

Discrete gauging and noninvertible selection rules

Jun Dong,^{1,*} Tim Jeric,^{1,†} Tatsuo Kobayashi^{1,‡}, Ryusei Nishida^{1,§} and Hajime Otsuka^{2,3,||}

¹Department of Physics, *Hokkaido University*, Sapporo 060-0810, Japan

²Department of Physics, *Kyushu University*, 744 Motoooka, Nishi-ku, Fukuoka 819-0395, Japan

³Quantum and Spacetime Research Institute (QuaSR), *Kyushu University*,
744 Motoooka, Nishi-ku, Fukuoka, 819-0395, Japan



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We clarify selection rules of conjugacy classes of several finite discrete groups where we deal with both gauged and ungauged cases. We find that the selection rules enjoy finite Abelian or non-Abelian discrete symmetries originating from the inner and/or outer automorphism of underlying discrete groups. Since the selection rules of conjugacy classes do not obey conventional grouplike selection rules, they open up new coupling selection rules of fields which are labeled by the conjugacy classes.

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I. INTRODUCTION

In string theory on orbifolds, some string states can be labeled by conjugacy classes of a group rather than its element due to the identification of string states under the orbifold twist. For example, suppose that the string has the boundary condition, $X(\sigma = \pi) = gX(\sigma = 0)$, where g denotes the space group action constructed by the point group action and a lattice translation. The same boundary condition may be satisfied by hX as $hX(\sigma = \pi) = ghX(\sigma = 0)$ with h being an additional space group action. That is, the boundary condition represented by g must be equivalent to $h^{-1}gh$, and equivalent elements belong to a single class. Similarly, if compact space and world-sheet theory have a symmetry, we can divide them by such a symmetry. Hence, we can gauge such a symmetry so as to obtain classes of string states and vertex operators. The selection rules of conjugacy classes do not obey a grouplike selection rule but a noninvertible one in general. Indeed, such noninvertible selection rules are known in heterotic string theory on toroidal orbifolds [1–7] and type II intersecting/magnetized D-brane models [8,9] (see, e.g., Refs. [10–14] for the fusion rules in two-dimensional conformal field theories). Furthermore, it can also be seen

in Calabi-Yau compactifications [15]. Indeed, a translation is invertible on the T^2 torus, but it is not a gauge-invariant quantity under an orbifold group [6]. Hence, in toroidal orbifolds, the momentum operators are constructed in a way to be invariant under the orbifold twists. In the type IIB magnetized D-brane models on $T^2/\mathbb{Z}_N$ with magnetic fluxes, the $\mathbb{Z}_N$ -invariant momentum operators obey noninvertible fusion rules under which degenerate chiral zero modes nontrivially transform under the noninvertible symmetries. See for reviews of noninvertible symmetries, e.g., Refs. [16,17]. Orbifolding is a way to obtain noninvertible fusion rules.

Each string state corresponds to each particle mode in the low-energy effective field theory. When string states obey the above noninvertible selection rules, the same coupling selection rules must also be satisfied in the low-energy effective field theory. For example, when each string state is labeled by a conjugacy class of a group G , each field in the effective theory is likewise labeled by a conjugacy class rather than by an element of G . The multiplication rules between group elements and conjugacy classes are different from each other, and they lead to distinct phenomenological aspects. Group-theoretical selection rules are often imposed in conventional field theory.

As a simple illustration, we consider an Abelian group G , whose elements are denoted by g_k, g_ℓ, g_m , etc. Suppose that they satisfy the following multiplication rule:

$$g_k g_\ell = g_m. \quad (1.1)$$

Obviously, a single element appears in the right-hand side of group-theoretical multiplication rules. Each field corresponds to a (representation of) element g_k of G . The process $\phi_k + \phi_\ell \rightarrow \phi_m$ can occur through their coupling terms only if the condition $g_k g_\ell = g_m$ is satisfied. That is the charge

*Contact author: j-dong@particle.sci.hokudai.ac.jp

†Contact author: t-jeric@particle.sci.hokudai.ac.jp

‡Contact author: kobayashi@particle.sci.hokudai.ac.jp

§Contact author: r-nishida@particle.sci.hokudai.ac.jp

||Contact author: otsuka.hajime@phys.kyushu-u.ac.jp

conservation law. If $g_k g_\ell \neq g_m$, this process, or equivalently the corresponding coupling, is forbidden by the group-theoretical selection rules.

Now, suppose that each field ϕ_k , corresponding to a string state, is labeled by a (conjugacy) class $C^{(k)}$, which may include more than one group element as mentioned above. The classes satisfy the following multiplication rules:

$$C^{(k)} C^{(\ell)} = \sum_m y_{k\ell m} C^{(m)}, \quad (1.2)$$

where $y_{k\ell m}$ are constants. Note that the classes appearing in the right-hand side are not necessarily unique. Analogously to the group theory, the process $\phi_k + \phi_\ell \rightarrow \phi_m$ can occur through their coupling term only if $y_{k\ell m} \neq 0$. However, an important difference is that $C^{(m)}$ and $\phi^{(m)}$ are not unique in general: more than one classes $C^{(m)}$ and corresponding fields $\phi^{(m)}$ can appear in the right-hand side. This feature leads to interesting phenomenological applications in particle physics [8,18–31].

A simple example of noninvertible selection rules is provided by $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries. They can be derived from magnetized D-brane models on $T^2/\mathbb{Z}_2$ orbifold [8,9]. Furthermore, the conjugacy classes of the dihedral group $D_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$ include the classes associated with the $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$. Thus, these selection rules can also be realized in heterotic string theory compactified on a non-Abelian D_N orbifold.¹ The $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries leads to new and interesting phenomenological aspects. For instance, let us suppose that quarks and leptons are labeled by classes obeying noninvertible selection rules in a family-dependent way. Then, it leads to the different structure of the textures of Yukawa matrix from models with conventional flavor symmetries. Indeed, one can realize the realistic fermion masses and mixings as well as CP phases in the quark [22,23] and lepton sectors [26]. Furthermore, one can realize the interesting Yukawa texture of quarks, addressing the strong CP problem without axion in the context of spontaneous CP violation [25,30,31]. Such noninvertible selection rules can be violated by radiative and/or stringy corrections [6,7,9]; thereby certain couplings are generated at loop levels. It will be the reason why the masses of active neutrinos are tiny. It was explicitly checked in the context of radiative neutrino mass models [27,28]. These new and interesting results have been derived through the use of $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$. It is quite important to extend these analyses to other types of noninvertible selection rules. Simple extensions include $\mathbb{Z}_3$ gauging of discrete symmetries such as $\mathbb{Z}_N$ and multiplication rules of conjugacy classes of discrete groups of the form $G = H \rtimes \mathbb{Z}_3$. Such structures can also be derived

from heterotic orbifold models and D-brane models on orbifolds.

The purpose of this paper is to clarify the selection rules of conjugacy classes of finite discrete groups and reveal remaining symmetries originating from the inner and/or outer automorphism of underlying discrete groups. Concerning the conjugacy classes of a finite discrete group G , we also discuss the gauging of G . The $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries are discussed in Refs. [8,22], corresponding to $\mathbb{Z}_2$ -invariant modes under the orbifold twist. It was found that such a $\mathbb{Z}_2$ gauging is phenomenologically useful to realize the interesting Yukawa textures. In this paper, we extend this analysis to $\mathbb{Z}_3$ and S_3 gaugings of a certain finite discrete group. Then, we apply the selection rules obtained from $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N$ symmetry to phenomenological models where fields are labeled by the elements in the algebra. It leads to different properties between $\mathbb{Z}_2$ and $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N$ symmetry.

This paper is organized as follows. In Sec. II, we first review the selection rules of classes for $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries and extend to the D_N conjugacy classes. This discussion is further extended to $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N$ symmetries in Sec. III. In Sec. IV, we discuss the conjugacy classes for $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$ and $\Delta(3N^2)$. In Sec. V, we deal with the selection rules of the conjugacy classes for $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N \times \mathbb{Z}''_N$ and $\Sigma(3N^3)$. The selection rules of the conjugacy classes for S_3 gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$ and $\Delta(6N^2)$ are discussed in Sec. VI. In Sec. VII, we discuss particle phenomenological aspects of multiplication rules, which are obtained in the previous sections. Finally, Sec. VIII is devoted to the conclusions. In Appendix A, we list the selection rules of classes for $\mathbb{Z}_3$ gauging of $\mathbb{Z}_9$ symmetry, and $\Delta(12) \cong A_4$ and $\Delta(24) \cong S_4$ cases are summarized in Appendix B. In Appendixes C and D, we discuss other types of gauging.

II. $\mathbb{Z}_2$ GAUGING OF $\mathbb{Z}_N$ AND D_N

This section serves as a review of the $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$, and we compare the resulting selection rule with that of D_N .

A. $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$

Here, we review $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries [8,22]. We start with the $\mathbb{Z}_N$ symmetry. We denote its generator by a , which satisfies

$$a^N = e, \quad (2.1)$$

where e is the identity. All of the $\mathbb{Z}_N$ elements are written by a^k ($k = 0, 1, \dots, N-1$) (mod N). Their multiplication rules are written as

$$a^{k_1} a^{k_2} = a^{k_1+k_2}. \quad (2.2)$$

¹See for heterotic non-Abelian models [32–40].

Next, we explain $\mathbb{Z}_2$ gauging. We introduce the $\mathbb{Z}_2$ element b , which satisfies

$$b^2 = e, \quad ba^k b^{-1} = a^{-k}. \quad (2.3)$$

That is, b corresponds to the outer automorphism of the $\mathbb{Z}_N$ group. Then, we define the class $C^{(k)}$ as

$$C^{(k)} = \{b^n a^k b^{-n} | n = 0, 1\} = \{a^k, a^{-k}\}. \quad (2.4)$$

That is $\mathbb{Z}_2$ gauging of the $\mathbb{Z}_N$ symmetry. These classes satisfy the following fusion rule:

$$C^{(k)} \cdot C^{(k')} = C^{(k+k')} + C^{(k-k')}. \quad (2.5)$$

They are commutable. Similarly, multiplication rules, which we will derive in the following sections, are commutable.

When N is even, this algebra has the $\mathbb{Z}_2$ symmetry $C^{(k=\text{odd})} \rightarrow -C^{(k=\text{odd})}$, because this leads the following relations:

$$\begin{aligned} C^{(k=\text{even})} \cdot C^{(k=\text{even})} &= C^{(k=\text{even})} + C^{(k=\text{even})}, \\ C^{(k=\text{even})} \cdot C^{(k=\text{odd})} &= C^{(k=\text{odd})} + C^{(k=\text{odd})}, \\ C^{(k=\text{odd})} \cdot C^{(k=\text{odd})} &= C^{(k=\text{even})} + C^{(k=\text{even})}. \end{aligned} \quad (2.6)$$

When N is odd, there is no such symmetry.

In general, the class $C^{(k)}$ includes two elements, a^k and a^{-k} , but obviously for $k = 0$, the class $C^{(0)}$ includes only one element $a^0 = e$. In addition, when N is even, the class $C^{(N/2)}$ also includes only one element. We denote them as

$$C_1 = \{e\}, \quad C_1^{(N/2)} = \{a^{N/2}\}, \quad C_2^{(k)} = C^{(k)}, \quad (2.7)$$

for $k \neq 0, N/2$. They satisfy

$$\begin{aligned} C_1 \cdot C_1 &= C_1^{(N/2)} \cdot C_1^{(N/2)} = C_1, \\ C_1 \cdot C_2^{(k)} &= C_1^{(N/2)} \cdot C_2^{(k)} = C_2^{(k)}, \\ C_2^{(k)} \cdot C_2^{(k')} &= C_2^{(k+k')} + C_2^{(k-k')}, \end{aligned} \quad (2.8)$$

where $k + k' \neq 0$ and $k - k' \neq 0 \pmod{N/2}$. When $k + k' = 0$ and $k - k' = 0 \pmod{N}$, $C_2^{(k+k')}$ and $C_2^{(k-k')}$ in the right-hand side of the last equation must be replaced by $2C_1$. Similarly, when $k + k' = N/2$ and $k - k' = N/2 \pmod{N}$, $C_2^{(k+k')}$ and $C_2^{(k-k')}$ in the right-hand side of the last equation must be replaced by $2C_1^{(N/2)}$. $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries can be realized by string compactifications [8,9]. Certain compactifications such as magnetized compactifications of higher-dimensional Yang-Mills theory and

string theory lead to the $\mathbb{Z}_N$ symmetry [41–43]. In these models, the massless modes φ_k transform as

$$\varphi_k \rightarrow a^k \varphi_k, \quad (2.9)$$

under the $\mathbb{Z}_N$ symmetry. The $\mathbb{Z}_N$ charge k corresponds to discrete momentum, winding number, and their mixture.

Now, we consider geometrical $\mathbb{Z}_2$ orbifolding of the above compactification. Then, the $\mathbb{Z}_2$ -invariant modes are written by [44]

$$\phi_k = \varphi_k + \varphi_{N-k}, \quad (2.10)$$

up to normalization. These modes ϕ_k correspond to $C^{(k)}$. On this $\mathbb{Z}_2$ orbifold compactification, the original $\mathbb{Z}_N$ symmetry is broken, but certain coupling selection rules remain in low-energy effective field theory. That corresponds to Eq. (2.5). The mode ϕ_k behaves as if it has both k and $N - k$ charges under $\mathbb{Z}_N$.

Each field ϕ_k corresponds to a class $C^{(k)}$. Note that the field ϕ_k and its conjugate ϕ_k^* correspond to the same class $C^{(k)}$. Their coupling selection rules are determined by fusion rules (2.5). For example, we consider the two-point coupling of ϕ_k and $\phi_{k'}$, which correspond to the classes $C^{(k)}$ and $C^{(k')}$. If their multiplication $C^{(k)} \cdot C^{(k')}$ includes the identity e , their two-point coupling is allowed. The condition is written by

$$\pm k \pm k' = 0 \pmod{N}. \quad (2.11)$$

We find that $C^{(k)} \cdot C^{(k)}$ always include e and the two-point coupling $\phi_k \phi_k$ such as mass terms and kinetic terms are allowed. That leads to the property that the diagonal entries of neutrino mass matrix induced by the Weinberg operators are always allowed [26]. Also, $2n$ point self-couplings $(\phi_k)^{2n}$ are allowed.

Similarly, we can study the n -point couplings, $\phi_{k_1} \cdots \phi_{k_n}$. Their couplings are allowed if $C^{(k_1)} \cdots C^{(k_n)}$ includes the identity. The condition can be written by

$$\sum_i \pm k_i = 0 \pmod{N}. \quad (2.12)$$

For example, when $N = 4$, there are three classes:

$$C_1 = \{e\}, \quad C_2^{(1)} = \{a, a^3\}, \quad C_1^{(2)} = \{a^2\}, \quad (2.13)$$

and they obey the fusion rules as presented in Table I. It is nothing but the Ising fusion rule realized in $c = \frac{1}{2}$ critical Ising model with three primary operators $\{\mathbb{I}, \epsilon, \sigma\}$:

$$e \cdot e = \mathbb{I}, \quad \epsilon \cdot \sigma = \sigma \cdot \epsilon = \sigma, \quad \sigma \cdot \sigma = \mathbb{I} + \epsilon, \quad (2.14)$$

TABLE I. Multiplication rules for $\mathbb{Z}_2$ gauging of $\mathbb{Z}_4$.

	C_1	$C_2^{(1)}$	$C_1^{(2)}$
C_1	C_1	$C_2^{(1)}$	$C_1^{(2)}$
$C_2^{(1)}$	$C_2^{(1)}$	$2C_1 + 2C_1^{(2)}$	$C_2^{(1)}$
$C_1^{(2)}$	$C_1^{(2)}$	$C_2^{(1)}$	C_1

corresponding to $C_1 = \mathbb{I}$, $C_1^{(2)} = \epsilon$, and $C_2^{(1)} = \sqrt{2}\sigma$. As mentioned in Eq. (2.6), their multiplication rules have the $\mathbb{Z}_2$ symmetry, where $C_2^{(1)}$ is $\mathbb{Z}_2$ odd, while C_1 and $C_1^{(2)}$ are $\mathbb{Z}_2$ even.²

It is very important to apply the above section rules due to $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries to particle phenomenology. For example, we can derive interesting texture structures by $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries [22,23,25,26]. We can also study radiative neutrino mass models [27] and further applications. In the following sections, we will extend $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries to $\mathbb{Z}_3$ gauging of discrete symmetries as well as S_3 gauging.

B. D_N

Before extending $\mathbb{Z}_2$ gauging to $\mathbb{Z}_3$ gauging, we discuss the relations of $\mathbb{Z}_2$ gauging of the $\mathbb{Z}_N$ symmetries to the $D_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$ symmetries. The classes, C_1 , $C_1^{(N/2)}$, and $C_2^{(k)}$ in the previous section correspond to the conjugacy classes of D_N , which do not include b [45,46].³ They also correspond to the $\mathbb{Z}_2$ -invariant modes (2.10).

The D_N group has other conjugacy classes. When $N = \text{odd}$, there is another conjugacy class:

$$B_1 \equiv \{b, ab, a^2b, \dots, a^{N-1}b\}. \quad (2.15)$$

There are trivial and nontrivial singlets and doublets.

When $N = \text{even}$, there are two conjugacy classes including b ,

$$B_1 \equiv \{b, a^2b, \dots, a^{N-2}b\}, \quad B_2 \equiv \{ab, a^3b, \dots, a^{N-1}b\}, \quad (2.16)$$

in addition to $C^{(k)}$. There are four singlets in addition to doublets.

For $C^{(k)}$, we have the realization by using the “bulk” in Eq. (2.10). We may need to introduce the twist fields, which change the boundary condition from $\mathbb{Z}_2$ even to $\mathbb{Z}_2$ odd [47,48]. For example, when $N = 4$, there are two conjugacy classes including b , $B_1 = \{b, a^2b\}$ and

TABLE II. Multiplication rules for conjugacy classes of D_4 .

	C_1	$C_2^{(1)}$	$C_1^{(2)}$	B_1	B_2
C_1	C_1	$C_2^{(1)}$	$C_1^{(2)}$	B_1	B_2
$C_2^{(1)}$	$C_2^{(1)}$	$2C_1 + 2C_1^{(2)}$	$C_2^{(1)}$	$2B_2$	$2B_1$
$C_1^{(2)}$	$C_1^{(2)}$	$C_2^{(1)}$	C_1	B_1	B_2
B_1	B_1	$2B_2$	B_1	$2C_1 + 2C_1^{(2)}$	$2C_2^{(1)}$
B_2	B_2	$2B_1$	B_2	$2C_2^{(1)}$	$2C_1 + 2C_1^{(2)}$

$B_2 = \{ab, a^3b\}$. Then, we introduce two twist fields, σ_0 and σ_1 , which correspond to $a^k b$ with $k = \text{even}$ and odd , respectively, where k corresponds to the momentum-winding number. This structure is consistent with the world-sheet theory on $S^1/\mathbb{Z}_2$. The fixed points on $S^1/\mathbb{Z}_2$ are represented by the conjugacy class of the space group (θ, ne) , where θ denotes $\mathbb{Z}_2$ twist and e denotes the lattice basis vector to construct S^1 . There are two conjugacy classes (θ, ne) with $n = 0, 1$. They correspond to σ_n with $n = 0, 1$, and B_1 and B_2 . There is the $\mathbb{Z}_2$ symmetry related to the generator a . Under this $\mathbb{Z}_2$, the twist fields transform as

$$\sigma_n \rightarrow (-1)^n \sigma_n. \quad (2.17)$$

Also there is the following permutation symmetry:

$$\sigma_0 \rightarrow \sigma_1, \quad \sigma_1 \rightarrow \sigma_0, \quad (2.18)$$

because the geometry does not change under the exchange of fixed points. Then, the full symmetry becomes the D_4 symmetry [1–7].

The multiplication rules of D_4 conjugacy classes shown in Table II have several symmetries. One is the permutation symmetry $B_1 \leftrightarrow B_2$, which corresponds to Eq. (2.18). In addition, recall that when $N = \text{even}$, there is the $\mathbb{Z}_2$ symmetry, where $C^{(k=\text{even})}$ and $C^{(k=\text{odd})}$ respectively have $\mathbb{Z}_2$ even and odd charges. To be consistent, B_1 and B_2 must have $\mathbb{Z}_2$ even and odd charges, respectively. That corresponds to Eq. (2.17). We combine them and find that the full symmetry is D_4 . The conjugacy classes B_1 and B_2 are D_4 doublets, while $C_2^{(1)}$ is nontrivial singlet and C_1 and $C_1^{(2)}$ are trivial singlets. On the other hand, when $N = \text{odd}$, we may introduce a single twist field σ in order to represent the conjugacy class B_1 . It may correspond to one of σ_0 and σ_1 , or both, but its geometrical meaning is not clear.

At any rate, $\mathbb{Z}_2$ -invariant elements correspond to $C^{(k)}$ and they satisfy the algebraic relations (2.5).

III. $\mathbb{Z}_3$ GAUGING OF $\mathbb{Z}_N$

As mentioned in Sec. I, it is important to extend the $\mathbb{Z}_2$ gauging discussed in the previous section to analyses of $\mathbb{Z}_3$ gauging of discrete groups and the multiplication rules of

²Similarly, one can realize the Fibonacci fusion rule for $N = 3$ and the fusion rule of $SU(2)_3$ for $N = 6$.

³Here, we follow the notation in Refs. [45,46]. In some references, D_N is written as D_{2N} .

conjugacy classes in groups of the form $G = H \rtimes \mathbb{Z}_3$. This construction is motivated by string compactifications on orbifolds, including $T_7 \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$. The simplest extension is $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N$ symmetries, whose generator is a , i.e., $a^N = e$. We consider the $\mathbb{Z}_3$ automorphism b of the $\mathbb{Z}_N$ group, i.e.,

$$b^{-1}ab = a^m, \quad b^3 = e, \quad (3.1)$$

for $m \neq 0$. Since the case with $m = 1 \pmod{N}$ is trivial, we study other numbers. Equation (3.1) can be written as

$$a = ba^mb^{-1}. \quad (3.2)$$

By use of this, we can derive

$$\begin{aligned} b^{-1}ab &= a^m = (ba^mb^{-1})^m \\ &= (ba^mb^{-1}) \cdots (ba^mb^{-1}) = ba^{m^2}b^{-1}. \end{aligned} \quad (3.3)$$

This relation can be written as

$$a = b^{-1}a^{m^2}b, \quad (3.4)$$

which leads to

$$\begin{aligned} b^{-1}ab &= a^m = (b^{-1}a^{m^2}b)^m \\ &= (b^{-1}a^{m^2}b) \cdots (b^{-1}a^{m^2}b) = b^{-1}a^{m^3}b. \end{aligned} \quad (3.5)$$

Through these consistency conditions, it is found that

$$a^{m^3} = a. \quad (3.6)$$

That is, m must satisfy

$$m^3 - 1 = (m - 1)(m^2 + m + 1) = 0 \pmod{N}. \quad (3.7)$$

One of the solution is $m = 1 \pmod{N}$, but that is trivial as mentioned above. Suppose that N is a prime number. Hence, m must satisfy

$$m^2 + m + 1 = 0 \pmod{N}. \quad (3.8)$$

The possible combinations of (N, m) are

$$(N, m) = (7, 2), (13, 3), \dots \quad (3.9)$$

For generic element a^k , we find

$$ba^kb^{-1} = a^{m^2k}, \quad b^2a^kb^{-2} = a^{m^4k}. \quad (3.10)$$

Then, we define the following classes:

TABLE III. Multiplication rules of classes for $\mathbb{Z}_3$ gauging of $\mathbb{Z}_7$.

	C_1	C_3^1	C_3^2
C_1	C_1	C_3^1	C_3^2
C_3^1	C_3^1	$C_3^1 + 2C_3^2$	$3C_1 + C_3^1 + C_3^2$
C_3^2	C_3^2	$3C_1 + C_3^1 + C_3^2$	$2C_3^1 + C_3^2$

$$\begin{aligned} C^{(k)} &= [a^k] = \{a^{m^l k} | l = 0, 2, 4\} \\ &= \{a^k, a^{km}, a^{km^2} = a^{-(m+1)k}\}. \end{aligned} \quad (3.11)$$

Their multiplication rules are written by

$$\begin{aligned} C^{(k)} \cdot C^{(\ell)} &= C^{(k+\ell)} + C^{(k+m^2\ell)} + C^{(k+m^4\ell)} \\ &= C^{(k+\ell)} + C^{(k-(m+1)\ell)} + C^{(k+m\ell)}. \end{aligned} \quad (3.12)$$

When $k = 0$, the class $C^{(0)}$ includes the only identity. Thus, we rewrite the classes as

$$C_1 = \{e\}, \quad (3.13)$$

$$C_3^k = \{a^k, a^{km}, a^{km^2}\}. \quad (3.14)$$

Their multiplication rules can be written by

$$C_1 \cdot C_3^k = C_3^k, \quad (3.15)$$

$$C_3^k \cdot C_3^\ell = C_3^{k+\ell} + C_3^{k+\ell m} + C_3^{k+\ell m^2}. \quad (3.16)$$

For example, when $N = 7$ and $m = 2$, there are three classes:

$$C_1 = \{e\}, \quad C_3^1 = \{a, a^2, a^4\}, \quad C_3^2 = \{a^3, a^5, a^6\}, \quad (3.17)$$

and they obey the fusion rules as presented in Table III. It indicates that there exists the S_2 permutation symmetry associated with $C_3^1 \leftrightarrow C_3^2$. The S_2 symmetry is isomorphic to $\mathbb{Z}_2$ symmetry. In order to emphasize the permutation of classes, we denote S_2 . This $S_2 \simeq \mathbb{Z}_2$ corresponds to the outer automorphism exchanging a and a^{-1} , which is used in $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries in the previous section.

$\mathbb{Z}_3$ gauging of $\mathbb{Z}_N$ can be realized by higher dimensional field theory in a way similar to $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$. We start with modes with definite $\mathbb{Z}_N$ charges as Eq. (2.9). Then, we introduce a proper $\mathbb{Z}_3$ identification, e.g.,

$$\phi_1 = \varphi_1 + \varphi_m + \varphi_{m^2}. \quad (3.18)$$

The classes in Eq. (3.17) correspond to the conjugacy classes of $\mathbb{Z}_7 \rtimes \mathbb{Z}_3 = T_7$ without b .⁴ The full conjugacy classes of T_7 include additional classes:

⁴Note that T_7 is the only simple subgroup of $SU(3)$ having a complex three-dimensional irreducible representation.

TABLE IV. Multiplication rules for conjugacy classes of T_7 .

	C_1	C_3^1	C_3^2	$C_7^{(1)}$	$C_7^{(2)}$
C_1	C_1	C_3^1	C_3^2	$C_7^{(1)}$	$C_7^{(2)}$
C_3^1	C_3^1	$C_3^1 + 2C_3^2$	$3C_1 + C_3^1 + C_3^2$	$3C_7^{(1)}$	$3C_7^{(2)}$
C_3^2	C_3^2	$3C_1 + C_3^1 + C_3^2$	$2C_3^1 + C_3^2$	$3C_7^{(1)}$	$3C_7^{(2)}$
$C_7^{(1)}$	$C_7^{(1)}$	$3C_7^{(1)}$	$3C_7^{(1)}$	$7C_7^{(2)}$	$7C_1 + 7C_3^1 + 7C_3^2$
$C_7^{(2)}$	$C_7^{(2)}$	$3C_7^{(2)}$	$3C_7^{(2)}$	$7C_1 + 7C_3^1 + 7C_3^2$	$7C_7^{(1)}$

$$C_7^{(1)} = \{b, ba, \dots, ba^6\}, \quad C_7^{(2)} = \{b^2, b^2a, \dots, b^2a^6\}. \quad (3.19)$$

Table IV shows multiplication rules of T_7 conjugacy classes, which have an S_2 symmetry associated with $C_3^1 \leftrightarrow C_3^2$ and $C_7^{(1)} \leftrightarrow C_7^{(2)}$ at the same time. This corresponds to the outer automorphism exchanging a and a^{-1} as mentioned above. At any rate, the classes in Eq. (3.17) correspond to $\mathbb{Z}_3$ -invariant ones.

Each field ϕ_k can correspond to a class $C^{(k)}$ of $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N$ symmetry. In $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$, the field ϕ_k and its conjugate ϕ_k^* correspond to the same class, but in $\mathbb{Z}_3$ gauging they are complex conjugate to each other. Thus, we can introduce complex representations in a theory with $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N$. These aspects are different from $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$.

The coupling selection rules can be derived in a way similar to coupling selection rules due to $\mathbb{Z}_2$ gauging to $\mathbb{Z}_N$ symmetries. That is, if the products of $\prod_i C^{(k_i)}$ include the identity e , the n -point coupling $\prod_i \phi_{k_i}$ is allowed. For example, the two-point coupling $\phi_k \phi_k$ is not always allowed, because $C^{(k)} \cdot C^{(k)}$ does not include the identity e for generic k . That is different from the coupling selection rules due to $\mathbb{Z}_2$ of $\mathbb{Z}_N$ symmetries. On the other hand, we find that $(C^{(k)})^3$ always includes the identity. Thus, the three-point self coupling $(\phi_k)^3$ is always allowed.

When we consider the case with nonprime N , more combinations of (N, m) are possible,⁵ but still $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N$ is possible for limited N . Thus, we extend $\mathbb{Z}_N$ to $\mathbb{Z}_N \times \mathbb{Z}'_N$ as well as $\mathbb{Z}_N \times \mathbb{Z}'_N \times \mathbb{Z}''_N$ in the following sections. Their $\mathbb{Z}_3$ gauging is possible for generic N .

IV. $\mathbb{Z}_3$ GAUGING OF $\mathbb{Z}_N \times \mathbb{Z}'_N$ AND $\Delta(3N^2)$

In this section, we consider the case in which $\mathbb{Z}_3$ acts as an automorphism of $\mathbb{Z}_N \times \mathbb{Z}'_N$, rather than of $\mathbb{Z}_N$ as discussed in the previous section, and compare the resulting selection rule with that of $\Delta(3N^2)$. This construction can be realized in string compactifications on orbifolds, as mentioned in the Introduction.

⁵For example, $\mathbb{Z}_3$ gauging of $\mathbb{Z}_9$ is possible. See Appendix A.

A. $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$

In this section, we extend the analysis of Sec. II A to $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$. The generators of $\mathbb{Z}_N \times \mathbb{Z}'_N$ are represented by a and a' which satisfy

$$a^N = a'^N = e, \quad aa' = a'a, \quad (4.1)$$

where e is the identity. All of the $\mathbb{Z}_N \times \mathbb{Z}'_N$ elements are written by $a^\ell a'^m$ ($\ell, m = 0, 1, \dots, N-1$) (mod N), and they obey the multiplication rules as in Eq. (2.2).

To describe $\mathbb{Z}_3$ gauging, let us introduce the $\mathbb{Z}_3$ element b satisfying

$$b^3 = e, \quad bab^{-1} = a^{-1}a'^{-1}, \quad ba'b^{-1} = a, \\ b(a^\ell a'^m)b^{-1} = a^{-\ell+m}a'^{-\ell}, \quad b^2(a^\ell a'^m)b^{-2} = a^{-m}a'^{\ell-m}. \quad (4.2)$$

It indicates that b corresponds to the outer automorphism of the $\mathbb{Z}_N \times \mathbb{Z}'_N$ group. Then, we define the $\mathbb{Z}_3$ -invariant class as follows:

$$C^{(k, \ell)} = \{b^n a^k a'^\ell b^{-n} | n = 0, 1, 2\} \\ = \{a^k a'^\ell, a^{-k+\ell} a'^{-k}, a^{-\ell} a'^{k-\ell}\}. \quad (4.3)$$

In this $\mathbb{Z}_3$ gauging of the $\mathbb{Z}_N \times \mathbb{Z}'_N$ symmetry, the classes obey the following fusion rule:

$$C^{(k, \ell)} \cdot C^{(m, n)} = C^{(k+m, \ell+n)} + C^{(k-m+n, \ell-m)} + C^{(k-n, \ell+m-n)}. \quad (4.4)$$

In general, the class $C^{(k, \ell)}$ includes three elements, as shown in (4.3), but obviously for $k = \ell = 0$, the class $C^{(0, 0)}$ includes only one element $a^0 a'^0 = e$. In the following, we consider two cases depending on the value of N :

(i) $N/3 \neq \text{integer}$

When $N/3$ is not an integer, we have two types of classes:

$$C_1 = \{e\}, \quad (4.5)$$

$$C_3^{(k, \ell)} = \{a^k a'^\ell, a^{-k+\ell} a'^{-k}, a^{-\ell} a'^{k-\ell}\}, \quad (4.6)$$

with $k, l \neq 0$, and they obey

$$C_1 \cdot C_1 = C_1, \quad (4.7)$$

$$C_1 \cdot C_3^{(k,\ell)} = C_3^{(k,\ell)}, \quad (4.8)$$

$$C_3^{(k,\ell)} \cdot C_3^{(m,n)} = C_3^{(k+m,\ell+n)} + C_3^{(k-m+n,\ell-m)} + C_3^{(k-n,\ell+m-n)}, \quad (4.9)$$

with $(k, \ell) \neq (0, 0)$, $(m, n) \neq (0, 0)$. Note that if any one of $(k+m, \ell+n)$, $(k-m+n, \ell-m)$, and $(k-n, \ell+m-n)$ are equal to $(0, 0)$, $C_3^{(k+m,\ell+n)}$, $C_3^{(k-m+n,\ell-m)}$, and $C_3^{(k-n,\ell+m-n)}$ in the right-hand side of the last equation must be replaced by $3C_1$, i.e., $C_3^{(0,0)} \Rightarrow 3C_1$.

(ii) $N/3 = \text{integer}$

On the other hand, when $N/3$ is an integer, we have three types of classes:

$$\begin{aligned} C_1 &= \{e\}, \\ C_1^s &= \{a^s a'^{-s}\} \quad (s = N/3, 2N/3), \\ C_3^{(k,\ell)} &= \{a^k a'^\ell, a^{-k+\ell} a'^{-k}, a^{-\ell} a'^{k-\ell}\} \quad (k, \ell) \neq (N/3, 2N/3), (2N/3, N/3), \end{aligned} \quad (4.10)$$

and they obey

$$C_1 \cdot C_1 = C_1, \quad (4.11)$$

$$C_1 \cdot C_1^s = C_1^s, \quad (4.12)$$

$$C_1 \cdot C_3^{(k,\ell)} = C_3^{(k,\ell)}, \quad (4.13)$$

$$C_1^s \cdot C_1^{s'} = \begin{cases} C_1^{2s} & (s = s') \\ C_1 & (s \neq s') \end{cases}, \quad (4.14)$$

$$C_1^s \cdot C_3^{(k,\ell)} = C_3^{(s+k, -s+\ell)}, \quad (4.15)$$

$$C_3^{(k,\ell)} \cdot C_3^{(m,n)} = C_3^{(k+m,\ell+n)} + C_3^{(k-m+n,\ell-m)} + C_3^{(k-n,\ell+m-n)}, \quad (4.16)$$

with $(k, \ell) \neq (0, 0)$, $(m, n) \neq (0, 0)$. When $(k+m, \ell+n)$, $(k-m+n, \ell-m)$, $(k-n, \ell+m-n)$ are equal to $(0, 0)$, $(N/3, 2N/3)$, or $(2N/3, N/3)$, $C_3^{(k,\ell)}$ in the right-hand side of the last equation is respectively replaced by $3C_1$, $3C_1^{N/3}$, or $3C_1^{2N/3}$, i.e.,

$$\begin{aligned} C_3^{(0,0)} &\Rightarrow 3C_1, \quad C_3^{(N/3, 2N/3)} \Rightarrow 3C_1^{N/3}, \\ C_3^{(2N/3, N/3)} &\Rightarrow 3C_1^{2N/3}. \end{aligned} \quad (4.17)$$

When $N/3 = \text{integer}$, this fusion algebra has $\mathbb{Z}_3$ symmetry, because if $k + \ell = \alpha \pmod{3}$ and $m + n = \beta \pmod{3}$, then $k + \ell + m + n$, $k + \ell - 2m + n$,

and $k + \ell + m - 2n$ are all equivalent to $\alpha + \beta \pmod{3}$, i.e.,

$$C^\alpha \cdot C^\beta = C^{\alpha+\beta} \pmod{3}. \quad (4.18)$$

For concreteness, we present the multiplication rules of the classes for cases with $N = 2$ in Table V, $N = 3$ in Table VI, and $N = 4$ in Table VII. The multiplication rules for $N = 3$ are equivalent to those of $\mathbb{Z}_3$ gauging of $\mathbb{Z}_9$ shown in Appendix A. It turns out that there exist two S_2 symmetries associated with $C_1^1 \leftrightarrow C_1^2$ and $C_3^{(1,0)} \leftrightarrow C_3^{(2,0)}$ as seen in Table VI, and two S_2 symmetries associated with $C_3^{(1,2)} \leftrightarrow C_3^{(2,1)}$ and $C_3^{(1,0)} \leftrightarrow C_3^{(3,0)}$ as seen in Table VII. They correspond to the outer automorphisms. One of them exchanges $a \leftrightarrow a^{-1}$ and $a' \leftrightarrow a'^{-1}$, and the other corresponds with exchanges $a \leftrightarrow a'$. In addition, recall that when $N/3 = \text{integer}$, there is the $\mathbb{Z}_3$ symmetry in Eq. (4.18), where $C_3^{(1,0)}$ and $C_3^{(2,0)}$ transform as

$$C_3^{(p,0)} \rightarrow \omega^p C_3^{(p,0)}, \quad (4.19)$$

with $\omega = e^{2\pi i/3}$. When we combine the above symmetry with one of S_2 permutation symmetries, i.e., $a \leftrightarrow a^{-1}$ and $a' \leftrightarrow a'^{-1}$, the symmetry is enhanced to $S_3 \simeq \mathbb{Z}_3 \rtimes S_2$. The full symmetry is $S_3 \times S_2$.

- (i) $N = 2$
- (ii) $N = 3$
- (iii) $N = 4$

TABLE V. Multiplication rules of classes for $\mathbb{Z}_3$ gauging of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

	C_1	$C_3^{(1,0)}$
C_1	C_1	$C_3^{(1,0)}$
$C_3^{(1,0)}$	$C_3^{(1,0)}$	$3C_1 + 2C_3^{(1,0)}$

TABLE VI. Multiplication rules of classes for $\mathbb{Z}_3$ gauging of $\mathbb{Z}_3 \times \mathbb{Z}'_3$.

	C_1	C_1^1	C_1^2	$C_3^{(1,0)}$	$C_3^{(2,0)}$
C_1	C_1	C_1^1	C_1^2	$C_3^{(1,0)}$	$C_3^{(2,0)}$
C_1^1	C_1^1	C_1^2	C_1	$C_3^{(1,0)}$	$C_3^{(2,0)}$
C_1^2	C_1^2	C_1	C_1^1	$C_3^{(1,0)}$	$C_3^{(2,0)}$
$C_3^{(1,0)}$	$C_3^{(1,0)}$	$C_3^{(1,0)}$	$C_3^{(1,0)}$	$3C_3^{(2,0)}$	$3C_1 + 3C_1^1 + 3C_1^2$
$C_3^{(2,0)}$	$C_3^{(2,0)}$	$C_3^{(2,0)}$	$C_3^{(2,0)}$	$3C_1 + 3C_1^1 + 3C_1^2$	$3C_3^{(1,0)}$

TABLE VII. Multiplication rules of classes for $\mathbb{Z}_3$ gauging of $\mathbb{Z}_4 \times \mathbb{Z}'_4$.

	C_1	$C_3^{(1,0)}$	$C_3^{(2,0)}$	$C_3^{(3,0)}$	$C_3^{(1,2)}$	$C_3^{(2,1)}$
C_1	C_1	$C_3^{(1,0)}$	$C_3^{(2,0)}$	$C_3^{(3,0)}$	$C_3^{(1,2)}$	$C_3^{(2,1)}$
$C_3^{(1,0)}$	$C_3^{(1,0)}$	$C_3^{(2,0)} + 2C_3^{(3,0)}$	$C_3^{(3,0)} + C_3^{(1,2)} + C_3^{(2,1)}$	$3C_1 + C_3^{(1,2)} + C_3^{(2,1)}$	$C_3^{(1,0)} + C_3^{(2,0)} + C_3^{(1,2)}$	$C_3^{(1,0)} + C_3^{(2,0)} + C_3^{(2,1)}$
$C_3^{(2,0)}$	$C_3^{(2,0)}$	$C_3^{(3,0)} + C_3^{(1,2)} + C_3^{(2,1)}$	$3C_1 + 2C_3^{(2,0)}$	$C_3^{(1,0)} + C_3^{(1,2)} + C_3^{(2,1)}$	$C_3^{(1,0)} + C_3^{(3,0)} + C_3^{(2,1)}$	$C_3^{(1,0)} + C_3^{(3,0)} + C_3^{(1,2)}$
$C_3^{(3,0)}$	$C_3^{(3,0)}$	$3C_1 + C_3^{(1,2)} + C_3^{(2,1)}$	$C_3^{(1,0)} + C_3^{(1,2)} + C_3^{(2,1)}$	$2C_3^{(1,0)} + C_3^{(2,0)}$	$C_3^{(2,0)} + C_3^{(3,0)} + C_3^{(1,2)}$	$C_3^{(2,0)} + C_3^{(3,0)} + C_3^{(2,1)}$
$C_3^{(1,2)}$	$C_3^{(1,2)}$	$C_3^{(1,0)} + C_3^{(2,0)} + C_3^{(1,2)}$	$C_3^{(1,0)} + C_3^{(3,0)} + C_3^{(2,1)}$	$C_3^{(2,0)} + C_3^{(3,0)} + C_3^{(1,2)}$	$C_3^{(2,0)} + 2C_3^{(2,1)}$	$3C_1 + C_3^{(1,0)} + C_3^{(3,0)}$
$C_3^{(2,1)}$	$C_3^{(2,1)}$	$C_3^{(1,0)} + C_3^{(2,0)} + C_3^{(2,1)}$	$C_3^{(1,0)} + C_3^{(3,0)} + C_3^{(1,2)}$	$C_3^{(2,0)} + C_3^{(3,0)} + C_3^{(2,1)}$	$3C_1 + C_3^{(1,0)} + C_3^{(3,0)}$	$C_3^{(2,0)} + 2C_3^{(1,2)}$

$\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$ can be realized by higher dimensional field theory and string theory similar to $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$. Certain compactifications can lead to $\mathbb{Z}_N \times \mathbb{Z}'_N$. Under such symmetry, massless modes $\varphi_{k,\ell}$ transform as

$$\varphi_{k,\ell} \rightarrow a^k \varphi_{k,\ell} \quad (4.20)$$

under $\mathbb{Z}_N$, and

$$\varphi_{k,\ell} \rightarrow a^{\ell} \varphi_{k,\ell} \quad (4.21)$$

under $\mathbb{Z}'_N$. Here, k and ℓ may correspond to two independent discrete momenta, winding numbers, and their mixtures in the compact space.

Here, we consider a proper $\mathbb{Z}_3$ orbifolding such that the $\mathbb{Z}_3$ -invariant mode is written by⁶

$$\phi_{k,\ell} = \varphi_{k,\ell} + \varphi_{-k+\ell,-k} + \varphi_{\ell,k-\ell}. \quad (4.22)$$

That corresponds to $C^{(k,\ell)}$. We study the model with $N = 3$ as an illustrating model. Suppose that the “momentum” $(k, \ell) = (1, 0)$ in the compact space corresponds to one of the simple roots of $SU(3)$, α_1 and the momentum $(k, \ell) = (0, 1)$ corresponds to the other α_2 . Then, the $\mathbb{Z}_3$ orbifolding

corresponds to dividing $T^2 = R^2/\Lambda_{SU(3)}$ by the $\mathbb{Z}_3$ Coxeter element of $SU(3)$, where $\Lambda_{SU(3)}$ denotes the $SU(3)$ root lattice. The fields $\phi_{1,0}$ and $\phi_{2,0}$ corresponding to $C_3^{(1,0)}$ and $C_3^{(2,0)}$ are written as

$$\phi_{1,0} = \varphi_{1,0} + \varphi_{2,2} + \varphi_{0,1}, \quad \phi_{2,0} = \varphi_{2,0} + \varphi_{1,1} + \varphi_{0,2}, \quad (4.23)$$

where the momenta (k, ℓ) appearing in the right-hand sides are six nonzero roots of $SU(3)$. They are $\mathbb{Z}_3$ twist-invariant modes [4].

$\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$ leads to field-theoretical properties different from $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$, but similar to $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N$. Each field $\phi_{k,\ell}$ can correspond to a class $C^{(k,\ell)}$. They can be complex representations different from $\mathbb{Z}_2$ gauging. The coupling selection rules can be derived in a way similar to coupling selection rules due to $\mathbb{Z}_2$ gauging to $\mathbb{Z}_N$ symmetries. That is, if the products of $\prod C^{(k,\ell)}$ includes the identity e , the n -point coupling $\prod \phi_{k,\ell}$ is allowed. For example, the two-point coupling $\phi_{k,\ell} \phi_{k,\ell}$ is not always allowed, because $C^{k,\ell} \cdot C^{k,\ell}$ does not include the identity e for generic (k, ℓ) . That is different from the coupling selection rules due to $\mathbb{Z}_2$ of $\mathbb{Z}_N$ symmetries. On the other hand, we find that $(C^{k,\ell})^3$ always includes the identity. Thus, the three-point self-coupling $(\phi_{k,\ell})^3$ is always allowed.

⁶See Refs. [49–51] for geometrical $\mathbb{Z}_N$ orbifolding of magnetized compactifications.

B. $\Delta(3N^2)$

In this section, we discuss the relations of $\mathbb{Z}_3$ gauging of the $\mathbb{Z}_N \times \mathbb{Z}'_N$ symmetries to the $\Delta(3N^2) = (\mathbb{Z}_N \times \mathbb{Z}'_N) \rtimes \mathbb{Z}_3$ symmetries. The classes, C_1 , C_1^s , and $C_3^{(k,\ell)}$ in the previous section correspond to the conjugacy classes of $\Delta(3N^2)$, which do not include b [45,46].⁷ The $\Delta(3N^2)$ group has other conjugacy classes. When $N/3 = \text{integer}$, there exist the following conjugacy classes:

$$B_{N^2/3}^{(1,p,0)} \equiv \left\{ ba^{p-n-3m} a'^m \mid m = 0, 1, \dots, \frac{N-3}{3}, \right. \\ \left. n = 0, 1, \dots, N-1 \right\}, \\ B_{N^2/3}^{(2,p,0)} \equiv \left\{ b^2 a^{p-n-3m} a'^m \mid m = 0, 1, \dots, \frac{N-3}{3}, \right. \\ \left. n = 0, 1, \dots, N-1 \right\}, \quad (4.24)$$

with $p = 0, 1, 2$. On the other hand, when $N/3 \neq \text{integer}$, there exist the following conjugacy classes:

$$C_{N^2}^1 \equiv \{ ba^\ell a'^m \mid \ell, m = 0, 1, \dots, N-1 \}, \\ C_{N^2}^2 \equiv \{ b^2 a^\ell a'^m \mid \ell, m = 0, 1, \dots, N-1 \}. \quad (4.25)$$

We discuss the realization of the conjugacy classes $B_{N^2/3}^{(1,p,0)}$ as well as $B_{N^2/3}^{(2,p,0)}$. We focus on $N = 3$.⁸ The other conjugacy classes $C^{k,\ell}$ can be realized by the “bulk” modes on $T^2/\mathbb{Z}_3$. We may need up to three twist fields σ_p to represent $B_{N^2/3}^{(1,p,0)}$. There are three fixed points on $T^2/\mathbb{Z}_3$, and they are represented by the conjugacy classes of space group $(\theta, p\alpha_1)$ with $p = 0, 1, 2$, where θ denotes the $\mathbb{Z}_3$ twist. Thus, the structure of the conjugacy classes $B_{N^2/3}^{(1,p,0)}$ is consistent with the geometrical property of $T^2/\mathbb{Z}_3$ orbifold. There exists the following $\mathbb{Z}_3$ symmetry:

$$\sigma_p \rightarrow \omega^p \sigma_p, \quad (4.26)$$

with $\omega = e^{2\pi i/3}$ and the cyclic symmetry:

$$\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \rightarrow \sigma_0. \quad (4.27)$$

Combining these symmetries, the full symmetry becomes $\Delta(27)$ [3,4]. The conjugacy class $B_{N^2/3}^{(2,p,0)}$ can be represented by σ_p^2 .

When $N/3 = \text{integer}$, introduction of the three twist fields σ_p can present the conjugacy classes $B_{N^2/3}^{(1,p,0)}$ as well as $B_{N^2/3}^{(2,p,0)}$. Table VIII shows multiplication rules of whole $\Delta(27)$ conjugacy classes:

$$C_1 = \{e\}, \quad (4.28)$$

$$C_1^1 = \{aa'^2\}, \quad (4.29)$$

$$C_1^2 = \{a^2a'\}, \quad (4.30)$$

$$C_3^{(1,0)} = \{a, a', a^2a'^2\}, \quad (4.31)$$

$$C_3^{(2,0)} = \{a^2, a'^2, aa'\}, \quad (4.32)$$

$$B_3^{(1,0,0)} = \{b, baa'^2, ba^2a'\}, \quad (4.33)$$

$$B_3^{(1,1,0)} = \{ba, ba', ba^2a'^2\}, \quad (4.34)$$

$$B_3^{(1,2,0)} = \{ba^2, ba'^2, baa'\}, \quad (4.35)$$

$$B_3^{(2,0,0)} = \{b^2, b^2aa'^2, b^2a^2a'\}, \quad (4.36)$$

$$B_3^{(2,1,0)} = \{b^2a, b^2a', b^2a^2a'^2\}, \quad (4.37)$$

$$B_3^{(2,2,0)} = \{b^2a^2, b^2a'^2, b^2aa'\}. \quad (4.38)$$

It was known that the outer automorphism of $\Delta(27)$ is generated by [52]

$$u_1: (aa', b) \rightarrow (aa'b(aa')^{-1}, b^{-1}aa'b), \\ u_2: (aa', b) \rightarrow (b^{-1}(aa')^{-1}, b^2), \quad (4.39)$$

which is isomorphic to $\text{GL}(2, 3) \cong Q_8 \rtimes S_3$. It is defined in terms of

$$(a_1)^4 = (a_3)^2 = (a_4)^2 = (a_3a_4)^3 = e, \\ (a_2)^2 = (a_1)^2, \quad (a_2)^{-1}a_1a_2 = (a_1)^{-1}, \\ a_3a_1a_3 = (a_1)^3, \quad a_3a_2a_3 = a_1a_2, \\ a_4a_1a_4 = a_2, \quad a_4a_2a_4 = a_1, \quad (4.40)$$

with $a_1 = u_1u_2u_1u_2$, $a_2 = u_2u_1u_2u_1$, $a_3 = u_2u_1(u_2)^2$, and $a_4 = u_1$, where $\{a_1, a_2\}$ generate Q_8 and $\{a_3, a_4\}$ generate S_3 . Specifically, u_1 and u_2 act on each of the conjugacy classes such as

⁷Here, we follow the notation in Refs. [45,46].

⁸The $N = 2$ case, i.e., $\Delta(12) \cong A_4$ conjugacy classes, is discussed in Appendix B.

TABLE VIII. Multiplication rules for conjugacy classes of $\Delta(27)$.

	C_1	C_1^1	C_1^2	$C_3^{(1,0)}$	$C_3^{(2,0)}$
C_1	C_1	C_1^1	C_1^2	$C_3^{(1,0)}$	$C_3^{(2,0)}$
C_1^1	C_1^1	C_1^2	C_1	$C_3^{(1,0)}$	$C_3^{(2,0)}$
C_1^2	C_1^2	C_1	C_1^1	$C_3^{(1,0)}$	$C_3^{(2,0)}$
$C_3^{(1,0)}$	$C_3^{(1,0)}$	$C_3^{(1,0)}$	$C_3^{(1,0)}$	$3C_3^{(2,0)}$	$3C_1 + 3C_1^1 + 3C_1^2$
$C_3^{(2,0)}$	$C_3^{(2,0)}$	$C_3^{(2,0)}$	$C_3^{(2,0)}$	$3C_1 + 3C_1^1 + 3C_1^2$	$3C_3^{(1,0)}$
$B_3^{(1,0,0)}$	$B_3^{(1,0,0)}$	$B_3^{(1,0,0)}$	$B_3^{(1,0,0)}$	$3B_3^{(1,1,0)}$	$3B_3^{(1,2,0)}$
$B_3^{(1,1,0)}$	$B_3^{(1,1,0)}$	$B_3^{(1,1,0)}$	$B_3^{(1,1,0)}$	$3B_3^{(1,2,0)}$	$3B_3^{(1,0,0)}$
$B_3^{(1,2,0)}$	$B_3^{(1,2,0)}$	$B_3^{(1,2,0)}$	$B_3^{(1,2,0)}$	$3B_3^{(1,0,0)}$	$3B_3^{(1,1,0)}$
$B_3^{(2,0,0)}$	$B_3^{(2,0,0)}$	$B_3^{(2,0,0)}$	$B_3^{(2,0,0)}$	$3B_3^{(2,1,0)}$	$3B_3^{(2,2,0)}$
$B_3^{(2,1,0)}$	$B_3^{(2,1,0)}$	$B_3^{(2,1,0)}$	$B_3^{(2,1,0)}$	$3B_3^{(2,2,0)}$	$3B_3^{(2,0,0)}$
$B_3^{(2,2,0)}$	$B_3^{(2,2,0)}$	$B_3^{(2,2,0)}$	$B_3^{(2,2,0)}$	$3B_3^{(2,0,0)}$	$3B_3^{(2,1,0)}$
	$B_3^{(1,0,0)}$	$B_3^{(1,1,0)}$	$B_3^{(1,2,0)}$	$B_3^{(2,0,0)}$	$B_3^{(2,1,0)}$
C_1	$B_3^{(1,0,0)}$	$B_3^{(1,1,0)}$	$B_3^{(1,2,0)}$	$B_3^{(2,0,0)}$	$B_3^{(2,2,0)}$
C_1^1	$B_3^{(1,0,0)}$	$B_3^{(1,1,0)}$	$B_3^{(1,2,0)}$	$B_3^{(2,0,0)}$	$B_3^{(2,2,0)}$
C_1^2	$B_3^{(1,0,0)}$	$B_3^{(1,1,0)}$	$B_3^{(1,2,0)}$	$B_3^{(2,0,0)}$	$B_3^{(2,2,0)}$
$C_3^{(1,0)}$	$3B_3^{(1,1,0)}$	$3B_3^{(1,2,0)}$	$3B_3^{(1,0,0)}$	$3B_3^{(2,1,0)}$	$3B_3^{(2,0,0)}$
$C_3^{(2,0)}$	$3B_3^{(1,2,0)}$	$3B_3^{(1,0,0)}$	$3B_3^{(1,1,0)}$	$3B_3^{(2,2,0)}$	$3B_3^{(2,1,0)}$
$B_3^{(1,0,0)}$	$3B_3^{(2,0,0)}$	$3B_3^{(2,1,0)}$	$3B_3^{(2,2,0)}$	$3C_1 + 3C_1^1 + 3C_1^2$	$3C_3^{(1,0)}$
$B_3^{(1,1,0)}$	$3B_3^{(2,1,0)}$	$3B_3^{(2,2,0)}$	$3B_3^{(2,0,0)}$	$3C_3^{(1,0)}$	$3C_3^{(2,0)}$
$B_3^{(1,2,0)}$	$3B_3^{(2,2,0)}$	$3B_3^{(2,0,0)}$	$3B_3^{(2,1,0)}$	$3C_3^{(2,0)}$	$3C_1 + 3C_1^1 + 3C_1^2$
$B_3^{(2,0,0)}$	$3C_1 + 3C_1^1 + 3C_1^2$	$3C_3^{(1,0)}$	$3C_3^{(2,0)}$	$3B_3^{(1,0,0)}$	$3B_3^{(1,2,0)}$
$B_3^{(2,1,0)}$	$3C_3^{(1,0)}$	$3C_3^{(2,0)}$	$3C_1 + 3C_1^1 + 3C_1^2$	$3B_3^{(1,1,0)}$	$3B_3^{(1,0,0)}$
$B_3^{(2,2,0)}$	$3C_3^{(2,0)}$	$3C_1 + 3C_1^1 + 3C_1^2$	$3C_3^{(1,0)}$	$3B_3^{(1,2,0)}$	$3B_3^{(1,1,0)}$

$$\begin{aligned}
u_1: C_1^1 &\rightarrow C_1^2, & C_1^2 &\rightarrow C_1^1, & C_3^{(1,0)} &\rightarrow B_3^{(2,0,0)}, & C_3^{(2,0)} &\rightarrow B_3^{(1,0,0)}, \\
B_3^{(1,0,0)} &\rightarrow C_3^{(2,0)}, & B_3^{(1,1,0)} &\rightarrow B_3^{(2,2,0)}, & B_3^{(2,0,0)} &\rightarrow C_3^{(1,0)}, & B_3^{(2,2,0)} &\rightarrow B_3^{(1,1,0)}, \\
u_2: C_3^{(1,0)} &\rightarrow B_3^{(1,2,0)}, & C_3^{(2,0)} &\rightarrow B_3^{(2,1,0)}, & B_3^{(1,0,0)} &\rightarrow B_3^{(2,0,0)}, & B_3^{(1,1,0)} &\rightarrow C_3^{(2,0)}, \\
B_3^{(1,2,0)} &\rightarrow B_3^{(1,1,0)}, & B_3^{(2,0,0)} &\rightarrow B_3^{(1,0,0)}, & B_3^{(2,1,0)} &\rightarrow B_3^{(2,2,0)}, & B_3^{(2,2,0)} &\rightarrow C_3^{(1,0)}.
\end{aligned} \tag{4.41}$$

It turns out that the multiplication rules of $\Delta(27)$ conjugacy classes in Table VIII are invariant under the $GL(2, 3)$ symmetry. In addition, there is the $\mathbb{Z}_3 \times \mathbb{Z}_3'$ symmetry. One $\mathbb{Z}_3$ symmetry corresponds to Eq. (4.19). To be consistent, the classes $B_3^{(n,p,0)}$ must also transform as

$$B_3^{(n,p,0)} \rightarrow \omega^{n+p} B_3^{(n,p,0)}. \tag{4.42}$$

The other $\mathbb{Z}_3'$ symmetry corresponds to the transformation $b \rightarrow \omega b$, i.e.,

$$B_3^{(n,p,0)} \rightarrow \omega^n B_3^{(n,p,0)}. \tag{4.43}$$

Hence, the multiplication rules of $\Delta(27)$ conjugacy classes have the $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathcal{S}_3$ symmetry.

When $N/3 \neq \text{integer}$, we may introduce a single twist field to represent the conjugacy class $C_{N^2}^1$, which may correspond to one of σ_p , or all. However, its geometrical interpretation is not clear. At any rate, $\mathbb{Z}_3$ -invariant elements of $\Delta(3N^2)$ correspond to the elements in $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$ in the previous section.

V. $\mathbb{Z}_3$ GAUGING OF $\mathbb{Z}_N \times \mathbb{Z}'_N \times \mathbb{Z}''_N$ AND $\Sigma(3N^3)$

To complete our analysis, we consider the case in which $\mathbb{Z}_3$ acts as an automorphism of $\mathbb{Z}_N \times \mathbb{Z}'_N \times \mathbb{Z}''_N$, and compare the resulting selection rule with that of $\Sigma(3N^2)$. This construction can be realized in string compactifications on orbifolds, as mentioned in the Introduction.

The generators of $\mathbb{Z}_N \times \mathbb{Z}'_N \times \mathbb{Z}''_N$ are chosen as

$$\begin{aligned} a^N &= a'^N = a''^N = e, \\ aa' &= a'a, \quad a'a'' = a''a', \quad a''a = aa''. \end{aligned} \quad (5.1)$$

Let us consider the outer automorphism of $\mathbb{Z}_N \times \mathbb{Z}'_N \times \mathbb{Z}''_N$ generated by b :

$$\begin{aligned} b^3 &= e, \quad b^2ab = a'', \quad b^2a'b = a, \\ b^2a''b &= a', \quad b(a^ma''a''^{\ell})b^{-1} = a^{\ell}a''^ma''^m, \\ b^2(a^ma''a''^{\ell})b^{-2} &= a^na''^{\ell}a''^m. \end{aligned} \quad (5.2)$$

Then, one can define the $\mathbb{Z}_3$ -invariant classes:

$$C^{(k,\ell,m)} = \{b^n a^k a'^{\ell} a''^m b^{-n} | n = 0, 1, 2\}. \quad (5.3)$$

They are written more explicitly as

$$C_1 = \{e\}, \quad (5.4)$$

$$C_1^s = \{(aa'a'')^s\}, \quad (5.5)$$

$$C_3^{(k,\ell,m)} = \{a^k a'^{\ell} a''^m, a^{\ell} a'^m a''^k, a^m a'^k a''^{\ell}\}, \quad (5.6)$$

with $(k, \ell, m) \neq (0, 0, 0)$, (s, s, s) . These classes satisfy the following fusion rules:

$$C_1 \cdot C_1 = C_1, \quad (5.7)$$

$$C_1 \cdot C_1^s = C_1^s, \quad (5.8)$$

$$C_1 \cdot C_3^{(k,\ell,m)} = C_3^{(k,\ell,m)}, \quad (5.9)$$

$$C_1^s \cdot C_1^t = C_1^{s+t}, \quad (5.10)$$

$$C_1^s \cdot C_3^{(k,\ell,m)} = C_3^{(k+s,\ell+s,m+s)}, \quad (5.11)$$

TABLE IX. Multiplication rules of classes for $\mathbb{Z}_3$ gauging of $\mathbb{Z}_2 \times \mathbb{Z}'_2 \times \mathbb{Z}''_2$.

	C_1	C_1^1	$C_3^{(1,0,0)}$	$C_3^{(1,1,0)}$
C_1	C_1	C_1^1	$C_3^{(1,0,0)}$	$C_3^{(1,1,0)}$
C_1^1	C_1^1	C_1	$C_3^{(1,1,0)}$	$C_3^{(1,0,0)}$
$C_3^{(1,0,0)}$	$C_3^{(1,0,0)}$	$C_3^{(1,1,0)}$	$3C_1 + 2C_3^{(1,1,0)}$	$3C_1^1 + 2C_3^{(1,0,0)}$
$C_3^{(1,1,0)}$	$C_3^{(1,1,0)}$	$C_3^{(1,0,0)}$	$3C_1^1 + 2C_3^{(1,0,0)}$	$3C_1 + 2C_3^{(1,1,0)}$

$$\begin{aligned} C_3^{(k,\ell,m)} \cdot C_3^{(p,q,r)} &= C_3^{(k+p,\ell+q,m+r)} + C_3^{(k+r,\ell+p,m+q)} \\ &\quad + C_3^{(k+q,\ell+r,m+p)}. \end{aligned} \quad (5.12)$$

Note that if C_1^0 and $C_3^{(s,s,s)}$ appear in the right-hand side in the above equations, we have to replace them by

$$C_1^0 \Rightarrow C_1, \quad C_3^{(s,s,s)} \Rightarrow 3C_1^s. \quad (5.13)$$

For concreteness, we present the multiplication rules of the classes for the $N = 2$ case in Table IX, from which there exists the $\mathbb{Z}_2$ symmetry associated with $C_1^1 \rightarrow -C_1^1$ and $C_3^{(1,0,0)} \rightarrow -C_3^{(1,0,0)}$.

$\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N \times \mathbb{Z}''_N$ can be realized by higher dimensional field theory as well as string theory similar to $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$. First, we consider the compactification leading to $\mathbb{Z}_N \times \mathbb{Z}'_N \times \mathbb{Z}''_N$. Under such symmetry, massless modes $\varphi_{k,\ell,m}$ transform as

$$\varphi_{k,\ell,m} \rightarrow a^k a'^{\ell} a''^m \varphi_{k,\ell,m}, \quad (5.14)$$

where k, ℓ, m may correspond to three independent discrete momenta, winding numbers, and their mixtures in the compact space. Then, we divide this compact space further by $A_3 \simeq \mathbb{Z}_3$ permutation. The $\mathbb{Z}_3$ -invariant mode is written by

$$\phi_{k,\ell,m} = \varphi_{k,\ell,m} + \varphi_{\ell,m,k} + \varphi_{m,k,\ell}. \quad (5.15)$$

That corresponds to $C^{(k,\ell,m)}$.

The above classes $C^{(k,\ell,m)}$ correspond to the conjugacy classes without b of $(\mathbb{Z}_N \times \mathbb{Z}'_N \times \mathbb{Z}''_N) \rtimes \mathbb{Z}_3 = \Sigma(3M^3)$. That is, $C^{(k,\ell,m)}$ are $\mathbb{Z}_3$ -invariant ones. Table X shows multiplication rules of whole $\Sigma(24)$ conjugacy classes:

TABLE X. Multiplication rules for conjugacy classes of $\Sigma(24)$.

	C_1	C_1^1	$C_3^{(1,1,0)}$	$C_3^{(1,0,0)}$	$B_4^{(1,0,0,0)}$	$B_4^{(1,1,0,0)}$	$B_4^{(2,0,0,0)}$	$B_4^{(2,1,0,0)}$
C_1	C_1	C_1^1	$C_3^{(1,1,0)}$	$C_3^{(1,0,0)}$	$B_4^{(1,0,0,0)}$	$B_4^{(1,1,0,0)}$	$B_4^{(2,0,0,0)}$	$B_4^{(2,1,0,0)}$
C_1^1	C_1^1	C_1	$C_3^{(1,0,0)}$	$C_3^{(1,1,0)}$	$B_4^{(1,1,0,0)}$	$B_4^{(1,0,0,0)}$	$B_4^{(2,1,0,0)}$	$B_4^{(2,0,0,0)}$
$C_3^{(1,1,0)}$	$C_3^{(1,1,0)}$	$C_3^{(1,0,0)}$	$3C_1 + 2C_3^{(1,1,0)}$	$3C_1^1 + 2C_3^{(1,0,0)}$	$3B_4^{(1,0,0,0)}$	$3B_4^{(1,1,0,0)}$	$3B_4^{(2,0,0,0)}$	$3B_4^{(2,1,0,0)}$
$C_3^{(1,0,0)}$	$C_3^{(1,0,0)}$	$C_3^{(1,1,0)}$	$3C_1^1 + 2C_3^{(1,0,0)}$	$3C_1 + 2C_3^{(1,1,0)}$	$3B_4^{(1,1,0,0)}$	$3B_4^{(1,0,0,0)}$	$3B_4^{(2,1,0,0)}$	$3B_4^{(2,0,0,0)}$
$B_4^{(1,0,0,0)}$	$B_4^{(1,0,0,0)}$	$B_4^{(1,1,0,0)}$	$3B_4^{(1,0,0,0)}$	$3B_4^{(1,1,0,0)}$	$4B_4^{(2,0,0,0)}$	$4B_4^{(2,1,0,0)}$	$4C_1 + 4C_3^{(1,1,0)}$	$4C_1^1 + 4C_3^{(1,0,0)}$
$B_4^{(1,1,0,0)}$	$B_4^{(1,1,0,0)}$	$B_4^{(1,0,0,0)}$	$3B_4^{(1,1,0,0)}$	$3B_4^{(1,0,0,0)}$	$4B_4^{(2,1,0,0)}$	$4B_4^{(2,0,0,0)}$	$4C_1^1 + 4C_3^{(1,0,0)}$	$4C_1 + 4C_3^{(1,1,0)}$
$B_4^{(2,0,0,0)}$	$B_4^{(2,0,0,0)}$	$B_4^{(2,1,0,0)}$	$3B_4^{(2,0,0,0)}$	$3B_4^{(2,1,0,0)}$	$4C_1 + 4C_3^{(1,1,0)}$	$4C_1^1 + 4C_3^{(1,0,0)}$	$4B_4^{(1,0,0,0)}$	$4B_4^{(1,1,0,0)}$
$B_4^{(2,1,0,0)}$	$B_4^{(2,1,0,0)}$	$B_4^{(2,0,0,0)}$	$3B_4^{(2,1,0,0)}$	$3B_4^{(2,0,0,0)}$	$4C_1^1 + 4C_3^{(1,0,0)}$	$4C_1 + 4C_3^{(1,1,0)}$	$4B_4^{(1,1,0,0)}$	$4B_4^{(1,0,0,0)}$

$$C_1 = \{e\},$$

$$C_1^1 = \{aa'a''\},$$

$$C_3^{(1,1,0)} = \{aa', a'a'', a''a\},$$

$$C_3^{(1,0,0)} = \{a, a', a''\},$$

$$B_4^{(1,0,0,0)} = \{b, baa', ba'a'', ba''a\},$$

$$B_4^{(1,1,0,0)} = \{ba, ba', ba'', baa'a''\},$$

$$B_4^{(2,0,0,0)} = \{b^2, b^2aa', b^2a'a'', b^2a''a\},$$

$$B_4^{(2,1,0,0)} = \{b^2a, b^2a', b^2a'', b^2aa'a''\}.$$

We find that there exists the $\mathbb{Z}_2$ symmetry:

$$\begin{aligned} C_1^1 &\rightarrow -C_1^1, & C_3^{(1,0,0)} &\rightarrow -C_3^{(1,0,0)}, \\ B_4^{(1,1,0,0)} &\rightarrow -B_4^{(1,1,0,0)}, & B_4^{(2,1,0,0)} &\rightarrow -B_4^{(2,1,0,0)}, \end{aligned} \quad (5.16)$$

and the others are unchanged.

The conjugacy classes without b can be realized by the bulk fields as Eq. (5.15). If the above $A_3 \simeq \mathbb{Z}_3$ permutation would induce twist fields, the conjugacy classes with b may be realized by such twist fields.

VI. S_3 GAUGING OF $\mathbb{Z}_N \times \mathbb{Z}'_N$ AND $\Delta(6N^2)$

So far, we have studied Abelian gauging of discrete groups, such as $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Similarly, one can consider gauging by non-Abelian groups. The smallest non-Abelian group is S_3 . Here, we study S_3 gauging of discrete groups and the multiplication rules of conjugacy classes in discrete groups of the form $G = H \rtimes S_3$. Our study is motivated by non-Abelian orbifolds, in which twisted modes are labeled by the conjugacy classes of the non-Abelian group.

A. S_3 gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$

In this section, we analyze another gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$, i.e., S_3 gauging. The generators of $\mathbb{Z}_N \times \mathbb{Z}'_N$ are the same as

in Sec. IV A, but we consider the outer automorphisms of $\mathbb{Z}_N \times \mathbb{Z}'_N$ generated by b and c :

$$\begin{aligned} b^3 &= c^2 = (bc)^2 = e, \\ bab^{-1} &= a^{-1}a'^{-1}, & ba'b^{-1} &= a, \\ cac^{-1} &= a'^{-1}, & ca'c^{-1} &= a^{-1}, \\ b(a^\ell a'^m)b^{-1} &= a^{-\ell+m}a'^{-\ell}, \\ b^2(a^\ell a'^m)b^{-2} &= a^{-m}a'^{\ell-m}, \\ c(a^\ell a'^m)c^{-1} &= a^{-m}a'^{-\ell}, \\ bc(a^\ell a'^m)(bc)^{-1} &= a^{m-\ell}a'^m, \\ b^2c(a^\ell a'^m)(b^2c)^{-1} &= a^{-\ell+m}a'^{-\ell}. \end{aligned} \quad (6.1)$$

Then, one can define the S_3 -invariant class:

$$\begin{aligned} C^{(k,\ell)} &\equiv \{b^n c^m a^k a'^\ell (b^n c^m)^{-1} | n = 0, 1, 2, m = 0, 1\} \\ &= \{a^k a'^\ell, a^{-k+\ell} a'^{-k}, a^{-\ell} a'^{k-\ell}, a^{-\ell} a'^{-k}, \\ &\quad a^{-k+\ell} a'^\ell, a^k a'^{k-\ell}\}. \end{aligned} \quad (6.2)$$

In this context of S_3 gauging of the $\mathbb{Z}_N \times \mathbb{Z}'_N$ symmetry, the classes obey the following fusion rule:

$$\begin{aligned} C^{(k,\ell)} \cdot C^{(m,n)} &= C^{(k+m,\ell+n)} + C^{(k-m+n,\ell-m)} + C^{(k-n,\ell+m-n)} \\ &\quad + C^{(k-n,\ell-m)} + C^{(k-m+n,\ell+n)} + C^{(k+m,\ell+m-n)}. \end{aligned} \quad (6.3)$$

In general, the class $C^{(k,\ell)}$ includes six elements, as shown in Eq. (6.2), but obviously for $k = \ell = 0$, the class $C^{(0,0)}$ includes only one element $a^0 a'^0 = e$. In the following, we consider two cases depending on the value of N :

(i) $N/3 \neq \text{integer}$

When $N/3$ is not an integer, we have three classes:

$$C_1 = \{e\}, \quad (6.4)$$

$$C_3^s = \{a^s a'^{-s}, a^{-2s} a'^{-s}, a^s a'^{2s}\}, \quad (6.5)$$

$$C_6^{(k,\ell)} = \{a^k a'^\ell, a^{-k+\ell} a'^{-k}, a^{-\ell} a'^{k-\ell}, a^{-\ell} a'^{-k}, a^{-k+\ell} a'^\ell, a^k a'^{k-\ell}\}, \quad (6.6)$$

with $l \neq -k$, and their fusion rules are given by

$$C_1 \cdot C_1 = C_1, \quad (6.7)$$

$$C_1 \cdot C_3^s = C_3^s, \quad (6.8)$$

$$C_1 \cdot C_6^{(k,\ell)} = C_6^{(k,\ell)}, \quad (6.9)$$

$$C_3^s \cdot C_3^t = C_3^{s+t} + C_6^{(s-2t, -s-t)}, \quad (6.10)$$

$$C_3^s \cdot C_6^{(k,\ell)} = C_6^{(s+k, -s+\ell)} + C_6^{(s-k+\ell, -s-k)} + C_6^{(s-\ell, -s+k-\ell)}, \quad (6.11)$$

$$C_6^{(k,\ell)} \cdot C_6^{(m,n)} = C_6^{(k+m, \ell+n)} + C_6^{(k-m+n, \ell-m)} + C_6^{(k-n, \ell+m-n)} + C_6^{(k-n, \ell-m)} + C_6^{(k-m+n, \ell+n)} + C_6^{(k+m, \ell+m-n)}. \quad (6.12)$$

Note that if C_3^0 , $C_6^{(0,0)}$, and $C_6^{(k,-k)}$ appear in the right-hand side in the above equations, we have to replace them by

$$C_3^0 \Rightarrow 3C_1, \quad C_6^{(0,0)} \Rightarrow 6C_1, \quad C_6^{(k,-k)} \Rightarrow 2C_3^k. \quad (6.13)$$

(ii) $N/3 = \text{integer}$

When $N/3$ is an integer, we have four classes:

$$\begin{aligned} C_1 &= \{e\}, \\ C_1^s &= \{a^s a'^{-s}\} & (s = N/3, 2N/3), \\ C_3^s &= \{a^s a'^{-s}, a^{-2s} a'^{-s}, a^s a'^{2s}\} & (s \neq N/3, 2N/3), \\ C_6^{(k,\ell)} &= \{a^k a'^\ell, a^{-k+\ell} a'^{-k}, a^{-\ell} a'^{k-\ell}, a^{-\ell} a'^{-k}, a^{-k+\ell} a'^\ell, a^k a'^{k-\ell}\} & (\ell \neq -k). \end{aligned} \quad (6.14)$$

Their fusion rules are given by

$$C_1 \cdot C_1 = C_1, \quad (6.15)$$

$$C_1 \cdot C_1^s = C_1^s, \quad (6.16)$$

$$C_1 \cdot C_3^s = C_3^s, \quad (6.17)$$

$$C_1 \cdot C_6^{(k,\ell)} = C_6^{(k,\ell)}, \quad (6.18)$$

$$C_1^s \cdot C_1^{s'} = \begin{cases} C_1^{2s} & (s = s') \\ C_1 & (s \neq s') \end{cases}, \quad (6.19)$$

$$C_1^s \cdot C_3^t = C_3^{(s+t)}, \quad (6.20)$$

$$C_1^s \cdot C_6^{(k,\ell)} = C_6^{(s+k, -s+\ell)}, \quad (6.21)$$

$$C_3^s \cdot C_3^t = C_3^{s+t} + C_6^{(s-2t, -s-t)}, \quad (6.22)$$

$$C_3^s \cdot C_6^{(k,\ell)} = C_6^{(s+k, -s+\ell)} + C_6^{(s-k+\ell, -s-k)} + C_6^{(s-\ell, -s+k-\ell)}, \quad (6.23)$$

$$C_6^{(k,\ell)} \cdot C_6^{(m,n)} = C_6^{(k+m, \ell+n)} + C_6^{(k-m+n, \ell-m)} + C_6^{(k-n, \ell+m-n)} + C_6^{(k-n, \ell-m)} + C_6^{(k-m+n, \ell+n)} + C_6^{(k+m, \ell+m-n)}. \quad (6.24)$$

Note that if C_3^0 , $C_6^{(0,0)}$, $C_6^{(N/3, 2N/3)}$, $C_6^{(2N/3, N/3)}$, and $C_6^{(k,-k)}$ appear in the right-hand side in the above equations, we have to replace them by

$$\begin{aligned} C_3^0 &\Rightarrow 3C_1, & C_6^{(0,0)} &\Rightarrow 6C_1, \\ C_6^{(N/3, 2N/3)} &\Rightarrow 6C_1^{N/3}, & C_6^{(2N/3, N/3)} &\Rightarrow 6C_1^{2N/3}, \\ C_6^{(k,-k)} &\Rightarrow 2C_3^k. \end{aligned} \quad (6.25)$$

For concreteness, we present the multiplication rules of the classes for cases with $N = 2$ in Table XI and $N = 3$ in Table XII.⁹ It turns out that there exists the S_2 symmetry associated with $C_1^1 \leftrightarrow C_1^2$ for $N = 3$.

⁹Multiplication rules of S_3 gauging of $\mathbb{Z}_3 \times \mathbb{Z}_3^l$ are similar to the $\mathbb{Z}_3$ Tambara-Yamagami fusion rules. See Appendix C.

TABLE XI. Multiplication rules of classes for S_3 gauging of $\mathbb{Z}_2 \times \mathbb{Z}'_2$.

	C_1	C_3^1
C_1	C_1	C_3^1
C_3^1	C_3^1	$3C_1 + 2C_3^1$

TABLE XII. Multiplication rules of classes for S_3 gauging of $\mathbb{Z}_3 \times \mathbb{Z}'_3$.

	C_1	C_1^1	C_1^2	$C_6^{(1,0)}$
C_1	C_1	C_1^1	C_1^2	$C_6^{(1,0)}$
C_1^1	C_1^1	C_1^2	C_1	$C_6^{(1,0)}$
C_1^2	C_1^2	C_1	C_1^1	$C_6^{(1,0)}$
$C_6^{(1,0)}$	$C_6^{(1,0)}$	$C_6^{(1,0)}$	$C_6^{(1,0)}$	$6C_1 + 6C_1^1 + 6C_1^2 + 3C_6^{(1,0)}$

TABLE XIII. Multiplication rules for conjugacy classes of $\Delta(54)$.

	C_1	C_1^1	C_1^2	$C_6^{(1,0)}$	$B_6^{(0,0)}$	$B_6^{(1,0)}$	$B_6^{(2,0)}$
C_1	C_1	C_1^1	C_1^2	$C_6^{(1,0)}$	$B_6^{(0,0)}$	$B_6^{(1,0)}$	$B_6^{(2,0)}$
C_1^1	C_1^1	C_1^2	C_1	$C_6^{(1,0)}$	$B_6^{(0,0)}$	$B_6^{(1,0)}$	$B_6^{(2,0)}$
C_1^2	C_1^2	C_1	C_1^1	$C_6^{(1,0)}$	$B_6^{(0,0)}$	$B_6^{(1,0)}$	$B_6^{(2,0)}$
$C_6^{(1,0)}$	$C_6^{(1,0)}$	$C_6^{(1,0)}$	$C_6^{(1,0)}$	$6C_1 + 6C_1^1 + 6C_1^2 + 3C_6^{(1,0)}$	$3B_6^{(1,0)} + 3B_6^{(2,0)}$	$3B_6^{(0,0)} + 3B_6^{(2,0)}$	$3B_6^{(0,0)} + 3B_6^{(1,0)}$
$B_6^{(0,0)}$	$B_6^{(0,0)}$	$B_6^{(0,0)}$	$B_6^{(0,0)}$	$3B_6^{(1,0)} + 3B_6^{(2,0)}$	$3B_6^{(0,0)} + 6C_1 + 6C_1^1 + 6C_1^2$	$3B_6^{(2,0)} + 3C_6^{(1,0)}$	$3B_6^{(1,0)} + 3C_6^{(1,0)}$
$B_6^{(1,0)}$	$B_6^{(1,0)}$	$B_6^{(1,0)}$	$B_6^{(1,0)}$	$3B_6^{(0,0)} + 3B_6^{(2,0)}$	$3B_6^{(2,0)} + 3C_6^{(1,0)}$	$3B_6^{(1,0)} + 6C_1 + 6C_1^1 + 6C_1^2$	$3B_6^{(0,0)} + 3C_6^{(1,0)}$
$B_6^{(2,0)}$	$B_6^{(2,0)}$	$B_6^{(2,0)}$	$B_6^{(2,0)}$	$3B_6^{(0,0)} + 3B_6^{(1,0)}$	$3B_6^{(1,0)} + 3C_6^{(1,0)}$	$3B_6^{(0,0)} + 3C_6^{(1,0)}$	$3B_6^{(2,0)} + 6C_1 + 6C_1^1 + 6C_1^2$
$B_9^{(0)}$	$B_9^{(0)}$	$B_9^{(2)}$	$B_9^{(1)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$
$B_9^{(1)}$	$B_9^{(1)}$	$B_9^{(0)}$	$B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$
$B_9^{(2)}$	$B_9^{(2)}$	$B_9^{(1)}$	$B_9^{(0)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$
	$B_9^{(0)}$			$B_9^{(1)}$			$B_9^{(2)}$
C_1	$B_9^{(0)}$			$B_9^{(1)}$			$B_9^{(2)}$
C_1^1	$B_9^{(2)}$			$B_9^{(0)}$			$B_9^{(1)}$
C_1^2	$B_9^{(1)}$			$B_9^{(2)}$			$B_9^{(0)}$
$C_6^{(1,0)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$			$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$			$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$
$B_6^{(0,0)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$			$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$			$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$
$B_6^{(1,0)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$			$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$			$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$
$B_6^{(2,0)}$	$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$			$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$			$2B_9^{(0)} + 2B_9^{(1)} + 2B_9^{(2)}$
$B_9^{(0)}$	$9C_1 + 3C_6^{(1,0)} + 3B_6^{(0,0)} + 3B_6^{(1,0)} + 3B_6^{(2,0)}$				$9C_1^1 + 3C_6^{(1,0)} + 3B_6^{(0,0)} + 3B_6^{(1,0)} + 3B_6^{(2,0)}$		
$B_9^{(1)}$	$9C_1^1 + 3C_6^{(1,0)} + 3B_6^{(0,0)} + 3B_6^{(1,0)} + 3B_6^{(2,0)}$				$9C_1^2 + 3C_6^{(1,0)} + 3B_6^{(0,0)} + 3B_6^{(1,0)} + 3B_6^{(2,0)}$		
$B_9^{(2)}$	$9C_1^2 + 3C_6^{(1,0)} + 3B_6^{(0,0)} + 3B_6^{(1,0)} + 3B_6^{(2,0)}$				$9C_1 + 3C_6^{(1,0)} + 3B_6^{(0,0)} + 3B_6^{(1,0)} + 3B_6^{(2,0)}$		

S_3 gauging of $(\mathbb{Z}_N \times \mathbb{Z}'_N)$ can be realized by higher dimensional theory in a way similar to $\mathbb{Z}_3$ gauging of $(\mathbb{Z}_N \times \mathbb{Z}'_N)$. The $(\mathbb{Z}_N \times \mathbb{Z}'_N)$ transformation is the same as Eqs. (4.20) and (4.21). Then, we consider a proper S_3 orbifolding. For example, when $N = 3$ and T^2 is the $T^2 = R^2/\Lambda_{SU(2)}$ in the previous section, we use a combination of the $\mathbb{Z}_3$ Coxeter element and $\mathbb{Z}_2$ reflection.

B. $\Delta(6N^2)$

In this section, we discuss the relations of S_3 gauging of the $\mathbb{Z}_N \times \mathbb{Z}'_N$ symmetries to the $\Delta(6N^2) = (\mathbb{Z}_N \times \mathbb{Z}'_N) \rtimes S_3$ symmetries. The classes C_1 , C_1^s , C_3^s , and $C_6^{(k,\ell)}$ in the previous section correspond to the conjugacy classes of $\Delta(6N^2)$, which do not include b and c [45,46].¹⁰

The full conjugacy classes of $\Delta(6N^2)$ include additional classes. For a concrete example, let us consider $\Delta(54)$ group whose conjugacy classes are listed as¹¹

$$C_1 = \{e\}, \quad (6.26)$$

$$C_1^1 = \{aa'^2\}, \quad (6.27)$$

$$C_1^2 = \{a^2a'\}, \quad (6.28)$$

$$C_6^{(1,0)} = \{a, a', a^2a'^2, a^2, a'^2, aa'\}, \quad (6.29)$$

$$B_6^{(0,0)} = \{b, ba^2a', ba'a'^2, b^2, b^2a^2a', b^2aa'^2\}, \quad (6.30)$$

$$B_6^{(1,0)} = \{ba, ba', ba^2a'^2, b^2a^2, b^2a'^2, b^2aa'\}, \quad (6.31)$$

$$B_6^{(2,0)} = \{ba^2, ba'a', ba'^2, b^2a, b^2a', b^2a^2a'^2\}, \quad (6.32)$$

$$B_9^{(m)} = \{ca^{m+p}a'^p, b^2ca^{-m}a'^{-m-p}, bca^{-p}a'^m | p = 0, 1, 2, (m = 0, 1, 2)\}. \quad (6.33)$$

Note that there exists the outer automorphism generated by [53]

$$\begin{aligned} u_1: (aa', b, c) &\rightarrow (aa', aa'baa', c), \\ u_2: (aa', b, c) &\rightarrow (aa'b^2aa', b, c), \end{aligned} \quad (6.34)$$

which is isomorphic to S_4 , i.e., $(u_1)^3 = (u_2)^2 = (u_1^{-1} \circ u_2)^4 = e$. Here, u_1 and u_2 act on each of the conjugacy classes such as

$$\begin{aligned} u_1: B_6^{(0,0)} &\rightarrow B_6^{(1,0)}, & B_6^{(1,0)} &\rightarrow B_6^{(2,0)}, & B_6^{(2,0)} &\rightarrow B_6^{(0,0)}, \\ u_2: C_1^1 &\rightarrow C_1^2, & C_1^2 &\rightarrow C_1^1, & C_6^{(1,0)} &\rightarrow B_6^{(2,0)}, \\ B_6^{(2,0)} &\rightarrow C_6^{(1,0)}, & B_9^{(1)} &\rightarrow B_9^{(2)}, & B_9^{(2)} &\rightarrow B_9^{(1)}. \end{aligned} \quad (6.35)$$

It turns out that the multiplication rules of $\Delta(54)$ conjugacy classes in Table XIII are invariant under the S_4 symmetry.

The $\Delta(54)$ can be realized by higher dimensional fields and string theory similar to $\Delta(27)$. We introduce twist fields σ_p in the same way as $\Delta(27)$. Then, we consider the reflection, e.g.,

$$\sigma_1 \rightarrow \sigma_2, \quad \sigma_2 \rightarrow \sigma_1, \quad (6.36)$$

in addition to $\mathbb{Z}_3$ symmetry (4.26) and the cyclic symmetry (4.27). The full symmetry becomes $\Delta(54)$ [3,4]. Hence, the whole symmetry for the multiplication rules of $\Delta(54)$ conjugacy classes is described by $\Delta(54) \rtimes S_4$.

VII. PARTICLE PHENOMENOLOGICAL ASPECTS

Here, we discuss particle phenomenological aspects of multiplication rules, which were obtained in the previous sections.

A. Coupling selection rules at tree and loop levels

Here, we study coupling selection rules derived from new multiplication rules. For comparison, we review briefly conventional coupling selection rules due to group theory. Suppose that g_k are elements in the group G , and they satisfy the following multiplication rule:

$$g_k g_\ell = g_m. \quad (7.1)$$

The element g_m appearing in the right-hand side is unique. Each field (particle) ϕ_k corresponds to a (representation of) element g_k of G . These fields can couple and the process $\phi_k + \phi_\ell \rightarrow \phi_m$ can occur only if $g_k g_\ell = g_m$. That is charge conservation law. These coupling selection rules are not violated in perturbation theory.¹²

We have derived several multiplication rules of classes. For simplicity, we denote the classes by $C^{(k)}$ and their multiplication rules by

$$C^{(k)} \cdot C^{(\ell)} = \sum_m y_{k\ell m} C^{(m)}, \quad (7.2)$$

where $y_{k\ell m}$ are constants and the elements $C^{(m)}$ appearing in the right-hand side are not always unique. Each field

¹⁰Here, we follow the notation in Refs. [45,46].

¹¹The multiplication rules of $\Delta(24) \cong S_4$ conjugacy classes are discussed in Appendix B.

¹²The group symmetry can be broken nonperturbatively if it is anomalous. See e.g., Refs. [54–59] for anomalies of discrete symmetries.

(particle) ϕ_k corresponds to a class $C^{(k)}$. Their couplings are allowed and the process $\phi_k + \phi_\ell \rightarrow \phi_m$ can occur only if $y_{k\ell m} \neq 0$. That can lead to new and interesting phenomenological aspects. An important difference is that the elements $C^{(m)}$ appearing in the right-hand side of Eq. (7.2) are not always unique.

The above new coupling selection rules hold at the tree level. In general, these coupling selection rules are violated at loop level [6,7,9]. Such a phenomenon also leads to another interesting aspect. That is, some couplings have loop suppression and can be small, if they are forbidden at the tree level.

The perturbation behavior of our selection rules includes more aspects. For example, the classes of $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ have the $\mathbb{Z}_2$ symmetry when $N = \text{even}$. That is, $C^{(k=\text{even})}$ and $C^{(k=\text{odd})}$ correspond to $\mathbb{Z}_2$ even and odd, respectively. Similarly, the classes of $\mathbb{Z}_3$ gauging of $\mathbb{Z}_N \times \mathbb{Z}'_N$ has the $\mathbb{Z}_3$ symmetry when $N/3 = \text{integer}$. We can find similar symmetries in other multiplication rules. For these types, we can write multiplication rules by

$$C_{r_1}^{(k)} \cdot C_{r_2}^{(\ell)} = \sum_m y_{k\ell m} C_{r_3}^{(m)}. \quad (7.3)$$

We have added the new index r_i to denote the $\mathbb{Z}_M$ charges. Note that r_3 in the right-hand side is unique because of $\mathbb{Z}_M$ charge conservation law. This is partially a group theoretical selection rule. Thus, the coupling selection rules due to these $\mathbb{Z}_M$ symmetries are not violated by loop effects. There are other symmetries in multiplication rules obtained in the previous sections. For example, there is the permutation symmetry $S_2 \simeq \mathbb{Z}_2$ in $\mathbb{Z}_3$ gauging of $\mathbb{Z}_7$, where C_1 is a singlet, and C_3^1 and C_3^2 are a doublet and they are permuted each other. This symmetry is not violated by loop effects, either. For example, we can take the basis to diagonalize this permutation and that is nothing but the above $\mathbb{Z}_2$ even and odd basis. Obviously, if the theory includes only fields with C_3^1 (C_3^2), but not C_3^2 (C_3^1), there is no $S_2 \simeq \mathbb{Z}_2$ symmetry from the beginning. We can find similar permutation symmetries in other multiplication rules. These symmetries in multiplication rules are shown in Table XIV. Although S_2 is isomorphic to $\mathbb{Z}_2$, we use this notation to show the permutation of conjugacy classes, i.e., the outer automorphism. On the other hand, the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ symmetries shown in Table XIV are represented by diagonal matrices on the classes. For example, the multiplication rules in $\mathbb{Z}_3$ gauging of $\mathbb{Z}_3 \times \mathbb{Z}'_3$ has the large symmetry $(\mathbb{Z}_3 \rtimes S_2) \times S_2$. This symmetry protects some parts of coupling selection rules from violations due to loop effects. The larger the symmetry is, the larger part is protected from violation due to loop effects. Obviously, as mentioned above, if the model at the tree level does not have the permutation symmetry like no partner of permutation, such symmetry does not exist from the beginning. Similarly, we can find the group theoretical symmetries in multiplication

TABLE XIV. Symmetries in multiplication rules.

Classes	Symmetry
$\mathbb{Z}_2$ gauging of $\mathbb{Z}_{N=\text{even}}$	$\mathbb{Z}_2$
$\mathbb{Z}_3$ gauging of $\mathbb{Z}_7$	S_2
$\mathbb{Z}_3$ gauging of $\mathbb{Z}_3 \times \mathbb{Z}'_3$	$(\mathbb{Z}_3 \rtimes S_2) \times S_2$
$\mathbb{Z}_3$ gauging of $\mathbb{Z}_4 \times \mathbb{Z}'_4$	$S_2 \times S_2$
$\mathbb{Z}_3$ gauging $(\mathbb{Z}_2 \times \mathbb{Z}'_2 \times \mathbb{Z}''_2)$	$\mathbb{Z}_2$

rules of conjugacy classes of discrete groups. Their conjugacy classes have outer automorphisms and inner automorphisms. Their combinations correspond to the full symmetries of multiplication rules. Those are quite large. For example, the multiplication rules of $\Delta(27)$ conjugacy classes have the $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes S_3$ symmetry.

B. $\mathbb{Z}_2$ gauging versus $\mathbb{Z}_3$ gauging

$\mathbb{Z}_2$ gauging and $\mathbb{Z}_3$ gauging have different properties. Here, we discuss some differences by using illustrating models. Note that the discussion of $\mathbb{Z}_2$ gauging serves as a review, while that of $\mathbb{Z}_3$ gauging presents a new analysis.

One of interesting applications of noninvertible selection rules is to realize interesting Yukawa textures [22,23]. First, let us compare Yukawa textures derived by $\mathbb{Z}_2$ and $\mathbb{Z}_3$ gauging. As a simple illustration, we briefly review the $\mathbb{Z}_2$ gauging of $\mathbb{Z}_5$, which leads to three distinct classes:

$$C^{(0)} = \{e\}, \quad C^{(1)} = \{a, a^{-1}\}, \quad C^{(2)} = \{a^2, a^{-2}\}, \quad (7.4)$$

where $C^{(k)}$ corresponds to $[g^k]$ in the notation in Refs. [22,23]. They satisfy Eq. (2.5) as multiplication rules. Suppose that three generations of both left-handed and conjugate of right-handed fermions respectively correspond to $C^{(0)}$, $C^{(1)}$, and $C^{(2)}$. Then, it was shown in Ref. [22] that the following Yukawa textures are obtained:

$$Y = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad (7.5)$$

for the Higgs field labeled by $C^{(0)}$,

$$Y = \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & * \end{pmatrix} \quad (7.6)$$

for the Higgs field labeled by $C^{(1)}$,

$$Y = \begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & 0 \end{pmatrix}, \quad (7.7)$$

for the Higgs field labeled by $C^{(2)}$, where the asterisk symbol $*$ denotes nonvanishing elements.

For a simple illustrating model to compare, we consider $\mathbb{Z}_3$ gauging of $\mathbb{Z}_7$, which has three classes C_1 , C_3^1 , and C_3^2 . Their multiplication rules are shown in Table III. Suppose that three generations of both left-handed and conjugate of right-handed fermions respectively correspond to C_1 , C_3^1 , and C_3^2 . Then, we can realize the following Yukawa textures:

$$Y = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}, \quad (7.8)$$

for the Higgs field labeled by C_1 ,

$$Y = \begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix}, \quad (7.9)$$

for the Higgs field labeled by C_3^1 ,

$$Y = \begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}, \quad (7.10)$$

for the Higgs field labeled by C_3^2 . These simple examples already lead to different texture realizations. $\mathbb{Z}_3$ gauging, including its various combinations, is expected to yield more intricate textures. It is important to study these systematically in a way similar to the $\mathbb{Z}_2$ gauging, which has been done in Ref. [23]. We leave such a study for future work.

Next, we study neutrino mass matrices. We briefly review the results in Ref. [26], where selection rules by $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries were studied. The class $C^{(k)}$ of $\mathbb{Z}_2$ gauging is assigned to each field of the Standard Model or its extension. Note that the product $C^{(k)} \cdot C^{(k)}$ always includes $C^{(0)}$. Furthermore, $C^{(k)} \cdot C^{(k)} \cdot C^{(\ell)} \cdot C^{(\ell)}$ always includes $C^{(0)}$. As a concrete example, let us first consider the Weinberg operators, which induce the neutrino masses. We discuss them within the framework of supersymmetric models and write them as

$$W = c_{ij} L_i H_u L_j H_u, \quad (7.11)$$

in the superpotential, where L_i ($i = 1, 2, 3$) denote three generations of lepton doublet superfields and H_u denotes the up-sector Higgs superfield.¹³ These fields are labeled by

¹³For the neutrino masses, supersymmetric and nonsupersymmetric models are the same, but later we will discuss the so-called μ -term, that is the supersymmetric mass of H_u and H_d , where H_d denotes the down-sector Higgs field.

the class $C^{(k)}$ of $\mathbb{Z}_2$ gauging. As mentioned before, the four-point coupling $C^{(k)} \cdot C^{(k)} \cdot C^{(\ell)} \cdot C^{(\ell)}$ always includes $C^{(0)}$, where L_i may correspond to $C^{(k)}$ and H_u may correspond to $C^{(\ell)}$.

Hence, the diagonal entries of c_{ij} are always allowed, and the possible textures are [26]

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \\ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}. \quad (7.12)$$

See Sec. 4.2.4 of Ref. [26] for more details, where we classify all the assignments of classes for fields in the case of $\mathbb{Z}_2$ gauging of $\mathbb{Z}_{3,4,5}$.

Now, let us discuss $\mathbb{Z}_3$ gauging of $\mathbb{Z}_7$ as an example. We assign its classes to L_i and H_u . For example, we assign such that (L_1, L_2, L_3) correspond to (C_1, C_3^1, C_3^2) . When the Higgs field H_u corresponds to C_1 , we obtain

$$c_{ij} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}. \quad (7.13)$$

When the Higgs field H_u corresponds to C_3^1 , we obtain

$$c_{ij} = \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix}. \quad (7.14)$$

Thus, $\mathbb{Z}_3$ gauging can lead to mass textures different from ones derived by $\mathbb{Z}_2$ gauging in Ref. [26]. Furthermore, if we can combine two selection rules, where one corresponds to $\mathbb{Z}_3$ gauging leading to Eq. (7.14) and $\mathbb{Z}_2$ gauging leading to the third matrix in Eq. (7.12), we can realize the following texture:

$$c_{ij} = \begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix}, \quad (7.15)$$

which corresponds to the neutrino mass in Refs. [60,61], i.e., the so-called A_1 texture. One can rederive previously known neutrino mass textures. We will study systematically mass matrices derived from $\mathbb{Z}_3$ gauging elsewhere.

To gain further insight into the features of $\mathbb{Z}_3$ gauging, we consider the μ -term in the supersymmetric standard model:

$$W_\mu = \mu H_u H_d \quad (7.16)$$

in the superpotential. For example, we assume that H_u corresponds to C_3^1 . In this case, if H_d corresponds to C_3^2 , the μ -term is allowed since the product $C_3^1 \cdot C_3^2$ includes the identity class C_1 , as seen in Table III. Otherwise, the μ -term is forbidden. In such a case, we may introduce a singlet field S such that the coupling $\lambda S H_u H_d$ are allowed like the next-to-minimal supersymmetric standard model (NMSSM). When H_d corresponds to C_1 , the singlet S must be C_3^2 . For this assignment, S^2 is forbidden, but S^3 is allowed. That is, the superpotential of the Higgs sector can be written as

$$W_H = \lambda S H_u H_d + \kappa S^3, \quad (7.17)$$

which can realize the Higgs sector in the $\mathbb{Z}_3$ -invariant NMSSM. Note that $\mathbb{Z}_3$ gauging allows the S^3 coupling for a singlet S with any class. When H_d corresponds to C_3^2 and the singlet S corresponds to C_1 , the S^2 term is also allowed. This behavior is different from $\mathbb{Z}_2$ gauging. In $\mathbb{Z}_2$ gauging, the S^2 term is always allowed, but the cubic term S^3 is not always allowed. This structure originates from the selection rules dictated by the fusion algebras. For $\mathbb{Z}_2$ gauging, the inverse class $[(C^{(k)})^{-1}]$, corresponding to the conjugate field ϕ^* , belongs to the same class $[C^{(k)}]$, since the product $[C^{(k)}] \cdot [C^{(\ell)}]$ always includes the identity. In contrast, for $\mathbb{Z}_3$ gauging, a class is not necessarily identical to its inverse class. For example, in the $\mathbb{Z}_3$ gauging of $\mathbb{Z}_7$, C_3^2 is regarded as the inverse class of C_3^1 , and conversely, C_3^1 is the inverse class of C_3^2 .¹⁴

VIII. CONCLUSIONS

By clarifying the selection rules of several finite discrete groups, we found that the selection rules of conjugacy classes do not have the conventional grouplike structure. When the group is gauged, the gauge-invariant classes enjoy finite Abelian or non-Abelian discrete symmetries, as summarized in Table XIV. The whole conjugacy classes of a discrete group G also enjoy the discrete symmetries determined by the inner and/or outer automorphism of G . In contrast to the previous studies of $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries, we have systematically analyzed $\mathbb{Z}_3$ gauging of discrete symmetries as well as S_3 gauging.

It turned out that $\mathbb{Z}_3$ gauging can lead to complex representations. That is different from $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries. Note that the field ϕ_k and its conjugate ϕ_k^* correspond to the same class C^k in $\mathbb{Z}_2$ gauging. The two-point coupling $\phi_k \phi_k$ as well as the $2n$ -point self-coupling $(\phi_k)^{2n}$ is always allowed. Also, the diagonal entries in the

neutrino mass matrix induced by the Weinberg operators are always allowed [26]. On the other hand, we have the phenomenologically interesting models, where up-sector and down-sector Higgs fields, H_u and H_d , correspond to different classes [22,23]. $\mathbb{Z}_2$ gauging does not allow the μ -term in the supersymmetric standard model when H_u and H_d have classes different from each other. $\mathbb{Z}_3$ gauging does not always allow the two-point coupling $(\phi_k)^2$ or diagonal entries of neutrino masses by the Weinberg operators, but allow the $3n$ -point self-coupling $(\phi_k)^{3n}$. That would lead to interesting neutrino mass textures. Also, $\mathbb{Z}_3$ gauging may allow the μ -term for proper combinations of classes of H_u and H_d , although they are different from each other. That can lead to a rich structure in mass textures. Thus, it is quite interesting to apply our $\mathbb{Z}_3$ gauging to particle phenomenology. We leave it for future work.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: $\mathbb{Z}_3$ GAUGING OF $\mathbb{Z}_9$ SYMMETRY

Here, we study $\mathbb{Z}_3$ gauging of $\mathbb{Z}_9$. We denote the generator of $\mathbb{Z}_9$ by a , i.e.,

$$a^9 = e. \quad (A1)$$

As in Sec. III, we consider the following $\mathbb{Z}_3$ automorphism b :

$$b^{-1}ab = a^7, \quad b^3 = e. \quad (A2)$$

TABLE XV. Multiplication rules of classes for $\mathbb{Z}_3$ gauging of $\mathbb{Z}_9$.

	C_1	C_3^1	C_3^6	$C_3^{(1)}$	$C_3^{(2)}$
C_1	C_1	C_3^1	C_3^6	$C_3^{(1)}$	$C_3^{(2)}$
C_3^1	C_3^1	C_1	C_1	$C_3^{(1)}$	$C_3^{(2)}$
C_3^6	C_3^6	C_1	C_1	$C_3^{(1)}$	$C_3^{(2)}$
$C_3^{(1)}$	$C_3^{(1)}$	$C_3^{(1)}$	$C_3^{(1)}$	$3C_3^{(2)}$	$3C_1 + 3C_3^1 + 3C_3^6$
$C_3^{(2)}$	$C_3^{(2)}$	$C_3^{(2)}$	$C_3^{(2)}$	$3C_1 + 3C_3^1 + 3C_3^6$	$3C_3^{(1)}$

¹⁴Such a structure indicates the presence of a CP transformation, as pointed out in Ref. [62].

TABLE XVI. Multiplication rules of classes for $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$.

	C_1	C_3^3	C_1^6	$C_3^{(1)}$	$C_3^{(2)}$	$B_3^{(1)}$	$B_3^{(2)}$	$B_3^{(3)}$	$B_3^{(4)}$	$B_3^{(5)}$	$B_3^{(6)}$
C_1	C_1	C_1^3	C_1^6	$C_3^{(1)}$	$C_3^{(2)}$	$B_3^{(1)}$	$B_3^{(2)}$	$B_3^{(3)}$	$B_3^{(4)}$	$B_3^{(5)}$	$B_3^{(6)}$
C_3^3	C_1^3	C_1^6	C_1	$C_3^{(1)}$	$C_3^{(2)}$	$B_3^{(1)}$	$B_3^{(2)}$	$B_3^{(3)}$	$B_3^{(4)}$	$B_3^{(5)}$	$B_3^{(6)}$
C_1^6	C_1^6	C_1	C_1^3	$C_3^{(1)}$	$C_3^{(2)}$	$B_3^{(1)}$	$B_3^{(2)}$	$B_3^{(3)}$	$B_3^{(4)}$	$B_3^{(5)}$	$B_3^{(6)}$
$C_3^{(1)}$	$C_3^{(1)}$	$C_3^{(1)}$	$C_3^{(1)}$	$3C_3^{(2)}$	$3C_1 + 3C_3^3 + 3C_1^6$	$3B_3^{(2)}$	$3B_3^{(3)}$	$3B_3^{(1)}$	$3B_3^{(5)}$	$3B_3^{(6)}$	$3B_3^{(4)}$
$C_3^{(2)}$	$C_3^{(2)}$	$C_3^{(2)}$	$C_3^{(2)}$	$3C_1 + 3C_3^3 + 3C_1^6$	$3C_3^{(1)}$	$3B_3^{(3)}$	$3B_3^{(1)}$	$3B_3^{(2)}$	$3B_3^{(6)}$	$3B_3^{(4)}$	$3B_3^{(5)}$
$B_3^{(1)}$	$B_3^{(1)}$	$B_3^{(1)}$	$B_3^{(1)}$	$3B_3^{(2)}$	$3B_3^{(3)}$	$3B_3^{(4)}$	$3B_3^{(5)}$	$3B_3^{(6)}$	$3C_1 + 3C_3^3 + 3C_1^6$	$3C_3^{(1)}$	$3C_3^{(2)}$
$B_3^{(2)}$	$B_3^{(2)}$	$B_3^{(2)}$	$B_3^{(2)}$	$3B_3^{(3)}$	$3B_3^{(1)}$	$3B_3^{(5)}$	$3B_3^{(6)}$	$3B_3^{(4)}$	$3C_3^{(1)}$	$3C_3^{(2)}$	$3C_1 + 3C_3^3 + 3C_1^6$
$B_3^{(3)}$	$B_3^{(3)}$	$B_3^{(3)}$	$B_3^{(3)}$	$3B_3^{(1)}$	$3B_3^{(2)}$	$3B_3^{(6)}$	$3B_3^{(4)}$	$3B_3^{(5)}$	$3C_3^{(2)}$	$3C_1 + 3C_3^3 + 3C_1^6$	$3C_3^{(1)}$
$B_3^{(4)}$	$B_3^{(4)}$	$B_3^{(4)}$	$B_3^{(4)}$	$3B_3^{(5)}$	$3B_3^{(6)}$	$3C_1 + 3C_3^3 + 3C_1^6$	$3C_3^{(1)}$	$3C_3^{(2)}$	$3B_3^{(1)}$	$3B_3^{(2)}$	$3B_3^{(3)}$
$B_3^{(5)}$	$B_3^{(5)}$	$B_3^{(5)}$	$B_3^{(5)}$	$3B_3^{(6)}$	$3B_3^{(4)}$	$3C_3^{(1)}$	$3C_3^{(2)}$	$3C_1 + 3C_3^3 + 3C_1^6$	$3B_3^{(3)}$	$3B_3^{(1)}$	$3B_3^{(2)}$
$B_3^{(6)}$	$B_3^{(6)}$	$B_3^{(6)}$	$B_3^{(6)}$	$3B_3^{(4)}$	$3B_3^{(5)}$	$3C_3^{(2)}$	$3C_1 + 3C_3^3 + 3C_1^6$	$3C_3^{(1)}$	$3B_3^{(2)}$	$3B_3^{(3)}$	$3B_3^{(1)}$

Note that the combination of $N = 9$ and $m = 7$ satisfies the condition (3.7), i.e., $m - 1 = 6 = 3 \times 2$ and $m^2 + m + 1 = 57 = 3 \times 19$. Then, we define the classes as Eq. (3.11), and they satisfy the multiplication rules as Eqs. (3.15) and (3.16). Note that if C_3^3 and C_3^6 appear in the multiplication rules, we have to replace them by $C_3^3 \Rightarrow 3C_1^3$, $C_3^6 \Rightarrow 3C_1^6$. More explicitly, we write the classes as follows:

$$\begin{aligned} C_1 &= \{e\}, \\ C_1^3 &= \{a^3\}, \\ C_1^6 &= \{a^6\}, \\ C_3^{(1)} &= \{a, a^4, a^7\}, \\ C_3^{(2)} &= \{a^2, a^5, a^8\}. \end{aligned} \quad (\text{A3})$$

Table XV shows their multiplication rules. That is equivalent to ones of $\mathbb{Z}_3$ of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

The above classes (A3) correspond to the conjugacy classes of $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ without b . The full conjugacy classes of $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ include additional classes:

$$\begin{aligned} B_3^{(1)} &= \{b, ba^3, ba^6\}, \\ B_3^{(2)} &= \{ba, ba^4, ba^7\}, \\ B_3^{(3)} &= \{ba^2, ba^5, ba^8\}, \\ B_3^{(4)} &= \{b^2, b^2a^3, b^2a^6\}, \\ B_3^{(5)} &= \{b^2a, b^2a^4, b^2a^7\}, \\ B_3^{(6)} &= \{b^2a^2, b^2a^5, b^2a^8\}. \end{aligned} \quad (\text{A4})$$

Their multiplication rules are shown in Table XVI.

APPENDIX B: $\Delta(12) \cong A_4$ AND $\Delta(24) \cong S_4$

In this section, we list the multiplication rules of $\Delta(12) \cong A_4$ and $\Delta(24) \cong S_4$ conjugacy classes. First, $\Delta(12)$ conjugacy classes are chosen as

$$C_1 = \{e\}, \quad (\text{B1})$$

TABLE XVII. Multiplication rules for conjugacy classes of $\Delta(12)$.

	C_1	$C_3^{(1,0)}$	C_4^1	C_4^2
C_1	C_1	$C_3^{(1,0)}$	C_4^1	C_4^2
$C_3^{(1,0)}$	$C_3^{(1,0)}$	$3C_1 + 2C_3^{(1,0)}$	$3C_4^1$	$3C_4^2$
C_4^1	C_4^1	$3C_4^1$	$4C_4^2$	$4C_1 + 4C_3^{(1,0)}$
C_4^2	C_4^2	$3C_4^2$	$4C_1 + 4C_3^{(1,0)}$	$4C_4^1$

$$C_3^{(1,0)} = \{a, a', aa'\}, \quad (\text{B2})$$

$$C_4^1 = \{b, ba, ba', baa'\}, \quad (\text{B3})$$

$$C_4^2 = \{b^2, b^2a, b^2a', b^2aa'\}, \quad (\text{B4})$$

which obey the multiplication rules in Table XVII. It turns out that there exists the $S_2 \cong \mathbb{Z}_2$ symmetry, $C_4^1 \leftrightarrow C_4^2$, corresponding to the outer automorphism exchanging b and b^2 . In addition, there exists $\mathbb{Z}_3$ symmetry, $C_4^1 \rightarrow \omega C_4^1$ and $C_4^2 \rightarrow \omega^2 C_4^2$, originating from $b \rightarrow \omega b$ with $\omega = e^{2\pi i/3}$. Then, the whole symmetry of the multiplication rules is described by $S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$.

Next, we deal with the $\Delta(24)$ conjugacy classes:

$$C_1 = \{e\}, \quad (\text{B5})$$

$$C_3^{(1,0)} = \{a, a', aa'\}, \quad (\text{B6})$$

$$C_8 = \{b, ba, ba', baa', b^2, b^2a, b^2a', b^2aa'\}, \quad (\text{B7})$$

$$B_6^{(0)} = \{ca^n a'^n, b^2 ca^{-n}, bca^{-n} | n = 0, 1\}, \quad (\text{B8})$$

$$B_6^{(1)} = \{ca^{n+1} a'^n, b^2 ca^{-1} a'^{-n-1}, bca^{-n} a' | n = 0, 1\}, \quad (\text{B9})$$

which obey the multiplication rules in Table XVIII. There exists the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry: $B_6^{(0)} \leftrightarrow B_6^{(1)}$ under $\mathbb{Z}_2$ and $B_6^{(0)} \rightarrow -B_6^{(0)}$, $B_6^{(1)} \rightarrow -B_6^{(1)}$ under $\mathbb{Z}_2'$.

TABLE XVIII. Multiplication rules for conjugacy classes of $\Delta(24)$.

	C_1	$C_3^{(1,0)}$	C_8	$B_6^{(0)}$	$B_6^{(1)}$
C_1	C_1	$C_3^{(1,0)}$	C_8	$B_6^{(0)}$	$B_6^{(1)}$
$C_3^{(1,0)}$	$C_3^{(1,0)}$	$3C_1 + 2C_3^{(1,0)}$	$3C_8$	$B_6^{(0)} + 2B_6^{(1)}$	$2B_6^{(0)} + B_6^{(1)}$
C_8	C_8	$3C_8$	$8C_1 + 8C_3^{(1,0)} + 4C_8$	$4B_6^{(0)} + 4B_6^{(1)}$	$4B_6^{(0)} + 4B_6^{(1)}$
$B_6^{(0)}$	$B_6^{(0)}$	$B_6^{(0)} + 2B_6^{(1)}$	$4B_6^{(0)} + 4B_6^{(1)}$	$6C_1 + 2C_3^{(1,0)} + 3C_8$	$4C_3^{(1,0)} + 3C_8$
$B_6^{(1)}$	$B_6^{(1)}$	$2B_6^{(0)} + B_6^{(1)}$	$4B_6^{(0)} + 4B_6^{(1)}$	$4C_3^{(1,0)} + 3C_8$	$6C_1 + 2C_3^{(1,0)} + 3C_8$

TABLE XIX. New multiplication rules.

	C_1	C_1^1	C_1^2	C_6'
C_1	C_1	C_1^1	C_1^2	C_6'
C_1^1	C_1^1	C_1^2	C_1	C_6'
C_1^2	C_1^2	C_1	C_1^1	C_6'
C_6'	C_6'	C_6'	C_6'	$C_1 + C_1^1 + C_1^2$

APPENDIX C: FURTHER DISCRETE GAUGING

As summarized in Table XIV, $\mathbb{Z}_3$ gauging of discrete groups as well as $\mathbb{Z}_2$ has symmetries in multiplication rules. We can gauge remaining symmetries to obtain other multiplication rules. For example, in $\mathbb{Z}_3$ gauging of $\mathbb{Z}_3 \times \mathbb{Z}_3'$, we have S_2 symmetry, where we exchange $C_3^{(1,0)}$ and $C_3^{(2,0)}$. When we gauge this S_2 , the two classes, $C_3^{(1,0)}$ and $C_3^{(2,0)}$ are combined to one, $C_6^{(1,0)} = C_3^{(1,0)} + C_3^{(2,0)}$. That is nothing but S_3 gauging of $\mathbb{Z}_3 \times \mathbb{Z}_3'$. Similarly, we gauge other cases by remaining symmetries.

Here, we reconsider the above process from $\mathbb{Z}_3$ gauging to S_3 gauging. We choose the basis of classes in $\mathbb{Z}_3$ gauging of $\mathbb{Z}_3 \times \mathbb{Z}_3'$ such that S_2 is diagonal, i.e.,

$$C_3^\pm = C_3^{(1,0)} \pm C_3^{(2,0)}. \quad (C1)$$

They have $\mathbb{Z}_2$ even and odd charges. The other classes also have even charges. The simple S_2 gauging is to restrict the representation space to be $\mathbb{Z}_2$ even and project out $\mathbb{Z}_2$ odd, i.e., $C_3^- = 0$. Then, the result is the same as S_3 gauging of $\mathbb{Z}_3 \times \mathbb{Z}_3'$ by identifying $C_6^{(1,0)} = C_3^+$. We may impose a different condition on the represented space such that $C_3^+ = 0$. Then, other classes satisfy the multiplication rules shown in Table XIX. Here we normalize C_3^- in a proper way such that all coefficients in multiplication rules are positive, i.e., $C_6' = \frac{i}{\sqrt{6}} C_3^-$. The result is equivalent to the $\mathbb{Z}_3$ Tambara-Yamagami fusion rules, i.e.,

$$\eta^3 = \mathbb{I}, \quad \eta \cdot \mathcal{N} = \mathcal{N}, \quad \mathcal{N}^2 = \mathbb{I} + \eta + \eta^2, \quad (C2)$$

where we identify $\mathbb{I} = C_1$, $\eta = C_1^1$, $\eta^2 = C_1^2$, and $\mathcal{N} = C_6'$. Thus, we have more variety in discrete gauging.

APPENDIX D: MORE ABOUT $\mathbb{Z}_2$ GAUGING

Here, we study another type of $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries.

Instead of $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ reviewed in Sec. II, we use the following automorphism:

$$b_m a^k b_m^{-1} = a^{m-k}. \quad (D1)$$

That is $\mathbb{Z}_2^{(m)}$ gauging. This is still the $\mathbb{Z}_2$ automorphism but includes a “shift.” The $\mathbb{Z}_2$ gauging in Sec. II with $m = 0$ corresponds to geometrical orbifolding around the origin.

On the other hand, $\mathbb{Z}_2^{(m)}$ gauging with $m \neq 0$ corresponds to the geometrical orbifolding at a different point. Thus we have the conjugacy class:

$$C_m^{(k)} = \{h a^k h^{-1} | h = e, b_m\} = \{a^k, a^{m-k}\}. \quad (D2)$$

Furthermore, we can extend this to $\mathbb{Z}_2^{(p)} \times \mathbb{Z}_2^{(q)}$. The outer automorphisms are defined as

$$\begin{aligned} b_p &: a^k \rightarrow a^{p-k}, \\ b_q &: a^k \rightarrow a^{q-k}, \end{aligned} \quad (D3)$$

where the exponents are mod N . Note that $b_p b_p = b_q b_q = e$. We can additionally define the composition of the two actions as

$$b_q b_p = r_\ell: g^k = g^{q-p+k}. \quad (D4)$$

Thus we can construct two types of elements:

$$\begin{aligned} b_\ell^n &: a^k = g^{k+n(q-p)}, \\ b_p r_\ell^n &: a^k = g^{p-k-n(q-p)}. \end{aligned} \quad (D5)$$

Note that when $n < 0$, we can get terms like $b_q b_p b_q$. The class is defined as

$$C_{pq}^{(k)} = \{a^{k+n(q-p)}, a^{p-k-n(q-p)} | n \in \mathbb{Z}\}. \quad (D6)$$

Note that the definition of the class in Eq. (D6) includes the $\Delta = (p - q)$ modulo structure. On top of that, recall that a is the generator of $\mathbb{Z}_N$. Thus, the truly meaningful modulo structure is given by $\Delta' = \text{LCD}(\Delta, N)$, where $\text{LCD}(k, \ell)$ denotes the least common divisor of k and ℓ . For example, when Δ and N are co-primes of each other and $\Delta' = 1$, all elements of the $\mathbb{Z}_N$ group appear in a single class.

The above types of gauging and classes would lead to a new type of selection rules and interesting phenomenology. We would study it elsewhere.

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