

Meson states in the ‘t Hooft model: Hamiltonian approach

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We point out that the masses of the highly excited bound quark-antiquark states in QCD₂ in the infinite N limit may be determined in the framework of a simple quantum-mechanical model with the potential $V(x) = \sigma|x|$. In the ultrarelativistic case, the masses follow the pattern $\mu_n^2 = \frac{\pi g^2 N}{2} n$, which coincides with the law derived by ‘t Hooft by solving the Bethe-Salpeter equation. In the nonrelativistic case, the levels follow the asymptotics $\epsilon_n = \mu_n - 2m = Cn^{2/3}$, where the constant C can be determined by solving the ‘t Hooft equation or, alternatively, the nonrelativistic Schrödinger equation.

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Introduction. In a seminal paper written half a century ago [1], ‘t Hooft found the particle spectrum in two-dimensional quantum chromodynamics with $SU(N)$ gauge group in the limit $N \rightarrow \infty$. In two dimensions, there is no gluon self-interaction (its absence is manifest if one imposes the light-cone gauge), so there are no glueballs. In the infinite N limit, there are no baryons either, and the only states are *mesons*—bound states of a quark and an antiquark: the production of extra $q\bar{q}$ pairs is also suppressed. The latter fact also means that the meson states are absolutely stable in this limit. For a finite large N , the widths are not zero, but they are suppressed $\propto 1/N$.

Note that ‘t Hooft determined the spectrum by solving the Bethe-Salpeter equation for the vertices relating the bound states to their quark and antiquark constituents.

As is the case for any pioneer paper, the notations in ‘t Hooft’s paper are somewhat sloppy. Thus, before going further, we are in a position to fix them.

The Lagrangian of the theory reads

$$\mathcal{L} = \frac{1}{g^2} \text{Tr}\{\hat{F}_{01}^2\} + i\bar{\psi}\gamma^\mu(\partial_\mu - i\hat{A}_\mu)\psi - m\bar{\psi}\psi, \quad (1)$$

where $\hat{A}_{\mu=0,1} = A_{\mu=0,1}^a t^a \in su(N)$ and $\hat{F}_{01} = \partial_0\hat{A}_1 - \partial_1\hat{A}_0 - i[\hat{A}_0, \hat{A}_1]$. Introducing the light-cone variables, $\hat{A}_\pm = (\hat{A}_0 \pm \hat{A}_1)/\sqrt{2}$ and similarly for $\partial_\pm$, and imposing the gauge $\hat{A}_- = 0$, the field density reduces to $\hat{F}_{01} = \partial_- \hat{A}_+$. The

fermion ψ (call it “quark”) belongs to the fundamental representation of $SU(N)$.

In the infinite momentum frame, the vertex of dissociation of a meson state into its constituents reduces to a simple function $\phi(x)$, with $x \in (0, 1)$ having the meaning of the fraction of the large meson momentum carried by the quark. Correspondingly, the antiquark carries the fraction $1 - x$. The Bethe-Salpeter equation reduces to a comparatively simple integral equation for $\phi(x)$. It can be written in the form¹:

$$\mu^2 \phi(x) = m^2 \left(\frac{1}{x} + \frac{1}{1-x} \right) \phi(x) + \frac{g^2 N}{2\pi} P \int_0^1 \frac{\phi(x) - \phi(y)}{(y-x)^2} dy, \quad (2)$$

where m is the quark mass and μ is the mass of the bound state. The singularity at $y = x$ is treated by the principal value prescription:

$$P \int_0^1 dy f(y, x) = \lim_{\epsilon \rightarrow 0} \left[\int_0^{x-\epsilon} dy f(y, x) + \int_{x+\epsilon}^1 dy f(y, x) \right]. \quad (3)$$

There is an infinite number of solutions for $\phi(x)$ and μ . Generically, this equation can be solved only numerically. But the asymptotics for high excitation levels can also be evaluated analytically. Note that ‘t Hooft did it for the excitations with masses μ_n much exceeding both relevant scales: $\mu_n^2 \gg m^2$ and $\mu_n^2 \gg g^2 N$. In such a case,

¹When comparing this with Ref. [1], note that a *nonstandard* definition of the coupling constant was used there:

$$g_{\text{Hooft}}^2 = \frac{g_{\text{standard}}^2}{2}.$$

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$$\mu_n^2 \approx \frac{\pi g^2 N}{2} n. \quad (4)$$

This behavior resembles Regge trajectories in hadron spectrum in the real four-dimensional world.²

Our own observation is that the law (4) may be derived in a very simple way as a solution of the relativistic quantum problem for the potential $V(x) = \sigma|x|$ in semiclassical approximation. This observation is not completely original: in Refs. [4,5] similar semiclassical formulas for the meson spectrum were derived in the string framework.³ However, no comparison with the results of Ref. [1] was made in those works.

Massive particles bound by a string.

- (i) Consider first the nonrelativistic case. The Hamiltonian describing two particles of mass m bound by the string is

$$H = \frac{p_q^2}{2m} + \frac{p_{\bar{q}}^2}{2m} + \sigma|x_1 - x_2|.$$

Separating out the center-of-mass motion, we may write⁴

$$H_{\text{c.m.}} = \frac{p^2}{m} + \sigma|x|. \quad (5)$$

The corresponding quantum Schrödinger problem can be solved exactly, the wave functions being expressed via Airy functions. But we are not so much interested in the exact solution. The asymptotics of the eigenstate energies at high n can be determined by semiclassical reasoning.

The classical momentum for a classical trajectory of energy ϵ is

$$p^2(x) = m(\epsilon - \sigma|x|).$$

There are two turning points, $x = \pm\epsilon/\sigma$, and the trajectory oscillates between them. The quantization condition reads

$$S(\epsilon_n) = 4 \int_0^{\epsilon_n/\sigma} dx \sqrt{m(\epsilon_n - \sigma x)} = \frac{8\sqrt{m\epsilon_n^3}}{3\sigma} \approx 2\pi n, \quad (6)$$

²The original finding was a linear dependence of the square of a hadron mass on the angular momentum [2]. But the dependence (4) resembles more a *radial* Regge trajectory when M^2 depends linearly on the radial quantum number [3].

³See also recent Ref. [6] where a related problem was treated in a semiclassical way.

⁴The reader understands, of course, that x here has nothing to do with x in Eq. (2).

giving⁵

$$\epsilon_n \approx \left(\frac{3\pi\sigma n}{4\sqrt{m}} \right)^{2/3}. \quad (7)$$

This nonrelativistic estimate holds while $\epsilon_n \ll m$, i.e., $n \ll m^2/\sigma$.

- (ii) To find the spectrum at larger energies, one has to solve the relativistic problem. The center-of-mass Hamiltonian now reads

$$\hat{H}_{\text{c.m.}}^{\text{rel}} = 2\sqrt{\hat{p}^2 + m^2} + \sigma|x|. \quad (8)$$

Again, we can evaluate the spectrum in the semiclassical approximation. The classical momentum is

$$p^2(x) = \frac{(E - \sigma|x|)^2}{4} - m^2$$

with $E = 2m + \epsilon$. There are now *four* turning points:

$$\pm x_- = \pm \frac{E - 2m}{\sigma} \quad \text{and} \quad \pm x_+ = \pm \frac{E + 2m}{\sigma}.$$

A particle may oscillate between $-x_-$ and x_- ; quantization of this motion gives us the energy levels, but it can also tunnel through the forbidden region between x_- and x_+ or between $-x_-$ and $-x_+$ and escape to infinity. This means that the levels will have a finite width.⁶ But let us find the real part of the energies first. The action integral is [5]

$$\begin{aligned} S(E) &= 4 \int_0^{x_-} \sqrt{\frac{(E - \sigma x)^2}{4} - m^2} dx \\ &= \frac{1}{\sigma} \left[E \sqrt{E^2 - 4m^2} - 4m^2 \ln \frac{E + \sqrt{E^2 - 4m^2}}{2m} \right]. \end{aligned} \quad (9)$$

In the nonrelativistic limit, $E - 2m \ll m$, this leads to the result (7) for the energy levels. In the ultrarelativistic limit, $E \gg m$, $S(E) \approx E^2/\sigma$, and we derive

⁵Cf., e.g., Eq. (16.2.38) in Ref. [7].

⁶This is an artefact of semiclassical approximation related to the fact that the Klein-Fock-Gordon (KFG) equation,

$$\left(\frac{(E - \sigma|x|)^2}{4} + \frac{\partial^2}{\partial x^2} - m^2 \right) \Psi = 0,$$

admits solutions with both positive and negative energies. If one sticks to the Hermitian Hamiltonian (8) and solves the Schrödinger equation *exactly*, then the eigenvalues stay real [8].

$$E_n \approx \sqrt{2\pi\sigma n}. \quad (10)$$

We go over to the widths now. The tunneling amplitude is proportional to $\exp\{-S_{\text{Eucl}}\}$, where

$$S_{\text{Eucl}} = \int_{(E_n-2m)/\sigma}^{(E_n+2m)/\sigma} dx \sqrt{m^2 - \frac{(E_n - \sigma x)^2}{4}} = \frac{\pi m^2}{\sigma}.$$

The width of a state with energy E is estimated as $\Gamma(E) \sim \omega e^{-2S_{\text{Eucl}}}$, where $\omega \sim \sigma/E$ is the frequency with which a particle collides with the walls at $x = \pm x_-$. We derive

$$\Gamma_n \sim \frac{\sigma}{E_n} \exp\left\{-\frac{2\pi m^2}{\sigma}\right\}. \quad (11)$$

It is very small for heavy quarks, $m^2 \gg \sigma$, while if the quarks are light, then the widths (appearing, as was mentioned, in semiclassical treatment of the linear potential problem, but not in its exact solution) are of the same order as the level spacings.

Comparison with QCD_2 . The spectra (7) and (10) that we derived depend on the parameter σ of the potential—the string tension. What is it in QCD_2 ?

Consider first the Abelian case. The one-dimensional analog of the Coulomb potential created by the charge e is

$$V_{\text{Coulomb}}^{d=1}(x) = \frac{e|x|}{2}. \quad (12)$$

Then $\partial^2 V_{\text{Coulomb}}^{d=1}(x)/\partial x^2 = e\delta(x)$. The Coulomb energy of a pair of Abelian charges separated by the distance R is

$$E_{\text{Coulomb}}^{\text{Ab}}(R) = \frac{e^2 R}{2}. \quad (13)$$

In the non-Abelian case, we have to replace e^2 by g^2 and multiply it by $c_F = (N^2 - 1)/(2N) \rightarrow N/2$. We derive

$$\sigma_{\text{QCD}_2} = \frac{g^2 N}{4}. \quad (14)$$

Substituting this into (10), we reproduce the spectrum (4).

An important difference between the semiclassical KFG analysis and QCD_2 is, however, the fact that in the latter case in the limit $N \rightarrow \infty$, the states have no width: they are absolutely stable.

The Wentzel-Kramers-Brillouin (WKB) asymptotics of the solution to the 't Hooft equation (2) in the non-relativistic limit $\mu_n - 2m \ll m$ has also been recently found [9]. The result coincides with (7) as it should.

The coincidence of leading WKB asymptotics for the spectra obtained by the solution of the 't Hooft equation (2) and by the solution of the quantum problem with the Hamiltonian (8) has a natural explanation in view of the fact that the classical counterparts of (8) and of the Hamiltonian in Eq. (2) are equivalent, being related by a canonical transformation [10]. If classical problems are equivalent, then there is no wonder that the quantum problems are equivalent too in the leading semiclassical order. The preasymptotic corrections need not coincide, however, and they do *not*, though the disagreement falls off rapidly as n increases [11].

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Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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