \dots, \tag{B5}$$

where $\Gamma^{(n)}$ may contain powers of Z^m with $m \leq n$ and logarithmic enhancements of the form $\log(\Lambda/2E_m)$ but, crucially, *does not contain* any terms of the form $\log(2E_m/m)$ or $\log(\Lambda/m)$.

Let us compare to the factorized form of Eq. (B2) prior to integration over phase space. Since $\hat{g}(E, \mu)$ contains a term proportional to $\log(m)$, its cancellation in Eq. (B5) is nontrivial. At $O(\bar{\alpha}_L)$, the cancellation proceeds via Eq. (B4). At $O(Z\bar{\alpha}_L^2)$, in the small electron-mass limit, the Fermi function reads $\tilde{F}(Z\bar{\alpha}_L, E, \mu_L) = (1 - \pi Z\bar{\alpha}_L + \dots)$, and Eq. (B4) thus implies that the $\log(m)$ cross term between $\tilde{F}(Z\bar{\alpha}_L, E, \mu_L)$ and $\hat{g}(E, \mu)$ again vanishes. A corollary of this observation is that the function appearing at $O(Z\bar{\alpha}_L^2)$ in Eq. (B5) is free of any $\log(m)$ singularity, in agreement with explicit computation.

Let us now turn to the undetermined functions $f^{(3,2)}$ and $f^{(2,0)}$. We may immediately conclude that $f^{(2,0)}$ contains no $\log(m)$ singularities $f^{(2,0)} = 0 + \dots$ with the ellipsis denoting $O(1)$ terms as $m \rightarrow 0$. For $f^{(3,2)}$, we instead find that a $\log(m)$ singularity must be present to cancel against the associated singularity generated by the cross term between the $Z^2\bar{\alpha}_L^2$ term in the Fermi function and the $\log(m)$ singularity from $\frac{\bar{\alpha}_L}{2\pi} \hat{g}(E, \mu)$. This arises because of the logarithmic energy dependence from

$$\begin{aligned}
\tilde{F}(Z\bar{\alpha}_L, E, \mu_L) = & 1 - \pi Z\bar{\alpha}_L + Z^2\bar{\alpha}_L^2 \left(\frac{\pi^2}{3} + \frac{11}{4} - \log \frac{2E}{\mu} \right) \\
& + \dots
\end{aligned} \tag{B6}$$

leads to the nonvanishing integral

$$\begin{aligned}
& \int_0^1 dx x^2 (1-x)^2 \left[\frac{4(1-x)}{3x} - 3 + \frac{(1-x)^2}{6x^2} + 4 \log\left(\frac{1-x}{x}\right) \right] \\
& \times [-\log(x)] = \frac{\pi^2}{45} - \frac{1}{12},
\end{aligned} \tag{B7}$$

which implies that¹⁰ [recall the tree-level phase space gives $\int_0^1 dx x^2 (1-x)^2 = 1/30$]

$$\begin{aligned}
f^{(3,2)} = & \frac{30}{2\pi} \log\left(\frac{2E_m}{m}\right) \left(\frac{1}{12} - \frac{\pi^2}{45} \right) + \dots \\
= & \frac{1}{4\pi} \log\left(\frac{m^2}{\mu^2}\right) \left(\frac{2\pi^2}{3} - \frac{5}{2} \right) + \dots,
\end{aligned} \tag{B8}$$

where the ellipses represent terms that are $O(1)$ in the $m \rightarrow 0$ limit, including $\log(2E_m/\mu)$ (hence why the two forms written above are equivalent). We have, thus, predicted the coefficient of $\log(m)$ in $f^{(2,3)}$ and $f^{(2,0)}$ (which vanishes) as introduced in Eq. (B2) on the basis of the KLN theorem.

Counting $\log(2E_m/m) \sim \log(\Lambda/2E_m) \sim Z \sim 1/\lambda$ and $\bar{\alpha}_L \sim \lambda^2$ as above, we have

$$\begin{aligned}
\Gamma = & |\mathcal{C}(\mu_H)|^2 \left| \frac{\mathcal{C}(\mu_L)}{\mathcal{C}(\mu_H)} \right|^2 \int d\Phi_e \tilde{F}(Z\bar{\alpha}_L, E, \mu_L) \\
& \times \left[1 + \frac{\bar{\alpha}_L}{2\pi} \left(\hat{g}(E, \mu) + \frac{11}{4} \right) + Z\bar{\alpha}_L^2 \left(-\frac{5}{3} L_E + \frac{8}{9} + \frac{11}{4} - \zeta_2 \right) \right. \\
& \left. + \frac{1}{4\pi} Z^2 \bar{\alpha}_L^3 L_m \left(\frac{2\pi^2}{3} - \frac{5}{2} \right) + O(\lambda^4) \right].
\end{aligned} \tag{B9}$$

Since we know $\tilde{F}(Z\bar{\alpha}_L, E, \mu_L)$ to all orders in perturbation theory and $|\mathcal{C}(\mu_L)/\mathcal{C}(\mu_H)|^2$ through $O(\lambda^3)$ [with the ability to compute through $O(\lambda^4)$ if desired], this result is

¹⁰Strictly speaking we can conclude only that the integral of $f^{(3,2)}$ over phase space yields $1/12 - \pi^2/45$; however, for the purposes of computing the superallowed lifetimes, these two statements are equivalent.

complete through $O(\lambda^3)$. One can use the relation between $\bar{\alpha}_L$ and α to rewrite this expression in terms of the on-shell coupling; however, we find it convenient to use the running coupling in our computations.

We use Eq. (B9) to compute $1 + \delta'_R(\mu_H)$ as presented in Table I, evaluating $\tilde{F}(Z\bar{\alpha}_L, E, \mu_L)$ at $\mu_L = 2E_m$ and using

$$\bar{\alpha}_L = \alpha \left(1 + \frac{2}{3\pi} \alpha \log\left(\frac{\mu_L}{m}\right) + \frac{4}{9\pi^2} \alpha^2 \log^2\left(\frac{\mu_L}{m}\right) + \dots \right). \quad (\text{B10})$$

Unknown corrections stem from the finite contributions to $f^{(2,0)}$ and $f^{(3,2)}$ which cannot be inferred from the general arguments given above.

Although not used explicitly in this work, it is useful to note that if one were to use an expansion in on-shell α rather than $\bar{\alpha}_L$, then higher-order terms must be kept to retain a complete $O(\lambda^3)$ counting:

$$\Gamma = |\mathcal{C}(\mu_H)|^2 \left| \frac{\mathcal{C}(\mu_L)}{\mathcal{C}(\mu_H)} \right|^2 \int d\Phi_e \tilde{F}(Z\alpha, E, \mu_L) [1 + \delta^{(1,0)}(E) + \delta^{(2,1)}(E) + \delta^{(3,2)}(E) + \delta^{(2,0)}(E) + \delta^{(3,1)}(E) + O(\lambda^4)], \quad (\text{B11})$$

where, through $O(\lambda^3)$, we have

$$\begin{aligned} \langle \delta^{(2,0)}(E) \rangle_0 &= \alpha^2 L_m \left(\frac{-n_e}{6\pi^2} \right) \left\langle \hat{g}(E, \mu) + \frac{11}{4} \right\rangle_0, \\ \langle \delta^{(3,2)}(E) \rangle_0 &= Z^2 \alpha^3 L_m \left\langle \frac{\pi}{6} - \frac{5}{8\pi} - \frac{2n_e}{3\pi} \left(-L_E + \frac{\pi^2}{3} - \frac{11}{4} \right) \right\rangle_0, \\ \langle \delta^{(3,1)}(E) \rangle_0 &= Z \alpha^3 L_m^2 \left\langle \frac{-n_e^2}{9\pi} \right\rangle_0, \end{aligned} \quad (\text{B12})$$

which can be inferred from the above expressions after relating $\bar{\alpha}_L$ to α . The angle brackets denote averages over tree-level phase space in the $m/E \rightarrow 0$ limit:

$$\langle \dots \rangle_0 = \frac{30}{E_m^5} \int_0^{E_m} E^2 (E_m - E)^2 [\dots]. \quad (\text{B13})$$

For completeness, we tabulate also the corrections to the quantity M through $O(\lambda^3)$, collecting terms at first, second, and third order in on-shell α as

$$M = 1 + \Delta^{(1,0)} + \Delta^{(2,1)} + \Delta^{(2,0)} + \Delta^{(3,2)} + \Delta^{(3,1)} + \dots, \quad (\text{B14})$$

with

$$\begin{aligned} \Delta^{(1,0)} &\rightarrow \frac{\alpha}{2\pi} \left\langle \hat{g}(E, \mu) + \frac{11}{4} \right\rangle_0 = \frac{\alpha}{2\pi} \left(3 \log \frac{\mu}{2E_m} + \frac{217}{20} - \frac{4\pi^2}{3} \right), \\ \Delta^{(2,1)} &\rightarrow Z \alpha^2 \left[\frac{n_e}{3} L_m - \left(\frac{2n_e}{3} + 1 \right) \left(\log \frac{2E_m}{\mu} - \frac{47}{60} \right) + \frac{8n_e}{9} + \frac{11}{4} - \zeta_2 \right], \\ \Delta^{(2,0)} &\rightarrow \alpha^2 L_m \left(\frac{-n_e}{6\pi^2} \right) \left(\frac{217}{20} - \frac{4\pi^2}{3} \right), \\ \Delta^{(3,2)} &\rightarrow Z^2 \alpha^3 L_m \left(-\frac{2n_e}{3} \right) \left[-\left(\log \frac{2E_m}{\mu} - \frac{47}{60} \right) + \frac{\pi^2}{3} + \frac{11}{4} \right], \\ \Delta^{(3,1)} &\rightarrow Z \alpha^3 L_m^2 \left(\frac{-n_e^2}{9\pi} \right), \end{aligned} \quad (\text{B15})$$

where the arrow denotes the limit $m/E_m \rightarrow 0$.

APPENDIX C: RELATING SCHEMES FROM TOWNER AND HARDY TO THIS WORK

The calculation done in this work has been performed in the $\overline{\text{MS}}$ scheme in a pointlike effective theory that uses the physical nuclei as external degrees of freedom. Past calculations of radiative corrections have been performed using models that include spectator protons with a relativistic scalar propagator [9,11,41] (we will refer to this as “Sirlin’s scheme” in what follows). This leads to some differences that must be accounted for when comparing results. For example, the Sirlin g function (which arises naturally in the Sirlin scheme) is defined by

$$g(E, E_m, m) = 3 \log\left(\frac{m_p}{m}\right) + \frac{1}{\beta} \tanh^{-1}(\beta) \left(2(\beta^2 + 1) - 4 \tanh^{-1}(\beta) + \frac{(E_m - E)^2}{6E^2} \right) + 4 \left(\frac{\tanh^{-1}(\beta)}{\beta} - 1 \right) \left(\log\left(\frac{2(E_m - E)}{m}\right) + \frac{E_m - E}{3E} - \frac{3}{2} \right) - \frac{4\text{Li}_2\left(\frac{2\beta}{\beta+1}\right)}{\beta} - \frac{3}{4}, \quad (\text{C1})$$

with $\beta = |\mathbf{p}|/E$ the electron velocity. In the EFT, no proton mass appears, and one instead finds the combination $g(E, E_m, m) + 3 \log(\mu/m_p) + \frac{11}{4}$, where the additive constant $11/4$ is specific to the $\overline{\text{MS}}$ scheme. Finite pieces, like $+11/4$, must be properly accounted for when comparing schemes.

Towner and Hardy define their total set of radiative corrections as

$$(1 + \text{RC}) = (1 + \delta'_R)_{\text{TH}} (1 + \delta_{\text{NS}} - \delta_C) (1 + \Delta_R^V), \quad (\text{C2})$$

where we have added a subscript of TH to distinguish from our $(1 + \delta'_R(\mu_H))$. Of particular interest in the present context is their “outer correction,” as defined in Eq. (37) of [2] motivated by the factorization ansatz from Eq. (15) of [27]:

$$(1 + \delta'_R)_{\text{TH}} = \left\{ 1 + \frac{\alpha}{2\pi} \left(\bar{g} - 3 \log\left(\frac{m_p}{2E_m}\right) \right) \right\} \times \left\{ L(2E_m, m_p) + \frac{\alpha}{2\pi} (\delta_2^{(\text{TH})} + \delta_3^{(\text{TH})}) \right\}, \quad (\text{C3})$$

where $\frac{\alpha}{2\pi} \bar{g}$ is taken from Table V of [34]. It is convenient to define

$$[\delta^{(1,0)}]_{\text{TH}} \equiv \frac{\alpha}{2\pi} \left(\bar{g} - 3 \log\left(\frac{m_p}{2E_m}\right) + \frac{11}{4} \right), \quad (\text{C4})$$

which involves only Towner and Hardy inputs and is closely related¹¹ to our $\langle \delta^{(1,0)}(E) \rangle_F$. The term

$$L(2E_m, m_p) = 1.026725 \left(1 - \frac{2}{3\pi} \hat{\alpha}(\mu = m) \log \frac{2E_m}{m} \right)^{9/4} \quad (\text{C5})$$

is analogous to $R(\mu_H, \mu_L)$ discussed above but containing only terms of $O(\alpha^n L^n)$. The coupling with a hat is related to α by $\hat{\alpha}^{-1}(\mu = m) = \alpha^{-1} + 1/(6\pi)$ [27]; following Towner and Hardy, we take $\hat{\alpha}(m) = 1/137.089$. Parts of $\frac{\alpha}{2\pi} \delta_2^{(\text{TH})}$ (taken from Table V of [34]) contain logarithms involving the nuclear scale Λ . Let us define $\delta_{\text{NS,TH}}^{(2,1)}$ via

$$\frac{\alpha}{2\pi} \delta_2^{(\text{TH})} = Z\alpha^2 \log\left(\frac{\Lambda}{2E_m}\right) + \langle \delta^{(2,1)}(E) \rangle_F + \delta_{\text{NS,TH}}^{(2,1)}, \quad (\text{C6})$$

where $\delta_{\text{NS,TH}}^{(2,1)}$ is an energy-independent constant related to the model calculation of Sirlin and Zucchini [9,10]. We have chosen the notation $\delta_{\text{NS,TH}}^{(2,1)}$ because it may be interpreted as the short-distance region of Towner and Hardy’s calculation [34] and, therefore, as their estimate for the $Z\alpha^2$ correction to $|\mathcal{C}(\mu)|^2$.

We may decompose δ_3 using formulas from Sirlin and Zucchini [9,10], which are given in terms of an energy-dependent correction. Averaging over phase space and isolating logarithms of Λ and m , we find

¹¹The numerical size of the difference ranges from 6×10^{-6} for ^{10}C to 1×10^{-4} for ^{54}Co . It stems from logarithmic terms in the Fermi function, $(\mu_H/\mu_L)^{2(\eta-1)} \sim 1 + Z^2 \alpha^2 \log(\mu_H/\mu_L)$, that are included in $R(\mu_H, \mu_L)$ in our analysis.

$$\frac{\alpha}{2\pi}\delta_3^{(\text{SZ})} = Z^2\alpha^3 \left[c_1 \log^2\left(\frac{\Lambda}{2E_m}\right) + c_2 \log\left(\frac{2E_m}{m}\right) \log\left(\frac{\Lambda}{2E_m}\right) + c_3 \log\left(\frac{\Lambda}{2E_m}\right) + c_4 \log\left(\frac{2E_m}{m}\right) + c_5 \right], \quad (\text{C7})$$

where “SZ” stands for Sirlin-Zucchini, c_5 is $O(\lambda^4)$ in our counting and can be neglected, whereas $c_1 = 2/(3\pi)$, $c_2 = 4/(3\pi)$,

$$c_3 = \frac{6\pi^2 - 67 - 12 \log(10)}{18\pi},$$

$$c_4 = -\frac{20\pi^2 + 48\gamma_E + 24 \log(10) - 177}{36\pi} - c_2 \left\langle \log\left(\frac{E}{E_m}\right) \right\rangle_0. \quad (\text{C8})$$

We have used $(2\pi)^{-1}(\frac{5}{2} - \frac{2\pi^2}{3}) \approx -0.649$ for Sirlin and Zucchini’s h from [9]. We have also accounted for the updates described in the footnotes of [10] which include adjusting to the 1986 Jaus-Rasche updated logarithmic coefficient $a = \pi/3 - 3/(2\pi)$ [11] (which is erroneous [4]) and using $\log(E/m)$ rather than $\log(E_m/m)$ for the term multiplying h as given in [9].

Towner and Hardy’s tabulation [34] disagrees with the tabulation found in Sirlin [10]. We have calculated $\frac{\alpha}{2\pi}\delta_3$ using the above formula and find that if we set $c_4 = c_5 = 0$ we reproduce all of Towner and Hardy’s tabulated results [34] except for $^{50}\text{Mn} \rightarrow ^{50}\text{Co}$, for which we obtain 6.3×10^{-4} as compared to Towner and Hardy’s 6.2×10^{-4} . We, therefore, conclude that the tabulated values in [34], $\frac{\alpha}{2\pi}\delta_3^{(\text{TH})}$, are equivalent to

$$\frac{\alpha}{2\pi}\delta_3^{(\text{TH})} \leftrightarrow Z^2\alpha^3 \left[c_1 \log^2\left(\frac{\Lambda}{2E_m}\right) + c_2 \log\left(\frac{2E_m}{m}\right) \log\left(\frac{\Lambda}{2E_m}\right) + c_3 \log\left(\frac{\Lambda}{2E_m}\right) \right], \quad (\text{C9})$$

which may be interpreted as part of the renormalization group flow.

Without using any new inputs, it is then natural to reorganize Towner and Hardy’s results as

$$(1 + \delta_{\text{NS}} - \delta_C)(1 + \Delta_R^V) \rightarrow \underbrace{L(\Lambda, m_p)(1 + \delta_{\text{NS}} + \delta_{\text{NS,TH}}^{(2,1)} - \delta_C)}_{\text{short-distance}} \left(1 + \Delta_R^V - \frac{11}{8\pi}\alpha\right), \quad (\text{C10})$$

$$(1 + \delta_R')_{\text{TH}} \rightarrow \underbrace{\{1 + [\delta^{(1,0)}]_{\text{TH}} + \langle \delta^{(2,1)}(E) \rangle_F\} \mathcal{R}(2E_m, \Lambda)}_{\text{long-distance}}, \quad (\text{C11})$$

where we have split up $L(2E_m, m_p) = L(2E_m, \Lambda)L(\Lambda, m_p)$ and introduce the analog of our RG factor:

$$\mathcal{R}(2E_m, \Lambda) = L(2E_m, \Lambda) + Z\alpha^2 \log\left(\frac{\Lambda}{2E_m}\right) + \frac{\alpha}{2\pi}\delta_3^{(\text{TH})}, \quad (\text{C12})$$

which now contains the $Z\alpha^2 L$ term and Towner and Hardy’s estimate for the $Z^2\alpha^3$ logarithmic terms taken explicitly from Table V of [34]. The quantity $\langle \delta^{(2,1)}(E) \rangle_F$ is expanded in terms of the on-shell coupling α and evaluated at $\mu_L = 2E_m$. The RG-running factor from Λ to $2E_m$ is computed following [27] for self-consistency with the Towner and Hardy result¹²:

$$L(2E_m, \Lambda) = 1.01882 \left(1 - \frac{2}{3\pi} \hat{\alpha}(m) \log \frac{2E_m}{m}\right)^{9/4} \left(\frac{1}{1 - \frac{22}{9\pi} \hat{\alpha}(m_\mu) \log(\Lambda/m_\mu)}\right)^{27/44}, \quad (\text{C13})$$

where $\hat{\alpha}(m_\mu) = 1/135.7896$. While Eq. (C13) matches [27] and includes constituent quarks and a dynamical muon, we have checked that the same numerical value is obtained, up to corrections of 10^{-5} , if one includes only dynamical electrons in the theory. The result, and, therefore, our comparison to Towner and Hardy, is insensitive to the detailed implementation.

¹²The numerical coefficient 1.01882 is obtained using Eq. (56) of Ref. [27] and includes the running from the electron-mass threshold up to the muon mass threshold in their given formalism.

The replacement rules defined above do not introduce any new ingredients to Towner and Hardy’s analysis. Instead, they simply rearrange existing inputs in accord with expectations from the factorization theorems discussed in the main text. We, therefore, refer to the resulting change as a “convention shift” defined via

$$\text{convention shift} = \frac{(\text{long-distance}) \times (\text{short-distance})}{(1 + \delta'_R)_{\text{TH}}(1 + \delta_{\text{NS}} - \delta_C)(1 + \Delta_R^V)}, \quad (\text{C14})$$

where the long- and short-distance pieces are defined by Eqs. (C10) and (C11), respectively. The convention shift receives a contribution of $\sim 0.85 \times 10^{-4}$ from induced cross terms between $\frac{11}{8\pi}\alpha$ and Δ_R^V or $L(2E_m, m_p)$. Other contributions are formally $O(\lambda^4)$ in our counting if $\log(m_p/\Lambda)$ is counted as $O(1)$, such as the cross term of $(Z\alpha^2 L) \times (\frac{3\alpha}{2\pi} \log \frac{m_p}{\Lambda})$ with the $\log(m_p)$ term coming from $L(\Lambda, m_p)$.

Next, we measure the impact of our new inputs by taking the ratio of our model-independent calculation of the long-distance corrections to that defined using Towner and

Hardy inputs:

$$\text{long-distance shift} = \frac{1 + \delta'_R(\mu_H = \Lambda)}{(\text{long-distance})}. \quad (\text{C15})$$

This comparison accounts for new information provided by our analysis beyond the structure of the amplitude’s factorization. For example, the coefficient of the $\log(\Lambda/2E_m)$ term in Eq. (C12) should read $(\frac{\pi}{6} - \frac{3}{\pi})$ [4]. Similarly, the $\frac{13}{6\pi} Z\alpha^2 L^2$ term in Eq. (A8) is a new effect not included in past analyses.

APPENDIX D: STRUCTURE-DEPENDENT EFFECTS

As a check of our averaging procedure, we have taken all of the numerical inputs listed in Towner and Hardy’s 2004 review [37] and attempted to reproduce the value of $\bar{g}$ quoted in their 2007 article [34]. This both validates that the averaging procedure used here is consistent with Towner and Hardy’s analysis and allows us to estimate errors associated with the factorization ansatz for one-loop corrections typically used in the literature. As discussed

TABLE III. Calculation of $\frac{\alpha}{2\pi} \bar{g}(E_m)$, in units of percent, with the conventions and inputs used for Table V of [34] (taken from [37]). The first column uses the pointlike Fermi function as defined in Eq. (D1). The second column uses instead $F_{\text{BJ}}(Z, E)$, the Behrens and Jänecke Fermi function defined in Eq. (D2). The third column uses the product of $F_{\text{BJ}}(Z, E)$ and the shape function $C(Z, E)$ as given in Ref. [35]. We find that agreement between our implementation and that tabulated in Towner and Hardy is, in general, improved after structure-dependent corrections are included, with this improvement more pronounced for heavier nuclei. The radiative corrections to higher-dimensional operators are not guaranteed to be captured by multiplication with the Sirlin- g function. We, therefore, use the difference between the Towner and Hardy values and our pointlike calculation as an estimate of the uncertainty due to the standard factorization ansatz used in the beta decay literature; if the difference is zero, we take the largest fluctuation in the row.

Beta emitter	$F_0(Z, E)$	$F_{\text{BJ}}(Z, E)$	$F_{\text{BJ}}(Z, E) \times C(Z, E)$	Towner and Hardy	Error estimate [10^{-4}]
^{10}C	1.467	1.467	1.467	1.468	0.1
^{14}O	1.286	1.285	1.285	1.286	0.1
^{18}Ne	1.204	1.204	1.203	1.204	0.1
^{22}Mg	1.122	1.121	1.121	1.122	0.1
^{26}Si	1.056	1.055	1.054	1.055	0.1
^{30}S	1.007	1.005	1.005	1.005	0.2
^{34}Ar	0.966	0.963	0.962	0.963	0.3
^{38}Ca	0.932	0.929	0.928	0.929	0.3
^{42}Ti	0.911	0.907	0.906	0.906	0.5
$^{26}\text{Al}^m$	1.110	1.109	1.109	1.110	0.1
^{34}Cl	1.004	1.002	1.002	1.002	0.2
$^{38}\text{K}^m$	0.967	0.964	0.963	0.964	0.3
^{42}Sc	0.943	0.940	0.939	0.939	0.4
^{46}V	0.909	0.905	0.903	0.903	0.6
^{50}Mn	0.880	0.875	0.873	0.873	0.7
^{54}Co	0.854	0.847	0.844	0.844	1.0
^{62}Ga	0.820	0.810	0.806	0.805	1.5
^{66}As	0.808	0.797	0.792	0.791	1.7
^{70}Br	0.796	0.783	0.777	0.776	2.0
^{74}Rb	0.785	0.770	0.763	0.761	2.4

in Sec. IV A, this ansatz turns out to be exact at leading power but is not guaranteed when including energy-dependent terms proportional to the nuclear radius which correspond to the inclusion of higher-dimensional operators in the effective theory.

Towner and Hardy compute their phase-space averages with numerical solutions of the Dirac equation [37]. In our comparison, we do not solve the Dirac equation numerically and instead use analytic formulas taken from [35]. The first step is to use the conventional pointlike Fermi function

$$F_0(Z, E) = 4(2pR)^{2(\eta-1)} e^{-\pi\xi} \frac{|\Gamma(\eta - i\xi)|^2}{|\Gamma(1 + 2\eta)|^2}. \quad (\text{D1})$$

The radius R is defined by $R = (5/3)^{1/2} \langle r^2 \rangle^{1/2}$, and we use the same root-mean-square radii as above (these can be inferred from Λ using $\langle r^2 \rangle^{1/2} = \sqrt{6}/\Lambda$). As we show in Table III and discuss below, this does a good job of replicating Towner and Hardy’s calculations, with some discrepancies at the level of $\sim 10^{-4}$ for heavier nuclei.

To investigate the origin of these discrepancies, we have tried three different methods for computing phase-space averages. The first involves only the Fermi function $F_0(Z, E)$ as describe above. Second, we replace F_0 by the “Behrens and Jänecke” [42] Fermi function as given in Eq. (10) of [35]:

$$F_{\text{BJ}}(Z, E) = F_0(Z, E) \left[1 + \frac{13}{15} Z\alpha ER \right], \quad (\text{D2})$$

which incorporates some nuclear structure effects. As a third approach, we multiply $F_{\text{BJ}}(Z, E)$ by the shape factor $C(Z, E)$, as parametrized in Eqs. (100) and (101) of [35].

The numerical results of these three averaging procedures are shown in Table III. The pointlike Fermi function $F_0(Z, E)$ fails to reproduce the full set of Towner and Hardy’s values of $\frac{a}{2\pi} \bar{g}$ with discrepancies of $\sim 10^{-4}$. Inclusion of structure-dependent effects, parametrized with the analytic formulas discussed above, improves agreement to $\sim 10^{-5}$. We conclude that there is a $\sim 10^{-4}$ uncertainty for heavier nuclei, due to the conventional factorization ansatz for radiative corrections.

APPENDIX E: COMPARISON TO CROSAS AND MEREGHETTI

As mentioned in the Note Added, an independent calculation [33] of the long-distance $Z\alpha^2$ corrections appeared concurrently as we finished this work (submitted to the arXiv on the same day, with private correspondences

in advance of posting). In this appendix, for the benefit of the interested reader, we emphasize the difference in methods between our work and theirs and the complementarity of the obtained results.

Reference [33] computes the $Z\alpha^2$ corrections by “brute force” working in the pointlike theory with no *a priori* simplifications from eikonal algebra [6]. Modern multiloop machinery was used to facilitate the reduction of the necessary loops into a basis of master integrals. After assembling the results, it was realized that (i) only a certain set of “cut” graphs were needed and (ii) the $Z\alpha^2$ real photon corrections can be written in a factorized form with the Fermi function. The results of Ref. [6] (see also [3,4]) explain observation (i), while the analysis presented above explains observation (ii). The results at fixed order for $Z\alpha^2$ are in complete agreement between our two works; this is a valuable cross-check.

Having worked directly in terms of master integrals, Ref. [33] was able to obtain finite- m_e results. Our work, based on existing results from the literature [8,43] and the relation between neutron decay amplitudes and the $Z\alpha^2$ corrections, is valid in the limit where $m_e/E_e \rightarrow 0$. We estimated that finite-mass effects were phenomenologically irrelevant in the context of superallowed decays, and the results of Ref. [33] confirm this conjecture (a possible exception is for ^{10}C , for which these corrections are marginally relevant).

Finally, our work goes beyond a fixed-order analysis and includes long-distance corrections that would only appear at three- and four-loop order in a fixed-order calculation. These contributions stem from the RG-improved results (first presented in Refs. [3,4]) and from the KLN constraints that are new and presented for the first time in this work. We systematize our analysis by applying a controlled power counting to logarithms and factors of Z [see the discussion above Eq. (A5)]. This is not only new input at the level of the pointlike effective theory, but also goes beyond the historical set of radiative corrections used in phenomenological analyses [2,9].

The most effective way to merge the analysis of Ref. [33] with the results of this work would be to define “power corrections” at $O(Z\alpha^2)$. These are obtained by writing

$$\delta^{(2,1)}(E, m) = \delta^{(2,1)}(E, 0) + [\delta^{(2,1)}(E, m) - \delta^{(2,1)}(E, 0)]. \quad (\text{E1})$$

The term in square brackets then defines the $O(m^2/E^2)$ terms in Eq. (21) and can be straightforwardly added to the analysis presented herein; as already emphasized, the numerical impact will be small.

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