

Phase diagram of the massless Gross-Neveu model with two components

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(Received 9 July 2026; accepted 3 September 2026; published 5 October 2026)

Recently, Benini *et al.* have presented exact results for the Gross-Neveu model at finite density, both at finite N and in the large N limit. Generalizing previous studies, they introduce a chemical potential acting only on a subset of fermion flavors—the two-component Gross-Neveu model. Here, we take up this idea and extend the semiclassical study to finite temperature. The full (large N) phase diagram including inhomogeneous phases is constructed without solving the thermal Hartree-Fock problem. All phase boundaries can be found on the basis of various kinds of stability analyses. As a result, we find a qualitative change of the phase diagram at some critical filling fraction.

DOI: [10.1103/nhgs-l4gn](https://doi.org/10.1103/nhgs-l4gn)

I. INTRODUCTION

The Gross-Neveu (GN) model [1] is the quantum field theory of N flavors of massless Dirac fermions in $1 + 1$ dimensions, interacting via a scalar-scalar four-fermion interaction,

$$\mathcal{L} = \sum_{f=1}^N \bar{\psi}_f i \gamma^\mu \partial_\mu \psi_f + \frac{g^2}{2} \left(\sum_{f=1}^N \bar{\psi}_f \psi_f \right)^2. \quad (1)$$

Its remarkable properties (renormalizability, dimensional transmutation, asymptotic freedom, discrete chiral symmetry breaking, bound states, integrability) have turned it into an iconic toy model over the years. Notably the large N limit has been explored thoroughly with semiclassical methods, both at zero and finite temperature and density [2–4].

Recently, a lot of progress has been achieved in solving the GN model at finite N , using either lattice Monte Carlo methods [5,6] or by exploiting the integrability of the model [7–9]. By and large, the semiclassical findings have been confirmed and limitations inherent in the large N treatment are by now better understood. An interesting generalization of the usual treatment of matter at finite density was proposed by Benini *et al.* [10,11]. These authors divide up the N fermion flavors into two groups. The first a fermions are called “charged,” the remaining $N - a$ ones “neutral,” and a chemical potential is assigned to charged fermions only. In this way, important insights

into renormalons and other features of the Bethe ansatz solution of the GN model with a finite number of flavors were obtained. For the present work, the implications for the semiclassical large N treatment are most relevant. The authors identified the ground state at finite density with a crystal composed of Dashen-Hasslacher-Neveu (DHN) baryons [2] with fermion number a . Previously, the chemical potential had always been assumed to act on all N flavors alike. In that case, the ground state crystal consists of baryons with maximal fermion number N (kinks and antikinks) only. Introducing two components thus enables one to generalize the notion of “partially filled bound states” from the DHN baryon to “partially filled valence bands” of a periodic mean field in dense matter.

That this concept is quite useful has already been confirmed in a preceding paper [12]. We have applied it to the zero temperature chiral GN model [with $U(1)$ chiral symmetry]. It turned out that the twisted kink crystal of Basar and Dunne [13,14] is the self-consistent mean field for fermion matter with filling fraction $\nu = a/N < 1$. The fact that the low density limit of this potential looks like an array of partially filled DHN baryons had been known before, but how to realize this thermodynamically was unclear. Apparently, the twisted kink crystal is exactly the mean field it takes to describe a system with filling fraction $\nu < 1$. For $\nu = 1$, it reduces to the simpler chiral spiral [15].

What happens if one heats up a system with a finite density of solitons? In the standard one-component GN model (1) with a common chemical potential for all flavors, this is well known. The phase diagram in the (μ, T) plane features three phases, a massless one, a massive one, and an inhomogeneous one, separated by second order phase boundaries [16]. We shall recall it below in more detail. By contrast, nothing is known yet about the phase diagram for $\nu < 1$, to the best of our knowledge. It is the purpose of the present paper to fill this gap.

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Judging from the experience with the massless and massive GN models [17,18], the mean field at finite T is likely to have the same functional form as at $T = 0$. There, the answer for the GN model is already known from Refs. [10,11]. The mean field has the same analytical expression as that of the massive GN model. It has long been known that partially filled DHN baryons of the massless GN model have the same shape as baryons with maximal fermion number in the massive model [19,20]. Apparently, a similar relationship holds for finite density systems.

As is known from the massive GN model, a full Hartree-Fock (HF) calculation with such a mean field is elaborate [18]. It is necessary to minimize the grand canonical potential with respect to three parameters at each point in (μ, T, γ) space, with γ the confinement parameter related to the bare mass [18]. Here, we have to replace γ by the filling fraction ν , so that a full solution of the generalized GN model would be equally challenging. Fortunately, if we are only interested in the phase diagram and not in observables off the critical lines, efficient shortcuts are available. In the case of the GN model with a standard chemical potential, this was demonstrated only recently [21]. The full phase diagram of both the massless and the massive GN models have been reproduced without HF calculation and without reference to the analytically known mean fields. The main idea is to use stability analyses with respect to different kinds of instabilities, both perturbative and nonperturbative ones. We shall explain these methods, some of which have been explored in the context of GN models before [22,23], and apply them to the present generalization to arbitrary filling parameter. This will enable us to present the full phase diagram of the massless GN model for different values of ν with a modest effort.

The paper is organized as follows. In Sec. II, we solve the generalized problem under the (unjustified) assumption of unbroken translational invariance. This is not only meant as a warm-up, but also serves to locate critical points and phase boundaries possibly shared by the full calculation. We will learn that, depending on ν , either all phase boundaries are connected or they exhibit two disconnected branches. Accordingly, we divide the study of the full phase diagram into two sections. Section III is devoted to the phase diagram of the full theory in the connected region, and Sec. IV in the disconnected region of ν values. It is followed by a summary in Sec. V.

II. PHASE DIAGRAM ASSUMING HOMOGENEOUS CONDENSATES

Let us first construct the phase diagram of the two-component GN model assuming unbroken translation invariance. As is known from the one-component case, this may give clues about the full solution exhibiting inhomogeneous condensates. The homogeneous phase

diagram is likely to share with the full one certain phase boundaries and critical points.

The renormalized grand canonical potential density for the GN model with homogeneous mean field M reads

$$N^{-1}\Psi_{\text{eff}}^{(1)}(\beta, \mu) = \frac{M^2}{4\pi}(\ln M^2 - 1) - \frac{1}{\beta\pi} \int_0^\infty dk \times \ln \left[\left(1 + e^{-\beta(E_k - \mu)}\right) \left(1 + e^{-\beta(E_k + \mu)}\right) \right]. \quad (2)$$

Here, M plays the role of the dynamical fermion mass (set equal to 1 in the vacuum), $E_k = \sqrt{k^2 + M^2}$ is the single particle energy, $\beta = 1/T$ is the inverse temperature, and μ is the chemical potential for fermion number. The superscript (1) on $\Psi_{\text{eff}}^{(1)}$ stands for one component, i.e., a common chemical potential is introduced for all fermion flavors. The generalization to the two-component system where the chemical potential acts only on a subset of $a = \nu N$ flavors is straightforward,

$$\Psi_{\text{eff}}^{(2)}(\nu; \beta, \mu) = (1 - \nu)\Psi_{\text{eff}}^{(1)}(\beta, 0) + \nu\Psi_{\text{eff}}^{(1)}(\beta, \mu). \quad (3)$$

Consider first the expected perturbative phase boundary between the chirally symmetric phase ($M = 0$) and the chirally broken phase ($M > 0$), defined by the condition

$$\left. \frac{\partial}{\partial M^2} \Psi_{\text{eff}}^{(2)}(\nu; \beta, \mu) \right|_{M=0} = 0. \quad (4)$$

Here, it is more convenient to turn to Ginzburg-Landau (GL) theory. For a uniform system, this is just a Taylor expansion in powers of M , sufficient to determine the exact perturbative phase boundary. We recall the well-known GL expression for the (homogeneous) one-component system to the order needed here [17],

$$\begin{aligned} N^{-1}\Psi_{\text{GL}}^{(1)}(T, \mu) &= \alpha_0(T, \mu) + \alpha_2(T, \mu)M^2 + \alpha_4(T, \mu)M^4, \\ \alpha_0(T, \mu) &= -\frac{\pi}{6}T^2 - \frac{\mu^2}{2\pi} - \frac{1}{4\pi}, \\ \alpha_2(T, \mu) &= \frac{1}{2\pi}[\ln(4\pi T) + \text{Re}\psi(z)], \\ \alpha_4(T, \mu) &= -\frac{1}{2^6\pi^3 T^2} \text{Re}\psi(2, z), \\ z &= \frac{1}{2} + \frac{i\eta}{2\pi}, \quad \eta = \frac{\mu}{T}. \end{aligned} \quad (5)$$

It is given in terms of standard di- and poly- γ functions, $\psi(z) = \psi(1, z)$ and $\psi(n, z)$ with $n = 2$. This yields the following GL effective action for the two-component system:

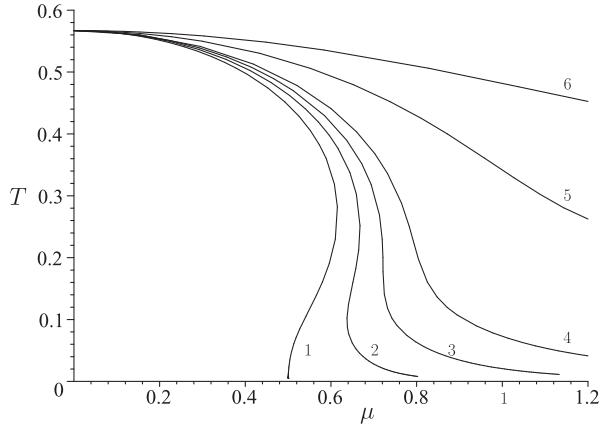


FIG. 1. Perturbative phase boundaries of two-component GN model with homogeneous condensates, separating the massless from the massive phase. Labels 1–6 correspond to filling fractions $\nu = 1.0, 0.90, 0.8271, 0.75, 0.50, 0.25$, respectively.

$$\begin{aligned}\Psi_{\text{GL}}^{(2)}(\nu; T, \mu) &= (1 - \nu)\Psi_{\text{GL}}^{(1)}(T, 0) + \nu\Psi_{\text{GL}}^{(1)}(T, \mu) \\ &= A_0(\nu; T, \mu) + A_2(\nu; T, \mu)M^2 \\ &\quad + A_4(\nu; T, \mu)M^4, \\ A_n(\nu; T, \mu) &= (1 - \nu)\alpha_n(T, 0) + \nu\alpha_n(T, \mu).\end{aligned}\quad (6)$$

The condition for the perturbative phase boundary, Eq. (4), amounts to $A_2(\nu; T, \mu) = 0$ or

$$\begin{aligned}0 &= \ln(4\pi T) + (1 - \nu)\psi(1/2) + \nu\text{Re}\psi(z), \\ \psi(1/2) &= -C - 2\ln 2\end{aligned}\quad (7)$$

(C = Euler constant). Since z depends only on $\eta = \mu/T$, Eq. (7) can be solved explicitly for $T(\eta)$. This is sufficient to plot T vs $\mu = T\eta$ with Maple using the parametric plot option, without need to solve transcendental equations. This step goes through for any ν . The resulting dependence of the perturbative phase boundary on ν is illustrated in Fig. 1. With decreasing ν the curves get smoother, until they would reach a horizontal line $T = T_c = e^C/\pi$ at $\nu = 0$ where nothing depends on μ anymore. From the standard treatment of the GN model ($\nu = 1$), it is known that the perturbative phase boundary ends at a tricritical point beyond which it goes over into a (nonperturbative) first order phase boundary [24]. The position of a tricritical point can be found by requiring $A_4 = 0$ in addition to $A_2 = 0$. Let us see what happens as we decrease ν . The function A_4 reads

$$A_4 = \frac{(1 - \nu)14\zeta(3) - \nu\text{Re}\psi(2, z)}{64\pi^3 T^2} := \frac{A'_4(\nu, \eta)}{64\pi^3 T^2}. \quad (8)$$

In Fig. 2, we have plotted A'_4 against η for all values of ν used in the present work. Starting from $\nu = 1$ and decreasing ν , one first finds two solutions of the condition $A'_4 = 0$.

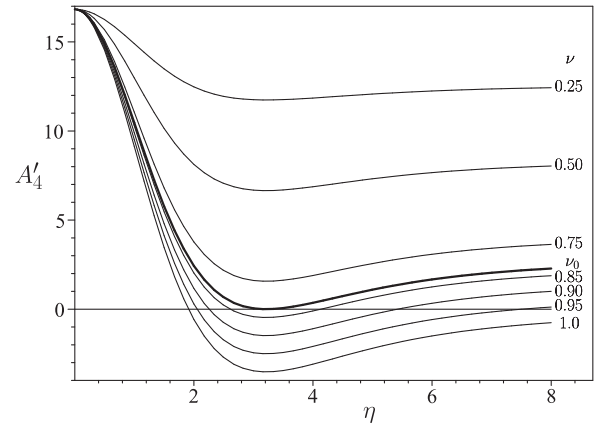


FIG. 2. Determination of the tricritical point using the zeros of A'_4 , Eq. (8), for a range of ν values. At $\nu = \nu_0 = 0.8271$, a bifurcation takes place and the tricritical point disappears.

The one at lower η defines a tricritical point, the upper solution being unphysical. At a critical value $\nu = \nu_0$, a bifurcation takes place—the two solutions coalesce and there is no solution for $\nu < \nu_0$. As can be inferred from Fig. 2, the condition determining ν_0 (in addition to $A_2 = 0$, $A_4 = 0$) is

$$\frac{\partial}{\partial \eta} A'_4(\nu, \eta) = 0 \quad (9)$$

or, equivalently,

$$\text{Im}\psi(3, z) = 0. \quad (10)$$

All three conditions together give the value ν_0 of the bifurcation point and the coordinates (μ_0, T_0) of the tricritical point there. One finds

$$\nu_0 = 0.8271, \quad \mu_0 = 0.7202, \quad T_0 = 0.2240. \quad (11)$$

For $\nu < \nu_0$, there is no tricritical point anymore and the curves shown in Fig. 1 continue forever.

In the Appendix of Ref. [12], the same model has already been solved at $T = 0$. A first order phase transition was found for any ν at a critical chemical potential μ_c depending on ν . The dynamical mass changes discontinuously from the vacuum value to a lower value. At $\nu = 1$, the mass drops to 0 at $\mu_c = 1/\sqrt{2}$, the base point of a first order phase boundary going upward in temperature and ending at the tricritical point. To study the corresponding curve for other values of ν , one has to go back to the full expression for the grand canonical potential, Eqs. (2) and (3), and locate points in the (μ, T) plane with two degenerate minima. The result of such an analysis is as follows. For $1 \geq \nu \geq \nu_0$, the situation stays similar to $\nu = 1$, confirming the existence of a tricritical point. For lower ν values, the first order line does not reach the perturbative phase

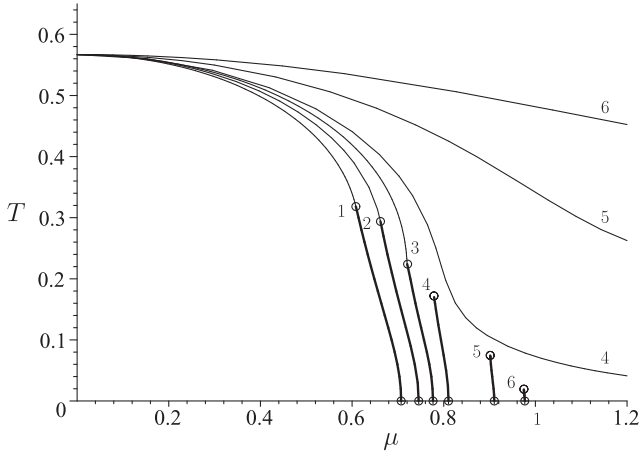


FIG. 3. Full phase diagram of the two-component GN model with homogeneous condensates only. The labeling 1–6 is the same as in Fig. 1. Starting from the bifurcation curve 3 ($\nu = 0.8271$), thin (second order) and fat (first order) curves get disconnected, the tricritical point being replaced by a critical end point. The base points at $T = 0$ are taken from [12].

boundary but terminates at a critical point within the homogeneous massive phase. Thus, at the bifurcation value ν_0 , the tricritical point moves away from the perturbative boundary, getting converted into a critical end point. Some representative phase boundaries are shown in Fig. 3. The base points are taken from Ref. [12]. Tricritical points are evaluated as discussed above. The critical end points and first order curves are numerical results. The coordinates of all end points of the first order curves determined in this work are listed in Table I.

Concluding this section, we have seen that the phase diagram of the homogeneous system features two qualitatively different regimes, depending on the value of ν . In the region $1 \geq \nu \geq \nu_0$, the first and second order phase boundaries are connected through a tricritical point. In the region $\nu < \nu_0$, they are disconnected. The first order phase boundary rises from $T = 0$ up to some critical point below the perturbative phase boundary. For simplicity, we refer to the two regions as “connected” and “disconnected” from

TABLE I. End points of all first order lines computed in this work ($\nu_0 = 0.8271$). The first order line shrinks to the point ($\mu = 1, T = 0$) in the limit $\nu \rightarrow 0$.

ν	μ at $T = 0$	μ_{crit}	T_{crit}
1.0	0.7071	0.6082	0.3183
0.95	0.7255	0.6330	0.3083
0.90	0.7453	0.6620	0.2938
0.85	0.7665	0.6978	0.2675
ν_0	0.7765	0.7202	0.2240
0.75	0.8104	0.779	0.172
0.50	0.9102	0.902	0.074
0.25	0.9768	0.975	0.019

here on. It is not evident that this topological distinction remains valid once inhomogeneous condensates are allowed. Nevertheless, it is useful to discuss possible crystal phases separately in these two distinct regions. This is the topic of the following two sections.

III. FULL SOLUTION IN THE CONNECTED REGION

To understand the GN phase diagram including inhomogeneous phases in the region $1 \geq \nu \geq \nu_0$, it is necessary to briefly recall the well-known case $\nu = 1$. Figure 4 shows the phase diagram of the massless GN model with the standard chemical potential. The curve AP is the phase boundary between massless and massive homogeneous phases already known from the translationally invariant calculation in Sec. II. Curve PC is the first order phase boundary discussed above. It does not exist anymore in the full phase diagram and has only been drawn to highlight the difference between homogeneous and full phase diagrams. Two new second order phase boundaries emerge, curve PD separating the massless from the inhomogeneous phase and curve PB separating the massive from the inhomogeneous phase. By abuse of language, we shall denote these two branches as “horizontal” and “vertical” phase boundaries. The horizontal one can be determined by a standard perturbative stability analysis. The vertical one can be inferred from the phase diagram of a single baryon by a different kind of stability analysis, based on the thermodynamics of a single baryon [12]. Hence, one does not need the full solution of the HF problem (available in this case) if one is only interested in mapping out the phase diagram,

Let us adapt the same procedures to the connected region $1 \geq \nu > \nu_0$. The curve AP has already been constructed in Sec. I and remains valid. The horizontal phase boundary

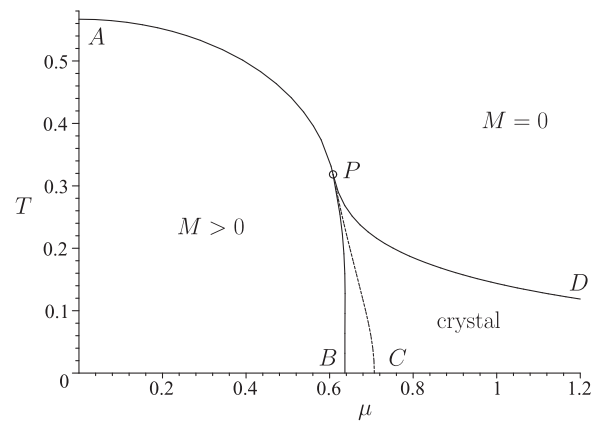


FIG. 4. Phase diagram of the one-component GN model ($\nu = 1$), adapted from [16]. The curve PC from the first order line of the translationally invariant solution is replaced by two second order phase boundaries PB and PD , whereas the second order line AP is unchanged. The tricritical point P is common to both phase diagrams.

PD can be obtained as follows. In the thermal HF approach, the central quantity is the HF potential or (Lorentz) scalar mean field $S(x)$ obeying the self-consistency equation

$$S(x) = -Ng^2 \langle \bar{\psi}\psi \rangle_{\text{th}}. \quad (12)$$

The right-hand side contains the thermal average over all single particle levels. It vanishes in the chirally symmetric phase. In order to test the stability against formation of an inhomogeneous broken phase, it is sufficient to consider only a single Fourier component with strength parameter S_1 vanishing at the phase boundary, since we are looking for a continuous phase transition. Thus, we shall use leading order (LO) perturbation theory in $S(x)$ using the ansatz

$$S(x) = 2S_1 \cos(2qx). \quad (13)$$

Because of the gaps in the spectrum of periodic potentials, this is not completely straightforward, as one has to use almost degenerate perturbation theory (ADPT). For the one-component case ($\nu = 1$), the technicalities can be taken over literally from the original work [16] and we can jump directly to the LO correction to the grand canonical potential density,

$$N^{-1} \delta\Psi^{(1)}(\beta, \mu) = \frac{S_1^2}{\pi} \left[\ln(\Lambda) - \oint_0^{\Lambda/2} dk \left(\frac{k}{k^2 - q^2} \right) \times \frac{\sinh \beta k}{\cosh \beta k + \cosh \beta \mu} \right]. \quad (14)$$

As shown in [16], the Cauchy principal value is the only relic from ADPT. Apart from this subtlety, the formula can be derived in naive second order perturbation theory. For fixed q , requiring that (14) vanishes yields the curve along which the massless phase is unstable against spatial modulations with wave number $2q$. The envelope of all these curves is the phase boundary PD . Rescaling of the variables according to

$$\kappa = \beta k, \quad \mu' = \beta \mu, \quad \Lambda' = \beta \Lambda, \quad q' = \beta q \quad (15)$$

allows one to solve the condition that (14) vanishes explicitly with the result

$$\beta_{\text{crit}}(\mu', q') = \lim_{\Lambda' \rightarrow \infty} \Lambda' \exp \Phi^{(1)}(\mu', q', \Lambda'), \quad (16)$$

where

$$\Phi^{(1)}(\mu', q', \Lambda') = \oint_0^{\Lambda'/2} d\kappa \left(\frac{\kappa}{(q')^2 - \kappa^2} \right) \frac{\sinh \kappa}{\cosh \kappa + \cosh \mu'}. \quad (17)$$

As in Sec. II, one can plot $T_{\text{crit}} = 1/\beta_{\text{crit}}(\mu', q')$ against $\mu_{\text{crit}} = \mu'/\beta_{\text{crit}}$ directly with Maple and draw a family of curves with varying q . The envelope of these curves can then easily be obtained numerically, leading to curve PD in Fig. 4. For the two-component GN model, we simply replace $\delta\Psi^{(1)}(\beta, \mu)$ in (14) by

$$\delta\Psi^{(2)}(\nu; \beta, \mu) = (1 - \nu) \delta\Psi^{(1)}(\beta, 0) + \nu \delta\Psi^{(1)}(\beta, \mu) \quad (18)$$

and $\Phi^{(1)}(\mu', q', \Lambda')$ in (16) and (17) by

$$\begin{aligned} \Phi^{(2)}(\nu; \mu', q', \Lambda') &= (1 - \nu) \Phi^{(1)}(0, q', \Lambda') \\ &+ \nu \Phi^{(1)}(\mu', q', \Lambda'). \end{aligned} \quad (19)$$

Everything else goes through as in the one-component case.

The vertical phase boundary PB is a different matter. Crossing this line in Fig. 4 from the left to the right, the homogeneous system in the massive phase develops an instability toward forming a single DHN baryon with maximal fermion number (or, equivalently, a kink). At $T = 0$, this happens at μ equal to the corresponding baryon mass divided by N ($\mu_c = 2/\pi$). As explained in Ref. [21], the (second order) phase boundary PB coincides with a first order phase boundary of a single baryon. It has been determined in [21] for the one-component GN model as follows. Take as ansatz for the mean field (inspired by the $T = 0$ DHN baryon)

$$\begin{aligned} S(x) &= M(1 + y[\tanh(yMx - c_0) - \tanh(yMx + c_0)]), \\ c_0 &= \frac{1}{2} \operatorname{artanh} y. \end{aligned} \quad (20)$$

Here, M is the thermal mass of the homogeneous solution, and y is a variational parameter. The grand canonical potential of a single DHN baryon is then given by

$$\begin{aligned} N^{-1} \Psi_B^{(1)}(\beta, \mu) &= \frac{2yM}{\pi} \left(1 + \ln \frac{m}{M} + \frac{\sqrt{1 - y^2}}{y} \arctan \frac{\sqrt{1 - y^2}}{y} \right) + \mu - \frac{1}{\beta} \ln \left[(1 + e^{-\beta(E_0 - \mu)})(1 + e^{\beta(E_0 + \mu)}) \right] \\ &+ \frac{1}{\pi\beta} \int_0^\infty dk \frac{2yM}{k^2 + y^2 M^2} \ln \left[(1 + e^{-\beta(E_k - \mu)})(1 + e^{-\beta(E_k + \mu)}) \right], \end{aligned} \quad (21)$$

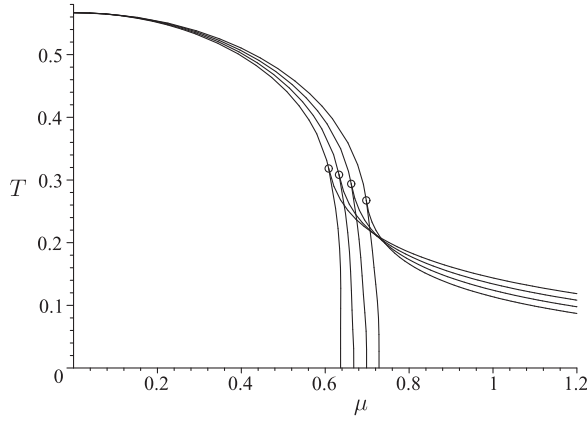


FIG. 5. Generalizing the phase diagram from Fig. 4 to values of ν in the connected region $1 \leq \nu \leq \nu_0$. From left to right: $\nu = 1.0, 0.95, 0.90, 0.85$.

with $E_0 = \sqrt{1 - y^2}$ the energy of the discrete state. The parameter y has to be determined by minimization. For the standard one-component Gross-Neveu model, one finds that y jumps from 0 to 1 across a phase boundary interpolating nicely between the base point M_B and the tricritical point of the homogeneous solution.

It is now easy to generalize this procedure to two components by setting

$$\Psi_B^{(2)}(\nu; \beta, \mu) = (1 - \nu)\Psi_B^{(1)}(\beta, 0) + \nu\Psi_B^{(1)}(\beta, \mu). \quad (22)$$

From here on, everything follows the one-component case. For a given T and ν one determines the chemical potential where the homogeneous minimum (at $y = 0$) and the second minimum at $1 \geq y > 0$ are degenerate. This gives the critical chemical potential on the vertical curve PB . Here we do not have the full HF solution so that we cannot directly check the result. The fact that the vertical phase boundary obtained in this way ends precisely at the tricritical point of the homogeneous calculation with filling fraction ν is a strong indication that the result is indeed correct.

It is interesting to take the limit $T \rightarrow 0$ of (21). In the region $\mu < 1$ of interest here, one finds

$$\begin{aligned} & \lim_{\beta \rightarrow \infty} N^{-1} \Psi_B^{(2)}(\beta, \mu) \\ &= \frac{2y}{\pi} + \frac{2\sqrt{1-y^2}}{\pi} \arctan \frac{\sqrt{1-y^2}}{y} - \sqrt{1-y^2} \\ &+ \nu \Theta\left(y - \sqrt{1-\mu^2}\right) \left(\sqrt{1-y^2} - \mu\right). \end{aligned} \quad (23)$$

The minimum with respect to y can be found by differentiation, yielding

$$\arctan \frac{\sqrt{1-y^2}}{y} = (1 - \nu) \frac{\pi}{2}. \quad (24)$$

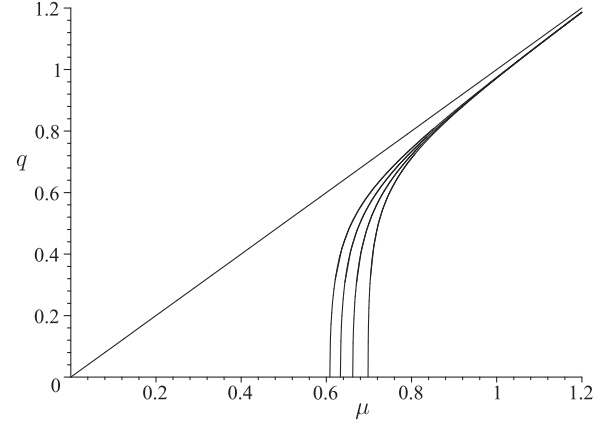


FIG. 6. Wave number q at which the massless phase is unstable. The result has been computed along the phase boundary, but is plotted against the chemical potential. The ν values are the same as in Fig. 5, again from left to right. The straight line is the asymptote $q = \mu$; the base points on the μ axis are the tricritical values.

This is precisely the relation between the parameter y in the potential and the occupation fraction ν of the valence state from the DHN baryon. The phase transition happens at $\mu = \mu_c$ where the right-hand side of (23) vanishes in addition to (24). Combining both conditions, we find

$$\mu_c = \frac{2y}{\pi\nu} = \frac{M_B(\nu)}{\nu}, \quad (25)$$

as expected. Thus, at $T = 0$, we see an instability against formation of a single DHN baryon with fermion number νN .

The result for several ν values between 1 and ν_0 (where the tricritical point of the homogeneous calculation leaves the perturbative phase boundary) is shown in Fig. 5 and shows no surprise. All the curves evolve in a gentle way as long as one does not cross the bifurcation point ν_0 . The wave number of the critical fluctuation changes with μ along the phase boundaries PD . It vanishes at the tricritical point and approaches $q = \mu$ asymptotically. This is illustrated in Fig. 6, showing again a smooth variation between the standard chemical potential ($\nu = 1$) and the bifurcation point ($\nu = \nu_0$).

IV. FULL SOLUTION IN THE DISCONNECTED REGION

If $\nu < \nu_0$, first and second order phase boundaries of the homogeneous phase diagram are disconnected. The tricritical point on the perturbative phase boundary no longer exists. Instead, a critical point appears at the end of a first order phase boundary. We have redrawn the homogeneous phase diagram at $\nu = 3/4$ in Fig. 7. How does this phase diagram change if we allow for inhomogeneous condensates? We expect that the perturbative phase boundary

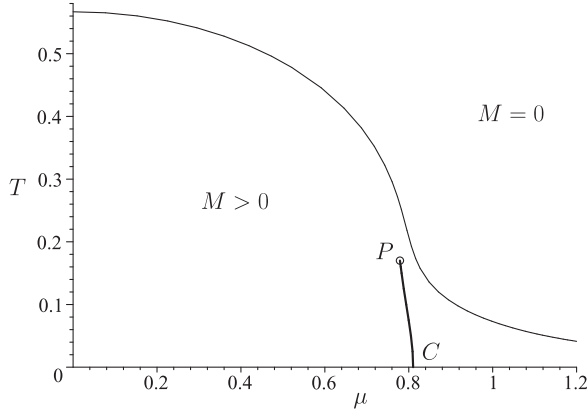


FIG. 7. Homogeneous phase diagram of two-component GN model with $\nu = 3/4$, same as curve 5 in Fig. 3. The (thin) perturbative phase boundary remains valid for all μ , the (fat) first order phase boundary stays within the massive region, ending at the critical point P . The order parameter M is discontinuous along curve PC .

remains valid. The critical point is now immersed into the massive homogeneous phase. This situation is reminiscent of the massive GN model, except that the massless phase does not exist there. In the massive GN model, it was found that the first order line disappears if one allows for crystal phases. It is replaced by two second order lines starting from the same critical point (μ_c, T_c) in a cusp and enclosing a crystal region. These lines are analogous to the horizontal and vertical branches in the connected region ($1 \geq \nu \geq \nu_0$). Let us once again try to construct these curves using appropriate stability analyses.

Consider first the horizontal curve. In the preceding section, the corresponding phase boundary signals instability of the massless phase against crystallization. Here, instead, we are dealing with the instability of the massive phase, at least close to the point P . This can again be dealt with by a perturbative stability analysis with minor changes due to the nonzero mass on the homogeneous side. Actually, the necessary formulas can be taken over from previous work on the massive GN model. The relevant formulas can be found in Ref. [25] on the massive chiral GN model [with U(1) chiral symmetry]. We take the condition for the instability, Eq. (37) of Ref. [25], and set the pseudoscalar potential P_1 and the confinement parameter γ equal to 0 with the result

$$N^{-1} \delta \Psi^{(1)}(\beta, \mu) = \frac{E_q^2 S_1^2}{\pi} \oint_0^\infty dk \frac{1}{E_k(k^2 - q^2)} \times \left(\frac{1}{e^{\beta(E_k - \mu)} + 1} + \frac{1}{e^{\beta(E_k + \mu)} + 1} \right) - \frac{E_q S_1^2}{2\pi q} \ln \left(\frac{E_q - q}{E_q + q} \right). \quad (26)$$

The step from here to the two-component system is once again trivial,

$$\delta \Psi^{(2)}(\nu; \beta, \mu) = (1 - \nu) \Psi^{(1)}(\beta, 0) + \nu \Psi^{(1)}(\beta, \mu). \quad (27)$$

Expression (27) and its derivative with respect to q must vanish along the phase boundary. Here one cannot solve the equation for T explicitly like in the massless case, Sec. III, but has to find the solution numerically.

Now turn to the vertical phase boundary. From the massive GN model and Ref. [21], we also know that it is closely related to a first order line in the thermodynamics of a single baryon. It should connect the critical point from the homogeneous calculation with the point

$$\mu_c = \frac{2 \sin(\pi\nu/2)}{\pi\nu} \quad (28)$$

at $T = 0$, i.e., the mass of a DHN baryon (fermion number νN) divided by ν . During the calculation, one has to minimize (21) and (22) with respect to the variational parameter y .

In Fig. 8, we show the result of such a calculation at $\nu = 3/4$. We find, indeed, the two second order lines starting at the critical point in a cusplike shape and enclosing a crystal phase, familiar from the massive GN model. However, there is one novel feature. The horizontal boundary intersects the perturbative homogeneous boundary in the point Q , giving rise to a new kind of bifurcation as a function of μ . Unlike P , the critical point Q does not exist in the homogeneous phase diagram. The curve segment PQ has been generated from the stability analysis of the massive phase and uses the thermal mass from the

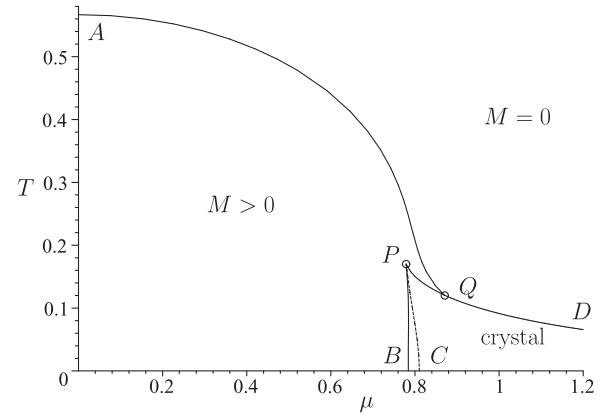


FIG. 8. Full phase diagram of two-component GN model with $\nu = 3/4$ including inhomogeneous region. The first order line PC of Fig. 7 is replaced by the second order line PB (instability against single baryon formation). Line PQ , perturbative instability of massive phase against periodic modulation; curve QD , similar instability of massless phase. The curve AQ and the point P are common with Fig. 7, point Q is a new bifurcation point where two critical lines intersect.

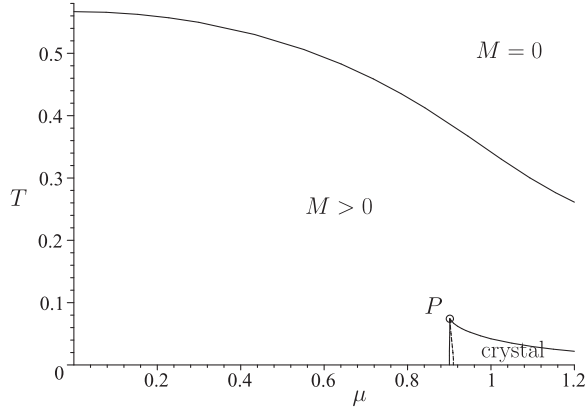


FIG. 9. Full phase diagram of two-component GN model with $\nu = 1/2$, showing that the crystal region shrinks rapidly with decreasing ν . The corresponding homogeneous phase boundary corresponds to curves 5 in Fig. 3. The point analog to Q in Fig. 8 cannot be seen in the region explored here.

homogeneous phase diagram. The part QD of the curve is the instability of the massless phase, as discussed in the connected region. To see how things evolve with decreasing ν , we also tried to find the bifurcation point Q for other values of ν . It turns out that the distance between P and Q increases very rapidly with decreasing ν , so that we could find the point Q only down to $\nu \approx 0.70$ with our calculational method.

If we proceed to smaller ν values, the inhomogeneous region shrinks rapidly, as shown in Figs. 9 and 10. The similarity of these plots with those of the massive GN model is striking. A small occupation fraction of the valence levels quenches the crystal region in the phase diagram in a similar way as a large confinement parameter (or bare fermion mass) does. We cannot rule out that an intersection point Q is present here as well, but it should move to very high chemical potentials where the width of the inhomogeneous island becomes tiny.

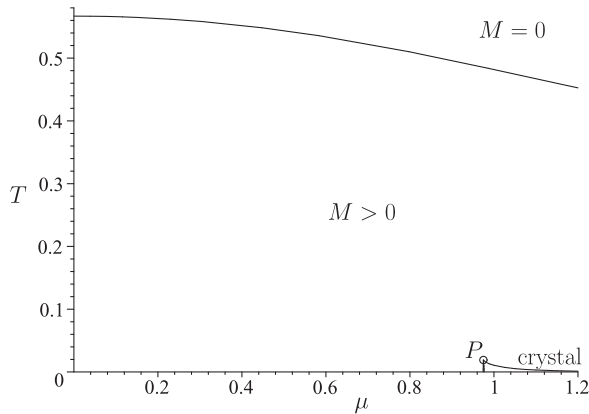


FIG. 10. Like Fig. 9, but for $\nu = 1/4$. The crystal region is now tiny, as could have been anticipated from the short first order line in curve 6 of Fig. 3.

Concluding this section, we remark that the classification into connected and disconnected phase diagrams from the homogeneous calculation is not valid anymore, once one allows for inhomogeneous condensates. At least in the vicinity of the bifurcation point ν_0 , we have seen that the phase diagram is again connected, see Fig. 8.

V. SUMMARY

In this paper, we have taken up a recent proposal by Benini *et al.* [10,11] of generalizing the (large N) GN model at finite density. The idea is to leave the Lagrangian of the GN model unchanged, but impose a chemical potential that acts on a fraction ν of the N fermion flavors only. We refer to this extension as the two-component GN model. It generalizes the notion of partially filled valence states familiar from the DHN baryon to partially filled bands for a periodic array of such baryons. Results are already available at $T = 0$, both for the GN model (Z_2 chiral symmetry [10,11]) and the chiral GN model [$U(1)$ chiral symmetry [12]]. The aim of the present work was to study the two-component GN model at finite temperature and chemical potential in the large N limit. Rather than performing a full thermal HF calculation, we have constructed the phase diagram exclusively with the help of different kinds of stability analyses. As is known from a recent study of the massless and massive one-component GN models, this is sufficient to yield the exact phase boundaries and critical points, both perturbative and nonperturbative.

In a first step, we had to generalize the thermodynamic calculation assuming a homogeneous condensate. This simple exercise already revealed a qualitatively new phase diagram as compared to the corresponding one-component calculation. As a function of ν , a bifurcation was identified at $\nu = \nu_0 = 0.8271$. For larger occupation fraction, the phase diagram looks similar to the one at $\nu = 1$. A second order phase boundary separating massless and massive phases ends at a tricritical point, where it goes over into a first order phase boundary with a discontinuous jump in the fermion mass. For $\nu < \nu_0$, the tricritical point disappears and the perturbative second order line does not end anymore. A first order line appears inside the massive region, terminating at a critical point. For $\nu = \nu_0$, this critical point touches the perturbative phase boundary and is converted into a tricritical point. The length of the first order line decreases rapidly with decreasing ν .

If one allows for inhomogeneous condensates, the distinction between the two regimes separated by the bifurcation at ν_0 stays clearly visible. For $\nu > \nu_0$, the stability analyses give a phase diagram looking very much like that of the one-component model. The perturbative phase boundary between massless and massive homogeneous phases remains valid. The tricritical point of the homogeneous phase diagram also stays in place. The first order line disappears, being replaced by a cusplike

structure of two second order lines, framing a crystal region. In the homogeneous phase diagram, for $\nu < \nu_0$, a first order line ends inside the massive region at a critical point. This is reminiscent of the massive (one-component) GN model, except that the massless region is absent there. The effect of inhomogeneous condensates is also similar to what happens in the massive GN model. The first order line disappears and gets replaced by a pair of second order lines emanating from the critical point of the homogeneous model and enclosing a crystal region. The size of this region shrinks rapidly with decreasing ν . An interesting new effect was found in the vicinity of the bifurcation, here at $\nu = 3/4$. The horizontal branch of the second order line intersects the perturbative phase boundary in a point Q where another type of bifurcation takes place, now as function of μ rather than ν . For the stability analysis, this means that one has to apply a perturbative

stability analysis both to the massive phase and the massless phase. Techniques formerly developed for massless and massive GN models have been used accordingly. Remarkably, in Fig. 8, one finds four distinct second order phase boundaries requiring all four kinds of stability analysis known from the GN model: massless to massive (AQ), massless to periodic (QD), massive to periodic (PQ), and massive to single soliton (PB). This underlines how much the extra freedom due to the parameter ν enriches the study of the phase diagram of the GN model.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [1] D. J. Gross and A. Neveu, *Phys. Rev. D* **10**, 3235 (1974).
 - [2] R. F. Dashen, B. Hasslacher, and A. Neveu, *Phys. Rev. D* **12**, 2443 (1975).
 - [3] J. Feinberg, *Ann. Phys. (Amsterdam)* **309**, 166 (2004).
 - [4] M. Thies, *J. Phys. A* **39**, 12707 (2006).
 - [5] J. Lenz, L. Pannullo, M. Wagner, B. Wellegehausen, and A. Wipf, *Phys. Rev. D* **101**, 094512 (2020).
 - [6] J. Lenz, M. Mandl, and A. Wipf, *Phys. Rev. D* **105**, 034512 (2022).
 - [7] R. Ciccone, L. Di Pietro, and M. Serone, *J. High Energy Phys.* **02** (2024) 211.
 - [8] V. Melin, Y. Sekiguchi, P. Wiegmann, and K. Zarembo, *Phys. Rev. Lett.* **133**, 101601 (2024).
 - [9] V. Melin, P. Wiegmann, and K. Zarembo, *arXiv:2512.19655*.
 - [10] F. Benini, O. Mamroud, T. Reis, and M. Serone, *Phys. Rev. Lett.* **135**, 141601 (2025).
 - [11] F. Benini, O. Mamroud, T. Reis, and M. Serone, *arXiv:2605.05307*.
 - [12] M. Thies, *Phys. Rev. D* **113**, 036016 (2026).
 - [13] G. Basar and G. V. Dunne, *Phys. Rev. Lett.* **100**, 200404 (2008).
 - [14] G. Basar and G. V. Dunne, *Phys. Rev. D* **78**, 065022 (2008).
 - [15] V. Schön and M. Thies, *Phys. Rev. D* **62**, 096002 (2000).
 - [16] M. Thies and K. Urlichs, *Phys. Rev. D* **67**, 125015 (2003).
 - [17] O. Schnetz, M. Thies, and K. Urlichs, *Ann. Phys. (Amsterdam)* **314**, 425 (2004).
 - [18] O. Schnetz, M. Thies, and K. Urlichs, *Ann. Phys. (Amsterdam)* **321**, 2604 (2006).
 - [19] J. Feinberg and A. Zee, *Phys. Lett. B* **411**, 134 (1997).
 - [20] M. Thies and K. Urlichs, *Phys. Rev. D* **71**, 105008 (2005).
 - [21] M. Thies, *Phys. Rev. D* **110**, 096012 (2024).
 - [22] J. Braun, S. Finkheimer, F. Karbstein, and D. Roscher, *Phys. Rev. D* **91**, 116006 (2015).
 - [23] A. Koenigstein, L. Pannullo, S. Rechenberger, M. Steil, and M. Winstel, *J. Phys. A: Math. Theor.* **55**, 375402 (2022).
 - [24] U. Wolff, *Phys. Lett. B* **157**, 303 (1985).
 - [25] C. Boehmer, U. Fritsch, S. Kraus, and M. Thies, *Phys. Rev. D* **78**, 065043 (2008).