

Resonance in elastic kink-meson scattering

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We analytically sum the leading bubble diagrams that contribute to the elastic scattering amplitude of a kink and a meson in the ϕ^4 double-well model. We find a single peak, corresponding to the unstable kink state in which the shape mode is excited twice. The peak has the usual Breit-Wigner form, and its imaginary part agrees with the shape mode decay rate found by Manton and Merabet.

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I. INTRODUCTION

A. Motivation

It is well established that nuclei are described by QCD and, to a lesser extent, electroweak interactions. However, this knowledge only allows one to learn about nuclei in special cases, such as interactions in which factorization theorems apply or static properties of small nuclei, which are amenable to lattice computations. When the number of colors, N , is large, the situation changes dramatically. It has been shown [1,2] that nuclei reduce to the skyrmions introduced in the Skyrme model of Refs. [3,4]. Real world nucleons are thus the solitons of the Skyrme model with some suitable $1/N$ corrections, and real world compact stars are configurations in this same deformed model.

It has been understood from the beginning [5] that these corrections are essential because the undeformed Skyrme model overpredicts the nuclear binding energies by an order of magnitude. Many deformations have been constructed over the years [6–11]. With each deformation of the Skyrme model, one calculates the nuclear binding energies, shapes, and excitation spectra. However, these three quantities are related; thus, altogether, when confronted with data, they do not provide a strong enough constraint to identify the deformation arising from the $1/N$ corrections, even qualitatively, and even including constraints from observations of compact stars [12].

The present work introduces a method to calculate a new and independent observable, the lifetime of a soliton's unstable excitations. Although the Skyrme model is not

renormalizable, the calculation introduced below can also be applied to that case, yielding the spectrum of nuclear excitations in any candidate deformation of the Skyrme model. In turn, these lifetimes can be used to confront each proposed deformation with experiments. Of course, this problem will be numerically challenging because of the need to understand and numerically manipulate the linearized perturbations about the Skyrmion [13,14].

Historically, kink-meson elastic scattering has had another application. The first calculations of this process in Refs. [15–18] were motivated by pion-nucleon scattering. In Ref. [19], the authors extended their kink-meson results to the Skyrme model to calculate the Born approximation to this process. For simplicity, they neglected the doubly excited intermediate states, even though they contribute to the scattering at the same leading order as the contributions they considered. It is this channel that leads to the resonance that is our main result. Thus, if our methods are also applied to Skyrme models, one would obtain the lifetimes of nucleon excitations, such as the Δ resonance, in those models.

B. Unstable excitations of solitons

Calculating the spectrum of perturbations of solitons is very important [20–24]; therefore, new soliton solutions require an analysis of this spectrum. The perturbations determine whether the soliton is perturbatively stable; they yield its spectrum of excitations and even affect its scattering with other solitons [25–30].

The calculations proceed as follows. One inserts a small perturbation of the soliton solution into the equations of motion. The solutions, at linear order in the perturbations, are the linearized perturbations. For a stable soliton, these will be stable. At nonlinear order, the situation is quite different, and such perturbations generally [31,32] decay by radiation. One example is the shape mode of the kink in the classical ϕ^4 double-well model, whose decay rate was calculated in Ref. [33].

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In quantum field theory, each such linearized perturbation corresponds to a mode that may be excited. The classical nonlinear instability is lifted to the statement in quantum field theory that the mode, when multiply excited,¹ lies above the mass gap; thus, it decays into bulk excitations, which we will call mesons. For example, in the case of the ϕ^4 model, the shape mode is stable if excited once; however, if excited twice, its energy lies above the mass gap, and its lifetime was calculated in Ref. [34]. That lifetime for a single pair of excitations, when applied to a coherent state, reproduced the decay rate in classical field theory of Ref. [33].

The existence of such unstable excitations of quantum solitons is therefore quite generic. For example, if $B^\dagger$ is an operator that excites a stable, bound perturbation, then some power of $B^\dagger$ generally excites an unstable perturbation. However, in general, how can the spectra of unstable perturbations be found? The present paper describes a novel way to find the energies and lifetimes of unstable excitations of solitons. Our strategy is to consider the scattering of a soliton with a meson. The locations of the peaks in the elastic scattering amplitude yield the energies of the unstable soliton excitations, while the widths of the peaks yield the lifetimes. Of course, in particle physics, this is all very standard, but the fact that it extends to kinks is nontrivial because, for example, the meson is a perturbation of a vacuum but the vacua themselves on the two sides of the kink are different.

C. Decay rates

The lifetime of an unstable perturbation in quantum field theory can be used to compute the corresponding radiation rate in classical field theory. Of course, this correspondence between classical and quantum decay rates only holds at tree level. In quantum field theory, one expects loop corrections to the decay rate. We would like to determine if these corrections can be calculated.

It has long been considered [35] that the lifetime of an unstable excitation is, beyond leading order in perturbation theory, ill defined, in general, because it depends on the arbitrary choice of initial state. For this reason, one is instead interested in the width of the resonance corresponding to the creation of the virtual excitation, which agrees with the inverse lifetime at leading orders in a certain adiabatic approximation [36]. In the present paper, we calculate, for the first time, the leading order width of a resonance corresponding to an excitation of a topological soliton, and we show that it agrees with the known leading order inverse lifetime.

¹In quantum field theory, the nonlinearity of the decay rate arises from the fact that multiple excitations of the bound mode are needed to create one quantum of radiation.

More specifically, we consider a model of a scalar subjected to a potential with degenerate minima. Such a model possesses kinks and unbound mesons, and in many cases, it also possesses shape modes, which are bound to the kinks. A shape mode is bound in the sense that its excitation energy ω_S is less than the mass gap of the theory. However, we restrict our attention to models such that $2\omega_S$ is more than the mass gap, so if the shape mode is excited with occupation number 2, then it may escape into the continuum. Although the generalization is rather trivial, for simplicity, we restrict our attention to classically reflectionless kinks such as those of the ϕ^4 double-well model mentioned above.

The resonance we study is a peak in the elastic kink-meson scattering amplitude. Classically, of course, there is no elastic scattering because we have restricted our attention to reflectionless kinks. However, elastic scattering appears at the next order in the semiclassical expansion. The corresponding amplitude was first considered in Ref. [37] using the collective coordinate approach of Ref. [38]. However, because of the complexity of the collective coordinate method [39,40], contributions from intermediate states with two excitations were explicitly dropped, although they contribute at the same order as the processes that were considered. It is precisely such an intermediate state that leads to the peak studied here.

Recently, a simpler framework to address such questions, linearized soliton perturbation theory (LSPT), was introduced [41,42]. It was used to address this problem at leading nonvanishing order in the case of a general potential in Refs. [43,44]. In Ref. [45], the authors restricted their attention to the ϕ^4 double-well model. They found that the S-channel process containing an intermediate state with a twice-excited shape mode contributes the usual S-channel pole as the incoming meson energy approaches twice the shape mode energy, $\omega_{k_0} \rightarrow 2\omega_S$, where ω_{k_0} is the energy of the incoming meson. Physically, this corresponds to virtually exciting a nearly on-shell intermediate kink configuration in which the shape mode is excited twice.

As always, S-channel poles resulting from on-shell virtual particles are smeared out by higher order corrections. The goal of this paper is to resum the leading bubble corrections that dress this twice-excited shape mode intermediate state in the near-resonant regime. We find that, as is usual when dealing with particles, bubble diagrams smooth the tree-level pole into a Breit-Wigner resonance.

The paper is organized as follows. In Sec. II, we review LSPT and fix our conventions. In Sec. III, we summarize the leading order elastic amplitude and isolate the analytic structure associated with the intermediate state [45]. In Sec. IV, we derive the bubble chain directly from the Schrödinger-picture Hamiltonian eigenvalue problem and extract the pole shift and width, comparing with Ref. [34]. In Sec. V, we evaluate the width and line shape in the ϕ^4 double-well model. We conclude in Sec. VI.

II. LINEARIZED SOLITON PERTURBATION THEORY

In this section, we briefly review LSPT as formulated in Refs. [41,42,46]. It is a formalism for lifting nonlinear solutions of classical field theory to states in quantum field theory and then computing observables associated with those states, such as form factors, scattering amplitudes, excitation spectra and their lifetimes, to name a few.

A. Displacement operator

In the present study, we apply LSPT to a $(1+1)$ -dimensional theory of a real scalar field $\phi(x)$ and its conjugate momentum $\pi(x)$ in the Schrödinger picture. Our model is defined by the Hamiltonian

$$H = \int dx : \mathcal{H}(x) :, \quad (2.1)$$

where the standard Schrodinger-picture plane-wave normal ordering $::$ removes the tadpole divergences, which are the only ultraviolet divergences in the model. The Hamiltonian density is

$$\mathcal{H}(x) = \frac{\pi^2(x)}{2} + \frac{\partial_x \phi(x) \partial_x \phi(x)}{2} + \frac{1}{\lambda} V(\sqrt{\lambda} \phi(x)). \quad (2.2)$$

We assume that the potential $V(\sqrt{\lambda} \phi(x))$ has two degenerate minima, so the classical equations admit a static kink solution $\phi(x) = f(x)$ which interpolates between the minima. We define

$$V^{(n)}(\sqrt{\lambda} \phi(x)) = \frac{\partial^n V(\sqrt{\lambda} \phi(x))}{\partial (\sqrt{\lambda} \phi(x))^n} \quad (2.3)$$

and the (vacuum) meson mass by the asymptotic second derivative along the kink profile,

$$m^2 = V^{(2)}(\sqrt{\lambda} f(x))|_{x=\pm\infty}. \quad (2.4)$$

We assume the masses in the two vacua at $x = \pm\infty$ coincide so that the kink does not experience a one-loop acceleration [47]. Choosing a particular static kink profile $f(x)$ singles out a definite position for the kink and thus makes translation invariance nonmanifest in intermediate steps. The underlying theory and also the kink ground state, however, remain translation invariant. In fact, translation invariance reappears order by order [46,48].

In the weak-coupling limit, kink states in quantum theory do not belong to the usual vacuum sector Fock space built by acting finite numbers of mesonic plane-wave creation operators on the vacuum. Instead, they live in a distinct one-kink sector. This is a Fock space which consists of the kink ground state, acted upon by a finite number of bound (shape) and continuum normal-mode (meson) creation operators, as well as an arbitrary function of the zero-mode operators. This arbitrary function is fixed for a momentum

eigenstate. As this one-kink sector Fock space lies outside of the vacuum n -meson Fock space, it is not directly accessible within the ordinary vacuum sector perturbation theory. Indeed, at weak coupling, in the vacuum sector, the expectation value of the field $\phi(x)$ is suppressed by a power of $\sqrt{\hbar}$, so it vanishes in the classical limit; however, in the kink sector, the expectation value is approximately equal to $f(x)$, which is independent of $\hbar$. In the Schrödinger wave functional approach, one would say that the one-kink sector is spanned by the field eigenvectors whose eigenvalues are equal to $f(x)$ plus small perturbations.

In classical field theory, we construct such small perturbations by decomposing the classical field $\phi(x, t)$ as

$$\phi(x, t) = f(x) + \mathbf{g}(x, t). \quad (2.5)$$

Then, we define a kink Hamiltonian H' by

$$H'[\mathbf{g}, \pi] \equiv H[\mathbf{g} + f, \pi] = H[\phi, \pi], \quad (2.6)$$

which generates the same classical evolution and fluctuation spectrum but has the practical advantage that configurations close to the kink can be treated perturbatively because the moments of $\mathbf{g}$ are small.

How does one apply the same idea in quantum field theory? The complication is that, in quantum field theory, the Hamiltonian must be regularized due to the ultraviolet divergences, and there is no general guarantee that a given regularization prescription is invariant under the shift $\phi \rightarrow \phi - f = \mathbf{g}$. The simplest possibility is using the H' and H defined in Eq. (2.6) but then regularizing H' and H with regulators Λ' and Λ , respectively, yielding $H'_{\Lambda'}$ and H_{Λ} . The problem is that the regularized Hamiltonians are no longer equal, $H'_{\Lambda'}[\mathbf{g}, \pi] \neq H_{\Lambda}[\phi, \pi]$, so quantities such as the kink mass will depend on both Λ' and Λ . More seriously, when Λ' and Λ are both taken to infinity, it was shown in Ref. [49] that the kink mass depends on the arbitrary choice of direction in the (Λ', Λ) plane in which this limit is taken. This is to be expected: As the theory is defined by the original Hamiltonian H_{Λ} , its spectrum is the physical one, and masses computed using the naively shifted Hamiltonian $H'_{\Lambda'}$ correspond to a different theory.

The displacement operator approach in the LSPT resolves this problem by defining the unitary displacement operator $\mathcal{D}_f$,

$$\mathcal{D}_f = \exp \left[-i \int dx f(x) \pi(x) \right], \quad (2.7)$$

which implements the shift by the classical profile,

$$\begin{aligned} \mathcal{D}_f^\dagger \phi(x) \mathcal{D}_f &= \phi(x) + f(x), \\ \mathcal{D}_f^\dagger : F[\phi(x), \pi(x)] : \mathcal{D}_f &=: F[\phi(x) + f(x), \pi(x)] :, \end{aligned} \quad (2.8)$$

of an arbitrary functional F .

Instead of Eq. (2.6), in LSPT one begins with the regularized Hamiltonian H_Λ . The kink Hamiltonian is defined to be

$$H'_\Lambda \equiv \mathcal{D}_f^\dagger H_\Lambda \mathcal{D}_f. \quad (2.9)$$

Now there is no need to arbitrarily define a regulator Λ' for H'_Λ because, as defined, H'_Λ is already regularized. As H'_Λ and H_Λ are unitarily equivalent, they have the same spectrum, so one may work with whichever is more convenient at a given step in a problem. This is in contrast with the H_Λ and H'_Λ that appear in the conventional approaches to quantum solitons, which have different spectra. As Λ is now the only regulator, for simplicity, we will leave it implicit from here on.

Note that we do not transform the field from $\phi(x)$ into $\mathbf{g}(x)$. The field operator $\phi(x)$ does not depend on the sector of the Hilbert space on which it acts. We always use the same operator. Furthermore, H is always the Hamiltonian which generates time evolution and whose eigenvalues are the energy, no matter which sector of Hilbert space is considered.

B. States

Let us consider a vacuum sector state $|\psi\rangle$, which consists of a linear combination of n -meson Fock space states up to some finite value of n . Then, the expectation value $\langle\psi|\phi(x)|\psi\rangle$ will be of order $O(\sqrt{\hbar})$ as a result of the $\hbar$ in the canonical commutation relations. Intuitively, the additional $\phi(x)$ leads to half of a commutator, yielding $\sqrt{\hbar}$.

Now, we consider the state $\mathcal{D}_f|\psi\rangle$? The expectation value in this state is

$$\begin{aligned} \langle\psi|\mathcal{D}_f^\dagger\phi(x)\mathcal{D}_f|\psi\rangle &= \langle\psi|\phi(x) + f(x)|\psi\rangle \\ &= \langle\psi|\phi(x)|\psi\rangle + f(x) \\ &= f(x) + O(\sqrt{\hbar}). \end{aligned} \quad (2.10)$$

In other words, this is a kink sector state. Inverting the argument, one writes a one-kink sector state as $\mathcal{D}_f|\psi\rangle$, where $|\psi\rangle$ is in the ordinary Fock space. The time evolution of such a state is quite simple. At time t , $\mathcal{D}_f|\psi\rangle$ becomes

$$e^{-iHt}\mathcal{D}_f|\psi\rangle = \mathcal{D}_f e^{-iH't}|\psi\rangle. \quad (2.11)$$

In other words, to find the evolution of $\mathcal{D}_f|\psi\rangle$, it suffices to evolve the ket $|\psi\rangle$ by the action of the kink Hamiltonian H' .

This is encouraging, but it leads to equations that are somewhat cluttered by the $\mathcal{D}_f$ at the beginning of every state. While it is certainly not necessary, to simplify, we change our notation to remove these $\mathcal{D}_f$ prefactors from the states. More formally, one can think of the ket vectors as coordinates on our Hilbert space of states. Every dictionary that identifies the kets with the states is called a frame. The defining frame is the usual one, in which H time evolves the states. We define the kink frame to be the identification of

the ket $|\psi\rangle$ with the one-kink sector state that was identified with $\mathcal{D}_f|\psi\rangle$ in the defining frame. In other words, we simply remove the $\mathcal{D}_f$ from the ket without changing the state to which the ket refers.

One needs to remember, however, that this $|\psi\rangle$ ket is evolved by the kink Hamiltonian H' . In general relativity, this is the familiar statement that, under a passive coordinate transformation, one also needs to transform the functions. In quantum theory, the statement is that a passive renaming of the kets which are Hilbert space coordinates requires a corresponding renaming of the operators acting on that Hilbert space.

In summary, we adapt the shorthand notation $|\psi\rangle$ to refer to the state normally called $\mathcal{D}_f|\psi\rangle$, which will simplify equations below, but the price of this simplification is that time evolution is generated by H' . Again, the notation $\phi(x)$ is always used to refer to the same operator, no matter which sector we consider and no matter which frame we choose to define our states. If desired, one could define a $\phi' = \phi + f$ analogously to H' ; this would allow us to recover the right expectation value of the field and form factors, but it would not be amenable to perturbative computations because the moments of ϕ' are large even in the vacuum sector.

The next step is to expand the kink Hamiltonian in powers of the fields [41]. More specifically, we decompose the kink Hamiltonian into a sum of terms H'_j which, when plane-wave normal ordered, each contain j factors of the operators ϕ and π ,

$$\begin{aligned} H'_0 &= Q_0, & H'_1 &= 0, \\ H'_2 &= \frac{1}{2} \int dx [\pi^2(x) + :(\partial_x \phi(x))^2: \\ &\quad + :V^{(2)}(\sqrt{\lambda}f(x))\phi^2(x):], \\ H'_{n>2} &= \lambda^{(n-2)/2} \int dx \frac{V^{(n)}(\sqrt{\lambda}f(x))}{n!} : \phi^n(x) :, \end{aligned} \quad (2.12)$$

where Q_0 denotes the classical kink mass.

Just as the free part of the defining Hamiltonian is diagonalized by expanding the fields in plane waves, the quadratic kink Hamiltonian H'_2 is diagonalized by expanding the fields in the normal modes of small fluctuations about the kink. These normal modes $\mathbf{g}$ are the constant frequency solutions of the classical equations derived from H'_2 , and they satisfy the Sturm-Liouville eigenvalue problem

$$\left(-\partial_x^2 + V^{(2)}(\sqrt{\lambda}f(x))\right)\mathbf{g}(x) = \omega^2 \mathbf{g}(x), \quad (2.13)$$

with a translational zero mode $\mathbf{g}_B$ at $\omega_B = 0$, continuum modes $\mathbf{g}_k$ with $\omega_k = \sqrt{m^2 + k^2}$, and possibly discrete bound (shape) modes $\mathbf{g}_S$ with $0 < \omega_S < m$, which we take to be real. We fix the normalization

$$\begin{aligned} \int dx \mathbf{g}_{k_1}^*(x) \mathbf{g}_{k_2}(x) &= 2\pi\delta(k_1 - k_2), \\ \int dx \mathbf{g}_{S_1}(x) \mathbf{g}_{S_2}(x) &= \delta_{S_1 S_2}, \quad \int dx \mathbf{g}_B^2(x) = 1. \end{aligned} \quad (2.14)$$

We fix the sign of the zero mode and also the decomposition into $\mathbf{g}_{\pm k}$ by the conventions

$$\mathbf{g}_B(x) = -\frac{f'(x)}{\sqrt{Q_0}}, \quad \mathbf{g}_k^* = \mathbf{g}_{-k}. \quad (2.15)$$

C. Mode expansion and kink sector Fock space

We now expand the field and its conjugate momentum in the basis of normal modes around the classical kink,

$$\begin{aligned} \phi(x) &= \phi_0 \mathbf{g}_B(x) + \sum \left(B_S^\dagger + \frac{B_S}{2\omega_S} \right) \mathbf{g}_S(x) \\ &\quad + \int \frac{dk}{2\pi} \left(B_k^\dagger + \frac{B_{-k}}{2\omega_k} \right) \mathbf{g}_k(x), \\ \pi(x) &= \pi_0 \mathbf{g}_B(x) + i \sum \left(\omega_S B_S^\dagger - \frac{B_S}{2} \right) \mathbf{g}_S(x) \\ &\quad + i \int \frac{dk}{2\pi} \left(\omega_k B_k^\dagger - \frac{B_{-k}}{2} \right) \mathbf{g}_k(x). \end{aligned} \quad (2.16)$$

Here, we have adopted the shorthand notation

$$B_k^\dagger = \frac{B_k}{2\omega_k}, \quad B_S^\dagger = \frac{B_S}{2\omega_S}, \quad (2.17)$$

which will remove some square root factors from the states computed below.

With the normalization conventions of Eq. (2.14), the canonical commutation relations obeyed by $\phi(x)$ and $\pi(x)$ imply that the coefficients satisfy the algebra

$$\begin{aligned} [B_S, B_{S'}^\dagger] &= \delta_{SS'}, \quad [B_k, B_{k'}^\dagger] = 2\pi\delta(k - k'), \\ [\phi_0, \pi_0] &= i, \end{aligned} \quad (2.18)$$

with all other commutators vanishing. We identify B_S and $B_S^\dagger$ as annihilation and creation operators for the discrete shape modes and $B_k, B_k^\dagger$ as the continuum, while ϕ_0 and π_0 are, at the linear level, proportional to the position and momentum of the kink center of mass.

In the normal-mode basis, the quadratic kink Hamiltonian takes the diagonal form

$$H'_2 = Q_1 + \frac{\pi_0^2}{2} + \sum_S \omega_S B_S^\dagger B_S + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k, \quad (2.19)$$

where Q_1 is the one-loop correction to the kink mass and (ϕ_0, π_0) are the canonical zero-mode variables. Thus, H'_2 describes the center-of-mass motion of the kink as a free particle, together with a collection of harmonic

oscillators, one for each shape mode S and each continuum mode k .

We define the kink sector ground state $|0\rangle_0$ to be the ground state of H'_2 , which is

$$B_S|0\rangle_0 = 0, \quad B_k|0\rangle_0 = 0, \quad \pi_0|0\rangle_0 = 0, \quad (2.20)$$

and construct multiparticle states by acting with creation operators, for example,

$$|k_1 \cdots k_n\rangle_0 \equiv B_{k_1}^\dagger \cdots B_{k_n}^\dagger |0\rangle_0, \quad (2.21)$$

where each k_i runs over real values k but may also be a shape mode index S . For example, we are interested in the twice-excited shape mode

$$|SS\rangle_0 = B_S^\dagger B_S^\dagger |0\rangle_0. \quad (2.22)$$

The zero subscript indicates that these are eigenstates of H'_2 , not the full kink Hamiltonian H' . We denote eigenstates $|\psi\rangle$ of the full Hamiltonian by removing the subscript but demanding that the leading order term is that above

$$|\psi\rangle = \sum_{i=0}^{\infty} \lambda^{i/2} |\psi\rangle_i. \quad (2.23)$$

Fixing the leading order part of the state $|\psi\rangle_0$, together with the fact that $|\psi\rangle$ is an eigenstate of the kink Hamiltonian H' and momentum operator $P' = \mathcal{D}_f^\dagger P \mathcal{D}_f$, is not enough to fix a state entirely. The ambiguity will appear below in the prescriptions for evaluating various poles.

A general kink sector ket, such as $|\psi\rangle_i$, may be expanded in this basis as

$$\begin{aligned} |\psi\rangle_i &= \sum_{m,n} |\psi\rangle_i^{mn}, \\ |\psi\rangle_i^{mn} &= \phi_0^m \oint \frac{d^n k}{(2\pi)^n} \gamma_i^{mn}(k_1, \dots, k_n) |k_1 \cdots k_n\rangle_0. \end{aligned} \quad (2.24)$$

Here, $|\psi\rangle_i^{mn}$ is the order $\lambda^{i/2}$ component of the state $|\psi\rangle$ that has m zero modes and n mesons or shape modes. Again, k runs over real momenta and also discrete shape modes, and the notation $\oint dk/(2\pi)$ represents an integral over continuum modes plus a sum over discrete shape modes S ,

$$\oint \frac{dk}{2\pi} = \int \frac{dk}{2\pi} + \sum_S. \quad (2.25)$$

The coefficients γ_i^{mn} are determined order by order by solving the Schrödinger equation for H' , yielding the usual old-fashioned perturbation theory structure with energy denominators from intermediate states [44].

Turning to interactions, we denote normal ordering with respect to the kink normal modes by $:\cdots_b$. This is the normal ordering scheme in which all $B^\dagger$ and ϕ_0 operators are placed on the left. Wick's theorem then relates plane-wave normal ordering to normal-mode normal ordering [50]. In the

present work, we only need the cubic interaction at leading order, which is the first term on the right-hand side in

$$H'_3 = \lambda^{1/2} \int dx \frac{V^{(3)}(\sqrt{\lambda}f(x))}{3!} : \phi^3(x) :_b + \text{tadpole}. \quad (2.26)$$

D. Constructing kink sector states

To determine states in the kink sector, we need to determine the coefficients γ_i^{mn} . This can be done using old-fashioned perturbation theory.

For example, consider the Hamiltonian eigenstate $|k_0\rangle$ consisting of one kink and one meson with momentum k_0 . Recall that we have defined this state such that, at leading order, it is an eigenstate of the free kink Hamiltonian H'_2 ,

$$|k_0\rangle_0 = B_{k_0}^\dagger |0\rangle_0. \quad (2.27)$$

The following discussion will apply equally well to the state $|S\rangle$ in which there are no mesons but instead the kink has an excited shape mode S ; one simply needs to set k_0 to be equal to the abstract index S . In other words, we will be able to construct excited kink states at the same time as ground-state kinks in the presence of a meson.

At the next order, the state $|k_0\rangle$ is defined to be a kink Hamiltonian eigenstate,

$$H'|k_0\rangle = E|k_0\rangle, \quad (2.28)$$

where the energy can be expanded in powers of the coupling

$$E = \sum_{i=-1}^{\infty} \lambda^i E_i, \quad E_{-1} = Q_0, \quad E_0 = Q_1 + \omega_{k_0}. \quad (2.29)$$

At order $O(\sqrt{\lambda})$, the eigenvalue equation (2.28) becomes

$$\sqrt{\lambda}(H'_2 - E_0)|k_0\rangle_1 = -H'_3|k_0\rangle_0. \quad (2.30)$$

Let us consider the right-hand side, where H'_3 is given in Eq. (2.26). It contains $: \phi^3(x) :_b$. This may be decomposed using Eq. (2.16). There are several terms which contribute to the right-hand side, each giving distinct contributions. For concreteness, let us restrict our attention, for the moment, to those terms with two $B^\dagger$ operators and one B operator. We use the $\supset$ symbol to indicate that we are restricting our attention to those terms. This restriction is

$$: \phi^3(x) :_b \supset 3 \int \frac{d^3k}{(2\pi)^3} \mathbf{g}_{k_1}(x) \mathbf{g}_{k_2}(x) \mathbf{g}_{k_3}(x) B_{k_1}^\dagger B_{k_2}^\dagger \frac{B_{-k_3}}{2\omega_{k_3}}. \quad (2.31)$$

The corresponding term in the interaction Hamiltonian is

$$H'_3 \supset \frac{\sqrt{\lambda}}{4} \int \frac{d^3k}{(2\pi)^3} \frac{V_{k_1 k_2 k_3}}{\omega_{k_3}} B_{k_1}^\dagger B_{k_2}^\dagger B_{-k_3}, \quad (2.32)$$

where we have defined the three-point interaction factor

$$V_{k_1 k_2 k_3} = \int dx V^{(3)}(\sqrt{\lambda}f(x)) \mathbf{g}_{k_1}(x) \mathbf{g}_{k_2}(x) \mathbf{g}_{k_3}(x). \quad (2.33)$$

This leads to

$$-H'_3|k_0\rangle_0 \supset -\frac{\sqrt{\lambda}}{4\omega_{k_0}} \int \frac{d^2k}{(2\pi)^2} V_{k_1, k_2, -k_0} |k_1 k_2\rangle_0 \quad (2.34)$$

on the right-hand side of the eigenvalue equation (2.30). Identifying this with a term on the left-hand side, we find the following contribution to the state $|k_0\rangle$:

$$|k_0\rangle_1 \supset \frac{1}{4\omega_{k_0}} \int \frac{d^2k}{(2\pi)^2} \frac{V_{k_1, k_2, -k_0}}{\omega_{k_0} - \omega_{k_1} - \omega_{k_2}} |k_1 k_2\rangle_0. \quad (2.35)$$

Thus, we are decomposing the Hamiltonian eigenstate $|k_0\rangle$ into components in the various n -particle Fock spaces with different numbers of mesons, shape modes, and zero modes. At leading order, $|k_0\rangle_0$ is just the expected state in the one-particle Fock space. However, at subleading order, written $|k_0\rangle_1$, the state receives, among others, contributions in the two-particle Fock space where the two particles are labeled k_1 and k_2 . There are several kinds of contributions. For example, k_1 and k_2 may be real numbers, leading to a kink and two mesons. These are virtual mesons in the meson cloud about the meson or shape mode excitation in the state $|k_0\rangle$. In addition, k_1 may be real, and k_2 may be a shape mode, corresponding to a virtual meson with momentum k_1 and a virtual shape mode excitation. In this work, we focus on the contribution in which k_1 and k_2 are both shape modes. In this case, the contribution to $|k_0\rangle_1$ is that of a virtual twice-excited shape mode. This is not in the spectrum of the theory because it is unstable, which will be the source of our resonance.

Note that when $\omega_{k_0} = \omega_{k_1} + \omega_{k_2}$, there is a pole. How is this pole to be interpreted? Should it be a principal value, or does one need to add $\pm i\epsilon$, for example? First, note that by the Sokhotski-Plemelj theorem, adding a $\pm i\epsilon$ to the denominator is the same as adding a term proportional to $|k_1 k_2\rangle_0$ to the right-hand side of Eq. (2.35). This state, when $\omega_{k_0} = \omega_{k_1} + \omega_{k_2}$, has the same energy as $|k_1\rangle_0$. Therefore, it is a degenerate Hamiltonian eigenvector. In other words, one is free to choose how the pole is to be evaluated, and the freedom is precisely the freedom to choose a Hamiltonian eigenvector $|k_0\rangle$ in a degenerate eigenspace. How it should be chosen in practice depends on the application. Below, we focus on Hamiltonian eigenvectors because of their appearance in the Lippmann-Schwinger equation, and in this context, the useful choice will be in-states and out-states. This choice determines how the pole is to be defined.

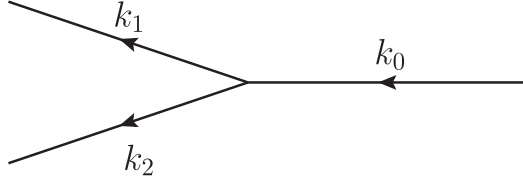


FIG. 1. State diagram contributing to the coefficient $\gamma_1^{02}(k_1, k_2)$ in the one-meson H' eigenstate $|k_0\rangle$. The right external line represents the leading state $|k_0\rangle_0$, while the two left external lines represent the components $|k_1 k_2\rangle_0$ of its leading correction $|k_0\rangle_1$. Here, the k_i can be real numbers, corresponding to continuum mesons, or abstract indices, corresponding to shape modes. The diagram calculates the decomposition of the H' eigenstate $|k_0\rangle$ into eigenstates of the free kink Hamiltonian H'_2 .

E. State diagrams

Above, we have calculated just one of many contributions to the leading correction $|k_0\rangle_1$ to the state $|k_0\rangle$. The number of corrections grows quickly with the order of perturbation theory, so to organize them, it is convenient to introduce state diagrams.

If we are interested in a state $|k'_1 \cdots k'_{n'}\rangle$ with a kink and n' mesons and shape modes, then on the right side of the diagram we need n' external lines. In the present case, $n' = 1$. Imagine that we are interested in the coefficient

$$\gamma_i^{mn}(k_1, \dots, k_n) \quad (2.36)$$

corresponding to the i th order correction to this state in Fock space with m zero modes and n mesons and shape modes. Then, on the left side of the diagram, we will include n solid external lines and m dashed external lines. In the example above, we calculate $\gamma_1^{02}(k_1, k_2)$, so there should be two solid lines on the left. These lines are connected using the vertices H'_n that appear in the perturbative calculation, keeping the convention that internal lines should be solid for shape modes and mesons and dashed for zero modes. With each order in perturbation theory, the corresponding vertices are placed further to the left. In the above example, there is a single application of H'_3 , corresponding to a single trivalent

vertex. This diagram is shown in Fig. 1. The arrows on the solid lines pointing to the left show the orientation of the diagram. These rules are summarized in Table I.

One advantage of the state diagram is that, by treating meson and shape modes on equal footing, a single diagram contains many distinct contributions $|k_1 k_2\rangle_0$ which can be written $|k_1 k_2\rangle_0$, $|Sk\rangle_0$, and $|SS\rangle_0$ without our condensed notation. For example, as we have seen, when k_1 and k_2 are both shape modes, this diagram describes the contribution to $|k_0\rangle$ arising from a virtual state in which the shape mode is excited twice but there are no mesons. The kink itself, on the other hand, does not appear. It is implied at all stages in the calculation that there is precisely one kink because the role of the diagram is to decompose a state in terms of the Fock space in the one-kink sector.

Another advantage of this diagrammatic approach is that it is possible to discuss a contribution without explicitly calculating it. For example, consider the one meson or shape mode contribution to $|k_0\rangle$

$$|k_0\rangle_2^{01} = \int \frac{dk}{2\pi} \gamma_2^{01}(k) |k\rangle_0. \quad (2.37)$$

Let us restrict our attention to contributions which are corrections to the process described above. These are all described by the same diagram, shown in Fig. 2.

In summary, the state diagrams allow one to decompose a Hamiltonian H' eigenstate in the one-kink sector into components which are in Fock space, consisting of a meson, shape modes, and zero mode excitations on top of the kink ground state. The poles are ambiguous, and the ambiguity reflects the degeneracy of the Hamiltonian eigenspaces.

F. Feynman diagrams

Given a model, two of the central problems in physics are the initial value problem and constructing the time-independent states. Corresponding to these two problems in quantum field theory, there are two kinds of diagrams. We have already seen that state diagrams construct the

TABLE I. Each term in a one-kink sector state can be described by a state diagram. The state diagrams do not explicitly show the kink, but they show shape modes and mesons on equal footing. The left-right position of the vertices yields the order in perturbation theory. On the other hand, Feynman diagrams calculate amplitudes involving any number of kinks. The left-right positions of the vertices are irrelevant.

	State diagram	Feynman diagram
Meson		
Ground state kink	<i>Not shown</i>	
Excited kink		
Twice excited kink		

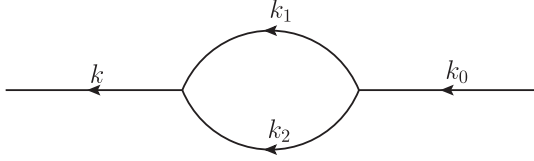


FIG. 2. State diagram contributing to the coefficient $\gamma_2^{01}(k)$ in the corrected one-meson state $|k_0\rangle$. The right external line represents the leading state $|k_0\rangle_0$, while the left external line represents the components $|k\rangle_0$ of its subleading correction $|k_0\rangle_2$. Here, k and k_i can be real numbers, corresponding to continuum modes, or abstract indices, corresponding to shape modes.

time-independent states. The more familiar Feynman diagrams calculate the amplitudes for physical processes.

While Feynman diagrams are quite versatile, in their most common form, they answer the following question. Given a state $|\phi\rangle$ at some initial time, what is the transition amplitude, heuristically $\langle\psi|e^{-iH'\infty}|\phi\rangle$, to arrive at the state $|\psi\rangle$ after an infinite time?

The Dyson formula² converts the operator $e^{-iH'\infty}$ into a time-ordered series expansion in interaction terms H'_n . The time ordering plays an important role. It implies that if one uses the Feynman prescription for the poles in the propagators, then there is no need to sum over the distinct time orderings of the vertices. LSZ reduction allows one to replace the quantum corrections to the initial and final states with relatively simple wave function renormalization factors for the external legs; however, at leading order, for a given interaction, both the quantum corrections and the wave function renormalization are trivial, so they will play no role here.

The fact that one need not separately consider distinct time orderings of vertices greatly simplifies calculations with Feynman diagrams. However, there is also a complication. Excited states are treated as separate particles. For example, the kink ground state $|0\rangle$, the kink with a once-excited shape mode $|S\rangle$, the kink with a twice-excited shape mode $|SS\rangle$, and the rest of the infinite tower each need their own propagators and interaction rules. Our conventions for the different propagators are shown in Table I. We do not know how to treat zero modes using Feynman diagrams, but they will not appear in the leading order sum of the bubble diagrams of interest in this work.

Kink-meson interactions are often lifted to baryon-meson interactions—for example, in Ref. [19]. In this case, for example, the ground-state kink may be lifted to a proton. The once-excited kink may be lifted to a neutron and the

twice-excited kink to a Δ^{++} . If we remove the leptons, then the neutron cannot decay to a proton because its mass is less than that of a proton plus a pion. However, the Δ^{++} is sufficiently massive to decay to a proton plus a π^+ . Similarly, the once-excited kink cannot decay because its mass is not sufficient, while the twice-excited kink can. Of course, all three of these baryons and also the pion require their own propagators and Feynman rules for their interactions.

In summary, the main difference between state diagrams and Feynman diagrams is that state diagrams treat shape modes and mesons on equal footing, whereas Feynman diagrams treat shape mode excitations as new varieties of kinks. As a result, state diagrams are less numerous, the number of loops determines the number of powers of $\lambda\hbar$ in a contribution, and they provide the simplest path to calculating the amplitudes below. However, Feynman diagrams are useful for understanding the pole structure arising from the kinematics and therefore the usual bubble diagram summation that leads to the Breit-Wigner resonance.

III. LEADING REFLECTION AMPLITUDE AND THE TWICE-EXCITED SHAPE MODE POLE

In this section, we review the leading nontrivial (one-loop) reflection amplitude for elastic kink-meson scattering in LSPT, following Refs. [43,45]. We refer to these processes as one-loop types, as they correspond to the state diagrams in Fig. 3.

Our goal in this section is to isolate the pole associated with the twice-excited shape mode ($|SS\rangle$) intermediate state. The goal of Sec. IV will be to eliminate this pole.

A. Lippmann-Schwinger equation

In Ref. [43], it was shown that the Lippmann-Schwinger equation allows one to read the amplitude for elastic kink-meson scattering directly off of the coefficients γ_i^{mn} in Eq. (2.24).

If the incoming momentum of a meson in the center-of-mass frame is k_0 , so that the outgoing momentum of a reflected meson is $-k_0$, then one may expand the state $|k_0\rangle$ corresponding to one kink and one meson as described above. Consider the coefficient at order j with a single meson and no zero modes,

$$\gamma_j^{01}(k') = F_j(k_0, k') + \frac{R_j(k')}{k_0 + k'}, \quad (3.1)$$

where the functions $F_j(k_0, k')$ and $R_j(k')$ are both continuous at $k = -k'$. There are many ways to define the pole, corresponding to the degenerate eigenstates of the kink momentum and energy, as was discussed above. It turns out that $-iR_j(k_0)$ is the order $O(\lambda^{j/2})$ contribution to the matrix element (i times the amplitude) for reflection. Therefore, finding the reflection amplitude is reduced to calculating the states using perturbation theory described in Sec. II.

²This is only true in the absence of zero modes ϕ_0 . As ϕ_0 does not annihilate the kink ground state, normal ordered operators, in general, have nonvanishing expectation values in the one-kink sector. This means that Dyson's approach does not apply to H' evolution in the kink frame, so we do not know how to treat Feynman diagrams including zero modes. On the other hand, in the defining frame, this scattering problem is nonperturbative.

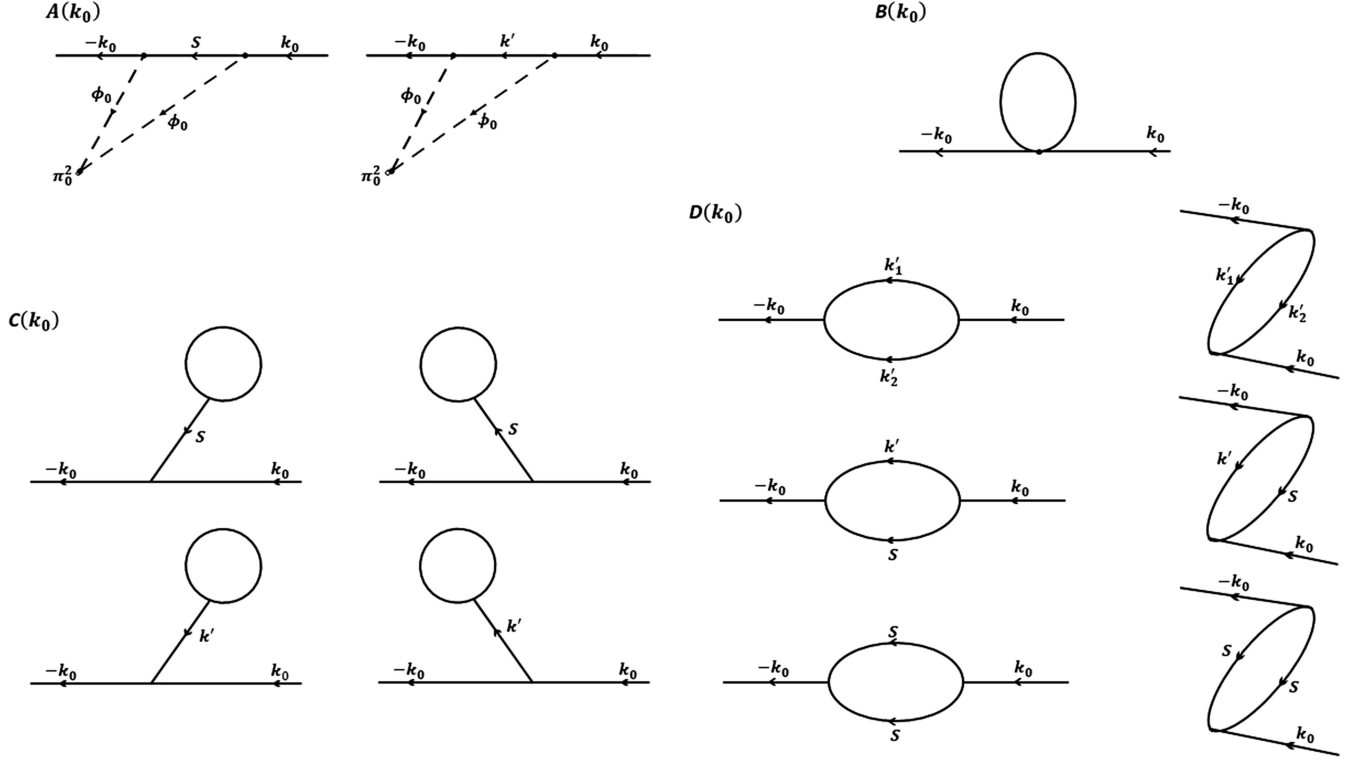


FIG. 3. Diagrams that compute the components A , B , C , and D of the elastic scattering amplitude R [45]. The $\pm k_0$ represent the incoming and outgoing meson momenta, S represents a shape mode, ϕ_0 is the zero mode, and π_0^2 insertions arise from the center-of-mass kinetic term in H'_2 . The tadpole represents the tadpole term in Eq. (2.26).

B. Amplitude decomposition at $\mathcal{O}(\lambda)$

The coefficient $\gamma_i^{mn}(k_1 \cdots k_n)$ corresponds to the component of the i th order correction to an eigenstate of the full interaction kink Hamiltonian H' , projected onto the Fock space corresponding to eigenstates of the free kink Hamiltonian H'_2 with one kink, m zero modes, and n mesons or shape modes with momenta or discrete shape mode labels k_i . These contributions may be computed diagrammatically [43] using state diagrams.

The one-loop reflection amplitude can be organized into four contributions, shown diagrammatically in Fig. 3,

$$R(k_0) = \lambda R_2(k_0) = \lambda(A(k_0) + B(k_0) + C(k_0) + D(k_0)). \quad (3.2)$$

Here, we have drawn additional diagrams to specify whether a given internal line is a shape mode or a meson. However, the benefit of state diagrams is that this is not necessary; one simply needs to sum over all meson and shape modes for each internal line.

For concreteness, consider a model with a single shape mode, although the generalization is obvious. We quote the explicit expressions from Refs. [43,45],

$$\begin{aligned} A(k_0) &= \frac{1}{4\lambda Q_0 k_0} \left[\left(\frac{\omega_{k_0}^2 + \omega_S^2}{\omega_S} \right) \Delta_{-k_0 S} \Delta_{-k_0 S} + \int \frac{dk'}{2\pi} \left(\frac{\omega_{k_0}^2 + \omega_{k'}^2}{\omega_{k'}} \right) \Delta_{-k_0 - k'} \Delta_{-k_0 k'} \right], \\ B(k_0) &= \frac{V_{I-k_0-k_0}}{4k_0}, \\ C(k_0) &= -\frac{1}{4k_0} \left[\frac{V_{IS} V_{S-k_0-k_0}}{\omega_S^2} + \int \frac{dk'}{2\pi} \frac{V_{Ik'} V_{-k'-k_0-k_0}}{\omega_{k'}^2} \right], \\ D(k_0) &= \frac{1}{8k_0} \int \frac{dk'_1 dk'_2}{(2\pi)^2} \frac{(\omega_{k'_1} + \omega_{k'_2}) V_{-k_0 k'_1 k'_2} V_{-k_0 - k'_1 - k'_2}}{\omega_{k'_1} \omega_{k'_2} (\omega_{k_0}^2 - (\omega_{k'_1} + \omega_{k'_2})^2 + i\epsilon)} \\ &\quad + \frac{1}{8k_0} \int \frac{dk'}{2\pi} \frac{(\omega_{k'} + \omega_S) V_{-k_0 k' S} V_{-k_0 - k' S}}{\omega_{k'} \omega_S (\omega_{k_0}^2 - (\omega_{k'} + \omega_S)^2 + i\epsilon)} + \frac{1}{4k_0} \frac{V_{-k_0 SS} V_{-k_0 SS}}{\omega_S (\omega_{k_0}^2 - 4\omega_S^2 + i\epsilon)}. \end{aligned} \quad (3.3)$$

The $+i\epsilon$ summands choose the eigenstates appropriate for the Lippmann-Schwinger equations, as will be shown below. If we again use the generalized notation in which k_i may run over the shape modes, then the matrix meson momentum operator is written in the normal mode basis as

$$\Delta_{k_1 k_2} = \int dx \mathbf{g}_{k_1}(x) \partial_x \mathbf{g}_{k_2}(x), \quad (3.4)$$

and the three-point vertex factor $V_{k_1 k_2 k_3}$ is defined in Eq. (2.33). In this condensed notation, the expressions for A , B , and D above would be simplified considerably; however, our pole involves two shape modes and thus would be obscured by this simplification.

Here, we give a few comments that will be useful later. First, Eq. (3.3) is written for a single discrete shape mode. If a kink supports multiple shape modes, then S should be summed. If there are no shape modes, the summands with S indices are not present.

Second, A , B , and C contribute a smooth, nonresonant background in the kinematic region of interest. Roughly, A is associated with recoil or zero-mode insertions, B is a local contact term, and C is a one-loop correction built from mode overlaps and interaction vertices. The term D is distinguished by the presence of explicit two-particle intermediate states and the corresponding energy denominators, and it is therefore responsible for threshold behavior and potential resonant enhancement. This term is the subject of the present work.

C. Channel decomposition of D and the $|SS\rangle$ pole

The $D(k_0)$ term, corresponding to the state diagram in Fig. 2 which was decomposed into three diagrams in Fig. 3, contains three distinct contributions,

$$D(k_0) = D_{kk}(k_0) + D_{Sk}(k_0) + D_{SS}(k_0), \quad (3.5)$$

corresponding to three intermediate states $|k'_1 k'_2\rangle$, $|Sk'\rangle$, and $|SS\rangle$. In the first term, the intermediate state is a

ground-state kink together with two virtual mesons. In the second, it is a kink with a once-excited shape mode, together with a virtual meson. We are interested in the third, in which the intermediate state is a kink with a twice-excited shape mode. Recall that while the once-excited shape mode $|S\rangle$ is stable because $\omega_S < m$, the twice-excited shape mode decays into a ground-state kink and a meson because $2\omega_S > m$.

The first two pieces control the onset of multiparticle channels: D_{kk} develops the standard two-particle threshold when $\omega_{k_0} \simeq 2m$, and D_{Sk} turns on when $\omega_{k_0} \simeq \omega_S + m$. By contrast, the $|SS\rangle$ channel contains the near-resonant denominator

$$\omega_{k_0}^2 - 4\omega_S^2 + i\epsilon, \quad (3.6)$$

so, as $\omega_{k_0} \rightarrow 2\omega_S$, the one-loop expression develops a pole. Physically, this indicates that the intermediate two-shape-mode configuration becomes nearly on shell and that the scattering amplitude is sensitive to the dynamics of the twice-excited kink state $|SS\rangle$.

D. Diagrammatic expansion of D

So far, we have used diagrams to compute the Hamiltonian eigenstates in terms of coefficients in a basis of states. The Lippmann-Schwinger equation (3.1) connects these coefficients with scattering amplitudes. These scattering amplitudes can also be computed using Feynman diagrams. Therefore, there must be some correspondence between the diagrams above and traditional Feynman diagrams.

For example, consider the three channels that we have just discussed, corresponding to the contributions $D(k_0)$ in Fig. 3. These all correspond to two diagrams in that figure, although each is drawn three times with different labels to make this clear. Reading time from right to left, they each correspond to the scattering of a ground-state kink and a meson, with three different intermediate states: a doubly

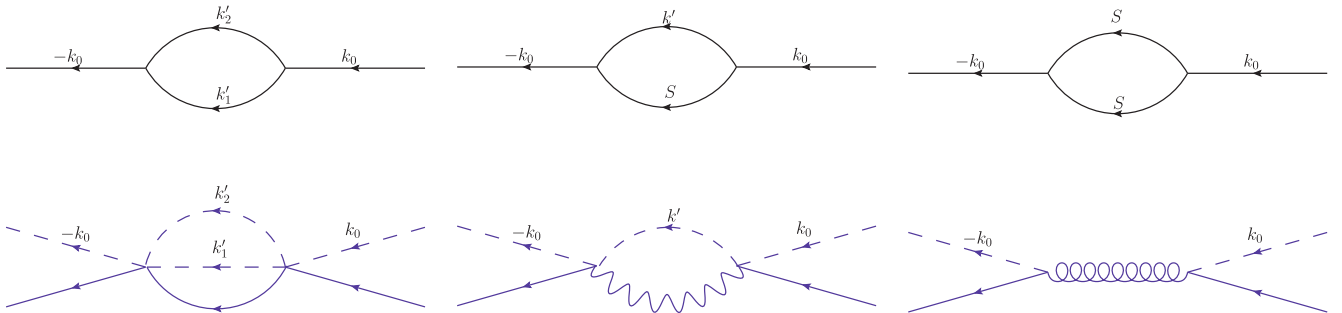


FIG. 4. Three channels of contributions to D . The intermediate states have an unexcited kink and two mesons (left), a shape mode excited kink and one meson (center), and a kink with an unstable twice-excited shape mode (right). The state diagrams are on top, and the corresponding Feynman diagrams on the bottom. The kink is not shown in the state diagrams. In the Feynman diagrams, the mesons are drawn as dashed lines; the kink curves are solid, and no momentum is given for them, as we work in the center-of-mass frame. The kink curve is drawn with more pronounced wiggles when more shape modes are excited. Note that all three channels correspond to the same state diagram, but the Feynman diagrams have different numbers of loops. Time always flows to the left.

excited kink, a singly excited kink with a virtual meson, and finally a ground-state kink with two virtual mesons. In Fig. 4, these same three identical diagrams are shown above the corresponding Feynman diagrams. Note that while the corresponding state diagrams each have one loop, reflecting the fact that they arise at the same order in our $\lambda\hbar$ expansion, the corresponding Feynman diagrams have different numbers of loops.

IV. SCHRÖDINGER PICTURE CALCULATION AND THE BUBBLES

In Sec. III, we isolated the two-shape-mode contribution $D_{SS}(k_0)$ and found the near-resonant enhancement when the incoming meson energy approaches twice the shape mode energy, $\omega_{k_0} \rightarrow 2\omega_S$. In this section, we rederive, in the Schrödinger picture (old-fashioned perturbation theory in the kink sector), virtual transitions of the intermediate $|SS\rangle$ configuration into continuum states, and we back shift the pole off the real axis. Equivalently, the doubly excited kink propagator acquires a self-energy with an imaginary part, producing a finite decay width.

We will ignore H'_0 , as it is a c -number that does not affect the eigenstates. We keep only the free H'_2 and the channel in the cubic interaction H'_3 that mixes the two-shape-mode state into the one-meson continuum and back. All other channels are analytic at $\omega_{k_0} = 2\omega_S$ and may be treated as a smooth background across the resonance. It should be understood that all equalities are up to such contributions, so we obtain only contributions from the leading order pole. The neglected contributions lead to subleading corrections to the decay width, to which we would like to return in future work.

A. Basis states and the role of projections

Recall that the amplitude at each order i is given by the decomposition of the state in Eq. (3.1). In Sec. III, we have already reviewed the leading contribution R_2 . For R_4 , we need to know the state $|k_0\rangle_4$. Most terms in $|k_0\rangle_4$ will, at weak coupling, vanish faster than the $O(\lambda)$ contributions that we have already discussed. The only exception is for values of k_0 near the pole, where the contribution to the amplitude may be enhanced by a factor of order $O(1/\lambda)$ or more. It is in this regime that the corrections discussed below are not subdominant to those reviewed in the previous section.

We recall that H'_2 is diagonal in the kink normal-mode basis,

$$\begin{aligned} H'_2|k\rangle_0 &= (Q_1 + \omega_k)|k\rangle_0, \\ H'_2|SS\rangle_0 &= (Q_1 + 2\omega_S)|SS\rangle_0, \\ \omega_k &= \sqrt{k^2 + m^2}. \end{aligned} \quad (4.1)$$

Equation (3.1) implies that we are only interested in a single coefficient in the state $|k_0\rangle_i$ at each order i .

B. Perturbative expansion of the eigenvalue equation

We construct the Hamiltonian eigenstate $|k_0\rangle$ describing one ground-state kink and one meson by solving the Schrödinger equation in the kink frame,

$$(H'_2 + H'_3 + H'_4 + \cdots)|k_0\rangle = E|k_0\rangle, \quad (4.2)$$

for an eigenstate whose leading term is $|k_0\rangle_0$. We recall that our eigenstate $|k_0\rangle$ is expanded in half-integer powers,

$$\begin{aligned} |k_0\rangle &= |k_0\rangle_0 + \lambda^{1/2}|k_0\rangle_1 + \lambda|k_0\rangle_2 + \lambda^{3/2}|k_0\rangle_3 \\ &\quad + \lambda^2|k_0\rangle_4 + O(\lambda^{5/2}). \end{aligned} \quad (4.3)$$

We also expand the energy but only in integer powers of λ ,

$$\begin{aligned} E &= E_0 + \lambda E_1 + \lambda^2 E_2 + O(\lambda^3), \\ E_0 &= Q_1 + \omega_{k_0}. \end{aligned} \quad (4.4)$$

The absence of an $O(\lambda^{n+1/2})$ energy shift follows from the fact that all half-integer powers of λ in the Hamiltonian come with odd powers of ϕ , which shift the zero mode number plus meson number $m+n$ by an odd amount. As a result, the coefficients γ_i^{mn} are nonvanishing only if $i+m+n$ is odd, meaning that the terms with m zero modes and n mesons are weighted by an even power of $\sqrt{\lambda}$ only if $m+n$ is odd. The left-hand side of the eigenvalue equation $H|k_0\rangle = E|k_0\rangle$, restricted to terms of order $O(\lambda^{i/2})$ with m zero modes and n mesons, is then also nonvanishing only for $i+m+n$ odd; thus, since $|k_0\rangle_j$ on the right-hand side of Eq. (4.2) is nonvanishing only for $j+m+n$ odd, the whole term is only nonvanishing if $i-j$ is even, corresponding to an integer power $\lambda^{(i-j)/2}$ in E .

The interaction relevant for the $|SS\rangle$ bubble is the cubic kink Hamiltonian which, dropping the tadpole term on the first line and the $|Sk\rangle$ and $|kk\rangle$ channels on the second, is

$$H'_3 = \frac{\sqrt{\lambda}}{6} \int dx V^{(3)}(\sqrt{\lambda}f(x)) : \phi^3(x) :_b \quad (4.5)$$

$$= \frac{\sqrt{\lambda}}{2} \int \frac{dk}{2\pi} V_{SSk} : \left(B_S^\dagger + \frac{B_S}{2\omega_S} \right)^2 \left(B_k^\dagger + \frac{B_{-k}}{2\omega_k} \right) :_b, \quad (4.6)$$

where k is a continuum mode and we remind the reader that the three-point interaction factor is

$$V_{SSk} = \int dx V^{(3)}(\sqrt{\lambda}f(x)) \mathbf{q}_S^2(x) \mathbf{q}_k(x). \quad (4.7)$$

It is convenient to decompose H'_3 into summands with fixed meson number,

$$H'_3 = H_3^3 + H_3^1 + H_3^{-1} + H_3^{-3}, \quad (4.8)$$

with

$$H_3^3 = \frac{\sqrt{\lambda}}{2} \int \frac{dk}{2\pi} V_{SSk} B_S^{\dagger 2} B_k^\dagger, \quad (4.9)$$

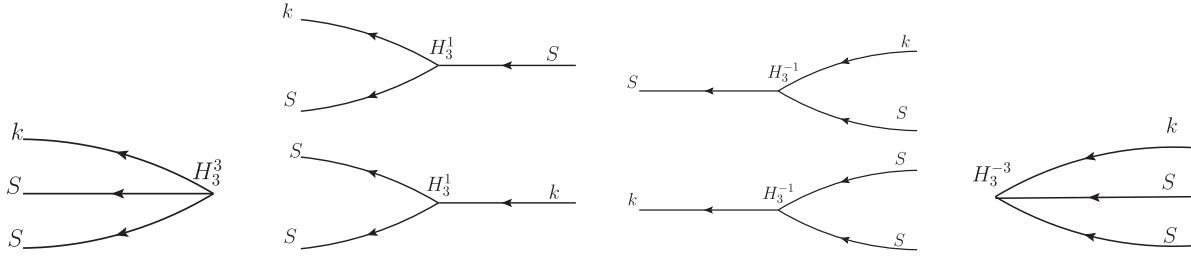


FIG. 5. Vertices representing the various interactions H_3^n in state diagrams, where n is the meson number. Note that for each n , the diagrams are the same up to the choice of which leg is a shape mode or a meson. The corresponding Feynman diagrams would depend on how excited the kink is at the beginning. Recall that we have restricted our attention to interactions with two shape modes and a meson because only these will contribute to the leading order pole.

$$H_3^1 = \sqrt{\lambda} \int \frac{dk}{2\pi} V_{SSk} \left(\frac{B_S^{\dagger 2} B_{-k}}{4\omega_k} + \frac{B_k^{\dagger} B_S^{\dagger} B_S}{2\omega_S} \right), \quad (4.10)$$

$$H_3^{-1} = \sqrt{\lambda} \int \frac{dk}{2\pi} V_{SSk} \left(\frac{B_k^{\dagger} B_S^2}{8\omega_S^2} + \frac{B_S^{\dagger} B_S B_{-k}}{4\omega_S \omega_k} \right), \quad (4.11)$$

$$H_3^{-3} = \frac{\sqrt{\lambda}}{16\omega_S^2} \int \frac{dk}{2\pi} \frac{V_{SSk}}{\omega_k} B_S^2 B_{-k} \quad (4.12)$$

as depicted in Fig. 5. The superscript indicates the net number of mesons and shape modes created minus those annihilated.

Substituting the expansions, Eqs. (4.3) and (4.4), into Eq. (4.2), matching powers of $\lambda^{1/2}$, and dropping³ all contributions arising from interactions other than H_3' yields the hierarchy

$$O(\lambda^0): (H_2' - E_0)|k_0\rangle_0 = 0, \quad (4.13)$$

$$O(\lambda^{1/2}): \sqrt{\lambda}(H_2' - E_0)|k_0\rangle_1 = -H_3'|k_0\rangle_0, \quad (4.14)$$

$$O(\lambda): \sqrt{\lambda}(H_2' - E_0)|k_0\rangle_2 = -H_3'|k_0\rangle_1 + E_1|k_0\rangle_0, \quad (4.15)$$

$$O(\lambda^{3/2}): \sqrt{\lambda}(H_2' - E_0)|k_0\rangle_3 = -H_3'|k_0\rangle_2 + E_1|k_0\rangle_1, \quad (4.16)$$

$$O(\lambda^2): \sqrt{\lambda}(H_2' - E_0)|k_0\rangle_4 = -H_3'|k_0\rangle_3 + E_1|k_0\rangle_2 + E_2|k_0\rangle_0. \quad (4.17)$$

The energy shifts E_1 and E_2 are smooth functions of ω_{k_0} and do not control the resonant enhancement near $\omega_{k_0} \approx 2\omega_S$. The Breit-Wigner behavior instead originates from the explicit two-particle denominators associated with intermediate $|SS\rangle$ states, which we will extract in the following subsections.

³The other interactions do not lead to a maximum number of bubbles and so will lead to subdominant corrections to the width of the resonance.

C. Solving the eigenvalue equation for $|k_0\rangle$

1. $O(\lambda^0)$: Fixing E_0

At leading order, we have defined

$$|k_0\rangle_0 = B_k^{\dagger}|0\rangle_0 \quad (4.18)$$

and our eigenvalue equation is Eq. (4.13),

$$H_2'|k_0\rangle_0 = (Q_1 + \omega_{k_0})|k_0\rangle_0 = E_0|k_0\rangle_0, \quad (4.19)$$

which is satisfied when we set

$$E_0 = Q_1 + \omega_{k_0}. \quad (4.20)$$

2. $O(\lambda^{1/2})$: The $|SS\rangle$ admixture and the resonant denominator

Let us begin with the right-hand side of Eq. (4.14), keeping only the terms in Eqs. (4.9) and (4.10),

$$\begin{aligned} -H_3'|k_0\rangle_0 &= -H_3^3|k_0\rangle_0 - H_3^1|k_0\rangle_0 \\ &= -\frac{\sqrt{\lambda}}{2} \int \frac{dk}{2\pi} V_{SSk} B_S^{\dagger 2} B_k^{\dagger} |k_0\rangle_0 \\ &\quad - \sqrt{\lambda} \int \frac{dk}{2\pi} V_{SSk} \left(\frac{B_S^{\dagger 2} B_{-k}}{4\omega_k} \right) |k_0\rangle_0 \\ &= -\frac{\sqrt{\lambda}}{2} \int \frac{dk}{2\pi} V_{SSk} |SSk k_0\rangle_0 - \frac{\sqrt{\lambda}}{4\omega_{k_0}} V_{SS-k_0} |SS\rangle_0. \end{aligned} \quad (4.21)$$

Other contributions would be finite as k approaches k_0 and so will not contribute to our residue R . To avoid clutter, we do not include them.

What about the left-hand side? As reported in Eq. (2.19), H_2' has several terms. The $\pi_0^2/2$ term is a kinetic term for the center of mass of the kink. It is not invertible. This issue is explained in Ref. [46]. We are interested in a state that is an eigenvector not only of the kink Hamiltonian H' but also of the kink momentum P' . This second condition fixes all

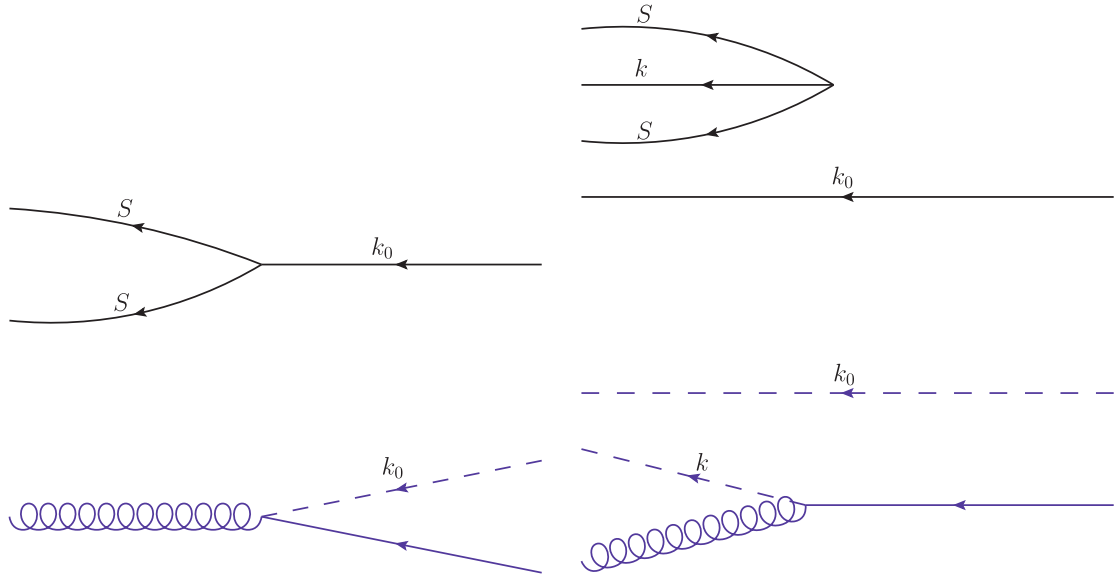


FIG. 6. Two channels contributing to $|k_0\rangle_1^{02}$ (left) and $|k_0\rangle_1^{04}$ (right), respectively. The state diagram is shown on top and the Feynman diagram on the bottom. Note that $|k_0\rangle_1^{04}$ arises from virtual shape modes and a virtual meson which are also present in the dressed kink. Here, $|k_0\rangle_1^{02}$ arises from a meson that is absorbed by a kink, twice exciting its shape mode.

terms γ_i^{mn} with zero mode number $m > 0$. With these fixed, H'_2 becomes invertible up to terms that fix the prescription for evaluating the poles.

On the right-hand side, we have only found terms with two shape modes, either zero or two mesons, and no zero modes. Let us match these to the same terms on the left-hand side. Since none of the terms in H'_2 changes the number of mesons, we can treat sectors with different numbers of mesons separately.

Let us start with the first term on the right-hand side. There are two such contributions on the left-hand side,

$$\begin{aligned} & \sqrt{\lambda} \left(Q_1 + \sum_S \omega_S B_S^\dagger B_S + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k - E_0 \right) |k_0\rangle_1^{04} \\ & + \sqrt{\lambda} \left(\frac{\pi_0^2}{2} \right) |k_0\rangle_1^{24}. \end{aligned} \quad (4.22)$$

In the first term, there are no zero modes in the operator or in the state $|k_0\rangle_1^{04}$, whereas the second term contains the part of the state that has two zero modes, which are annihilated by the kink center-of-mass kinetic term $\pi_0^2/2$.

As we know E_0 and that the right-hand side is proportional to $|SSkk_0\rangle_0$, the left-hand side can be simplified somewhat to

$$\sqrt{\lambda} \left(2\omega_S - \omega_{k_0} + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k \right) |k_0\rangle_1^{04} + \sqrt{\lambda} \left(\frac{\pi_0^2}{2} \right) |k_0\rangle_1^{24}. \quad (4.23)$$

The first term already has a suggestive coefficient, suggesting that it may give rise to a pole. The second does not, and so it will not contribute to R . Moreover, the momentum

operator produces a total shift of m and n of two units at each order, so $|k_0\rangle_1^{24}$ vanishes for a state which is annihilated by P' , in other words, by a momentum eigenstate in the center-of-mass frame. Therefore, the second term is equal to zero. We are left with

$$\begin{aligned} & \left(2\omega_S - \omega_{k_0} + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k \right) |k_0\rangle_1^{04} \\ & = -\frac{1}{2} \int \frac{dk}{2\pi} V_{SSk} |SSkk_0\rangle_0, \end{aligned} \quad (4.24)$$

which is easily solved to yield

$$|k_0\rangle_1^{04} = -\frac{1}{2} \int \frac{dk}{2\pi} \frac{V_{SSk}}{(2\omega_S + \omega_k)} |SSkk_0\rangle_0 \quad (4.25)$$

or, equivalently,

$$\gamma_1^{04}(k) = -\frac{V_{SSk}}{2(2\omega_S + \omega_k)}, \quad (4.26)$$

where we remind the reader that we have dropped many terms that will not eventually lead to the desired pole. Note that the denominator is always positive, as the two virtual shape modes and the virtual meson of momentum k never conserve energy and thus are off shell. The calculation above is summarized diagrammatically in the right panel of Fig. 6.

Now, let us proceed to the other term on the right-hand side, with two shape modes and no mesons, which is shown in the left panel of Fig. 6. Again, $|k_0\rangle_1^{22}$ vanishes, and proceeding as above, we find

$$\begin{aligned} & \left(2\omega_S - \omega_{k_0} + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k \right) |k_0\rangle_1^{02} \\ &= -\frac{1}{4\omega_{k_0}} V_{SS-k_0} |SS\rangle_0, \end{aligned} \quad (4.27)$$

which is solved by

$$|k_0\rangle_1^{02} = -\frac{V_{SS-k_0}}{4\omega_{k_0}(2\omega_S - \omega_{k_0})} |SS\rangle_0 \quad (4.28)$$

in agreement with Eq. (2.35) in the case $k_1 = k_2 = S$, or, equivalently,

$$\gamma_1^{02}(k) = -\frac{V_{SS-k_0}}{4\omega_{k_0}(2\omega_S - \omega_{k_0})}. \quad (4.29)$$

This coefficient is the amplitude corresponding to the projection onto a virtual twice-excited shape mode inside a system consisting of a ground-state kink and a meson. As usual, it has a pole when the total mass of the virtual system is equal to that of the original system. The position of the pole is that of the pole in D_{SS} .

How is the pole to be defined, and do we need to add $i\epsilon$? This is a rather subtle point. The state $|SS\rangle_0$ in fact is not the leading order expansion of any Hamiltonian eigenstate $|SS\rangle$ because it is unstable. As a result of this instability, $|SS\rangle_0$ cannot be the initial or final state of such an interaction, although it is not evident at this order in perturbation theory; thus, the usual causality argument requiring $i\epsilon$ does not apply.

3. $O(\lambda)$: $|SS\rangle$ goes into the one-meson continuum

Now, let us proceed to order $i = 2$, keeping only the terms from Eqs. (4.25) and (4.28), as only these correspond to the production of virtual pairs of shape modes, although the former also contains two virtual mesons.

First, we try to reproduce the pole in Sec. III C. For this, we need to evaluate γ_2^{01} , so we need the one-meson part of $|k_0\rangle_2$.

On the right-hand side of Eq. (4.15), there are two terms that have meson number equal to 1,

$$-H_3^{-3} |k_0\rangle_1^{04} - H_3^{-1} |k_0\rangle_1^{02}. \quad (4.30)$$

In the first term, there are two possible contractions. The meson annihilation operator in H_3^{-3} could annihilate the meson with momentum k or that with momentum k_0 . The first case would create a disconnected diagram, so it will certainly not shift our pole. We therefore restrict our attention to the second contraction, simply dropping the first.

With this simplification, the first term on the right-hand side becomes

$$\frac{\sqrt{\lambda}}{16\omega_S^2} \frac{V_{SS-k_0}}{\omega_{k_0}} \int \frac{dk}{2\pi} \frac{V_{SSk}}{(2\omega_S + \omega_k)} |k\rangle_0 \quad (4.31)$$

while the second is

$$\frac{\sqrt{\lambda} V_{SS-k_0}}{16\omega_{k_0}(2\omega_S - \omega_{k_0})} \int \frac{dk}{2\pi} \frac{V_{SSk}}{\omega_S^2} |k\rangle_0. \quad (4.32)$$

They are quite similar; however, the second has a pole, and the first does not. This is because the first term describes a system that not only has a virtual twice-excited shape mode but also two virtual mesons, so it can never be on shell. Clearly, this first term will be subdominant by one power of $(2\omega_S - \omega_{k_0})$; thus, at leading order, it will not affect our residue R . Therefore, we will keep only the second term.

This leads us to an important simplification which will help us at higher orders. Even though the diagrams leading to both terms contain a single doubly excited meson, that in which the double excitation is created when the meson is destroyed, corresponding to H_3^1 , is the most divergent because it has the least virtual particles, as is that in which the double excitation is destroyed when a meson is created, corresponding to H_3^{-1} . The interactions H_3^3 and H_3^{-3} lead to lower order poles, so we will drop them.

Next, we find the left-hand side of Eq. (4.15) which is equal to Eq. (4.32). Again, there are two contributions,

$$\begin{aligned} & \sqrt{\lambda} \left(Q_1 + \sum_S \omega_S B_S^\dagger B_S + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k - E_0 \right) |k_0\rangle_2^{01} \\ & + \sqrt{\lambda} \left(\frac{\pi_0^2}{2} \right) |k_0\rangle_2^{21}. \end{aligned} \quad (4.33)$$

As we are interested in terms on the right-hand side with a single meson and no shape modes, we may simplify this to

$$\sqrt{\lambda} \left(-\omega_{k_0} + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k \right) |k_0\rangle_2^{01} + \sqrt{\lambda} \left(\frac{\pi_0^2}{2} \right) |k_0\rangle_2^{21}. \quad (4.34)$$

One can see that the first term will lead to another pole for $k \sim \pm k_0$, which is necessary for this to contribute to leading order residue. The second term is proportional to $|k_0\rangle_2^{21}$, which is fixed by translation invariance, not dynamics, and can be shown to have no such pole. Therefore, we will drop the second term.

The eigenvalue equation (4.15) then becomes

$$\begin{aligned} & \left(-\omega_{k_0} + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k \right) |k_0\rangle_2^{01} \\ &= \frac{V_{SS-k_0}}{16\omega_{k_0}(2\omega_S - \omega_{k_0})} \int \frac{dk}{2\pi} \frac{V_{SSk}}{\omega_S^2} |k\rangle_0, \end{aligned} \quad (4.35)$$

which is solved by

$$|k_0\rangle_2^{01} = \frac{V_{SS-k_0}}{16\omega_{k_0}\omega_S^2(\omega_{k_0}-2\omega_S)} \int \frac{dk}{2\pi} \frac{V_{SSk}}{\omega_{k_0}-\omega_k+i\epsilon} |k\rangle_0, \quad (4.36)$$

corresponding to

$$\gamma_2^{01}(k) = \frac{V_{SS-k_0}}{16\omega_{k_0}\omega_S^2(\omega_{k_0}-2\omega_S)} \frac{V_{SSk}}{\omega_{k_0}-\omega_k+i\epsilon}. \quad (4.37)$$

This contribution corresponds to the third panel in Fig. 4. As can be seen in that Feynman diagram, the $\omega_{k_0}-2\omega_S$ denominator corresponds to the usual S-channel pole in kink-meson scattering not to the self-energy of the twice-excited kink. Indeed, this Feynman diagram does not have any loops.

Here, $i\epsilon$ follows the usual in-state prescription, which chooses the initial condition

$$|t=0\rangle = \int \frac{dk'}{2\pi} e^{-\sigma^2(k'-k_0)^2-i(k'-k_0)x_0} |k'\rangle, \quad (4.38)$$

$$0 < 1/k_0 \ll \sigma \ll -x_0,$$

corresponding to a wave packet of Hamiltonian eigenstates such that there is no reflection before the scattering. To see this, note that the $O(\lambda)$ one-meson part of the initial wave packet is

$$\begin{aligned} |t=0\rangle_2^{01} &= \int \frac{dk'}{2\pi} e^{-\sigma^2(k'-k_0)^2-i(k'-k_0)x_0} |k'\rangle_2^{01} \\ &= \int \frac{dk}{2\pi} \left[\int \frac{dk'}{2\pi} e^{-\sigma^2(k'-k_0)^2-i(k'-k_0)x_0} \right. \\ &\quad \times \left. \frac{V_{SS-k'}}{16\omega_{k'}\omega_S^2(\omega_{k'}-2\omega_S)} \frac{V_{SSk}}{\omega_{k'}-\omega_k+i\epsilon} \right] |k\rangle_0. \end{aligned} \quad (4.39)$$

As x_0 is very negative, the k' integration can be closed in the upper imaginary part of the complex plane. The wave packet can be taken to be sufficiently monochromatic so that the $\omega_{k'}-2\omega_S$ pole's residue is suppressed by the Gaussian. However, as k is itself a dummy variable, there will always be values of k in the integrand such that the $\omega_{k'}-\omega_k$ poles need to be considered. As k_0 is positive, the Gaussian will suppress the residue when k' is negative. However, the $\omega_{k'}-\omega_k$ term yields poles at both $k = \pm k'$. When k is negative, this denominator is proportional to $k' + k + i\epsilon$, so the pole is in the negative imaginary part of the complex plane and thus is not in our contour. Therefore, this $+i\epsilon$ prescription guarantees that we do not have any left-going mesons in our initial state, as desired. This is the reason that we choose $+i\epsilon$. Right-going mesons appear at leading order in the initial state. However, at order $O(\lambda)$, another contribution arises from Eq. (4.39), as the corresponding pole is proportional to $k' - k + i\epsilon$. Again, the pole is in the negative imaginary part of the complex plane,

so there is no contribution at initial times. If there was such a contribution, it could be absorbed into the normalization of the initial state.

This concludes our discussion of the initial wave packet. Now, we return to the construction of the Hamiltonian eigenstate.

Near the $k = -k_0$ pole, we may approximate

$$\frac{1}{\omega_{k_0}-\omega_k+i\epsilon} = \frac{\omega_{k_0}}{k_0} \frac{1}{k_0+k+i\epsilon}, \quad (4.40)$$

so the residue of the $(k+k_0)$ pole, when $2\omega_S \sim \omega_{k_0}$, is therefore

$$R_2(k_0) = \frac{V_{SS-k_0}^2}{16k_0\omega_S^2(\omega_{k_0}-2\omega_S)} = \frac{V_{SS-k_0}^2}{4k_0\omega_S(\omega_{k_0}^2-4\omega_S^2)}, \quad (4.41)$$

in agreement with the pole term in Eq. (3.3), which was taken from Ref. [43].

We will drop the terms which are finite at the $k = \pm k_0$ poles and simply write

$$\begin{aligned} |k_0\rangle_2 &= \int \frac{dk}{2\pi} \gamma_2^{01}(k) |k\rangle_0, \\ \gamma_2^{01}(k) &= \left(\frac{k_0 R_2(k_0)}{\omega_{k_0}} \right) \frac{1}{\omega_{k_0}-\omega_k+i\epsilon}. \end{aligned} \quad (4.42)$$

Let us consider, for the moment, the pole at $k = -k_0$, corresponding to reflection. Keeping only the pole term, we find

$$\gamma_2^{01}(k) = \frac{R_2(k_0)}{k+k_0+i\epsilon}. \quad (4.43)$$

As usual, there are also terms that are finite at $k = k_0$, which, for brevity, are not included in our expressions.

4. $O(\lambda^{i+1/2})$: Return to $|SS\rangle$

Imagine that at some order i we have found

$$\begin{aligned} |k_0\rangle_{2i} &= \int \frac{dk}{2\pi} \gamma_{2i}^{01}(k) |k\rangle_0, \\ \gamma_{2i}^{01}(k) &= \left(\frac{k_0 R_{2i}(k_0)}{\omega_{k_0}} \right) \frac{1}{\omega_{k_0}-\omega_k+i\epsilon} \end{aligned} \quad (4.44)$$

plus terms that are finite at the poles. Here, the $k = -k_0$ pole is defined, as above, by the $k+k_0+i\epsilon$ prescription. Imagine further that at each lower value of i , there is an i th order pole at $k = \pm k_0$. The case $i=1$ appears in Eq. (4.42).

Now, let us proceed to calculate γ_{2i+1} using the eigenvalue equation

$$\sqrt{\lambda}(H'_2 - E_0)|k_0\rangle_{2i+1} = -H'_3|k_0\rangle_{2i}. \quad (4.45)$$

We have learned above that the highest order pole will arise from the H_3^1 term in H'_3 . The leading pole on the right-hand side is therefore

$$\begin{aligned} -H_3^1 |k_0\rangle_{2i} &= -\int \frac{dk}{2\pi} \gamma_{2i}^{01}(k) H_3^1 |k_0\rangle_0 \\ &= -\frac{\sqrt{\lambda}}{4} \int \frac{dk}{2\pi} \frac{\gamma_{2i}^{01}(k) V_{SS-k}}{\omega_k} |SS\rangle_0. \end{aligned} \quad (4.46)$$

The eigenvalue equation (4.45) then yields

$$\begin{aligned} |k_0\rangle_{2i+1} &= \int \frac{dk}{2\pi} \gamma_{2i}^{01}(k) H_3^1 |k_0\rangle_0 \\ &= \frac{1}{4(\omega_{k_0} - 2\omega_S)} \int \frac{dk}{2\pi} \frac{\gamma_{2i}^{01}(k) V_{SS-k}}{\omega_k} |SS\rangle_0, \end{aligned} \quad (4.47)$$

as is drawn in Fig. 7.

5. $\mathcal{O}(\lambda^{i+1})$

At the next order, we will apply the eigenvalue equation

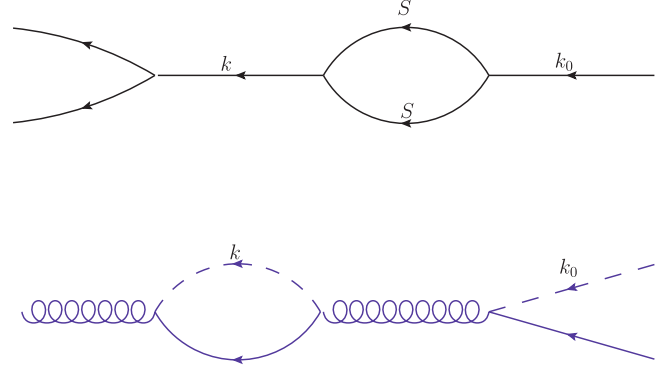


FIG. 7. State (top) and Feynman (bottom) diagrams describing the contributions to $|k_0\rangle_3$ in Eq. (4.47). The intermediate kink plus meson configuration corresponds to a loop in the Feynman diagram, which leads to the $\omega_k - \omega_{k_0}$ pole. However, it corresponds to a line in the state diagram, as the kink is not drawn. The loop in the state diagram leads to the $\omega_{k_0} - 2\omega_S$ pole. More generally, the contribution to $|k_0\rangle_{2i+1}$ would consist of i loops separated by internal lines, repeating the pattern seen here.

$$\sqrt{\lambda}(H'_2 - E_0)|k_0\rangle_{2i+2} = -H'_3|k_0\rangle_{2i+1} + \sum_{j=1}^{i-1} E_j |k_0\rangle_{2(i+1-j)}. \quad (4.48)$$

By assumption, the E_j terms have lower order poles; thus, they will not affect the leading pole calculation here, so we will drop them. Now, we have learned that the dominant pole on the right-hand side will arise from H_3^{-1} , so the right-hand side is

$$-H_3^{-1} |k_0\rangle_{2i+1} = \frac{\sqrt{\lambda}}{16\omega_S^2(2\omega_S - \omega_{k_0})} \int \frac{dk'}{2\pi} \int \frac{dk}{2\pi} \frac{\gamma_{2i}^{01}(k) V_{SS-k} V_{SSk'}}{\omega_k} |k'\rangle_0. \quad (4.49)$$

The eigenvalue equation then implies

$$|k_0\rangle_{2i+2} = \frac{1}{16\omega_S^2(\omega_{k_0} - 2\omega_S)} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \frac{\gamma_{2i}^{01}(k) V_{SS-k} V_{SSk'}}{\omega_k(\omega_{k_0} - \omega_{k'} + i\epsilon)} |k'\rangle_0 \quad (4.50)$$

corresponding to

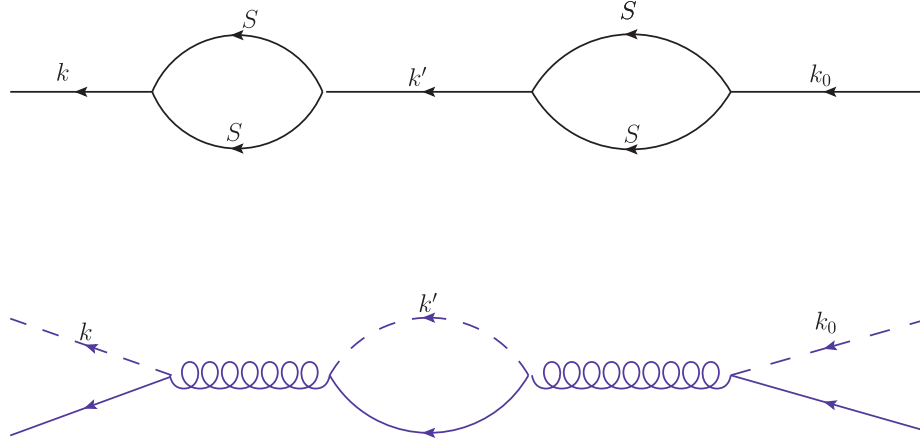
$$\begin{aligned} \gamma_{2i+2}^{01}(k) &= \frac{V_{SSk}}{16\omega_S^2(\omega_{k_0} - 2\omega_S)(\omega_{k_0} - \omega_k + i\epsilon)} \int \frac{dk'}{2\pi} \frac{V_{SS-k'}}{\omega_{k'}} \gamma_{2i}^{01}(k') \\ &= \frac{V_{SSk}}{4\omega_S(\omega_{k_0}^2 - 4\omega_S^2)(\omega_{k_0} - \omega_k + i\epsilon)} \int \frac{dk'}{2\pi} \frac{V_{SS-k'}}{\omega_{k'}} \gamma_{2i}^{01}(k'), \end{aligned} \quad (4.51)$$

where in the last line we made the approximation $\omega_{k_0} \sim 2\omega_S$, which is valid close to the pole.

Note that

$$\gamma_0^{01}(k') = 2\pi\delta(k' - k_0) \quad (4.52)$$

as a result of the definition of $|k_0\rangle_0$ in Eq. (4.18), so in the case $i = 0$ this reproduces Eq. (4.37). The case $i = 1$ is drawn in Fig. 8.


 FIG. 8. State (top) and Feynman (bottom) diagrams describing the contributions to $|k_0\rangle_4$ in Eq. (4.50).

Inserting Eq. (4.51) into itself recursively, one finds

$$\begin{aligned}\gamma_{2i+2}^{01}(k) &= \frac{V_{SSk}}{4\omega_S(\omega_{k_0}^2 - 4\omega_S^2)(\omega_{k_0} - \omega_k + i\epsilon)} I(k_0)^j \int \frac{dk'}{2\pi} \frac{V_{SS-k'}}{\omega_{k'}} \gamma_{2(i-j)}^{01}(k'), \\ I(k_0) &= \int \frac{dk}{2\pi} \frac{|V_{SSk}|^2}{4\omega_S\omega_k(\omega_{k_0}^2 - 4\omega_S^2)(\omega_{k_0} - \omega_k + i\epsilon)}\end{aligned}\quad (4.53)$$

for any non-negative integer $j \leq i$. Here, we have used $V_{SSk}^* = V_{SS-k}$, which results from the reality of the $V^{(3)}$ and $\mathbf{g}_S$ together with the convention $g_k^* = g_{-k}$.

Setting $j = i$, this reduces to

$$\gamma_{2i+2}^{01}(k) = \frac{V_{SSk} V_{SS-k_0}}{4\omega_{k_0}\omega_S(\omega_{k_0}^2 - 4\omega_S^2)(\omega_{k_0} - \omega_k + i\epsilon)} I(k_0)^i. \quad (4.54)$$

The corresponding total correction to the one-meson Fock space component of $|k_0\rangle$ is the sum of these contributions,

$$\sum_{i=1}^{\infty} \lambda^i \gamma_{2i}^{01}(k) = \frac{\lambda V_{SSk} V_{SS-k_0}}{4\omega_{k_0}\omega_S(\omega_{k_0}^2 - 4\omega_S^2)(\omega_{k_0} - \omega_k + i\epsilon)} \frac{1}{1 - \lambda I(k_0)}. \quad (4.55)$$

D. Decay rate

Next, we study $I(k_0)$, which looks real except for the $i\epsilon$ in the pole prescription. Therefore, the imaginary contribution corresponding to reflection arises from the residues at the $k' = \pm k_0$ poles,

$$\begin{aligned}\frac{1}{\omega_{k_0} - \omega_k + i\epsilon} &= \frac{\omega_{k_0}}{k_0} \left(\frac{1}{k_0 + k + i\epsilon} + \frac{1}{k_0 - k + i\epsilon} \right) + \text{finite} \\ &= -\frac{\omega_{k_0}}{k_0} i\pi (\delta(k_0 + k) + \delta(k_0 - k)) + \text{finite},\end{aligned}\quad (4.56)$$

where the finite part is to be integrated using the Cauchy principal value prescription around the $k = \pm k_0$ poles,

$$\Re[I(k_0)] = \text{PV}[I(k_0)]. \quad (4.57)$$

The imaginary part is

$$\Im[I(k_0)] = -\frac{2|V_{SSk_0}|^2}{8\omega_S k_0(\omega_{k_0}^2 - 4\omega_S^2)}, \quad (4.58)$$

where the factor of 2 in the numerator comes from the sum over the two poles. The factor of 2 in the denominator comes from $i\pi$ in the Sokhotski-Plemelj theorem divided by 2π in $dk/(2\pi)$. The sum of the geometric series is

$$\begin{aligned} & \frac{1}{1 - \lambda(\text{PV}[I(k_0)] + i\lambda\Im[I(k_0)])} \\ &= \frac{\omega_{k_0}^2 - 4\omega_S^2}{\omega_{k_0}^2 - 4\omega_S^2 + i\lambda|V_{SSk_0}|^2/(4\omega_S k_0)} \\ &= \frac{\omega_{k_0} - 2\omega_S}{\omega_{k_0} - 2\omega_S + i\Gamma/2}, \end{aligned} \quad (4.59)$$

where in the first equality we have dropped the principal value of $I(k_0)$, which does not shift the imaginary part of the pole. The last equality matches this complex pole with the Breit-Wigner form, with the decay rate

$$\Gamma = \frac{\lambda|V_{SSk_0}|^2}{8\omega_S^2 k_0}. \quad (4.60)$$

This is the usual Breit-Wigner pole shift corresponding to a decay of the $|SS\rangle$ state into a left-traveling meson, as it arises from the $k' = \pm k_0$ poles and $k_0 > 0$. This indeed is the decay rate found in quantum field theory in Ref. [34], which agrees precisely with the classical result of Ref. [33].

E. Residue

The total correction to the one-meson state is now

$$\begin{aligned} \sum_{i=1}^{\infty} \lambda^i \gamma_{2i}^{01}(k) &= \frac{\lambda V_{SSk} V_{SS-k_0}}{\omega_{k_0}(\omega_{k_0} - \omega_k + i\epsilon)} \\ &\times \left(\frac{k_0}{4k_0\omega_S(\omega_{k_0}^2 - 4\omega_S^2) + i\lambda|V_{SSk_0}|^2} \right). \end{aligned} \quad (4.61)$$

Matching this to the definition in Eq. (3.1) of the residue R at the $k_0 + k$ pole, one finds

$$R = \frac{\lambda|V_{SSk_0}|^2}{4k_0\omega_S(\omega_{k_0}^2 - 4\omega_S^2) + i\lambda|V_{SSk_0}|^2}. \quad (4.62)$$

The imaginary part of the residue is then

$$\Im[R] = -\frac{\lambda^2|V_{SSk_0}|^4}{16k_0^2\omega_S^2(\omega_{k_0}^2 - 4\omega_S^2)^2 + \lambda^2|V_{SSk_0}|^4} + O(\lambda^3). \quad (4.63)$$

Note that the $O(\lambda^0)$ term in the denominator has a zero, near which the $O(\lambda^2)$ term in the denominator dominates. This is the usual Γ^2 term, which cuts off the pole reported in Ref. [45].

The probability P of elastic scattering is $|R|^2$. As there are no other imaginary contributions to the residue at this order with which this process may interfere, the leading contribution arising from the imaginary part of the residue is

$$\begin{aligned} \Delta P &= (\Im[R])^2 \\ &= \frac{\lambda^4|V_{SSk_0}|^8}{(16k_0^2\omega_S^2(\omega_{k_0}^2 - 4\omega_S^2)^2 + \lambda^2|V_{SSk_0}|^4)^2} + O(\lambda^5). \end{aligned} \quad (4.64)$$

Notice that, at the pole, the first term in the denominator vanishes, so we find a probability of reflection which is, in this leading pole approximation, equal to unity. In other words, there is total reflection.

The contribution to forward scattering from $\gamma_{2i}^{01}(k \sim k_0)$ follows from the same calculation as above and so is given by the same residue, Eq. (4.63), which is equal to $-i$ at the pole. As in backward scattering, one arrives at the forward scattering amplitude by integrating around this pole, closing the contour in the negative imaginary part of the complex plane. This yields a forward scattering matrix element which, like the backward scattering matrix element, is equal to -1 at the Breit-Wigner peak. However, the forward scattering amplitude, unlike the backward scattering amplitude, interferes coherently with the unscattered meson, which has a matrix element of 1. Altogether, the forward scattering matrix element is then $1 - 1 = 0$, which is consistent with complete reflection at the peak.

How does this affect the pole in R reported above? Recall that the pole arises from the D contribution in the last term of Eq. (3.3). Instead, we have found the following expression for D ,

$$\begin{aligned} D(k_0) &= \frac{1}{8k_0} \int \frac{dk'_1 dk'_2}{(2\pi)^2} \frac{(\omega_{k'_1} + \omega_{k'_2})V_{-k_0 k'_1 k'_2} V_{-k_0 -k'_1 -k'_2}}{\omega_{k'_1} \omega_{k'_2} (\omega_{k_0}^2 - (\omega_{k'_1} + \omega_{k'_2})^2 + i\epsilon)} \\ &+ \frac{1}{8k_0} \int \frac{dk'}{2\pi} \frac{(\omega_{k'} + \omega_S)V_{-k_0 k' S} V_{-k_0 -k' S}}{\omega_{k'} \omega_S (\omega_{k_0}^2 - (\omega_{k'} + \omega_S)^2 + i\epsilon)} \\ &+ \frac{1}{4k_0} \frac{|V_{SSk_0}|^2}{\omega_S(\omega_{k_0}^2 - 4\omega_S^2) + i\lambda|V_{SSk_0}|^2}. \end{aligned} \quad (4.65)$$

Now that R has a finite imaginary part, one must remember [43] that the probability is $|R|^2$.

V. EXAMPLE: THE ϕ^4 DOUBLE WELL

We now specialize our general expressions to the ϕ^4 double-well model in $(1+1)$ dimensions, defined by the potential

$$V(\sqrt{\lambda}\phi) = \frac{1}{4}(\sqrt{\lambda}\phi)^2(\sqrt{\lambda}\phi - \sqrt{2}m)^2, \quad (5.1)$$

so the static kink solution is

with classical kink mass

$$Q_0 = \frac{m^3}{3\lambda}. \quad (5.3)$$

The translational zero mode, the (single) shape mode, and the continuum normal modes are, respectively,

$$\begin{aligned} g_B(x) &= -\frac{\sqrt{3m}}{2\sqrt{2}} \operatorname{sech}^2\left(\frac{mx}{2}\right), & g_S(x) &= \frac{3\sqrt{m}}{2\sqrt{3}} \tanh\left(\frac{mx}{2}\right) \operatorname{sech}\left(\frac{mx}{2}\right), \\ g_k(x) &= \frac{2e^{-ikx}}{\omega_k \sqrt{m^2 + 4k^2}} \left(k^2 - \frac{m^2}{2} + \frac{3m^2}{4} \operatorname{sech}^2\left(\frac{mx}{2}\right) - i\frac{3m}{2} k \tanh\left(\frac{mx}{2}\right) \right), \end{aligned} \quad (5.4)$$

with corresponding frequencies

$$\omega_B = 0, \quad \omega_S = \frac{\sqrt{3}}{2}m, \quad \omega_k = \sqrt{m^2 + k^2}. \quad (5.5)$$

The cubic vertex that mixes the two-shape-mode state into the one-meson continuum is [45]

$$V_{SSk} = -i\pi \frac{3\sqrt{\lambda} k^2 \omega_k (m^2 - 2k^2)}{\sqrt{2} m^3 \sqrt{m^2 + 4k^2}}. \quad (5.6)$$

In the ϕ^4 model, there is only one real shape mode. Reference [45] provided the corresponding amplitude from Eq. (3.1). However, we have seen that the sum of bubble diagrams implies that we must change the D_{SS} contribution. Equation (4.65) implies that the new D_{SS} term is

$$D_{SS}(k_0) = \frac{9\pi^2 \lambda k_0^4 \omega_{k_0}^2 (3m^2 - 2\omega_{k_0}^2)^2}{8m^6 k_0 \omega_S (4\omega_{k_0}^2 - 3m^2)(\omega_{k_0}^2 - 4\omega_S^2) + i\lambda 9\pi^2 k_0^4 \omega_{k_0}^2 (3m^2 - 2\omega_{k_0}^2)^2} \operatorname{csch}^2\left(\frac{\pi k_0}{m}\right), \quad (5.7)$$

while all the other contributions to the reflection amplitude, $A(k_0)$, $B(k_0)$, and $C(k_0)$, are untouched since these terms do not have a pole corresponding to the twice-excited shape mode state. Note that the intermediate states $|k'_1 k'_2\rangle$ and $|Sk'\rangle$ corresponding to the terms $D_{kk}(k_0)$ and $D_{Sk}(k_0)$ in the full $D(k_0)$ are also unchanged but have branch cuts [45]; therefore,

$$\begin{aligned} D(k_0) &= D_{SS}(k_0) + D_{Sk}(k_0) + D_{kk}(k_0) \\ &= \frac{9\pi^2 \lambda k_0^4 \omega_{k_0}^2 (3m^2 - 2\omega_{k_0}^2)^2}{8m^6 k_0 \omega_S (4\omega_{k_0}^2 - 3m^2)(\omega_{k_0}^2 - 4\omega_S^2) + i\lambda 9\pi^2 k_0^4 \omega_{k_0}^2 (3m^2 - 2\omega_{k_0}^2)^2} \operatorname{csch}^2\left(\frac{\pi k_0}{m}\right) \\ &\quad + \frac{27\pi^2 \lambda}{2^{14} k_0 m^3 \omega_{k_0}^2 \omega_S (4\omega_{k_0}^2 - 3m^2)} \end{aligned}$$

$$\begin{aligned}
& \times \int \frac{dk'}{2\pi} \frac{(\omega_{k'} + \omega_S)[m^2(17m^4 + 68m^2k_+ + 32k_+^2) - 16k_-^2(3m^2 + 4k_+)]^2}{\omega_{k'}^3(4\omega_{k'}^2 - 3m^2)(\omega_{k_0}^2 - (\omega_{k'} + \omega_S)^2 + i\epsilon)} \\
& \times \operatorname{sech}\left(\frac{(k_0 - k')\pi}{m}\right) \operatorname{sech}\left(\frac{(k_0 + k')\pi}{m}\right) \\
& + F_{10}(k_0) \int \frac{dk'_1}{2\pi} F_{11}(k_0, k'_1)(Q_-(k_0, k'_1) - Q_+(k_0, k'_1)) \\
& + F_{21}(k_0) \int \frac{dk'_1 dk'_2}{(2\pi)^2} F_{22}(k_0, k'_1, k'_2). \tag{5.8}
\end{aligned}$$

The functions $F_{10}(k_0)$, $F_{11}(k_0, k'_1)$, and $Q_{\mp}(k_0, k'_1)$ are

$$\begin{aligned}
F_{10}(k_0) &= \frac{27m^2\pi\lambda}{2\omega_{k_0}^2(4\omega_{k_0}^2 - 3m^2)} \operatorname{csch}\left(\frac{2k_0\pi}{m}\right), \\
F_{11}(k_0, k'_1) &= \frac{k'_1}{(4\omega_{k'_1}^2 - 3m^2)\omega_{k'_1}^3}, \\
Q_{\mp}(k_0, k'_1) &= \frac{F_{\mp}(k_0, k'_1)P_{\mp}(k_0, k'_1)}{D_{\mp}(k_0, k'_1)}, \tag{5.9}
\end{aligned}$$

with

$$\begin{aligned}
F_{\mp}(k_0, k'_1) &= (k_0 \mp k'_1)(4(k_0^2 \mp k_0k'_1 + k_1'^2) + 3m^2), \\
P_{\mp}(k_0, k'_1) &= (S_{\mp}(k_0, k'_1) + L_{\mp}(k_0, k'_1))(\omega_{k'_1} + \omega_{k_0 \mp k'_1}), \\
S_{\mp}(k_0, k'_1) &= 8k_0^2(k_0 \mp k'_1)^2k_1'^2 - 4(k_0^2 \mp k_0k'_1 + k_1'^2)^2m^2, \\
L_{\mp}(k_0, k'_1) &= -5(k_0^2 \mp k_0k'_1 + k_1'^2)m^4 - m^6, \\
D_{\mp}(k_0, k'_1) &= (4\omega_{k_0 \mp k'_1}^2 - 3m^2)\omega_{k_0 \mp k'_1}^3 \\
& \times (\omega_{k_0}^2 - (\omega_{k'_1} + \omega_{k_0 \mp k'_1})^2 + i\epsilon). \tag{5.10}
\end{aligned}$$

The functions $F_{21}(k_0)$ and $F_{22}(k_0, k'_1, k'_2)$ in the $D_{kk}(k_0)$ are

$$\begin{aligned}
F_{21}(k_0) &= \frac{-9\pi^2\lambda}{4k_0\omega_{k_0}^2(4\omega_{k_0}^2 - 3m^2)}, \\
F_{22}(k_0, k'_1, k'_2) &= \frac{[f(k_0, k'_1, k'_2)]^2(\omega_{k'_1} + \omega_{k'_2})}{D_0(k'_1, k'_2)D_1(k'_1)D_2(k'_2)D_3(k_0, k'_1, k'_2)} \\
& \times \operatorname{csch}\left(\frac{(k_0 - k'_1 - k'_2)\pi}{m}\right) \\
& \times \operatorname{csch}\left(\frac{(k_0 + k'_1 + k'_2)\pi}{m}\right). \tag{5.11}
\end{aligned}$$

The energy denominators are

$$\begin{aligned}
D_0(k'_1, k'_2) &= \omega_{k'_1}^3\omega_{k'_2}^3, \quad D_{1,2}(k'_{1,2}) = (4\omega_{k'_{1,2}}^2 - 3m^2), \\
D_3(k_0, k'_1, k'_2) &= (\omega_{k_0}^2 - (\omega_{k'_1} + \omega_{k'_2})^2 + i\epsilon). \tag{5.12}
\end{aligned}$$

The numerator function $f(k_0, k'_1, k'_2)$ and its coefficients are

$$\begin{aligned}
f(k_0, k'_1, k'_2) &= 3a^3 - 3a^2s - a(3s^2 - 4p + 8sm^2 + 5m^4) \\
& + 3s^3 - 12sp - 8pm^2 - 5sm^4 - 2m^6, \\
a &= k_0^2, \quad s = k_1'^2 + k_2'^2, \\
p &= k_1'^2k_2'^2, \quad k_{\pm} = k_0^2 \pm k'^2. \tag{5.13}
\end{aligned}$$

The resonance occurs when the incoming meson is on shell with twice the shape mode energy, $\omega_{k_0} = 2\omega_S$, which gives

$$k_0 = \sqrt{2}m, \quad \omega_S = \frac{\sqrt{3}}{2}m. \tag{5.14}$$

The twice-excited shape mode has energy $2\omega_S = \sqrt{3}m$, which is above the one-meson energy but below the two-meson energy; therefore, near the $|SS\rangle$ state, the single-meson-bubble energy dominates. The decay width of the twice-excited shape mode state is

$$\Gamma_{\phi^4} = \frac{9\pi^2}{\sqrt{2}} \operatorname{csch}^2(\sqrt{2}\pi) \frac{\lambda}{m}, \tag{5.15}$$

which also agrees with the decay width found in Ref. [34].

Figure 9 shows the probability $P(k_0)$ with the enhanced D_{SS} contribution, Eq. (5.7), where the probability $P(k_0)$ and reflection amplitude $R(k_0)$ are

$$\begin{aligned}
P(k_0) &= (\Re[R(k_0)])^2 + (\Im[R(k_0)])^2, \\
R(k_0) &= \lambda(A(k_0) + B(k_0) + C(k_0) + D(k_0)). \tag{5.16}
\end{aligned}$$

VI. REMARKS

In the classical ϕ^4 double-well field theory, at linear level, radiation passes through the kink without interacting.⁴ In quantum field theory, this is not the case [43]. A meson striking a kink has a finite probability of elastic scattering, so the scattered power is linear in the power of the incoming

⁴At the nonlinear level, kink-radiation interactions have a rich phenomenology, exhibiting phenomena such as the negative radiation pressure of Refs. [51,52], which has since been extended to other classical solutions [53,54].

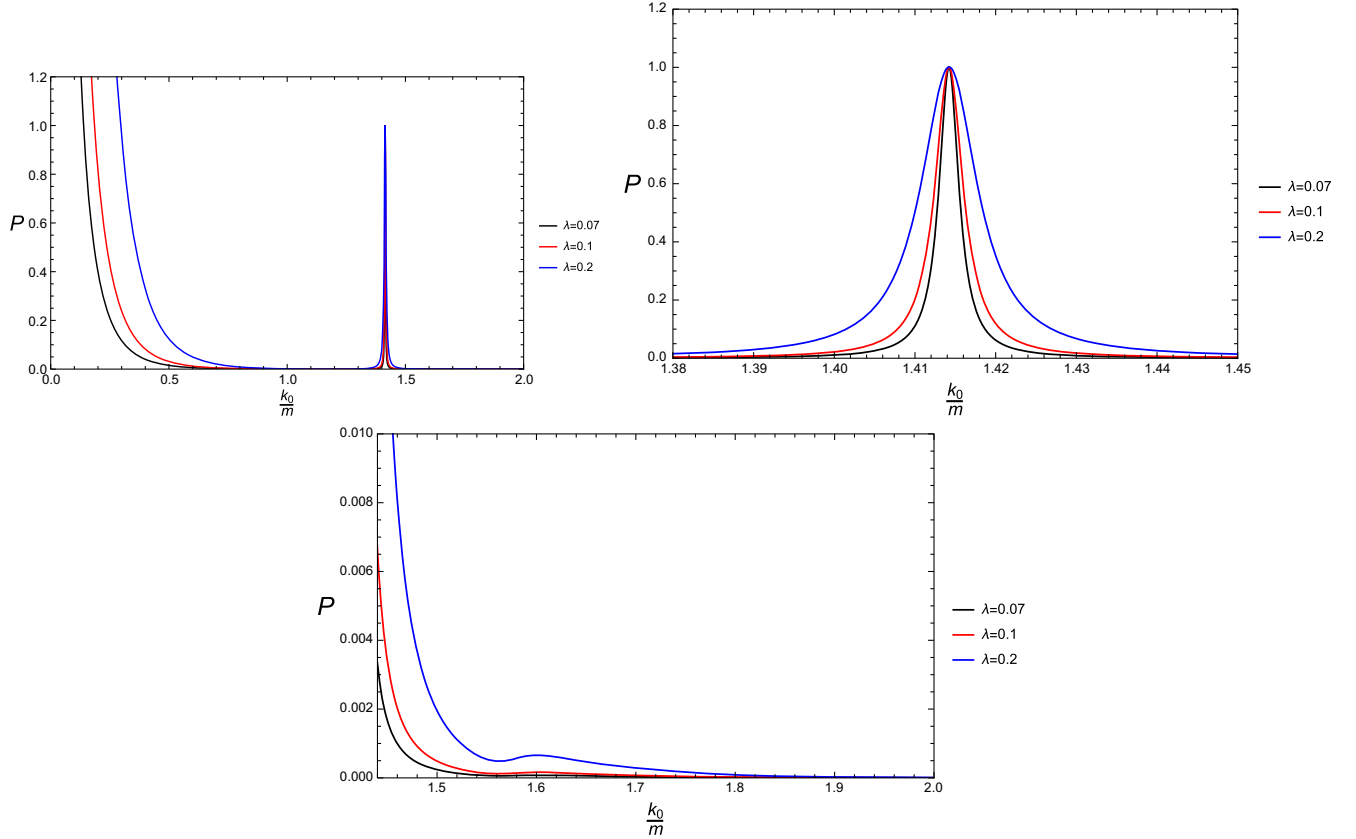


FIG. 9. Leading reflection probability as a function of the initial momentum k_0 . The bottom panels are, respectively, zoomed near the resonance in D_{SS} and near the branch point in D_{Sk} , which is the threshold for an intermediate state with a once-excited shape mode and a virtual meson. Subleading corrections will make the total probability less than unity.

radiation when both are small. In the present paper, we have seen that this elastic scattering process, which is only allowed in the quantum theory, opens the window to new observables: the energies and lifetimes of the kink's unstable excitations. Strictly speaking, these new observables also only exist in the quantum theory. However, at tree level the lifetime of the unstable state can be used, and indeed has been used [34], to calculate the decay rate that results from nonlinear dynamics in the classical field theory [33]. In other words, a resonance in a linear process in quantum field theory tells us about the nonlinear dynamics of the resonant state in classical field theory and also about its quantum corrections.

In this work, we obtained the leading order lifetime of the twice-excited shape mode state by summing the bubble contributions that dominate elastic kink-meson scattering in the vicinity of the two-shape-mode threshold. At leading order in the coupling, the reflection amplitude contains an explicit pole associated with an intermediate kink state

carrying two-shape-mode quanta [45]. In the near-resonant regime, successive bubble insertions are not parametrically suppressed and must be summed. The result is a Dyson-type dressing of the two-shape-mode denominator governed by the self-energy, transforming the pole into a Breit-Wigner peak with width fixed by Eq. (4.60).

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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