

Fierz analyses on the decay properties of two- and three-gluon glueballs

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The Fierz rearrangement, based on the various internal symmetries of hadrons, can be used to study their decay properties in a largely model-independent way. We have developed a semiautomatic *Mathematica* program to implement this method, with which we calculate the relative branching ratios of two-gluon glueballs with $J^{PC} = 0^{++}/0^{-+}$ and three-gluon glueballs with $J^{PC} = 0^{++}/1^{+-}$. In total, we derive nearly 100 ratios for these glueballs. Our results suggest that the $f_0(1710)$ and $\eta(2370)$ likely contain a significant gluon component, whereas the gluon component in $f_0(1500)$ appears to be small. Furthermore, we propose observing the three-gluon glueball with $J^{PC} = 0^{++}$ in the $\pi\pi\omega$ and $K\bar{K}\phi$ channels, and the three-gluon glueball with $J^{PC} = 1^{+-}$ in the $\pi\pi\omega$, $\pi\pi\phi$, and $K\bar{K}\phi$ channels. This study enhances our understanding of the gluonic structure of exotic hadrons and will assist future experimental searches in high-energy physics.

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I. INTRODUCTION

Glueballs, being color-singlet states formed solely through the self-interaction of gluon fields, are of significant importance for the study of nonperturbative quantum chromodynamics (QCD) [1]. According to relevant theoretical predictions, the mass of the lightest scalar glueball is approximately 1.6 GeV, while that of the lightest pseudo-scalar glueball is around 2.3 GeV [2–23]. Consequently, several glueball candidates have been discovered experimentally over the past several decades, including the $f_0(1500)$, $f_0(1710)$, and $\eta(2370)$, among others [24–26]. Extensive theoretical studies have been devoted to interpreting the nature of these states, and we refer to the review papers [27–44] and references therein for detailed discussions. However, most of these studies are model-dependent, making it difficult to definitively identify these states as glueballs.

In this paper, we apply the Fierz rearrangement [45] to study the decay properties of pure glueball states. This method, grounded in the internal symmetries of hadrons, is largely model-independent, as it relies on the fundamental symmetries of QCD. A key property of glueballs, if they exist, is that they couple equally to mesons composed of quarks with different flavors. Furthermore, if glueballs mix with meson states composed of quarks, the mixing coefficients should be comparable for mesons containing

quarks of different flavors [46–48]. The model-independent nature of the Fierz rearrangement offers a distinct advantage in describing this property. In the present analysis, we adopt a set of simplifying assumptions: the decay of glueballs is assumed to be dominated by two- or three-constituent gluons; each gluon is converted into a quark-antiquark pair; mesons are treated as quark-antiquark bound states; and final-state interactions among the produced mesons are neglected. These assumptions effectively correspond to a constituent picture, in the spirit of the constituent quark model [49–54] extended to include gluonic degrees of freedom. On this basis, and by making use of the universal symmetries of QCD, we construct a framework for analyzing the algebraic structure of glueball decay channels. It should be emphasized that the present approach remains a model-based approximation. However, since it primarily relies on symmetry constraints rather than specific dynamical inputs, the dependence on particular dynamical models is reduced to some extent, which allows the resulting relations to retain a certain degree of generality across different model settings.

We first apply the Fierz rearrangement to investigate the two-gluon glueballs with $J^{PC} = 0^{++}$ and 0^{-+} . As depicted in Fig. 1(a), a two-gluon glueball decays via the excitation of its two gluon fields into two vector quark-antiquark currents. We describe this process through a single Fierz rearrangement by interchanging the quark fields q_2 and q_4 . We examine eight decay channels of the two-gluon glueball with $J^{PC} = 0^{++}$ and subsequently derive seven relative branching ratios. Our results suggest that the $f_0(1710)$ is a promising candidate, whereas the $f_0(1500)$ is not favored, based on the relative branching ratios of its decays into the $K\bar{K}$, $\pi\pi$, and $\eta\eta$ channels [1,55,56]. We also derive 20 relative branching ratios for the two-gluon glueball with

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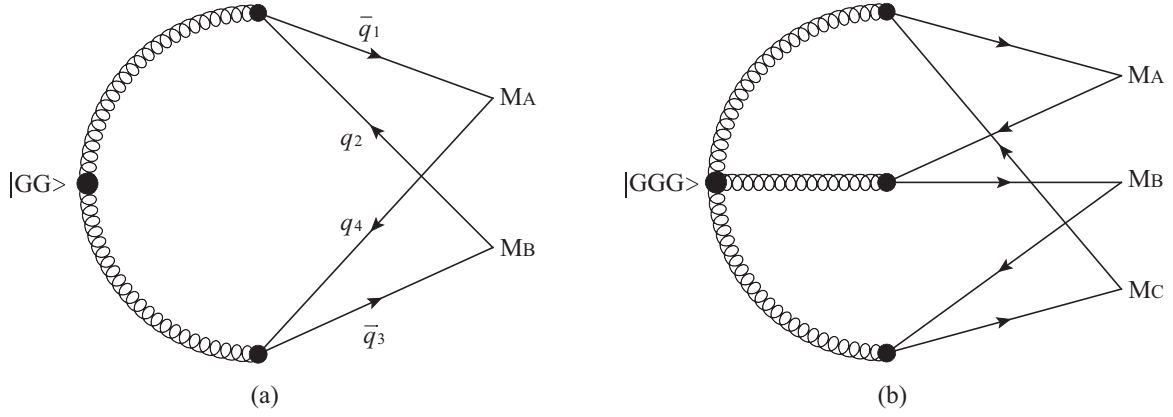


FIG. 1. (a) The schematic diagram of a two-gluon glueball decaying into two mesons via a single Fierz rearrangement. (b) The schematic diagram of a three-gluon glueball decaying into three mesons through two successive Fierz rearrangements.

TABLE I. Relative branching ratios of two-gluon glueballs with $J^{PC} = 0^{++}/0^{-+}$ and three-gluon glueballs with $J^{PC} = 0^{++}/1^{+-}$, derived via the Fierz rearrangement, where σ denotes $f_0(500)$, κ denotes $K_0^*(700)$, f_0 denotes $f_0(980)$, and K^* denotes $K^*(892)$.

$ GG; 0^{++}\rangle$	$\frac{\mathcal{B}(M_A M_B)}{\mathcal{B}(\pi\pi)}$
$\pi\pi$	1.00
$K\bar{K}$	1.93
$\kappa\bar{\kappa}$	1.19
$\eta\eta$	0.43
$\rho\rho$	6.49×10^{-2}
$\sigma\sigma$	5.76×10^{-2}
$\eta\eta'$	2.27×10^{-2}
$\omega\omega$	1.02×10^{-2}

$ GG; 0^{-+}\rangle$	$\frac{\mathcal{B}(M_A M_B)}{\mathcal{B}(f_0\eta)}$
$f_0\eta$	1.00
$\omega\phi$	46.38
$\omega\omega$	36.72
$\rho b_1(1235)$	13.51
$\phi\phi$	11.39
$K^*\bar{K}_1(1270)$	11.28
$\omega h_1(1170)$	5.45
$\eta f_0(1770)$	4.37
$\pi a_2(1320)$	4.14
$K\bar{K}_0^*(1430)$	3.83
$K\bar{K}_2^*(1430)$	1.88
$\pi a_0(1450)$	1.85
$\kappa\bar{\kappa}$	1.31
$\eta f_2(1270)$	1.26
$f_0\eta'$	0.31
$\eta f_2'(1525)$	0.23
$\eta' f_0(1370)$	0.10
$\eta f_0(1370)$	5.79×10^{-2}
$\eta' f_2(1270)$	4.47×10^{-2}
$\sigma\eta$	6.63×10^{-4}
$\sigma\eta'$	1.31×10^{-4}

$ GGG; 0^{++}\rangle$	$\frac{\mathcal{B}(M_A M_B M_C)}{\mathcal{B}(\pi\pi\omega)}$
$\pi\pi\omega$	1.00
$K\bar{K}\phi$	0.75
$K\bar{K}\rho$	0.69
$\kappa\bar{\kappa}\rho$	0.35
$K\bar{K}\omega$	0.35
$\rho\rho\omega$	0.35
$\kappa\bar{\kappa}\phi$	0.34
$K^*\bar{K}^*\rho$	0.24
$K^*\bar{K}^*\phi$	0.21
$\kappa\bar{\kappa}\omega$	0.18
$\pi\pi f_2(1270)$	0.14
$K^*\bar{K}^*\omega$	0.12
$K\bar{K}a_2(1320)$	7.59×10^{-2}
$K\bar{K}f_2(1270)$	4.72×10^{-2}
$K\bar{K}f_2'(1525)$	4.34×10^{-2}
$\kappa\bar{\kappa}a_2(1320)$	2.43×10^{-2}
$\eta\eta\phi$	2.24×10^{-2}
$\eta'\eta'\phi$	1.78×10^{-2}
$f_0 f_0 \phi$	1.76×10^{-2}
$\kappa\bar{\kappa}f_2(1270)$	1.52×10^{-2}
$\kappa\bar{\kappa}f_2'(1525)$	1.31×10^{-2}
$\eta\eta\omega$	8.82×10^{-3}
$\kappa\bar{\kappa}f_0$	7.41×10^{-3}
$\phi\phi\phi$	6.88×10^{-3}
$\omega\omega\omega$	5.64×10^{-3}
$\sigma\sigma\omega$	4.76×10^{-3}
$\eta\eta' f_2'(1525)$	1.68×10^{-3}
$\eta\eta f_2(1270)$	1.47×10^{-3}
$\eta'\eta'\omega$	1.43×10^{-3}
$\eta\eta f_2'(1525)$	1.35×10^{-3}
$\eta\eta' f_2(1270)$	9.02×10^{-4}
$\kappa\bar{\kappa}\sigma$	7.69×10^{-4}
$\sigma\sigma f_2(1270)$	4.85×10^{-4}
$\eta'\eta' f_2'(1525)$	3.59×10^{-4}
$f_0 f_0 f_2'(1525)$	2.59×10^{-4}

(Table continued)

TABLE I. (Continued)

$ \text{GGG}; 0^{++}\rangle$	$\frac{\mathcal{B}(M_A M_B M_C)}{\mathcal{B}(\pi\pi\omega)}$
$f_0 f_0 f_0$	2.24×10^{-4}
$\eta' \eta' f_2(1270)$	1.10×10^{-4}
$\sigma\sigma\sigma$	4.67×10^{-8}
$ \text{GGG}; 1^{+-}\rangle$	$\frac{\mathcal{B}(M_A M_B M_C)}{\mathcal{B}(\pi\pi\omega)}$
$\pi\pi\omega$	1.00
$\pi\pi\phi$	1.49
$K\bar{K}\omega$	0.83
$\kappa\bar{\kappa}\omega$	0.29
$K\bar{K}\phi$	0.27
$\eta\eta\omega$	0.16
$K\bar{K}\rho$	0.15
$\kappa\bar{\kappa}\phi$	0.14
$\eta\eta\phi$	7.62×10^{-2}
$\sigma\sigma\omega$	4.69×10^{-2}
$\sigma\sigma\phi$	2.88×10^{-2}
$\eta'\eta'\omega$	4.62×10^{-3}
$K\bar{K}f_0$	3.65×10^{-3}
$\eta\eta'\omega$	3.51×10^{-3}
$f_0 f_0 \omega$	5.38×10^{-4}
$\eta\eta'\phi$	4.95×10^{-4}
$\eta'\eta'\phi$	4.12×10^{-4}
$\eta\eta f_0$	8.77×10^{-5}
$\eta\eta' f_0$	4.58×10^{-5}
$f_0 f_0 \phi$	3.61×10^{-5}
$\pi\pi\sigma$	9.35×10^{-6}
$K\bar{K}\sigma$	2.80×10^{-6}
$\eta'\eta' f_0$	1.27×10^{-6}
$\eta\eta\sigma$	8.05×10^{-8}
$\eta\eta'\sigma$	3.40×10^{-8}
$\eta'\eta'\sigma$	2.38×10^{-9}
$K\bar{K}a_1(1260)$	1.62×10^{-15}
$K\bar{K}f_1(1420)$	7.18×10^{-16}
$K\bar{K}f_1(1285)$	6.25×10^{-16}
$\pi\pi f_1(1285)$	3.58×10^{-16}

$J^{PC} = 0^{-+}$, and the results indicate that the $\eta(2370)$ is a strong candidate. This structure was observed in the $f_0(980)\eta'$ channel [57], and we propose further investigation in the $\phi\phi$, $\omega\omega$, and $\phi\omega$ channels.

If two-gluon glueballs exist, three-gluon glueballs might also exist, and there have been some experimental signals [58]. As depicted in Fig. 1(b), we can describe their decay processes through two successive Fierz rearrangements. We apply this method to investigate three-gluon glueballs with $J^{PC} = 0^{++}$ and 1^{+-} , for which we derive 37 and 29 relative branching ratios, respectively. We propose observing the $J^{PC} = 0^{++}$ state in the $\pi\pi\omega$ and $K\bar{K}\phi$ channels, and the $J^{PC} = 1^{+-}$ state in the $\pi\pi\omega$, $\pi\pi\phi$, and $K\bar{K}\phi$ channels. The relevant relative branching ratios are derived and summarized in Table I. This study provides a theoretical

foundation for designing targeted search strategies in both current and future high-luminosity experiments, such as BESIII, GlueX, LHCb, and PANDA, among others.

II. GLUEBALL INTERPOLATING CURRENTS

The interpolating currents for two- and three-gluon glueballs have been systematically constructed in Ref. [23]. In the present study, we investigate two-gluon glueballs with $J^{PC} = 0^{++}/0^{-+}$ and three-gluon glueballs with $J^{PC} = 0^{++}/1^{+-}$, denoted by $|\text{GG}; 0^{++}/0^{-+}\rangle$ and $|\text{GGG}; 0^{++}/1^{+-}\rangle$, respectively. The corresponding interpolating currents are given by

$$J_0 = g_s^2 G_i^{\mu\nu} G_{\mu\nu}^i, \quad (1)$$

$$\tilde{J}_0 = g_s^2 G_i^{\mu\nu} \tilde{G}_{\mu\nu}^i, \quad (2)$$

$$\eta_0 = f^{ijk} g_s^3 G_i^{\mu\nu} G_{j,\nu\rho} G_{k,\mu}^\rho, \quad (3)$$

$$\eta_1^{\alpha\beta} = d^{ijk} g_s^3 G_i^{\mu\nu} G_{j,\mu\nu} G_k^{\alpha\beta}, \quad (4)$$

where i, j, k are color indices (running from 1 to 8), and $\mu, \nu, \rho, \alpha, \beta$ are Lorentz indices; f^{ijk} and d^{ijk} are the antisymmetric and symmetric structure constants, respectively. The gluon field-strength tensor $G_{\mu\nu}^i$ can be expressed in terms of the gauge field A_μ^i as

$$G_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g_s f^{ijk} A_\mu^j A_\nu^k. \quad (5)$$

Its dual field is $\tilde{G}_{\mu\nu}^i = G^{i,\rho\sigma} \times \epsilon_{\mu\nu\rho\sigma}/2$.

We assume that a glueball decays via the excitation of its gluon fields into vector quark-antiquark currents,

$$A_\mu^i \rightarrow \lambda_{ab}^i \times \bar{q}^a \gamma_\mu q^b, \quad (6)$$

where λ_{ab}^i are the Gell-Mann matrices, with a and b as color indices (running from 1 to 3). Accordingly, the decay processes of two- and three-gluon glueballs are depicted in Fig. 1, which will be discussed separately as follows.

III. FIERZ REARRANGEMENT ON TWO-GLUON CURRENTS

As an example, we study the decay properties of the two-gluon glueball $|\text{GG}; 0^{++}\rangle$ through the current J_0 . We need to perform the Fierz rearrangement once, but in both color and Lorentz spaces, as schematically illustrated below:

$$\begin{aligned}
|GG; 0^{++}\rangle &\longleftrightarrow J_0 = G_{i_1}^{\mu\nu} \times G_{\mu\nu}^i \times g_s^2 \\
&\longrightarrow (M_A)_\alpha^i \times (M_B)_\beta^j \times \mathbb{C}_1 + \dots \\
&\longrightarrow \lambda_{ab}^i \lambda_{cd}^j \times \bar{q}_1^a \gamma_\alpha q_2^b \times \bar{q}_3^c \gamma_\beta q_4^d \times \mathbb{C}_2 + \dots \\
&\xrightarrow{\text{color}} \delta_{ad} \delta_{cb} \times \bar{q}_1^a \gamma_\alpha q_2^b \times \bar{q}_3^c \gamma_\beta q_4^d \times \mathbb{C}_3 + \dots \\
&\xrightarrow{\text{Fierz}} \delta_{ad} \delta_{cb} \times \bar{q}_1^a \Gamma_1 q_4^d \times \bar{q}_3^c \Gamma_2 q_2^b \times \mathbb{C}_4 + \dots \\
&\longrightarrow [\bar{q}_1^a \Gamma_1 q_4^a]_A \times [\bar{q}_3^c \Gamma_2 q_2^c]_B \times \mathbb{C}_4 + \dots \quad (7)
\end{aligned}$$

Here, $\mathbb{C}_{1,2,3,4}$ and $\Gamma_{1,2}$ are the relevant coefficients and Dirac structures to be determined. The quark fields $\bar{q}_1^a$ and q_2^b carry the same flavor, and likewise $\bar{q}_3^c$ and q_4^d carry the same flavor. The subscripts 1, 2, 3, 4, labeling the four quark fields, and the subscripts M_A, M_B , labeling the two meson components, are introduced for convenience in the discussion below. In particular, we employ the color identity

$$\lambda_{ab}^i \lambda_{cd}^i = 2\delta_{ad} \delta_{cb} - \frac{2}{3} \delta_{ab} \delta_{cd}, \quad (8)$$

and we also require the Fierz rearrangement in Lorentz space, for which the relevant formulas can be found in Ref. [59].

The detailed result of Eq. (7) will be presented in the Appendix. In total, there are 19 terms, and we display only seven of them:

$$\begin{aligned}
J_0 = & + \frac{1}{4} [\bar{u}_1^a \gamma_\mu \gamma_5 d_4^a]_A [\bar{d}_3^b \gamma_\mu \gamma_5 u_2^b]_B \times (q_A^\nu + q_B^\nu)^2 \\
& + \frac{1}{2} [\bar{u}_1^a \gamma_\mu \gamma_5 d_4^a]_A [\bar{d}_3^b \gamma_\nu \gamma_5 u_2^b]_B \times (q_A^\mu + q_B^\mu)(q_A^\nu + q_B^\nu) \\
& + \frac{3}{4} [\bar{u}_1^a \gamma_5 d_4^a]_A [\bar{d}_3^b \gamma_5 u_2^b]_B \times (q_A^2 + q_B^2 + 2q_A \cdot q_B) \\
& + \frac{1}{4} [\bar{u}_1^a \gamma_\mu d_4^a]_A [\bar{d}_3^b \gamma_\mu u_2^b]_B \times (q_A^\nu + q_B^\nu)^2 \\
& + \frac{1}{2} [\bar{u}_1^a \gamma_\mu d_4^a]_A [\bar{d}_3^b \gamma_\nu u_2^b]_B \times (q_A^\mu + q_B^\mu)(q_A^\nu + q_B^\nu) \\
& - \frac{1}{2} [\bar{u}_1^a \sigma_{\mu\alpha} d_4^a]_A [\bar{d}_3^b \sigma_{\nu\alpha} u_2^b]_B \times (q_A^\mu + q_B^\mu)(q_A^\nu + q_B^\nu) \\
& + \frac{1}{8} [\bar{u}_1^a \sigma_{\mu\nu} d_4^a]_A [\bar{d}_3^b \sigma_{\mu\nu} u_2^b]_B \times (q_A^2 + q_B^2 + 2q_A \cdot q_B). \quad (9)
\end{aligned}$$

Here, we have replaced the partial derivative operator acting on a quark-antiquark current with the momentum of the corresponding meson component: e.g., $\partial^\mu [\bar{u}_1^a \gamma_5 d_4^a]_A = p_A^\mu \times [\bar{u}_1^a \gamma_5 d_4^a]_A$. This result will be used to evaluate the relative branching ratio $\mathcal{B}(|GG; 0^{++}\rangle \rightarrow \rho^+ \rho^-) / \mathcal{B}(|GG; 0^{++}\rangle \rightarrow \pi^+ \pi^-)$ as described below.

IV. DECAY ANALYSIS ON TWO-GLUON GLUEBALLS

The couplings of various quark-antiquark currents to meson states are well-established in the literature [1,60]; e.g.,

$$\begin{aligned}
\langle 0 | \bar{u}_a i \gamma_5 d_a | \pi^-(q) \rangle &= \lambda_\pi, \\
\langle 0 | \bar{u}_a \gamma_\mu \gamma_5 d_a | \pi^-(q) \rangle &= i q_\mu f_\pi, \\
\langle 0 | \bar{u}_a \gamma_\mu d_a | \rho^-(q, \epsilon) \rangle &= m_\rho f_\rho \epsilon_\mu, \\
\langle 0 | \bar{u}_a \sigma_{\mu\nu} d_a | \rho^-(q, \epsilon) \rangle &= i f_\rho^T (q_\mu \epsilon_\nu - q_\nu \epsilon_\mu). \quad (10)
\end{aligned}$$

Using these couplings together with Eq. (9), we obtain the following:

- (1) The decay amplitude of $|GG; 0^{++}\rangle$ into the $\pi^+ \pi^-$ channel receives contributions from both $[\bar{u}_a \gamma_5 d_a][\bar{d}_b \gamma_5 u_b]$ and $[\bar{u}_a \gamma_\mu \gamma_5 d_a][\bar{d}_b \gamma_\mu \gamma_5 u_b]$. Altogether, it can be written as

$$\begin{aligned}
\mathcal{M}_{\pi\pi} \propto & -\frac{3}{4} \lambda_\pi^2 (q_1^2 + q_2^2 + 2q_1 \cdot q_2) \\
& -\frac{3}{4} f_\pi^2 (q_1^2 + q_2^2)(q_1 \cdot q_2) \\
& -\frac{1}{2} f_\pi^2 (q_1^2 \times q_2^2 + 2(q_1 \cdot q_2)^2), \quad (11)
\end{aligned}$$

where $q_{1,2}$ are the momenta of the two pions.

- (2) The decay amplitude of $|GG; 0^{++}\rangle$ into the $\rho^+ \rho^-$ channel receives contributions from both $[\bar{u}_a \gamma_\mu d_a][\bar{d}_b \gamma_\mu u_b]$ and $[\bar{u}_a \sigma_{\mu_1 \nu_1} d_a][\bar{d}_b \sigma_{\mu_2 \nu_2} u_b]$. Altogether, it can be written as

$$\begin{aligned}
\mathcal{M}_{\rho\rho} \propto & + \frac{1}{2} m_\rho^2 f_\rho^2 (\epsilon(q_1) \cdot q_2)(\epsilon(q_2) \cdot q_1) \\
& + \frac{1}{4} m_\rho^2 f_\rho^2 (\epsilon(q_1) \cdot \epsilon(q_2))(q_1^2 + q_2^2 + 2q_1 \cdot q_2) \\
& + \frac{1}{4} (f_\rho^T)^2 (\epsilon(q_1) \cdot \epsilon(q_2))(q_1^2 + q_2^2)(q_1 \cdot q_2) \\
& + \frac{1}{2} (f_\rho^T)^2 (\epsilon(q_1) \cdot \epsilon(q_2)) q_1^2 \times q_2^2 \\
& - \frac{1}{4} (f_\rho^T)^2 (\epsilon(q_1) \cdot q_2)(\epsilon(q_2) \cdot q_1)(q_1^2 + q_2^2), \quad (12)
\end{aligned}$$

where $q_{1,2}$ are the momenta of the two ρ mesons, and $\epsilon_{1,2}$ are their polarization vectors.

The amplitudes above can be used to compute the partial decay widths of $|GG; 0^{++}\rangle$ into the $\pi^+ \pi^-$ and $\rho^+ \rho^-$ channels. Notably, in the present work, the structure of the decay amplitudes is obtained through Fierz rearrangements of the interpolating currents, followed by an identification of the resulting quark operators with the corresponding meson states. In this sense, the amplitudes should be understood as effective couplings dominated by algebraic structures within a constituent picture. It should be noted that dynamical effects such as spatial wave-function [61–63] overlaps are not included at this stage, and therefore the overall normalization, as well as possible channel-dependent

dynamical corrections, remains undetermined. After factoring out the overall common factor appearing in all amplitude expressions, we obtain the relative branching ratio between these two channels:

$$\frac{\mathcal{B}(|\text{GG}; 0^{++}\rangle \rightarrow \rho\rho)}{\mathcal{B}(|\text{GG}; 0^{++}\rangle \rightarrow \pi\pi)} = 6.49 \times 10^{-2}. \quad (13)$$

Here, the input masses and decay constants of the two-gluon glueball, as well as those of the ρ and π mesons, will introduce uncertainties into the aforementioned results. Since we primarily adopt the experimentally measured values in this study, the relative errors in the branching ratios across different decay channels are relatively small for two-gluon glueballs, while they are expected to be somewhat larger for three-gluon glueballs.

Similarly, we compute the relative branching ratios for various decay channels of two-gluon glueballs with $J^{PC} = 0^{++}$ and 0^{-+} :

- (1) Our results suggest that the $f_0(1710)$ is a good candidate for a two-gluon glueball with $J^{PC} = 0^{++}$. Using its mass, 1733 MeV [1], as input, we derive relative branching ratios for eight decay channels, which are summarized in Table I. In particular, the $K\bar{K}$ and $\pi\pi$ channels exhibit relatively large branching fractions. The relative branching ratio of these two channels is found to be

$$\frac{\mathcal{B}(\text{GG}; 0^{++} \rightarrow K\bar{K})}{\mathcal{B}(\text{GG}; 0^{++} \rightarrow \pi\pi)} = 1.93, \quad (14)$$

which is consistent with the experimental measurements listed in Ref. [1]. It should be noted that light pseudoscalar mesons are subject to chiral symmetry constraints [64–66] and may exhibit additional dynamical suppressions, which are not captured in the present framework. We note that the standard PCAC relation [67–70] is an operator identity relating the divergence of the axial current to the pion field. In our framework, we effectively use its matrix element between the vacuum and a single-pion state, which leads to the well-known relation proportional to $f_\pi p^\mu$. However, we do not incorporate the full implications of PCAC at the level of interaction dynamics, such as derivative couplings and Adler zeros in soft-pion limits. These effects are therefore beyond the scope of the present analysis. Regarding the $f_0(1500)$, its relative branching ratios measured in experiments—especially those for the $K\bar{K}$, $\pi\pi$, and $\eta\eta$ channels—differ significantly from the results obtained in this work, suggesting that the $f_0(1500)$ is unlikely to be a pure glueball and may, in fact, contain only a minor gluon component.

- (2) Our results suggest that the $\eta(2370)$ is a good candidate for a two-gluon glueball with $J^{PC} = 0^{-+}$. Using its mass, 2377 MeV [1], as input, we derive relative branching ratios for 20 decay channels, which are also summarized in Table I. In particular, the $\phi\phi$, $\omega\omega$, and $\phi\omega$ channels exhibit relatively large branching fractions. Since the $\eta(2370)$ has been observed in the $f_0(980)\eta'$ channel [57], it is interesting to further investigate it in these three decay channels.

V. DECAY ANALYSIS ON THREE-GLUON GLUEBALLS

The decay properties of three-gluon glueballs can be studied in a similar manner. However, two successive Fierz rearrangements are required, making the calculation significantly more complex. To address this, we have developed a semiautomatic *Mathematica* program to implement this method. In particular, we need the following color identities:

$$f^{ijk}\lambda_i^{ab}\lambda_j^{cd}\lambda_k^{ef} = +2i\delta^{af}\delta^{bc}\delta^{de} - 2i\delta^{ad}\delta^{be}\delta^{cf}, \quad (15)$$

$$\begin{aligned} d^{ijk}\lambda_i^{ab}\lambda_j^{cd}\lambda_k^{ef} &= +2\delta^{af}\delta^{bc}\delta^{de} + 2\delta^{ad}\delta^{be}\delta^{cf} \\ &\quad - \frac{4}{3}\delta^{ab}\delta^{de}\delta^{cf} - \frac{4}{3}\delta^{af}\delta^{cd}\delta^{be} \\ &\quad - \frac{4}{3}\delta^{ad}\delta^{bc}\delta^{ef} + \frac{8}{9}\delta^{ab}\delta^{cd}\delta^{ef}. \end{aligned} \quad (16)$$

We apply the Fierz rearrangement to compute the relative branching ratios for various decay channels of three-gluon glueballs with $J^{PC} = 0^{++}$ and 1^{+-} :

- (1) The mass of the three-gluon glueball with $J^{PC} = 0^{++}$ was estimated in Ref. [23] to be 4210 MeV. Using this value as input, we derive relative branching ratios for 37 decay channels, which are summarized in Table I. In particular, the $\pi\pi\omega$ and $K\bar{K}\phi$ channels exhibit relatively large branching fractions.
- (2) The mass of the three-gluon glueball with $J^{PC} = 1^{+-}$ was estimated in Ref. [23] to be 3190 MeV. Using this value as input, we derive relative branching ratios for 29 decay channels, which are summarized in Table I. In particular, the $\pi\pi\omega$, $\pi\pi\phi$, and $K\bar{K}\phi$ channels exhibit relatively large branching fractions.

VI. SUMMARY

In this paper, we employ the Fierz rearrangement to compute the relative branching ratios for various decay channels of two-gluon glueballs with $J^{PC} = 0^{++}$ and 0^{-+} . Our results suggest that the $f_0(1710)$ and $\eta(2370)$ likely contain a significant gluon

component, whereas the gluon contribution in $f_0(1500)$ appears to be relatively small. We extend this approach to three-gluon glueballs with $J^{PC} = 0^{++}$ and 1^{+-} . For the $J^{PC} = 0^{++}$ state, we propose observing it in the $\pi\pi\omega$ and $K\bar{K}\phi$ channels, while the $J^{PC} = 1^{+-}$ state can be studied in the $\pi\pi\omega$, $\pi\pi\phi$, and $K\bar{K}\phi$ channels.

Our findings provide a theoretical framework to assist the search for glueball states in high-luminosity experiments, such as BESIII, GlueX, LHCb, and PANDA. These results also contribute to the ongoing efforts to identify exotic hadrons and enhance our understanding of QCD dynamics.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: FULL RESULT OF THE FIERZ REARRANGEMENT

In this appendix, we aim to make the derivation of Eq. (7) more explicit and present the complete result of the Fierz transformation with the quark component $\bar{u}d$:

$$\begin{aligned}
|GG; 0^{++}\rangle &\longleftrightarrow J_0 = G_i^{\mu\nu} \times G_{\mu\nu}^i \times g_s^2 \\
&\longrightarrow \mathbb{C}_1 \times (\partial^\mu A_i^\nu - \partial^\nu A_i^\mu + g_s f_{ijk} A_j^\mu A_k^\nu) \times (\partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g_s f^{ijk} A_\mu^j A_\nu^k) \\
&\longrightarrow \mathbb{C}_2 \times \lambda_{ab}^i \lambda_{cd}^i \times (\partial^\mu \bar{q}_1^a \gamma^\nu q_2^b + \bar{q}_1^a \gamma^\nu \partial^\mu q_2^b) \times (\partial_\mu \bar{q}_3^c \gamma_\nu q_4^d + \bar{q}_3^c \gamma_\nu \partial_\mu q_4^d) + \dots \\
&\xrightarrow{\text{color}} \mathbb{C}_3 \times \delta_{ad} \delta_{cb} \times \partial^\mu \bar{q}_1^a \gamma^\nu q_2^b \times \partial_\mu \bar{q}_3^c \gamma_\nu q_4^d + \dots \\
&\xrightarrow{\text{Fierz}} \mathbb{C}_4 \times \delta_{ad} \delta_{cb} \times \left(-\partial^\mu \bar{q}_1^a q_4^d \times \partial_\mu \bar{q}_3^c q_2^b + \frac{1}{2} \partial^\mu \bar{q}_1^a \gamma^\nu q_4^d \times \partial_\mu \bar{q}_3^c \gamma_\nu q_2^b \right. \\
&\quad \left. + \frac{1}{2} \partial^\mu \bar{q}_1^a \gamma^\nu \gamma_5 q_4^d \times \partial_\mu \bar{q}_3^c \gamma_\nu \gamma_5 q_2^b + \partial^\mu \bar{q}_1^a \gamma_5 q_4^d \times \partial_\mu \bar{q}_3^c \gamma_5 q_2^b \right) + \dots \\
&\longrightarrow \mathbb{C}_5 \times -(\partial^\mu [\bar{q}_1^a q_4^a]_A - [\bar{q}_1^a \overleftrightarrow{\partial}^\mu q_4^a]_A) \times (\partial_\mu [\bar{q}_3^b q_2^b]_B - [\bar{q}_3^b \overleftrightarrow{\partial}_\mu q_2^b]_B) + \dots \\
&\longrightarrow +\frac{1}{4} [\bar{u}_1^a \gamma_\mu \gamma_5 d_4^a]_A [\bar{d}_3^b \gamma_\mu \gamma_5 u_2^b]_B \times (q_A^\nu + q_B^\nu)^2 + \frac{1}{2} [\bar{u}_1^a \gamma_\mu \gamma_5 d_4^a]_A [\bar{d}_3^b \gamma_\mu \gamma_5 u_2^b]_B \times (q_A^\mu + q_B^\mu)(q_A^\nu + q_B^\nu) \\
&\quad + \frac{3}{4} [\bar{u}_1^a \gamma_5 d_4^a]_A [\bar{d}_3^b \gamma_5 u_2^b]_B \times (q_A^2 + q_B^2 + 2q_A \cdot q_B) + \frac{1}{4} [\bar{u}_1^a \gamma_\mu d_4^a]_A [\bar{d}_3^b \gamma_\mu u_2^b]_B \times (q_A^\nu + q_B^\nu)^2 \\
&\quad + \frac{1}{2} [\bar{u}_1^a \gamma_\mu d_4^a]_A [\bar{d}_3^b \gamma_\nu u_2^b]_B \times (q_A^\mu + q_B^\mu)(q_A^\nu + q_B^\nu) - \frac{1}{2} [\bar{u}_1^a \sigma_{\mu\alpha} d_4^a]_A [\bar{d}_3^b \sigma_{\nu\alpha} u_2^b]_B \times (q_A^\mu + q_B^\mu)(q_A^\nu + q_B^\nu) \\
&\quad + \frac{1}{8} [\bar{u}_1^a \sigma_{\mu\nu} d_4^a]_A [\bar{d}_3^b \sigma_{\mu\nu} u_2^b]_B \times (q_A^2 + q_B^2 + 2q_A \cdot q_B) - \frac{3}{4} [\bar{u}_1^a d_4^a]_A [\bar{d}_3^b u_2^b]_B \times (q_A^2 + q_B^2 + 2q_A \cdot q_B) \\
&\quad - \frac{3}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu u_2^b]_B + \frac{3}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_5 d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma_5 u_2^b]_B + \frac{1}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\mu d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma^\mu u_2^b]_B \\
&\quad + \frac{1}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\nu d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma^\nu u_2^b]_B + \frac{1}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\nu d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\nu \gamma^\mu u_2^b]_B + \frac{1}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\mu \gamma_5 d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma^\mu \gamma_5 u_2^b]_B \\
&\quad + \frac{1}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\nu \gamma_5 d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \gamma^\nu \gamma_5 u_2^b]_B + \frac{1}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \gamma_\nu \gamma_5 d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\nu \gamma^\mu \gamma_5 u_2^b]_B - \frac{1}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \sigma_{\mu\alpha} d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\alpha \sigma^{\nu\alpha} u_2^b]_B \\
&\quad - \frac{1}{2} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \sigma_{\nu\alpha} d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\alpha \sigma^{\mu\alpha} \gamma_5 u_2^b]_B + \frac{1}{4} [\bar{u}_1^a \overleftrightarrow{\partial}^\mu \sigma_{\nu\alpha} d_4^a]_A [\bar{d}_3^b \overleftrightarrow{\partial}_\mu \sigma^{\nu\alpha} \gamma_5 u_2^b]_B.
\end{aligned} \tag{A1}$$

Here, the global coefficients $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$, $\mathbb{C}_4$ and $\mathbb{C}_5$ are not independent, but cannot be fully determined within the present framework. This is due to the fact that the gluon field is here mapped onto an effective diquark vertex in the form of a vector

current, while the corresponding dynamical strength of the correlation is not explicitly specified. This undetermined overall normalization does not enter our analysis of relative branching ratios, which depend only on the relative weights among different decay channels.

The ellipses in the intermediate expressions indicate additional terms arising at the corresponding stage of the expansion. Their explicit forms are not shown for brevity. The complete expressions obtained after performing the full Fierz transformation are presented in the final step.

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