

Quasiparticle hydrodynamics with momentum-dependent relaxation time

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(Received 23 April 2025; accepted 22 August 2025; published 16 September 2025)

We formulate the relativistic dissipative hydrodynamics of a system of quasiparticles from the Boltzmann equation within the ambit of relaxation time approximation with modified collision kernels. We focus on two specific scenarios with single quasiparticle species, (i) the extended relaxation time approximation, and (ii) the novel relaxation time approximation. We find that both approaches lead to equivalent results up to first order in spacetime gradients. We generalize the extended relaxation time approach to incorporate multiple quasiparticle species and obtain the corresponding expressions for the shear (η_s) and bulk (ζ_s) viscous coefficients. As an application, we study the temperature dependence of the transport coefficients of hot QCD medium with quasisgluon and (light and strange) quasisquark sectors considering the power law ansatz for the momentum dependence of the relaxation time. We explore the impact of the power law exponent on the ratio ζ_s/η_s . Our study suggests that in comparison to a constant exponent, a temperature dependent exponent in the power law ansatz is more suitable for modeling the quasiparticle dynamics in the relevant temperature regime of heavy ion collision.

DOI: [10.1103/nkx2-bxys](https://doi.org/10.1103/nkx2-bxys)

I. INTRODUCTION

Relativistic viscous hydrodynamics plays a crucial role in the space-time evolution of the strongly interacting extreme state of matter produced in the ultrarelativistic heavy ion collisions [1–5]. The computational modeling of the collision dynamics from the initial preequilibrium stage to the final stage of kinetic freeze-out requires a multistage approach of which a major part is the hydrodynamic evolution of the hot and dense quark-gluon plasma (QGP) [4,6]. For the correct description of the dynamical evolution, it is necessary to have a detailed understanding of the transport properties of the QGP medium. However, precise estimation of transport coefficients based on the first principle lattice quantum chromodynamics (LQCD) approach remains challenging [7]. On the other hand, in the relevant temperature regime of heavy ion collision experiments, the results from phenomenological modeling [8–10] suggest a significantly smaller specific shear viscosity and a significantly larger specific bulk viscosity compared to the

perturbative estimations [11–13]. There has been a conscientious effort to estimate the QGP transport using holographic QCD framework (see Refs. [4,14] for an overview) as well as effective approaches (see Refs. [15–31] and the references therein), which has enriched our understanding considerably. In the present work we consider the effective kinetic theory approach based on the recent formulation of quasiparticle hydrodynamics from the relativistic Boltzmann equation with a modified collision kernel [32,33].

The conventional Anderson and Witting relaxation time approximation (RTA) of the collision kernel [34] has been extremely successful in the order by order formulation of the dissipative hydrodynamics and has been widely used to derive the transport coefficients. However, if the energy (momentum) dependence of the relaxation time is taken into consideration, the microscopic conservation law in the conventional RTA is violated. Similar discrepancy arises if one considers a general matching condition. Recent formulations based on the modified collision kernel [32,33,35,36] have addressed this issue. In Ref. [37], the novel relaxation time approximation (NRTA) has been implemented to study the transport properties of QGP considering a single quasiparticle species with temperature dependent effective mass. The formulation is based on Maxwell-Boltzmann statistics. The quasiparticles in this formulation do not correspond to gluon or quark degrees of freedom. Rather, they correspond to an effective degree of freedom that satisfies the QCD thermodynamics. On the other hand, the formulation based on extended relaxation

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time approximation (ERTA) [33] analyzes the implications of the power law ansatz in the transport coefficients for both classical and quantum statistics. However, the ERTA formulation does not consider the quasiparticle picture. Thus, to implement either of the frameworks in case of $2 + 1$ flavor QCD, one needs a generalization, which we attempt in the present work.

The article is organized as follows: In Sec. II we first discuss the thermodynamic consistency of the single-species quasiparticle model and formulate the corresponding relativistic hydrodynamics from kinetic theory considering two types of collision kernels—(i) ERTA, and (ii) NRTA. Up to first-order in spacetime gradient, we show that both of the collision kernels result in the same out-of-equilibrium corrections to phase-space distribution and hence the transport coefficients. We obtain expressions for the frame and matching invariant transport coefficients from the entropy production. The results obtained with single quasiparticle species are generalized for the multispecies system in Sec. III with the ERTA collision kernel. The multi-species framework is then utilized in Sec. IV to explore the impact of the power law momentum dependence of the relaxation time on the shear and bulk viscous coefficient of $2 + 1$ flavor QCD medium with quasigluon and light and strange quark (antiquark) sectors. Finally, in Sec. V, we conclude with a brief summary and future outlook.

Notations and conventions: Throughout this article, natural units have been adopted, i.e., $k_B = \hbar = c = 1$. For metric, the mostly negative convention has been chosen, i.e., $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. A shorthand notation is introduced for the phase-space distribution function, $f(x, p)$, as $f_{\mathbf{p}}$ where p is the absolute value of the particle four-momentum, p^μ , i.e., $p^2 \equiv p^\mu p_\mu$ and x^μ is the spacetime coordinate. Any macroscopic and microscopic variable at equilibrium will carry a subscript of “0,” e.g., $f_{0\mathbf{p}}$, whereas the nonequilibrium corrections will be denoted by adding a “ δ ” in front of the variable, e.g., $\delta f_{\mathbf{p}}$. Three-vectors will be denoted by bold letters. The scalar product of two Lorentz-vectors is denoted by, $A \cdot B \equiv A^\mu B_\mu$ where Einstein’s summation convention is assumed.

II. SINGLE SPECIES QUASIPARTICLE

In this section, we will discuss in detail how to obtain the transport coefficients of a relativistic fluid with a single quasiparticle degree of freedom. We will first establish the thermodynamics and then go on to evaluate the transport coefficients from two newly proposed collision kernels, ERTA and NRTA. We will show that when truncated at a first order, both the collision kernels yield the same result. In the next section, using this, we will determine the transport coefficients of a relativistic fluid with multiple quasiparticle species.

A. Quasiparticle thermodynamics

To provide an effective kinetic theoretical description of the strongly interacting matter, we will consider a system with single quasiparticle species without any conserved charge. The effect of the interactions among the quasiparticles is modeled with a temperature-dependent effective mass, $m(T)$. Such an effective mass allows the incorporation of a desired equation of state in the hydrodynamic framework. However, it should be noted that for a system of quasiparticles with temperature dependent masses, the formulation of hydrodynamics with usual definition of the conserved currents can not be adopted directly as it destroys the thermodynamic consistency [15,38,39]. To restore the thermodynamic consistency, we covariantly introduce a Bag pressurelike term $B(T)$ ¹ into the energy-momentum tensor as,

$$T^{\mu\nu} = (\varepsilon_0 + \delta\varepsilon)u^\mu u^\nu - (P_0 + \delta P)\Delta^{\mu\nu} + 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu} \\ \equiv \langle p^\mu p^\nu \rangle + B(T)g^{\mu\nu}, \quad (1)$$

where ε_0 , P_0 are the equilibrium energy density and pressure, $\delta\varepsilon$, δP are the out-of-equilibrium corrections to energy density and pressure, and h^μ and $\pi^{\mu\nu}$ are the energy diffusion current and shear viscous pressure, respectively. The fluid four-velocity, u^μ is normalized to unity as $u \cdot u = 1$, and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projection operator orthogonal to u^μ . The Bag term, in general, can have contributions from equilibrium as well as nonequilibrium effects so that we can write $B(T) = B_0(T) + \delta B(T)$. Here the parentheses around the indices are defined as, $A^{(\mu}B^{\nu)} = (1/2)(A^\mu B^\nu + A^\nu B^\mu)$, whereas the angular brackets represent the momentum integral weighted with the nonequilibrium phase-space distribution function $f_{\mathbf{p}}$, defined by,

$$\langle \dots \rangle \equiv \int_{\mathbf{p}} (\dots) f_{\mathbf{p}}. \quad (2)$$

We have used a Lorentz invariant integral measure defined as [45,47],

$$\int_{\mathbf{p}} \equiv \int \frac{gd^3\mathbf{p}}{(2\pi)^3 E_{\mathbf{p}}} \quad (3)$$

where g is the degeneracy factor, and $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ is the particle energy in the local rest frame.

We want our system to abide by the thermodynamic relation, $s_0 = \beta(\varepsilon_0 + P_0)$, where s_0 is the equilibrium

¹It should be noted that there exists thermodynamically consistent alternative approaches (see for example [40–44]). However, in the present work, we follow the procedure introduced in Ref. [38], which has been widely used in the formulation of quasiparticle hydrodynamics [19,21,45–48].

entropy density and $\beta = 1/T$ is the inverse temperature. This requires us to impose the condition,

$$\partial_\mu B_0 = -\frac{1}{2}(\partial_\mu m^2)\langle 1 \rangle_0 \quad (4)$$

where we defined,

$$\langle \cdots \rangle_0 \equiv \int_{\mathbf{p}} (\cdots) f_{0\mathbf{p}}, \quad \langle \cdots \rangle_1 \equiv \int_{\mathbf{p}} (\cdots) f_{0\mathbf{p}} (1 - r f_{0\mathbf{p}}). \quad (5)$$

Here r can take values 0, +1, -1 for particles obeying Maxwell-Boltzmann (MB), Fermi-Dirac (FD), and Bose-Einstein (BE) statistics. The equilibrium phase-space distribution function has been defined as,

$$f_{0\mathbf{p}} = (e^{\beta E_{\mathbf{p}}} + r)^{-1}. \quad (6)$$

Additionally, we can ensure the conservation of the energy-momentum tensor by requiring,

$$\partial_\mu B = -\frac{1}{2}(\partial_\mu m^2)\langle 1 \rangle. \quad (7)$$

We can coincide the two conditions from Eqs. (4) and (7) by requiring,

$$\langle 1 \rangle = \langle 1 \rangle_0, \quad (8)$$

which can be treated as a matching condition, as we will see in later sections. This matching condition, together with Eqs. (4) and (7), provides $\partial_\mu \delta B(T) = 0$, which allows one to set $\delta B(T) = 0$.

B. Quasiparticle hydrodynamics and relativistic kinetic theory

Having fixed the thermodynamics, we focus on building the kinetic theory framework to be used for obtaining the transport properties of the system. We can define the hydrodynamic variables appearing in Eq. (1) as moments of the distribution function as,

$$\varepsilon_0 = \langle E_{\mathbf{p}}^2 \rangle_0 + B_0(T), \quad P_0 = -\langle (1/3)(p \cdot \Delta \cdot p) \rangle_0 - B_0(T), \quad (9a)$$

$$\delta\varepsilon = \langle E_{\mathbf{p}}^2 \phi_{\mathbf{p}} \rangle_1, \quad \delta P = -\langle (1/3)(p \cdot \Delta \cdot p) \phi_{\mathbf{p}} \rangle_1, \quad (9b)$$

$$h^\mu = \langle E_{\mathbf{p}} p^{(\mu)} \phi_{\mathbf{p}} \rangle_1, \quad \pi^{\mu\nu} = \langle p^{(\mu} p^{\nu)} \phi_{\mathbf{p}} \rangle_1, \quad (9c)$$

where $A^{(\mu} B^{\nu)} = \Delta_{\alpha\beta}^{\mu\nu} A^\alpha B^\beta$, with $\Delta_{\alpha\beta}^{\mu\nu} = \Delta_\alpha^{(\mu} \Delta_\beta^{\nu)} - (1/3)\Delta^{\mu\nu} \Delta_{\alpha\beta}$ being the rank-4 doubly symmetric, traceless projection operator. The hydrodynamic equations can be obtained from the conservation of the energy-momentum tensor, i.e., $\partial_\mu T^{\mu\nu} = 0$ by taking projection along and orthogonal to u_μ . These lead to,

$$D\varepsilon_0 + D\delta\varepsilon + (\varepsilon_0 + \delta\varepsilon + P_0 + \delta P)\theta - \pi^{\mu\nu} \sigma_{\mu\nu} + \partial_\mu h^\mu + u_\mu D h^\mu = 0, \quad (10a)$$

$$(\varepsilon_0 + \delta\varepsilon + P_0 + \delta P) D u^\mu - \nabla^\mu (P_0 + \delta P) + h^\mu \theta + h^\alpha \Delta^{\mu\nu} \partial_\alpha u_\nu + \Delta^{\mu\nu} D h_\nu + \Delta^{\mu\nu} \partial_\alpha \pi_\nu^\alpha = 0 \quad (10b)$$

where the spacetime derivative, “ ∂_μ ” has been split into two components as $\partial_\mu \equiv u_\mu D + \nabla_\mu$, with $D = u_\mu \partial^\mu$ being the comoving derivative and $\nabla_\mu = \Delta_\mu^\alpha \partial_\alpha$ being the orthogonal derivative. In the nonrelativistic limit/fluid rest frame, the former reduces to time derivative and the latter reduces to spatial derivatives. These two equations also contain the scalar expansion rate, $\theta \equiv \partial_\mu u^\mu$ and fluid stress tensor, $\sigma^{\mu\nu} = \partial^{(\mu} u^{\nu)}$.

Using the conservation of the energy-momentum tensor from Eq. (1) along with Eq. (7) we can arrive at the equation,

$$\int_{\mathbf{p}} p^\nu \left[(p \cdot \partial) f_{\mathbf{p}} + \frac{1}{2} (\partial_\mu m^2) \partial_{(p)}^\mu f_{\mathbf{p}} \right] = 0, \quad (11)$$

where $\partial_{(p)}^\mu \equiv \partial/\partial p_\mu$ is the derivative with respect to momentum four-vector. With the help of Eq. (11) we can write the relativistic Boltzmann equation as,

$$(p \cdot \partial) f_{\mathbf{p}} + \frac{1}{2} (\partial_\mu m^2) \partial_{(p)}^\mu f_{\mathbf{p}} = C[f_{\mathbf{p}}], \quad (12)$$

where $C[f_{\mathbf{p}}]$ is the collision kernel that carries the information of the interactions among the medium constituents. One can retrieve the conservation of the energy momentum tensor from the first moment of Eq. (12) provided $\int_{\mathbf{p}} p^\mu C[f] = 0$. While the RTA of the collision kernel as proposed by Anderson-Witting [34] is a widely used approach [47–52] it can not ensure a vanishing first moment of the collision kernel in an arbitrary frame with a momentum-dependent relaxation time. As already mentioned, successful formulation of relativistic hydrodynamics with momentum-dependent relaxation time has been achieved in recent times using modified collision kernels [32,33]. In the following, we will consider these modified frameworks to obtain the out-of-equilibrium corrections to the phase-space distribution function and the transport coefficients.

C. Extended relaxation time approximation

The collision kernel of ERTA is given by,

$$C_1[f_{\mathbf{p}}] = -(E_{\mathbf{p}}/\tau_R)(f_{\mathbf{p}} - f_{0\mathbf{p}}^*), \quad (13)$$

where $\tau_R = \tau_R(x, p)$ is the relaxation time defining the characteristic timescale in which the particle with four-momentum p^μ relaxes toward the equilibrium state

momentum. A common practice in the existing literature is to parametrize this relaxation time as, $\tau_R(x, p) = t_R(x)(\beta E_p)^\ell$, where ℓ is some exponent determined by the nature of the theory [33,53]. We also denote f_{0p}^* as the phase-space distribution function at thermodynamic equilibrium and it is given by,

$$f_{0p}^* = (e^{\beta^* \cdot p} + r)^{-1}, \quad (14)$$

for chargeless massive particles (for a detailed discussion on the difference between thermodynamic and hydrodynamic frames, see Ref. [33]). Here $\beta_\mu^* = u_\mu^*/T^*$ with u_μ^* and T^* being the fluid four-velocity (normalized to unity as $u^* \cdot u^* = 1$) and temperature at thermodynamic equilibrium. These thermodynamic variables can be related with the hydrodynamic ones as,

$$u_\mu^* \equiv u_\mu + \delta u_\mu, \quad T^* \equiv T + \delta T. \quad (15)$$

We can expand the thermodynamic equilibrium around the hydrodynamic equilibrium as, $f_{0p}^* = f_{0p} + \delta f_p^* = (1 + \phi_p^* \tilde{f}_{0p}) f_{0p}$, which means $\delta f_p^* = \phi_p^* f_{0p} \tilde{f}_{0p}$, with $\tilde{f}_{0p} \equiv 1 - r f_{0p}$. Here, r controls the statistics of the particle and takes the values, 0, +1, -1 corresponding to Maxwell-Boltzmann, Fermi-Dirac, and Bose-Einstein statistics, respectively. The nonequilibrium distribution function can also be expanded as, $f_p = f_{0p} + \delta f_p = (1 + \phi_p \tilde{f}_{0p}) f_{0p}$, which gives $\delta f_p = \phi_p f_{0p} \tilde{f}_{0p}$. Using

$$(p \cdot \partial) f_{0p} + \frac{1}{2} \partial_\mu m^2 (\partial_{(p)}^\mu f_{0p}) = (A_p \theta - \beta p^\mu p^\nu \sigma_{\mu\nu}) f_{0p} \tilde{f}_{0p}, \quad (16)$$

we can reexpress the Boltzmann equation as,

$$(A_p \theta - \beta p^\mu p^\nu \sigma_{\mu\nu}) f_{0p} \tilde{f}_{0p} = -(E_p / \tau_R) (\phi_p - \phi_p^*) f_{0p} \tilde{f}_{0p}, \quad (17)$$

where,

$$A_p = -\beta \left[\left(c_s^2 - \frac{1}{3} \right) E_p^2 + \frac{m^2}{3} + \frac{\beta c_s^2}{2} \frac{\partial m^2}{\partial \beta} \right]. \quad (18)$$

To obtain Eqs. (16) and (18), we have replaced the comoving derivatives in terms of other gradients using the relations

$$\dot{\beta} = \beta c_s^2 \theta, \quad u^\mu = -\nabla^\mu \ln \beta, \quad (19)$$

that follow from the hydrodynamic equations (10a) and (10b). Here, c_s^2 in Eq. (19) is the square of the speed of sound, given by,

$$c_s^2 = \frac{J_{31}}{J_{30} + \frac{\beta}{2} \frac{\partial m^2}{\partial \beta} J_{10}}. \quad (20)$$

In order to derive these relations, we have used two types of thermodynamic integrals defined as

$$I_{nq} \equiv \frac{(-1)^q}{(2q+1)!!} \langle E_p^{n-2q} (p \cdot \Delta \cdot p)^q \rangle_0, \quad (21a)$$

$$J_{nq} \equiv \frac{(-1)^q}{(2q+1)!!} \langle E_p^{n-2q} (p \cdot \Delta \cdot p)^q \rangle_1. \quad (21b)$$

For further details on the properties of these momentum integrals see Appendix. Keeping terms up to first-order in spacetime gradient we write,

$$\phi_p^* = \beta [-p_\mu \delta u^\mu + \beta E_p \delta T]. \quad (22)$$

To determine ϕ_p , we expand it using a set of orthogonal polynomials, $P_n^{(l)}$ as [54],

$$\phi_p = \sum_{n,l=0}^{\infty} \Phi_n^{(\mu_1 \dots \mu_l)} p_{\langle \mu_1} \dots p_{\mu_l \rangle} P_n^{(l)}, \quad (23)$$

where $p^{\langle \mu_1} \dots p^{\mu_l \rangle} = \Delta_{\alpha_1 \dots \alpha_l}^{\mu_1 \dots \mu_l} p^{\alpha_1} \dots p^{\alpha_l}$ are the irreducible tensors for $l = 0, 1, \dots$, with $\Delta_{\alpha_1 \dots \alpha_l}^{\mu_1 \dots \mu_l}$ defined in Ref. [55]. The label “ l ” corresponds to the rank of the term under consideration, e.g., $l = 0, 1, 2, \dots$ correspond to scalars, vectors, rank-2 tensors, and so on. The label “ n ” corresponds to the n -th term of some rank- ℓ quantity. The polynomials $P_n^{(l)}$ can be determined from the Gram-Schmidt orthogonalization process. In the present case, we have chosen $P_0^{(0)} = \beta E_p$ and $P_0^{(l \geq 1)} = 1$. Apparently it seems straightforward to obtain the coefficients, $\Phi_n^{(\mu_1 \dots \mu_l)}$ by solving the Boltzmann equation given in Eq. (17). However, it should be noted that the complete determination of $\Phi_n^{(\mu_1 \dots \mu_l)}$ will require the knowledge of ϕ_p^* given in Eq. (22). At this point we need to introduce the matching and frame conditions. The matching and frame conditions, respectively, fix the definitions of temperature and fluid four-velocity, which essentially determines δu^μ and δT given in Eq. (22) and hence completely specifies ϕ_p^* . In this work we will use the following matching and frame conditions,

$$\langle g_p \phi_p \rangle_1 = 0, \quad \langle g_p p^{(\mu} \phi_p \rangle_1 = 0, \quad (24)$$

which gives us,²

²In Eq. (25) we have used the general matching condition but for the present work, we will set $g_p = 1$.

$$\delta T = \frac{\langle g_{\mathbf{p}} \tau_{\mathbf{R}} A_{\mathbf{p}} / E_{\mathbf{p}} \rangle_1 \theta}{\beta^2 \langle g_{\mathbf{p}} E_{\mathbf{p}} \rangle_1}, \quad \delta u^\mu = 0. \quad (25)$$

We can use these in the Boltzmann equation, (17) to determine $\Phi_n^{(\mu_1 \dots \mu_\ell)}$ of l -th rank, by multiplying both sides of Eq. (17) by $P_q^{(l)} p^{\mu_1} \dots p^{\mu_\ell}$ and integrating over momentum with the measure defined in Eq. (3). This allows us to write,³

$$\Phi_n = \frac{1}{A_n^{(0)}} [\beta^2 \delta T \langle P_n^{(0)} E_{\mathbf{p}}^2 / \tau_{\mathbf{R}} \rangle_1 - \langle A_{\mathbf{p}} P_n^{(0)} \rangle_1 \theta], \quad \text{for } n \geq 0 \quad (26a)$$

$$\Phi_n^{(\mu_1)} = 0, \quad \text{for } n \geq 0 \quad (26b)$$

$$\Phi_n^{(\mu_1 \mu_2)} = \frac{2\beta}{15A_n^{(2)}} \langle (p \cdot \Delta \cdot p)^2 P_n^{(2)} \rangle_1 \sigma^{\mu_1 \mu_2}, \quad \text{for } n \geq 0 \quad (26c)$$

$$\Phi_n^{(\mu_1 \dots \mu_\ell)} = 0, \quad \text{for } n \geq 0, \ell \geq 3. \quad (26d)$$

To obtain the above result, we have used the orthogonality property of $P_n^{(l)}$,

$$\frac{l!}{(2l+1)!!} \langle (E_{\mathbf{p}} / \tau_{\mathbf{R}}) (p \cdot \Delta \cdot p)^l P_n^{(l)} P_q^{(l)} \rangle_1 = A_n^{(l)} \delta_{nq}, \quad (27)$$

where $A_n^{(l)}$ is a normalization constant and we will treat Eq. (27) as its definition after setting $n = q$. The orthogonal polynomials allow us to express any function of phase-space variables, $\mathcal{G}_{\mathbf{p}}$ as,

$$\mathcal{G}_{\mathbf{p}} = \sum_{n=0}^{\infty} \frac{1}{A_n^{(l)}} \frac{l!}{(2l+1)!!} \langle (E_{\mathbf{p}} / \tau_{\mathbf{R}}) \mathcal{G}_{\mathbf{p}} P_n^{(l)} (p \cdot \Delta \cdot p)^l \rangle_1 P_n^{(l)}. \quad (28)$$

Using this property we can substitute Eqs. (26a)–(26d) into Eq. (23) to obtain,

$$\phi_{\mathbf{p}} = \beta^2 E_{\mathbf{p}} \delta T - \left(\frac{\tau_{\mathbf{R}} A_{\mathbf{p}}}{E_{\mathbf{p}}} \right) \theta + \left(\frac{\beta \tau_{\mathbf{R}}}{E_{\mathbf{p}}} \right) p_{\langle \alpha} p_{\beta \rangle} \sigma^{\alpha \beta}. \quad (29)$$

With this, we can determine the out-of-equilibrium correction to distribution, $\delta f_{\mathbf{p}}$ for ERTA and hence the transport coefficients. Our next task is to determine the same for the case of NRTA.

³Note that we obtained $\Phi_n^{(\alpha)} = 0$ in Eq. (26b) as the system under consideration consists of quasiparticles of zero chemical potential, giving us $\delta u^\mu = 0$. Furthermore, it is possible to show $\Phi_0 = \beta \delta T$ and $\Phi_0^{(\mu)} = -\beta \delta u^\mu$. This fact indicates that $\phi_{\mathbf{p}}^*$ is related to the homogeneous part of the solution to the Boltzmann equation.

D. Novel relaxation time approximation

In this section, we use a collision kernel inspired from Ref. [37], which conserves the energy-momentum tensor by construction. We have rederived the collision kernel of NRTA to be suitable for quantum statistics. The linear collision kernel⁴ is found to be,

$$C_2[f_{\mathbf{p}}] = -\frac{E_{\mathbf{p}}}{\tau_{\mathbf{R}}} \left[\phi_{\mathbf{p}} - \frac{\langle (E_{\mathbf{p}}^2 / \tau_{\mathbf{R}}) \phi_{\mathbf{p}} \rangle_1 E_{\mathbf{p}}}{\langle (E_{\mathbf{p}}^3 / \tau_{\mathbf{R}}) \rangle_1} - \frac{\langle (E_{\mathbf{p}} / \tau_{\mathbf{R}}) \phi_{\mathbf{p}} p^{(\mu)} \rangle_1 P_{\langle \mu \rangle}}{\langle (1/3) (E_{\mathbf{p}} / \tau_{\mathbf{R}}) (p \cdot \Delta \cdot p) \rangle_1} \right] f_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}}. \quad (30)$$

Similar to Sec. II C, to solve for $\phi_{\mathbf{p}}$, we will expand $\phi_{\mathbf{p}}$ according to Eq. (23). Substituting Eq. (30) in Eq. (12) we can write the Boltzmann equation under NRTA with the help of Eq. (16) as,

$$\begin{aligned} & (A_{\mathbf{p}} \theta - \beta p^\mu p^\nu \sigma_{\mu\nu}) f_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}} \\ &= -\frac{E_{\mathbf{p}}}{\tau_{\mathbf{R}}} \left[\phi_{\mathbf{p}} - \frac{\langle (E_{\mathbf{p}}^2 / \tau_{\mathbf{R}}) \phi_{\mathbf{p}} \rangle_1 E_{\mathbf{p}}}{\langle (E_{\mathbf{p}}^3 / \tau_{\mathbf{R}}) \rangle_1} - \frac{\langle (E_{\mathbf{p}} / \tau_{\mathbf{R}}) \phi_{\mathbf{p}} p^{(\mu)} \rangle_1 P_{\langle \mu \rangle}}{\langle (1/3) (E_{\mathbf{p}} / \tau_{\mathbf{R}}) (p \cdot \Delta \cdot p) \rangle_1} \right] f_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}} \end{aligned} \quad (31)$$

where $A_{\mathbf{p}}$ is defined in Eq. (18). Upon substitution of Eq. (23), one finds Φ_0 and $\Phi_0^{(\mu)}$ gets removed from the equation by virtue of the second and third terms in Eq. (31). Therefore, they cannot be determined from the Boltzmann equation, rather we have to use the matching and frame conditions. Thus, using Eq. (23) in Eq. (24) and setting $g_{\mathbf{p}} = 1$ we find,

$$\Phi_0 = -\sum_{n=1}^{\infty} \Phi_n \frac{\langle P_n^{(0)} \rangle_1}{\langle P_0^{(0)} \rangle_1}, \quad (32a)$$

$$\Phi_0^{(\mu)} = 0. \quad (32b)$$

Thus, Eqs. (32a) and (32b) describe the $n = 0$ elements of $\Phi_n^{(\mu_1 \dots \mu_\ell)}$ for $\ell = 0, 1$, i.e., the homogeneous part of the solution of Eq. (31) (similar to $\phi_{\mathbf{p}}^*$ in the case of ERTA). To determine rest of the $\Phi_n^{(\mu_1 \dots \mu_\ell)}$, we substitute Eq. (23) into Eq. (31) to obtain,

⁴Note that the linearized collision kernel can be related to a linear collision operator, $\hat{L}$ such that, $C_2[f_{\mathbf{p}}] = f_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}} \hat{L} \phi_{\mathbf{p}}$. In the case of uncharged particles, $\hat{L}$ can have four degenerate eigenmodes with zero eigenvalues. Therefore, we may approximate this as,

$$\hat{L} \propto \mathbb{1} - \sum_{n=0}^3 |\lambda_n\rangle \langle \lambda_n|.$$

$$\Phi_n = -\frac{\langle A_{\mathbf{p}} P_n^{(0)} \rangle_1 \theta}{A_n^{(0)}}, \quad \text{for } n \geq 1 \quad (33a)$$

$$\Phi_n^{(\mu)} = 0, \quad \text{for } n \geq 1 \quad (33b)$$

$$\Phi_n^{(\mu\nu)} = \frac{2\beta}{15A_n^{(2)}} \langle (p \cdot \Delta \cdot p)^2 P_n^{(2)} \rangle_1 \sigma^{\mu\nu}, \quad \text{for } n \geq 0 \quad (33c)$$

$$\Phi_n^{(\mu_1 \dots \mu_l)} = 0, \quad \text{for } n \geq 0, l \geq 3. \quad (33d)$$

This fixes all the unknown coefficients $\Phi_n^{(\mu_1 \dots \mu_l)}$ and hence, we are ready to write down the expression of $\phi_{\mathbf{p}}$. Examining Eqs. (32a)–(33d) we can conclude that $\phi_{\mathbf{p}}$ can have the following structure,

$$\phi_{\mathbf{p}} = F_{\mathbf{p}}^{(0)} \theta + F_{\mathbf{p}}^{(2)} p^{(\mu} p^{\nu)} \sigma_{\mu\nu}, \quad (34)$$

where $F_{\mathbf{p}}^{(0)}$ and $F_{\mathbf{p}}^{(2)}$ are the coefficients of scalar and tensor structures. Recalling Eqs. (23) and (32a)–(33d) we can write,

$$F_{\mathbf{p}}^{(0)} = \sum_{n=0}^{\infty} \frac{\langle A_{\mathbf{p}} P_n^{(0)} \rangle_1}{A_n^{(0)}} \left(\frac{\langle P_n^{(0)} \rangle_1}{\langle P_0^{(0)} \rangle_1} P_0^{(0)} - P_n^{(0)} \right), \quad (35a)$$

$$F_{\mathbf{p}}^{(2)} = \sum_{n=0}^{\infty} \frac{2\beta}{15A_n^{(2)}} \langle (p \cdot \Delta \cdot p)^2 P_n^{(2)} \rangle_1 P_n^{(2)}, \quad (35b)$$

where we note the $n=0$ term on the right-hand side of Eq. (35a) is zero. To simplify these expressions, as in the case of ERTA, we will use Eq. (28). By choosing $\mathcal{G}_{\mathbf{p}} = A_{\mathbf{p}} \tau_{\mathbf{R}} / (u \cdot p)$ for $l=0$ we can write, $A_{\mathbf{p}} \tau_{\mathbf{R}} / (u \cdot p) = \sum_{n=0}^{\infty} \frac{1}{A_n^{(0)}} \langle A_{\mathbf{p}} P_n^{(0)} \rangle_1 P_n^{(0)}$ and consequently, $\langle A_{\mathbf{p}} \tau_{\mathbf{R}} / E_{\mathbf{p}} \rangle_1 = \sum_{n=0}^{\infty} \langle A_{\mathbf{p}} P_n^{(0)} \rangle_1 \langle P_n^{(0)} \rangle_1 / A_n^{(0)}$. Therefore we can write $F_{\mathbf{p}}^{(0)}$ as,

$$F_{\mathbf{p}}^{(0)} = \left\langle \left(\frac{A_{\mathbf{p}} \tau_{\mathbf{R}}}{E_{\mathbf{p}}} \right) \right\rangle_1 \frac{E_{\mathbf{p}}}{J_{10}} - \left(\frac{A_{\mathbf{p}} \tau_{\mathbf{R}}}{E_{\mathbf{p}}} \right). \quad (36)$$

And, similarly, choosing $\mathcal{G}_{\mathbf{p}} = \tau_{\mathbf{R}} / (u \cdot p)$ with $l=2$, we can write $F_{\mathbf{p}}^{(2)}$ as,

$$F_{\mathbf{p}}^{(2)} = \left(\frac{\beta \tau_{\mathbf{R}}}{E_{\mathbf{p}}} \right). \quad (37)$$

Thus, we can use Eqs. (36) and (37) in Eq. (34) to obtain the solution of the Boltzmann equation with the collision kernel of Eq. (30) as,

$$\phi_{\mathbf{p}} = \left\{ \left\langle \frac{A_{\mathbf{p}} \tau_{\mathbf{R}}}{(u \cdot p)} \right\rangle_1 \frac{(u \cdot p)}{J_{10}} - \frac{A_{\mathbf{p}} \tau_{\mathbf{R}}}{(u \cdot p)} \right\} \theta + \frac{\beta \tau_{\mathbf{R}}}{(u \cdot p)} p^{(\mu} p^{\nu)} \sigma_{\mu\nu}. \quad (38)$$

Interestingly, we note that the phase-space correction, $\phi_{\mathbf{p}}$ for the case of ERTA with the matching condition $g_{\mathbf{p}} = 1$ in (24) gives us exactly Eq. (38) by using δT from Eq. (25). In the next part, we will use this information to obtain the transport coefficients.

E. Transport coefficients

In the first-order theory of uncharged fluids, the dissipative currents are given by Navier-Stokes equations as

$$\delta \varepsilon = e \theta, \quad \delta P = \rho \theta, \quad h^{\mu} = \kappa \nabla^{\mu} \alpha, \quad \pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}, \quad (39)$$

where e , ρ , κ , and η are the transport coefficients, although only two frame independent transport coefficients exist that give rise to dissipation. To determine the transport coefficients appearing in Eq. (39), we substitute Eq. (34) using (36) and (37) into Eqs. (9b) and (9c) and compare with Eq. (39) to find,

$$e = \langle (A_{\mathbf{p}} \tau_{\mathbf{R}} / E_{\mathbf{p}}) \rangle_1 (J_{30} / J_{10}) - \langle A_{\mathbf{p}} \tau_{\mathbf{R}} E_{\mathbf{p}} \rangle_1, \quad (40a)$$

$$\rho = \langle (A_{\mathbf{p}} \tau_{\mathbf{R}} / E_{\mathbf{p}}) \rangle_1 (J_{31} / J_{10}) + \langle (1/3) (A_{\mathbf{p}} \tau_{\mathbf{R}} / E_{\mathbf{p}}) (p \cdot \Delta \cdot p) \rangle_1, \quad (40b)$$

$$\kappa = 0, \quad (40c)$$

$$\eta = \frac{\beta}{15} \langle (p \cdot \Delta \cdot p)^2 (\tau_{\mathbf{R}} / E_{\mathbf{p}}) \rangle_1. \quad (40d)$$

The fact that here we have, $\kappa = 0$ can be understood from the absence of any vector gradient in the expressions of $\phi_{\mathbf{p}}$, i.e., Eqs. (29) and (38). To determine the matching and frame invariant transport coefficients, we employ the second law of thermodynamics, i.e., we will determine the divergence of entropy four-current. From Boltzmann's H theorem we can write the entropy four-current of particles obeying quantum statistics to be given by,

$$S^{\mu} = - \int_{\mathbf{p}} p^{\mu} (f_{\mathbf{p}} \ln f_{\mathbf{p}} + r \tilde{f}_{\mathbf{p}} \ln \tilde{f}_{\mathbf{p}}). \quad (41)$$

Taking the four-divergence of Eq. (41) and keeping terms up to quadratic order in spacetime gradient we can write,

$$\partial_{\mu} S^{\mu} = - \int_{\mathbf{p}} C[f_{\mathbf{p}}] \phi_{\mathbf{p}}. \quad (42)$$

Now we substitute the expression of $\phi_{\mathbf{p}}$ from Eq. (23) and for the collision kernel, we use both ERTA and NRTA from Eqs. (13) and (30). Using appropriate expressions of $\Phi_n^{(\mu_1 \dots \mu_l)}$ for both the cases, we arrive at,

$$\begin{aligned} \partial_\mu S^\mu &= \sum_{n=1}^{\infty} \Phi_n^2 A_n^{(0)} + \sum_{n=1}^{\infty} \Phi_n^{(\mu)} \Phi_{n,(\mu)} A_n^{(1)} \\ &+ \sum_{n=0}^{\infty} \Phi_n^{(\mu_1 \mu_2)} \Phi_{n,(\mu_1 \mu_2)} A_n^{(2)} \\ &+ \sum_{l=3}^{\infty} \sum_{n=0}^{\infty} \Phi_n^{(\mu_1 \dots \mu_l)} \Phi_{n,(\mu_1 \dots \mu_l)} A_n^{(l)}. \end{aligned} \quad (43)$$

Following the same argument as in Ref. [37], we may conclude that the right-hand side of Eq. (43) is positive. Then, recalling the relation (28) and using that in Eq. (43) along with Eqs. (26a)–(26d) and (33a)–(33d) we can write,

$$\partial_\mu S^\mu = \beta \zeta_s \theta^2 + 2\beta \eta_s \sigma^{\mu\nu} \sigma_{\mu\nu} \quad (44)$$

where,

$$\zeta_s = \frac{1}{\beta} \langle (\tau_R/E_{\mathbf{p}}) A_{\mathbf{p}}^2 \rangle_1, \quad (45a)$$

$$\eta_s = \frac{\beta}{15} \langle (\tau_R/E_{\mathbf{p}}) (p \cdot \Delta \cdot p)^2 \rangle_1, \quad (45b)$$

are the transport coefficients producing entropy and are frame invariant.

III. MULTISPECIES QUASIPARTICLE

In the previous section, we have established that up to first-order in spacetime gradients, the two modified collision kernels of ERTA and NRTA, produce identical results. Therefore, in the present section, we will only work with the ERTA framework due to its relatively simpler structure and discuss the scenario with multiple quasiparticle species.

A. Quasiparticle thermodynamics

We consider a system with “ N ” number of quasiparticle species. We denote the phase-space distribution function of the j -th species by $f_{\mathbf{p}}^{(j)} = f_{0\mathbf{p}}^{(j)} + \delta f_{\mathbf{p}}^{(j)}$, where $f_{0\mathbf{p}}^{(j)}$ corresponds to the equilibrium part and $\delta f_{\mathbf{p}}^{(j)} = \phi_{\mathbf{p}}^{(j)} f_{0\mathbf{p}}^{(j)} \tilde{f}_{0\mathbf{p}}^{(j)}$ represents the out-of-equilibrium correction where $\tilde{f}_{0\mathbf{p}}^{(j)} = 1 - r_j f_{0\mathbf{p}}^{(j)}$. The parameter r_j controls the statistics of the j -th particle species and takes the discrete values, 0, +1, -1 corresponding to Maxwell-Boltzmann, Fermi-Dirac, and Bose-Einstein statistics, respectively. Then we may define the brackets for multispecies as,

$$\begin{aligned} \langle (\dots) \rangle^{(j)} &\equiv \int_{\mathbf{p}_j} (\dots) f_{\mathbf{p}}^{(j)}, & \langle (\dots) \rangle_0^{(j)} &\equiv \int_{\mathbf{p}_j} (\dots) f_{0\mathbf{p}}^{(j)}, \\ \langle (\dots) \rangle_1^{(j)} &\equiv \int_{\mathbf{p}_j} (\dots) f_{0\mathbf{p}}^{(j)} (1 - r_j f_{0\mathbf{p}}^{(j)}) \end{aligned} \quad (46)$$

where the integral is defined as,

$$\int_{\mathbf{p}_j} \equiv \int \frac{g_j d^3 \mathbf{p}_j}{(2\pi)^3 E_{\mathbf{p}_j}}, \quad (47)$$

where g_j is the degeneracy factor of j -th species of particle, and $E_{\mathbf{p}_j} = \sqrt{\mathbf{p}_j^2 + m_j^2}$ is the particle energy of j -th species in the local rest frame with m_j being the temperature-dependent mass of the j -th particle. From this point onward, we will drop the subscript, “ j ” in momenta [except in Eq. (54)] and its species dependence can be understood from the context. While the tensor decomposition of the energy-momentum tensor remains the same as Eq. (1), the kinetic theory definition is modified to,

$$T^{\mu\nu} = \sum_{j=1}^N \langle p^\mu p^\nu \rangle^{(j)} + B(T) g^{\mu\nu}, \quad (48)$$

where thermodynamic consistency requires the bag pressure to satisfy the relation,

$$(\partial^\nu B_0) = -\frac{1}{2} \sum_j (\partial^\nu m_j^2) \langle 1 \rangle_0^{(j)}. \quad (49)$$

In this case of multispecies quasiparticles, we demand the following matching condition,

$$\sum_j (\partial^\nu m_j^2) \langle 1 \rangle^{(j)} = \sum_j (\partial^\nu m_j^2) \langle 1 \rangle_0^{(j)}. \quad (50)$$

Similar to the single-species quasiparticle case, such a matching condition can be used to show that the out-of-equilibrium part of the Bag pressure vanishes.

B. Quasiparticle hydrodynamics and relativistic kinetic theory

Consequently, the components of $T^{\mu\nu}$ are given by,

$$\begin{aligned} \epsilon_0 &= \sum_{j=1}^N \langle E_{\mathbf{p}}^2 \rangle_0^{(j)} + B_0(T), \\ P_0 &= -\sum_{j=1}^N \langle (1/3) (p \cdot \Delta \cdot p) \rangle_0^{(j)} - B_0(T), \end{aligned} \quad (51a)$$

$$\delta \epsilon = \sum_{j=1}^N \langle E_{\mathbf{p}}^2 \phi_{\mathbf{p}}^{(j)} \rangle_1^{(j)}, \quad \delta P = -\sum_{j=1}^N \langle (1/3) (p \cdot \Delta \cdot p) \phi_{\mathbf{p}}^{(j)} \rangle_1^{(j)}, \quad (51b)$$

$$h^\mu = \sum_{j=1}^N \langle E_{\mathbf{p}} p^{(\mu} \phi_{\mathbf{p}}^{(j)} \rangle_1^{(j)}, \quad \pi^{\mu\nu} = \sum_{j=1}^N \langle p^{(\mu} p^{\nu)} \phi_{\mathbf{p}}^{(j)} \rangle_1^{(j)}, \quad (51c)$$

where the correction to the phase-space distribution function of the j -th species, i.e., $\phi_{\mathbf{p}}^{(j)}$ can be determined from the Boltzmann's equation for j -th species given by,

$$p^\mu \partial_\mu f_{\mathbf{p}}^{(j)} + \frac{1}{2} (\partial_\mu m_j^2) (\partial_{(p)}^\mu f_{\mathbf{p}}^{(j)}) = -(E_{\mathbf{p}}/\tau_R^{(j)}) (f_{\mathbf{p}}^{(j)} - f_{0\mathbf{p}}^{(j)*}), \quad (52)$$

where we have used the extended relaxation time approximation for the collision kernel.⁵ Here we have defined,

$$f_{0\mathbf{p}}^{(j)*} = (e^{\beta^* \cdot p} + r_j)^{-1}, \quad (53)$$

as the equilibrium distribution function in terms of thermodynamic variables (starred ones). The unstarred quantities correspond to hydrodynamic variables. As in Refs. [33,35] we can Taylor expand each of the “ N ” types of distribution functions (starred) as, $f_{0\mathbf{p}}^{(j)*} = f_{0\mathbf{p}}^{(j)} + \delta f_{\mathbf{p}}^{(j)*}$. Expressing the correction part as, $\delta f_{\mathbf{p}}^{(j)*} = \phi_{\mathbf{p}}^{(j)*} f_{0\mathbf{p}}^{(j)}$ the Taylor expansion mentioned in the previous line allows us to write,

$$\phi_{\mathbf{p}}^{(j)*} = \beta [-p_j^\mu \delta u_\mu + \beta E_{\mathbf{p}_j} \delta T]. \quad (54)$$

Recalling the relation, $\delta f_{\mathbf{p}}^{(j)} = \phi_{\mathbf{p}}^{(j)} f_{0\mathbf{p}}^{(j)}$ we can rewrite the Boltzmann's equation up to first-order in spacetime gradient as,

$$\begin{aligned} (A_{\mathbf{p}}^{(j)} \theta - \beta p^\mu p^\nu \sigma_{\mu\nu}) f_{0\mathbf{p}}^{(j)} \tilde{f}_{0\mathbf{p}}^{(j)} \\ = -(E_{\mathbf{p}}/\tau_R) (\phi_{\mathbf{p}}^{(j)} - \phi_{\mathbf{p}}^{(j)*}) f_{0\mathbf{p}}^{(j)} \tilde{f}_{0\mathbf{p}}^{(j)}, \end{aligned} \quad (55)$$

where,

$$A_{\mathbf{p}}^{(j)} = -\beta \left\{ E_{\mathbf{p}}^2 \left(c_s^2 - \frac{1}{3} \right) + \frac{m_j^2}{3} + \frac{\beta c_s^2}{2} \left(\frac{\partial m_j^2}{\partial \beta} \right) \right\}. \quad (56)$$

The expression for the speed of sound square of the “ N ” species fluid is modified to,

$$c_s^2 = \frac{\sum_{j=1}^N J_{31}^{(j)}}{\sum_{j=1}^N \left[\frac{\beta}{2} \left(\frac{\partial m_j^2}{\partial \beta} \right) J_{10}^{(j)} + J_{30}^{(j)} \right]}, \quad (57)$$

where the thermodynamic integral for the j -th species is defined as,

⁵In principle, the relaxation time of the j -th particle species, $\tau_R^{(j)}$ should be different for each species. However, following the arguments in Ref. [56] it may be shown that we end up having $\tau_R^{(j)} = \tau_R$ for all j species and this is used from this point onward. The study of the more general case where the relaxation times of each species can be different is an interesting topic and is left for future investigations.

$$I_{nq}^{(j)} \equiv \frac{(-1)^q}{(2q+1)!!} \langle E_{\mathbf{p}}^{n-2q} (p \cdot \Delta \cdot p)^q \rangle_0^{(j)}, \quad (58a)$$

$$J_{nq}^{(j)} \equiv \frac{(-1)^q}{(2q+1)!!} \langle E_{\mathbf{p}}^{n-2q} (p \cdot \Delta \cdot p)^q \rangle_1^{(j)}. \quad (58b)$$

We can solve the counter term, $\phi_{\mathbf{p}}^{(j)*}$ with the help of frame and matching conditions, which in the present case is assumed to be given by,

$$\sum_j \langle g_{\mathbf{p}} \phi_{1\mathbf{p}}^{(j)} \rangle_1^{(j)} = 0, \quad \sum_j \langle q_{\mathbf{p}} p^{(\mu)} \phi_{1\mathbf{p}}^{(j)} \rangle_1^{(j)} = 0. \quad (59)$$

Using Eqs. (54)–(56) and defining $\delta T \equiv C_2 \theta$, we obtain,⁶

$$\delta u^\mu = 0, \quad C_2 \equiv \frac{\sum_j \langle (\tau_R/E_{\mathbf{p}}) g_{\mathbf{p}} A_{\mathbf{p}}^{(j)} \rangle_1^{(j)}}{\beta^2 \sum_j \langle g_{\mathbf{p}} E_{\mathbf{p}} \rangle_1^{(j)}}, \quad (60)$$

where we used the fact that due to the normalization of fluid four-velocity, $u \cdot \delta u = \mathcal{O}(\partial^2)$. We are now ready to evaluate the solution, $\phi_{\mathbf{p}}^{(j)}$. For this, we consider the expansion,

$$\phi_{\mathbf{p}}^{(j)} = \sum_{n,l=0}^{\infty} \Phi_{j,n}^{(\mu_1 \dots \mu_l)} p_{(\mu_1} \dots p_{\mu_l)} P_n^{(j,l)}, \quad (61)$$

where, $P_n^{(j,l)}$ is the n -th order orthogonal polynomial associated with the l -rank irreducible tensor for j -th species. These polynomials satisfy the orthogonality condition,

$$\frac{l!}{(2l+1)!!} \langle (E_{\mathbf{p}}/\tau_R) (p \cdot \Delta \cdot p)^l P_n^{(j,l)} P_q^{(j,l)} \rangle_1^{(j)} = A_n^{(j,l)} \delta_{nq}, \quad (62)$$

where the expression of $A_n^{(j,l)}$ can be obtained by substituting $q \rightarrow n$. As in the case of single species in Sec. II, we define $P_0^{(j,0)} = \beta E_{\mathbf{p}}$ and $P_0^{(j,l \geq 1)} = 1$ for all “ N ” types of species. Substituting $\phi_{\mathbf{p}}^{(j)}$ from Eq. (61) in the Boltzmann equation of j -th species in Eq. (55) and taking moments with appropriate weights of the form $P_q^{(j,l)} p^{(\nu_1} \dots p^{\nu_l)}$ along with the orthogonality property of the polynomials, $P_q^{(j,l)}$ we can write,

$$\begin{aligned} \Phi_{j,n} = \frac{1}{A_{j,n}^{(0)}} [\beta^2 C_2 \langle (E_{\mathbf{p}}^2/\tau_R) P_n^{(j,0)} \rangle_1^{(j)} \\ - \langle A_{\mathbf{p}}^{(j)} P_n^{(j,0)} \rangle_1^{(j)}] \theta \quad \text{for } n \geq 0. \end{aligned} \quad (63a)$$

$$\Phi_{j,n}^{(\mu_1)} = 0 \quad \text{for } n \geq 0, \quad (63b)$$

⁶The subscript “2” in C_2 is used to be consistent with earlier works [33,57].

$$\Phi_{j,n}^{\langle\mu_1\mu_2\rangle} = \frac{\beta\sigma^{\mu_1\mu_2}}{A_n^{(j,2)}} \langle(2/15)P_n^{(j,2)}(p \cdot \Delta \cdot p)^2\rangle_1^{(j)} \quad \text{for } n \geq 0, \quad (63c)$$

$$\Phi_{j,n}^{\langle\mu_1 \dots \mu_l\rangle} = 0 \quad \text{for } n \geq 0, l \geq 3. \quad (63d)$$

One may further notice that, due to the relation $\langle A_{\mathbf{p}}^{(j)} P_n^{(j,0)} \rangle_1^{(j)} = 0$, it is possible to prove $\Phi_{(j),0} P_0^{(j,0)} = \phi_{1\mathbf{p}}^{*(j)}$. Similarly to Eq. (28) we can write for the j -th species,

$$\mathcal{G}_{\mathbf{p}}^{(j)} = \sum_{n=0}^{\infty} \frac{1}{A_n^{(j,l)}} \frac{l!}{(2l+1)!!} \langle (E_{\mathbf{p}}/\tau_R) \mathcal{G}_{\mathbf{p}}^{(j)} P_n^{(j,l)} (p \cdot \Delta \cdot p)^l \rangle_1 P_n^{(j,l)}, \quad (64)$$

allowing us to rewrite $\phi_{\mathbf{p}}^{(j)}$ by substituting Eqs. (63a)–(63d) in Eq. (61) as,

$$\phi_{\mathbf{p}}^{(j)} = \beta^2 E_{\mathbf{p}} \delta T - \left(\frac{\tau_R A_{\mathbf{p}}^{(j)}}{E_{\mathbf{p}}} \right) \theta + \left(\frac{\beta \tau_R}{E_{\mathbf{p}}} \right) p_{\langle \alpha} p_{\beta \rangle} \sigma^{\alpha\beta}. \quad (65)$$

C. Transport coefficients

Considering the matching condition, $g_{\mathbf{p}} = 1$, and using Eq. (65) in Eqs. (51b) and (51c) along with Eq. (39) we can write down the transport coefficients as,

$$e = \sum_{j=1}^N \left[\langle (\tau_R/E_{\mathbf{p}}) A_{\mathbf{p}}^{(j)} \rangle_1^{(j)} \left(\frac{\sum_{j=1}^N J_{30}^{(j)}}{\sum_{j=1}^N J_{10}^{(j)}} \right) - \langle \tau_R E_{\mathbf{p}} A_{\mathbf{p}}^{(j)} \rangle_1^{(j)} \right], \quad (66a)$$

$$\rho = - \sum_{j=1}^N \left[\langle (\tau_R/E_{\mathbf{p}}) A_{\mathbf{p}}^{(j)} \rangle_1^{(j)} \left(\frac{\sum_{j=1}^N J_{31}^{(j)}}{\sum_{j=1}^N J_{10}^{(j)}} \right) - \langle (1/3) (\tau_R/E_{\mathbf{p}}) (p \cdot \Delta \cdot p) A_{\mathbf{p}}^{(j)} \rangle_1^{(j)} \right], \quad (66b)$$

$$\kappa = 0, \quad (66c)$$

$$\eta = \frac{\beta}{15} \sum_{j=1}^N \langle (p \cdot \Delta \cdot p)^2 (\tau_R/E_{\mathbf{p}}) \rangle_1^{(j)}. \quad (66d)$$

The frame invariant transport coefficients are determined through the entropy production, which for N -particle species is given by,

$$\partial_{\mu} S_1^{\mu} = - \sum_{j=1}^N \int_{\mathbf{p}} C[f_{\mathbf{p}}^{(j)}] \phi_{\mathbf{p}}^{(j)}, \quad (67a)$$

$$\begin{aligned} &= \sum_{j=1}^N \left[\sum_{n=1}^{\infty} \Phi_{j,n}^2 A_n^{(j,0)} + \sum_{n=0}^{\infty} \Phi_{j,n}^{\langle\mu\rangle} \Phi_{n,\langle\mu\rangle} A_n^{(j,1)} \right. \\ &\quad + \sum_{n=0}^{\infty} \Phi_{j,n}^{\langle\mu_1\mu_2\rangle} \Phi_{j,n,\langle\mu_1\mu_2\rangle} A_n^{(j,2)} \\ &\quad \left. + \sum_{l=3}^{\infty} \sum_{n=0}^{\infty} \Phi_{j,n}^{\langle\mu_1 \dots \mu_l\rangle} \Phi_{j,n,\langle\mu_1 \dots \mu_l\rangle} A_n^{(j,l)} \right]. \quad (67b) \end{aligned}$$

Substituting the expressions of $\Phi_{j,n}^{\langle\mu_1 \dots \mu_l\rangle}$ from Eqs. (63a)–(63d) and comparing the results with Eq. (44) we find the frame-invariant transport coefficients are given by,

$$\zeta_s = \frac{1}{\beta} \sum_{j=1}^N \left\langle \frac{A_{\mathbf{p}}^{(j)} A_{\mathbf{p}}^{(j)} \tau_R}{E_{\mathbf{p}}} \right\rangle_1^{(j)}, \quad (68a)$$

$$\eta_s = \sum_{j=1}^N \frac{\beta}{15} \langle (\tau_R/E_{\mathbf{p}}) (p \cdot \Delta \cdot p)^2 \rangle_1^{(j)}. \quad (68b)$$

These results for multiple species are consistent with the recent findings of Ref. [58], within the appropriate limit. In the next section, we will explore the temperature dependence of the obtained transport coefficients numerically.

IV. RESULTS AND DISCUSSIONS

To analyze the temperature variation of the shear and bulk viscous coefficients obtained in Eqs. (68a) and (68b), we require two ingredients, (i) the dispersion relation for the quasiparticle species and (ii) the specific functional form of the relaxation time. We consider the on-shell dispersion relation, $E_{\mathbf{p}_j} = \sqrt{\mathbf{p}_j^2 + m_j^2(T)}$ for the quasiparticles with temperature dependent effective mass given by

$$m_j(T) = \sqrt{(m_j^B)^2 + \Pi_j^{\text{int}}(T)}, \quad (69)$$

where the temperature independent constant m_j^B is the bare mass of the quasiparticle and $\Pi_j^{\text{int}}(T)$ effectively incorporates the interactions. For the description of the strongly interacting $2 + 1$ ($= N_f$) flavor hot QCD matter we consider nondegenerate strange and light (up and down) quark-antiquark sectors with bare masses $m_{s,\bar{s}}^B = 0.150$ GeV and $m_{u,\bar{u}}^B = m_{d,\bar{d}}^B = m_s^B/28.15$, respectively, along with massless gluons. The interaction part is motivated from the asymptotic form of the hard thermal loop (HTL) self-energy [59], which is given by [31]

$$\Pi_j^{\text{int}}(T) = 2 \left[m_j^B \sqrt{\frac{g_{\text{eff}}^2(T)}{6} T^2} + \frac{g_{\text{eff}}^2(T)}{6} T^2 \right], \quad (70)$$

for $j = \{u, d, s\}$ and their antiparticles whereas for gluons,

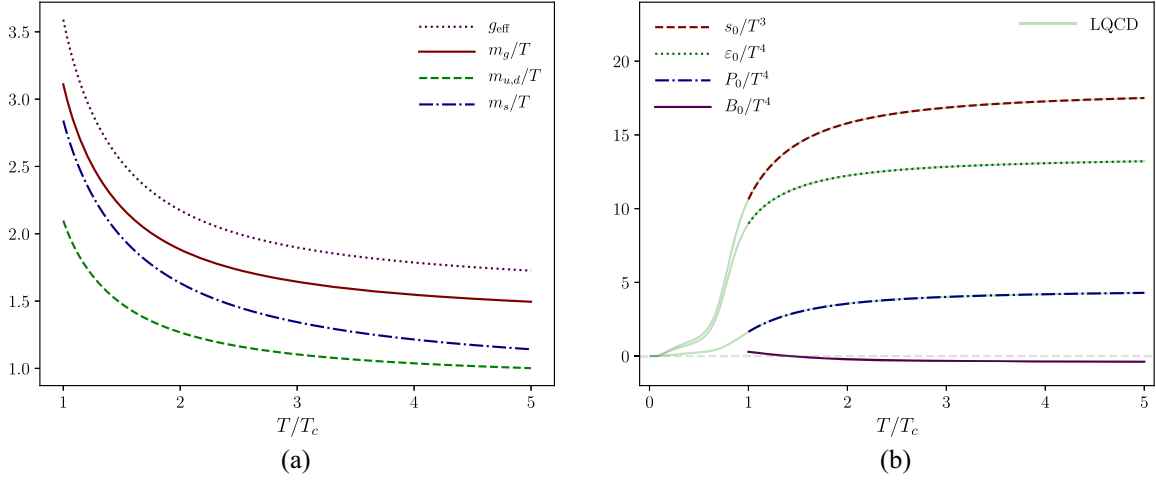


FIG. 1. (a) Variation of the effective coupling (purple dotted) and scaled quasiparticle masses (m_g/T , $m_{u,d}/T$, and m_s/T) are shown (in red solid, green dashed, and blue dot-dashed style, respectively) as a function of T/T_c where $T_c = 0.155$ GeV. (b) The equilibrium thermodynamic quantities obtained in the quasiparticle approach, namely the scaled entropy density (s_0/T^3), the scaled energy density (ϵ_0/T^4), and the scaled pressure (P_0/T^4) are shown (in red dashed, green dotted, and blue dot-dashed style) as a function of T/T_c . The corresponding LQCD results (light-green solid) obtained from the parametrization of the trace anomaly are also plotted for comparison. The solid-purple curve corresponds to the variation of the equilibrium bag contribution (B_0/T^4) in the quasiparticle approach.

$$\Pi_g^{\text{int}}(T) = \left(N_c + \frac{N_f}{2}\right) \frac{g_{\text{eff}}^2(T)}{6} T^2, \quad (71)$$

with number of colors, $N_c = 3$. The temperature dependence of the effective coupling can be fixed from the $2 + 1$ flavor QCD thermodynamics obtained in lattice calculation following the procedure outlined in Ref. [37]. For this purpose, we consider the parametrized form of the trace anomaly given by [60,61]

$$\frac{I(T)}{T^4} = \left(\frac{h_0}{1 + h_3 t^2} + f_0 \frac{\tanh(f_1 t + f_2) + 1}{1 + g_1 t + g_2 t^2} \right) \times \exp \left[-(h_1/t) - (h_2/t^2) \right], \quad (72)$$

with $t = T/(0.2 \text{ GeV})$ and we consider the following parameter set for $N_f = 2 + 1$:

$$(h_0, h_1, h_2, h_3) = (0.1396, -0.18, 0.035, 0.01), \quad (73a)$$

$$(f_0, f_1, f_2) = (2.76, 6.79, -5.29), \quad (73b)$$

$$(g_1, g_2) = (-0.47, 1.04). \quad (73c)$$

From the interaction measure, one can extract the equilibrium pressure corresponding to the lattice data as [37,60]

$$\frac{P_{\text{LQCD}}}{T^4} = \int_0^T \frac{dT'}{T'} \frac{I(T')}{T'^4}, \quad (74)$$

where we have considered $P_{\text{LQCD}}(T \rightarrow 0) \rightarrow 0$. Once the pressure is obtained, the energy density can be extracted

using the relation

$$\epsilon_{\text{LQCD}}(T) = I(T) + 4P_{\text{LQCD}}(T). \quad (75)$$

With Eqs. (74) and (75), one can obtain the entropy density $s_{\text{LQCD}}(T) = [\epsilon_{\text{LQCD}}(T) + P_{\text{LQCD}}(T)]/T$. To extract the temperature dependence of the effective coupling $g_{\text{eff}}(T)$, we simply equate $s_{\text{LQCD}}(T)$ with the equilibrium entropy density $s_0(T) = [\epsilon_0(T) + P_0(T)]/T$ and solve for g_{eff} at a given T . Note that, in $s_0(T)$, the bag contribution from the equilibrium energy density and pressure cancels out [see Eq. (51a) for the definition] whereas the effective coupling enters into the thermodynamic integrals via the temperature dependent mass terms. Considering the pseudocritical temperature $T_c = 0.155$ GeV, the extracted effective coupling is shown in Fig. 1(a) as a function of the scaled variable T/T_c in the range $1 \leq T/T_c \leq 5$. As the temperature is increased, a rapid decrease in the effective coupling near T_c is observed, which eventually almost saturates at higher temperature. This is similar to the observations obtained in quasiparticle models where a parametrization of the effective coupling is considered based on the functional form of one and two loop QCD running coupling [17,20]. This trend in effective coupling is also reflected in the temperature variation of the scaled effective masses of the gluon (m_g/T), light ($m_{u,d}/T$), and strange (m_s/T) quark sectors as shown in Fig. 1(a). In fact, the bare mass m_g^B being zero, the temperature variation of the scaled effective mass for gluon quasiparticle becomes identical to that of the effective coupling up to an overall constant factor $\sqrt{3/4}$. Similar reasoning also applies to up and down quarks with small $m_{u,d}^B$ (with an overall factor around $1/\sqrt{3}$). However, the

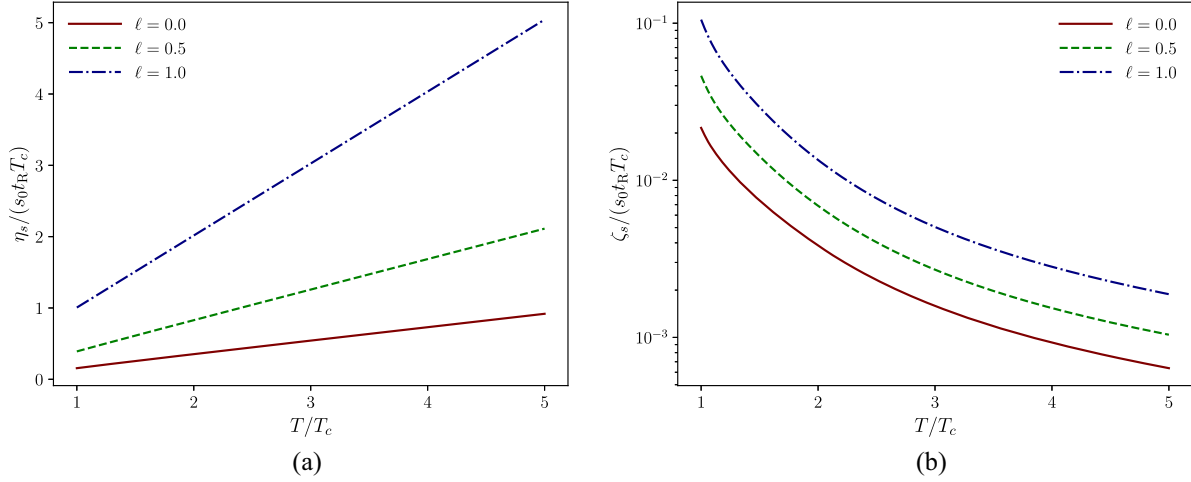


FIG. 2. (a) Variation of the ratio of the frame independent shear viscosity coefficient (η_s scaled by $t_R T_c$) and equilibrium entropy density (s_0) is shown (in red solid, green dashed, and blue dot-dashed style, respectively) as a function of T/T_c (where $T_c = 0.155$ GeV) for fixed set of values of the power law exponent $\ell = (0.0, 0.5, 1.0)$. (b) Variation of the ratio of the frame independent bulk viscosity coefficient (ζ_s scaled by $t_R T_c$) and equilibrium entropy density (s_0) is shown (in red solid, green dashed, and blue dot-dashed style, respectively) as a function of T/T_c (where $T_c = 0.155$ GeV) for fixed set of values of the power law exponent $\ell = (0.0, 0.5, 1.0)$.

bare mass term does play a role for the strange quarks resulting in a qualitatively different temperature profile in comparison to the lighter counterparts. Also note that the hierarchy of the effective masses $m_g \geq m_s \geq m_{u,d}$ throughout the considered temperature range is consistent with other quasiparticle approaches with $N_f = 2 + 1$ [30,31,62].

To obtain the thermodynamics of the system of quasiparticles we require the temperature evolution of the equilibrium bag pressure. It should be noted that this temperature evolution arises due to the temperature dependence of the quasiparticle masses. Therefore, utilizing the thermodynamic consistency relation given in Eq. (49), one obtains the temperature evolution of the bag pressure given by [37]

$$\frac{dB_0}{dT} = -\frac{1}{2} \sum_j \frac{dm_j^2}{dT} \langle 1 \rangle_0^{(j)}. \quad (76)$$

The solution of the differential equation is given by

$$B_0(T) = B_0(T_c) - \sum_j \int_{T_c}^T dT' m_j(T') \frac{dm_j(T')}{dT'} \langle 1 \rangle_0^{(j)}, \quad (77)$$

where the constant $B(T_c)$ is fixed as $B(T_c) = [-\langle (1/3)(p \cdot \Delta \cdot p) \rangle_0 - P_0]_{T=T_c}$. It is worth mentioning that the equilibrium bag pressure so obtained ensures $s_0 = dP_0/dT = [\epsilon_0(T) + P_0(T)]/T$ and provides a thermodynamically consistent description of the system of quasiparticles with temperature dependent masses.

The equilibrium thermodynamic quantities of the QCD medium, namely the scaled entropy density (s_0/T^3), the scaled energy density (ϵ_0/T^4), and the scaled pressure (P_0/T^4) from the lattice parametrization are shown in

Fig. 1(b) and compared with those obtained in the quasiparticle model with temperature dependent effective masses. As can be seen from the figure, the energy density and pressure defined in Eq. (51a) with the bag contribution (B_0) reproduce the results obtained from the parametrized interaction measure [given in Eq. (72)] within the considered range of temperature $1 \leq T/T_c \leq 5$. The temperature variation of the scaled bag contribution B_0/T^4 is also shown in Fig. 1(b). As can be noticed, the bag contribution (apart from a tiny interval close to T_c) is negative and it remains an order of magnitude smaller compared to the energy density and pressure throughout the considered temperature range. This is consistent with the results obtained in Ref. [37].

For the momentum dependence, we restrict ourselves to the simple power law ansatz, which in the fluid rest frame is given by $\tau_R = t_R (\beta E_p)^\ell$. We explore the influence of the exponent in the range $0 \leq \ell \leq 1$. The lower limit describes the momentum independent case whereas the $\ell = 1$ corresponds to the scalar field theory [63]. Note that $\ell = 0.5$ is commonly quoted as the value of the exponent for QCD effective kinetic theory and it lies in the midway between the lower and upper limit of the power law exponent considered here.

Apart from the power law exponent, the functional form of the relaxation time also depends on the parameter t_R which though momentum independent can be a function of temperature. As our primary objective is to analyze the influence of the power law exponent in the QCD transport, in the present work we do not attempt to obtain the temperature dependence of t_R explicitly. Rather, we scale the transport coefficients by an overall dimensionless factor $t_R T_c$ and study the effect of the power law exponent on the residual temperature dependence. The temperature variation of the scaled shear ($\eta_s/s_0 t_R T_c$) and bulk ($\zeta_s/s_0 t_R T_c$)

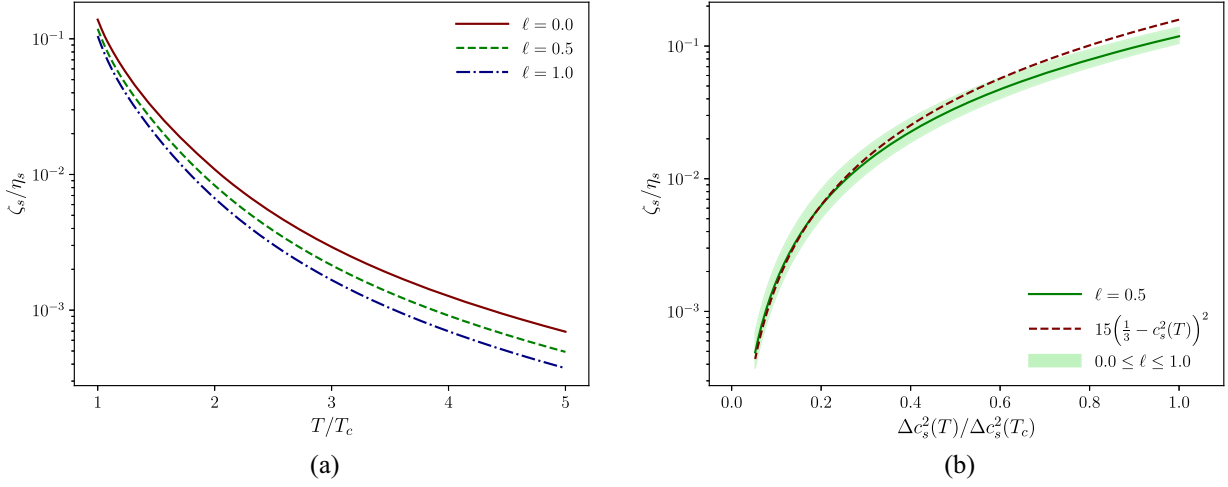


FIG. 3. (a) Variation of the ratio of the frame independent transport coefficients (ζ_s/η_s) is shown (in red solid, green dashed, and blue dot-dashed style, respectively) as a function of T/T_c (where $T_c = 0.155$ GeV) for a fixed set of values of the power law exponent $\ell = (0.0, 0.5, 1.0)$. (b) Variation of the ratio of the frame independent transport coefficients (ζ_s/η_s) is shown as a function of $\Delta c_s^2(T) = \frac{1}{3} - c_s^2(T)$ [scaled by $\Delta c_s^2(T_c)$]. The green patch corresponds to the ζ_s/η_s values for $0.0 \leq \ell \leq 1.0$. The variation corresponding to $\ell = 0.5$ (shown in green solid style) and the case $\zeta_s/\eta_s = 15\left(\frac{1}{3} - c_s^2(T)\right)^2$ (shown in red dashed style) are also plotted for comparison.

viscous coefficients are shown in Figs. 2(a) and 2(b), respectively, for fixed set of values of the power law exponent $\ell = (0.0, 0.5, 1.0)$. It can be observed that as the temperature increases from T_c , $\eta_s/s_0 t_R T_c$ increases whereas $\zeta_s/s_0 t_R T_c$ shows a decreasing trend for all the ℓ values considered. Moreover, at a fixed temperature, with larger ℓ the specific shear coefficient increases whereas the specific bulk coefficient decreases. This opposite trend of shear and bulk for different values of the power law exponent is consistent with the transport coefficients obtained with single quasiparticle species [37].

The temperature dependence of the ratio of the frame independent transport coefficients (ζ_s/η_s) is shown in Fig. 3(a) for $\ell = (0.0, 0.5, 1.0)$. Apart from the overall decreasing trend, it can be noticed that higher values of the power law exponent result in a reduced ζ_s/η_s throughout the temperature range. Also note that this reduction in the ζ_s/η_s ratio for $\ell > 0.0$ is more prominent at higher values of temperature compared to the region close to T_c . It should be mentioned here that for weakly coupled QCD, a scaling law exists for the bulk to shear ratio, which is given by [11–13]

$$\zeta_s/\eta_s = 15\left(\frac{1}{3} - c_s^2(T)\right)^2, \quad (78)$$

which is in fact similar to the estimation for a gas of photon first obtained in [64]. On the other hand, in the strongly coupled scenario, $\zeta_s/\eta_s \sim \left(\frac{1}{3} - c_s^2(T)\right)$ [65] (see Refs. [14,66] for a detailed discussion on transport properties in the strongly coupled regime). As already discussed, in the present formulation the ratio of the frame independent transport coefficients depends on the power law exponent

parameter ℓ . Thus it is interesting to analyze the dependence of ζ_s/η_s on ℓ as the conformal measure $\Delta c_s^2(T) = \frac{1}{3} - c_s^2(T)$ varies with temperature. In Fig. 3(b), the variation of ζ_s/η_s is shown as a function of $\Delta c_s^2(T)$ [scaled by $\Delta c_s^2(T_c)$] for $0.0 \leq \ell \leq 1.0$. The ratio increases with the scaled conformal measure and it can be observed that it is more suppressed for higher ℓ values compared to the momentum independent scenario $\ell = 0.0$. ζ_s/η_s for $\ell = 0.5$ as well as the scaling relation of the weakly coupled QCD given in Eq. (78) are also plotted in Fig. 3(b) for comparison. It can be seen that at higher temperature [close to the origin in the $\Delta c_s^2(T)$ axis], the ratio in the quasiparticle formulation with $\ell = 0.5$ is similar to the estimate from the weakly coupled scenario. However we find that close to $T = T_c$, the weakly coupled approach overestimates the ratio.

So far, we have analyzed the influence of the power law exponent on the transport coefficient considering a fixed set of ℓ values in the range $[0, 1]$. However, to achieve a better understanding of the applicability of the quasiparticle model as formulated in the present study, it is desirable to compare the results with realistic scenarios. For this purpose, let us focus on the results obtained in the data driven approach based on the Bayesian analysis [8]. In Ref. [8], the temperature dependence of the specific shear viscosity has been parametrized by a three parameter linear ansatz whereas a three-parameter unnormalized Cauchy distribution is used to parametrize the temperature dependence of the specific bulk viscosity. Further improvement of the parametrization is obtained in Ref. [10] utilizing larger statistics as well as additional observables. In the present work we consider the best-fit values obtained in [10] to parametrize the frame independent bulk to shear ratio as

$$(\zeta_s/\eta_s)_{\text{data}} = \frac{(\zeta_s/s_0)_{\text{max}}}{\left[1 + \left(\frac{T - (\zeta_s/s_0)_{\text{peak}}}{(\zeta_s/s_0)_{\text{width}}}\right)^2\right] \left[(\eta_s/s_0)(\tilde{T}_c) + (\eta_s/s_0)_{\text{slope}}(T - \tilde{T}_c)\left(\frac{T}{\tilde{T}_c}\right)^{(\eta_s/s_0)_{\text{crv}}}\right]}, \quad (79)$$

with $\tilde{T}_c = 0.147$ GeV and

$$((\zeta_s/s_0)_{\text{max}}, (\zeta_s/s_0)_{\text{peak}}, (\zeta_s/s_0)_{\text{width}}) = (0.010, 0.170, 0.057), \quad (80a)$$

$$((\eta_s/s_0)(\tilde{T}_c), (\eta_s/s_0)_{\text{slope}}, (\eta_s/s_0)_{\text{crv}}) = (0.104, 0.425, -0.738). \quad (80b)$$

In Fig. 4, we compare the above parametrization with the ζ_s/η_s obtained from the quasiparticle approach with fixed values of the power law exponent in the range $0.0 \leq \ell \leq 0.5$ shown in the light green band. The scaling relation $\zeta_s/\eta_s = 15(\Delta c_s^2(T))^2$ given in Eq. (78) is also shown. To compare the results with the perturbative approach we further consider the next-to-leading log results from Refs. [11,12] given by

$$\zeta_s/\eta_s = \left(\frac{A g_{\text{eff}}^4}{16\pi^2 \log(\mu_2^*/m_D)}\right) / \left(\frac{\eta_1}{g_{\text{eff}}^4 \log(\mu_1^*/m_D)}\right), \quad (81)$$

where, $m_D = \sqrt{3/2} g_{\text{eff}}(T) T$, $\eta_1 = 106.66$, $\mu_1^* = 2.957T$, $A = 0.657$, and $\mu_2^* = 7.77T$ and the coupling is replaced by $g_{\text{eff}}(T)$ [30,31]. It can be observed that the order of

magnitude for ζ_s/η_s obtained from the quasiparticle approach is consistent with the next-to-leading-log perturbative estimate with effective coupling at high temperatures. On the other hand, the scaling relation $\zeta_s/\eta_s = 15(\frac{1}{3} - c_s^2)^2$ is in line with the quasiparticle result throughout the considered range of the temperature. Moreover, it can be noticed that beyond $T \sim 2.5T_c$, the best-fit parametrization from the Bayesian analysis lies well within the $0.0 \leq \ell \leq 0.5$ band and in fact never reaches the edge corresponding to $\ell = 0.5$ up to $T = 5T_c$. One can extract the power law exponent corresponding to the best-fit parametrization (say $\ell = \ell_{\text{data}}$) in the overlap region by equating the ratio in the two scenarios as

$$(\zeta_s/\eta_s)_{\ell=\ell_{\text{data}}} = (\zeta_s/\eta_s)_{\text{data}}, \quad (82)$$

for a given temperature. In Fig. 4, the temperature dependence of the extracted $\ell_{\text{data}}(T)$ is shown in the inset. It can be noticed that the power law exponent smoothly increases with temperature reaching up to $\ell = 0.3$ at $T = 5T_c$. Also, it should be noted that at temperatures close to T_c , negative values of the power law exponent are required to match the best-fit result.

V. CONCLUSION AND OUTLOOK

In this work, we formulate the first order relativistic dissipative hydrodynamics of a system of quasiparticles from kinetic theory with modified relaxation time approximation. The primary objective being the incorporation of the momentum dependence of the relaxation time, we consider two approaches based on the modified collision kernels (i) NRTA and (ii) ERTA. For single quasiparticle species, we have shown that both approaches are consistent with each other providing the same out-of-equilibrium corrections to phase-space distribution and hence the transport coefficients. We generalize the extended relaxation time approach to incorporate multiple quasiparticle species and obtain the frame and matching invariant transport coefficients from the entropy production. The formalism is then implemented to study the shear and bulk viscosity coefficients of the strongly interacting $2+1$ flavor hot QCD medium with quasigluon and (light and strange) quasiquark (antiquark) sectors. The quasiparticle masses are temperature-dependent due to the interaction term present in the dispersion relation. In the present study we consider the interaction term with temperature dependent effective coupling $g_{\text{eff}}(T)$. The functional form of the interaction term is motivated from the asymptotic form of

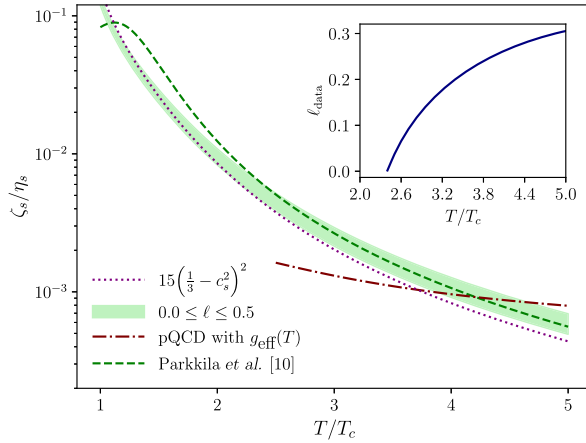


FIG. 4. Best-fit parametrization $(\zeta_s/\eta_s)_{\text{data}}$ obtained in Ref. [10] is shown (in green dashed style) as a function of T/T_c . The scaling relation $\zeta_s/\eta_s = 15(\frac{1}{3} - c_s^2)^2$ as well as the next-to-leading log perturbative QCD (pQCD) result [11,12] with the effective coupling is shown (in purple dotted and red dot-dashed style, respectively) for comparison. The light green band corresponds to the ζ_s/η_s values in the quasiparticle model for $0.0 \leq \ell \leq 0.5$. The temperature dependence of the power law exponent extracted from the best-fit parametrization is shown in the inset.

the HTL self-energy. However, in the present study, we do not consider the perturbative running of the coupling. Rather the temperature dependence of the effective coupling is fixed using the parametrized form of the trace anomaly obtained in the $2+1$ flavor LQCD simulation [61] and the extracted $g_{\text{eff}}(T)$ essentially fixes the temperature dependence of the equilibrium thermodynamics.

To study the transport coefficient of the hot QCD medium, we consider the power law ansatz $\tau_R = t_R(\beta E_p)^\ell$ for the momentum dependence of the relaxation time. We focus on the influence of the power law exponent ℓ on the frame choice and matching condition independent shear and bulk viscosity coefficient and also on the bulk to shear ratio ζ_s/η_s where the momentum independent part t_R cancels. For the power law exponent, we consider the range of values $0.0 \leq \ell \leq 1.0$ and find that specific shear viscosity coefficient (scaled by $t_R T_c$) significantly increases for higher ℓ compared to the $\ell = 0.0$ case whereas the specific bulk viscous coefficient (scaled by $t_R T_c$) gets significantly reduced. As a consequence, an overall reduction in the ratio ζ_s/η_s is observed for higher ℓ throughout the considered temperature range $1 \leq T/T_c \leq 5$. We also analyze the variation of ζ_s/η_s as a function of the conformal measure $\Delta c_s^2(T)$. We find that at higher temperature the estimation of the bulk to shear ratio for $\ell = 0.5$ is well described by the scaling relation $\zeta_s/\eta_s = 15[(1/3) - c_s^2(T)]^2$. However, significant deviation is observed close to T_c . Finally, we compare the ζ_s/η_s with the best-fit parametrization obtained in the data-driven Bayesian analysis [10]. We find that for temperatures higher than $T \sim 2.5T_c$, the parametrized result can be well described using the power law ansatz with a temperature-dependent power law exponent. The exponent extracted from the best-fit result shows a gradual increasing trend with temperature reaching up to $\ell = 0.3$ at $T = 5T_c$. This is less than the commonly quoted value of the exponent for QCD, which is $\ell = 0.5$. It should be mentioned that the present hydrodynamic formulation for the multiple quasiparticle species considers a single relaxation time as a simplification, which needs to be generalized for a realistic scenario. However, the assumption of a single relaxation time facilitates the extraction of the temperature dependence of the power law exponent simply from the ratio of transport coefficients. Moreover, the compatibility of the ζ_s/η_s results from the data-driven analysis and the quasiparticle scenario beyond $T \sim 2.5T_c$ is indeed promising and motivates one to analyze the impact of the momentum-dependent relaxation time in the presence of nonzero chemical potential [67]. It will also be interesting to implement this framework to study the dynamical evolution of the strongly interacting matter in the presence of the background magnetic field [68,69]. Furthermore, looking into the growing interest in the spin

physics of heavy-ion collisions, one may consider formulating a spin hydrodynamics [57] for spin-1/2 quasiparticles with temperature-dependent effective mass. We leave such analysis for future exploration.

ACKNOWLEDGMENTS

S. B. would like to thank Gabriel S. Rocha for some clarifying discussions. The authors acknowledge Amaresh Jaiswal for insightful correspondence. S. B. acknowledges the support of the Faculty of Physics, Astronomy, and Applied Computer Science, Jagiellonian University via Grant No. LM/36/BS. P. S. is funded by the EU's NextGenerationEU instrument through the National Recovery and Resilience Plan of Romania—Pillar III-C9-I8, managed by the Ministry of Research, Innovation and Digitization, within the project entitled “Facets of Rotating Quark-Gluon Plasma” (FORQ), Contract No. 760079/23.05.2023 code CF 103/15.11.2022.

DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: THERMODYNAMIC INTEGRALS AND PROPERTIES OF δ FUNCTION

The thermodynamic integrals obey the following identities,

$$i_{nq} = \dot{\beta} \left[\frac{1}{2} \frac{\partial m^2}{\partial \beta} \{ (n-1)I_{n-2,q} - \beta J_{n-1,q} \} - J_{n+1,q} \right], \quad (\text{A1})$$

$$X_{nq} = \frac{1}{(2q+1)} [m^2 X_{n-2,q-1} - X_{n,q-1}] \quad (\text{A2})$$

where X_{nq} can be either of I_{nq} or J_{nq} . In deriving Eq. (A1) we have used the relation,

$$D\delta(p^2 - m^2) = -\frac{m(\partial_\mu m)}{(u \cdot p)} u^\mu D_{(p)} \delta(p^2 - m^2), \quad (\text{A3})$$

where $D \equiv (u \cdot \partial)$ and $D_{(p)} \equiv u \cdot \partial_{(p)}$. One can also show that,

$$(p \cdot \partial) \delta(p^2 - m^2) = -m(\partial_\mu m) \partial_{(p)}^\mu \delta(p^2 - m^2). \quad (\text{A4})$$

Note that the relations in Eqs. (A1)–(A4) remain true for any j -th particle species, i.e., we can use the subscript j in these equations and they will be satisfied.

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