

Bremsstrahlung photon polarization observables in Bethe-Heitler process: Calculation in an exact kinematics

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The effects of polarization transfer from the initial electron to the bremsstrahlung photon in electron-nucleus scattering (Bethe-Heitler process) are considered. The calculation is carried out without the assumption of smallness of the electron mass. A detailed comparison with previous well-known works is made. Some issues related to neglecting the electron mass are shown and commented on. The results are applicable to the modeling of polarized cross sections at energies down to the electron mass scale.

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I. INTRODUCTION

Bremsstrahlung, a fundamental process in quantum electrodynamics (QED), has been studied since the early development of the theory, with foundational work dating back to the 1930s [1–3]. Polarization phenomena in this process were first calculated in the 1950s [4–6]. Subsequent detailed investigations into the polarization of bremsstrahlung radiation produced by an electron beam near an atomic nucleus [7,8] led to the concept of generating polarized photon beams.

It was demonstrated that an unpolarized electron beam produces linearly polarized photons, whereas a polarized electron beam yields circularly polarized photons—with the degree of polarization directly proportional to that of the initial electron beam. These characteristics have been extensively exploited at various accelerator facilities. Notably, they were utilized in Hall B at the Thomas Jefferson National Accelerator Facility (JLab) to operate a high-energy polarized photon beam [9]. More recently, the technique has been applied in Hall D at JLab, which hosts the GlueX experiment [10]. This experiment employs a 9 GeV linearly polarized photon beam in the search for hybrid mesons.

Scientific interest in photon polarization extends beyond its role as a selective filter for certain reaction mechanisms, encompassing also the generation of

secondary beams—most notably, polarized positron beams. Theory predicts that a sufficiently energetic circularly polarized photon beam can produce a polarized e^+e^- pair, with both longitudinal and transverse polarization components scaling proportionally to the initial photon beam's polarization [8]. However, polarization transfer proves markedly less efficient in the transverse plane than in the longitudinal one, yielding a positron beam that is predominantly longitudinally polarized. Although the underlying calculations date back many years, experimental confirmation of circular-to-longitudinal polarization transfer has been achieved only recently. This milestone was reached through independent efforts at three major facilities, each employing distinct methods to generate the initial polarized photon beam: the High Energy Accelerator Research Organization [11], the Stanford Linear Accelerator Center [12], and the Continuous Electron Beam Accelerator Facility (CEBAF) [13]. The proof-of-principle PEPPo experiment [14] successfully demonstrated efficient transfer of polarization from the initial electron beam to a secondary positron beam. This secondary beam is generated via polarized bremsstrahlung produced by the initial electrons [15]. This technique, particularly suitable for continuous electron beam facilities, promises to extend the reach of polarized positron capabilities—scaling from MeV to GeV electron accelerators. As a result, it will open new avenues for physics research with polarized positron beams, making such studies accessible to a broad scientific community.

The prospect of operating a high-intensity polarized positron beam at CEBAF has spurred significant scientific activity, as summarized in a recent Topical Issue of the European Physical Journal [16]. This initiative has given rise to an extensive experimental program [17] centered around a novel positron source. The source is designed to

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produce a continuous positron beam—either polarized or unpolarized—with a maximum polarization of up to 60% and an intensity ranging from 100 nA to 3 μ A. Notably, the beam intensity exhibits an inverse relationship with the degree of polarization. This capability will be realized through a new positron injector [18] that leverages the PEPPo technique, marking a substantial advancement in the field.

Optimizing the design of such a facility is a complex process that begins with a precise and comprehensive description of polarization transfer in bremsstrahlung and pair-creation reactions. Early theoretical work [8], relying on ultrarelativistic and small-angle approximations, was later found to have significant limitations [19]. The core issue lies in the unjustified neglect of the electron mass across the full kinematic range. This shortcoming becomes especially evident in two critical scenarios. First, near the bremsstrahlung endpoint, where the electron transfers virtually all its kinetic energy to the photon. Second, in pair production, when one of the resulting particles emerges with minimal kinetic energy. These limitations grow more pronounced at lower energies of the initial electron or photon [20].

To overcome these deficiencies, subsequent research incorporated finite lepton mass effects [21], specifically tailored for the peripheral kinematics regime. However, other kinematic domains also exist where the finite electron mass plays a crucial role. As demonstrated in [22], certain regions are essential for properly accounting for mass dependence in unpolarized observables, underscoring the need for more complete theoretical treatments.

The present work aims to derive and analyze complete expressions for polarization observables in the bremsstrahlung reaction. Crucially, these expressions are free from kinematic approximations and are obtained at the first order of perturbation theory. The primary motivation behind these calculations is to optimize the production of polarized photon beams. A key application lies in developing low-energy PEPPo-like sources of polarized positrons, where precise control over photon polarization is essential for efficient positron beam generation.

The structure of the paper is as follows. Section II introduces the key kinematic parameters governing the scattering process. In Sec. III, we present the process amplitude and discuss the spin states of initial and final particles. This section also defines the cross sections for both unpolarized and polarized particle collisions. Section IV examines aspects of the process that emerge beyond the Born approximation. Section V offers a comparative analysis of our precise calculations with previously published results. Section VI explores various spectral distributions and demonstrates how cross sections vary with different kinematic variables across multiple energy regimes. Section VII addresses a range of physical challenges that arise when applying established formulas

outside their intended scope of applicability. The paper includes two Appendixes with Supplementary Material. Appendix A details the construction of pure photon polarization states, while Appendix B provides the complete, exact expressions for both unpolarized and polarized cross sections.

II. KINEMATICS

Let us consider the process of single-photon bremsstrahlung in an electron–nucleus collision (see Fig. 1). The four-momentum conservation law for this process is given by

$$p_1 + p_2 = p_3 + p_4 + k, \quad (1)$$

where $p_{1,3} = (E_{1,3}, \mathbf{p}_{1,3})$ are the four-momenta of the initial and scattered electrons, with energies $E_{1,3}$ and three-momenta $\mathbf{p}_{1,3}$; $p_{2,4} = (E_{2,4}, \mathbf{p}_{2,4})$ are the four-momenta of the initial and recoil nucleus, with energies $E_{2,4}$ and three-momenta $\mathbf{p}_{2,4}$; $k = (\omega, \mathbf{k})$ is the four-momentum of the bremsstrahlung photon, where ω denotes its energy and $\mathbf{k}$ its three-momentum. We develop our formalism in the laboratory frame, assuming the target nucleus is at rest—i.e., $\mathbf{p}_2 = \mathbf{0}$ and $E_2 = M_A$, where M_A is the mass of the target nucleus. Consequently, the total energy of the reaction is determined by the energy of the initial electron E_1 , which has mass m_e . We introduce the standard notation $\gamma = E_1/m_e$, which will be used throughout this paper.

The square of the total center-of-mass energy, s , is

$$s = (p_1 + p_2)^2 = M_A(2E_1 + M_A) + m_e^2. \quad (2)$$

The four-momentum transferred from the incoming electron to the target is given by $q = p_4 - p_2$, and its square is

$$q^2 = 2M_A(M_A - E_4) = -\frac{2M_A}{M_A + E_4} |\mathbf{p}_4|^2. \quad (3)$$

We note that the second form of this expression is better suited for numerical treatment, as it avoids the subtraction of two large but nearly equal quantities. Specifically, at low momentum transfer, the final nucleus moves slowly, so that $E_4 \approx M_A$.

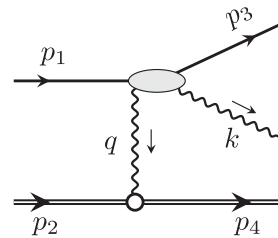


FIG. 1. Notation of momenta for the bremsstrahlung process.

The phase space volume for the reaction depicted in Fig. 1 takes the standard form

$$d^9\Phi_3 = \frac{1}{(2\pi)^5} \delta(p_1 + p_2 - p_3 - p_4 - k) \frac{d\mathbf{p}_3}{2E_3} \frac{d\mathbf{p}_4}{2E_4} \frac{d\mathbf{k}}{2\omega}. \quad (4)$$

Taking into account the δ function and integrating over the trivial variables, we obtain

$$d\Phi_3 = \frac{1}{2^8 \pi^5} F_b \omega d\omega d\Omega_{\mathbf{k}} d\Omega_{\mathbf{p}_3}, \quad (5)$$

where $d\Omega_{\mathbf{k}} = dC_{1k} d\varphi_k$ and $d\Omega_{\mathbf{p}_3} = dC_{13} d\varphi_3$ are the solid-angle elements for the final photon and the scattered electron, respectively. We use the following notation for the cosines of the angles: $C_{1k} \equiv \cos \theta_{1k}$ and $C_{13} \equiv \cos \theta_{13}$. The reaction angles θ_{1k} and θ_{13} are defined as the angles between the initial electron beam momentum $\mathbf{p}_1$ and the momenta of the bremsstrahlung photon $\mathbf{k}$ and the scattered electron $\mathbf{p}_3$, respectively (see Fig. 2). All azimuthal angles are measured from the x axis, as illustrated in Fig. 2. It is worth noting that in the literature it is customary to select a certain reference plane for measuring all azimuthal angles. For instance, when studying the final electron spectra or angular distribution, it is convenient to define the *scattering plane* as the plane spanned by the initial electron momentum $\mathbf{p}_1$ and the scattered electron momentum $\mathbf{p}_3$. In Ref. [8], the momenta $\mathbf{p}_1$ and $\mathbf{k}$ lie in the *radiation plane* (or *plane of emission*). If the initial electron beam lacks transverse polarization, there is no preferred azimuthal direction, and the dependence on the overall azimuthal orientation of the process becomes trivial. For example, the integration over the final photon's azimuthal angle yields

$$\int_0^{2\pi} d\varphi_k \cdots \rightarrow 2\pi \cdots. \quad (6)$$

We discuss this aspect in greater detail in Sec. VI. For now, we retain this freedom for convenience: the orientation of the x axis is arbitrary and chosen for overall simplicity, with

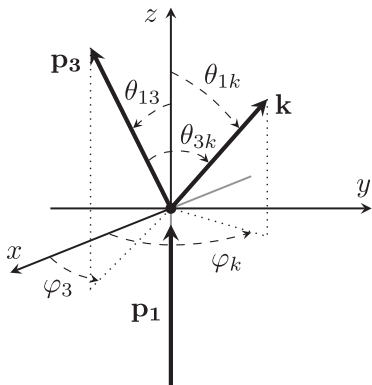


FIG. 2. Disposition of the 3-momenta in the laboratory frame.

all azimuthal angles subsequently measured relative to this axis.

The dimensionless quantity F_b in Eq. (5) is the Jacobian of the transformation to the new variables, and it takes the form

$$F_b = \frac{|\mathbf{p}_3|^2}{|\mathbf{p}_3|E_4 - E_3|\mathbf{p}_4|C_{34}}, \quad (7)$$

where C_{34} is the cosine of the angle between the momenta $\mathbf{p}_3$ and $\mathbf{p}_4$, given by

$$C_{34} = \frac{|\mathbf{p}_1|C_{13} - |\mathbf{p}_3| - |\mathbf{k}|C_{3k}}{|\mathbf{p}_4|}. \quad (8)$$

For completeness, we also present the cosine C_{14} between the momenta $\mathbf{p}_1$ and $\mathbf{p}_4$,

$$C_{14} = \frac{|\mathbf{p}_1| - |\mathbf{p}_3|C_{13} - |\mathbf{k}|C_{1k}}{|\mathbf{p}_4|}. \quad (9)$$

The energy of the scattered electron, E_3 , is determined by the energy-momentum conservation law (1)—or, equivalently, by the δ -function constraint in (4). It is given by

$$E_3 = \frac{-B + c_2\sqrt{D}}{A}, \quad (10)$$

where the coefficients are defined as $A = c_1^2 - c_2^2$, $B = c_1c_3$, $D = c_3^2 - Am_e^2$ and the auxiliary quantities c_1 , c_2 , and c_3 read

$$\begin{aligned} c_1 &= E_1 - \omega + M_A, & c_2 &= |\mathbf{p}_1|C_{13} - |\mathbf{k}|C_{3k}, \\ c_3 &= E_1(\omega - M_A) + \omega M_A - |\mathbf{p}_1||\mathbf{k}|C_{1k} - m_e^2. \end{aligned} \quad (11)$$

The expression for C_{3k} (the cosine of the angle θ_{3k} between the momenta $\mathbf{p}_3$ and $\mathbf{k}$) can be derived using the cosine law,

$$C_{3k} = C_{13}C_{1k} + S_{13}S_{1k} \cos(\varphi_3 - \varphi_k), \quad (12)$$

where $S_{13} \equiv \sin \theta_{13}$ and $S_{1k} \equiv \sin \theta_{1k}$.

All integrations over angles and energies were performed using numerical methods.

III. CROSS SECTION

In the lowest-order approximation of perturbation theory, the Bethe-Heitler process is described by the one-photon exchange mechanism. This approximation corresponds to two Feynman diagrams (see Fig. 3). We neglect diagrams involving photons emission from the nucleus, as the nuclear mass M_A is much larger than the electron mass m_e , and our primary focus is on the low-energy regime.

Since our goal is to obtain the cross section for the emission of a polarized photon—parametrized by the

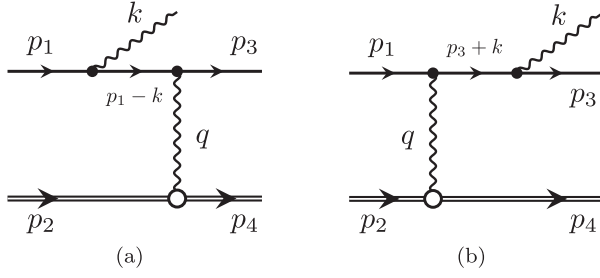


FIG. 3. The Feynman diagrams of the Bethe-Heitler process: (a) - emission from initial electron, (b) - emission from final electron.

Stokes parameters—we must calculate the bremsstrahlung cross section for each of two specific pure polarization states, $\epsilon_{1,2}^\mu$, and then combine these results to construct the Stokes parameters [23]. The explicit construction of these pure polarization states $\epsilon_{1,2}^\mu$ is provided in Appendix A. To obtain the invariant amplitudes of the process, we apply the Feynman rules of QED to the diagrams in Fig. 3, yielding the expression

$$\mathcal{M}_i = \frac{e^3 Z}{q^2} [\bar{u}(p_3) O_{\mu\nu} u(p_1)] [\bar{u}(p_4) \gamma^\mu u(p_2)] \epsilon_i^\nu, \quad (13)$$

where Z is the electric charge of the nucleus in units of the proton charge $e = \sqrt{4\pi\alpha} > 0$. The fine-structure constant takes the value $\alpha \approx 1/137$, and other parameter values are taken from the Particle Data Group summary [24]. We note that q^2 , which stems from the intermediate photon propagator, largely determines key observables in the process—such as the angular distribution of the cross section and the linear polarization of the final photon. In the following analysis, we study distributions over the angles and energies of the final particles. Consequently, q^2 is not fixed but is integrated over the kinematically allowed domain.

The quantity $O_{\mu\nu}$ in Eq. (13) takes the form

$$O_{\mu\nu} = \frac{\gamma_\mu (\hat{p}_1 - \hat{k} + m_e) \gamma_\nu}{-2(p_1 k)} + \frac{\gamma_\nu (\hat{p}_3 + \hat{k} + m_e) \gamma_\mu}{2(p_3 k)}, \quad (14)$$

where $\hat{p} \equiv \gamma_\mu p^\mu$ denotes the Feynman slash notation. The two terms in this expression correspond to the two Feynman diagrams shown in Fig. 3.

The cross section of the process can be expressed in the standard form

$$d\sigma_{ij} = \frac{1}{4F} \sum_s \overline{\mathcal{M}_i \mathcal{M}_j^+} d\Phi_3, \quad (15)$$

where the phase-space volume of the final particles, $d\Phi_3$, was defined in Eq. (5), and the invariant flux F of the initial particles is given by

$$F = \sqrt{(p_1 p_2)^2 - m_e^2 M_A^2} = \frac{1}{2} \lambda^{\frac{1}{2}}(s, m_e^2, M_A^2). \quad (16)$$

Here $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is the well-known triangle function.

The overline summation operator $\bar{\Sigma}$ in Eq. (15) denotes the averaging over the initial-particle spin states and the summation over the final-particle spin states (excluding photon polarizations). It is defined as

$$\bar{\Sigma} \cdots = \frac{1}{2} \sum_{s_1} \left[\frac{1}{2} \sum_{s_2} \left[\sum_{s_3, s_4} \cdots \right] \right]. \quad (17)$$

To perform this operation for fermions, we employ the spin density matrices in the following form:

$$\begin{aligned} \frac{1}{2} \sum_s u^s(p_1) \bar{u}^s(p_1) &= \frac{1}{2} (\hat{p}_1 + m_e) (1 - \gamma_5 \hat{a}_1), \\ \frac{1}{2} \sum_s u^s(p_2) \bar{u}^s(p_2) &= \frac{1}{2} (\hat{p}_2 + M_A), \\ \sum_s u^s(p_3) \bar{u}^s(p_3) &= \hat{p}_3 + m_e, \\ \sum_s u^s(p_4) \bar{u}^s(p_4) &= \hat{p}_4 + M_A, \end{aligned} \quad (18)$$

where $a_1 = (a_1^0, a_1^x, a_1^y, a_1^z)$ is the polarization vector of the initial electron beam. For a *longitudinally* polarized electron beam, the polarization vector is given by

$$a_1 = \lambda_1 \left(\frac{|\mathbf{p}_1|}{m_e}, 0, 0, \frac{E_1}{m_e} \right), \quad (19)$$

where $\lambda_1 = \pm 1$ represents the doubled projection of the electron spin onto the z axis for an electron at rest. In the case of *transverse* polarization, the vector a_1 is a unit vector perpendicular to the beam direction (lying in the xy plane). Its orientation is determined by the azimuthal angle φ_a , measured from the x axis,

$$a_1 = (0, \cos \varphi_a, \sin \varphi_a, 0). \quad (20)$$

The product of the amplitudes in Eq. (15), after applying the summation operator $\bar{\Sigma}$, takes the form

$$\bar{\Sigma} \mathcal{M}_i \mathcal{M}_j^+ = \frac{(4\pi\alpha)^3 Z^2}{4(q^2)^2} S_{ij}, \quad (21)$$

where S_{ij} is expressed in terms of traces of products of Dirac γ matrices,

$$\begin{aligned} S_{ij} &= \epsilon_i^\alpha (\epsilon_j^\beta)^* \text{Tr}[(\hat{p}_1 + m_e)(1 - \gamma_5 \hat{a}_1) \tilde{O}_{\nu\beta} (\hat{p}_3 + m_e) O_{\mu\alpha}] \\ &\quad \times \text{Tr}[(\hat{p}_2 + M_A) \gamma^\nu (\hat{p}_4 + M_A) \gamma^\mu]. \end{aligned} \quad (22)$$

The tilde symbol denotes Dirac conjugation, i.e., $\tilde{O} \equiv \gamma_0 O^\dagger \gamma_0$, where $O^\dagger$ is the Hermitian conjugation of O . Evaluating the traces in Eq. (22) yields rather cumbersome expressions. The explicit forms for the quantities S_{ij} are provided in Appendix B.

We now derive the final expression for the cross section suitable for numerical estimation. Substituting the spin-averaged amplitudes from Eq. (21) and the phase-space volume $d\Phi_3$ from Eq. (5) into the general formula Eq. (15), we obtain,

$$d\sigma_{ij} = \frac{\alpha^3 Z^2}{2^6 \pi^2 F} \frac{\omega F_b}{(q^2)^2} S_{ij} d\omega d\Omega_{\mathbf{k}} d\Omega_{\mathbf{p}_3}. \quad (23)$$

It is convenient to parametrize the polarization state of the final photon using the Stokes parameters $P_{1,2,3}$ [23]. To achieve this, we combine the differential cross sections $d\sigma_{ij}$ into the following quantities:

$$d\sigma_I = d\sigma_{11} + d\sigma_{22}, \quad (24)$$

$$d\sigma_{P_1} = d\sigma_{11} - d\sigma_{22}, \quad (25)$$

$$d\sigma_{P_2} = d\sigma_{12} + d\sigma_{21}, \quad (26)$$

$$d\sigma_{P_3} = i(d\sigma_{21} - d\sigma_{12}), \quad (27)$$

where we follow the notation of [23] [see their formula (27)]. The quantity $d\sigma_I$ [Eq. (24)] represents the cross section for unpolarized photon production, corresponding to the total intensity I of the emitted photon beam. The Stokes parameters P_i of the polarized photon beam are then defined by the relations

$$d\sigma_{P_i} = P_i d\sigma_I, \quad i = 1, 2, 3. \quad (28)$$

IV. POSSIBLE IMPROVEMENTS

The model-dependent part of the above formulas is derived within the Born approximation, which corresponds to single-photon exchange between the incident electron and the target nucleus. However, the actual dynamical picture of this process can be significantly modified by various subtle effects. While most of these effects are prominent at low energies, they may also manifest under specific high-energy conditions.

A. Screening of the nucleus charge

In experimental setups, the target typically consists of atoms. Consequently, the incident electron does not interact solely with the nucleus (of charge Ze) but also experiences the influence of the surrounding electron cloud. This leads to an effective reduction of the nuclear charge perceived by the incoming particle—a phenomenon commonly referred

to as *screening*. To account for this effect, the atomic form factor $F(q)$ is incorporated into the scattering formalism. Specifically, the bare Coulomb propagator $1/q^2$ is modified as follows:

$$\frac{1}{-q^2} \longrightarrow \frac{1 - F(q)}{-q^2} = \sum_{i=1}^3 \frac{\alpha_i}{\beta_i^2 - q^2}. \quad (29)$$

We employ the parametrization of the atomic form factor based on the Molière approach [25], which relies on the Thomas-Fermi screening function. The empirical parameters in Eq. (29) encode the atomic structure and are given by,

$$\alpha_1 = 0.1, \quad \alpha_2 = 0.55, \quad \alpha_3 = 0.35, \quad (30)$$

$$b_1 = 6.0, \quad b_2 = 1.2, \quad b_3 = 0.3, \quad (31)$$

with $\beta_i = (Z^{1/3}/121)m_e b_i$. While the Molière approach provides a convenient parametrization, it remains a rather approximate treatment. For instance, certain finer details of atomic structure—such as electron correlation effects, shell-specific screening, or relativistic corrections—can induce sizeable deviations in the form factor, as discussed in [26]. In such cases, more refined models for the atomic form factors become necessary. Notable examples include the approach of [27], which incorporates explicit treatment of electron-density distributions, relativistic corrections, and shell-resolved screening contributions. Nevertheless, we retain the Molière formalism for the present analysis. The choice is justified by our focus on the low-energy scattering regime, where $E_1 \sim \text{few } m_e$.

It is important to note that a noticeable difference between the observable quantities in the screened and unscreened cases arises only at small values of q^2 . This is significant because q^2 enters the cross section as $1/(q^2)^2$, meaning the screening effect becomes crucially important during the integration over the angles of the final particles and consequently over the range of q^2 values.

B. Nuclear form factors

Another important aspect concerns nucleus form factors. At moderate and high photon energies—much larger than the average binding energy of a nucleon within the nucleus—the photon interacts predominantly with individual nucleons. At lower energies, however, the photon probes the dynamical structure of the nucleus as a whole. Our formalism assumes that the intermediate photon interacts with a nucleus of spin-1/2, which remains on shell after scattering. For simplicity, we employ a minimal form factor that approaches Z in the limit of small q^2 . We present numerical results for a lead target, whose nucleus predominantly has spin zero. Notably, our formalism remains valid in this case at small q^2 , as the form factor's leading behavior is still governed by the atomic number Z . A more

refined treatment could incorporate advanced models of the nuclear form factor, accounting for detailed nuclear structure and spin effects. We defer such improvements to future work, particularly in the context of Monte Carlo simulations.

C. Coulomb corrections

When the electron velocity v is sufficiently small, the standard perturbative treatment of the Coulomb interaction breaks down [28]. In this regime, the dimensionless parameter $\kappa = Ze^2/\hbar v$ becomes of order unity or larger, signaling that nonperturbative effects dominate.

For slow electrons or heavy nuclei (large Z), the long-range nature of the Coulomb force cannot be adequately captured within perturbative quantum field theory. Instead, a rigorous approach requires solving the Dirac equation in the external Coulomb field of the nucleus. This framework naturally incorporates relativistic effects, spin-orbit coupling, and bound-state contributions that are missed in the Born approximation.

The necessity of such a nonperturbative treatment has been demonstrated in specific physical scenarios. For instance, studies of low-energy electron scattering off heavy ions [29] have shown that Coulomb corrections can modify differential cross sections by tens of percent. Similarly, analyses of bremsstrahlung photon emission in strong Coulomb fields [30] reveal significant shifts in spectral features when exact solutions of the Dirac equation are employed. These examples underscore that in the low-velocity regime, where κ is not negligible, only a treatment rooted in the Dirac equation can provide reliable predictions for observables such as cross sections and photon spectra.

It is also worth noting that the dimensionless parameter κ can approach order unity not only at low velocities v , but also for heavy nuclei with large atomic number $Z \sim 100$. For lead ($Z = 82$), which serves as a representative example in the present work (as well as in [8]), κ reaches values near unity even at relativistic speeds $v \approx c$ —that is, across the entire energy range considered.

Despite this, we neglect Coulomb corrections in the following analysis for the sake of simplicity. This approximation simplifies the treatment of the scattering process and allows us to focus on the dominant mechanisms within the Born approximation framework. However, a more accurate quantitative description would necessarily require inclusion of these nonperturbative Coulomb effects.

D. Nonperturbative effects

It is worth noting that nonperturbative effects become significant when the final electron's kinetic energy approaches zero. In this regime, the electron cannot be adequately described by a plane wave. Consequently, the bremsstrahlung cross section at the high-frequency end of the spectrum does not vanish—instead, it exhibits a

nonzero limit that is intimately related to the photoelectric effect. This connection has been discussed in several foundation works, e.g., [31–33]. A different perspective on the same reaction is presented in [34], which offers additional arguments and extends the discussion. In our work, we also consider low final electron energies; however, we remain within the perturbative regime—our energies are not sufficiently low to trigger nonperturbative dynamics.

E. Landau-Pomeranchuk-Migdal effect

Several additional effects influence the bremsstrahlung process. One prominent example is the Landau-Pomeranchuk-Migdal (LPM) effect [35,36], which leads to a substantial suppression of the cross section under specific kinematic conditions. The underlying mechanism—even for a single scattering event—involves destructive interference in the forward region between two contributions shown in Fig. 3. The full LPM effect, however, arises from the coherent interference of bremsstrahlung radiation emitted during multiple scattering off different atoms. This coherence can lead to additional cancellation. In general, the LPM effect manifests when the spatial separation between successive scatterings is small compared to the photon formation length. In such a case, the photon does not have sufficient space to fully materialize, thereby suppressing the radiation.

For a comprehensive review of the LPM effect and other mechanisms affecting bremsstrahlung cross sections in thick targets, we refer the reader to Ref. [37].

V. COMPARISON WITH PREVIOUS RESULTS

The work of Olsen and Maximon [8] is widely employed to correct experimental data for radiative effects. Notably, their formalism incorporates polarization transfer observables, making it a key reference for studies involving spin-dependent phenomena. Hereafter, we shall refer to this paper as “OM.”

We begin by comparing our calculation of the unpolarized cross section—derived from Eq. (24)—with the results presented in Fig. 1 of Ref. [8]. Specifically, we focus on the differential cross section $d\sigma/d\omega d\xi$. The two formulations are related via the following transformation:

$$\frac{d\sigma}{d\omega d\xi} = J_\xi \frac{d\sigma}{d\omega dC_{1k}}, \quad (32)$$

where $\xi = 1/(1 + u^2)$ and $u = (|\mathbf{p}_1|/m_e) \sin \theta_{1k}$. The quantity J_ξ in Eq. (32) is the Jacobian that accounts for the change of variables from C_{1k} to ξ . It takes the explicit form,

$$J_\xi = \frac{dC_{1k}}{d\xi} = \frac{m_e^2}{2|\mathbf{p}_1|\xi^2 \sqrt{|\mathbf{p}_1|^2 - m_e^2(\frac{1}{\xi} - 1)}}. \quad (33)$$

Moreover, the cross section in Fig. 1 of Ref. [8] is rendered dimensionless through the introduction of an additional scaling factor,

$$\phi = (eZ)^2 \frac{e^4}{m_e}. \quad (34)$$

In their work [8], Olsen and Maximon present plots of the cross section and polarization observables as functions of the photon emission angle θ_{1k} . The authors carefully select the displayed angles to coincide with the peak regions of these observables. Specifically, the cross section attains its maximum value at an emission angle $\theta_{1k} \sim 1/(2\gamma)$. Meanwhile, both the linear polarization P_1 and the longitudinal circular polarization P_3^L reach their peaks at a slightly larger angle, $\theta_{1k} \sim 1/\gamma$. The transverse circular polarization P_3^T , however, exhibits a more complex angular dependence, displaying two distinct peaks: a negative peak located at $\theta_{1k} \sim 1/(2\gamma)$ and a positive peak at $\theta_{1k} \sim 2/\gamma$. To ensure a direct and meaningful comparison with the results of Olsen and Maximon, we adopt the same angular values in our analysis.

In Fig. 4, we reproduce Fig. 1 from Ref. [8], where $T_1 \equiv E_1 - m_e$ denotes the kinetic energy of the incident electron. Our exact numerical results are represented by solid lines: the solid black line corresponds to the cross section calculated without accounting for nuclear screening effects, whereas the thick grey line shows the same cross section

when nuclear screening is included [according to the modification introduced in Eq. (29)]. It is evident that the screened cross section lies below the unscreened one, as screening effectively reduces the electric charge of the scattering center. For comparison, we also plot the results from Eq. (7.2) of Ref. [8] using dashed lines: long dashes represents the unscreened case, and short dashes correspond to the screened cross section. A close inspection reveals that our calculation aligns well with the Olsen-Maximon curve for the unscreened scenario. However, agreement with their screened curve is observed only in the low-photon-energy region ($\omega/T_1 < 0.5$). This discrepancy highlights the first issue we wish to emphasize: Eq. (7.2) of Ref. [8] fails to accurately describe hard photon emission. Furthermore, we note that the dotted line extracted from Fig. 1 of Ref. [8] (represented by black circles in Fig. 4) agrees with our present calculation but deviates from Eq. (7.2) in the same paper [8]. This inconsistency suggests that the analytical approximation in Eq. (7.2) has limited validity, particularly in the high-energy regime.

Good agreement between the Olsen-Maximon results and our present calculation is observed only in the regime of high electron energies and small scattering angles. To illustrate this limitation, we reexamine the comparison by plotting Fig. 4 under markedly different kinematic conditions: specifically, for a low-energy electron beam with $E_1 = 1$ MeV and a relatively large photon emission angle $\theta_{1k} = 0.2$ rad (see Fig. 5). Under these conditions, a significant discrepancy becomes apparent between the OM results and our calculation. This divergence underscores the restricted validity of the OM approach outside its intended kinematic domain.

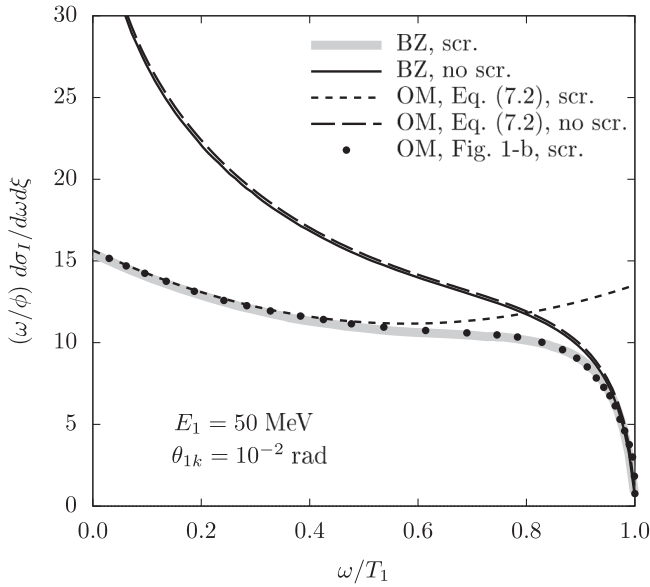


FIG. 4. Unpolarized cross section for bremsstrahlung of 50 MeV electrons scattered on lead at the photon emission angle $\theta_{1k} = 10^{-2}$ rad.

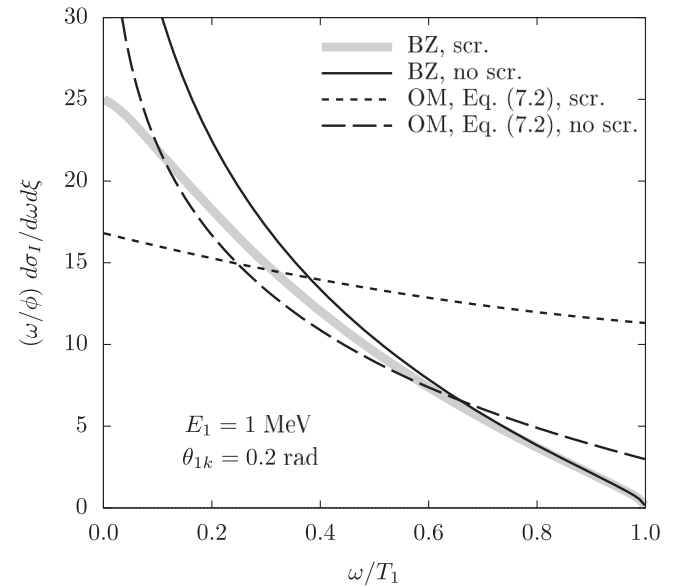


FIG. 5. Unpolarized cross section for bremsstrahlung of 1 MeV electrons scattered on lead at the photon emission angle $\theta_{1k} = 0.2$ rad.

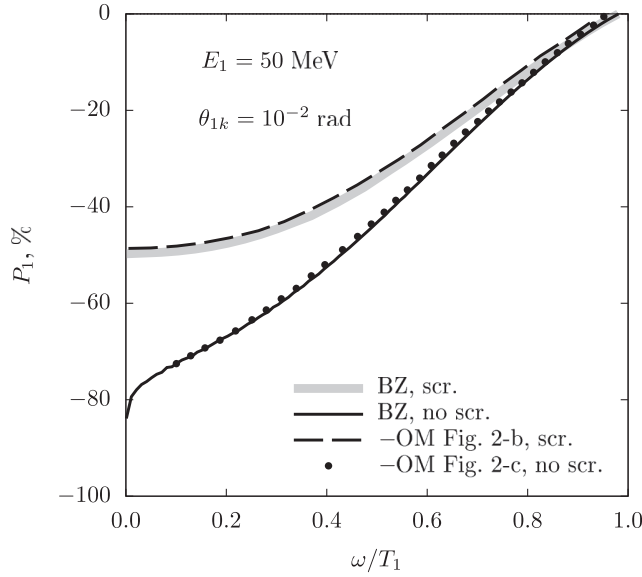


FIG. 6. Linear polarization of bremsstrahlung, P_1 , of 50 MeV electrons in lead at the photon emission angle $\theta_{1k} = 10^{-2}$ rad (we integrate over $d\Omega_{\mathbf{p}_3}$ and $d\varphi_k$), comparing with Fig. 2 of paper [8].

We now turn to the discussion of polarized observables. Following the conventions established in Refs. [8,38] the Stokes parameter P_1 is referred to as the *linear polarization*, while P_3 denotes the *circular polarization*.

In Fig. 6, we reproduce Fig. 2 of Ref. [8], which illustrates the scenario where the initial electron is unpolarized and the final photon is linearly polarized (characterized by the Stokes parameter P_1). The figure demonstrates that both the screened and unscreened cases are accurately reproduced in our calculation. However, a notable difference emerges in the overall sign of the effect: our plot displays the OM curves with an opposite sign relative to those presented in Ref. [8]. This discrepancy stems from differing conventions in defining the purely polarized photon states $\varepsilon_{1,2}^\mu$. Specifically, we define the three-vector ε_1 to lie within the photon scattering plane, while the three-vector ε_2 is oriented perpendicular to this plane. In contrast, Olsen and Maximon adopt the inverse convention (as detailed in the text and equations preceding Eq. (7.3) of Ref. [8]).

In Fig. 7, we reproduce Fig. 3 of Ref. [8], which illustrates the transfer of longitudinal polarization from the initial electron (with helicity $\lambda_1 = +1$) to the final photon, measured via the circular polarization observable (Stokes parameter P_3). The figure demonstrates excellent agreement between our results—represented by the solid black and grey lines—and those of Olsen and Maximon, shown as black dashed lines. Additionally, Fig. 7 includes data points (black dots) corresponding to the calculation by E. A. Kuraev *et al.* [see Eq. (46) in Ref. [21]] for the polarization transfer observable. This result was derived using the infinite momentum technique [39], which relies

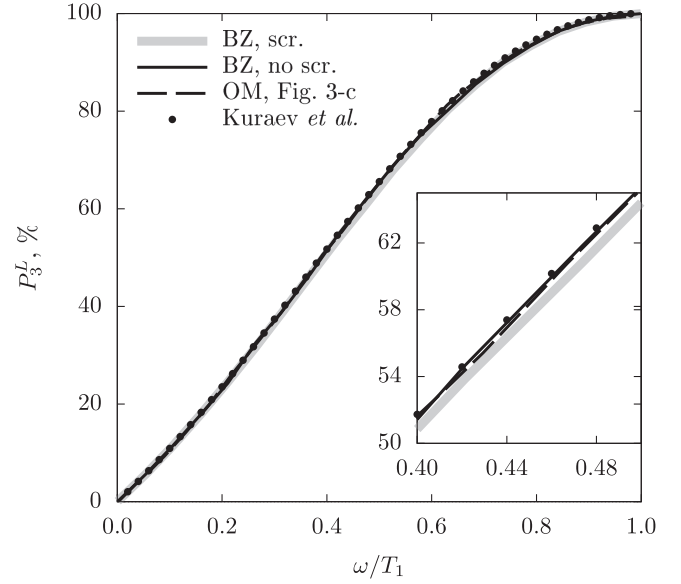


FIG. 7. Circular polarization of bremsstrahlung, P_3^L , of longitudinally polarized ($\lambda_1 = +1$) electrons with energy $E_1 = 50$ MeV in lead at the photon emission angle $\theta_{1k} = 10^{-2}$ rad (we integrate over $d\Omega_{\mathbf{p}_3}$ and $d\varphi_k$).

on the Sudakov parametrization of vectors—also known as light-cone decomposition [40]. This method is applicable in the peripheral kinematic regime, characterized by the hierarchy $s \gg -q^2 \approx M_A^2 \gg m_e^2$. Within this domain, we observe very good agreement among the E. A. Kuraev *et al.* calculation, the Olsen-Maximon result, and our precise numerical evaluation. However, it is important to note that such agreement should not be expected in kinematic regions involving low energies or where electron mass effects become significant. We will address this limitation in greater detail below.

Moreover, all these calculations—those by Olsen and Maximon as well as those by Kuraev's group—pertain to the unscreened scenario. Nevertheless, as the data demonstrate, the difference between the screened and unscreened cases for the observable P_3^L is remarkably small.

In Fig. 8, we reproduce Fig. 4 from Ref. [8], which illustrates the transfer of transverse polarization from the initial electron to the final photon, as measured by the circular polarization observable (Stokes parameter P_3). Once again, our calculation—represented by the solid black and grey lines—shows excellent agreement with the results of Olsen and Maximon, depicted by the thick black dashed line.

It should be noted that Fig. 8 is generated under specific kinematic setup: the photon is emitted in the plane containing the electron's transverse polarization vector, i.e., $\varphi_k = \varphi_a$. However, this configuration does not generally reflect actual experimental conditions. In practice, the photon is typically emitted with a nonzero azimuthal angle

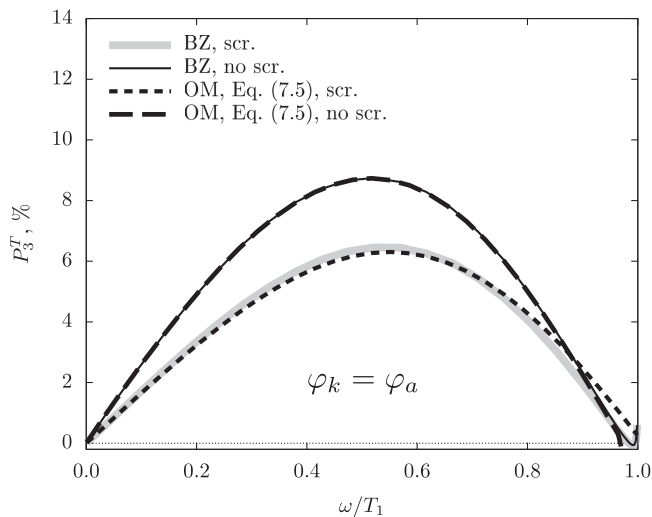


FIG. 8. Circular polarization of bremsstrahlung, P_3^T , of transversely polarized electrons with energy $E_1 = 50$ MeV in lead at the photon emission angle $\theta_{1k} = 4.1$ mrad (we integrate over $d\Omega_{p_3}$ and $d\varphi_k$).

relative to the fixed electron polarization vector. We address this more general scenario in Sec. VI.

In Table I, we present a comparative analysis of our calculations for all three polarizations components against the two most widely cited reference results: those of Olsen and Maximon [8] and those of Haug and Nakel [38]. The Haug–Nakel values were derived as follows (with equation numbers referring to Ref. [38]): the linear polarization P_1 is computed as the ratio of the integrated cross section from Eq. (3.362) to that from Eq. (3.363); the longitudinal circular polarization P_3^L is given by the ratio of Eq. (3.373) to Eq. (3.81); the transverse circular polarization P_3^T is obtained as the ratio of Eq. (3.376) to Eq. (3.81). It is worth emphasizing that for the linear polarization P_1 , we again report both the OM and HN results with an inverted sign. This adjustment stems from differing conventions in defining the pure photon polarization states $\epsilon_{1,2}^\mu$. At moderate beam energies E_1 , photon energies ω , and small emission angles θ_{1k} , our results show good agreement with both OM and HN predictions. However, the level of agreement deteriorates toward the kinematic boundaries, particularly at high photon energies ω . This discrepancy is examined in greater detail in the subsequent Section.

VI. NUMERICAL ANALYSIS

In this Section, we present a series of angular and energy distributions at moderately high energies to elucidate key general features of the bremsstrahlung process. Specific issues, as well as the behavior in the low-energy regime, are examined in greater detail in the subsequent section.

The angular dependence of photon emission is illustrated in Fig. 9, which reveals the well-known peaklike behavior

aligned with the initial electron beam direction. Integration over this peak region gives rise to the so-called large collinear logarithms of the form $\log \gamma$ [3]. The majority of the contribution is concentrated within a narrow cone defined by the angle $\theta_{1k} \sim 1/\gamma$; the position of this boundary is marked in Fig. 9 by a vertical dotted line. It is important to emphasize that the intensity maximum lies closer to the initial electron beam direction than the cone boundary. The precise locations of this maximum exhibits a mild dependence on the emitted photon energy ω , but it can be approximated as $\theta_{1k} \approx 1/(2\gamma)$. This approximate position is indicated in Fig. 9 by vertical dashed lines. In subsequent analysis, we will plot curves fixing the photon emission angle θ_{1k} at this approximate maximum—referred to as the “peak value.” The vertical dashed line Fig. 9 explicitly marks this “peak value.” Additionally, Fig. 9 includes data points corresponding to the results published in Olsen and Maximon’s work [8]. A comparison shows that at high electron energies ($E_1 \gg m_e$), our calculation agrees well with the OM results. However, this agreement deteriorates significantly in the low-energy regime—for instance, at $E_1 = 1$ MeV [see Fig. 9(a)]. This pattern of reduced agreement at low energies is consistently observed across other cases as well.

In Fig. 10, we present the unpolarized cross section as a function of the emitted photon energy ω , evaluated for various beam energies E_1 . All the curves are plotted at the peak emission angle, i.e., $\theta_{1k} \approx 1/(2\gamma)$. Once again, the results demonstrate that the screened cross sections (represented by thick grey lines) systematically lie below the unscreened counterparts (shown as black lines).

In Figs. 11 and 12, we present the dependence of the final photon’s polarization on the photon energy ω . The results reveal a marked difference in how the two polarization components respond to screening effects: while the circular polarization P_3^L exhibits only a weak sensitivity to screening, the linear polarization P_1 demonstrates a significant dependence—particularly at low photon energies ($\omega \rightarrow 0$).

We now proceed to examine the azimuthal dependence of the polarization degrees. Figures 13(a) and 13(b) illustrate the angular dependencies for the linear polarization degree P_1 . In Fig. 13(a), we present the dependence of the emitted photon’s P_1 Stokes parameter on the scattered electron’s azimuthal angle φ_3 , with integration performed over the photon’s azimuthal angle φ_k . The calculation corresponds to a fixed electron beam energy $E_1 = 50$ MeV and a photon energy $\omega = 0.2E_1$. Figure 13(b), in turn, displays the same observable— P_1 polarization degree—as a function of the emitted photon’s azimuthal angle φ_k . The results show that the polarization values remain constant (i.e., independent of the azimuthal angles) and are finite across the entire angular range.

Next, we consider the azimuthal dependencies of the circular polarization degree P_3 , which are presented in

TABLE I. The comparison of our precise calculation with the results of OM [8] and of Haug-Nakel (HN) [38]. We perform the integration of the unscreened cross sections over the solid angle element $d\Omega_{\mathbf{p}_3}$ and the azimuthal angle $d\varphi_k$.

ω/T_1	P_1			P_3^L			P_3^T		
	BZ	–OM	–HN	BZ	OM	HN	BZ	OM	HN
$E_1 = 1 \text{ MeV}, \theta_{1k} = 0.25 \text{ rad}$									
0.1	–0.1885	–0.2173	–0.1703	0.0490	0.0506	0.0473	0.0161	0.0104	–0.0059
0.2	–0.1133	–0.1844	–0.0908	0.1007	0.1045	0.0964	0.0298	0.0176	–0.0185
0.3	–0.0395	–0.1536	–0.0125	0.1548	0.1618	0.1468	0.0420	0.0221	–0.0374
0.4	0.0384	–0.1217	0.0703	0.2104	0.2223	0.1977	0.0530	0.0233	–0.0631
0.5	0.1232	–0.0870	0.1606	0.2666	0.2857	0.2477	0.0634	0.0208	–0.0964
0.6	0.2176	–0.0483	0.2612	0.3218	0.3517	0.2947	0.0747	0.0139	–0.1381
0.7	0.3258	–0.0039	0.3757	0.3740	0.4197	0.3355	0.0893	0.0013	–0.1889
0.8	0.4549	0.0484	0.5100	0.4201	0.4892	0.3654	0.1123	–0.0185	–0.2494
0.9	0.6190	0.1123	0.6740	0.4553	0.5594	0.3768	0.1539	–0.0484	–0.3192
$E_1 = 3 \text{ MeV}, \theta_{1k} = 0.1 \text{ rad}$									
0.1	–0.3794	–0.3838	–0.3726	0.0878	0.0883	0.0871	0.0199	0.0183	0.0139
0.2	–0.3205	–0.3314	–0.3111	0.1855	0.1868	0.1838	0.0353	0.0316	0.0217
0.3	–0.2630	–0.2798	–0.2504	0.2913	0.2940	0.2887	0.0467	0.0400	0.0238
0.4	–0.2023	–0.2246	–0.1857	0.4025	0.4072	0.3991	0.0537	0.0429	0.0194
0.5	–0.1379	–0.1646	–0.1161	0.5149	0.5225	0.5109	0.0562	0.0393	0.0076
0.6	–0.0701	–0.0996	–0.0414	0.6235	0.6350	0.6185	0.0549	0.0285	–0.0124
0.7	0.0005	–0.0296	0.0387	0.7229	0.7391	0.7143	0.0516	0.0099	–0.0413
0.8	0.0750	0.0478	0.1284	0.8075	0.8295	0.7873	0.0519	–0.0182	–0.0805
0.9	0.1650	0.1465	0.2460	0.8721	0.9018	0.8155	0.0741	–0.0629	–0.1335
$E_1 = 10 \text{ MeV}, \theta_{1k} = 0.025 \text{ rad}$									
0.1	–0.3262	–0.3273	–0.3266	0.1014	0.1014	0.1014	0.0243	0.0242	0.0238
0.2	–0.2971	–0.2985	–0.2972	0.2151	0.2154	0.2153	0.0444	0.0441	0.0433
0.3	–0.2657	–0.2672	–0.2653	0.3393	0.3396	0.3397	0.0597	0.0593	0.0579
0.4	–0.2290	–0.2307	–0.2281	0.4696	0.4701	0.4708	0.0691	0.0682	0.0663
0.5	–0.1867	–0.1886	–0.1853	0.5999	0.6005	0.6024	0.0711	0.0697	0.0671
0.6	–0.1402	–0.1419	–0.1380	0.7224	0.7232	0.7268	0.0652	0.0629	0.0595
0.7	–0.0914	–0.0926	–0.0880	0.8287	0.8298	0.8356	0.0518	0.0479	0.0433
0.8	–0.0432	–0.0428	–0.0377	0.9120	0.9133	0.9207	0.0323	0.0253	0.0192
0.9	0.0026	0.0070	0.0132	0.9673	0.9693	0.9745	0.0113	–0.0046	–0.0133
$E_1 = 50 \text{ MeV}, \theta_{1k} = 0.0041 \text{ rad}$									
0.1	–0.2503	–0.2513	–0.2512	0.1054	0.1055	0.1056	0.0260	0.0260	0.0260
0.2	–0.2365	–0.2371	–0.2370	0.2233	0.2236	0.2237	0.0490	0.0491	0.0491
0.3	–0.2196	–0.2194	–0.2194	0.3516	0.3521	0.3522	0.0680	0.0681	0.0681
0.4	–0.1975	–0.1967	–0.1967	0.4855	0.4867	0.4869	0.0813	0.0815	0.0814
0.5	–0.1685	–0.1685	–0.1684	0.6195	0.6206	0.6212	0.0871	0.0872	0.0872
0.6	–0.1358	–0.1353	–0.1352	0.7436	0.7455	0.7465	0.0839	0.0840	0.0840
0.7	–0.0988	–0.0985	–0.0984	0.8504	0.8521	0.8538	0.0714	0.0714	0.0713
0.8	–0.0605	–0.0605	–0.0604	0.9316	0.9327	0.9350	0.0503	0.0501	0.0500
0.9	–0.0244	–0.0242	–0.0241	0.9819	0.9822	0.9851	0.0231	0.0225	0.0223
$E_1 = 100 \text{ MeV}, \theta_{1k} = 0.0025 \text{ rad}$									
0.1	–0.3616	–0.3636	–0.3635	0.1063	0.1067	0.1067	0.0280	0.0281	0.0281
0.2	–0.3398	–0.3438	–0.3438	0.2266	0.2269	0.2269	0.0531	0.0532	0.0532
0.3	–0.3149	–0.3184	–0.3184	0.3583	0.3581	0.3582	0.0739	0.0739	0.0739
0.4	–0.2838	–0.2851	–0.2851	0.4948	0.4952	0.4955	0.0882	0.0882	0.0882
0.5	–0.2441	–0.2437	–0.2437	0.6307	0.6307	0.6311	0.0942	0.0943	0.0943
0.6	–0.1956	–0.1954	–0.1953	0.7533	0.7554	0.7562	0.0904	0.0907	0.0907
0.7	–0.1432	–0.1425	–0.1425	0.8584	0.8603	0.8614	0.0770	0.0771	0.0772
0.8	–0.0888	–0.0886	–0.0886	0.9356	0.9378	0.9393	0.0548	0.0548	0.0549
0.9	–0.0381	–0.0379	–0.0379	0.9829	0.9843	0.9862	0.0265	0.0264	0.0263

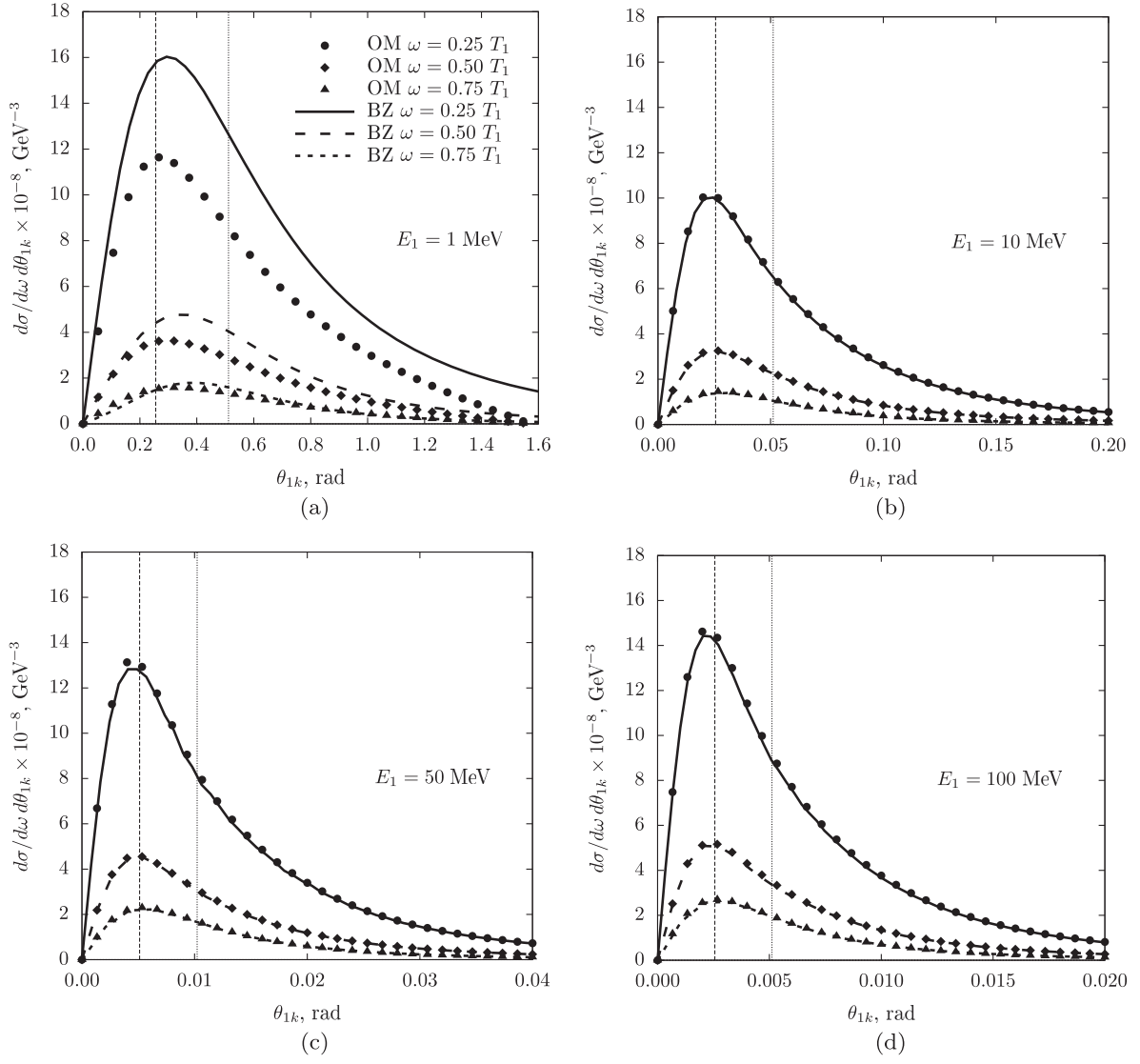


FIG. 9. Cross sections without screening as functions of the photon emission angle θ_{1k} for different initial unpolarized electron beam energies E_1 [(a) $E_1 = 1\text{ MeV}$, (b) $E_1 = 10\text{ MeV}$, (c) $E_1 = 50\text{ MeV}$, (d) $E_1 = 100\text{ MeV}$] and for different photon energies ω . The vertical dotted line shows the position of the cone open angle $\theta_{1k} = 1/\gamma$, while the vertical dashed line shows the position of the peak open angle $\theta_{1k} = 1/(2\gamma)$. The curves are our exact results and the dots correspond to the calculations from paper [8]. We integrate over $d\Omega_{p_3}$ and $d\varphi_k$.

Figs. 13(c) and 13(d). These plots were generated under the condition of a transverse initial electron spin oriented along the x axis. The photon emission occurs at a definite azimuthal angle relative to the initial beam's spin orientation. When integrating over this orientation, the resulting value of P_3^T vanishes. Consequently, the physically meaningful plot in Fig. 8 must yield zero.

With regard to the photon polarization component P_2 , we note that it was not depicted in Ref. [8], as the authors assumed it to vanish identically. This assumption holds true for the integrated observable; however, P_2 is nonzero when both azimuthal angles are held fixed. For instance, Fig. 14 displays the transverse component P_2^T as a function of the scattered electron's azimuthal angle φ_3 , with the photon's

azimuthal angle φ_k held constant. It is evident that upon integration over φ_3 , the quantity P_2^T averages to zero for any fixed value of φ_k . Similarly, integrating over φ_k at a fixed φ_3 also yields a vanishing result. This behavior underscores the crucial distinction between differential and integrated observables in the context of azimuthal angular dependencies.

VII. ISSUES AND PROBLEMS

In this Section, we elucidate the challenges that emerge from a naïve application of the preceding results. This issues often remain concealed during the modeling of the bremsstrahlung process, only becoming apparent under specific configurations and experimental settings:

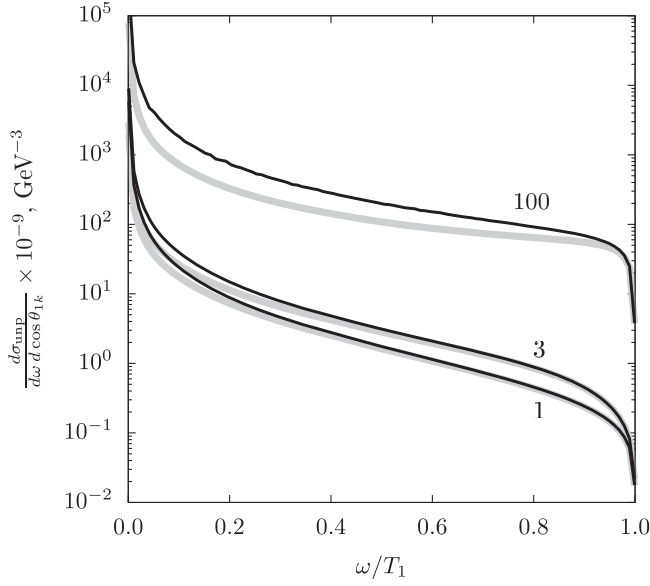


FIG. 10. Cross section as a function of emitted photon energy ω for different initial unpolarized electron beam energies E_1 (numbers near the curves in MeVs). The photon emission angle is fixed at peaks (see the explanation in the text). The black curves are without screening and the grey ones are with screening. We integrate over $d\Omega_{\mathbf{p}_3}$ and $d\varphi_k$.

- (1) As previously noted in Sec. V, the application of Eq. (7.2) from Ref. [8] to calculate the unpolarized bremsstrahlung cross section—including nuclear screening effects—leads to erroneous results at high emitted photon energies ω , specifically when $\omega > 0.5T_1$. This discrepancy persists even for

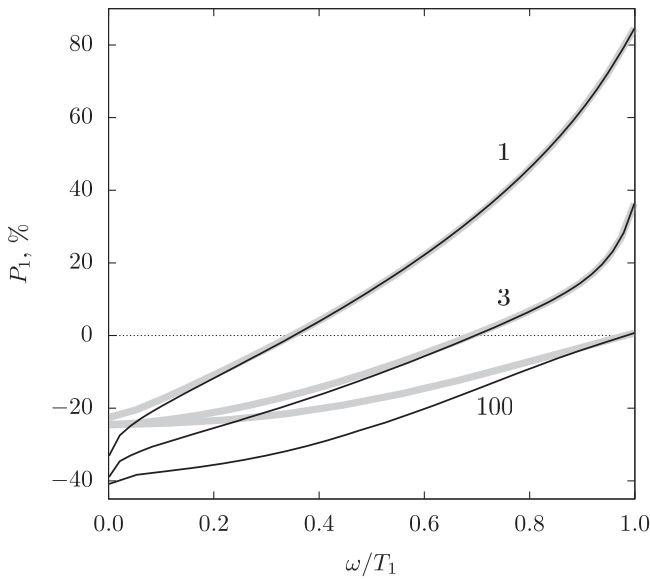


FIG. 11. Linear polarization of the emitted photon, P_1 , versus its energy ω for the unpolarized initial electron beam with different energies E_1 . The notation is as in Fig. 10.

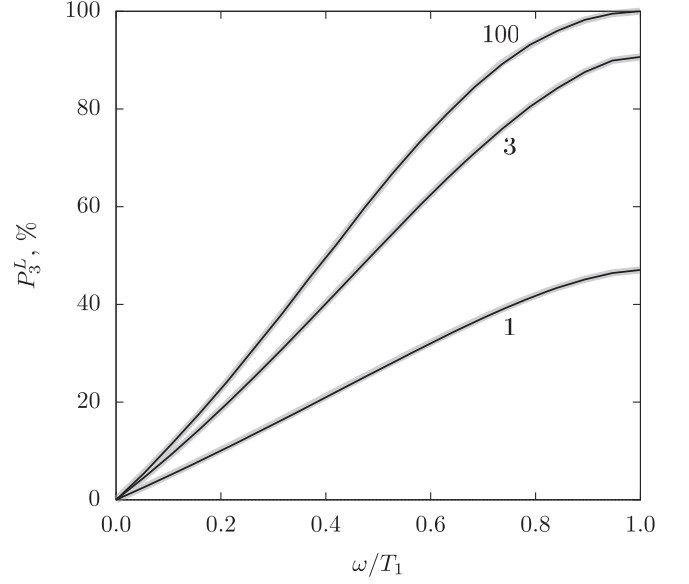


FIG. 12. Circular polarization of the emitted photon, P_3^L , versus its energy ω for the longitudinally polarized ($\lambda_1 = +1$) initial electron beam with different energies E_1 . The notation is as in Fig. 10.

relatively high electron beam energies such as $E_1 = 50$ MeV, as clearly demonstrated Fig. 4. However, it is evident that the authors of Ref. [8] employed a correct formula when generating the curve on Fig. 1 of their original paper. This reference curve is reproduced in Fig. 4 using black dots and shows excellent agreement with our own calculations. Consequently, we conclude that Eq. (7.2) of Ref. [8] is not suitable for describing hard-photon emission in the bremsstrahlung process.

- (2) A systematic discrepancy emerges in the low-energy regime ($E_1 \sim \text{few } m_e$) when comparing the approximate unpolarized cross section from Ref. [8] with our present results. As demonstrated in Fig. 9(a), a substantial deviation is observed between the two calculations. This discrepancy can reach 30%–40% even for low-energy bremsstrahlung photons, specifically in the range $\omega < 0.5T_1$. The magnitude of the difference underscores the limitations of the approximation used in Ref. [8] in this kinematic region.
- (3) Problems also arise in the context of polarization observables. Figure 15 reproduces Fig. 7 for a low-energy electron beam ($E_1 = 1$ MeV) and a relatively large photon emission angle [$\theta_{1k} = 1/(2\gamma) = 0.2$ rad]. A noticeable discrepancy between the two sets of results is evident in this kinematic regime. Moreover, the magnitude of this discrepancy grows as the initial electron energy E_1 decreases. The underlying cause lies in the systematic omission of two critical physical effects in the OM calculation: the electron mass and the nucleus recoil. These effects become particularly

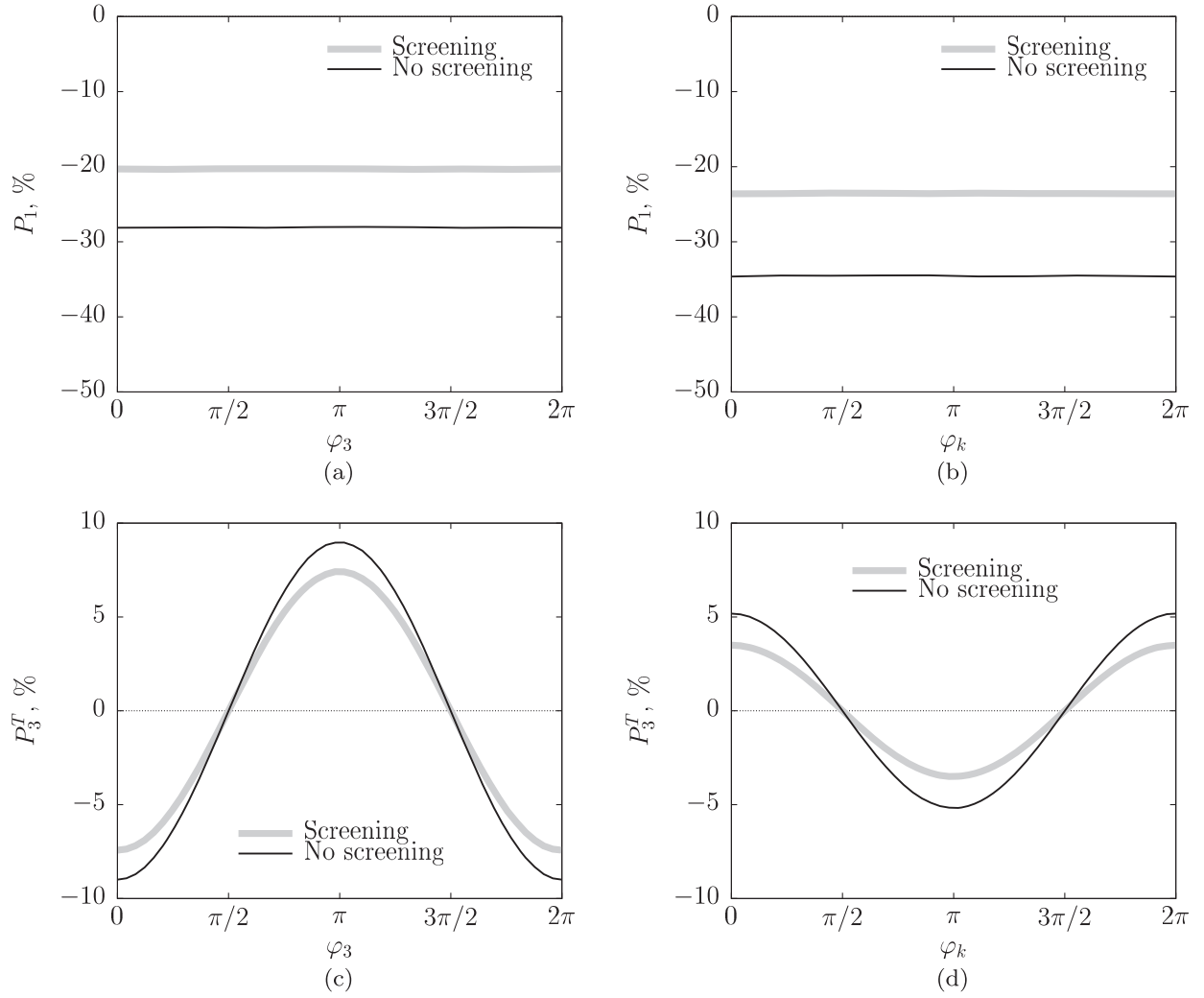


FIG. 13. Polarization degrees of emitted photons as functions of the electron azimuthal emission angle φ_3 and the photon azimuthal emission angle φ_k for the fixed photon emission angle θ_{1k} at peak value. The initial electron beam energy is $E_1 = 50$ MeV and photon energy is fixed at $\omega = 0.4E_1$ (left plots) and $\omega = 0.2E_1$ (right plots). We integrate over dC_{13} and $d\varphi_k$ in (a) and (c) and over dC_{13} and $d\varphi_3$ in (b) and (d).

significant in the low-energy domain, where their neglect leads to a progressively larger deviation from the more complete treatment presented here.

- (4) Moreover, a noticeable nonlinearity in the energy dependence of the bremsstrahlung photon's circular polarization P_3 becomes evident. This is illustrated in Fig. 7 for a high-energy electron beam ($E_1 = 50$ MeV) and in Fig. 16 for a lower beam energy ($E_1 = 3$ MeV). Figure 16 reveals a clear trend: the larger the photon emission angle θ_{1k} , the more significant the deviation from linear behavior. In contrast, the Olsen-Maximon result remains independent of θ_{1k} , as it is derived under the small-angle approximation. This discrepancy carries practical significance because many studies rely on the linear approximation for quick estimates of bremsstrahlung polarization. For instance, Ref. [15]

employs such a linearized approach to approximate photon polarization. While this approximation performs adequately at low photon energies and small emission angles, it breaks down severely when the photon energy approaches the upper kinematic limit, or the emission angle becomes large. In such cases, the linear approximation can fail dramatically, leading to substantial inaccuracies in predicted polarization values.

- (5) Lastly, but by no means least, we address a critical shortcoming of the approximate equations in the Olsen-Maximon paper [8]: their unphysical behavior near the boundaries of the kinematically allowed region. This issue is illustrated in Fig. 17, which displays the circular polarization P_3 of the emitted photon as a function of its energy ω . The calculation assumes a relatively high beam energy ($E_1 = 10$ MeV) and a large emission

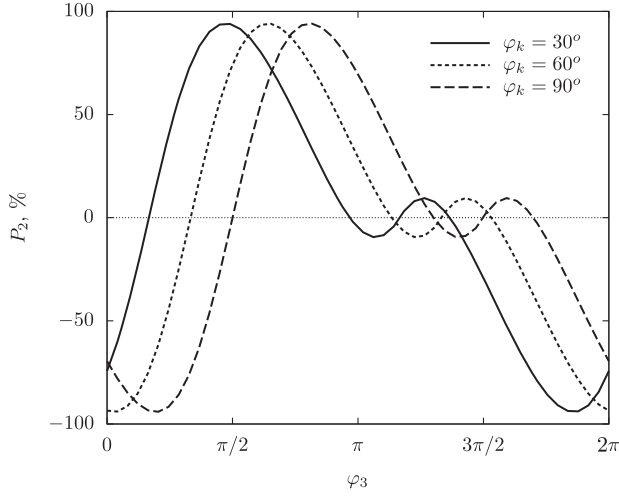


FIG. 14. Linear polarization P_2 of bremsstrahlung on an unscreened nucleus versus the electron azimuthal emission angle φ_3 (for fixed φ_k). The photon emission angle θ_{1k} is fixed at peak value. The initial electron beam energy is $E_1 = 50$ MeV and photon energy is fixed at $\omega = 0.2E_1$. We integrate over dC_{13} .

angle ($\theta_{1k} = 1$ rad). Strikingly, the Olsen-Maximon result becomes unphysical in this regime: it predicts polarization values exceeding 100% for hard photons that carry away nearly the entire energy of the initial electron. The root cause of this anomaly lies once again in the systematic omission of the electron mass m_e in the OM formalism. While this approximation simplifies the mathematical expressions, it severely restricts the domain of their validity. Although such approximations

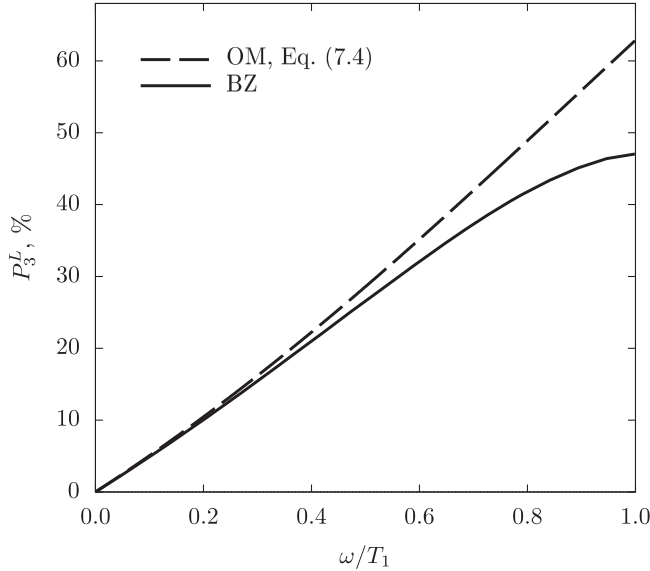


FIG. 15. Circular polarization P_3^L of bremsstrahlung on an unscreened nucleus versus photon energy ω for the initial longitudinally polarized ($\lambda_1 = +1$) electron beam with the energy $E_1 = 1$ MeV and peak emission angle. The comparison with the OM result is given. We integrate over $d\Omega_{\mathbf{p}_3}$ and $d\varphi_k$.

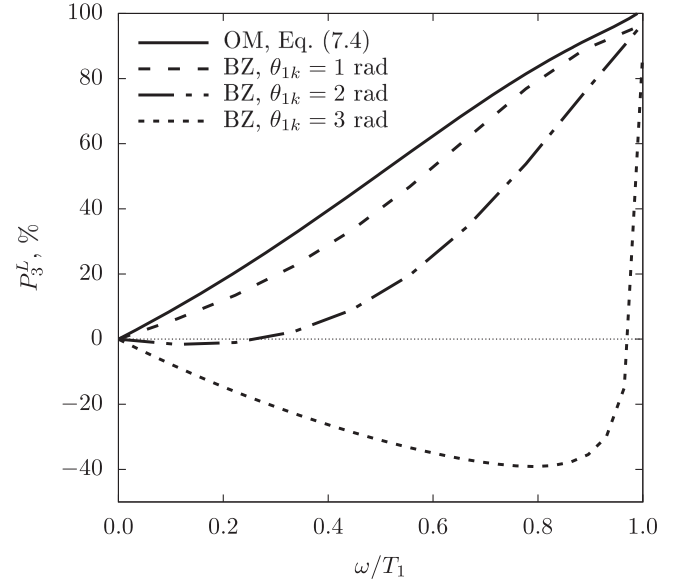


FIG. 16. Circular polarization P_3^L of bremsstrahlung on an unscreened nucleus versus photon energy ω for the initial longitudinally polarized ($\lambda_1 = +1$) electron beam with energy $E_1 = 3$ MeV. The emission angles are large ($\theta_{1k} = 1, 2, 3$ rad). We integrate over $d\Omega_{\mathbf{p}_3}$ and $d\varphi_k$.

may occasionally prove convenient for analytical tractability, it is essential to remain mindful of their inherent limitations.

In modern experimental physics, Monte Carlo generators play an indispensable role in data analysis. A comprehensive understanding and reconstruction of the kinematics

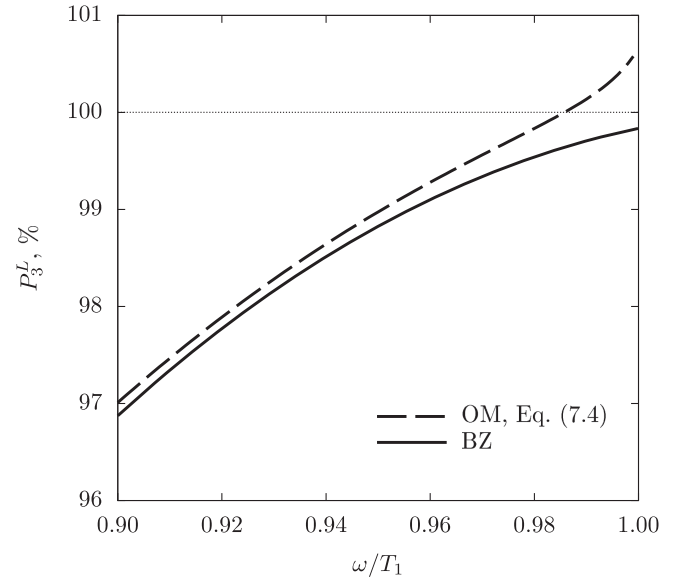


FIG. 17. Circular polarization of the emitted photon, P_3 , versus its energy ω at the upper limit for the initial longitudinally polarized ($\lambda_1 = +1$) electron beam with energy $E_1 = 10$ MeV and the emission angle is $\theta_{1k} = 1$ rad. We integrate over $d\Omega_{\mathbf{p}_3}$ and $d\varphi_k$.

across the entire phase space of final-state particles are essential. Consequently, the infinite-momentum approximation employed in the calculations by Kuraev *et al.* [21] proves inadequate for this purpose. This approach is restricted to the peripheral kinematic regime characterized by $s \gg -q^2 \approx M_A^2 \gg m_e^2$, which fails to cover the full phase space required for realistic event simulation.

Similarly, caution is warranted when relying on the widely cited monograph on bremsstrahlung by Haug and Nakel [38]. Many of the results presented therein are approximate and largely follow the framework established in Ref. [3]. A comparative analysis with our present calculations confirms the limitations outlined above, as summarized in Table I.

To facilitate future development of Monte Carlo event generators, we provide explicit formulae for the exact differential cross sections in Appendix B. These expressions are designed to ensure accurate phase-space sampling across the full kinematic domain, thereby addressing the shortcomings of earlier approximations.

VIII. CONCLUSION

In this study, we have computed the unpolarized cross section and polarization observables for electron bremsstrahlung in nuclear scattering—the Bethe-Heitler process. The calculation was performed at leading order in quantum electrodynamics perturbation theory.

Our approach systematically retains the electron mass m_e and incorporates nuclear recoil effects. We neglect radiation from the nucleus itself, as this contribution remains negligible at relatively low energies. In a dedicated section, we discuss potential extensions of our leading-order treatment to account for finer effects, such as nuclear screening and Coulomb corrections near threshold.

We have conducted an extensive comparison of our results with established formulas commonly employed to estimate radiative effects in experimental setups. Section VII outlines key caveats to bear in mind when applying these conventional expressions in practice. We demonstrate that the systematic use of our precise formulas resolves the shortcomings inherent in prior approximations. Consequently, we recommend their adoption—particularly for the development of robust Monte Carlo generators.

The exact differential cross sections for both unpolarized and polarized bremsstrahlung, derived in this work, are compiled in Appendix B. These expressions provide a reliable description of the Bethe-Heitler process at moderately low energies. Our results hold particular relevance for upcoming polarization experiments and for the design of PEPPo-like polarized positron sources.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [41].

APPENDIX A: PHOTON PURE POLARIZATION STATES

In this appendix, we construct two independent polarization states of the photon emitted in a specific direction. We adhere to the fundamental approach and notation introduced in Ref. [23].

The polarization state of a photon is conventionally characterized by the vibration direction of the electric field vector $\mathbf{E}$, which is always transverse to the photon’s propagation direction. This electric vector can be expressed as a linear combination of two mutually orthogonal unit vectors $\mathbf{E}_1$ and $\mathbf{E}_2$,

$$\mathbf{E} = a_1 \mathbf{E}_1 + a_2 \mathbf{E}_2, \quad (\text{A1})$$

where $a_{1,2}$ are generally complex scalars that encode both the amplitude and phase information of the two orthogonal vibrational modes. From the two expansion coefficients a_1 and a_2 , we can construct a 2×2 density matrix (see Ref. [42]),

$$\rho = \begin{pmatrix} a_1 a_1^* & a_1 a_2^* \\ a_2 a_1^* & a_2 a_2^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + P_1 & P_2 + iP_3 \\ P_2 - iP_3 & 1 - P_1 \end{pmatrix}, \quad (\text{A2})$$

where the matrix is normalized to unity and parameterized in terms of the Stokes parameters P_i . To determine the polarization state of the emitted photon, it is necessary to construct two orthogonal unit vectors $\mathbf{E}_{1,2}$ in the plane transverse to the photon’s propagation direction, and subsequently compute the corresponding projection coefficients a_i .

We begin by assuming that the photon propagates along the z axis, such that its four-momentum take the form,

$$k^\mu = (k^0, k^x, k^y, k^z) = (\omega, 0, 0, \omega).$$

Given this choice, the polarization vectors $\mathbf{E}_{1,2}$ can be conveniently identified with the unit vectors along the x - and y -axes, respectively (see Fig. 18). Accordingly, we

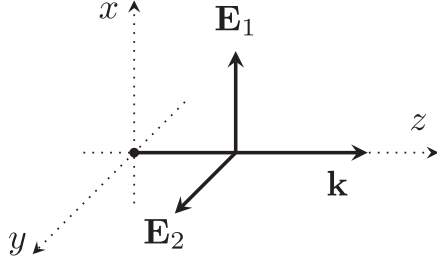


FIG. 18. Initial disposition of the pure polarization photon states.

construct the corresponding polarization four-vectors $\epsilon_{1,2}^\mu = (0, \mathbf{E}_{1,2})$ as,

$$\epsilon_1^\mu = (e_1^0, e_1^x, e_1^y, e_1^z) = (0, 1, 0, 0),$$

$$\epsilon_2^\mu = (e_2^0, e_2^x, e_2^y, e_2^z) = (0, 0, 1, 0).$$

We now rotate the photon momentum $\mathbf{k}$ and the polarization vectors $\mathbf{E}_{1,2}$ by the angle θ_{1k} about the y axis in the clockwise direction, as illustrated in Fig. 19. After this rotation, the vector $\mathbf{E}_1$ lies in the xz plane and it remains perpendicular to the y axis. Consequently, its four-vector representation becomes,

$$\epsilon_1^\mu = (0, C_{1k}, 0, -S_{1k}).$$

Meanwhile, the polarization vector $\mathbf{E}_2$ is unaffected by this rotation and retains its original orientation. Following the rotation, the photon four-momentum k^μ takes the form,

$$k^\mu = \omega(1, S_{1k}, 0, C_{1k}). \quad (\text{A3})$$

Finally, we apply a clockwise rotation about the z axis by the azimuthal angle φ_k , as illustrated in Fig. 20 (where the z axis is oriented from top to bottom). Under this transformation, the z components of all the vectors remain unchanged, while their x - and y components are rotated accordingly. The resulting four-vectors are,

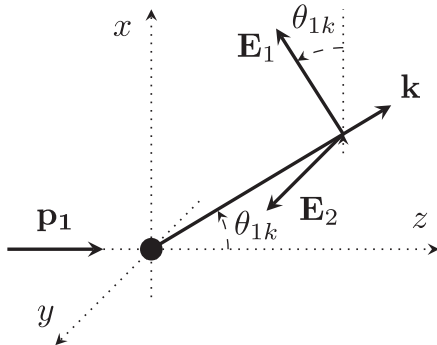


FIG. 19. Pure polarization photon states after polar rotation.

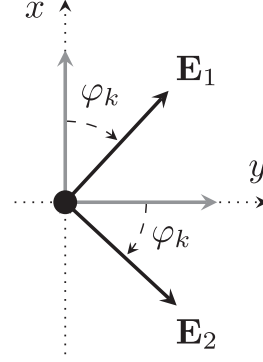


FIG. 20. Pure polarization photon states after azimuthal rotation.

$$\epsilon_1^\mu = (0, C_{1k} \cos \varphi_k, C_{1k} \sin \varphi_k, -S_{1k}), \quad (\text{A4})$$

$$\epsilon_2^\mu = (0, -\sin \varphi_k, \cos \varphi_k, 0), \quad (\text{A5})$$

$$k^\mu = \omega(1, S_{1k} \cos \varphi_k, S_{1k} \sin \varphi_k, C_{1k}). \quad (\text{A6})$$

It is straightforward to verify that the derived vectors satisfy all required physical conditions,

$$(\epsilon_1 \epsilon_2) = 0, \quad \epsilon_i^2 = -1, \quad (\epsilon_i k) = 0, \quad i = 1, 2. \quad (\text{A7})$$

These vectors are employed in our numerical calculations.

APPENDIX B: EXPLICIT EXPRESSIONS FOR CROSS SECTIONS

In this appendix, we present explicit expressions for the unpolarized and polarized cross sections of the bremsstrahlung process. The figures shown above (with the exception of Figs. 13 and 14) display cross sections as functions of the photon energy ω for fixed values of the cosine of the emission angle C_{1k} , or vice versa. The differential cross section is given by,

$$\frac{d\sigma_i}{d\omega dC_{1k}} = \frac{\alpha^3 Z^2}{2^6 \pi^2 F} \int_{-1}^1 dC_{13} \int_0^{2\pi} d\varphi_3 \int_0^{2\pi} d\varphi_k \frac{\omega F_b}{(q^2)^2} S_i, \quad (\text{B1})$$

where the index i specifies the type of cross section, and the quantity S_i is related to the S_{ij} terms defined in Eq. (22) through the following relations:

$$S_I = S_{11} + S_{22} = 16S_I, \quad (\text{B2})$$

$$S_{P_1} = S_{11} - S_{22} = 8S_{P_1}, \quad (\text{B3})$$

$$S_{P_2} = S_{12} + S_{21} = 8S_{P_2}, \quad (\text{B4})$$

$$S_{P_3} = i(S_{21} - S_{12}) = 4S_{P_3}. \quad (\text{B5})$$

1. Unpolarized cross section

Here we present the quantity $\mathcal{S}_I$ introduced in Eq. (B2), which corresponds to the unpolarized cross section. It exhibits the following structure:

$$\mathcal{S}_I = \frac{A_1}{w_1 w_3} + \frac{A_2}{w_1} + \frac{A_3}{w_3} + 2m_e^2 \left(\frac{A_4}{w_1^2} + \frac{A_5}{w_3^2} \right), \quad (\text{B6})$$

where the coefficients A_i take the form,

$$\begin{aligned} A_1 &= -4M_A^2 p_{13}(p_{13} - 4m_e^2) \\ &\quad - p_{13}(w_2(p_{34} - p_{14}) - w_4(p_{12} - p_{23})) \\ &\quad + 2p_{13}(p_{12}p_{34} + p_{23}p_{14}) - 4m_e^2(w_2w_4 + p_{13}p_{24}), \\ A_2 &= w_2p_{34} + w_4p_{23} + 2M_A^2(2p_{13} - w_3) \\ &\quad - p_{12}p_{34} - 2p_{12}p_{14} - p_{23}p_{14} + 4m_e^2M_A^2, \\ A_3 &= w_2p_{14} + w_4p_{12} - 2M_A^2(2p_{13} + w_1) \\ &\quad + p_{12}p_{34} + 2p_{23}p_{34} + p_{23}p_{14} - 4m_e^2M_A^2, \\ A_4 &= w_2p_{34} + w_4p_{23} - p_{12}p_{34} - p_{14}p_{23} \\ &\quad + 2M_A^2(p_{13} - w_3) + 2m_e^2(p_{24} - 4M_A^2), \\ A_5 &= -w_2p_{14} - w_4p_{12} - p_{12}p_{34} - p_{23}p_{14} \\ &\quad + 2M_A^2(p_{13} + w_1) + 2m_e^2(p_{24} - 4M_A^2). \end{aligned}$$

In this derivation, we employ several radiative invariants defined as $w_i \equiv 2(kp_i)$ and scalar products given by $p_{ij} \equiv 2(p_i p_j)$. These invariants are evaluated using the explicit four-vector components in the laboratory frame, which are summarized in Table II.

2. Linear polarization

Here we present the expressions for the quantities $\mathcal{S}_{P_{1,2}}$ introduced in Eqs. (B3) and (B4), which describe the cross sections for the emission of linearly polarized photons. These quantities take the following form:

$$\mathcal{S}_{P_1} = \frac{A_6}{w_1 w_3} + \frac{A_7}{w_1} + \frac{A_8}{w_3} + \frac{A_9}{w_1^2} + \frac{A_{10}}{w_3^2} + A_{11}, \quad (\text{B7})$$

TABLE II. Components of 4-momenta.

	0	x	y	z
p_1	E_1	0	0	$ \mathbf{p}_1 $
p_2	M_A	0	0	0
p_3	E_3	$ \mathbf{p}_3 S_{13} \cos \varphi_3$	$ \mathbf{p}_3 S_{13} \sin \varphi_3$	$ \mathbf{p}_3 C_{13}$
k	ω	$\omega S_{1k} \cos \varphi_k$	$\omega S_{1k} \sin \varphi_k$	ωC_{1k}
$a_1^\parallel$	$\lambda_1 \mathbf{p}_1 /m_e$	0	0	$\lambda_1 E_1/m_e$
$a_1^\perp$	0	$\cos \varphi_a$	$\sin \varphi_a$	0
ε_1	0	$C_{1k} \cos \varphi_k$	$C_{1k} \sin \varphi_k$	$-S_{1k}$
ε_2	0	$-\sin \varphi_k$	$\cos \varphi_k$	0

$$\mathcal{S}_{P_2} = \frac{A_{12}}{w_1 w_3} + \frac{A_{13}}{w_1} + \frac{A_{14}}{w_3} + \frac{A_{15}}{w_1^2} + \frac{A_{16}}{w_3^2} + A_{17}, \quad (\text{B8})$$

where the coefficients A_i are given by,

$$\begin{aligned} A_6 &= (x_1 x_3 - y_1 y_3)(2(w_2 w_4 - p_{12} p_{34} - p_{23} p_{14}) \\ &\quad - w_2(p_{14} - p_{34}) - w_4(p_{12} - p_{23}) \\ &\quad + 4(M_A^2 p_{13} + m_e^2 p_{24} - 4m_e^2 M_A^2)), \\ A_7 &= (x_1 x_4 - y_1 y_4)(p_{12} + p_{23} - w_2) \\ &\quad + (x_1 x_2 - y_1 y_2)(p_{14} + p_{34} - w_4) \\ &\quad - 2M_A^2(2(x_1 x_3 - y_1 y_3) + x_1^2 - y_1^2), \\ A_8 &= -(x_3 x_4 - y_3 y_4)(p_{12} + p_{23} + w_2) \\ &\quad - (x_2 x_3 - y_2 y_3)(p_{14} + p_{34} + w_4) \\ &\quad + 2M_A^2(2(x_1 x_3 - y_1 y_3) + x_3^2 - y_3^2), \\ A_9 &= (x_1^2 - y_1^2)(p_{12} p_{34} + p_{23} p_{14} \\ &\quad - w_2 p_{34} + 2w_3 M_A^2 - w_4 p_{23} \\ &\quad - 2(M_A^2 p_{13} + m_e^2 p_{24} - 4m_e^2 M_A^2)), \\ A_{10} &= (x_3^2 - y_3^2)(p_{12} p_{34} + p_{23} p_{14} \\ &\quad - 2w_1 M_A^2 + w_2 p_{14} + w_4 p_{12} \\ &\quad - 2(M_A^2 p_{13} + m_e^2 p_{24} - 4m_e^2 M_A^2)), \\ A_{11} &= 2(x_2 x_4 - y_2 y_4), \\ A_{12} &= (x_1 y_3 + x_3 y_1)(2w_2 w_4 - 2(p_{14} p_{23} + p_{12} p_{34}) \\ &\quad - w_2(p_{14} - p_{34}) - w_4(p_{12} - p_{23}) \\ &\quad + 4M_A^2(p_{13} - 4m_e^2) + 4m_e^2 p_{24}), \\ A_{13} &= (x_1 y_4 + x_4 y_1)(p_{12} + p_{23} - w_2) \\ &\quad + (x_1 y_2 + x_2 y_1)(p_{14} + p_{34} - w_4) \\ &\quad - 4M_A^2(x_1 y_1 + x_1 y_3 + x_3 y_1), \\ A_{14} &= -(x_3 y_4 + x_4 y_3)(p_{12} + p_{23} + w_2) \\ &\quad - (x_2 y_3 + x_3 y_2)(p_{14} + p_{34} + w_4) \\ &\quad + 4M_A^2(x_1 y_3 + x_3 y_1 + x_3 y_3), \\ A_{15} &= 2x_1 y_1(p_{12} p_{34} + p_{14} p_{23} - p_{34} w_2 - p_{23} w_4 \\ &\quad - 2M_A^2(p_{13} - w_3 - 4m_e^2) - 2m_e^2 p_{24}), \\ A_{16} &= 2x_3 y_3(p_{12} p_{34} + p_{14} p_{23} + p_{14} w_2 + p_{12} w_4 \\ &\quad - 2M_A^2(p_{13} + w_1 - 4m_e^2) - 2m_e^2 p_{24}), \\ A_{17} &= 2(x_2 y_4 + x_4 y_2). \end{aligned} \quad (\text{B9})$$

In this derivation, we introduce additional abbreviations to simplify the notation: $x_i \equiv 2(\varepsilon_1 p_i)$ and $y_i \equiv 2(\varepsilon_2 p_i)$.

3. Circular polarization

In this section, we present the quantity $\mathcal{S}_{P_3}$ introduced in Eq. (B5), which describes the cross section for the emission of a circularly polarized photon. The expression takes the following form:

$$\mathcal{S}_{P_3} = \frac{A_{18}}{w_1 w_3} + 2 \left(\frac{A_{19}}{w_1} + 2 \frac{A_{20}}{w_3} \right) + 4 \left(\frac{A_{21}}{w_1^2} + \frac{A_{22}}{w_3^2} \right), \quad (\text{B10})$$

where coefficients A_{18} — A_{22} are given by,

$$\begin{aligned} A_{18} = & 4E_{2kxy}(2a_{1k}p_{23} - m_e^2(a_{12} + 3a_{14})) - 4E_{13xy}((a_{13} + a_{14})p_{23} + a_{12}p_{34} + (a_{13} - a_{1k})w_2 + a_{12}m_e^2) \\ & + 4E_{1kxy}(2a_{13}M_A^2 + 2a_{1k}p_{23} - a_{13}(p_{23} + p_{24}) - (a_{13} + a_{14} - a_{1k})w_2 - a_{12}w_4 - 2a_{12}m_e^2) \\ & + 2E_{3kxy}(a_{14}p_{12} + a_{12}p_{14} - 4a_{1k}p_{23} + 4a_{12}w_4 + 2a_{12}m_e^2) \\ & + 4E_{12xy}((a_{12} + a_{14})m_e^2 + a_{13}(p_{23} + p_{34}) + (a_{13} - a_{1k})(w_2 + w_4)) \\ & + 2E_{3xya}(-2((p_{13} + p_{14})p_{23} + p_{12}p_{34}) - (p_{13} + p_{14} + 2m_e^2)w_2 - p_{12}w_4 - 2m_e^2p_{12}) \\ & + 2E_{1xya}(2p_{13}p_{23} + (p_{13} - p_{34} + 2m_e^2)w_2 - p_{23}w_4 + 2m_e^2(p_{12} - 2p_{24})) \\ & - 2E_{kxya}(2p_{13}p_{23} + (p_{13} + 2(p_{14} + m_e^2))w_2 + 2(p_{12} - 2p_{23})w_4 + 2m_e^2(p_{12} + 4p_{24})) \\ & + 2E_{2xya}(2p_{13}(p_{23} + p_{34}) + (p_{13} + 2m_e^2)w_2 + (p_{13} + 6m_e^2)w_4 + 2m_e^2(p_{12} + p_{14})) \\ & - E_{23ya}(4(p_{23} + p_{34})x_1 + 2(p_{12} + p_{14})x_3 - w_2(x_1 - 2x_3) - w_4(x_1 - 6x_3)) \\ & - 2E_{12ya}((p_{23} + p_{34})x_3 - w_2(2x_1 + x_3) - w_4(2x_1 + x_3) - 2m_e^2(x_2 + x_4)) \\ & - E_{13ya}(4p_{23}x_1 + 2(p_{12} - p_{23} - 2p_{24})x_3 + w_2(3x_1 + 4x_3 - x_4) - w_4x_2 + 4m_e^2x_2) \\ & - E_{2kya}((4(p_{12} + p_{14}) - p_{23} - p_{34})x_1 + 2p_{13}(x_2 + x_4) + (p_{12} + p_{14} + 8p_{23})x_3 + 4(w_2 + w_4)x_3 + 4m_e^2(x_2 + 3x_4)) \\ & - E_{1kya}(32M_A^2x_3 + (2p_{13} + p_{34})x_2 - 10p_{24}x_3 + p_{12}(4x_1 + x_3) - p_{23}(x_1 - 6x_3 - x_4) + 2w_2(2x_1 + 3x_3 - 2x_4) \\ & - 4w_4x_2 + 8m_e^2x_2) - E_{3kya}(16M_A^2(x_1 - x_3) + 5p_{23}x_1 - 2(p_{13} + p_{14})x_2 - 8(p_{23} - p_{24})x_3 - p_{12}(4x_1 - x_3 + 2x_4) \\ & - w_2(x_1 + 2x_3 + 4x_4) - 4w_4x_2 - 4m_e^2x_2) - 2E_{123y}((a_{12} + a_{14})x_3 + 2(p_{23} + p_{34})x_a) \\ & - 2E_{12ky}(2a_{1k}x_3 + a_{12}(2x_1 - x_3) + a_{14}(2x_1 - x_3) + a_{13}(2(x_2 + x_4) - x_3) - (p_{23} + p_{34})x_a) \\ & + E_{13ky}(8M_A^2x_a + (4(a_{13} + a_{1k}) - a_{14})x_2 - a_{12}(x_1 + 6x_3 + x_4) - (p_{12} + 6p_{23} + 2p_{24} - 4w_2)x_a) \\ & + E_{23ky}(4a_{1k}x_2 + 2a_{13}x_3 - a_{12}(5x_1 + 2x_3) - a_{14}(5x_1 + 6x_3) - 4a_{1k}(x_3 + x_4) - (p_{12} + p_{14} - 4(w_2 + w_4))x_a) \\ & + E_{13ka}(4(x_1y_2 - x_2y_1) - 7(x_2y_3 - x_3y_2)) + E_{23xa}(4(p_{23} + p_{34})y_1 + 2(p_{12} + p_{14})y_3 - w_2(y_1 - 2y_3) - w_4(y_1 - 6y_3)) \\ & + E_{13xa}(4p_{23}y_1 + 2(p_{12} - p_{23} - 2p_{24})y_3 + w_2(3y_1 + 4y_3 - y_4) - w_4y_2 + 4m_e^2y_2) \\ & + 2E_{12xa}((p_{23} + p_{34})y_3 - w_2(2y_1 + y_3) - w_4(2y_1 + y_3) - 2m_e^2(y_2 + y_4)) \\ & - 2(E_{123a} + 2E_{23ka})((x_2 + x_4)y_3 - x_3(y_2 + y_4)) + E_{12ka}((x_2 + x_4)(4y_1 + y_3) - (4x_1 + x_3)(y_2 + y_4)) \\ & + E_{1kxa}(2p_{13}y_2 + p_{34}y_2 + p_{23}(-y_1 + 6y_3 + y_4) - 10p_{24}y_3 + p_{12}(4y_1 + y_3) + 32M_A^2y_3 + 2w_2(2(y_1 - y_4) + 3y_3) \\ & - 4w_4y_2 + 8m_e^2y_2) + E_{2kxa}((4(p_{12} + p_{14}) - p_{23} - p_{34})y_1 + (p_{12} + p_{14} + 8p_{23} + 4(w_2 + w_4))y_3 + 2p_{13}(y_2 + y_4) \\ & + 4m_e^2(y_2 + 3y_4)) + E_{3kxa}(16M_A^2(y_1 - y_3) + 5p_{23}y_1 - 2(p_{13} + p_{14})y_2 - 8(p_{23} - p_{24})y_3 - p_{12}(2(2y_1 + y_4) - y_3) \\ & - w_2(y_1 + 2(y_3 + 2y_4)) - 4w_4y_2 - 4m_e^2y_2) + 2E_{12kx}(2((a_{12} + a_{14})y_1 + a_{13}y_2 + a_{13}y_4) - (a_{12} + a_{13} + a_{14} - 2a_{1k})y_3 \\ & - (p_{23} + p_{34})y_a) + 2E_{123x}((a_{12} + a_{14})y_3 + 2(p_{23} + p_{34})y_a) + E_{13kx}((a_{14} - 4(a_{13} + a_{1k}))y_2 + a_{12}(y_1 + 6y_3 + y_4) \\ & - 8M_A^2y_a + (p_{12} + 2(3p_{23} + p_{24} - 2w_2))y_a) - E_{23kx}(2a_{13}y_3 - a_{12}(5y_1 + 2y_3) - a_{14}(5y_1 + 6y_3) \\ & + 4a_{1k}(y_2 - y_3 - y_4) - (p_{12} + p_{14} - 4(w_2 + w_4))y_a) + E_{123k}(x_a(y_2 + 2y_3 + y_4) - (x_2 + 2x_3 + x_4)y_a), \end{aligned}$$

$$\begin{aligned}
A_{19} &= -4a_{12}(E_{1kxy} + E_{2kxy} - E_{3kxy}) - E_{kxya}(p_{12} + 2p_{23}) - E_{12ya}(x_2 - x_3 + x_4) + (E_{1xya} - E_{3xya})(p_{12} + 2(p_{23} + 2M_A^2)) \\
&\quad + E_{2xya}(p_{12} + p_{14} + 2(p_{23} + p_{34})) + (E_{13ya} - E_{1kya} + 2E_{3kya})x_2 + E_{23ya}x_3 - E_{2kya}(2(x_2 - x_4) - x_3) \\
&\quad + 2(E_{12ky} + E_{23ky})x_a - E_{23xa}y_3 - (E_{13xa} - E_{1kxa} + 2E_{3kxa})y_2 + E_{12xa}(y_2 - y_3 + y_4) + E_{2kxa}(2(y_2 - y_4) - y_3) \\
&\quad - 2(E_{12kx} + E_{23kx})y_a, \\
A_{20} &= -(E_{1xya} - E_{kxya} + E_{2xya} - E_{3xya})w_2 - w_4E_{2xya} + 2M_A^2E_{3xya}, \\
A_{21} &= -2((a_{12} + a_{14})E_{2kxy} - a_{12}E_{3kxy})m_e^2 + E_{13xy}(a_{1k}(p_{23} - 2M_A^2) - a_{12}m_e^2) \\
&\quad + E_{1kxy}(a_{12}p_{34} + (a_{14} + a_{1k})p_{23} - 2a_{13}M_A^2 - 3a_{12}m_e^2) - E_{12xy}(a_{1k}(p_{23} + p_{34}) - (a_{12} + a_{14})m_e^2) \\
&\quad + E_{3xya}m_e^2(4M_A^2 - p_{12} + 2(2p_{23} - w_2)) + E_{1xya}(2(p_{12}p_{34} + p_{14}p_{23}) - 2M_A^2(p_{13} - 2w_3) \\
&\quad + m_e^2(p_{12} - 2(2p_{23} + p_{24} - w_2))) + E_{2xya}m_e^2(p_{12} + p_{14} - 2(2(p_{23} + p_{34}) - w_2 - w_4)) \\
&\quad - E_{kxya}((p_{14} - 2w_4)p_{23} + (p_{12} - 2w_2)p_{34} + 2M_A^2(p_{13} + 2w_3) + m_e^2(p_{12} - 2(2(p_{23} - p_{24}) - w_2 - 4M_A^2))) \\
&\quad + E_{13ya}(2x_1(p_{23} + M_A^2) - m_e^2x_2) - E_{3kya}(x_1(p_{23} + 2M_A^2) - 2m_e^2x_2) \\
&\quad + E_{1kya}((3p_{23} - 2p_{24})x_1 + 4M_A^2(x_1 - x_3) - 3m_e^2x_2) + E_{2kya}(x_1(p_{23} + p_{34}) - 2m_e^2(x_2 + x_4)) \\
&\quad - E_{12ya}(2x_1(p_{23} + p_{34}) - m_e^2(x_2 + x_4)) + E_{12ky}(p_{23} + p_{34})x_a - E_{13ky}(p_{23} - 2M_A^2)x_a \\
&\quad + E_{3kxa}((p_{23} + 2M_A^2)y_1 - 2m_e^2y_2) - E_{13xa}(2(p_{23} + M_A^2)y_1 - m_e^2y_2) \\
&\quad - E_{1kxa}((3p_{23} - 2p_{24})y_1 + 4M_A^2(y_1 - y_3) - 3m_e^2y_2) + E_{12xa}(2(p_{23} + p_{34})y_1 - m_e^2(y_2 + y_4)) \\
&\quad - E_{2kxa}((p_{23} + p_{34})y_1 - 2m_e^2(y_2 + y_4)) - E_{12kx}(p_{23} + p_{34})y_a + E_{13kx}(p_{23} - 2M_A^2)y_a, \\
A_{22} &= E_{3kxy}(2(a_{14}w_2 + a_{12}w_4) - 4a_{1k}M_A^2) + (E_{12ky} + 2E_{23ky})(a_{12} + a_{14})x_3 - (E_{12kx} + 2E_{23kx})(a_{12} + a_{14})y_3 \\
&\quad + a_{12}(E_{13ky}x_3 - E_{13kx}y_3) + (E_{2kya}x_3 - E_{2kxa}y_3)(p_{12} + p_{14}) + (E_{13ya}x_3 - E_{13xa}y_3)w_2 \\
&\quad + (E_{3kya}x_3 - E_{3kxa}y_3)(4M_A^2 - p_{12}) + (E_{1kya}x_3 - E_{1kxa}y_3)(p_{12} + w_2) - (E_{12ya}x_3 - E_{12xa}y_3)(w_2 + w_4) \\
&\quad + E_{13ka}(x_2y_3 - x_3y_2) - E_{12ka}((x_2 + x_4)y_3 - x_3(y_2 + y_4)). \tag{B11}
\end{aligned}$$

The following invariants are used in these expressions:

$$\begin{aligned}
x_a &\equiv 2(e_1a_1), & y_a &\equiv 2(e_2a_1), \\
a_{12} &\equiv 2(a_1p_2), & a_{13} &\equiv 2(a_1p_3), \\
a_{14} &\equiv 2(a_1p_4), & a_{1k} &\equiv 2(a_1k). \tag{B12}
\end{aligned}$$

The expressions for the traces involve numerous contractions of vectors with the antisymmetric Levi-Civita tensor, which are compactly denoted as,

$$E_{abcd} \equiv m_e \epsilon_{\mu\nu\alpha\beta} a^\mu b^\nu c^\alpha d^\beta. \tag{B13}$$

These contractions can be explicitly evaluated using the components of the relevant four-vectors, as listed in Table II.

Below we illustrate the calculation of the quantity E_{abcd} to clarify the rationale for including the electron mass m_e in its definition. Although the total expression for $\mathcal{S}_{P_3}$ is formally proportional to m_e , this does not imply that the entire contribution is suppressed. In particular, for a longitudinally polarized initial electron beam, if one of the vectors in E_{abcd} is the spin vector a_1 , the contribution

remains significant and is not suppressed by the smallness of m_e . To demonstrate this point, let us examine the expression for E_{2xya} ,

$$E_{2xya} = m_e \begin{vmatrix} p_{20} & p_{2x} & p_{2y} & p_{2z} \\ \epsilon_{10} & \epsilon_{1x} & \epsilon_{1y} & \epsilon_{1z} \\ \epsilon_{20} & \epsilon_{2x} & \epsilon_{2y} & \epsilon_{2z} \\ a_{10} & a_{1x} & a_{1y} & a_{1z} \end{vmatrix}.$$

Substituting the explicit components of the four-vectors from Table II we obtain,

$$E_{2xya} = \begin{vmatrix} M_A & 0 & 0 & 0 \\ 0 & C_{1k} \cos \varphi_k & C_{1k} \sin \varphi_k & -S_{1k} \\ 0 & -\sin \varphi_k & \cos \varphi_k & 0 \\ \lambda_1 |\mathbf{p}_1| & 0 & 0 & \lambda_1 E_1 \end{vmatrix}.$$

Computing the determinant explicitly, we find that the quantity reduces to,

$$E_{2xya} = \lambda_1 C_{1k} M_A E_1.$$

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