

$\pi^+\pi^-$ and $D_s^+\pi^\pm$ mass spectra in the $D_{s1}(2536) \rightarrow D_s^+\pi^+\pi^-$ decaysJorgivan Morais Dias^{1,*}, Yi-Yao Li^{2,3,4,5,†} and Eulogio Oset^{5,6,‡}¹*Departamento de Física, Universidade Federal do Piauí, 64049-550 Teresina, Piauí, Brazil*²*State Key Laboratory of Nuclear Physics and Technology, Institute of Quantum Matter, South China Normal University, Guangzhou 510006, China*³*Key Laboratory of Atomic and Subatomic Structure and Quantum Control (MOE), Guangdong-Hong Kong Joint Laboratory of Quantum Matter, Guangzhou 510006, China*⁴*Guangdong Basic Research Center of Excellence for Structure and Fundamental Interactions of Matter, Guangdong Provincial Key Laboratory of Nuclear Science, Guangzhou 510006, China*⁵*Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC Institutos de Investigación de Paterna, 46071 Valencia, Spain*⁶*Department of Physics, Guangxi Normal University, Guilin 541004, China*

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We have carried out an evaluation of the $\pi^+\pi^-$ and $D_s^+\pi^+$ mass distributions in the $D_{s1}(2536)$ decay to $D_s^+\pi^+\pi^-$, from the perspective that the $D_{s1}(2536)$ is a molecular state, mostly made from K^*D in $I = 0$. We are able to obtain not only the mass distributions, but also the branching ratio of this decay. The shape of the mass distributions differs appreciably from those of the analogous reaction $D_{s1}(2460) \rightarrow D_s^+\pi^+\pi^-$, which has been measured by the LHCb Collaboration and analyzed theoretically from the perspective that the $D_{s1}(2460)$ is a molecular state of D^*K , showing good agreement with the data. In spite of the analogy with the $D_{s1}(2460)$ decay, the dynamical differences in the decay mechanism are important, since now the $f_0(500)$ resonance is not generated, while it was the dominant mechanism in the $D_{s1}(2460) \rightarrow D_s^+\pi^+\pi^-$ decay. Nonetheless, we find striking differences in the mass distributions compared with phase space as a consequence of the decay mechanism. The branching ratio obtained is an order of magnitude bigger than the one of the $D_{s1}(2460) \rightarrow D_s^+\pi^+\pi^-$ reaction, mostly due to the larger available phase space. We also show that the shape of the distributions obtained from the molecular picture is quite different from those obtained based on a $q\bar{q}$ picture. We conclude that measuring the shape of the mass distributions and the total strength of the decay mode should be very valuable to learn about the structure of the $D_{s1}(2536)$.

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I. INTRODUCTION

Over the past two decades, numerous new hadronic states have been observed, many of which do not fit into the conventional quark configurations predicted by the quark model, that is, $q\bar{q}$ for mesons and qqq for baryons [1–5]. While these discoveries challenge our understanding of the strong interaction, they also provide a natural laboratory to explore the complexity and richness of quantum chromodynamics (QCD) in the low-energy regime [6,7].

Most of the experimental observations, as well as the corresponding theoretical studies, have focused on hadrons

in the heavy-quark sector, involving either charm or bottom. In contrast, systems containing both charm and strangeness remain relatively unexplored, despite the existence of some experimental data available for quite some time—as is the case of the $D_{s1}(2536)$ resonance, with quantum numbers $J^P = 1^+$.

In particular, the $D_{s1}(2536)$ state was first observed in 1989 by the ARGUS Collaboration in e^+e^- collisions at the Deutsches Elektronen-Synchrotron storage ring [8]. Its mass was measured as $(2535.9 \pm 0.6 \pm 2.0)$ MeV, with a narrow width of less than 4.6 MeV. Since then, this state has been confirmed by other collaborations, including CLEO [9], BABAR [10,11], Belle [12], and, more recently, LHCb [13].

There have been some attempts not only to describe the structure of the $D_{s1}(2536)$ meson, but also to establish a broader understanding of the spectroscopy of mesons containing both charm and strangeness—namely, the D_s mesons [14–18]. Specifically, the $D_{s1}(2536)$ has quantum numbers $J^P = 1^+$ and is, therefore, an axial-vector resonance [11,19,20]. In principle, this structure is interpreted

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as a quark configuration of the type $c\bar{s}$. However, mass predictions from potential models based on heavy-quark effective theory [21] are in disagreement with experimental data. This discrepancy also extends to other D_{s1} structures, such as the $D_{s1}(2460)$ [22]. In particular, the predicted values from quark models [17] lie well above the D^*K and K^*D thresholds, corresponding to the $D_{s1}(2460)$ and the $D_{s1}(2536)$, respectively. On the other hand, hadronic structures near meson-meson thresholds can be well described as molecular-type states using the molecular model [6]. In Refs. [23–27], both the $D_{s1}(2460)$ and the $D_{s1}(2536)$ are interpreted as dynamically generated states arising from the interactions between pseudoscalar and vector mesons, with the D^*K and K^*D channels playing a dominant role in the formation of the $D_{s1}(2460)$ and $D_{s1}(2536)$, respectively. The $D_{s1}(2536)$ might be different, since its mass lies closer to conventional $c\bar{s}$ quark models [28]. Yet, from the perspective of the molecular nature of the $D_{s1}(2460)$ and the similar dynamics that goes into the D^*K and K^*D interactions, one also expects a sizable K^*D component for the $D_{s1}(2536)$. The present calculation assumes the state as pure molecular, but we discuss how the experiment could eventually tell us about the size of this molecular component of the $D_{s1}(2536)$ state.

The molecular interpretation finds strong support in the successful description of the hadronic properties of several resonances, both in the heavy-quark sector and in the light sector [29–46]. As a result, this framework has been extensively explored, with many studies proposing reactions in which such properties can be measured, putting the molecular picture to the test [47]. This may shed light on the current debate surrounding the interpretation of these states as more complex structures, often referred to by the term *exotic*.

From this perspective, the reaction $D_{s1}(2460) \rightarrow D_s^+ \pi^+ \pi^-$ was studied in Ref. [48], assuming a molecular D^*K interpretation for the $D_{s1}(2460)$ meson, and also in Ref. [49] including contributions from the $D_s\eta$ channel, and using a different formalism. In this approach, the decay proceeds via triangle loop diagrams. In particular, the configuration of the intermediate states in the loop allows for the manifestation of resonances such as the $f_0(500)$ and even the $f_0(980)$ in the unitarized amplitudes associated with the interactions between pseudoscalar mesons [32,50,51]. A key point in favor of this mechanism, as applied to the reaction discussed in Refs. [48,49], is its good agreement with the corresponding experimental data reported by the LHCb Collaboration for the decay $D_{s1}(2460) \rightarrow D_s^+ \pi^+ \pi^-$ [52]. Indeed, such agreement supports the molecular interpretation of the $D_{s1}(2460)$ meson.

Following this line of thought, we assume in the present work that the $D_{s1}(2536)$ structure is also interpreted as a molecular state, dynamically generated by the K^*D interaction, as suggested in Ref. [24]. In order to test its molecular nature, we compute the invariant mass spectra

of the $D_s^+ \pi^\pm$ and $\pi^+ \pi^-$ pairs, which are among the final products of the decay $D_{s1}(2536) \rightarrow D_s^+ \pi^+ \pi^-$. In particular, we adopt the mechanism in which the $D_{s1}(2536)$ meson decays into $D_s^+ \pi^+ \pi^-$ via a triangular loop, with D , K^* , and K mesons as intermediate states, in analogy to the successful mechanism employed in Refs. [48,49]. Through the interactions between the D and K mesons, the final states $D_s^+ \pi^+$ and $\pi^+ \pi^-$ are produced, enabling the calculation of the relevant spectra. Assuming that the $D_{s1}(2536)$ state is pure molecular, this mechanism is rather predictive. The couplings needed stem from the molecular hypothesis, as well as a momentum cutoff factor which enters the loop, tied to the same K^*D interaction [24]. It contains an extra cutoff, tied to the $KD \rightarrow D_s^+ \pi^+$ transition, which is governed by the range of the light vector mesons exchanged to obtain this transition, and we quantify uncertainties from this source. In Ref. [49], the absolute rate of $D_{s1}(2460) \rightarrow D_s^+ \pi^+ \pi^-$ was also evaluated and found compatible with experiment within errors. This gives us confidence in the predictions for the rate of the $D_{s1}(2536) \rightarrow D_s^+ \pi^+ \pi^-$ decay, yet, accepting that the $D_{s1}(2536)$ state could have a large nonmolecular component, as claimed in [28], we suggest that the measurement of the absolute rate for this reaction could give us the molecular component of the state.

This work is organized as follows: In Sec. II, we present in detail the formalism employed to evaluate the decay $D_{s1}(2536) \rightarrow D_s^+ \pi^+ \pi^-$. In particular, we describe the mechanism through which the reaction proceeds and introduce the relevant triangle diagrams that contribute to this reaction. In the same section, we explicitly evaluate the triangle amplitudes associated with each diagram. Next, in Sec. III, we present the numerical results for the invariant mass spectra of the $\pi^+ \pi^-$ and $D_s^+ \pi^\pm$ pairs, discussing in detail the physical implications of the corresponding line shapes. Moreover, we compute the decay width of the $D_{s1}(2536)$ into the $D_s^+ \pi^+ \pi^-$ channel without the use of free parameters. We devote two sections to numerical improvements and a critical discussion of the results. Finally, in Sec. VI, we summarize our main results and present our conclusions.

II. FORMALISM

In our model, the decay $D_{s1}(2536) \rightarrow D_s^+ \pi^+ \pi^-$ occurs via triangle loops, illustrated in Fig. 1. In particular, we assume that the meson $D_{s1}(2536)$ is a molecular state K^*D , whose coupling to this channel is given in Ref. [53]. In Fig. 1, as a further step in the process, the meson K^* decays into a pair of pseudoscalar mesons, namely, K and π . The meson K , thus produced, interacts with the meson D originally generated at the vertex $D_{s1}DK^*$, and the DK pair undergoes a transition to the final $D_s\pi$ meson state, whose distribution interests us. The $D_{s1}(2536)$ has $I = 0$. With the isospin phase convention

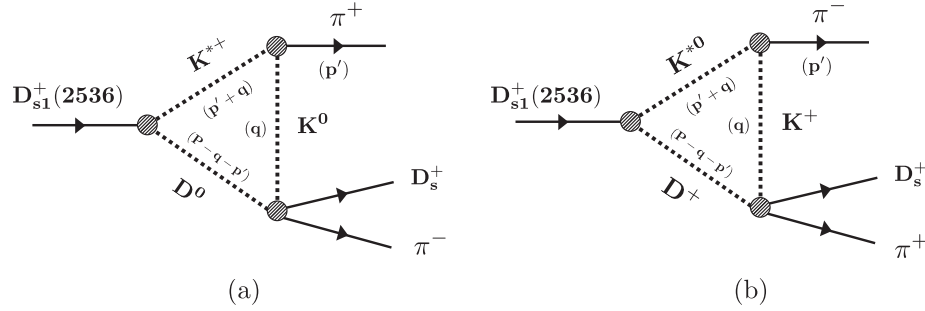


FIG. 1. D_{s1} decay mechanism: (a) decay through the $K^{*+}D^0$ component, while in (b), the D_{s1} meson decays through the $K^{*0}D^+$ component.

$$K = \begin{Bmatrix} K^{*+} \\ K^{*0} \end{Bmatrix}, \quad D = \begin{Bmatrix} D^+ \\ -D^0 \end{Bmatrix}, \quad (1)$$

the K^*D wave function of the $D_{s1}(2536)$ is $|K^*D, I=0\rangle = -\frac{1}{\sqrt{2}}(K^{*+}D^0 + K^{*0}D^+)$. The coupling of $D_{s1}(2536)$ to the $|K^*D, I=0\rangle$ state is $g_{D_{s1}(2536), K^*D, I=0} = 20234 \text{ MeV}$ [53], and then $g_{D_{s1}(2536), K^{*+}D^0} = g_{D_{s1}(2536), K^{*0}D^+} = -\frac{1}{\sqrt{2}}g_{D_{s1}(2536), K^*D, I=0}$. The vertex of $D_{s1}(2536) \rightarrow K^{*+}D^0$ corresponding to Fig. 1(a) is given by

$$V_{D_{s1}(2536) \rightarrow K^{*+}D^0} = g_{D_{s1}(2536), K^{*+}D^0} \vec{\epsilon}_A \vec{\epsilon}_{K^*}, \quad (2)$$

with $\vec{\epsilon}_A$ the polarization vector for the axial meson state $D_{s1}(2536)$. We have neglected the ϵ^0 component of the K^* , but we will show in Sec. IV that this is a very good approximation.

To calculate the amplitudes in Fig. 1, we first need to compute the upper vertices of the VPP type (V for vector and P for pseudoscalar mesons). This can be easily obtained with the help of the effective Lagrangians of the local hidden gauge approach [54–57]. Specifically, for the VPP vertices, we have

$$\mathcal{L} = -ig \langle [P, \partial_\mu P] V^\mu \rangle, \quad (3)$$

where $g = M_V/2f$, with $M_V = 800 \text{ MeV}$, an average vector meson mass, while f is the pion decay constant, equal to 93 MeV , and the symbol $\langle \cdots \rangle$ denotes the trace of the SU(3) matrices. Additionally, P and V^μ correspond to the matrices $q_i \bar{q}_j$ in terms of quarks, given in terms of the physical pseudoscalar and vector mesons, respectively, by

$$P = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} & \pi^+ & K^+ & \bar{D}^0 \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} & K^0 & D^- \\ K^- & \bar{K}^0 & -\frac{\eta}{\sqrt{3}} & D_s^- \\ D^0 & D^+ & D_s^+ & \eta_c \end{pmatrix}, \quad (4)$$

$$V = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} & \bar{D}^{*0} \\ \rho^{*-} & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} & D^{*-} \\ K^{*-} & \bar{K}^{*0} & \phi & D_s^{*-} \\ D^{*0} & D^{*+} & D_s^{*+} & J/\psi \end{pmatrix}, \quad (5)$$

where we take into account the standard $\eta - \eta'$ mixing, as discussed in Ref. [58]. Thus, considering the momenta defined in Fig. 1, we obtain the following expression for the relevant VPP vertices:

$$V_{VPP} = g\epsilon_\mu (q - p')^\mu, \quad (6)$$

where ϵ_μ is the polarization vector of the meson K^* .

Next, we need to calculate the transitions $D^0 K^0 \rightarrow D_s^+ \pi^-$, $D^+ K^+ \rightarrow D_s^+ \pi^+$, which appear at the lower vertices of the diagrams in Fig. 1. These $PP \rightarrow PP$ transitions (with P standing for pseudoscalar meson) can be easily computed using an extension of the local hidden gauge approach [54–57]. They were calculated in Ref. [49]. Specifically, for the $D^0 K^0 \rightarrow D_s^+ \pi^-$ we have

$$V_{K^0 D^0 \rightarrow D_s^+ \pi^-} = \frac{1}{4f^2} \frac{1}{2} \left[3s - (m_{K^0}^2 + m_{\pi^-}^2 + m_{D^0}^2 + m_{D_s^+}^2) - \frac{1}{s} (m_{D^0}^2 - m_{K^0}^2)(m_{D_s^+}^2 - m_{\pi^-}^2) \right], \quad (7)$$

where $s \equiv M_{\text{inv}}^2(D_s^+ \pi^-)$.

Because of isospin symmetry, the transition potentials for the processes $D^0 K^0 \rightarrow D_s^+ \pi^-$ and $D^+ K^+ \rightarrow D_s^+ \pi^+$ are identical in both magnitude and sign. This follows from the fact that both initial two-meson states couple to a total isospin $I = 1$, differing only in their third component I_3 . Furthermore, by carefully considering the conventional phase assignments of the isospin multiplets—particularly the $(-\pi^+, \pi^0, \pi^-)$ basis for pions and the D, K isospin phase conventions given above—the relative phases between the physical states ensure that no additional sign difference arises between these transitions. Consequently,

the interaction potential derived for the $D^0 K^0 \rightarrow D_s^+ \pi^-$ channel in Eq. (7) can be directly applied to the $D^+ K^+ \rightarrow D_s^+ \pi^+$ channel.

Once the relevant vertices for calculating the diagrams in Figs. 1(a) and 1(b) have been obtained, we can write the corresponding amplitudes for the triangle loops under consideration. Next, we will discuss in detail the loop in Fig. 1(a), while the calculation for the diagram in Fig. 1(b) follows analogously. With the upper vertices $K^{*+} \pi^+ K^0$ and $K^{*0} \pi^- K^+$ at hand and considering Eq. (7) for the transition $D^0 K^0 \rightarrow D_s^+ \pi^-$, the diagram in Fig. 1(a) can be written as

$$\begin{aligned} -it_{\text{loop}}^{(a)} = & \int \frac{d^4 q}{(2\pi)^4} (-i) g_{D_{s1}(2536), K^{*+} D^0} \epsilon_{Ai} \epsilon_{K^* i} \\ & \times \frac{i}{(p' + q)^2 - m_{K^*}^2 + i\epsilon} (-i) g \epsilon_{K^* \alpha} (q - p')^\alpha \\ & \times \frac{i}{q^2 - m_K^2 + i\epsilon} \frac{i}{(P - p' - q)^2 - m_D^2 + i\epsilon} \\ & \times (-i) V_{D^0 K^0 \rightarrow D_s^+ \pi^-}, \end{aligned} \quad (8)$$

with P standing for the momentum of the $D_{s1}(2536)$.

Assuming we are dealing with small three-momenta compared to the masses of the particles involved, it implies that only the spatial components of the polarization vector of K^* meson are nonvanishing, that is,

$$\epsilon_{K^*}^0 = \frac{|\vec{p}' + \vec{q}|}{m_{K^*}} \simeq 0, \quad (9)$$

and the completeness relation for the corresponding polarization vectors, which reads

$$\sum_{\text{pol}} \epsilon_{K^* \mu} \epsilon_{K^* \alpha} = -g_{\mu\alpha} + \frac{(p' + q)_\mu (p' + q)_\alpha}{m_{K^*}^2}, \quad (10)$$

can now be written as

$$\sum_{\text{pol}} \epsilon_{K^* \mu} \epsilon_{K^* \alpha} \simeq \sum_{\text{pol}} \epsilon_{K^* i} \epsilon_{K^* j} \simeq \delta_{ij}. \quad (11)$$

In Sec. IV, we will actually see that the approximation of Eq. (9) is indeed very good.

Therefore, using Eq. (11), the triangle amplitude reduces to

$$\begin{aligned} -it_{\text{loop}}^{(a)} = & \int \frac{d^4 q}{(2\pi)^4} (-i) g_{D_{s1}(2536), K^{*+} D^0} \epsilon_{Ai} \epsilon_{K^* i} \frac{i}{(p' + q)^2 - m_{K^*}^2 + i\epsilon} (-i) g \epsilon_{K^* j} \\ & \times (p' - q)_j \frac{i}{q^2 - m_K^2 + i\epsilon} \frac{i}{(P - p' - q)^2 - m_D^2 + i\epsilon} (-i) V_{D^0 K^0 \rightarrow D_s^+ \pi^-}. \end{aligned} \quad (12)$$

For the heavy mesons K^{*+} and D^0 , coming from the decay of a particle, it is a good approximation to take the positive energy part of their full propagators, that is,

$$\begin{aligned} \frac{1}{(p' + q)^2 - m_{K^*}^2 + i\epsilon} & \simeq \frac{1}{2\omega_{K^*}(\vec{p}' + \vec{q})} \frac{1}{p'^0 + q^0 - \omega_{K^*}(\vec{p}' + \vec{q}) + i\epsilon}, \\ \frac{1}{(P - q - p')^2 - m_D^2 + i\epsilon} & \simeq \frac{1}{2\omega_D(\vec{p}' + \vec{q})} \frac{1}{P^0 - p'^0 - q^0 - \omega_D(\vec{p}' + \vec{q}) + i\epsilon}, \end{aligned} \quad (13)$$

where $\omega_{K^*}(\vec{p}' + \vec{q}) = \sqrt{(\vec{p}' + \vec{q})^2 + m_{K^*}^2}$, $\omega_D(\vec{p}' + \vec{q}) = \sqrt{(\vec{p}' + \vec{q})^2 + m_D^2}$, $\omega_K(\vec{q}) = \sqrt{\vec{q}^2 + m_K^2}$ (for later use), and $\vec{P} = 0$ in the D_{s1} rest frame. On the other hand, for the K meson, we keep both positive and negative energy parts, since it is a lighter meson and can be more off shell. In Sec. IV, we shall see the goodness of the approximation. Hence, Eq. (12) can be rewritten as

$$\begin{aligned} t_{\text{loop}}^{(a)} = & i g g_{D_{s1}(2536), K^{*+} D^0} V_{D^0 K^0 \rightarrow D_s^+ \pi^-} \epsilon_{Ai} \int \frac{d^4 q}{(2\pi)^4} (p' - q)_i \frac{1}{2\omega_{K^*}} \frac{1}{2\omega_K} \frac{1}{2\omega_D} \\ & \times \frac{1}{p'^0 + q^0 - \omega_{K^*}(\vec{p}' + \vec{q}) + i\epsilon} \frac{1}{P^0 - p'^0 - q^0 - \omega_D(\vec{p}' + \vec{q}) + i\epsilon} \\ & \times \left(\frac{1}{q^0 - \omega_K(\vec{q}) + i\epsilon} - \frac{1}{q^0 + \omega_K(\vec{q}) - i\epsilon} \right). \end{aligned} \quad (14)$$

Now, we can perform an analytical integration over the q^0 variable, using Cauchy's residue theorem, so that Eq. (14) now assumes the following form:

$$t_{\text{loop}}^{(a)} = gg_{D_{s1}(2536), K^{*+}D^0} V_{D^0 K^0 \rightarrow \pi^- D_s^+} \vec{\epsilon}_{Ai} \int \frac{d^3 q}{(2\pi)^3} (p' - q)_i \frac{1}{2\omega_{K^*}} \frac{1}{2\omega_K} \frac{1}{2\omega_D} \\ \times \frac{1}{P^0 - \omega_D(\vec{p}' + \vec{q}) - \omega_{K^*}(\vec{p}' + \vec{q}) + i\epsilon} \left(\frac{1}{P^0 - p'^0 - \omega_D(\vec{p}' + \vec{q}) - \omega_K(\vec{q}) + i\epsilon} \right. \\ \left. + \frac{1}{p'^0 - \omega_{K^*}(\vec{p}' + \vec{q}) - \omega_K(\vec{q}) + i\epsilon} \right). \quad (15)$$

Next, considering that

$$\int d^3 q F(\vec{q}, \vec{p}') q_\ell = p'_\ell \int d^3 q F(\vec{q}, \vec{p}') \frac{\vec{q} \cdot \vec{p}'}{|\vec{p}'|^2}, \quad (16)$$

we finally write Eq. (15) as

$$t_{\text{loop}}^{(a)} = \mathcal{I}_{\text{loop}}^{(a)}(\vec{\epsilon}_A \cdot \vec{p}'), \quad (17)$$

with $\mathcal{I}_{\text{loop}}^{(a)}$ given by

$$\mathcal{I}_{\text{loop}}^{(a)} = gg_{D_{s1}(2536), K^{*+}D^0} V_{D^0 K^0 \rightarrow D_s^+ \pi^-} \int \frac{d^3 q}{(2\pi)^3} \Theta(q_{\text{max}} - |\vec{q} + \vec{p}'|) \Theta(\Lambda - |q^*|) \left(1 - \frac{\vec{q} \cdot \vec{p}'}{|\vec{p}'|^2} \right) \\ \times \frac{1}{2\omega_{K^*}(\vec{p}' + \vec{q})} \frac{1}{2\omega_D(\vec{p}' + \vec{q})} \frac{1}{2\omega_K(\vec{q})} \frac{1}{P^0 - \omega_D(\vec{p}' + \vec{q}) - \omega_{K^*}(\vec{p}' + \vec{q}) + i\frac{\Gamma_{K^*}}{2}} \\ \times \left(\frac{1}{P^0 - p'^0 - \omega_D(\vec{p}' + \vec{q}) - \omega_K(\vec{q}) + i\epsilon} + \frac{1}{p'^0 - \omega_{K^*}(\vec{p}' + \vec{q}) - \omega_K(\vec{q}) + i\frac{\Gamma_{K^*}}{2}} \right), \quad (18)$$

where $\vec{p}'$ in this case corresponds to the π^+ momentum, namely,

$$\vec{p}' = \frac{\lambda^{\frac{1}{2}}(m_{D_{s1}}^2, m_{\pi^+}^2, M_{\text{inv}}^2(D_s^+ \pi^-))}{2m_{D_{s1}}}. \quad (19)$$

In Eq. (18), we have implemented two changes: the introduction of two Θ functions and the explicit consideration of the K^* width. The appearance in Eq. (18) of a Heaviside theta function Θ , which sets the upper limit in the integral, comes naturally from the chiral unitary approach, which considers a separable potential of the form $V(\vec{q}', \vec{q}) = V\Theta(q_{\text{max}} - |\vec{q}'|)\Theta(q_{\text{max}} - |\vec{q}|)$, leading to $T(\vec{q}', \vec{q}) = T\Theta(q_{\text{max}} - |\vec{q}'|)\Theta(q_{\text{max}} - |\vec{q}|)$, where q_{max} stands for the cutoff used to regularize the loops encoded in the G function. In Ref. [53], the $D_{s1}(2536)$ state is dynamically generated from K^*D interaction with $q_{\text{max}} = 1025$ MeV, which is the value we are considering in Eq. (18). Another theta function $\Theta(\Lambda - |q^*|)$ is for the vertex $D^0 K^0 \rightarrow D_s^+ \pi^-$, where q^* is the three-momentum of K^0 boosted to $D_s^+ \pi^-$ rest frame

$$q^* = \left[\left(\frac{E_{D_s^+ \pi^-}}{M_{\text{inv}}(D_s^+ \pi^-)} - 1 \right) \frac{\vec{q} \cdot \vec{p}'}{|\vec{p}'|^2} + \frac{E_{K^0}}{M_{\text{inv}}(D_s^+ \pi^-)} \right] \cdot \vec{p}' + \vec{q}, \quad (20)$$

where $E_{D_s^+ \pi^-} = \sqrt{M_{\text{inv}}^2(D_s^+ \pi^-) + |\vec{p}'|^2}$ and $E_{K^0} = \sqrt{m_{K^0}^2 + |\vec{q}|^2}$. We calculate with $\Lambda \equiv 600$ MeV and vary it within the range $\Lambda \in [500, 700]$ MeV to estimate the uncertainty in our result as done in Ref. [49] (the changes induced by changing q_{max} are very small compared to those changing Λ). The values of q_{max} or Λ are not arbitrary; they are tied to the range of the interaction in momentum space, which is governed by the exchange of light vector mesons [59]. The values 500–700 MeV are common in most problems. The fact that q_{max} tied to the K^*D interaction is somewhat bigger when forcing the $D_{s1}(2536)$ to be fully molecular is an indication that other components should also play a role in the building up of this state.

The triangle loop in Fig. 1(b) is readily obtained from Eq. (17) by setting $\vec{p}' \rightarrow \vec{p}'_{\pi^-}$, that is,

$$t_{\text{loop}}^{(b)} = \mathcal{I}_{\text{loop}}^{(b)}(\vec{\epsilon}_A \cdot \vec{p}'_{\pi^-}), \quad (21)$$

where $t_{\text{loop}}^{(b)}$ is

$$\begin{aligned} \mathcal{I}_{\text{loop}}^{(b)} = & gg_{D_{s1}(2536), K^*0 D^+} V_{D^+ K^+ \rightarrow D_s^+ \pi^+} \int \frac{d^3 q}{(2\pi)^3} \Theta(q_{\text{max}} - |\vec{q} + \vec{p}'_{\pi^-}|) \Theta(\Lambda - |q^*|) \left(1 - \frac{\vec{q} \cdot \vec{p}'_{\pi^-}}{|\vec{p}'_{\pi^-}|^2}\right) \\ & \times \frac{1}{2\omega_{K^*}(\vec{p}'_{\pi^-} + \vec{q})} \frac{1}{2\omega_D(\vec{p}'_{\pi^-} + \vec{q})} \frac{1}{2\omega_K(\vec{q})} \frac{1}{P^0 - \omega_D(\vec{p}'_{\pi^-} + \vec{q}) - \omega_{K^*}(\vec{p}'_{\pi^-} + \vec{q}) + i\frac{\Gamma_{K^*}}{2}} \\ & \times \left(\frac{1}{P^0 - p_{\pi^-}^0 - \omega_D(\vec{p}'_{\pi^-} + \vec{q}) - \omega_K(\vec{q}) + i\varepsilon} + \frac{1}{p_{\pi^-}^0 - \omega_{K^*}(\vec{p}'_{\pi^-} + \vec{q}) - \omega_K(\vec{q}) + i\frac{\Gamma_{K^*}}{2}} \right). \end{aligned} \quad (22)$$

As a result, the total amplitude will be

$$t_{\text{total}} = \mathcal{I}_{\text{loop}}^{(a)}(\vec{\epsilon}_A \cdot \vec{p}'_{\pi^+}) + \mathcal{I}_{\text{loop}}^{(b)}(\vec{\epsilon}_A \cdot \vec{p}'_{\pi^-}), \quad (23)$$

where $\vec{p}'_{\pi^+}$ and $\vec{p}'_{\pi^-}$ are the π^+ and π^- momenta in the D_{s1} rest frame:

$$\begin{aligned} \vec{p}'_{\pi^+} &= \frac{\lambda^{\frac{1}{2}}(m_{D_{s1}}^2, m_{\pi^+}^2, m_{23}^2)}{2m_{D_{s1}}}, \\ \vec{p}'_{\pi^-} &= \frac{\lambda^{\frac{1}{2}}(m_{D_{s1}}^2, m_{\pi^-}^2, m_{13}^2)}{2m_{D_{s1}}}, \end{aligned} \quad (24)$$

with m_{23} and m_{13} standing for the invariant masses associated with the $D_s^+ \pi^-$ and $D_s^+ \pi^+$ pairs, respectively [$\pi^+(1)$, $\pi^-(2)$, $D_s^+(3)$]. Furthermore, these masses are not all independent, and, by using

$$m_{12}^2 + m_{23}^2 + m_{13}^2 = m_{D_{s1}}^2 + m_{D_s^+}^2 + m_{\pi^+}^2 + m_{\pi^-}^2, \quad (25)$$

we can evaluate Eq. (23) as a function of the invariant masses that we are interested in.

Averaging over the D_{s1} polarizations, and taking the modulus squared of the amplitude in Eq. (23), that is,

$$\begin{aligned} \overline{\sum \sum} |t_{\text{total}}|^2 = & \frac{1}{3} \left\{ |\mathcal{I}_{\text{loop}}^{(a)}|^2 |\vec{p}'_{\pi^+}|^2 + |\mathcal{I}_{\text{loop}}^{(b)}|^2 |\vec{p}'_{\pi^-}|^2 \right. \\ & \left. + 2\text{Re}[\mathcal{I}_{\text{loop}}^{(a)} \cdot (\mathcal{I}_{\text{loop}}^{(b)})^*](\vec{p}'_{\pi^+} \cdot \vec{p}'_{\pi^-}) \right\}, \end{aligned} \quad (26)$$

we are able to obtain the double differential invariant mass distribution [60], which is

$$\frac{d\Gamma}{dm_{12}dm_{23}} = \frac{1}{(2\pi)^3} \frac{2m_{12}2m_{23}}{32m_{D_{s1}}^3} \overline{\sum \sum} |t_{\text{total}}|^2. \quad (27)$$

Since all the momenta involved and the scalar products in Eq. (26) are given in terms of the invariant masses m_{12} , m_{23} , and m_{13} , by using Eq. (25) we can obtain the two-particle mass distribution in terms of any one of those invariant masses. For instance, for the m_{12} (the $\pi^+ \pi^-$) distribution, we integrate Eq. (27) over m_{23} , such that

$$\frac{d\Gamma}{dm_{12}} = \int_{m_{23}^{\min}}^{m_{23}^{\max}} \frac{d\Gamma}{dm_{12}dm_{23}} dm_{23}, \quad (28)$$

with $m_{23}^{\min}$ and $m_{23}^{\max}$ given, respectively, by

$$\begin{aligned} m_{23}^{\min} &= \sqrt{(E_2 + E_3)^2 - \left(\sqrt{E_2^2 - m_2^2} + \sqrt{E_3^2 - m_3^2} \right)^2}, \\ m_{23}^{\max} &= \sqrt{(E_2 + E_3)^2 - \left(\sqrt{E_2^2 - m_2^2} - \sqrt{E_3^2 - m_3^2} \right)^2}. \end{aligned} \quad (29)$$

Moreover, E_2 and E_3 are given, respectively, by

$$\begin{aligned} E_2 &= \frac{m_{12}^2 - m_1^2 + m_2^2}{2m_{12}}, \\ E_3 &= \frac{m_{D_{s1}}^2 - m_{12}^2 - m_3^2}{2m_{12}}. \end{aligned} \quad (30)$$

All the other two-particle mass distribution are easily derived by taking cyclical permutations of the 1,2,3 indices in the expressions above.

III. NUMERICAL RESULTS

The numerical results are presented in Figs. 2 and 3. Specifically, Fig. 2 illustrates the $\pi^+ \pi^-$ mass distribution. We can observe a smooth line shape throughout the invariant mass range allowed by kinematics, including a broad peak in the range of 300–400 MeV. It is interesting to mention that, even if we do not have a direct $\pi^+ \pi^-$

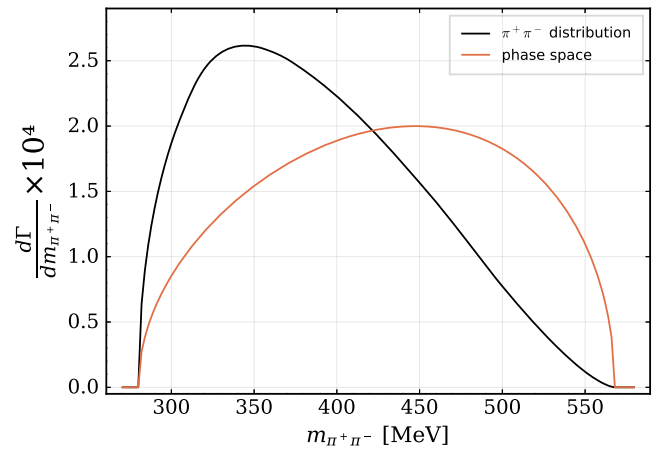


FIG. 2. $\pi^+ \pi^-$ invariant mass spectrum and phase space.

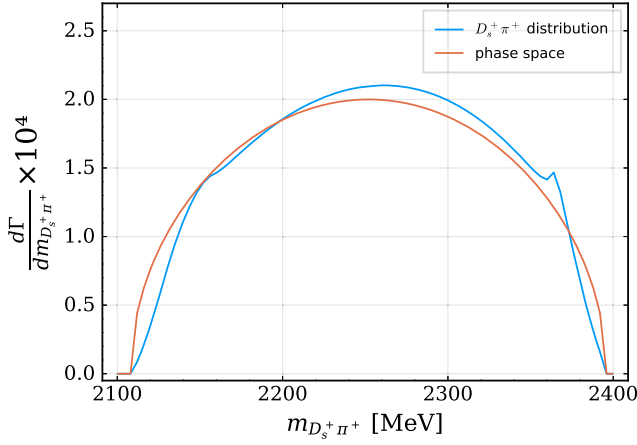


FIG. 3. $D_s^+\pi^+$ invariant mass spectrum and phase space.

interaction here, the shape of the mass distribution is quite different from one of pure phase space. This is due to the particular mechanism of $D_s^+\pi^+\pi^-$ production, tied to the molecular nature of the $D_{s1}(2536)$ and the triangle mechanism that follows as a consequence.

In Fig. 3, on the other hand, we notice a smooth curve throughout most of the distribution, except at the extremes, where two cusp structures are clearly visible: one around 2140 MeV and another, more pronounced, in the high-energy region of the spectrum, at approximately 2363 MeV. This latter value is associated with a threshold singularity, arising from the contribution of the $D^0K^0 \rightarrow D_s^+\pi^-$ transition exactly at the D^0K^0 production threshold, located at 2363 MeV. This threshold condition consequently generates a cusp at this energy.

Similarly, the small enhancement observed around 2140 MeV can be understood as a result of the on-shell contribution of the D^+K^+ pair in the $D^+K^+ \rightarrow D_s^+\pi^+$ transition. In particular, we expect the appearance of a cusp in the mass spectrum shown in Fig. 3, precisely at 2143 MeV, when the invariant mass of the $D_s^+\pi^+$ system, $m_{D_s^+\pi^+}$, reaches approximately 2363 MeV.

This correlation between the two values can be understood by noting that, in the rest frame of the $D_s^+\pi^+$ pair, the invariant mass $m_{D_s^+\pi^-}$ can be written as a function of $m_{D_s^+\pi^+}$ through the relation

$$m_{D_s^+\pi^-}^2 = m_{\pi^-}^2 + m_{D_s^+}^2 + 2E_{\pi^-}E_{D_s^+} - 2(\vec{p}_{\pi^-} \cdot \vec{p}_{D_s^+}), \quad (31)$$

where $\vec{p}_{\pi^-}$ and $\vec{p}_{D_s^+}$ are the three-momenta of the π^- and D_s^+ mesons, respectively, evaluated in the $D_s^+\pi^+$ rest frame:

$$\begin{aligned} \vec{p}_{\pi^-} &= \frac{\lambda^{\frac{1}{2}}(m_{D_{s1}}^2, m_{\pi^-}^2, M_{\text{inv}}^2(D_s^+\pi^+))}{2M_{\text{inv}}(D_s^+\pi^+)}, \\ \vec{p}_{D_s^+} &= \frac{\lambda^{\frac{1}{2}}(M_{\text{inv}}^2(D_s^+\pi^+), m_{\pi^-}^2, m_{D_s^+}^2)}{2M_{\text{inv}}(D_s^+\pi^+)}. \end{aligned} \quad (32)$$

Both E_{π^-} and $E_{D_s^+}$ and the magnitudes of the momenta depend on $m_{D_s^+\pi^+}$. Therefore, by setting $m_{D_s^+\pi^+} = 2363$ MeV, the above equation yields $m_{D_s^+\pi^-} \approx 2143$ MeV, precisely where the corresponding cusp is expected to appear. It should be noticed that in this case the phase space distribution is similar to the actual one, except for the two cusps discussed above.

We do not plot the $D_s^+\pi^-$ mass distribution which is very similar to the $D_s^+\pi^+$ one (there is a slight difference because of different masses of the intermediate states).

As we have mentioned, with the assumption of $D_{s1}(2536)$ as a pure K^*D molecule, we can make absolute predictions of the mass distributions and the integrated decay width, up to the dependence on Λ . This width is obtained by determining the area under the spectral curves in Figs. 2 and 3, which should, of course, yield the same value. Consequently, we obtain the following value for the decay width of the $D_{s1}(2536) \rightarrow D_s^+\pi^+\pi^-$ process:

$$\Gamma(D_{s1}(2536) \rightarrow D_s^+\pi^+\pi^-) = 44 \pm 15 \text{ keV}, \quad (33)$$

where the uncertainty arises from the variation of the cutoff parameter Λ . This value is about an order of magnitude bigger than the one obtained in the $D_{s1}(2460) \rightarrow D_s^+\pi^+\pi^-$ decay [49], which should make its observation feasible.

IV. CONSIDERATION OF $\epsilon^0(K^*)$ AND NEGATIVE ENERGY PARTS OF THE PROPAGATORS

A. Consideration of $\epsilon^0(K^*)$

We should start with a covariant $V_{D_{s1}(2536) \rightarrow K^{*+}D^0}$ vertex substituting in Eq. (2):

$$\vec{\epsilon}_A \vec{\epsilon}_{K^*} = -\epsilon_{A\mu} \epsilon_{K^*}^\mu. \quad (34)$$

Actually, since the $D_{s1}(2536)$ state is at rest, $\epsilon_A^0 = 0$, and we would have the vertex

$$\epsilon_{Ai} \epsilon_{K^*i}. \quad (35)$$

Then we would have for the combination of Eq. (8)

$$\begin{aligned} &\sum_{K^*\text{pol}} \epsilon_{Ai} \epsilon_{K^*i} \epsilon_{K^*a} (q - p')^\alpha \\ &= \epsilon_{Ai} \left(-g_{ia} + \frac{(p' + q)_i (p' + q)_a}{m_{K^*}^2} \right) (q - p')^\alpha. \end{aligned} \quad (36)$$

The first term in Eq. (36) gives rise to what we have calculated, and we get a new term

$$\frac{\epsilon_{Ai} (p' + q)_i (q^2 - p'^2)}{m_{K^*}^2}, \quad (37)$$

which should be compared to

$$\varepsilon_{Ai}(q - p')_i. \quad (38)$$

With the value of $q^0 = P^0 - p'^0 - \omega_D(\vec{p}' + \vec{q})$ that we get from the Cauchy integration, we find that $(q^2 - p'^2)/m_{K^*}^2 \approx 0.03$. Furthermore, the term of Eq. (37) goes with $\varepsilon_{Ai}(p' + q)_i$, where $\vec{p}' + \vec{q}$ will be zero in average, while the term in Eq. (38) goes as $(q - p')_i$, which does not vanish as an average. We can conclude that the new term is basically negligible.

B. Consideration of negative energy component of the propagators

In Eq. (14), we will now consider the negative energy component of the K^* propagator:

$$-\frac{1}{2\omega_{K^*}(\vec{p}' + \vec{q})} \frac{1}{p'^0 + q^0 + \omega_{K^*}(\vec{p}' + \vec{q}) - i\varepsilon}, \quad (39)$$

and we apply again Cauchy's theorem to evaluate the term. We obtain in Eq. (15) a propagator

$$\frac{1}{p'^0 + \omega_K(\vec{q}) + \omega_{K^*}(\vec{p}' + \vec{q}) - i\varepsilon} \quad (40)$$

instead of the term in Eq. (15):

$$\frac{1}{P^0 - \omega_D(\vec{p}' + \vec{q}) - \omega_{K^*}(\vec{p}' + \vec{q}) + i\varepsilon}. \quad (41)$$

We find numerically that the corrections from inclusion of the new term are of the order of 10% as an average. These uncertainties have to be assumed in our result but are still small compared to the uncertainties in Eq. (33) from other sources.

V. DISCUSSION

Recently, a work appeared [61] evaluating the $D_{s1}(2460)$ and $D_{s1}(2536)$ decay to $D_s^+ \pi^+ \pi^-$. This work is timely and illustrating and allows us to give a new perspective to our work.

Contrary to our work, that assumes the $D_{s1}(2536)$ state to be mostly molecular from the K^*D component, Ref. [61] considers the state mostly of $q\bar{q}$ type, based on the findings of Ref. [28]. The molecular components are K^*D in S and D wave and barely amount to about 2%. Interestingly, it is observed in [61] that the $c\bar{s}(P)$ quark model component of the D_{s1} state cannot decay to $c\bar{s}(S) + f_0(500) \rightarrow \pi\pi$, that is Okubo–Zweig–Iizuka (OZI) forbidden, which forces the $D_{s1} \rightarrow D_s^+ \pi^+ \pi^-$ decay through a loop mechanism involving the molecular components, even if they are small.

Since the molecular components for $D_{s1}(2536)$ assumed in [61] are different from those considered in the present work, it is not surprising that the line shapes of the $\pi\pi$ and $D_s\pi$ mass distributions are quite different in the two pictures, which is a welcome feature, because it means

that the experimental implementation of the reaction can tell us about the basic molecular components present in the $D_{s1}(2536)$ state. This should be an incentive to measure the reaction.

From our point of view, the presence of the K^*D component in the $D_{s1}(2536)$ is necessary, although we cannot quantify it. This statement is based on the observation that the $D_{s1}(2460)$ contains a large D^*K molecular component, $(57 \pm 21 \pm 6)\%$ according to [62] based on lattice QCD data, compatible with that found in [28] of $(1 - (52.4^{+5.1}_{-3.8}))\%$. The D^*K component is obtained because the D^*K interaction is found attractive and strong enough to bind by itself a D^*K molecule [53]. The K^*D system also has a strong attractive interaction. Indeed, using the local hidden gauge approach with the exchange of vector mesons, the D^*K interaction at threshold goes as $2m_{D^*}2m_K$ and the K^*D interaction as $2m_D2m_{K^*}$ with the same coefficient. This means that the K^*D interaction has a strength 1.7 times bigger than that of the D^*K and should generate some important K^*D component in a state orthogonal to the $D_{s1}(2460)$ with the same quantum numbers, in this case the $D_{s1}(2536)$. It is true that the binding of the $D_{s1}(2536)$ is about 220 MeV below the K^*D threshold, while that of the $D_{s1}(2460)$ is only about 40 MeV below the D^*K threshold, but given the much larger strength of the K^*D interaction, as pointed above, it is natural to expect a much larger binding. Such large bindings are not absent in hadron molecules, and, for instance, the $f_2(1270)$ resonance appears as a bound state of two ρ mesons in the work of [63,64]. From the discussion above, it seems natural to expect a sizable K^*D component in the $D_{s1}(2536)$ state, and this has been our starting point. This idea might be in conflict with the zero K^*D component of [28], but we think there is no conflict since in the analysis of the lattice QCD data done in [28] the interpolating K^*D fields were not considered and, hence, a solution is found without the K^*D component.

From our point of view, there should be a sizable K^*D component in the $D_{s1}(2536)$ state, but experiments like the one studied here and others will tell us in the future. However, we should also not expect a 100% molecular state as we have assumed so far. The $D_{s1}(2460)$ has about 60% D^*K component, and we should not expect more than that. However, in the present calculation, we have assumed the whole $D_{s1}(2536)$ to be of molecular nature, and this provided the coupling $g_{D_{s1}(2536),K^*D^0}$ extracted from Ref. [53] under this assumption. Should the molecular K^*D component be smaller than 100%, this coupling would be smaller, and the probability would roughly be proportional to this coupling squared [59]. This means that the measurement of the total width of $D_{s1}(2536) \rightarrow D_s^+ \pi^+ \pi^-$ would serve as a measure of this component. If this experimental rate would be about 1/2 of our predictions and the shape of the distributions the same that we predict, we could come to the conclusion that the K^*D molecular component is about 50%.

After the former discussion, it is clear that future measurements of the $D_{s1}(2536) \rightarrow D_s^+\pi^+\pi^-$ reaction with precise mass distributions and integrated width should provide excellent information to learn about the molecular components of the $D_{s1}(2536)$ and their strength.

VI. SUMMARY

In summary, our study highlights the relevance of triangle loop mechanisms in the $D_{s1}(2536) \rightarrow D_s^+\pi^+\pi^-$ decay process, where intermediate meson-meson interactions play a crucial role in shaping the final-state mass distributions. Within our model, we assume that the decay proceeds via such loops, adopting a molecular picture for the axial meson $D_{s1}(2536)$, where this state is dynamically generated through the K^*D interaction.

We have also emphasized the impact of dynamical effects driven by the proximity of intermediate thresholds. In the present study, these effects are particularly significant, giving rise to nontrivial structures in the spectra. Specifically, in the invariant mass distribution of the $D_s^+\pi^+$ pair, the manifestation of threshold singularities becomes evident at values that coincide with the on-shell conditions of the intermediate D^0K^0 and D^+K^+ states.

We have provided the mass distributions and the total decay width in absolute values, within a range of uncertainty. The calculations are based on the assumption of the $D_{s1}(2536)$ as a pure molecular K^*D state. Interestingly, the $q\bar{q}$ components that one also expects for this state do not contribute to this process, since it is OZI forbidden as noted in [61] and, hence, the shape of the distribution is tied to the molecular components. Yet, the coupling of $D_{s1}(2536)$ to K^*D would be smaller if there is a sizable $q\bar{q}$ component in the state, which would revert to a smaller integrated width

than the one that we have calculated. This is telling us that the actual measurement of the $D_{s1}(2536) \rightarrow D_s^+\pi^+\pi^-$ mass distributions and decay rate can tell us much about the nature of the $D_{s1}(2536)$.

Overall, these findings offer further insight into the role of hadronic loops and final-state interactions in shaping observable features—such as cusps and enhancements—in multibody decay processes of heavy mesons.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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