

Black-hole–neutron-star mergers: New numerical-relativity simulations and multipolar effective-one-body model with spin precession and eccentricity

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In this paper, we present 52 new numerical-relativity (NR) simulations of black-hole–neutron-star (BHNS) mergers and employ the data to inform TEOBResumS-Dalí: a multipolar effective-one-body model also including precession and eccentricity. Our simulations target quasicircular mergers and the parameter space region characterized by significant tidal disruption of the star. Convergent gravitational waveforms are produced with a detailed error budget after extensive numerical tests. We study in detail the multipolar amplitude hierarchy and identify a characteristic tidal signature in the $(\ell, m) = (2, 0)$, and $(3, 0)$ modes. We also develop new NR-informed models for the remnant black hole and for the recoil velocity. The numerical data are then used to inform next-to-quasicircular corrections and the ringdown of TEOBResumS-Dalí for BHNS mergers. We show an overall order of magnitude improvement in the waveform’s amplitude at merger and more consistent multipoles over our older TEOBResumS-GIOTTO for BHNS mergers. TEOBResumS-Dalí is further validated with a new 12-orbit precessing simulation, showing phase and relative amplitude differences below ~ 0.5 (rad) throughout the inspiral. The computed mismatches including all the modes lie at the 1% level for low inclinations. Finally, we demonstrate for the first time that TEOBResumS-Dalí can produce robust waveforms with both eccentricity and precession, and use the model to identify the most urgent BHNS mergers to simulate for waveform development. Our new numerical data are publicly released as part of the CoRe database.

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I. INTRODUCTION

Recent years have been marked by the gravitational wave (GW) observations of binary black-hole (BBH), binary neutron star (BNS) and most recently black-hole–neutron-star (BHNS) mergers by the Laser Interferometer Gravitational-Wave Observatory (LIGO)-Virgo interferometers [1,2]. With the inclusion of Kamioka Gravitational Wave Detector (KAGRA) in the ongoing fourth GW Observing Run (O4), the estimated merger rate for BHNSs is $\mathcal{R} = 94^{+109}_{-64} \text{ Gpc}^{-3} \text{ yr}^{-1}$ [3]. Nevertheless, detection of such binaries remains a challenge as current

waveform models do not reach the sophistication that is available for BBH. This is mainly due to tidal effects playing a role on the dynamics, especially towards merger and postmerger. Coalescences from BHNSs where the neutron star (NS) is tidally disrupted are of physical interest, as these are expected to be the source of electromagnetic (EM) signals such as kilonovae and γ -ray bursts. However, (poorly constrained) population studies favor cases with no detectable EM signal, corresponding to low spinning and/or highly asymmetric BHNS systems [4–7]. This type of BHNS binary produces GW signals that resemble those from BBH with a dimmer EM counterpart, thus making the detection of these BHNS mergers particularly challenging.

Thus far, the identification of observed GW events as BHNSs has been possible through the inferred component masses. During the LIGO-Virgo-KAGRA (LVK) O3 run, two BHNS events have been observed, namely GW200105 and GW200115 [8]. Possible evidence for eccentricity and precession has been recently discussed [9]. These

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detections were then followed by the event GW230529 in O4 [3]. The observation was the first of its kind, providing evidence of the existence of compact objects in the mass range between the heaviest NSs and the lightest black holes. Its low mass ratio has also hinted at the possibility of an EM counterpart [5,10], and has sparked interest in the binary's origin [11,12] and implications to future detections [6,13].

Numerical-relativity simulations have become an essential tool to better understand the physical properties and dynamics of BHNS mergers [14–20]. Simulations in full general relativity have reached a significant level of sophistication and currently comprise studies with microphysical equations of state (EOSs) [21–23], magnetic fields [24–31], neutrino transport [32–34], and evolutions with extremal black-hole (BH) parameters [35,36].

Simulations make possible a detailed exploration of the merger physics; in particular, they show how the tidal disruption of the NS influences the merger dynamics and the radiated gravitational waveforms. Kyutoku *et al.* [37] identify three merger classes: type I corresponds to the scenario where the NS is tidally disrupted before merging with the BH; type II involves the NS directly plunging into the BH without tidal disruption; in type III, the BH's tidal field induces unstable mass transfer from the NS during mass shedding. Each of these cases has a distinctive imprint on the remnant BH and ringdown part of the GW spectrum [4,38]. Another effect of astrophysical interest is the kick velocity on the remnant BH, which is due to both the anisotropic GW emission and the mass ejecta. For highly asymmetric binaries (type II), the resulting kick velocity has shown to be consistent with BBH fits and simulations [39]. Moreover, the ejecta velocity is expected to be dominant over the one due to GWs for type I BHNSs [40].

The waveforms extracted from numerical relativity (NR) simulations crucially allowed us to fine-tune and validate waveform models for GW astronomy. Error-controlled simulations with a long inspiral, although scarce, are necessary for the development of waveform models. Significant efforts have been done with the SACRA code [41,42], that offers a catalog containing more than 100 configurations. Quasicircular initial data are simulated for a variety of spins and mass ratios, with the NS modeled employing a piecewise polytropic EOS [37,40,43,44]. The SACRA waveforms cover about five orbits and do not reach convergence completely (see Appendix of Ref. [43]). Only one resolution with the dominant (2,2) mode is made publicly available for each simulation. The Simulating eXtreme Spacetimes (SXS) Collaboration's catalog offers longer waveforms produced with the SpEC code [45] at three different grid resolutions. Their catalog comprises six configurations evolved for up to 12 orbits, three simulations with ~ 15 –16 orbits (including one with a precessing spin BH), and a longer BHNS simulation with 16.6 orbits [46–48]. These binaries have both spinning and nonspinning components and an NS modeled with an

ideal gas EOS. Errors of 1% on the amplitude and 0.01 rad on the phase have been achieved when comparing current extrapolation methods with Cauchy-characteristic extraction (CCE) in Ref. [48]. Nevertheless, detailed waveform accuracy and convergence studies for BHNS waveforms are lacking in the literature. Contrary to BNS (see, e.g., Refs. [49–52]), achieving convergent waveforms for BHNSs still appears a challenging task.

The availability of accurate gravitational waveform templates is essential for identifying the source of the GW signal. For BHNS coalescences, early development started with the calibration of a phenomenological BBH model with numerical BHNS data [53,54]. Further progress went into adding amplitude corrections to account for the different physics at merger [55,56] based on the merger classification of Pannarale *et al.* [57], and a remnant model built upon estimates of the remnant's disk baryon mass [38]. Following on this early work, an updated phenomenological model was developed by Thompson *et al.* [58] including an NR-informed GW phase with tidal contributions. Effective one body (EOB) waveforms in the frequency domain have been computed by Matas *et al.* [59] by combining a BBH EOB baseline with the NRTidal model [60] and the remnant model of Zappa *et al.* [4]. In Ref. [61], we presented TEOBResumS-GIOTTO, an EOB time domain waveform model for BHNSs consisting of three main building blocks: (i) an NR-informed remnant BH model, (ii) BHNS-specific next-to-quasicircular corrections (NQC), and (iii) a “deformable” ringdown model. Compared to other models, TEOBResumS-GIOTTO includes subdominant modes and describes also precessing binaries. A main limitation of all of these models is the availability of NR data to design accurate prescriptions for the merger.

In this paper, we present 52 new BHNS simulations together with error-controlled waveforms and use them to improve the EOB model of Gonzalez *et al.* [61]. In Sec. II, we describe the numerical methods employed for the simulations and the set of simulated binaries. An extensive study of the grid configuration, convergence tests, and data quality are presented in Appendix A. In Sec. III, we discuss the main simulation results. First, we assess initial data by considering quasiequilibrium sequences and comparing to EOB predictions. Second, we give an overview of the dynamics and present an updated model for the mass and spin of the remnant BH. Third, we investigate in detail the multipolar structure of BHNS waveforms and identify the hierarchy of modes contributing to the waveforms. Fourth, we discuss a model for GW recoil of the final BH. Finally, one of our simulations is compatible with GW230529; we thus discuss some consequences for the interpretation of the event. In Sec. IV, we present TEOBResumS-Dalí for BHNSs. First, we describe the new ringdown model for the (2, 2) and subdominant modes together with NQC and other new design choices for the inspiral-merger-ringdown waveform. Explicit NR-driven models and the fitting coefficients

developed for all modes are collected in Appendix B. In Sec. V, we validate our model by comparing it with a new precessing NR simulation comprising 12 orbits. In Sec. VI, we showcase the use of `TEOBResumS-Dalí` for guiding future NR simulations and for predicting GW merger signals from arbitrary orbits. First, we apply `TEOBResumS-Dalí` to identify the most urgent regions of the parameter space where more quasicircular nonprecessing simulations are necessary. Second, we present the very first BHNS waveforms with eccentricity and precession. Finally, we compare the prediction of our model with some of the recent waveforms best-fitting LVK events.

Notation. Throughout this work, we employ geometric units $c = G = 1$ and solar masses $M_\odot$, unless explicitly indicated. The binary mass is indicated as M , $q = m_1/m_2 \geq 1$ is the mass ratio, and $\nu = q/(1+q)^2$ is the symmetric mass ratio. We use M_{BH} and $a_{\text{BH}} = \chi_1$ for the mass and dimensionless spin of the BH in the binary system, with the latter defined as $\chi_i = S_i/M_i^2$; M_{NS} is the NS mass (here we consider irrotational NS, $\chi_2 = 0$). We denote with $M_\bullet$ and $a_\bullet = S_\bullet/M_\bullet^2$ the remnant BH's mass and dimensionless spin, respectively. We also employ the effective spin $\tilde{a}_0 = \tilde{a}_1 - \tilde{a}_2 = X_1\chi_1 - X_2\chi_2$ and $\tilde{a}_{12} \equiv \tilde{a}_1 + \tilde{a}_2$, where $X_{1,2}$ are the mass fractions: $X_i = m_i/M$ and $X_{12} \equiv X_1 - X_2 = \sqrt{1-4\nu}$; and χ_i the dimensionless spins. Additionally, we use the following spins combination:

$$\hat{S} \equiv \frac{S_1 + S_2}{M^2} = \frac{1}{2}(\tilde{a}_0 + X_{12}\tilde{a}_{12}), \quad (1)$$

with the dimensionful spins S_i along the direction of the orbital momentum. Finally, it is useful to define the dimensionless precession spin parameter as in Ref. [62]:

$$\chi_p = \max \left(|\chi_{1,\perp}|, \frac{4+3q}{4q^2+3q} |\chi_{2,\perp}| \right). \quad (2)$$

The strain is defined as

$$h \equiv h_+ - ih_\times = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} {}_{-2}Y_{\ell m}, \quad (3)$$

where ${}_{-2}Y_{\ell m}$ are the $s = -2$ spin-weighted spherical harmonics. Multipoles are decomposed in amplitude and phase as follows:

$$h_{\ell m} = A_{\ell m} e^{-i\phi_{\ell m}}, \quad (4)$$

where the instantaneous GW frequency is defined as the time derivative of the phase, $\omega_{\ell m} \equiv \dot{\phi}_{\ell m}$. We present the NR waveforms in terms of the retarded time $u = t - r_*$, where r_* corresponds to the associated tortoise Schwarzschild coordinate at the extraction radius R in the simulation.

II. NUMERICAL-RELATIVITY METHODS AND SIMULATIONS

A. Initial data

The initial data solution is obtained with the publicly available and open-source pseudospectral code `Elliptica` [63,64]. In contrast to other available initial data solvers, `Elliptica` constructs BHNS initial data with dimensionless spin magnitudes up to ~ 0.8 with arbitrary spin orientations.

Using the extended conformal thin sandwich method formalism [65,66], and the velocity potential method [67], `Elliptica` efficiently solves the coupled elliptic partial differential equations of the Einstein-Euler system in a multicore parallelism paradigm. This approach leverages employing the divide-and-conquer method of Schur complement domain decomposition.

`Elliptica` uses excision boundary conditions [68] to solve for the BH in the BHNS system. Consequently, before transferring the initial data to the evolution codes such as the `BAM` code, `Elliptica` fills the excised region by applying a C^2 continuous extrapolation of the metric fields from the surrounding area of the BH.

B. 3+1 evolution

The initial data in this work are evolved with the `BAM` code [69,70] using the Z4c formulation of Einstein's equations coupled to relativistic hydrodynamics.

The computational domain in `BAM` consists of cell-centered nested Cartesian grids with n points per direction in L refinement levels, labeled as $l = 0, \dots, L-1$. Each refinement level l is composed by one or more overlapping grids with a constant grid spacing h_l . These are related by a factor of 2 as $h_l = h_0/2^l$, where h_0 is the grid spacing at the coarsest level $l = 0$. The refinement level grids always stay within the coarser levels. Refinement levels above a given user-defined threshold l^{mv} , can be dynamically moved, as to follow the orbits of the two objects, adopting a “moving boxes” technique with a number of points per direction n^{mv} arbitrarily selected. The time evolution of the grid fields relies on the method of lines and Runge-Kutta time integrators with a Courant-Friedrichs-Lewy factor of 0.25. The metric variables are approximated employing fourth-order finite differencing stencils.

For hydrodynamics, we employ the local Lax-Friedrichs central scheme [71,72] for the interface fluxes and fifth-order weighted essentially nonoscillatory method scheme [73] for the primitive reconstruction. For the NS EOS, we employ the piecewise polytropic models `SLy` and `MS1b` [74], and the hybrid model `ALF2` [75] that accounts for deconfined quark matter.

In order to obtain the most accurate possible waveforms with the least computational cost, several grid configurations employing additional refinement levels on the BH with respect to the NS were tested. This is described in Appendix A 1. For the production runs, the configuration

TABLE I. BHNS configurations simulated in this work: M_b , the baryonic mass of the NS; Ω_{BHNS} , the angular velocity of the BHNS system; M_{ADM} , the total Arnowitt–Deser–Misner (ADM) mass; and J_{ADM} , the total ADM angular momentum. For all configurations, we consider a nonspinning NS, and for the BH spin, the subscript p refers instead to χ_p . The last column refers to the merger type according to our classification; see text.

Name	EOS	q	M_{BH}	a_{BH}	M_b	M_{NS}	Ω_{BHNS}	M_{ADM}	J_{ADM}	Type
BAM:0177	ALF2	2.2	2.73	−0.2991	1.35	1.24	0.0069	3.9309	10.9535	I
BAM:0178	ALF2	2.3	2.8	−0.4971	1.35	1.24	0.007	3.9978	9.7176	I
BAM:0179	ALF2	2.3	2.85	−0.5944	1.35	1.24	0.0071	4.0541	9.0849	I
BAM:0180	ALF2	2.4	2.93	−0.6897	1.35	1.24	0.0071	4.1381	8.4227	I
BAM:0176	ALF2	2.2	2.73	0.3008	1.35	1.24	0.0069	3.9299	14.9955	I
BAM:0181	ALF2	2.2	3.24	−0.2991	1.6	1.44	0.0074	4.6327	14.3109	III
BAM:0182	ALF2	2.3	3.38	−0.597	1.6	1.44	0.0075	4.7807	11.7118	III
BAM:0185	ALF2	3.4	4.86	−0.299	1.6	1.44	0.0061	6.246	18.7861	III
BAM:0186	ALF2	3.5	5.08	−0.5974	1.6	1.44	0.0062	6.471	12.2521	III
BAM:0183	ALF2	3.3	4.8	0.001	1.6	1.44	0.006	6.1868	24.8999	III
BAM:0184	ALF2	3.4	4.86	0.2996	1.6	1.44	0.006	6.2423	31.5341	III
BAM:0187	ALF2	3.5	5.09	0.5974	1.6	1.44	0.006	6.4753	40.1652	III
BAM:0224	ALF2	2.2	3.24	0.3005	1.6	1.44	0.0073	4.631	19.8464	I
BAM:0188	MS1b	2.2	2.73	−0.2991	1.35	1.25	0.0084	3.9423	10.6347	I
BAM:0189	MS1b	2.2	2.8	−0.4971	1.35	1.25	0.007	4.0117	9.8693	I
BAM:0192	MS1b	2.3	2.85	−0.5944	1.35	1.25	0.0071	4.068	9.2412	I
BAM:0193	MS1b	2.3	2.93	−0.6897	1.35	1.25	0.0072	4.1521	8.5846	I
BAM:0191	MS1b	1.9	2.83	−0.2992	1.6	1.46	0.0072	4.2504	13.3969	I
BAM:0194	MS1b	2.0	2.96	−0.5956	1.6	1.46	0.0073	4.3785	11.5023	I
BAM:0190	MS1b	1.9	2.8	0.0013	1.6	1.46	0.0071	4.2174	15.3542	I
BAM:0196	MS1b	2.2	3.24	−0.2991	1.6	1.46	0.0063	4.6546	15.0722	I
BAM:0198	MS1b	2.3	3.38	−0.5951	1.6	1.46	0.0064	4.8024	12.4563	I
BAM:0195	MS1b	2.2	3.2	0.0012	1.6	1.46	0.0073	4.6134	17.1394	I
BAM:0197	MS1b	2.2	3.24	0.3004	1.6	1.46	0.0073	4.6499	20.0667	I
BAM:0200	MS1b	3.3	4.86	−0.2989	1.6	1.46	0.0061	6.2649	19.123	III
BAM:0203	MS1b	3.5	5.08	−0.5974	1.6	1.46	0.0062	6.49	12.6149	III
BAM:0199	MS1b	3.3	4.8	0.0011	1.6	1.46	0.006	6.2058	25.2409	III
BAM:0201	MS1b	3.3	4.86	0.3007	1.6	1.46	0.006	6.2624	31.9143	III
BAM:0202	MS1b	3.5	5.08	0.5988	1.6	1.46	0.006	6.4818	40.433	I
BAM:0225	MS1b	2.3	2.94	0.6936	1.35	1.25	0.007	4.1533	19.3349	I
BAM:0226	MS1b	2.0	2.96	0.5975	1.6	1.46	0.0071	4.3767	20.7158	I
BAM:0204	SLy	2.0	2.8	0.0013	1.6	1.43	0.0071	4.1913	15.0943	III
BAM:0205	SLy	2.0	2.83	0.3009	1.6	1.43	0.0071	4.2236	17.3788	III
BAM:0208	SLy	2.5	3.24	−0.2991	1.4	1.27	0.0062	4.4713	12.9242	III
BAM:0210	SLy	2.3	3.24	−0.2991	1.6	1.43	0.0074	4.6256	14.1989	III
BAM:0211	SLy	2.7	3.38	−0.5952	1.4	1.27	0.0063	4.6184	10.1689	III
BAM:0206	SLy	2.2	3.2	0.0012	1.6	1.43	0.0073	4.5873	16.8223	III
BAM:0207	SLy	2.5	3.24	0.3007	1.4	1.27	0.0061	4.4702	18.6489	III
BAM:0209	SLy	2.3	3.24	0.3004	1.6	1.43	0.0073	4.624	19.7719	III
BAM:0212	SLy	2.7	3.38	0.5976	1.4	1.27	0.0062	4.6164	22.5826	I
BAM:0213	SLy	2.4	3.38	0.5989	1.6	1.43	0.0073	4.7694	23.7182	III
BAM:0216	SLy	3.4	4.86	−0.299	1.6	1.43	0.0053	6.241	19.1908	II
BAM:0219	SLy	4.0	5.08	−0.5973	1.4	1.27	0.0062	6.3087	9.1763	III
BAM:0220	SLy	3.5	5.08	−0.5964	1.6	1.43	0.0054	6.469	12.5726	II
BAM:0221	SLy	4.1	5.23	−0.6946	1.4	1.27	0.0062	6.4669	6.516	III
BAM:0214	SLy	3.3	4.8	0.0011	1.6	1.43	0.006	6.1799	24.7838	II
BAM:0215	SLy	3.4	4.86	0.2996	1.6	1.43	0.006	6.235	31.4135	III
BAM:0217	SLy	4.0	5.08	0.5987	1.4	1.27	0.0059	6.3012	37.3741	III
BAM:0218	SLy	3.5	5.08	0.5988	1.6	1.43	0.006	6.4557	39.9675	III
BAM:0222	SLy	4.1	5.23	0.6971	1.4	1.27	0.006	6.4622	41.3462	III
BAM:0223	ALF2	2.5	3.57	0.613 ^p	1.6	1.44	0.0042	4.9824	27.0892	I

M8 (see Table IV) is chosen as it provides high quality data comparable to higher resolutions with the least amount of computational resources. Convergence and comparison of these grid choices are presented in Appendix A 1.

C. Simulations

For this work, we simulated 51 quasicircular and non-precessing BHNSs and one precessing configuration. Table I summarizes the initial data parameters of each simulation, which are chosen so that the configurations are close to our estimated tidal disruption boundary [61]. Since we are interested in extracting information from the merger and postmerger, we evolve the 51 configurations for 3–4 orbits which are used to inform our EOB model. Given the small length of the waveforms, we do not perform any eccentricity reduction procedure on them, which usually stays below $e \sim 0.02$. Similarly for our precessing configuration, evolved for 12 orbits and employed for validation, we do not eccentricity reduce the initial data as it stays at a similar low value. Future work will focus on low eccentricity data for precessing BHNSs.

Figure 1 shows the parameter space that current publicly available simulations (including the ones produced in this work as circular markers) cover in terms of the tidal parameter Λ , symmetric mass ratio ν , and BH dimensionless spin a_{BH} . A key challenge in waveform modeling of mixed binaries is classifying the different merger types according to the initial binary parameters. Hence, we focus our attention on the region of the parameter space between the boundary of tidal disruption, Type I binaries, (gray shaded area on Fig. 1), and type III cases where the quasinormal modes (QNM) are still excited and present in the ringdown, but are slightly damped due to mass shedding close to merger. Contrary to earlier simulations, we also consider a variety of antialigned spins down to $a_{\text{BH}} \geq -0.7$. Low values of

retrograde spin are consistent with the expected $|\chi_{\text{eff}}| \approx 0$ from astrophysical studies [76], and higher values help us inform and extend our analytical models.

III. SIMULATION RESULTS

A. Quasiequilibrium sequences

We start discussing quasiequilibrium configurations of BHNS initial data with different mass ratios and spins. The study of these sequences serves as a tool to diagnose the consistency of the initial data constructed for our evolutions to predictions from analytical approximations (e.g., post-Newtonian [PN] and EOB), e.g., Refs. [17,77–79]. For BHNS binaries, sequences of this type have served especially to study the onset of mass shedding and tidal disruption [16,79].

We produce quasiequilibrium data for BHNS configurations employing the SLy EOS [74]. The baryonic mass of the NS is set to be $M_b = 1.6$ in all cases, and we show the comparison with $q = 2$ and $q = 3$. We consider a non-rotating NS, whereas for the BH we choose both aligned and antialigned spins as well as nonspinning punctures with $\chi_z^{\text{BH}} = a_{\text{BH}} = -0.3, 0.0$, and 0.3 . In the following, we focus on the relation between the binding energy $E_b/M = M_{\text{ADM}}/M - 1$ and orbital angular velocity $M\Omega_{\text{BHNS}}$.

Figure 2 shows the energy curves for the adiabatic configuration and compares them to the BHNS model in TEOBResumS-GIOTTO [61] and the post-Newtonian (3PN) point-mass nonspinning prediction. Larger values of orbital angular velocity indicate smaller separations between the BH and NS (closer to merger). The EOB prediction agrees with the produced initial data with errors around $\sim 0.01\%$. As expected, tidal effects take over the dynamics with increasing $\Omega_{\text{BHNS}}M$ (towards the late inspiral) as the EOB and 3PN curves start deviating from each other. As the two objects approach each other, the effect of the spin-orbit coupling starts increasing, thus making the system more or less bound according to the spin magnitudes and orientation as seen in Fig. 2. This effect is also captured by the prediction from EOB in agreement with the numerical sequences.

B. Dynamics

Here, we give an overview of the results from the NR evolutions produced in this work. As described in Sec. II, these simulations compose for the most part the parameter space close to the boundary between tidal disruption and mass shedding. Earlier systematic BHNS studies have performed similar evolutions with different a_{BH} and EOSs and have focused on studying the ejecta properties of these systems [34,37,80–82].

Figure 3 illustrates the merger dynamics of fiducial simulations. The plot shows the rest mass density profile during the merger and postmerger of three different merger scenarios, including the corresponding GW signal for the

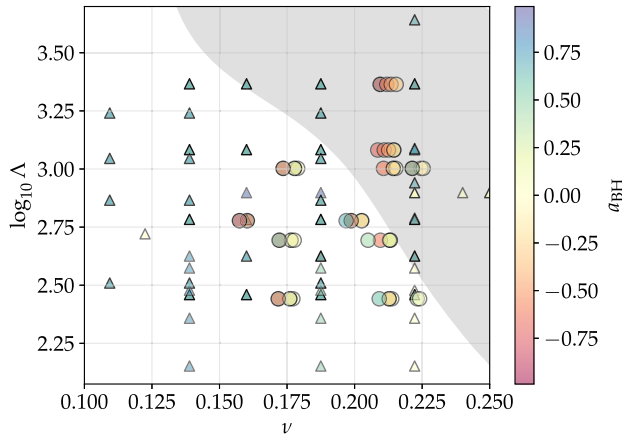


FIG. 1. Available NR simulations for different BHNS configurations. Circle markers show the simulations done for this paper. The gray shaded area covers the binaries where we estimate tidal disruption to occur; see Sec. IV B.

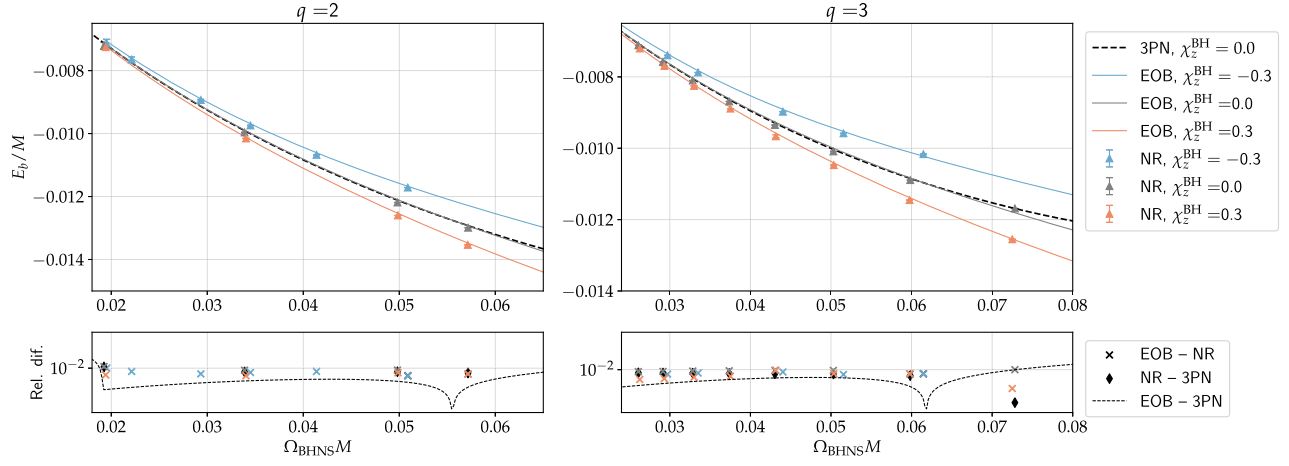


FIG. 2. Quasiequilibrium sequences of BHNS configurations with the SLy EOS. These are computed for two mass ratios $q = 2$ and $q = 3$ with aligned (blue) and antialigned (orange) spins. The results from our simulations are compared directly with those of EOB and 3PN (see text). We see an agreement between the initial data and the prediction from EOB, thus asserting the correctness of the numerically produced data.

(2,2) mode for each case. According to our previously defined classification in Ref. [61] (we present here an update on its boundaries in Sec. IV B), the top panels correspond to BAM:0225, a type I merger. Binaries of this type experience tidal disruption from the interplay of the effects due to masses, spins, and tides. Parallel to our discussion on the quasiequilibrium sequences in Sec. III A, attractive tidal effects decrease the orbital separation more rapidly for stiffer EOSs. Higher-aligned spins, on the other hand, have a repulsive effect. Consequently, the encounter of the two objects is delayed, prompting the NS to reach the onset of tidal disruption before the innermost stable circular orbit (ISCO). The star is thus deformed while leaving most of its material outside the BH, seen as a bright disk of mass. Consequently, the gravitational radiation does not show a ringdown, as the material outside the BH dampens it. The disk formed for this simulation reached a mass of $M_{\text{disk}} \approx 1 \times 10^{-6} M_{\odot}$ (rightmost top panel) and ejected mass of $M_{\text{ej}} \approx 0.02 M_{\odot}$ with velocities around $v_{\text{ej}} \approx 0.1c - 0.2c$.

The middle panels show the merger of BAM:0214 (type II), with the NS directly plunging into the BH. Here, the mass ratio's repulsive effect and the low magnitude (or zero) spins contribute to the NS reaching the ISCO first, leaving no ejected material outside the BH and increasing the mass of the remnant BH (notice the size increase of the apparent horizon [AH] from the left panel, where the two objects are merging to the middle one after merger). The perturbed remnant is thus responsible for the clear ringdown signal of the gravitational waveform.

The last panels correspond to BAM:0206. This is an intermediate case, close to (but not reaching) the onset of tidal disruption: a type III merger. As seen in the figure, after the NS is swallowed by the BH, leftover (low density) material from the shed outer layers surrounds the remnant. The GW thus shows a partially dampened ringdown, where

excited QNM are still present but are suppressed compared to type I.

C. Remnant BH

The properties of the remnant BH from a BHNS merger are of great interest for waveform modeling and inference. We therefore extract this information from the AH data of BAM [69]. The remnant mass is obtained from the Christodoulou formula [83] that involves the irreducible mass from the AH area, $M_{\text{irr}} = \sqrt{A/16\pi}$, and a contribution from the BH spin,

$$M_{\bullet}^2 = (M_{\text{irr}})^2 + \frac{S_{\bullet}^2}{4(M_{\text{irr}})^2}, \quad (5)$$

where $S_{\bullet}$ is the spin of the puncture. The results from our simulations are summarized in Table II. Note that the AH finder was not successful in finding the AH in all cases, hence we present measured values of fewer configurations than the ones shown in Table I.

Models to predict the mass and spin of the remnant have been developed as more NR data of BHNS evolutions have been proposed in various works [4,37,38,61]. Here, we present yet another updated version of our previous model from Refs. [4,61] by including the newly produced data. Similarly to these previous works, we factorize the BBH contribution in order to obtain a model that smoothly connects to the BBH case. The remnant mass can be well represented by the parameters $\{\Lambda, \nu, a_{\text{BH}}\}$ as

$$\frac{M_{\bullet}^{\text{BHNS}}}{M_{\bullet}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{1 + \Lambda^2 c_{312} \nu^2}, \quad (6)$$

with the low-order polynomials

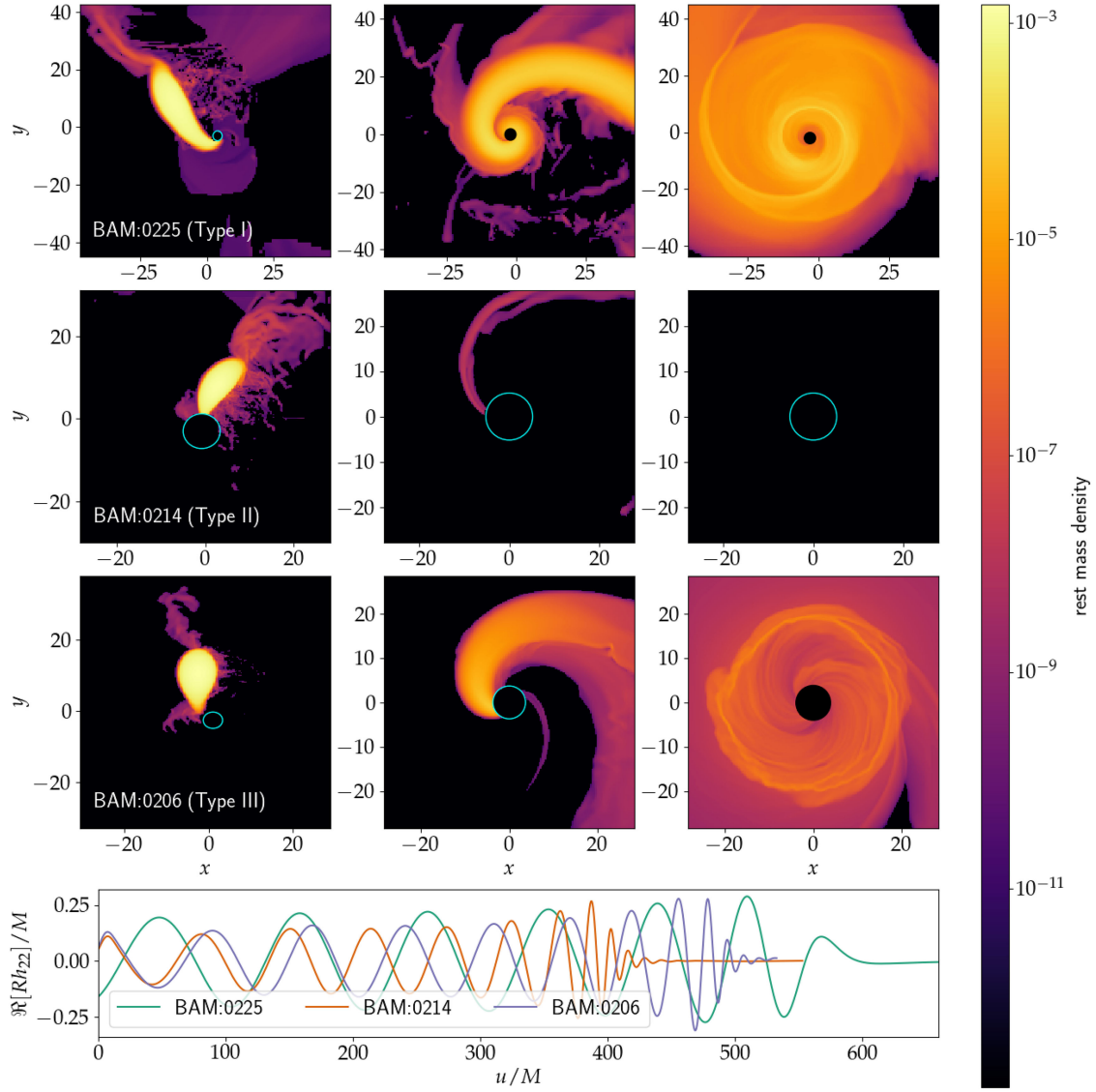


FIG. 3. Density profile at three different moments of the merger and postmerger of BAM:0225 (top), BAM:0214 (middle), and BAM:0206 (bottom), corresponding to types I, II, and III, respectively. The cyan contour indicates the location of the AH. The bottom panel shows the resulting gravitational radiation from each coalescence. Note that the retarded time axis is shifted for better visualization of the waveforms.

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2, \quad (7a)$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (7b)$$

The model clearly captures the BBH remnant mass for $\Lambda \rightarrow 0$. The polynomial coefficients fitting the NR data can be found in Table V in Appendix A 2.

Figure 4 shows the remnant BH mass model together with the data employed for the fitting. For nonspinning BHs, NSs with $\Lambda \lesssim 1000$ directly plunge with no significant tidal disruption (type II). NSs with larger Λ also directly plunge as far as the mass ratio remains larger than $q \sim 3$. Aligned BH spins trigger the tidal disruption of the NS at a mass ratio as high as $q \sim 5$. This is because the ISCO radius is smaller for

larger positive a_{BH} and the NS is disrupted well before merger (type II). Small deviations from $M_{\bullet}^{\text{BBH}}$, corresponding to type III where some material from the NS is shedded before merger, are highly sensitive to the spin's alignment. Namely, for a nonspinning $q \sim 2$ and $\Lambda \sim 1000$ binary the remnant mass would be identical to that of a BBH. A spin of $a_{\text{BH}} = 0.5$ would imply a smaller remnant mass than in the BBH case for $\Lambda \gtrsim 500$, whereas if it is antialigned $a_{\text{BH}} = -0.5$ binaries with Λ as high as $\Lambda \sim 1200$ would have instead a remnant mass as in type I. For $a_{\text{BH}} \sim +0.75$, only an NS with a very soft EOS ($\Lambda \lesssim 500$) directly plunges into the companion BH.

For the remnant's final spin $a_{\bullet} \equiv S_{\bullet}/M_{\bullet}^2$, we developed a model in a similar fashion as the final mass,

$$\frac{a_{\bullet}^{\text{BHNS}}}{a_{\bullet}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(3)}(\nu, a_{\text{BH}})}{(1 + \Lambda^2 c_{412} \nu)^2}, \quad (8)$$

where the polynomials are defined as

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (9a)$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (9b)$$

The resulting fits for the remnant's spin are shown in Fig. 4. Due to the remaining material outside of the AH influencing the angular momentum of the system, the final spin decreases or increases depending on the alignment of the initial spin. When tidal disruption occurs, the formed hot disk surrounding the remnant can increase the magnitude of the final spin with respect to $a_{\bullet}^{\text{BBH}}$. These are the regions in the plot where $a_{\bullet}^{\text{BHNS}}/a_{\bullet}^{\text{BBH}} > 1$, especially for mass ratios close to equal mass and high Λ values. Lower remnant spins are expected above $\Lambda \gtrsim 2000$ and $q > 2$ for nonspinning cases, whereas higher aligned spins and lower values of Λ are enough to deviate from BBH. The same can be said for $a_{\text{BH}} = 0.75$ for any Λ value and $q > 4$. BHNSs

TABLE II. Remnant BH properties obtained for the different BHNS configurations.

Name	$M_{\bullet}$	$a_{\bullet}$
BAM:0176	3.7167	0.684
BAM:0177	3.8163	0.5313
BAM:0178	3.9095	0.464
BAM:0179	3.9745	0.4249
BAM:0184	6.132	0.6639
BAM:0189	3.8021	0.4359
BAM:0190	4.0024	0.6259
BAM:0192	3.9061	0.416
BAM:0193	4.025	0.3791
BAM:0194	4.3095	0.4852
BAM:0195	4.4366	0.6195
BAM:0196	4.5437	0.5318
BAM:0197	4.392	0.6951
BAM:0199	6.091	0.5304
BAM:0200	6.218	0.3919
BAM:0201	6.0721	0.6503
BAM:0202	6.1971	0.7759
BAM:0204	4.1177	0.6676
BAM:0205	4.1218	0.7553
BAM:0206	4.5299	0.6254
BAM:0210	4.5515	0.5117
BAM:0214	6.0871	0.5199
BAM:0215	6.1254	0.6591
BAM:0216	6.1551	0.3716

with antialigned spins will remain close to the BBH's remnant spin for the most part, as long as $\Lambda \lesssim 3000$.

D. Gravitational waves

Waveforms are extracted from the Weyl's scalar Ψ_4 curvature modes ψ_{lm} which are then integrated to obtain the strain, $\ddot{h}_{\ell m} = \psi_{lm}$, using the fix-frequency integration method [84]. We consider modes up to $\ell = 4$. Convergence and error budget of our waveforms are discussed in detail in Appendix A 1.

Since it is the first time so many (publicly available) multipolar waveforms are extracted from BHNS simulations, we study the contribution each mode has to the waveform. In particular, earlier works have suggested that the BBH waveform amplitude peaks of each mode, $A_{\ell m}^{\text{peak}} = \max(A_{\ell m})$, show a structured behavior of the form [85,86]

$$\frac{A_{\ell m}^{\text{peak}}}{\nu |c_{\ell+\epsilon}(\nu)|} \approx e^{c_1(\ell)m + c_2(\ell)\ell}, \quad (10)$$

where the leading ν dependence is factorized in the denominator $\nu |c_{\ell+\epsilon}(\nu)|$. The functions $c_1(\ell), c_2(\ell)$ are quasiuniversal and can be computed in the test-mass limit; see Table VI of Ref. [85]. Identifying this kind of pattern proves useful when modeling the ringdown part of the waveform.

Figure 5 compares the multipolar hierarchy structure between a BBH and BHNS of the same parameters ($q = 2$, nonspinning), namely BAM:0190 and BAM:0204. As a reference, we also add the corresponding test mass case as a thin solid line on the plot. For the BBH data, we employ the simulation SXS:BBH:0184 from the SXS catalog [46,87,88]. The BHNS amplitude peaks are smaller than BBH for all the $\ell = m$ modes, and decrease according to the stiffness of the EOS. As we go to the $m < 2$ and $m < 3$ cases for $\ell = 3$ and 4, respectively, the amplitude peaks tend to surpass those of the BBH. Particularly, the $(\ell = 3, 4, m = 0)$ peaks tend to approach those of $(\ell + 1, 0)$ and correspond to a significant contribution to the waveform comparable to that of the $(3, 2)$ and $(4, 4)$ modes.

In order to quantify the multipolar amplitude peak structure's dependence on mass ratio, spin, and tidal effects, we employ the following rescaling of the peak amplitude [89]:

$$\hat{A}_{22} \equiv A_{22}^{\text{peak}} / [\nu(1 - \hat{S}\omega_{22})], \quad (11a)$$

$$\hat{A}_{21} \equiv A_{21}^{\text{peak}} / \nu, \quad (11b)$$

$$\hat{A}_{33} \equiv A_{33}^{\text{peak}} / \nu, \quad (11c)$$

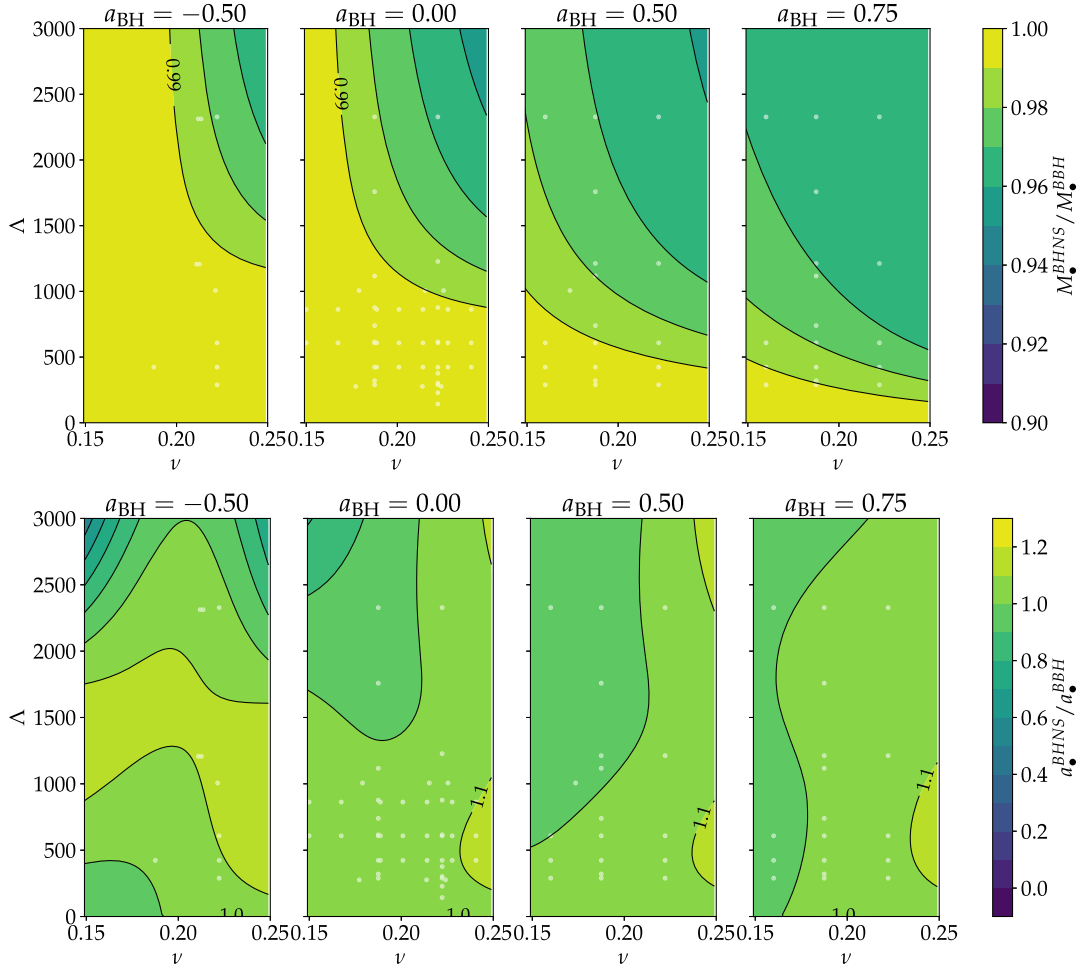


FIG. 4. Remnant BH mass (top) and spin (bottom) model as a function of the tidal polarizability Λ and the symmetric mass ratio ν for different BH initial spin values a_{BH} . White markers are the NR data extracted from the AH.

$$\hat{A}_{32} \equiv A_{32}^{\text{peak}} / [\nu(1 - \tilde{a}_0(\omega_{32}/2)^{1/3})], \quad (11d)$$

$$\hat{A}_{44} \equiv A_{44}^{\text{peak}} / \left[\nu \left(1 - \frac{1}{2} \hat{S} \omega_{44} \right) \right]. \quad (11e)$$

Figure 6 summarizes the contribution of each mode as a function of the symmetric mass ratio ν . From this plot and Fig. 5, we identify the most significant subdominant modes for a GW coming from a BHNS merger: (2,1), (3,3), (4,4), and (3,2) (in order of higher contribution). In contrast, the modes (4,1) and (4,0) contribute the least to the full waveform (dark blue squares and circles, respectively, on the figure). Noteworthy is the mode (3,0) (red circles on middle panel) as mentioned above, with magnitudes comparable to those of the (4,4) and (3,2) modes, which decrease towards equal mass cases ($\nu \rightarrow 1/4$). This is a significant difference from what one would expect in the BBH case, where the (3,0) has a much lower contribution. In Fig. 6, for instance, the (3,0) amplitude is of the same order as the (3,2) mode for $q \gtrsim 2$. Similarly, the contribution of the (2,0) approaches that of the (2,1) for the same mass ratio range.

The fact that the dominance of the (2,0) and (3,0) modes in BHNS waveforms is larger for increasing mass ratio could potentially help us in distinguishing the source of binaries where no electromagnetic counterparts are observed. Therefore, the modeling of these modes in analytical waveform templates would prove useful for future potential observations. The $m = 0$ modes are characterized by the nonlinear memory effects arising from general relativity. However, further study on this effect is only possible with additional NR simulations with waveforms extracted at null infinity employing methods such as CCE [90–92].

E. Kick velocity from GWs

Gravitational waves carry energy, angular, and linear momentum away from a system. The kick velocity $v_{\text{kick}}^{\text{GW}}$ imparted to the final BH is a response to the loss of the latter. Estimating these velocities started with the work of Fitchett [93] and has been studied for BHNS systems [37,80]. In the following, we discuss the effects that the initial spin and tidal polarizability has on $v_{\text{kick}}^{\text{GW}}$ and present a fitting model to estimate kick velocity values for BHNS mergers.

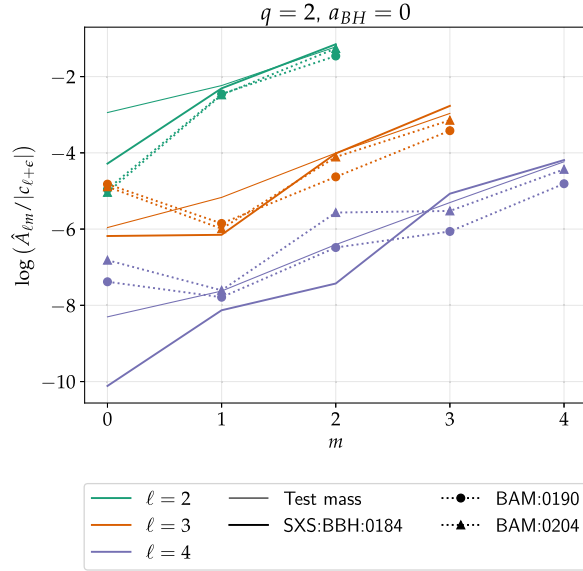


FIG. 5. Multipolar amplitude hierarchy comparing $q = 2$ nonspinning configurations: SXS:BBH:0184, BAM:0190, BAM:0204, and the test mass case. Note that the BHNS simulations are not evolved from the same initial data (different EOS), and as such their mass ratios are not exactly the same: $q = 1.91$ for BAM:0190 and $q = 1.95$ for BAM:0204. Although small differences, the amplitude is extremely sensitive to these changes, thus showing a low degree of scatter on the plot. These differences do not affect the multipolar pattern described in the main text.

Results focus solely on the recoil due to radiation of GWs, and we leave the discussion of the ejecta velocity for future studies. Effects due to resolution on the measurement of $v_{\text{kick}}^{\text{GW}}$ are discussed in Appendix A 3 a.

We obtain $v_{\text{kick}}^{\text{GW}}$ directly from the extracted waveforms to compute the linear momentum fluxes $\dot{P}_x$ and $\dot{P}_y$ [94,95], as follows:

$$\dot{P}_i = \frac{R^2}{16\pi} \int d\Omega (\dot{h}_+^2 + \dot{h}_\times^2) n_i, \quad (12)$$

and then integrate them to obtain the kick velocity vector v (see Ref. [96] for the full expression of the integrand), as follows:

$$v \equiv v_x + i v_y = -\frac{1}{M} \int_{-\infty}^t (\dot{P}_x + i \dot{P}_y) dt, \quad (13)$$

where v is a complex quantity with modulus $v_{\text{kick}}^{\text{GW}} = |v| = \sqrt{v_x^2 + v_y^2}$ and $n_i = x_i/r$ is the unit radial vector pointing from the source to the observer. By fixing an initial time t_0 for integration (when the two objects are at a large separation), we are not taking into account the net linear momentum of the binary from $t \rightarrow \infty$ to $t = t_0$. As earlier works have shown [95,97], finding an appropriate vectorial integration constant v_0 can reduce the error of the kick velocity measurement. This fix is of special relevance for our data since we are evolving binaries for just a few orbits. The procedure is described in detail in Appendix A 3 b.

Our simulations allow us to illustrate the behavior of $v_{\text{kick}}^{\text{GW}}$ with respect to the initial spins of the BH showing consistency with our estimates for BBH. Figure 7 shows the GW kick velocities for different configurations with $q \approx 3$ and $\Lambda = 494$: BAM:0186, BAM:0185, BAM:0183, BAM:0184, and BAM:0187 with spins $a_{\text{BH}} = -0.6, -0.3, 0.0, 0.3, \text{ and } 0.6$, respectively. The binaries with initial antialigned spin, and increasing spin magnitude, induce a significantly larger kick on the remnant, reaching velocities of almost $v_{\text{kick}}^{\text{GW}} \approx 120$ km/s. This result goes on par with earlier BBH where they measure superkicks for antialigned configurations [98–100]. However, contrary to our estimates for BBH where the recoil on the remnant increases significantly with the initial spin, for a BHNS with this mass ratio and tidal polarizability, the kick velocity stays

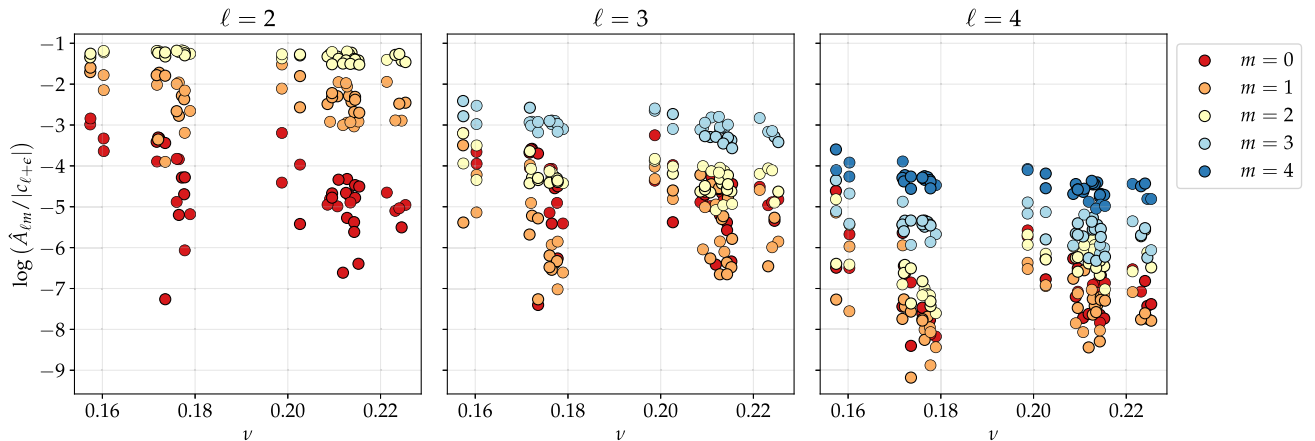


FIG. 6. Overview of the multipolar amplitudes of the data produced for this work as a function of ν . Each color represents a different value of m for each ℓ .

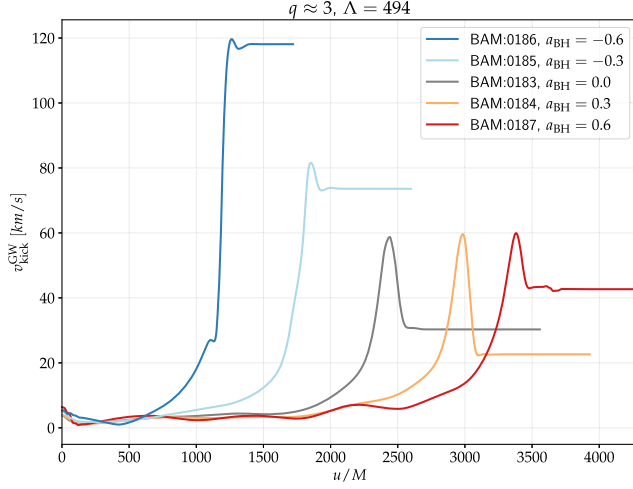


FIG. 7. Kick velocity obtained for different configurations with $q \approx 3$, $\Lambda = 494$, and a variety of spins. The antialigned spins have a higher impact on the resulting recoil velocity of the remnant.

below $v_{\text{kick}}^{\text{GW}} \approx 60$ km/s for both nonspinning and aligned spins up to $a_{\text{BH}} = 0.6$. This suggests that tides can have a “suppressing” effect on the remnant’s GW kick.

For future estimates of the remnant’s kick velocities, we develop a fitting model based on the one developed in Varma *et al.* for BBH mergers. We again factorize the BBH kick, $v_{\text{kick}}^{\text{BBH}}$, and represent the data as

$$\frac{v_{\text{kick}}^{\text{BHNS}}}{v_{\text{kick}}^{\text{BBH}}} = \frac{\Lambda p_1^{(2)}(a_{\text{BH}}, \nu)}{(1 + \Lambda p_2^{(1)}(a_{\text{BH}})\nu^2)^2}, \quad (14)$$

where we define the polynomials as

$$p_1^{(2)}(\nu, a_{\text{BH}}) = p_{11}^{(3)}(a_{\text{BH}})\nu + p_{12}^{(3)}(a_{\text{BH}})\nu^2, \quad (15a)$$

$$p_{1j}^{(3)}(a_{\text{BH}}) = c_{1j3}a_{\text{BH}}^3 + c_{1j2}a_{\text{BH}}^2 + c_{1j1}a_{\text{BH}} + c_{k0}, \quad (15b)$$

$$p_2^{(1)}(a_{\text{BH}}) = c_{221}a_{\text{BH}} + c_{220}. \quad (15c)$$

The coefficients c_{kji} are listed in Table VI in Appendix A 3. Figure 8 summarizes the behavior of the kick velocity values on a (Λ, ν) parameter space for different initial spins, with the white dots representing the NR data employed for fitting. For nonspinning and low spin cases, the remnant’s kick velocity from a BHNS significantly differs from the estimated for BBH, dramatically decreasing in value with increasing values of Λ regardless of the mass ratio. However, with higher-aligned spins, the $v_{\text{kick}}^{\text{BHNS}}$ starts approaching that of BBH for $q \gtrsim 2$ and $\Lambda \lesssim 1000$.

For $q \lesssim 2$ and $\Lambda \gtrsim 500$ the BHNS kick velocity is remarkably lower than for the nontidal case. We could then argue on the impact of the ejecta in these scenarios; it has been shown in earlier works that tidal disruption suppresses the kick coming from GWs, and consequently the backreaction of the mass ejection will result in a higher magnitude for the kick velocity due to the ejecta’s momentum [40,80]. The GW recoil will thus be more suppressed for binaries presenting earlier tidal disruption than those when it occurs later closer to merger. This correlates with the fact that highly spinning BHs and more compact NSs experience tidal disruption early in the evolution, thus inducing a higher ejecta velocity. Nonetheless, since we compare directly with the BBH case, we only take into account the recoil due to GWs.

F. Simulation compatible with GW230529

During the completion of the present work, a BHNS GW event, labeled GW230529, was detected in the O4a observation run [3]. Among the simulations produced for this campaign, the 12-orbit evolution, namely BAM: 0223, has a chirp mass of $\mathcal{M}_c = 1.94$, a total mass of $M = 5M_\odot$, $q = 2.47$, $\chi_1 = a_{\text{BH}} = 0.74$, and precessing spin of $\chi_p = 0.6$. These parameters coincide exactly within the

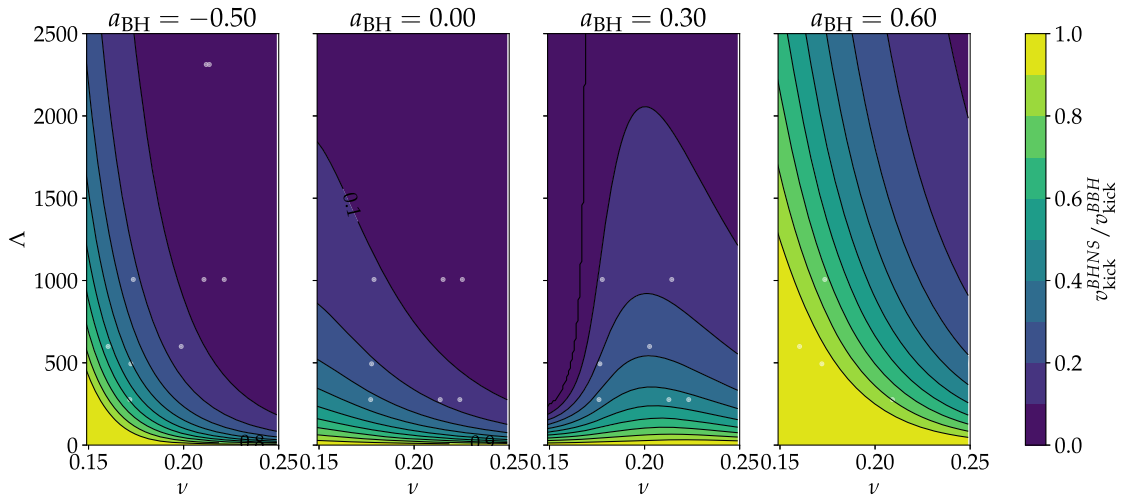


FIG. 8. Kick velocity model as a function of symmetric mass ratio ν and tidal polarizability Λ for different initial spins a_{BH} .

best values to 90% CI found in the LVK analysis. We compute the mismatch between the NR waveform and the best waveform obtained in Ref. [3] with the SEOBNRv5PHM model [101,102] (see Sec. VB for a description of how we obtain this quantity). The mismatch is calculated from $f_{\text{low}} = 276$ Hz, corresponding to the initial frequency of the simulation, up to $f_{\text{high}} = 2048$ Hz; and employing the noise curve of aLIGOZeroDetHighPower [103]. The resulting mismatches lie around ~ 0.3 including subdominant modes for different inclinations. In the same way, we obtain similar mismatches of ~ 0.2 with TEOBResumS-Dalí and no tides (see Ref. [104] for a description of the model). Given the symmetric nature of this event, we obtain again the mismatch against our BHNS model within TEOBResumS-Dalí (see Sec. IV). These results are presented in Sec. VB, which include tides ($\Lambda \neq 0$). The mismatches stay around the order of ~ 0.01 . These results highlight the need for accurate modeling of the high frequency regime for upcoming next-generation detectors, which will be able to uncover the physics at higher frequencies.

Finally, we obtain the kick velocity directly from our numerical simulation data and obtain a recoil of $v_{\text{kick}}^{\text{GW}} = 211$ km/s. This is a much higher value than our results from our spin-aligned configurations, as one expects precessing systems to produce large kicks [105–107].

IV. TEOBRESUMS MODEL

In this section, we describe a new EOB model for BHNSs. Our earlier work [61] was based on TEOBResumS-GIOTTO, a version of TEOBResumS for spin-aligned quasi-circular compact binary coalescences [89,108–113]. The present model has been implemented within the framework of TEOBResumS-Dalí [104,113–116], an EOB model for generic compact binaries and arbitrary orbits, including eccentricity, precession, and scattering. Similarly to our previous work, we focus on developing a ringdown model for the merger and postmerger part of the BHNS waveform. This is achieved by employing the same strategy: extract information from the new NR data presented in Sec. II together with the old simulations (used in Ref. [61]), and represent the deviation of relevant quantities to the BBH case. From our discussion in Sec. IIID, we model explicitly the modes (2,1), (2,2), (3,2), (3,3), and (4,4), which contribute the most to the overall waveform. At the time of this work's development, TEOBResumS did not provide a description for the $m = 0$ modes, making it impossible to build a BHNS extension. However, the implementation of these modes into TEOBResumS-Dalí is ongoing [117,118]. We will leave its extension for BHNSs for future improvement of the model.

A. Ringdown model

Following the usual recipe to model the ringdown based on the procedure from Ref. [119] and used in

TEOBResumS-Dalí, we employ the QNM rescaled waveform seen in Eq. (4) of Ref. [61]. Furthermore, for the modes (2,1), (3,3), and (4,4), we use instead the strategy implemented in the SEOBNR waveform family [120,121], where the coefficients $c_i^{A,\phi}$ are constrained in the following manner:

$$c_1^A = (\dot{A}_{\ell m}^{\text{match}} + \alpha_1 A_{\ell m}^{\text{match}}) \cosh^2 c_3^A / c_2^A, \quad (16a)$$

$$c_4^A = A_{\ell m}^{\text{match}} - (\dot{A}_{\ell m}^{\text{match}} + \alpha_1 A_{\ell m}^{\text{match}}) \cosh c_3^A \sinh c_3^A / c_2^A, \quad (16b)$$

$$c_1^\phi = (\omega_1 - \omega_{\ell m}^{\text{match}}) \frac{1 + c_3^\phi}{c_2^\phi c_3^\phi}, \quad (16c)$$

$$c_4^\phi = 0, \quad (16d)$$

and using $(c_2^A, c_3^A, c_2^\phi, c_3^\phi)$ as free coefficients. Note that for these modes, the matching time corresponds to the time of the amplitude peak of the (2,2) mode, $t_{\ell m}^{\text{match}} \equiv t_{22}^{\text{peak}}$. Additionally, we account for the time delay between the time of merger t_{mrg} and the time where the amplitude of each peak occurs, by defining it as [89]

$$\Delta t_{\ell m} \equiv t_{\ell m}^{\text{peak}} - t_{\text{mrg}}. \quad (17)$$

For all the modes, we fit the quantities $(\alpha_{\ell m 1}, \omega_{\ell m 1}, A_{\ell m}^{\text{peak}}, \omega_{\ell m}^{\text{peak}}, \Delta t_{\ell m})$ as a function of $(\nu, \Lambda, a_{\text{BH}})$ as described in Appendix B. The fits for the peak amplitudes employ the same rescaling as in Eq. (11).

The new ringdown waveform is shown in Fig. 9, where we compare the (2,2) mode with the new NR simulations and our previous model from Ref. [61]. We show fiducial simulations BAM:0190 (left) and BAM:0200 (right). Our new model shows amplitude differences at merger well below 10% and a phase that qualitatively agrees to that of the NR waveform for both configurations. The ringdown is accurately modeled with the new fits employed to classify the different BHNS merger types within the code (see next subsection). This is particularly the case for BAM:0200, where the fits adequately deform the BBH waveform to model the suppressed BHNS ringdown. In contrast, the old model (shown in gray) misclassifies this configuration and produces a ringdown with very excited QNM as opposed to the damped signal from NR. Furthermore, for BAM:0190, one can notice a small attachment artifact at merger, which is no longer present for this case in TEOBResumS-Dalí.

We show the ringdown waveform for the subdominant modes in Fig. 10 for two different binary configurations. We find qualitatively good agreement between the NR and EOB amplitudes towards merger and a ringdown signal approaching the numerical one.

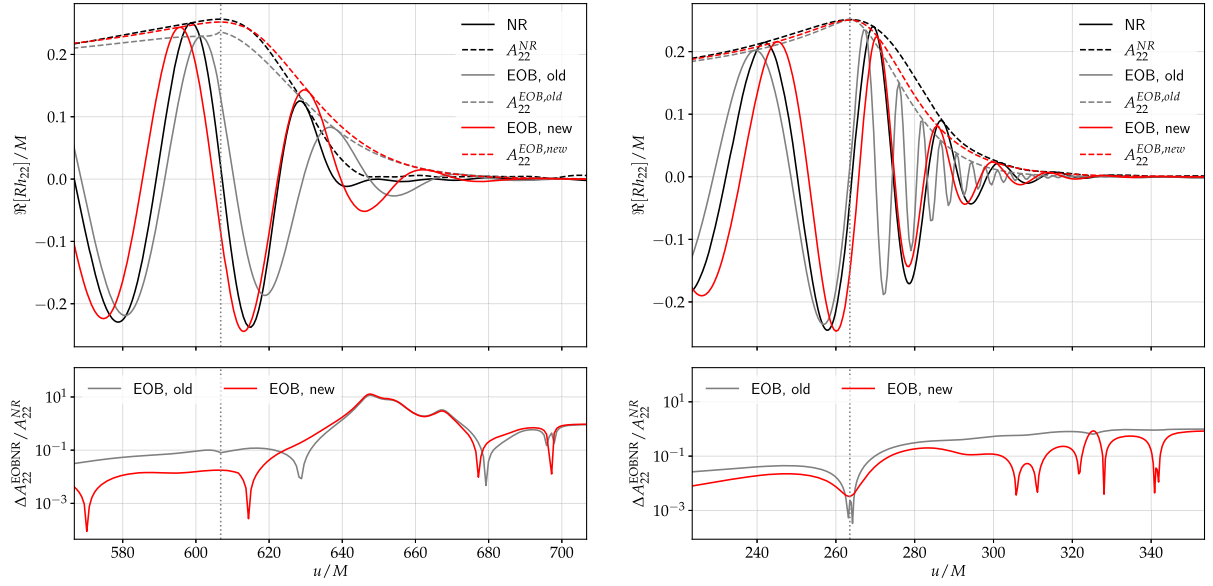


FIG. 9. Ringdown waveform of the (2,2) mode for different BHNS binary configurations from the fitting set: BAM: 0190 (left) and BAM: 0200 (right). The solid lines show the real part of the strain while the dashed lines indicate their respective amplitudes. Black lines represent the NR waveform, the gray ones come from the old model of Ref. [61], and the red lines are produced with the new model presented in this paper. The smaller bottom panels show the amplitude differences with respect to the NR simulation for both models. As reference, we add a dotted vertical line to indicate the moment of merger.

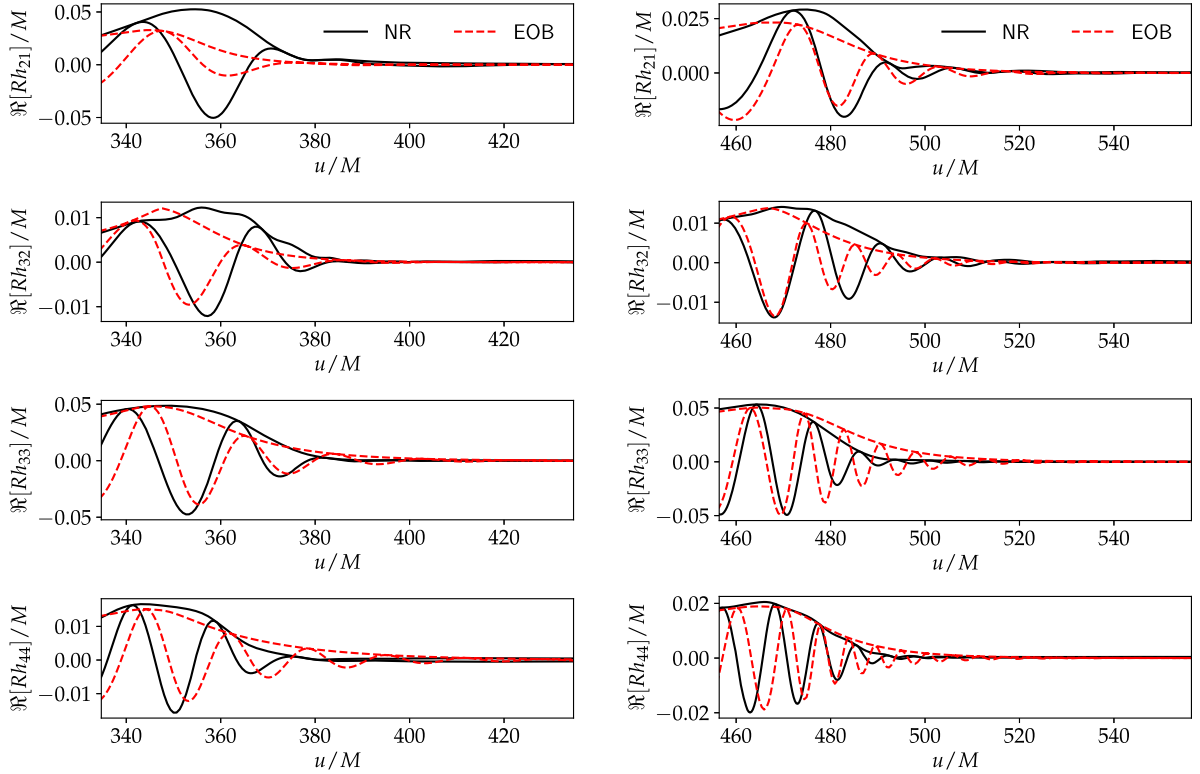


FIG. 10. Similar to Fig. 9 but showing the corresponding subdominant modes' ringdown waveforms and amplitudes for NR (black solid lines) and the new model (red dashed lines).

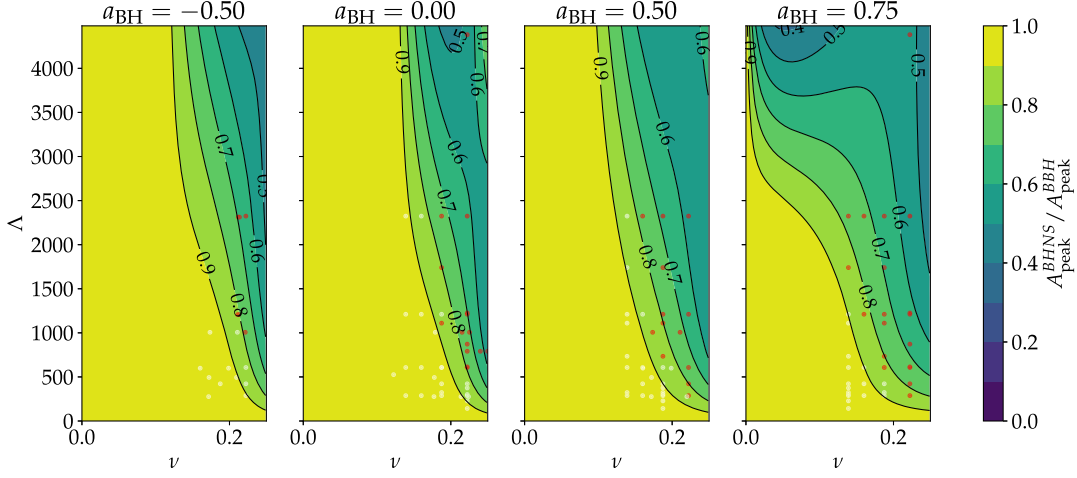


FIG. 11. Fits for the peak of the (2,2) amplitude A^{BHNS}/A^{BBH} as a function of (ν, Λ, a_{BH}) . The dots represent the NR data used to inform the fits; the red dots indicate the presence of tidal disruption (see text).

B. Inspiral-merger-ringdown waveform

The main new elements entering TEOBResumS-Dalí inspiral-merger-ringdown for BHNSs are: (i) the remnant model presented in Sec. III C; (ii) an updated NQC to the (2,2) waveform and new, specific NQC for the other multipoles; (iii) the ringdown template described in Sec. IV A; and (iv) an update classification of the binary in types I, II, and III. Tidal effects are incorporated in the same fashion as for BNSs, namely using 2PN and General self-force with 3+ tidal terms (GSF3) models for gravitomagnetic and gravitoelectric tidal effects, respectively [122–124]. Spin, spin-precession, and eccentricity effects are incorporated in the same way as for BBH (and BNS). We discuss in the following items (ii) and (iv).

NQC are fixed by extracting from NR multipolar waveform the quantities $(A_{\ell m}^{NQC}, \dot{A}_{\ell m}^{NQC}, \omega_{\ell m}^{NQC}, \dot{\omega}_{\ell m}^{NQC})$ at times

$$t_{\ell m}^{NQC} \equiv t_{\ell m}^{\text{peak}} + 2, \quad (18)$$

with exception of the (2,1), (3,3), and (4,4) modes that are extracted instead at $t_{\ell m}^{\text{peak}} \equiv t_{22}^{\text{peak}}$. These quantities are fitted as described in Appendix B; i.e., by factorizing the BBH values and representing them as a functions of $\{\nu, \Lambda, a_{BH}\}$. In TEOBResumS, the time shift between t_{22}^{NQC} (time at which the NQC parameters are computed on the EOB time axis) and the peak of the orbital frequency $t_{\Omega_{\text{orb}}}^{\text{peak}}$ is defined as [110]

$$t_{\text{NQC}}^{\text{EOB}} = t_{\Omega_{\text{orb}}}^{\text{peak}} - \Delta t_{\text{NQC}}, \quad (19)$$

where $\Delta t_{\text{NQC}} = 1$ inspired by test-particle results. We find, however, that setting this quantity to $\Delta t_{\text{NQC}} = 4$ for BHNSs yields better results. We note that this choice is purely technical and has no physical meaning in the dynamics.

The classification in types I, II, and III BHNSs is used in TEOBResumS to select the ringdown waveform and obtain the best possible description with minimal NR information.

Contrary to our earlier work, where we employ the QNM inverse damping time α_{221} fit to approximately identify a binary among the three types of mergers, we now make use of A_{22}^{peak} . This fit is shown in Fig. 11, where red dots indicate the binaries going through tidal disruption, type I. We therefore consider a contour around $A^{BHNS}/A^{BBH} < 0.85$ to identify a binary as type I and use all fits developed in this work. All points falling above the contour $A^{BHNS}/A^{BBH} > 0.99$ (type II) go through the BBH pipeline of TEOBResumS but employ the remnant model developed for BHNSs. For the cases falling in the middle of these limits, type III, we use all fits in this work except for $\alpha_{\ell m 1}$ and $\omega_{\ell m 1}$ as we saw that the ones designed for BBH do a better job at simulating the ringdown produced in NR data.

V. WAVEFORM MODEL VALIDATION

To validate our new model, we use BAM:0223: a 12-orbit inspiral-merger-ringdown waveform with a precessing spin BH. The binary has a mass ratio of $q = 2.3$, a spin parameter of $\chi_p = 0.61$ (indicating a highly precessing spin) and $\Lambda = 494$. Waveform convergence and error budget for this specific waveform is discussed in Appendix A 4. We discuss below time-domain phasing and faithfulness with TEOBResumS-Dalí.

A. Phasing

We follow the same phasing procedure as performed in our previous work [61]: minimize the functional $\Xi^2(\delta t, \delta \phi)$ of the EOB and NR waveform phases,

$$\Xi^2(\delta t, \delta \phi) = \int_{t_i}^{t_f} [\phi_{\text{NR}}(t) - \phi_{\text{EOB}}(t + \delta t) + \delta \phi]^2 dt, \quad (20)$$

in an alignment window $[t_i, t_f]$ and extract the optimal values for the time δt and phase $\delta \phi$ shift. These are then

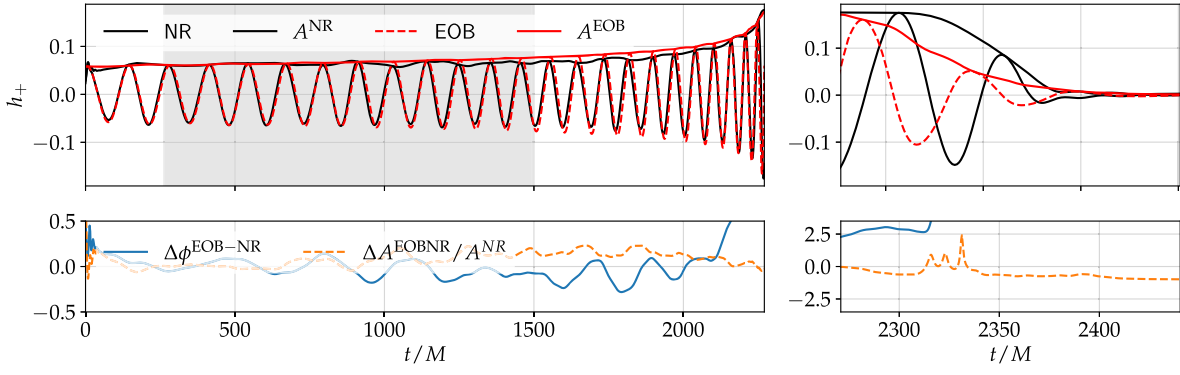


FIG. 12. Alignment against EOB of the real part of the waveform h_+ of BAM:0223 computed with the modeled modes. Amplitude (orange dashed line) and phase differences (blue solid line) are shown in the bottom panels. The gray area indicates the alignment window.

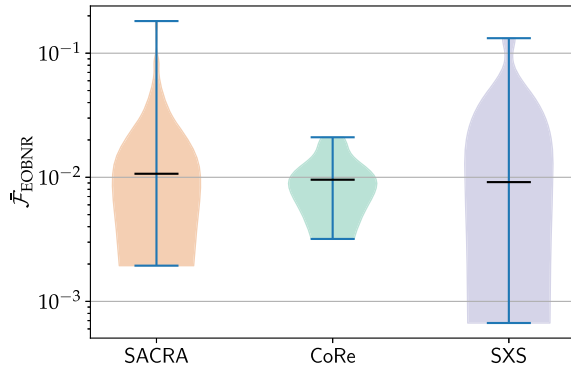


FIG. 13. Mismatches for all NR simulations on the three available datasets: SACRA (orange), CoRe (green), and SXS (purple). The black line indicates the median for each case.

used to shift the EOB waveform, which is then compared to the waveform from NR.

The phasing using this alignment is shown in Fig. 12. One can see the consistency between the EOB and the NR waveforms, especially during the inspiral with minimal phase and amplitude deviations towards merger and ringdown. Despite this, the model accurately deforms the BBH ringdown to accommodate the morphology resulting from the numerical simulation. The phase difference stays below 0.5 rad throughout the inspiral and increases to ~ 2.3 rad at merger.

B. Mismatch

As a check for validity of the model, we employ the unfaithfulness (or mismatch) defined as

$$\bar{\mathcal{F}} \equiv 1 - \mathcal{F} = 1 - \max_{t_0, \phi_0} \frac{\langle h^{\text{EOB}}, h^{\text{NR}} \rangle}{\|h^{\text{EOB}}\| \|h^{\text{NR}}\|}, \quad (21)$$

where t_0 and ϕ_0 are the initial time and phase, and $\|h\| \equiv \sqrt{\langle h, h \rangle}$. We define the inner product with the power

spectral density (PSD) of the detector $S_n(f)$ and the Fourier transformed waveform $\tilde{h}(f)$, as follows:

$$\langle h_1, h_2 \rangle \equiv 4\Re \int \frac{\tilde{h}_1(f) \tilde{h}_2^*(f)}{S_n(f)} df. \quad (22)$$

Since the binary experiences precession, the unfaithfulness will also depend on the extrinsic parameters of the binary. Hence, we report the sky-maximized faithfulness including all modes at different inclinations, and optimizing over the coalescence angle ϕ and the rotations on the in-plane component of the spin, as described in Refs. [125,126]. The mismatch is obtained from the initial frequency of the simulation $f_{\text{low}} = 276$ Hz up to $f_{\text{high}} = 2048$ Hz, and employing the PSD from both the Einstein Telescope [127] and the zero-detuned high-power Advanced LIGO [103] noise curves.

We present the resulting mismatches in Table III for each noise curve employed. The lowest mismatches are obtained with $\iota = \pi/8$, with values as low as ~ 0.01 . For $\iota = 0$ and $\iota = \pi/4$, the mismatch stays well below ~ 0.1 , whereas for $\iota = \pi/2$, it reaches ~ 0.1 for both detectors. These values showcase the performance of the model for precessing binaries with spins as high as $\chi_p = 0.6$, especially at low inclinations.

Finally, we comment on the mismatches previously computed in Ref. [104] for 184 BHNS simulations across the available datasets including the CoRe data produced in this work. These were obtained in a frequency range of $f \in [10, 4096]$ Hz for the (2,2) mode. The results are shown in Fig. 13. We mark the median values for each dataset, which lie around $\sim 1\%$ for all of them. The lowest mismatch corresponds to SXS: BHNS:0001 with $\bar{\mathcal{F}} = 0.07\%$, whereas the highest reaches $\bar{\mathcal{F}} = 18\%$ for 2H-Q2M12a75 from the SACRA catalog. The latter is a configuration with a highly spinning BH $a_{\text{BH}} = 0.75$ and $\Lambda = 4392$.

TABLE III. Unfaithfulness of TEOBResumS including Higher Modes with NR waveform from BAM:0223 for different inclinations ι and noise curves.

PSD	$\iota = 0$	$\iota = \pi/8$	$\iota = \pi/4$	$\iota = \pi/2$
EinsteinTelescopeP1600143	4.7×10^{-2}	1.3×10^{-2}	3.4×10^{-2}	9.9×10^{-2}
aLIGOZeroDetHighPower	4.9×10^{-2}	1.4×10^{-2}	3.8×10^{-2}	1.1×10^{-1}

VI. BHNS PARAMETER SPACE

In this section, we discuss how TEOBResumS-Dalí can guide the development of future NR simulations for GW modeling. We also demonstrate, for the first time, the capability of our model to generate eccentric and precessing waveforms for BHNSs.

A. Where to further simulate?

Despite having simulated 52 binaries, our work clearly highlights the necessity of more NR data in order to be able to develop faithful waveforms. Just focusing on the non-precessing and quasicircular merger, a main issue is to identify the binaries that maximize the information required by EOB. We are here able to address this issue by leveraging on the predictions of TEOBResumS-Dalí and using a greedy algorithm [128,129]. The latter selects points in the parameter space by identifying a waveform basis among those that are “most different” according to a mismatch-based metric. Points in the parameter space are sequentially added to the basis until either a desired number of points or a certain accuracy threshold is reached.

We run the greedy algorithm on a reduced parameter space described by mass ratio, effective spin, and tidal polarizability parameter $\{q, \chi_{\text{eff}}, \Lambda\}$, which adequately capture the relevant binary interactions and effects impacting the waveform morphology. The parameter space sampled ranges from $q \in [1, 5]$ for masses $M_{\text{NS}} \in [1, 2]M_{\odot}$; spins $\chi_1 \in [-0.8, 0.8]$, $\chi_2 \in [-0.2, 0.2]$, and 10 different EOSs: 2B, 2H, ALF2, APR4, ENG, H4, MPA1, MS1, MS1b, and

SLy [74], corresponding to a range of $\Lambda \in [1, 10000]$. The signals produced with TEOBResumS-Dalí start from a frequency of $f_0 = 0.0055$, corresponding to roughly ~ 4000 M before reaching merger. The mismatch is obtained within a range of $f \in [0.007, 0.1]$ using a flat PSD. The algorithm is stopped when the number of basis functions reaches $N_{\text{sim}} = 200$. We note that these results have been anticipated in Albanesi *et al.* [104].

The result is shown in Fig. 14. The greedy algorithm identifies 200 configurations, resulting in residuals with mismatches below ~ 0.05 , where 50% are less than ~ 0.02 . The algorithm’s results suggest that future NR simulations should focus on systems with $\Lambda > 3000$ and mass ratio $q \leq 2$. These waveforms make up $\sim 50\%$ of the total points and physically correspond to binaries where matter effects and tidal disruption are significant. Therefore, they are more likely to produce significant differences in the GW signal. The distribution of the effective spin χ_{eff} of the systems selected by the greedy algorithm pushes towards the upper limit of the parameter space considered. Note that also in this case, binaries with large aligned spins are more likely to produce significant tidal disruption.

B. Eccentric and precessing waveforms

TEOBResumS-Dalí is a physically complete EOB model that allows us to generate BHNS waveforms from eccentric and precessing BHNS mergers [104]. We have carefully checked the robustness of our eccentric BHNS waveform by generating more than 1000 different binaries with $q \in [1, 5]$, $a_{\text{BH}} \in [-0.8, 0.8]$, $\Lambda \in [1, 5000]$, and $e_0 \in [0.01, 0.2]$. All the waveforms generated are smooth and showed no evident unphysical features; see Appendix C. As an example, we show in Fig. 15 a gravitational waveform with initial eccentricity $e_0 = 0.5$, $q = 2$ and $\Lambda = 500$ (top, red line); and a second one also including spin precessing effects, $q = 3$, $\Lambda = 1000$ (bottom, blue line) with $\vec{\chi}_1 = (0.2, 0.2, 0.2)$. Interestingly, one can notice small oscillating features early in the inspiral which are not present for eccentric BBH waveforms. These appear for the $q = 2$ (eccentric) configuration, where tidal disruption takes place. These oscillations are still present but significantly damped for the $q = 3$ (eccentric + precessing) waveform, corresponding to a type III from our classification (intermediate case). Later in the inspiral, the waveforms show morphologies consistent with our predictions for BBH but with the dampened ringdown that characterizes BHNS binaries.

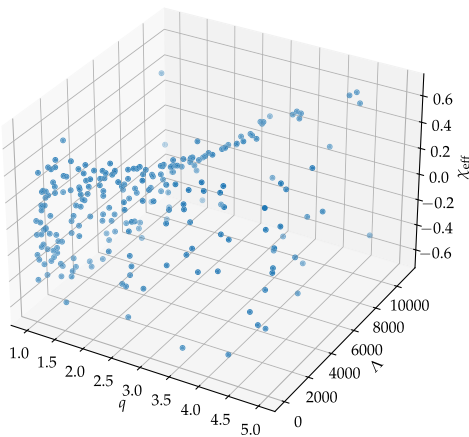


FIG. 14. Greedy basis waveforms obtained in terms of Λ , q and χ_{eff} for nonprecessing quasicircular orbits.

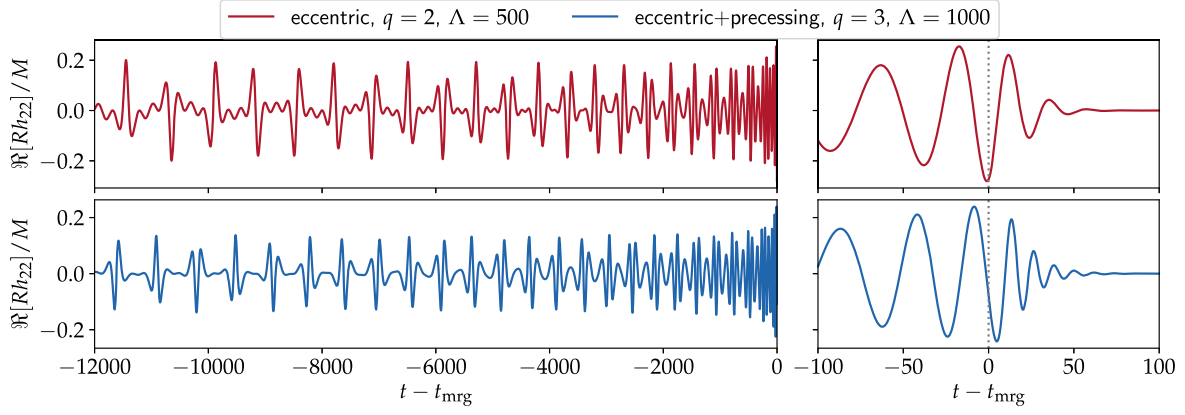


FIG. 15. Gravitational waveforms produced by TEOBResumS-Dalí with eccentricity $e_0 = 0.5$ and no corresponding NR data. Top (red line): binary in an eccentric orbit with nonspinning components, $q = 2$ and $\Lambda = 500$, (type I). Bottom (blue line): binary in an eccentric orbit with $\vec{\chi}_1 = (0.2, 0.2, 0.2)$, $q = 3$ and $\Lambda = 1000$, (type III).

The early inspiral oscillations for eccentric configurations are consistently present throughout the parameter space of TEOBResumS-Dalí, and get significantly more pronounced with higher positive spins and increasing Λ , corresponding to tidal disruption cases. See Appendix C for more details on the behavior of the eccentric model. Given the lack of publicly available NR simulations in elliptical orbits, we cannot directly assess the faithfulness of our model against numerical data.

Recent work has suggested the event GW200105 shows evidence of both eccentricity and precession [9]. The analysis was carried out employing a post-Newtonian model, pyEFPE [130], which incorporates eccentricity and precession, and considering a frequency range of $f \in [20, 280]$ Hz. They find an orbital eccentricity of $e_{20} = 0.145^{+0.007}_{-0.097}$ at a

GW frequency of 20 Hz within the 90% credibility interval and evidence of precession with $\chi_p = 0.06^{+0.13}_{-0.04}$ at the 95% credible upper limit.

Inspired by these claims, we put at test TEOBResumS-Dalí on the best waveform inferred in Ref. [9]. In Fig. 16, we compare the best pyEFPE waveform against a TEOBResumS-Dalí waveform produced with the same parameters. We include in the plot TEOBResumS-Dalí waveforms with different Λ values as illustration, but perform the phase alignment between the $\Lambda = 0$ and the pyEFPE waveforms since the latter model does not account for tides. The PN-based waveform aligns well in the early inspiral with the EOB-based one, but rapidly accumulates tidal dephasing starting at GW frequencies of $f \sim 760$ Hz (outside of the considered frequency range of the analysis in Ref. [9]).

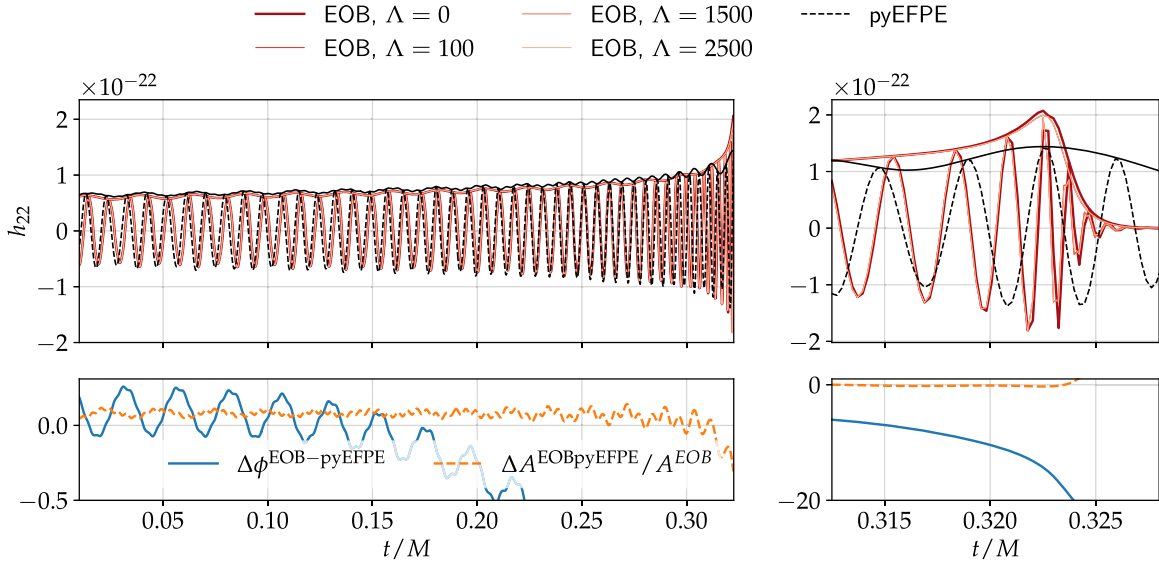


FIG. 16. Alignment between the best waveform obtained in the analysis of Ref. [9] with the pyEFPE model (black) and the equivalent waveform produced with TEOBResumS-Dalí (dark red). Their corresponding phase and relative amplitude differences are shown at the bottom of the figure. As a reference, we show equivalent EOB waveforms with different Λ values on the top panel.

By merger time, the PN approximant has dephased of several cycles in comparison to `TEOBResumS-Dalí`.

We compute mismatches among these waveforms in the same way described in Sec. VB but employing the same frequency range as in Ref. [9] and considering the (2,2) mode only. To avoid complications arising from different definitions of eccentricity among models, we optimize the `TEOBResumS-Dalí` waveform over the eccentricity to find the equivalent system to that of Ref. [9]. We find a mismatch of $\bar{\mathcal{F}} = 0.28$, indicating the PN model is affected by large systematics towards the upper limit of the analysis frequency range. This result suggests that an analysis of GW200105 with `TEOBResumS-Dalí` with eccentricity and precession is likely to deliver different parameters than the PN analysis. The mismatch computation with tidal effects ($\Lambda \neq 0$) yields very similar mismatches, which is expected for this highly asymmetric event (type II). Therefore, using waveform models without tides for the inference of this event appears to be a robust choice.

As a further check, we compute the mismatches between the best waveform obtained by the LVK analysis [8] with `IMRPhenomXPHM` [131] against the waveform produced with `TEOBResumS-Dalí`, both with and without tides, higher modes, and same other parameters. The mismatches stay at values ~ 0.001 and similarly as before, we find no difference in the mismatch when including different values of Λ , which is expected from the high mass ratio of this event. This suggests that a `TEOBResumS-Dalí` analysis of the same data (and without tidal effects) is likely to deliver source parameters compatible with the LIGO-Virgo-KAGRA results.

VII. CONCLUSIONS

In the first part of this paper, we presented 52 new NR simulations of circularized BHNSs with different configurations of mass ratios, spins, and employing three different EOSs. We validated our simulations with an extensive systematic study including initial data quasiequilibrium sequences, grid setups, and convergence studies. The simulation data were employed to quantitatively model BH remnants and gravitational waveforms. About the former, we provided updated formulas for the remnant mass and spin which smoothly deform NR-driven models for binary black holes. About the latter, we studied for the first time the multipolar structure of the GW modes up to $\ell = 4$ and devised a quantitative estimate for the GW recoil.

We found that the most relevant GW subdominant modes are (2,1), (3,2), (3,3), and (4,4), as expected from the hierarchy of binary black holes. Contrary to the latter, however, the (2,0) and (3,0) amplitudes contribute more to the whole multipolar amplitude for BHNSs and are related to the memory part of the GW. If these multipoles will be accurately modeled in future waveform templates, GW observations in these channels could potentially help in distinguishing between BBHs and BHNSs, in addition to enhancing the effects of GW memory for these binaries.

With the numerical data, we develop a model to estimate the GW recoil of the remnant and find that tidal effects are more prominent for more comparable masses and antialigned spins, i.e. type I (see Fig. 8.) The net effect is to suppress the kick velocity with respect to BBH due to the correspondingly lower radiated momentum. This is not the case for binaries with increasing spin magnitude, mass ratios $q \gtrsim 2$ and $\Lambda \lesssim 500$, which approach more the estimates for BBHs.

Further, we discussed the results from a 12-orbit spin precessing simulation (BAM: 0223, type I) compatible with the LIGO-Virgo-KAGRA event GW230529 and the implication for the interpretation of that event. Comparing the numerical simulation to the best waveform obtained in Ref. [3], we find a mismatch of ~ 0.3 . The discrepancy is mainly ascribable to tidal effects that are not modeled in the best inferred waveform.

In the second part of our work, we presented `TEOBResumS-Dalí` for BHNSs. The latter EOB model is an extension of our previous work [61], which uses the improved NR-information developed here to reduce systematic uncertainties. `TEOBResumS-Dalí` specifically models NQC and ringdown for multipoles (2,2), (2,1), (3,2), (3,3), and (4,4). The (2,2) waveform amplitude at merger is improved by an order of magnitude by the new NQC prescriptions. Such amplitude is employed to phenomenologically classify types I, II, and III binaries and make a design choice for the EOB waveforms. The new multipolar ringdown allows us to better capture the key morphological features that distinguish BHNS from BBH waveforms. Moreover, `TEOBResumS-Dalí` can make predictions for precessing and eccentric binaries.

We validated our model with BAM: 0223, which was not used to inform `TEOBResumS-Dalí`. We obtain a phase difference throughout the whole inspiral waveform below ~ 0.5 rad. The mismatches are the order of ~ 0.01 for low inclinations $\iota \leq \pi/4$, thus highlighting the model's capacity to accurately reproduce the waveforms of highly precessing configurations. Additionally, we obtained a median mismatch of 0.01 against all available simulations, demonstrating the accurate performance of the model.

Our work clearly emphasizes the necessity of more NR data in order to be able to develop faithful waveforms for advanced and next-generation observations. Focusing on quasicircular nonprecessing BHNSs, we employ `TEOBResumS-Dalí` in a greedy search to identify an optimal set of future simulations. About 200 simulations with initial frequency of $f_0 = 0.0055$ appear sufficient to describe BHNS waveforms with mismatches $\lesssim 0.02$ – 0.05 . This results sets the task for the immediate future: new simulations should be performed for large tidal polarizability parameters, mass ratio $q \sim 2$, and large effective spins. Similar studies also including precession and eccentricity are being conducted to identify the most relevant parameters for those cases.

Finally, we verified the robustness of TEOBResumS-Dalí in producing the first eccentric (+precessing) BHNS waveforms. In absence of NR simulations, we verified the model produces sane waveforms for more than 1000 different binaries with parameters in ranges $q \in [1, 5]$, $a_{\text{BH}} \in [-0.8, 0.8]$, $\Lambda \in [1, 5000]$, and $e_0 \in [0.01, 0.2]$. This is possibly the parameter space region which more urgently needs specific BHNS waveforms.

Furthermore, we computed mismatches between TEOBResumS-Dalí and the best waveforms for the event GW200105 as obtained by the authors of Ref. [9] and LIGO-Virgo-KAGRA [8]. The former analysis was conducted with a PN model incorporating eccentricity and precession. For the best parameters, we found large mismatches corresponding to phase differences of more than 14 rad accumulated over the frequency interval analyzed. This suggests that the inference with TEOBResumS-Dalí including eccentricity and precession is likely to deliver different source parameters. The latter analysis was conducted with IMRPhenomXPHM, including higher modes and precession. TEOBResumS-Dalí shows mismatches of order ~ 0.001 for the same inferred parameters, indicating consistency of the results independently on the inclusion of tidal effects.

The primary challenge facing contemporary BHNS modeling is the limited suite of NR simulations suitable for calibration. While the analytical description of the inspiral is robust, the strong-field dynamics represent a critical vulnerability across all existing models. Much of the available NR data lack the convergence and length (often fewer than four orbits) necessary for high-fidelity modeling, underscoring the importance of the convergent simulations released in this study. In particular, the tidal disruption regime remains the most significant source of inaccuracy, despite its importance for multimessenger observations. Although recent NR efforts have integrated sophisticated microphysics, including temperature-dependent EOS and neutrino transport, a gap remains in connecting the complex merger dynamics and the resulting gravitational waveforms. Future progress will require a quantitative assessment of how matter and ejecta suppress the QNM of the remnant black hole to better characterize the ringdown in these systems.

Simulations were performed on SuperMUC-NG at the Leibniz-Rechenzentrum (LRZ) Munich and on the national HPE Apollo Hawk at the High Performance Computing Center Stuttgart (HLRS).

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DATA AVAILABILITY

The data that support the findings of this article are openly available [133,134].

APPENDIX A: NUMERICAL RELATIVITY DATA

In this Appendix, we show the details of the configurations simulated for this work. In Appendix A 1, we describe the accuracy of the waveforms extracted. Appendix A 2 presents the obtained remnant properties and the coefficients necessary for the remnant BH model. Finally, resolution effects and errors on the measurement of the GW kick velocity are discussed in Appendix A 3.

1. Data quality

In this section, we assess the convergence of our simulations and justify our grid configuration choices for the production runs. We consider four resolutions and three different refinement levels on the BH; see Table IV. For these studies, we employ the configuration of BAM:0206 to find the “cheapest,” highest resolution configuration to employ for the rest of the simulations.

Figure 17 shows the norm of the Hamiltonian constraint for all the resolutions, with eight refinement levels for the BH. The norm stays at low values for most of the simulation time and decreases notably as we increase the resolution.

TABLE IV. Grid configurations employed for the evolutions with the BAM code.

Name	L	l^{mv}	n^{mv}	h_{L-1}	n	h_0
L6	6	2	64	0.209	96	13.38
L8	8	2	64	0.052	128	13.38
L9	9	2	64	0.026	128	13.38
M6	6	2	96	0.139	144	8.92
M8	8	2	96	0.035	163	8.92
M9	9	2	96	0.017	163	8.92
H6	6	2	128	0.104	192	6.69
H8	8	2	128	0.026	218	6.69
H9	9	2	128	0.013	384	6.69
F6	6	2	192	0.069	288	4.46
F8	8	2	192	0.017	384	4.46
F9	9	2	192	0.009	326	4.46

The resulting waveforms are presented in Fig. 18, where we also show the merger time difference (gray area). From this plot, one can already tell that the set L8-M8-H8-F8 does not build a convergent series. Indeed, the following convergence study in Sec. A 1 a does show that the lowest resolution (L8), although the cheapest to simulate, does not produce accurate enough data.

The first thing we want to check is the uncertainty due to extracting the waveform at a finite radius. These errors affect mostly the amplitude and phase, which are critical quantities for waveform modeling. We show the differences between each extraction radius in Figs. 19 and 20 for the two highest resolutions. Differences in both amplitude and phase are obtained as the difference between two consecutive radii, e.g., $\Delta^* \phi(R_i) = \phi(R_i) - \phi(R_{i-1})$. For both resolutions, differences in amplitude stay below 1%, whereas for the phase, we see a clear decrease as we extract at higher radii. Furthermore, we notice how significant the phase errors are early in the evolution and that they are less relevant as we approach merger.

a. Convergence tests

We look again at the amplitude and phase difference among the different resolutions with eight refinement levels on the BH in Fig. 21. As stated earlier, we note that the differences with the lowest resolution (green lines) are too high for them to belong to the convergent series M8-H8-F8. We obtain the convergence rate r experimentally with the scaling factor SF , as follows:

$$SF = \frac{h_M^r - h_H^r}{h_H^r - h_F^r}, \quad (\text{A1})$$

where h corresponds to the minimum grid spacing for each resolution (M,H,F). As seen in the figure, our data show second-order convergence throughout the inspiral, which

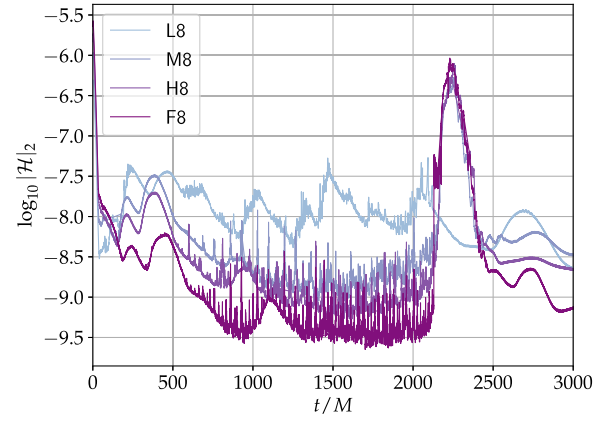


FIG. 17. Hamiltonian constraint of BAM:0206 with L8, M8, H8, and F8 settings.

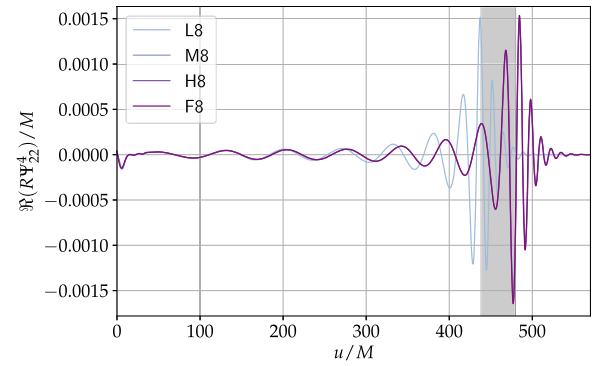
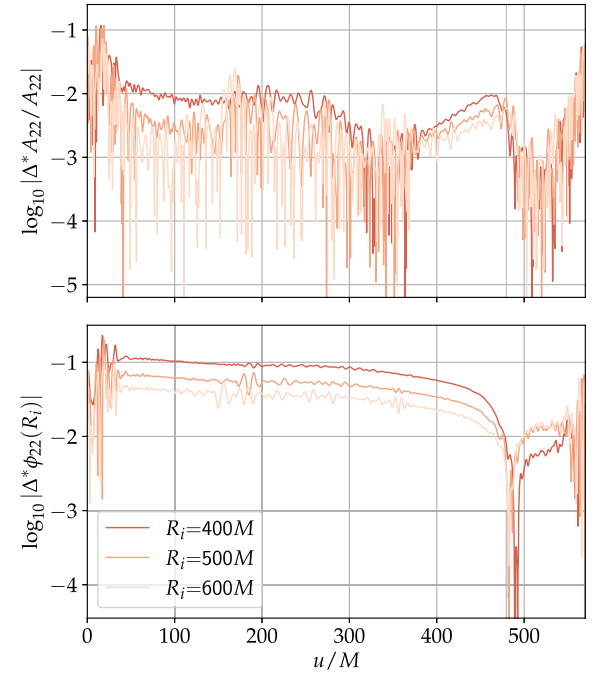
FIG. 18. Real part of $R\Psi_{22}^4/M$ of BAM:0206 with L8, M8, H8, and F8 settings. The gray area indicates the different merger times.

FIG. 19. Uncertainties due to finite extraction radii for BAM:0206 with H8 settings.

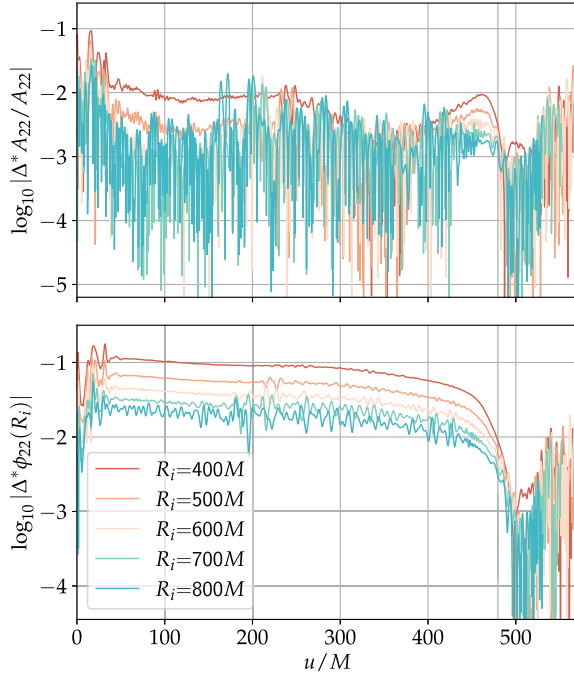


FIG. 20. Uncertainties due to finite extraction radii for BAM:0206 with F8 settings.

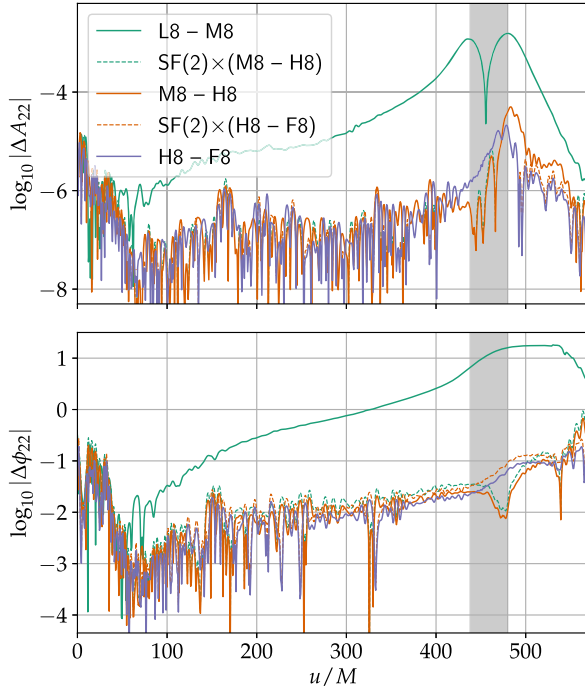


FIG. 21. Uncertainties due to grid resolution for BAM:0206. Amplitude (top) and phase differences (bottom) of $R\Psi_{22}^4/M$ among the L8, M8, H8, and F8 settings.

decreases slightly around merger for the series M8-H8-F8. Truncation errors such as these tend to accumulate throughout the simulation, hence increasing with simulation time.

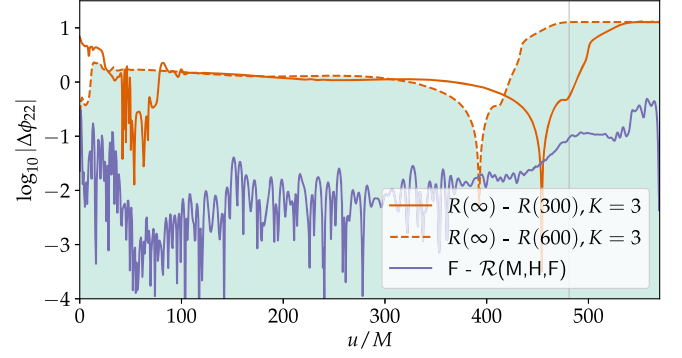


FIG. 22. Error budget for $R\Psi_{22}^4/M$ using the L8, M8, H8, and F8 convergent series for BAM:0206.

At merger, uncertainties in the amplitude and phase stay below $\sim 0.01\%$ and $\sim 10\%$, respectively, for the convergent series.

Additionally, we obtain an error budget for our simulations as shown in Fig. 22. To estimate the error from finite radius extraction $\delta\phi_{(r)}$, we compute an extrapolation to null infinity $R(\infty)$ by employing a polynomial in $1/R$ of order $K = 3$ as described in Ref. [49]. In the figure, we compare the extrapolated waveform phase to that of the lowest and highest available extraction radius (orange solid and dashed lines, respectively). We notice no significant difference throughout the inspiral, but lower values towards merger for the highest radius. To account for resolution (or truncation) errors, we obtain an improved dataset $\mathcal{R}(M, H, F)$ through a Richardson extrapolation [49,50]. For this procedure, we employ the convergence factor found experimentally and the convergent datasets we have (M8-H8-F8). The difference between the highest resolution and extrapolated values will provide an error estimate $\delta\phi_{(h)}$ for said value. Figure 22 shows this estimate as a solid purple line, which typically increases as the evolution advances. The total error budget is obtained as the sum in quadrature of both error estimates (shaded green area in the figure), $\delta\phi = (\delta\phi_{(h)}^2 + \delta\phi_{(r)}^2)^{1/2}$.

b. Grid setups comparisons

In the previous discussion, we presented the convergence of a simulation with three different simulations. Our goal is to select the most optimal grid configuration that both saves more computer time and still produces accurate results. We choose M8 to be our “base” configuration and in the following, we assess its accuracy by comparing its performance with higher resolutions and with more refinement levels.

Figure 23 compares three refinement levels for the BH at medium resolution. Our base configuration (M8) has clearly lower differences in amplitude and phase with the more refined grid of M9 than with the less refined M6, with an order of magnitude difference at merger among the two.

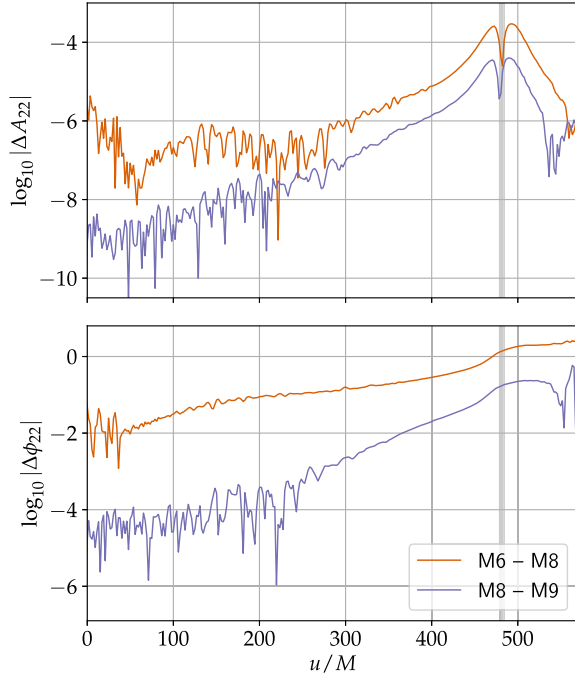


FIG. 23. Amplitude and phase comparisons between the different refinement levels on the BH for medium resolution. Differences between $L = 6$ and $L = 8$ (here, M6 and M8) are shown in orange, those between $L = 8$ and $L = 9$ (M8 and M9) are shown in purple. The gray area indicates the different merger times.

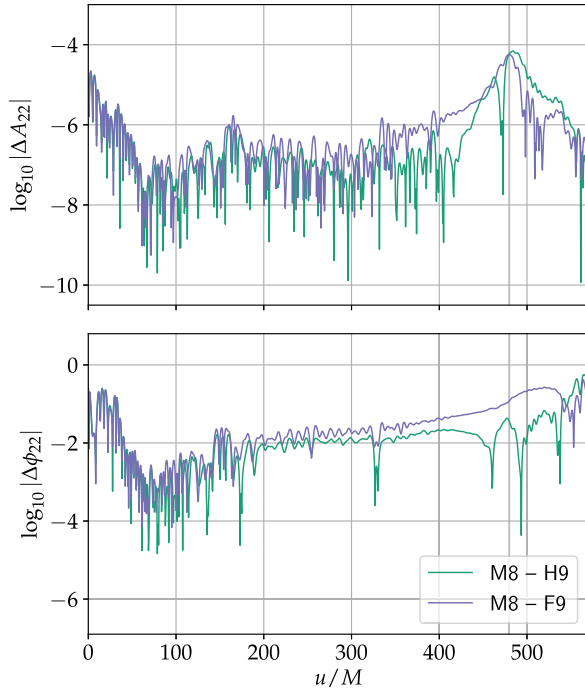


FIG. 24. Amplitude and phase differences between M8 and the two highest resolutions, H9 (green) and F9 (purple). The gray area indicates the different merger times.

If we now compare M8 with a higher resolution run with an extra refinement level, H9 (green line), we keep showing promising results as seen in Fig. 24. Overall, the inspiral, amplitude, and phase differences stay around $\sim 0.00001\%$ and $\sim 1\%$, respectively, with a phase difference at merger close to 1% , similarly as in the previous case with M9. Analogously, we show the same but comparing M8 with our highest simulation available, F9 (purple line). The amplitude differences show no considerable difference in comparison; however, we do see a slight increase of the phase difference at merger, reaching almost $\sim 10\%$.

In general, we show good performance for the M8 configuration, presenting small enough differences with the most refined and highest resolutions. We therefore select this grid configuration as our minimal setup to produce high quality waveforms with the least computer resources.

2. Remnant black hole

In this appendix, we present the fitting parameters for the updated remnant BH model. Table V shows the best coefficients obtained for $a_\bullet$ and $M_\bullet$ as a deviation from the BBH case.

3. Kick velocity

In the following, we describe the effects of resolution and integration errors on the computation of $v_{\text{kick}}^{\text{GW}}$. The coefficients of the kick velocity fits presented in Sec. III E are shown in Table VI.

a. Effects due to resolution

In this subsection, we discuss the effects of resolution on the measurement of the recoil velocity. Firstly, we focus on the different refinement levels for the BH considered in this work. Figure 25 shows the obtained kick velocity for BAM:0206 produced with the (H)igh resolution (see Table IV) and different refinement levels. We notice a considerable gap between the velocity computed employing six refinement levels (~ 5 km/s) with the others (~ 8 km/s), thus showcasing the importance of resolving the BH for accurate measurements of the remnant BH. On the other hand, choosing between eight or nine refinement levels does not seem to make a significant difference in the final kick velocity of the remnant.

Additionally, we also see effect that resolution has in Fig. 26 with eight refinement levels (top) and nine (bottom). For both figures, we see similar effects as were expected from our previous discussion. The configurations L8 ad L9 stand out from the rest as they are not part of the convergent series; however, their resulting $v_{\text{kick}}^{\text{GW}}$ lies within the values obtained with higher resolutions. From the figure, we notice that the recoil velocity tends to converge to a value of ~ 10 km/s. We remind the reader that for the results presented in this work, our chosen configuration for most simulations (with the exception of binaries with higher mass

TABLE V. Fitting parameters for $a_\bullet$ and $M_\bullet$ with $R^2 = 0.914$ and $R^2 = 0.930$, respectively. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

$\mathcal{F}$	k	c_{k12}	c_{k11}	c_{k10}	c_{k22}	c_{k21}	c_{k20}	c_{k32}	c_{k31}	c_{k30}
$a_\bullet$	1	1.4×10^{-3}	4.6×10^{-3}	9.1×10^{-4}	-5.0×10^{-2}	-2.3×10^{-2}	-2.5×10^{-2}	1.8×10^{-1}	1.5×10^{-2}	1.3×10^{-1}
	2	2.1×10^{-5}	-3.1×10^{-5}	8.1×10^{-6}	-1.7×10^{-4}	2.8×10^{-4}	-4.4×10^{-5}	3.9×10^{-4}	-6.3×10^{-4}	2.1×10^{-5}
	3	-1.6×10^{-8}	1.7×10^{-8}	-3.1×10^{-9}	1.5×10^{-7}	-1.6×10^{-7}	2.4×10^{-8}	-3.9×10^{-7}	4.0×10^{-7}	-2.5×10^{-8}
	4	7.9×10^{-7}	...	...	...	...	...	...	...	...
$M_\bullet$	1	-9.4×10^{-4}	1.7×10^{-3}	-5.9×10^{-4}	1.9×10^{-3}	-9.0×10^{-3}	4.2×10^{-3}	...	...	...
	2	1.6×10^{-7}	-1.5×10^{-6}	1.0×10^{-6}	9.4×10^{-7}	6.6×10^{-6}	4.0×10^{-5}	...	...	...
	3	4.7×10^{-5}	...	...	...	...	...	...	...	...

TABLE VI. Fitting parameters for $v_{\text{kick}}^{\text{GW}}$ yielding $R^2 = 0.926$.

Coefficient	$k = 1$	$k = 2$
c_{k13}	2.143	...
c_{k12}	0.096	...
c_{k11}	-0.820	...
c_{k10}	-0.051	...
c_{k23}	-11.668	...
c_{k22}	0.339	...
c_{k21}	4.109	1.272
c_{k20}	0.400	-2.297

ratios) is M8. It is therefore expected to obtain errors of $\sim 3\text{--}4$ km/s in the results presented in this paper according to what is shown in Fig. 26.

b. Reducing measurement error

As described in the main text, the linear momentum of the binary is not taken into account for earlier times when integrating the kick velocity vector. This effect adds a substantial amount of error on the measurement of $v_{\text{kick}}^{\text{GW}}$, which can be reduced by estimating an adequate integration constant, i.e.,

$$v = v_0 - \frac{1}{M} \int_{t_0}^t (\dot{P}_x + i\dot{P}_y) dt. \quad (\text{A2})$$

We obtain v_0 by means of a hodograph as shown in the left panel of Fig. 27, where we want to shift the center of the spiral to the origin. The resulting correction is presented in the right panel, showing a more monotonic increase of the recoil velocity. There is a significant difference of 9% between the integration done from t_0 ($v_0 = 0$) and the one accounting for $v_0 \neq 0$. This procedure is thus applied to all of our simulations.

4. Long simulation accuracy

In Sec. V, we compared our model to a new 12-orbit precessing simulation, BAM:0223, performed in three different resolutions: M8, H8, and F8 (see Table IV). We inspect the self-convergence of the configuration in

Fig. 28, where we show the amplitude and phase differences between the available datasets. The amplitude differences for all resolutions stay well below 10^{-5} before reaching merger, whereas the phase differences lie below 1 rad up until $\sim 2000M$ for the finest grids and until $\sim 1700M$ for the coarser ones. The medium and high resolutions show a clear $\sim$ third-order convergence throughout all evolution. Although computationally expensive, additional simulations with even higher resolution would help us determine the convergence behavior of the highest resolutions presented here.

Here, we present the total phase error budget on the phase, as a crucial quantity sensitive to numerical errors. This error needs to be considered when employing this simulation for calibration or validation of a model. We use the same strategy as in Appendix A 1 a; see Fig. 29. Furthermore, we show two different Richardson extrapolated datasets, one including all resolutions and another only employing the two highest ones. As seen in the figure, we find the most optimal dataset to be $\mathcal{R}(M, F)$, resulting in a significant decrease in the truncation errors. To account for finite extraction radii, we extrapolate to infinity employing a $(1/R)^K$ polynomial with $K = 3$, since other K orders resulted in high phase variations over time. Nevertheless, although errors due to resolution slowly increase over simulation time and dominate briefly before merger, we notice that extrapolation errors are the ones that contribute the most to the total phase error throughout most of the evolution.

APPENDIX B: WAVEFORM FIT MODELS

In this appendix, we discuss the fitting models developed for the waveform model presented in this work for each individual mode. Appendix B 1 collects the fitting functions and coefficients for all the quantities related to building the ringdown waveform, i.e., $(A_{\ell m}^{\text{peak}}, \omega_{\ell m}^{\text{peak}}, \alpha_{\ell m1}, \omega_{\ell m1})$, and in Appendix B 2, one can find the models for all the parameters needed for the NQC, $(A_{\ell m}^{\text{NQC}}, \dot{A}_{\ell m}^{\text{NQC}}, \omega_{\ell m}^{\text{NQC}}, \dot{\omega}_{\ell m}^{\text{NQC}})$.

Due to the low quantity of higher modes data, some fits present higher-order terms. In order to mitigate extreme

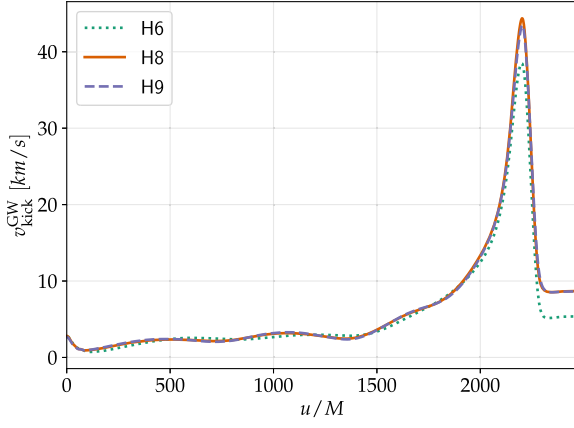


FIG. 25. Computed recoil velocity of BAM:0206 using the (H)igh resolution and different refinement levels for the BH.

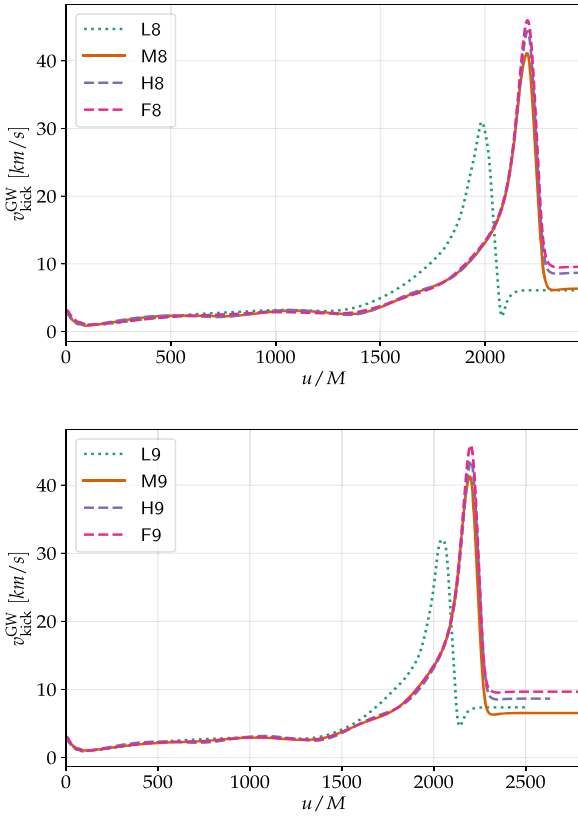


FIG. 26. Same as Fig. 25 but for different resolutions and employing eight (top) and nine (bottom) refinement levels on the BH.

values, we include artificial BBH data in the relevant regions of the parameter space and ensure a physical behavior of the fits. Results out of the data sample region could reach up to a 10% error.

1. Ringdown model parameters

Here, we present the fit models and corresponding coefficients for the quantities needed to build the ringdown model

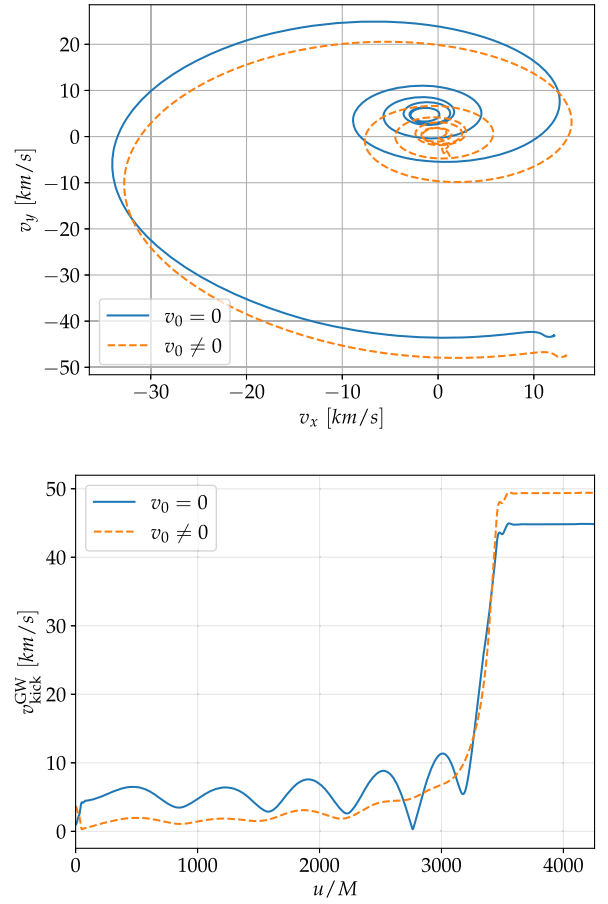


FIG. 27. Left: Hodograph of the component recoil velocities v_x and v_y . Right: Evolution in time of the kick velocity magnitude. The blue lines ($v_0 = 0$) are computed by integrating from a finite t_0 , and the orange ones by accounting for a nonzero integration constant ($v_0 \neq 0$); see text.

described in Sec. IV A. All fitting models for each quantity $\mathcal{F}$, are based on a deviation from the BBH fits from Ref. [89] as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$ and modeled using a Padé approximant function with dependence on masses, spins, and tides. For the multipolar amplitude peaks, we employ the rescaling shown in Eq. (11) (see Sec. III D) and represented as $\hat{A}_{\ell m}$.

The ringdown part of the waveform requires for each multipole the quantities $(A_{\ell m}^{\text{peak}}, \omega_{\ell m}^{\text{peak}}, \alpha_{\ell m 1}, \omega_{\ell m 1})$ extracted from NR data. We evaluate the reliability of our fits by computing the coefficient of determination R^2 , which we report below for every quantity.

$$\mathbf{a.} \quad (\ell, m) = (2, 2)$$

We first develop the fits for the dominant (2,2) mode and model the peak of the amplitude as

$$\hat{A}_{22}^{\text{BHNS}}/\hat{A}_{22}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{1 + \Lambda p_3^{(2)}(\nu)}, \quad (\text{B1})$$

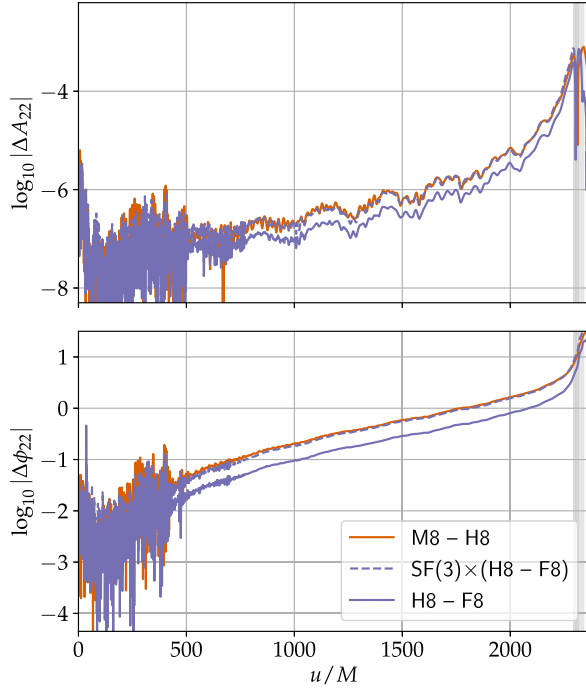


FIG. 28. Same as Fig. 21 but considering the M8, H8, and F8 settings of BAM:0223. The vertical gray area shows the merger time differences among the three resolutions.

with the following polynomials:

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2 + p_{k3}^{(3)}(a_{\text{BH}})\nu^3, \quad (\text{B2a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B2b})$$

$$p_3^{(2)}(\nu) = c_{311}\nu + c_{312}\nu^2. \quad (\text{B2c})$$

With this fitting model for $\hat{A}_{22}^{\text{BHNS}}/\hat{A}_{22}^{\text{BBH}}$, we achieve $R^2 = 0.97$.

Next, we extract the frequency at merger from our numerical data and use the form

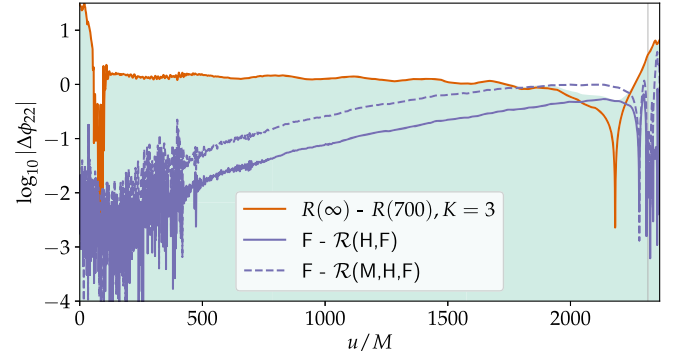


FIG. 29. Same as Fig. 22 but considering the M8, H8, and F8 settings of BAM:0223 used for validation of the model. The vertical gray line indicated the merger time difference between the extrapolated $\mathcal{R}(H, F)$ and the finest resolution.

$$\hat{\omega}_{22}^{\text{BHNS}}/\hat{\omega}_{22}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{(1 + \Lambda p_3^{(2)}(\nu))^2}, \quad (\text{B3})$$

to model it using

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B4a})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B4b})$$

$$p_3^{(2)}(\nu) = c_{311}\nu^2, \quad (\text{B4c})$$

thus also reaching $R^2 = 0.97$.

To model the characteristic ringdown form, we need to fit the inverse damping time [Eq. (B5)] and the frequency [Eq. (B7)] of the QNM. For our $\alpha_{221}^{\text{BHNS}}/\alpha_{221}^{\text{BBH}}$ model, we obtain a $R^2 = 0.95$, whereas for the QNM frequency the coefficient of determination is $R^2 = 0.98$, as follows:

$$\frac{\alpha_{221}^{\text{BHNS}}}{\alpha_{221}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(2)}(\nu, a_{\text{BH}})}{1 + \Lambda p_{41}^{(1)}(a_{\text{BH}})\nu}, \quad (\text{B5})$$

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(1)}(a_{\text{BH}})\nu + p_{k2}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B6a})$$

$$p_{kj}^{(1)}(a_{\text{BH}}) = c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B6b})$$

$$\omega_{221}^{\text{BHNS}}/\omega_{221}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{[1 + \Lambda(p_3^{(2)}(\nu, a_{\text{BH}}))^2]^2} \quad (\text{B7})$$

TABLE VII. Fitting coefficients for ringdown paramters from the (2,2) waveform. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

$\mathcal{F}$	k	c_{k12}	c_{k11}	c_{k10}	c_{k22}	c_{k21}	c_{k20}	c_{k32}	c_{k31}	c_{k30}
$\hat{A}_{22}$	1	2.5×10^{-2}	1.1×10^{-2}	-2.5×10^{-3}	-3.1×10^{-1}	-1.3×10^{-1}	1.4×10^{-1}	8.7×10^{-1}	3.1×10^{-1}	-5.1×10^{-1}
	2	-1.9×10^{-5}	-2.8×10^{-6}	7.7×10^{-6}	2.2×10^{-4}	2.5×10^{-5}	-9.7×10^{-5}	-5.8×10^{-4}	-5.1×10^{-5}	2.8×10^{-4}
	3	-1.6×10^{-2}	8.7×10^{-3}	...	...	...	...	...	...	...
$\hat{\omega}_{22}$	1	1.1×10^{-1}	3.1×10^{-2}	-6.2×10^{-2}	-1.5×10^0	-2.3×10^{-1}	9.0×10^{-1}	4.4×10^0	2.3×10^{-1}	-2.4×10^0
	2	-1.3×10^{-4}	4.7×10^{-5}	3.1×10^{-5}	1.6×10^{-3}	-6.8×10^{-4}	-3.1×10^{-4}	-4.4×10^{-3}	2.1×10^{-3}	9.1×10^{-4}
	3	...	5.1×10^{-2}	...	...	...	...	...	...	...
α_{221}	1	...	7.0×10^{-2}	7.1×10^{-3}	...	-2.9×10^{-1}	7.2×10^{-2}	...	...	...
	2	...	-5.2×10^{-5}	5.0×10^{-5}	...	1.8×10^{-4}	-2.4×10^{-4}	...	...	...
	3	...	6.4×10^{-9}	-1.1×10^{-8}	...	-1.2×10^{-8}	5.2×10^{-8}	...	...	...
	4	...	-2.7×10^{-3}	5.2×10^{-2}	...	...	...	...	...	...
ω_{221}	1	...	-4.8×10^6	4.0×10^6	...	1.5×10^7	-1.2×10^7	...	...	...
	2	...	6.3×10^3	-4.3×10^3	...	-2.8×10^4	2.0×10^4	...	...	...
	3	...	6.2×10^1	-1.6×10^1	...	-3.0×10^2	1.8×10^2	...	...	...

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(1)}(a_{\text{BH}})\nu + p_{k2}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B8a}) \quad \text{with the polynomials}$$

$$p_{kj}^{(1)}(a_{\text{BH}}) = c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B8b})$$

The coefficients for the (2,2) mode fit models are shown in Table VII.

b. $(\ell, m) = (2, 1)$

Contrary to the dominant mode, we employ the BBH fits from Ref. [121] to model the amplitude and frequency at merger for the (2,1) mode (see Sec. IV A). The amplitude peak is fitted in the form

$$\hat{A}_{21}^{\text{BHNS}}/\hat{A}_{21}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{1 + \Lambda(p_{31}^{(1)}(a_{\text{BH}})\nu)^2}, \quad (\text{B9})$$

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B10a})$$

$$p_{31}^{(1)}(a_{\text{BH}}) = c_{311}a_{\text{BH}} + c_{310}, \quad (\text{B10b})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B10c})$$

thus, obtaining a determination coefficient of $R^2 = 0.94$. On the other hand, for the frequency at merger, we get $R^2 = 0.93$ employing the model

$$\frac{\hat{\omega}_{21}^{\text{BHNS}}}{\hat{\omega}_{21}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(3)}(\nu, a_{\text{BH}})}{1 + \Lambda p_4^{(2)}(\nu, a_{\text{BH}})}, \quad (\text{B11})$$

with the following expressions:

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2 + p_{k3}^{(3)}(a_{\text{BH}})\nu^3, \quad (\text{B12a})$$

$$p_4^{(2)}(\nu, a_{\text{BH}}) = p_{41}^{(2)}(a_{\text{BH}})\nu + p_{42}^{(2)}(a_{\text{BH}})\nu^2, \quad (\text{B12b})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{4j1}a_{\text{BH}} + c_{4j0}. \quad (\text{B12c})$$

The QNM quantities α_{211} and ω_{211} are modeled with Eqs. (B13) and (B15), respectively, as follows:

$$\alpha_{211}^{\text{BHNS}}/\alpha_{211}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{[1 + \Lambda(p_3^{(2)}(\nu, a_{\text{BH}}))^2]^2}, \quad (\text{B13})$$

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2, \quad (\text{B14a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B14b})$$

$$p_3^{(2)}(\nu, a_{\text{BH}}) = p_{31}^{(1)}(a_{\text{BH}})\nu + p_{32}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B14c})$$

TABLE VIII. Fitting coefficients for ringdown parameters from the (2,1) waveform. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

$\mathcal{F}$	k	c_{k12}	c_{k11}	c_{k10}	c_{k22}	c_{k21}	c_{k20}	c_{k32}	c_{k31}	c_{k30}
$\hat{A}_{21}$	1	8.0×10^1	-6.4×10^1	4.8×10^2	1.4×10^4	-1.7×10^4	1.9×10^2	-4.7×10^4	5.6×10^4	-3.7×10^3
	2	-3.7×10^0	4.1×10^0	-1.8×10^0	4.5×10^1	-4.2×10^1	1.6×10^1	-1.4×10^2	1.1×10^2	-3.7×10^1
	3	...	-7.2×10^1	4.4×10^1	...	...	...	...	...	...
$\hat{\omega}_{21}$	1	...	-1.6×10^7	2.9×10^7	...	1.9×10^8	-3.0×10^8	...	-5.6×10^8	7.8×10^8
	2	...	6.6×10^4	-2.9×10^4	...	-7.2×10^5	2.0×10^5	...	1.9×10^6	-2.0×10^5
	3	...	-8.0×10^1	4.2×10^1	...	8.7×10^2	-4.1×10^2	...	-2.3×10^3	9.9×10^2
	4	...	3.5×10^1	-2.1×10^1	...	-1.7×10^2	2.3×10^2	...	...	...
		c_{k13}	c_{k12}	c_{k11}	c_{k10}	c_{k23}	c_{k22}	c_{k21}	c_{k20}	
α_{211}	1	1.0×10^0	-6.6×10^{-2}	-1.8×10^{-1}	-4.1×10^{-2}	-4.8×10^0	6.4×10^{-1}	1.2×10^0	2.5×10^{-1}	
	2	-6.5×10^{-4}	2.0×10^{-4}	4.0×10^{-5}	-1.5×10^{-5}	3.6×10^{-3}	-3.7×10^{-4}	5.6×10^{-5}	1.1×10^{-4}	
	3	...	...	8.3×10^{-1}	7.7×10^{-2}	...	...	-2.4×10^0	4.9×10^{-1}	
ω_{211}	1	3.3×10^0	1.7×10^0	6.4×10^{-1}	1.5×10^{-1}	-1.7×10^1	-9.6×10^0	-4.0×10^0	-9.2×10^{-1}	
	2	-4.3×10^{-3}	-6.7×10^{-3}	-4.1×10^{-3}	-6.3×10^{-4}	2.5×10^{-2}	3.9×10^{-2}	2.3×10^{-2}	3.8×10^{-3}	
	3	...	...	-3.7×10^{-2}	-7.5×10^{-3}	...	...	2.6×10^{-1}	9.2×10^{-2}	

$$p_{3j}^{(1)}(a_{\text{BH}}) = c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B14d})$$

$$p_3^{(2)}(\nu, a_{\text{BH}}) = p_{31}^{(2)}(a_{\text{BH}})\nu + p_{32}^{(2)}(a_{\text{BH}})\nu^2, \quad (\text{B16c})$$

$$\omega_{211}^{\text{BHNS}}/\omega_{211}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{[1 + \Lambda^2(p_3^{(3)}(\nu, a_{\text{BH}}))^2]^2}, \quad (\text{B15})$$

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(4)}(a_{\text{BH}})\nu + p_{k2}^{(4)}(a_{\text{BH}})\nu^2, \quad (\text{B16a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B16b})$$

For $\alpha_{211}^{\text{BHNS}}/\alpha_{211}^{\text{BBH}}$, we get $R^2 = 0.90$, and for the QNM frequency rational between the BHNS and BBH cases, one obtains $R^2 = 0.98$. All the corresponding coefficients for the (2,1) mode models are reported in Table VIII.

c. $(\ell, m) = (3, 2)$

Similarly to the dominant (2,2), we fit the amplitude peak of the (3,2) mode using the expression

$$\frac{\hat{A}_{32}^{\text{BHNS}}}{\hat{A}_{32}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(3)}(\nu, a_{\text{BH}})}{1 + \Lambda(p_4^{(2)}(\nu, a_{\text{BH}}))^2}, \quad (\text{B17})$$

with

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B18a})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B18b})$$

$$p_4^{(2)}(\nu, a_{\text{BH}}) = p_{41}^{(1)}(a_{\text{BH}})\nu + p_{42}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B18c})$$

$$p_{4j}^{(1)}(a_{\text{BH}}) = c_{4j1}a_{\text{BH}} + c_{4j0}. \quad (\text{B18d})$$

The frequency at merger is instead modeled with

$$\hat{\omega}_{32}^{\text{BHNS}}/\hat{\omega}_{32}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{1 + \Lambda(p_3^{(2)}(\nu, a_{\text{BH}}))^2}, \quad (\text{B19})$$

where the polynomials are expressed as

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(1)}(a_{\text{BH}})\nu + p_{k2}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B20a})$$

$$p_{kj}^{(1)}(a_{\text{BH}}) = c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B20b})$$

TABLE IX. Fitting coefficients for ringdown paramters from the (3,2) waveform. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

$\mathcal{F}$	k	c_{k12}	c_{k11}	c_{k10}	c_{k22}	c_{k21}	c_{k20}	c_{k32}	c_{k31}	c_{k30}
$\hat{A}_{32}$	1	1.6×10^1	6.4×10^0	-4.0×10^0	-1.6×10^2	-6.8×10^1	4.0×10^1	4.2×10^2	1.9×10^2	-9.8×10^1
	2	-1.3×10^{-2}	-5.0×10^{-3}	5.1×10^{-3}	1.2×10^{-1}	4.5×10^{-2}	-5.0×10^{-2}	-2.6×10^{-1}	-1.0×10^{-1}	1.2×10^{-1}
	3	2.2×10^{-6}	6.0×10^{-7}	-1.1×10^{-6}	-1.1×10^{-5}	-3.0×10^{-7}	9.9×10^{-6}	2.8×10^{-6}	-1.1×10^{-5}	-2.2×10^{-5}
	4	$\cdots$	2.8×10^0	1.9×10^{-1}	$\cdots$	-1.9×10^1	-4.3×10^0	$\cdots$	$\cdots$	$\cdots$
$\hat{\omega}_{32}$	1	$\cdots$	-1.9×10^{-1}	1.7×10^{-1}	$\cdots$	8.0×10^{-1}	-7.4×10^{-1}	$\cdots$	$\cdots$	$\cdots$
	2	$\cdots$	3.5×10^{-5}	-4.5×10^{-5}	$\cdots$	-1.5×10^{-4}	2.0×10^{-4}	$\cdots$	$\cdots$	$\cdots$
	3	$\cdots$	1.1×10^0	-1.2×10^0	$\cdots$	-4.4×10^0	4.3×10^0	$\cdots$	$\cdots$	$\cdots$
α_{321}	1	$\cdots$	$\cdots$	$\cdots$	1.4×10^1	1.1×10^1	-4.0×10^0	-6.1×10^1	-5.1×10^1	1.8×10^1
	2	$\cdots$	$\cdots$	$\cdots$	-6.6×10^{-2}	-5.2×10^{-2}	2.1×10^{-2}	3.0×10^{-1}	2.4×10^{-1}	-9.6×10^{-2}
	3	$\cdots$	$\cdots$	$\cdots$	5.5×10^{-5}	4.2×10^{-5}	-1.6×10^{-5}	-2.5×10^{-4}	-2.0×10^{-4}	7.4×10^{-5}
	4	$\cdots$	2.4×10^{-1}	7.7×10^{-1}	$\cdots$	-3.9×10^{-1}	-3.0×10^0	$\cdots$	$\cdots$	$\cdots$
ω_{321}	1	$\cdots$	2.7×10^{-2}	-2.9×10^{-2}	$\cdots$	-1.6×10^{-1}	1.0×10^{-1}	$\cdots$	$\cdots$	$\cdots$
	2	$\cdots$	4.4×10^{-6}	5.8×10^{-5}	$\cdots$	2.7×10^{-5}	-2.4×10^{-4}	$\cdots$	$\cdots$	$\cdots$
	3	$\cdots$	-2.8×10^{-8}	-3.0×10^{-8}	$\cdots$	1.2×10^{-7}	1.3×10^{-7}	$\cdots$	$\cdots$	$\cdots$
	4	$\cdots$	$\cdots$	$\cdots$	$\cdots$	5.3×10^{-1}	6.1×10^{-1}	$\cdots$	$\cdots$	$\cdots$
$\Delta t_{32}^{\text{BHNS}}$		c_{k13}	c_{k12}	c_{k11}	c_{k10}					
	1	-1.5×10^0	-1.5×10^0	1.2×10^{-1}	2.8×10^{-1}					
	2	8.3×10^{-6}	2.0×10^{-5}	1.4×10^{-5}	2.9×10^{-6}					
	3	$\cdots$	$\cdots$	-1.4×10^{-1}	-1.3×10^{-1}					
		c_{k23}	c_{k22}	c_{k21}	c_{k20}					
	1	1.4×10^1	1.4×10^1	-1.9×10^0	-2.9×10^0					
	2	-9.9×10^{-5}	-2.3×10^{-4}	-1.6×10^{-4}	-3.4×10^{-5}					
	3	$\cdots$	$\cdots$	1.1×10^0	8.9×10^{-1}					
		c_{k33}	c_{k32}	c_{k31}	c_{k30}					
	1	-3.3×10^1	-2.9×10^1	7.1×10^0	7.7×10^0					
	2	3.0×10^{-4}	6.9×10^{-4}	4.7×10^{-4}	1.0×10^{-4}					

The quality of the amplitude and frequency peak fits is proved by the coefficients of determination $R^2 = 0.92$ and $R^2 = 0.91$, respectively. However, given the high-order nature of the resulting amplitude fit, we perform an out-of-sample test to discard overfitting. Employing a simulation not used for calibration, BHNS data on Fig. 1 of Ref. [104], we obtain a relative error of 9.92%.

The QNM inverse damping time of the (3,2) mode is fitted with a template in the form

$$\frac{\alpha_{321}^{\text{BHNS}}}{\alpha_{321}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(3)}(\nu, a_{\text{BH}})}{[1 + \Lambda(p_4^{(2)}(\nu, a_{\text{BH}}))^2]^2}, \quad (\text{B21})$$

using the polynomials

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k2}^{(3)}(a_{\text{BH}})\nu^2 + p_{k3}^{(3)}(a_{\text{BH}})\nu^3, \quad (\text{B22a})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B22b})$$

$$p_4^{(2)}(\nu, a_{\text{BH}}) = p_{41}^{(1)}(a_{\text{BH}})\nu + p_{42}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B22c})$$

$$p_{4j}^{(1)}(a_{\text{BH}}) = c_{4j1}a_{\text{BH}} + c_{4j0}, \quad (\text{B22d})$$

which give $R^2 = 0.97$. For the QNM frequency, we obtain as well $R^2 = 0.97$, employing the expression

$$\frac{\omega_{321}^{\text{BHNS}}}{\omega_{321}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(2)}(\nu, a_{\text{BH}})}{1 + \Lambda(p_{42}^{(1)}(a_{\text{BH}})\nu^2)^2}, \quad (\text{B23})$$

with

$$p_{3j}^{(1)}(a_{\text{BH}}) = c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B28d})$$

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(1)}(a_{\text{BH}})\nu + p_{k2}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B24a})$$

$$p_{kj}^{(1)}(a_{\text{BH}}) = c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B24b})$$

Considering that we make use of the fits from Ref. [121] for (2,1), (3,3), and (4,4), where the matching time is set to be the amplitude peak of (2,2), we only need to fit the time shift $\Delta t_{\ell m}$ for the (3,2) mode. We do this with the model

$$\Delta t_{32}^{\text{BHNS}}/\Delta t_{32}^{\text{BBH}} = \frac{1 + \Lambda^2 p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^4 p_2^{(3)}(\nu, a_{\text{BH}})}{(1 + \Lambda^2 (p_3^{(2)}(\nu, a_{\text{BH}}))^2)^2}, \quad (\text{B25})$$

with $R^2 = 0.91$, and the polynomials are expressed as

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2 + p_{k3}^{(3)}(a_{\text{BH}})\nu^3, \quad (\text{B26a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B26b})$$

$$p_3^{(2)}(\nu, a_{\text{BH}}) = p_{31}^{(1)}(a_{\text{BH}})\nu + p_{32}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B26c})$$

$$p_{3j}^{(1)}(a_{\text{BH}}) = c_{3j1}a_{\text{BH}} + c_{3j0}. \quad (\text{B26d})$$

Table IX presents the coefficients of all the fit models made for the (3,2) mode.

d. $(\ell, m) = (3, 3)$

Equations (B27) and (B29) below with their corresponding polynomials are used as templates to fit the amplitude and frequency at merger of the (3,3) mode, as follows:

$$\hat{A}_{33}^{\text{BHNS}}/\hat{A}_{33}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^3 p_2^{(3)}(\nu, a_{\text{BH}})}{[1 + \Lambda^2 (p_3^{(2)}(\nu, a_{\text{BH}}))^2]^2} \quad (\text{B27})$$

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(4)}(a_{\text{BH}})\nu + p_{k2}^{(4)}(a_{\text{BH}})\nu^2 + p_{k3}^{(4)}(a_{\text{BH}})\nu^3, \quad (\text{B28a})$$

$$p_{kj}^{(4)}(a_{\text{BH}}) = c_{kj4}a_{\text{BH}}^4 + c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B28b})$$

$$p_3^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(1)}(a_{\text{BH}})\nu + p_{k2}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B28c})$$

$$\hat{\omega}_{33}^{\text{BHNS}}/\hat{\omega}_{33}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{[1 + \Lambda^2 (p_3^{(3)}(\nu))^2]}, \quad (\text{B29})$$

with

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(1)}(a_{\text{BH}})\nu + p_{k2}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B30a})$$

$$p_{kj}^{(1)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B30b})$$

$$p_3^{(2)}(\nu) = c_{310}\nu + c_{320}\nu^2. \quad (\text{B30c})$$

The fit of these two quantities results in determination coefficients of $R^2 = 0.89$ and $R^2 = 0.91$, respectively.

To model the BBH to BHNS deviation of the QNM's inverse damping time, we use the template

$$\alpha_{331}^{\text{BHNS}}/\alpha_{331}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{(1 + \Lambda (p_{31}^{(1)}(a_{\text{BH}})\nu)^2)}, \quad (\text{B31})$$

with the polynomials

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2, \quad (\text{B32a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B32b})$$

$$p_{31}^{(1)}(a_{\text{BH}}) = c_{311}a_{\text{BH}} + c_{310}. \quad (\text{B32c})$$

For the QNM frequency of the (3,3) mode, we employ instead

$$\omega_{331}^{\text{BHNS}}/\omega_{331}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{1 + \Lambda^2 (p_{32}^{(1)}(a_{\text{BH}})\nu^2)^2}, \quad (\text{B33})$$

with

$$p_1^{(2)}(\nu, a_{\text{BH}}) = p_{11}^{(1)}(a_{\text{BH}})\nu + p_{12}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B34a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B34b})$$

TABLE X. Fitting coefficients for ringdown paramters from the (3,3) waveform. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

$\hat{A}_{33}$	k	c_{k14}	c_{k13}	c_{k12}	c_{k11}	c_{k10}					
	1	9.9×10^1	2.3×10^1	-2.9×10^1	1.5×10^0	1.5×10^{-1}					
	2	-1.0×10^3	-2.5×10^2	3.0×10^2	-1.4×10^1	-1.4×10^0					
	3	...	...	...	-5.4×10^{-2}	9.6×10^{-3}					
							c_{k24}	c_{k23}	c_{k22}	c_{k21}	c_{k20}
	1	2.6×10^3	6.6×10^2	-7.7×10^2	3.5×10^1	3.2×10^0					
	2	-2.3×10^{-4}	-8.6×10^{-5}	7.1×10^{-5}	1.3×10^{-5}	6.8×10^{-7}					
	3	...	...	...	2.9×10^{-1}	9.8×10^{-3}					
							c_{k34}	c_{k33}	c_{k32}	c_{k31}	c_{k30}
	1	2.4×10^{-3}	8.9×10^{-4}	-7.5×10^{-4}	-1.5×10^{-4}	-6.7×10^{-6}					
	2	-6.2×10^{-3}	-2.3×10^{-3}	2.0×10^{-3}	4.2×10^{-4}	1.9×10^{-5}					
$\mathcal{F}$	k	c_{k13}	c_{k12}	c_{k11}	c_{k10}	c_{k23}	c_{k22}	c_{k21}	c_{k20}		
$\hat{\omega}_{33}$	1	-1.8×10^{-1}	-6.3×10^{-2}	3.9×10^{-2}	-1.3×10^{-2}	8.6×10^{-1}	3.1×10^{-1}	-1.9×10^{-1}	-2.2×10^{-4}		
	2	2.6×10^{-4}	7.3×10^{-5}	-4.7×10^{-5}	7.8×10^{-5}	-1.3×10^{-3}	-3.9×10^{-4}	2.2×10^{-4}	-2.9×10^{-4}		
	3	...	...	3.2×10^{-5}	-1.2×10^{-4}	...	...	...	...		
α_{331}	1	1.6×10^0	9.1×10^{-1}	1.1×10^{-1}	-3.4×10^{-2}	-7.7×10^0	-1.7×10^0	1.1×10^0	3.6×10^{-1}		
	2	-1.6×10^{-3}	9.1×10^{-5}	3.8×10^{-4}	9.3×10^{-5}	7.1×10^{-3}	-1.4×10^{-3}	-2.1×10^{-3}	-4.3×10^{-4}		
	3	...	...	2.1×10^0	6.6×10^{-1}	...	...	...	...		
ω_{331}	1	-2.0×10^{-2}	-2.4×10^{-2}	2.7×10^{-2}	8.1×10^{-3}	-7.6×10^{-1}	-1.8×10^{-1}	4.6×10^{-1}	2.1×10^{-1}		
	2	4.5×10^{-6}	5.3×10^{-6}	-8.2×10^{-6}	-2.8×10^{-6}	-1.4×10^{-4}	1.8×10^{-3}	2.4×10^{-3}	7.3×10^{-4}		
	3	...	...	3.9×10^{-1}	2.3×10^{-1}	...	...	...	...		

$$p_{32}^{(1)}(a_{\text{BH}}) = c_{321}a_{\text{BH}} + c_{320}. \quad (\text{B34c})$$

With these QNM fit models, we obtain $R^2 = 0.90$ and $R^2 = 0.95$, respectively. The fitting paramaters for all the (3,3) mode models are reported in Table X.

e. $(\ell, m) = (4, 4)$

The amplitude peak of the (4,4) mode is obtained by fitting the following expression:

$$\frac{\hat{A}_{44}^{\text{BHNS}}}{\hat{A}_{44}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(3)}(\nu, a_{\text{BH}})}{1 + \Lambda(p_3^{(2)}(\nu, a_{\text{BH}}))^2}, \quad (\text{B35})$$

with the polynomials

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B36a})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B36b})$$

$$p_4^{(2)}(\nu, a_{\text{BH}}) = p_{41}^{(1)}(a_{\text{BH}})\nu + p_{42}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B36c})$$

$$p_{4j}^{(1)}(a_{\text{BH}}) = c_{4j1}a_{\text{BH}} + c_{4j0}. \quad (\text{B36d})$$

Using this template to fit our model, we obtain a determination coefficient of $R^2 = 0.93$.

For the frequency at merger, we employ a template of the form

$$\hat{\omega}_{44}^{\text{BHNS}}/\hat{\omega}_{44}^{\text{BBH}} = \frac{1 + \Lambda(p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda p_2^{(2)}(\nu, a_{\text{BH}}))}{1 + \Lambda(p_3^{(2)}(\nu, a_{\text{BH}}))^2}, \quad (\text{B37})$$

where

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2, \quad (\text{B38a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B38b})$$

$$p_3^{(2)}(\nu, a_{\text{BH}}) = p_{31}^{(1)}(a_{\text{BH}})\nu + p_{32}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B38c})$$

$$p_{3j}^{(1)}(a_{\text{BH}}) = c_{3j1}a_{\text{BH}} + c_{3j0}. \quad (\text{B38d})$$

TABLE XI. Fitting coefficients for ringdown paramters from the (4,4) waveform. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

$\mathcal{F}$	k	c_{k12}	c_{k11}	c_{k10}	c_{k22}	c_{k21}	c_{k20}	c_{k32}	c_{k31}	c_{k30}
$\hat{A}_{44}$	1	3.5×10^3	3.3×10^3	8.1×10^2	-4.1×10^4	-3.7×10^4	-8.6×10^3	1.2×10^5	1.1×10^5	2.4×10^4
	2	2.6×10^{-1}	-5.6×10^0	-2.7×10^0	-5.5×10^0	5.7×10^1	2.8×10^1	1.9×10^1	-1.4×10^2	-7.1×10^1
	3	-1.6×10^{-3}	2.3×10^{-3}	1.8×10^{-3}	1.9×10^{-2}	-2.2×10^{-2}	-1.8×10^{-2}	-5.5×10^{-2}	5.4×10^{-2}	4.7×10^{-2}
	4	...	-1.2×10^2	-1.0×10^1	...	7.4×10^2	1.3×10^2	...	...	...
$\hat{\omega}_{44}$		c_{k13}	c_{k12}	c_{k11}	c_{k10}	c_{k23}	c_{k22}	c_{k21}	c_{k20}	
	1	-4.8×10^{-1}	-2.7×10^{-2}	7.4×10^{-1}	3.7×10^{-1}	2.2×10^0	5.3×10^{-1}	-2.8×10^0	-1.5×10^0	
	2	7.5×10^{-5}	9.0×10^{-5}	-7.8×10^{-5}	-3.1×10^{-5}	-5.9×10^{-4}	-8.8×10^{-4}	8.4×10^{-5}	8.2×10^{-5}	
	3	...	...	2.3×10^0	2.5×10^0	...	...	-6.3×10^0	-7.5×10^0	
α_{441}		c_{k12}	c_{k11}	c_{k10}	c_{k22}	c_{k21}	c_{k20}	c_{k32}	c_{k31}	c_{k30}
	1	1.1×10^1	-1.7×10^1	2.7×10^0	-9.8×10^1	1.8×10^2	-2.6×10^1	2.2×10^2	-4.5×10^2	6.2×10^1
	2	8.8×10^{-2}	-1.2×10^{-2}	1.9×10^{-2}	-8.9×10^{-1}	7.2×10^{-2}	-1.7×10^{-1}	2.3×10^0	-7.5×10^{-2}	3.9×10^{-1}
	3	...	-3.4×10^0	3.7×10^0	...	1.8×10^1	-1.7×10^1	...	...	...
ω_{441}	1	...	-8.3×10^{-1}	-3.7×10^{-1}	...	8.4×10^0	3.8×10^0	...	-2.1×10^1	-9.8×10^0
	2	...	8.9×10^{-4}	4.9×10^{-4}	...	-9.3×10^{-3}	-5.0×10^{-3}	...	2.4×10^{-2}	1.3×10^{-2}
	3	...	...	...	...	-6.5×10^{-2}	6.5×10^{-1}	...	...	...

We obtain $R^2 = 0.94$ as a measure of the fit model's performance.

Finally, the QNM quantities α_{441} and ω_{441} are modeled as Eqs. (B39) and (B41) below with their respective polynomials:

$$\alpha_{441}^{\text{BHNS}}/\alpha_{441}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{[1 + \Lambda(p_3^{(2)}(\nu, a_{\text{BH}}))^2]^2}, \quad (\text{B39})$$

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B40a})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B40b})$$

$$p_3^{(2)}(\nu, a_{\text{BH}}) = p_{31}^{(1)}(a_{\text{BH}})\nu + p_{32}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B40c})$$

$$p_{3j}^{(1)}(a_{\text{BH}}) = c_{3j1}a_{\text{BH}} + c_{3j0}. \quad (\text{B40d})$$

$$\omega_{441}^{\text{BHNS}}/\omega_{441}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{[1 + \Lambda(p_{32}^{(1)}(a_{\text{BH}})\nu^2)^2]^2} \quad (\text{B41})$$

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B42a})$$

$$p_{kj}^{(1)}(a_{\text{BH}}) = c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B42b})$$

These result in a coefficient of determination of $R^2 = 0.95$ and $R^2 = 0.99$, correspondingly. The coefficients for all

the fitted quantities of the (4,4) mode are presented in Table XI.

2. NQC extraction points

In this subsection, we present the fits developed for the NQC extraction points for all multipoles available [not available for the (3,2) mode], namely the quantities ($A_{\ell m}^{\text{NQC}}, \dot{A}_{\ell m}^{\text{NQC}}, \omega_{\ell m}^{\text{NQC}}, \dot{\omega}_{\ell m}^{\text{NQC}}$); see Sec. IV B for more details on how these fits are used. Note that both the amplitude $A_{\ell m}^{\text{NQC}}$ and its time derivative $\dot{A}_{\ell m}^{\text{NQC}}$ are normalized by $1/\sqrt{(\ell+2)(\ell+1)\ell(\ell-1)}$ in all cases.

a. $(\ell, m) = (2, 2)$

The (2,2) mode amplitude extracted at t_{22}^{NQC} is fitted with the template

$$A_{22}^{\text{BHNS}}/A_{22}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{1 + \Lambda p_3^{(2)}(\nu, a_{\text{BH}})}, \quad (\text{B43})$$

with the polynomials

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B44a})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B44b})$$

$$p_3^{(2)}(\nu) = c_{310}\nu + c_{320}\nu^2. \quad (\text{B44c})$$

This results in a value of $R^2 = 0.99$ for the determination coefficient.

TABLE XII. Fitting coefficients for NQC extraction points from the (2,2) waveform. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

$\mathcal{F}$	k	c_{k12}	c_{k11}	c_{k10}	c_{k22}	c_{k21}	c_{k20}	c_{k32}	c_{k31}	c_{k30}
A_{22}^{NQC}	1	8.5×10^{-4}	1.1×10^{-3}	6.2×10^{-2}	2.1×10^{-1}	-2.6×10^0	1.7×10^0	-1.3×10^{-1}	7.1×10^0	-3.5×10^0
	2	-4.6×10^{-7}	3.0×10^{-7}	3.3×10^{-7}	2.4×10^{-4}	-1.1×10^{-4}	-1.7×10^{-5}	-1.2×10^{-3}	1.1×10^{-3}	-6.2×10^{-4}
	3	...	...	6.4×10^{-2}	...	...	1.1×10^0	...	...	...
$\dot{A}_{22}^{\text{NQC}}$	1	...	6.5×10^{-3}	-2.9×10^{-3}	...	9.1×10^{-2}	1.5×10^{-1}	...	...	...
	2	...	-1.7×10^{-6}	8.7×10^{-7}	...	1.0×10^{-5}	-2.2×10^{-5}	...	...	...
	3	...	6.3×10^{-1}	1.9×10^{-1}	...	-1.7×10^0	4.3×10^{-1}	...	...	...
ω_{22}^{NQC}	1	3.1×10^{-3}	-1.3×10^{-3}	4.4×10^{-2}	-8.5×10^{-1}	6.5×10^{-1}	3.4×10^{-1}	3.8×10^0	-3.5×10^0	-5.7×10^{-1}
	2	-8.7×10^{-7}	7.7×10^{-7}	6.7×10^{-7}	2.2×10^{-4}	-1.3×10^{-3}	1.1×10^{-3}	-7.0×10^{-4}	5.6×10^{-3}	-4.2×10^{-3}
	3	...	...	2.3×10^{-2}	...	...	...	...	...	...
$\dot{\omega}_{22}^{\text{NQC}}$	1	-2.1×10^{-3}	-1.4×10^{-4}	3.6×10^{-2}	-3.3×10^{-1}	-5.0×10^{-1}	4.8×10^{-1}	2.0×10^0	1.1×10^0	-2.3×10^0
	2	1.3×10^{-7}	6.0×10^{-7}	-3.1×10^{-7}	1.2×10^{-3}	-1.3×10^{-3}	2.5×10^{-4}	-5.3×10^{-3}	6.4×10^{-3}	-1.3×10^{-3}
	3	...	...	1.7×10^{-2}	...	...	...	...	...	...

For the first-time derivative of the NQC amplitude, we obtain instead $R^2 = 0.87$ using the expression

$$\dot{A}_{22}^{\text{BHNS}}/\dot{A}_{22}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{[1 + \Lambda(p_3^{(2)}(\nu, a_{\text{BH}}))^2]^2} \quad (\text{B45})$$

as fitting model, where the polynomials are defined as

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(1)}(a_{\text{BH}})\nu + p_{k2}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B46a})$$

$$p_{kj}^{(1)}(a_{\text{BH}}) = c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B46b})$$

Additionally, the (2,2) mode frequency and its time derivative extracted at the NQC point are modeled employing Eqs. (B47) and (B49), respectively, as follows:

$$\omega_{22}^{\text{BHNS}}/\omega_{22}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{(1 + \Lambda p_3^{(1)}(\nu))^2} \quad (\text{B47})$$

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B48a})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B48b})$$

$$p_3^{(1)}(\nu) = c_{310}\nu. \quad (\text{B48c})$$

$$\dot{\omega}_{22}^{\text{BHNS}}/\dot{\omega}_{22}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{(1 + \Lambda c_{310}\nu)^2} \quad (\text{B49})$$

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B50a})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B50b})$$

Thus resulting in the following corresponding determination coefficients: $R^2 = 0.96$ and $R^2 = 0.98$. The fit parameters for all the NQC quantities modeled for the (2,2) are reported in Table XII.

b. $(\ell, m) = (2, 1)$

For the (2,1) mode, we extract all NQC quantities at $t_{21}^{\text{NQC}} = t_{22}^{\text{peak}}$ according to the fits from Ref. [121] (see Sec. IV B). We model the NQC amplitude of the (2,1) mode with the following expression:

$$A_{21}^{\text{BHNS}}/A_{21}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{1 + \Lambda(p_3^{(2)}(\nu, a_{\text{BH}}))^2}, \quad (\text{B51})$$

where

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2, \quad (\text{B52a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B52b})$$

$$p_3^{(2)}(\nu, a_{\text{BH}}) = p_{32}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B52c})$$

$$p_{31}^{(1)}(a_{\text{BH}}) = c_{311}a_{\text{BH}} + c_{310}. \quad (\text{B52d})$$

With this fit model, we obtain $R^2 = 0.95$.

For the time derivative of the NQC amplitude, we employ instead

TABLE XIII. Fitting coefficients for NQC extraction points from the (2,1) waveform. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

$\mathcal{F}$	k	c_{k13}	c_{k12}	c_{k11}	c_{k10}	c_{k23}	c_{k22}	c_{k21}	c_{k20}	
A_{21}^{NQC}	1	-9.3×10^1	-1.2×10^3	1.1×10^3	-1.8×10^2	1.9×10^3	9.0×10^3	-9.5×10^3	2.1×10^3	
	2	8.7×10^{-1}	1.4×10^0	-1.4×10^0	1.5×10^{-1}	-7.1×10^0	-6.6×10^0	8.6×10^0	-1.3×10^0	
	3	...	...	-3.0×10^2	1.8×10^2	...	...	...	...	
		c_{k12}	c_{k11}	c_{k10}	c_{k22}	c_{k21}	c_{k20}	c_{k32}	c_{k31}	c_{k30}
$\dot{A}_{21}^{\text{NQC}}$	1	-3.7×10^5	3.3×10^5	-1.3×10^5	3.7×10^6	-3.4×10^6	1.4×10^6	-9.1×10^6	8.5×10^6	-3.5×10^6
	2	-7.4×10^2	-1.7×10^2	4.2×10^2	8.9×10^3	9.9×10^2	-4.2×10^3	-2.6×10^4	-1.1×10^3	1.1×10^4
	3	...	...	...	...	-9.3×10^0	5.3×10^0	...	...	...
ω_{21}^{NQC}		c_{k13}	c_{k12}	c_{k11}	c_{k10}					
	1	3.5×10^2	3.7×10^2	4.0×10^1	2.5×10^0					
	2	-4.3×10^{-1}	-1.5×10^0	-4.6×10^{-1}	-1.5×10^{-2}					
	3	-1.8×10^{-4}	1.4×10^{-3}	6.6×10^{-4}	1.4×10^{-5}					
	4	...	...	-7.1×10^{-1}	3.8×10^{-1}					
		c_{k23}	c_{k22}	c_{k21}	c_{k20}					
	1	-3.8×10^3	-3.9×10^3	-3.7×10^2	-2.3×10^1					
	2	5.1×10^0	1.6×10^1	4.6×10^0	1.4×10^{-1}					
	3	1.4×10^{-3}	-1.5×10^{-2}	-6.7×10^{-3}	-1.3×10^{-4}					
		c_{k33}	c_{k32}	c_{k31}	c_{k30}					
	1	1.0×10^4	1.0×10^4	8.6×10^2	5.1×10^1					
	2	-1.5×10^1	-4.1×10^1	-1.1×10^1	-3.1×10^{-1}					
	3	-2.3×10^{-3}	3.9×10^{-2}	1.7×10^{-2}	3.0×10^{-4}					
	k	c_{k13}	c_{k12}	c_{k11}	c_{k10}	c_{k23}	c_{k22}	c_{k21}	c_{k20}	
$\dot{\omega}_{21}^{\text{NQC}}$	1	1.1×10^2	2.8×10^2	1.8×10^2	3.0×10^1	7.5×10^2	2.7×10^2	-1.8×10^2	-4.8×10^1	
	2	-5.6×10^0	-3.5×10^0	4.3×10^{-1}	3.2×10^{-1}	2.7×10^1	1.7×10^1	-1.9×10^0	-1.5×10^0	
	3	1.8×10^2	4.6×10^1	-7.7×10^1	-2.1×10^1	-8.3×10^2	-2.0×10^2	3.8×10^2	1.0×10^2	

$$\dot{A}_{21}^{\text{BHNS}}/\dot{A}_{21}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{1 + \Lambda^2 (p_3^{(1)}(a_{\text{BH}})\nu)^2}, \quad (\text{B53})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B54b})$$

$$p_{32}^{(1)}(a_{\text{BH}}) = c_{311}a_{\text{BH}} + c_{310}, \quad (\text{B54c})$$

with

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B54a})$$

which gives us a determination coefficient of $R^2 = 0.99$.

Moreover, for the NQC frequency and its time derivative, we employ Eqs. (B55) and (B57), which result correspondingly in $R^2 = 0.85$ and $R^2 = 0.91$, as follows:

$$\frac{\omega_{21}^{\text{BHNS}}}{\omega_{21}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(3)}(\nu, a_{\text{BH}})}{[1 + \Lambda (p_{41}^{(1)}(a_{\text{BH}})\nu)^2]^2}, \quad (\text{B55})$$

with

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2 + p_{k3}^{(3)}(a_{\text{BH}})\nu^3, \quad (\text{B56a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B56b})$$

$$p_{41}^{(2)}(a_{\text{BH}}) = c_{411}a_{\text{BH}} + c_{410}, \quad (\text{B56c})$$

$$\dot{\omega}_{21}^{\text{BHNS}}/\dot{\omega}_{21}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{(1 + \Lambda (p_3^{(2)}(\nu, a_{\text{BH}}))^2)^2}, \quad (\text{B57})$$

where

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2, \quad (\text{B58a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}. \quad (\text{B58b})$$

Table [XIII](#) summarizes all fit parameters obtained for each NQC extraction point of the (2,1) mode.

c. $(\ell, m) = (3, 3)$

Similarly as before, we set the NQC extraction point at $t_{33}^{\text{NQC}} = t_{22}^{\text{peak}}$ for the (3,3) mode, and model the amplitude as

$$A_{33}^{\text{BHNS}}/A_{33}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{[1 + \Lambda(p_3^{(2)}(\nu, a_{\text{BH}}))^2]^2}, \quad (\text{B59})$$

with the expressions detailed as

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{11}^{(3)}(a_{\text{BH}})\nu + p_{12}^{(3)}(a_{\text{BH}})\nu^2 + p_{13}^{(3)}(a_{\text{BH}})\nu^3, \quad (\text{B60a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B60b})$$

$$p_3^{(2)}(\nu, a_{\text{BH}}) = p_{31}^{(1)}(a_{\text{BH}})\nu + p_{32}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B60c})$$

$$p_{3j}^{(1)}(a_{\text{BH}}) = c_{3j1}a_{\text{BH}} + c_{3j0}. \quad (\text{B60d})$$

We thus obtain a coefficient of determination of $R^2 = 0.89$. On the other hand, for the amplitude's time derivative, we obtain $R^2 = 0.90$ with the template in Eq. [\(B61\)](#), as follows:

$$\dot{A}_{33}^{\text{BHNS}}/\dot{A}_{33}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{[1 + \Lambda(p_{32}^{(1)}(a_{\text{BH}})\nu^2)^2]^2}, \quad (\text{B61})$$

where the polynomials $p_{kj}^{(o)}$ are

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2 + p_{k3}^{(3)}(a_{\text{BH}})\nu^3, \quad (\text{B62a})$$

$$p_{kj}^{(4)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B62b})$$

$$p_{32}^{(1)}(a_{\text{BH}}) = c_{321}a_{\text{BH}} + c_{320}. \quad (\text{B62c})$$

We fit the NQC frequency with the template

$$\omega_{33}^{\text{BHNS}}/\omega_{33}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{(1 + \Lambda(p_{32}^{(1)}(a_{\text{BH}})\nu^2)^2)^2}, \quad (\text{B63})$$

where

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2, \quad (\text{B64a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B64b})$$

$$p_{32}^{(1)}(a_{\text{BH}}) = c_{321}a_{\text{BH}} + c_{320}. \quad (\text{B64c})$$

For the time derivative of the NQC frequency, we employ instead

$$\frac{\dot{\omega}_{33}^{\text{BHNS}}}{\dot{\omega}_{33}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(2)}(\nu, a_{\text{BH}})}{[1 + \Lambda^2(p_{42}^{(1)}(a_{\text{BH}})\nu^2)^2]^2}, \quad (\text{B65})$$

with

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2, \quad (\text{B66a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B66b})$$

$$p_{42}^{(1)}(a_{\text{BH}}) = c_{421}a_{\text{BH}} + c_{420}. \quad (\text{B66c})$$

For both fitting models, we obtain a determination coefficient of $R^2 = 0.89$. The fit parameters for all NQC quantities of the (3,3) are listed in Table [XIV](#).

d. $(\ell, m) = (4, 4)$

In the case of the (4,4) mode, we also make use of the NQC fits of Ref. [\[121\]](#) and extract at $t_{44}^{\text{NQC}} = t_{22}^{\text{peak}}$. The amplitude is modeled by employing the following expression with its corresponding polynomials below:

TABLE XIV. Fitting coefficients for NQC extraction points from the (3,3) waveform. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

A_{33}^{NQC}	k	c_{k13}	c_{k12}	c_{k11}	c_{k10}				
	1	-8.6×10^1	-1.1×10^1	1.7×10^1	5.1×10^0				
	2	3.7×10^{-2}	-2.3×10^{-2}	-3.2×10^{-2}	1.7×10^{-2}				
	3	...	...	-1.1×10^0	2.4×10^0				
		c_{k23}	c_{k22}	c_{k21}	c_{k20}				
	1	8.5×10^2	8.8×10^1	-1.7×10^2	-4.4×10^1				
	2	-3.3×10^{-1}	2.7×10^{-1}	3.2×10^{-1}	-1.5×10^{-1}				
	3	...	...	6.9×10^0	-9.8×10^0				
		c_{k33}	c_{k32}	c_{k31}	c_{k30}				
	1	-2.1×10^3	-1.8×10^2	4.1×10^2	9.6×10^1				
	2	7.2×10^{-1}	-7.7×10^{-1}	-8.0×10^{-1}	3.4×10^{-1}				
$\dot{A}_{33}^{\text{NQC}}$		c_{k13}	c_{k12}	c_{k11}	c_{k10}				
	1	-1.0×10^8	-2.8×10^7	-1.0×10^7	7.8×10^6				
	2	1.2×10^5	7.2×10^4	3.3×10^4	-2.0×10^4				
		c_{k23}	c_{k22}	c_{k21}	c_{k20}				
	1	1.1×10^9	2.9×10^8	9.8×10^7	-7.8×10^7				
	2	-1.3×10^6	-7.4×10^5	-3.3×10^5	2.0×10^5				
	3	...	...	-2.7×10^1	4.9×10^1				
		c_{k33}	c_{k32}	c_{k31}	c_{k30}				
	1	-3.0×10^9	-7.2×10^8	-2.3×10^8	1.9×10^8				
	2	3.4×10^6	1.9×10^6	8.1×10^5	-5.1×10^5				
		c_{k13}	c_{k12}	c_{k11}	c_{k10}	c_{k23}	c_{k22}	c_{k21}	c_{k20}
ω_{33}^{NQC}	1	-2.0×10^1	-4.3×10^1	-2.8×10^1	-5.5×10^0	1.3×10^2	2.7×10^2	1.7×10^2	3.3×10^1
	2	-2.5×10^{-2}	-3.9×10^{-2}	-2.6×10^{-2}	-6.9×10^{-3}	1.8×10^{-1}	3.1×10^{-1}	2.1×10^{-1}	5.2×10^{-2}
	3	...	...	...	...	...	...	6.3×10^0	4.8×10^0
$\dot{\omega}_{33}^{\text{NQC}}$	1	-1.6×10^9	1.2×10^9	4.0×10^7	-1.6×10^8	9.0×10^9	-6.8×10^9	-1.9×10^8	8.9×10^8
	2	7.9×10^6	-6.0×10^6	9.6×10^4	6.7×10^5	-4.5×10^7	3.4×10^7	-5.3×10^5	-3.8×10^6
	3	-7.0×10^3	5.2×10^3	-4.3×10^2	-4.0×10^2	3.8×10^4	-2.7×10^4	1.5×10^3	2.4×10^3
	4							-1.1×10^1	9.9×10^0

$$\frac{A_{44}^{\text{BHNS}}}{A_{44}^{\text{BBH}}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}}) + \Lambda^3 p_3^{(2)}(\nu, a_{\text{BH}})}{(1 + \Lambda^2 p_4^{(2)}(\nu, a_{\text{BH}}))^2}, \quad (\text{B67})$$

where

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2, \quad (\text{B68a})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B68b})$$

$$p_4^{(2)}(\nu, a_{\text{BH}}) = p_{41}^{(1)}(a_{\text{BH}})\nu + p_{42}^{(1)}(a_{\text{BH}})\nu^2, \quad (\text{B68c})$$

$$p_{4j}^{(1)}(a_{\text{BH}}) = c_{4j1}a_{\text{BH}} + c_{4j0}. \quad (\text{B68d})$$

By employing Eq. (B67) to fit our NQC amplitude with our NR data, we get $R^2 = 0.91$. For its time derivative, we use

$$\dot{A}_{44}^{\text{BHNS}}/\dot{A}_{44}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(3)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(3)}(\nu, a_{\text{BH}})}{(1 + \Lambda^2 (p_3^{(2)}(\nu, a_{\text{BH}}))^2)^2}, \quad (\text{B69})$$

with

$$p_k^{(3)}(\nu, a_{\text{BH}}) = p_{k1}^{(2)}(a_{\text{BH}})\nu + p_{k2}^{(2)}(a_{\text{BH}})\nu^2 + p_{k3}^{(2)}(a_{\text{BH}})\nu^3, \quad (\text{B70a})$$

$$p_3^{(2)}(\nu, a_{\text{BH}}) = p_{31}^{(2)}(a_{\text{BH}})\nu + p_{32}^{(2)}(a_{\text{BH}})\nu^2, \quad (\text{B70b})$$

$$p_{kj}^{(2)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B70c})$$

thus giving us a coefficient of determination of $R^2 = 0.87$.

The NQC frequency and its time derivative are fitted with the templates in Eqs. (B71) and (B73) below, where we obtain $R^2 = 0.94$ and $R^2 = 0.95$, respectively.

TABLE XV. Fitting coefficients for NQC extraction points from the (4,4) waveform. Here, we make a fit of a quantity $\mathcal{F}$ as $\mathcal{F}^{\text{BHNS}}/\mathcal{F}^{\text{BBH}}$.

$\mathcal{F}$	k	c_{k12}	c_{k11}	c_{k10}	c_{k22}	c_{k21}	c_{k20}	c_{k32}	c_{k31}	c_{k30}
A_{44}^{NQC}	1	-3.8×10^8	-4.4×10^8	-1.1×10^8	2.2×10^9	2.6×10^9	6.6×10^8	$\dots$	$\dots$	$\dots$
	2	2.1×10^6	2.4×10^6	6.3×10^5	-1.3×10^7	-1.4×10^7	-3.8×10^6	$\dots$	$\dots$	$\dots$
	3	-2.9×10^3	-3.5×10^3	-1.0×10^3	1.7×10^4	2.1×10^4	6.4×10^3	$\dots$	$\dots$	$\dots$
	4	$\dots$	-3.0×10^0	-2.0×10^0	$\dots$	2.2×10^1	1.5×10^1	$\dots$	$\dots$	$\dots$
$\dot{A}_{44}^{\text{NQC}}$	1	1.4×10^4	2.2×10^3	-3.4×10^3	-2.0×10^5	-2.3×10^4	5.7×10^4	6.5×10^5	6.0×10^4	-1.9×10^5
	2	1.7×10^2	3.7×10^0	-5.9×10^1	-1.5×10^3	-4.1×10^1	5.2×10^2	3.4×10^3	1.1×10^2	-1.2×10^3
	3	6.3×10^0	3.0×10^{-1}	-2.2×10^0	-3.0×10^1	-1.4×10^0	1.1×10^1	$\dots$	$\dots$	$\dots$
		c_{k13}	c_{k12}	c_{k11}	c_{k10}	c_{k23}	c_{k22}	c_{k21}	c_{k20}	
ω_{44}^{NQC}	1	-1.1×10^{-1}	-1.6×10^{-1}	1.0×10^{-1}	-6.8×10^{-2}	7.2×10^{-1}	9.2×10^{-1}	-6.2×10^{-1}	6.1×10^{-1}	
	2	1.1×10^{-3}	8.7×10^{-4}	-1.6×10^{-4}	-3.2×10^{-4}	-6.0×10^{-3}	-4.9×10^{-3}	6.5×10^{-4}	1.9×10^{-3}	
	3	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	-1.3×10^{-1}	1.8×10^0	
$\dot{\omega}_{44}^{\text{NQC}}$	1	$\dots$	1.8×10^5	-1.4×10^5	2.1×10^4	$\dots$	-5.6×10^5	2.4×10^5	3.7×10^4	
	2	$\dots$	-2.9×10^2	1.3×10^2	1.3×10^1	$\dots$	1.4×10^3	-4.8×10^2	-1.6×10^2	
	3	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	-3.3×10^3	2.3×10^3	

Table XV shows the best-fit coefficients for all the fitted NQC quantities of the (4,4) mode, as follows:

$$\omega_{44}^{\text{BHNS}}/\omega_{44}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{(1 + \Lambda(p_{32}^{(1)}(a_{\text{BH}})\nu^2)^2)^2}, \quad (\text{B71})$$

with

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2, \quad (\text{B72a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj3}a_{\text{BH}}^3 + c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B72b})$$

$$p_{32}^{(2)}(a_{\text{BH}}) = c_{321}a_{\text{BH}} + c_{320}. \quad (\text{B72c})$$

$$\dot{\omega}_{44}^{\text{BHNS}}/\dot{\omega}_{44}^{\text{BBH}} = \frac{1 + \Lambda p_1^{(2)}(\nu, a_{\text{BH}}) + \Lambda^2 p_2^{(2)}(\nu, a_{\text{BH}})}{1 + \Lambda(p_{32}^{(1)}(a_{\text{BH}})\nu^2)^2}, \quad (\text{B73})$$

where the polynomials are expressed as

$$p_k^{(2)}(\nu, a_{\text{BH}}) = p_{k1}^{(3)}(a_{\text{BH}})\nu + p_{k2}^{(3)}(a_{\text{BH}})\nu^2, \quad (\text{B74a})$$

$$p_{kj}^{(3)}(a_{\text{BH}}) = c_{kj2}a_{\text{BH}}^2 + c_{kj1}a_{\text{BH}} + c_{kj0}, \quad (\text{B74b})$$

$$p_{32}^{(1)}(a_{\text{BH}}) = c_{320}a_{\text{BH}} + c_{321}. \quad (\text{B74c})$$

APPENDIX C: WAVEFORM ROBUSTNESS FOR GENERIC ORBITS

In this appendix, we highlight the model's robustness and consistency for a variety of configurations with eccentricity. We generate waveforms from more than 1000 different binaries with parameters $q \in [1, 5]$, $a_{\text{BH}} \in [-0.8, 0.8]$, $\Lambda \in [1, 5000]$, and $e_0 \in [0.01, 0.2]$. In Fig. 30, we show the amplitude of the model's waveforms produced with different q , a_{BH} , Λ , and eccentricity e . This figure demonstrates the model's capacity to produce a smooth amplitude and proper ringdown attachment for eccentricities as high as $e \sim 0.5$.

As a second consistency check, we computed mismatches between the BHNS waveforms against their corresponding BBH cases in order to verify the behavior that we expect between the two models: almost identical

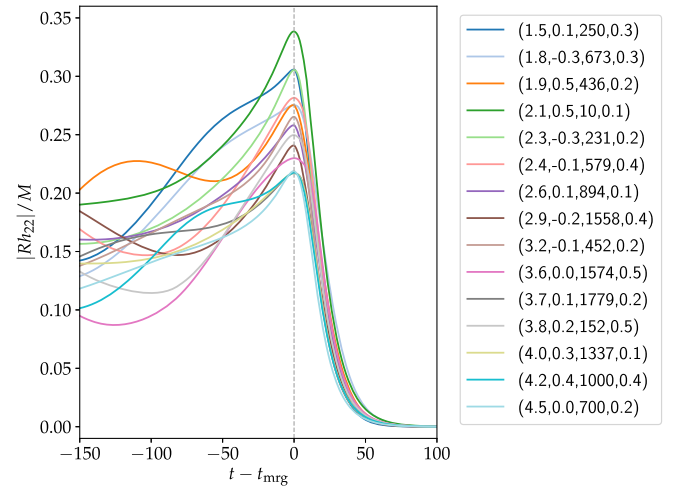


FIG. 30. Sanity check for eccentric waveform amplitudes employing different parameters (q , a_{BH} , Λ , e). The gray dashed line indicates the moment of merger for all configurations.

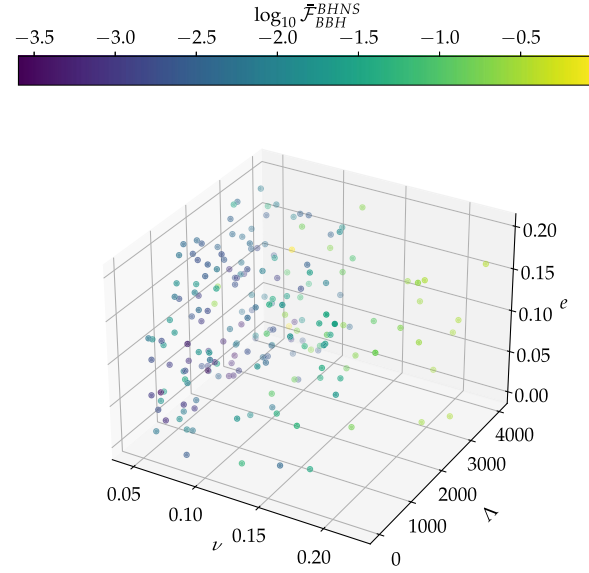


FIG. 31. Mismatches between BHNS and its corresponding BBH waveform, $\bar{\mathcal{F}}_{\text{BBH}}^{\text{BHNS}}$, in the parameter space of (ν, Λ, e) .

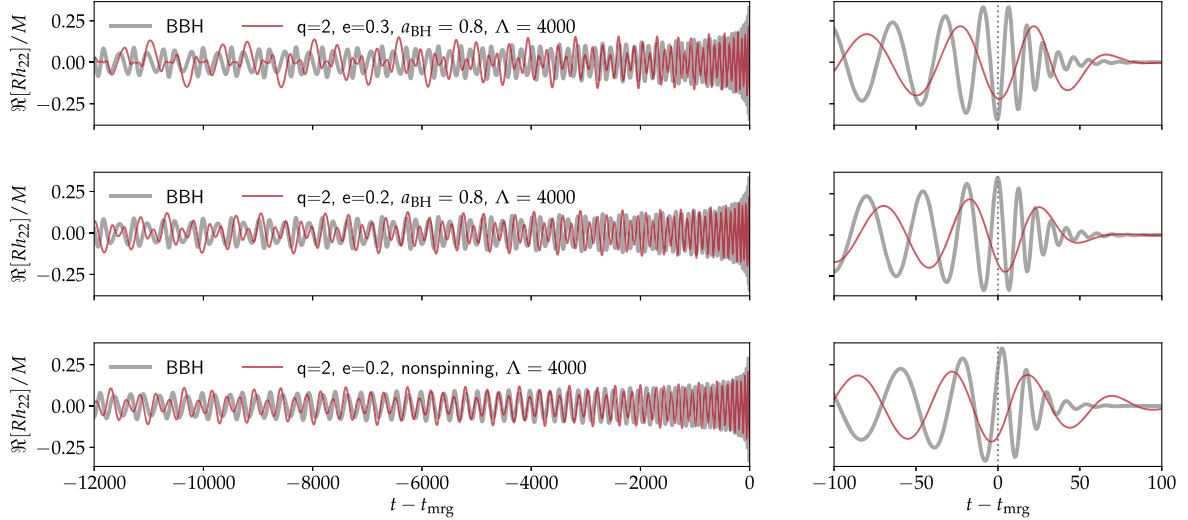


FIG. 32. Gravitational waveforms from eccentric binary configurations with increasing initial eccentricity and spin. The corresponding BBH ($\Lambda = 0$) waveform is added as reference as a gray solid line.

waveforms towards $\nu \rightarrow 0$ and $\Lambda \rightarrow 0$, and increasing deviations with higher values of Λ , $\nu \rightarrow \frac{1}{4}$, together with increasing eccentricities. This behavior is shown by the mismatches in Fig. 31, demonstrating the robustness of our new model across the vast parameter space. Overall, the model is sufficiently stable to serve in future parameter estimation studies.

Finally, we discuss further the effect of eccentricity, spins, and tides in our model. In comparison to the effect from the spins aligned to the orbital plane, precessing spins have little

to no noticeable effect on the eccentric-tidal waveforms. The combination of increasing Λ and a_{BH} has a significant effect on the eccentric waveform, as seen in Fig. 32. The oscillating features in the early inspiral get more pronounced with higher spin magnitudes and are especially sensible to increasing eccentricity. On the other hand, amplitude deviations from the BBH case on the inspiral are already significant when tides are turned on (bottom panel of Fig. 32). A further analytical investigation would be necessary to understand the interplay between eccentricity, tides, and spin.

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