

Exclusive decays of fully heavy tetraquarks into light mesons

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(Received 16 January 2026; accepted 19 February 2026; published 6 March 2026)

In this work, we investigate the exclusive decays of the fully-heavy tetraquark states $T_{4c,b}$ into light mesons, specifically π and K , using the framework of nonrelativistic quantum chromodynamics (NRQCD) and collinear QCD factorization for hard exclusive processes. We estimate the decay widths to be 10^{-9} GeV and 10^{-14} GeV for the decays $T_{4c} \rightarrow \pi^+\pi^-$ and $T_{4b} \rightarrow \pi^+\pi^-$ (and similarly for K^+K^-), respectively. The branching ratio for $T_{4c} \rightarrow \pi^+\pi^-(K^+K^-)$ is on the order of 10^{-8} , making it currently unobservable in existing experiments. The factorization of $T_{4c,b}$ into light hadrons shares similarities with the decay $J/\psi \rightarrow p\bar{p}$. However, unlike the latter process, the decay $T_{4c,b} \rightarrow \pi^+\pi^-(K^+K^-)$ exhibits unique features that arise only in processes involving multiple incoming or outgoing particles. One such feature is the necessity of maintaining the $i\epsilon$ -prescription for the denominators or propagators due to the divergences in the kinematic region of interest. Employing the sector decomposition method, along with the aid of the Cheng-Wu theorem and the QHULL program, we present a systematic approach to handle the convolutions and phase-space integrations. This method can also be extended to similar processes, such as $T_{4c,b} \rightarrow p\bar{p}$, as well as to phase-space integrations where each denominator is a linear combination of integration variables.

DOI: [10.1103/n6n2-d8mv](https://doi.org/10.1103/n6n2-d8mv)

I. INTRODUCTION

In recent years, experimental searches for fully charmed tetraquarks have progressed significantly. In 2020, the LHCb Collaboration reported a narrow structure, denoted as X(6900), with the significance larger than 5σ [1], together with a broad enhancement in the di- J/ψ invariant-mass spectrum around the double J/ψ threshold region. Subsequently, the ATLAS and CMS Collaborations confirmed the existence of X(6900) and observed additional broad structures in the di- J/ψ and $J/\psi\psi(2S)$ channels [2,3]. More recently, ATLAS Collaboration measured the ratio of partial decay widths between $J/\psi\psi(2S)$ and di- J/ψ channels [4], and CMS Collaboration determined the quantum numbers of the X(6900) to be $J^{PC} = 2^{++}$ [5]. In the bottom quark sector, the CMS Collaboration reported an observation of a Υ pair [6]; however, this structure was not yet confirmed by subsequent analyses from CMS [7]. Additionally, the LHCb Collaboration investigated the $\Upsilon\mu^+\mu^-$ invariant mass spectrum but did not observe any reliable signals [8].

Although the search for exotic states presents numerous challenges, the potential existence of exotic hadrons—especially those composed of four or five quarks—has been extensively studied in the literature (for recent reviews, see, e.g., Refs. [9,10]). The pioneering ideas regarding fully charmed tetraquark configurations near the J/ψ -pair threshold can be traced back to the 1970s and 1980s [11–14]. Subsequently, a broad range of approaches have been proposed to explore and understand the nature of these tetraquarks. These include constituent quark models [15–25], QCD sum rules [26–38], diffusion Monte Carlo simulations [39,40], lattice QCD calculations [41–44], and studies based on partial wave analyses [45], the color evaporation model [46,47], and the Bethe–Salpeter framework [48–50]. Related extensions to other fully heavy tetraquark systems can be found in Refs. [51–54]. On the phenomenological side, the production of fully heavy tetraquarks in various collision environments—such as pp , pA , AA , ep , e^+e^- , and $\gamma\gamma$ —has been considered in Refs. [46,47,51,55–75]. Studies on the electromagnetic and hadronic decays of fully heavy tetraquarks can be found in Refs. [76–84].

In this work, we investigate the exclusive decays of fully heavy tetraquarks into light mesons, specifically $T_{4c,b} \rightarrow \pi^+\pi^-$ and $T_{4c,b} \rightarrow K^+K^-$. These exclusive decay processes bear notable similarities to the hadronic exclusive decays of charmonia. The first calculation of such exclusive decays in charmonia, exemplified by the decay $J/\psi \rightarrow p\bar{p}$, was

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performed in Ref. [85]. Subsequently, various decays of S - and P -wave charmonia have been extensively studied in numerous publications (see, for example, Refs. [86,87] and references therein). The factorization approach used in hard exclusive processes for charmonia decays into light hadrons can also be extended to the processes $T_{4c,b} \rightarrow \pi^+\pi^-$ and $T_{4c,b} \rightarrow K^+K^-$, which we aim to explore in this work.

The paper is organized as follows. In Sec. II, we present the theoretical framework, including the factorization formula for the exclusive decay of the T_{4c} into light hadrons $\pi^+\pi^-$ within the context of NRQCD and collinear QCD factorization for hard exclusive processes. We also outline the matching procedure for calculating the hard kernels. Additionally, we describe the computational techniques and present the analytical expressions for the hard kernels, along with the master formula for the decay amplitudes, which can be directly utilized for making predictions. Section III is dedicated to phenomenological predictions and discussions, while Sec. IV provides a summary and outlook. In the Appendix, we discuss the underlying ideas of numerical integration using sector decomposition, with the assistance of the Cheng-Wu theorem and the QHULL program.

II. THEORETICAL FRAMEWORK

A. Factorization formula

Similar to the process of charmonium decay into baryon-antibaryon pairs [85], we consider the exclusive decays of fully heavy tetraquarks into light mesons. In this section, we take the decay $T_{4c} \rightarrow \pi^+\pi^-$ as a concrete example; a similar analysis can be applied to the decays $T_{4c} \rightarrow K^+K^-$ and $T_{4b} \rightarrow \pi^+\pi^-(K^+K^-)$. Within the NRQCD factorization framework [59,60,88] for T_{4c} and the collinear factorization approach for hard exclusive reactions [89–94], the amplitude for the process $T_{4c}^{(J)}(P) \rightarrow \pi^+(Q) + \pi^-(Q')$ can be factorized as

$$\begin{aligned} \mathcal{A}(T_{4c}^{(J)}(P) \rightarrow \pi^+(Q) + \pi^-(Q')) \\ = \sqrt{2m_H} \int_0^1 dx \int_0^1 dy \Phi_\pi(x) \left[\frac{\langle \mathcal{O}_{3\otimes 3}^{(J)} \rangle}{4} T_{3\otimes 3}^{(J)}(x, y) \right. \\ \left. + \frac{\langle \mathcal{O}_{6\otimes 6}^{(J)} \rangle}{4} T_{6\otimes 6}^{(J)}(x, y) \right] \Phi_\pi^*(y), \end{aligned} \quad (1)$$

where P , Q and Q' are the momenta of T_{4c} , π^+ and π^- respectively, the factor $\sqrt{2m_H}$ is explicitly factored out to indicate the nonrelativistic normalization for the hadron T_{4c} , with $m_H (=4m_c)$ representing the mass of T_{4c} and m_c denoting the charm quark mass.

The long distance matrix elements (LDME) $\langle \mathcal{O}_{3\otimes 3}^{(J)} \rangle$ and $\langle \mathcal{O}_{6\otimes 6}^{(J)} \rangle$ in (1) are defined as

$$\langle \mathcal{O}_{3\otimes 3}^{(J)} \rangle = \langle 0 | \mathcal{O}_{3\otimes 3}^{(J)} | T_{4c} \rangle, \quad \text{and} \quad \langle \mathcal{O}_{6\otimes 6}^{(J)} \rangle = \langle 0 | \mathcal{O}_{6\otimes 6}^{(J)} | T_{4c} \rangle, \quad (2)$$

where the operators $\mathcal{O}_{3\otimes 3}^{(J)}$ and $\mathcal{O}_{6\otimes 6}^{(J)}$ are defined as [59,64]

$$\begin{aligned} \mathcal{O}_{3\otimes 3}^{(0)} &= -\frac{1}{\sqrt{3}} [\psi_a^T(i\sigma^2)\sigma^i\psi_b] [\chi_c^\dagger\sigma^i(i\sigma^2)\chi_d^*] \mathcal{C}_{3\otimes 3}^{ab;cd}, \\ \mathcal{O}_{3\otimes 3}^{(1)} &= -\frac{i}{\sqrt{2}} \epsilon_H^{i*} [\psi_a^T(i\sigma^2)\sigma^j\psi_b] [\chi_c^\dagger\sigma^k(i\sigma^2)\chi_d^*] \epsilon^{ijk} \mathcal{C}_{3\otimes 3}^{ab;cd}, \\ \mathcal{O}_{3\otimes 3}^{(2)} &= \frac{1}{2} \epsilon_H^{ij*} [\psi_a^T(i\sigma^2)\sigma^k\psi_b] [\chi_c^\dagger\sigma^l(i\sigma^2)\chi_d^*] \Gamma^{ijkl} \mathcal{C}_{3\otimes 3}^{ab;cd}, \\ \mathcal{O}_{6\otimes 6}^{(0)} &= [\psi_a^T(i\sigma^2)\psi_b] [\chi_c^\dagger(i\sigma^2)\chi_d^*] \mathcal{C}_{6\otimes 6}^{ab;cd}, \end{aligned} \quad (3)$$

where ψ and $\chi^\dagger$ are Pauli spinor fields that annihilate a heavy quark and antiquark, respectively. The symbol σ^i denotes the 2-dimensional Pauli matrix, ϵ_H^{ij} and ϵ_H^{ij} refer to the polarization vector and tensor of T_{4c} with quantum numbers $J^{\text{PC}} = 1^{+-}$ and $J^{\text{PC}} = 2^{++}$, respectively. The color projectors $\mathcal{C}$ and symmetric traceless tensor Γ are defined as

$$\begin{aligned} \mathcal{C}_{3\otimes 3}^{ab;cd} &\equiv \frac{1}{2\sqrt{3}} (\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}), \\ \mathcal{C}_{6\otimes 6}^{ab;cd} &\equiv \frac{1}{2\sqrt{6}} (\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}), \\ \Gamma^{ijkl} &= \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \frac{2}{3}\delta^{ij}\delta^{kl} \end{aligned} \quad (4)$$

with i, j, k denoting the Cartesian indices and a, b, c, d representing the color indices.

The function $\Phi_\pi(x)$ in Eq. (1) represents the nonperturbative yet universal leading-twist pion light-cone distribution amplitude (LCDA). This amplitude indicates the probability of finding the valence u and $\bar{d}$ quarks inside the π^+ meson, carrying fractional momenta x and $\bar{x} \equiv 1 - x$, respectively. The leading-twist pion LCDA is defined as [86,95]

$$\begin{aligned} \Phi_\pi(x, \mu_F) &= \int \frac{dz^-}{2\pi i} e^{iz^-xP^+} \\ &\times \langle 0 | \bar{d}(0) \gamma^+ \gamma_5 \mathcal{W}(0, z^-) u(z^-) | \pi^+(P) \rangle, \end{aligned} \quad (5)$$

where P is the momentum of the pion and $\mathcal{W}(0, z^-)$ denotes the lightlike gauge link that ensures gauge invariance. By renormalizing the nonlocal operator in (5), one recovers the well-known Efremov-Radyushkin-Brodsky-Lepage (ERBL) evolution equation [91,93]:

$$\frac{d\Phi_\pi(x, \mu_F)}{d \ln \mu_F^2} = \int_0^1 dy V(x, y) \Phi_\pi(y, \mu_F). \quad (6)$$

Finally, the hard kernels $T_{3\otimes 3}^{(J)}(x, y)$ and $T_{6\otimes 6}^{(J)}(x, y)$ are insensitive to the internal structure of the pion and can be

calculated perturbatively. The computation of these hard kernels is one of the primary aims of this work.

B. Perturbative matching

By virtue of the factorization theorem, the hard kernels $T_{\bar{3}\otimes 3}^{(J)}(x, y)$ and $T_{6\otimes \bar{6}}^{(J)}(x, y)$ are insensitive to the internal structure of the hadrons T_{4c} and $\pi^+\pi^-$. Therefore, these hard kernels can be deduced by replacing the hadrons with free partonic states that have the same quantum numbers as the incoming or outgoing hadrons. A typical leading-order Feynman diagram for the process $T_{4c} \rightarrow \pi^+\pi^-$ is depicted in Fig. 1.

To be more specific, for the incoming fully heavy tetraquark T_{4c} , we replace the hadronic state $|T_{4c}\rangle$ with the free partonic state $|T_{4c}\rangle^{\text{free}}$, which consists of free heavy quarks and antiquarks, represented as $cc\bar{c}\bar{c}$. In this configuration, all quarks and antiquarks share the same momentum $p = \frac{1}{4}P$, where P is the momentum of T_{4c} and satisfies $p^2 = m_c^2$. For a specific Feynman diagram, we obtain four Dirac spinors associated with the four incoming free quarks or antiquarks, represented as $u_i^a u_j^b \bar{v}_k^c \bar{v}_l^d$. To project the same quantum numbers as the incoming state $|T_{4c}\rangle$, we can perform the following replacement for the $\bar{3} \otimes 3$ case (it is similar for the $6 \otimes \bar{6}$ case)

$$\begin{aligned} u_i^a u_j^b \bar{v}_k^c \bar{v}_l^d &\rightarrow (uu^T)_{ij}(\bar{v}\bar{v}^T)_{lk} C_{\bar{3}\otimes 3}^{ab;cd} \\ &= (u\bar{v}C)_{ij}(C\bar{u}\bar{v})_{lk} C_{\bar{3}\otimes 3}^{ab;cd} \\ &= (\Pi_s C)_{ij}(C\Pi_{s'})_{lk} C_{\bar{3}\otimes 3}^{ab;cd} \end{aligned} \quad (7)$$

where C is the charge conjugation matrix, and the diquark cc and diantiquark $\bar{c}\bar{c}$ have spin s and s' , respectively. The spin s and s' can be 0 for the spin singlet or 1 for the spin triplet. The corresponding spin projectors are given by [96]

$$\Pi_0 = \frac{\gamma^5(\not{p} - m_c)}{2\sqrt{2}m_c}, \quad \Pi_1 = \frac{\gamma_\mu(\not{p} - m_c)}{2\sqrt{2}m_c}, \quad (8)$$

Note that we adopt nonrelativistic normalization for the incoming free quarks or antiquarks.

Similarly, by replacing the hadronic state $|T_{4c}\rangle$ with the partonic state $|T_{4c}\rangle^{\text{free}}$, we can explicitly compute the corresponding LDMEs. We express them as follows:

$$\begin{aligned} \sqrt{2m_H} \langle 0 | \mathcal{O}_{\bar{3}\otimes 3}^{(J)} | T_{4c} \rangle &\rightarrow \langle 0 | \mathcal{O}_{\bar{3}\otimes 3}^{(J)} | T_{4c} \rangle^{\text{free}} = 4\epsilon_H^{(J)}, \\ \sqrt{2m_H} \langle 0 | \mathcal{O}_{6\otimes \bar{6}}^{(0)} | T_{4c} \rangle &\rightarrow \langle 0 | \mathcal{O}_{6\otimes \bar{6}}^{(0)} | T_{4c} \rangle^{\text{free}} = 4. \end{aligned} \quad (9)$$

where $\epsilon_H^{(0)} = 1$, $\epsilon_H^{(1)}$ and $\epsilon_H^{(2)}$ represent the polarization vector of the tetraquark $T_{4c}^{(1)}$ and polarization tensor of the tetraquark $T_{4c}^{(2)}$, respectively.

Furthermore, for the spin $s = s' = 1$, one needs to combine both spin angular momenta to obtain the total angular

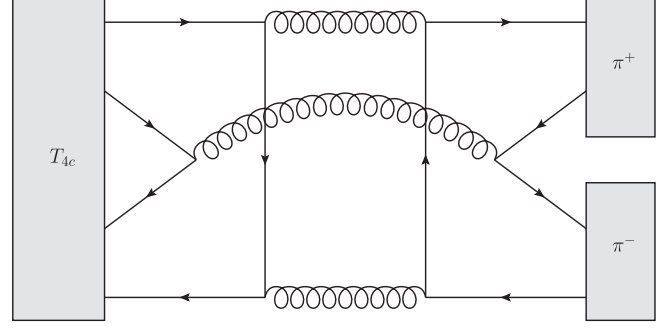


FIG. 1. A typical Feynman diagram for $T_{4c} \rightarrow \pi^+\pi^-$.

momentum $j = 0, 1, 2$, corresponding to the tetraquark T_{4c} with quantum numbers $J^{\text{PC}} = 0^{++}, 1^{+-}, 2^{++}$, respectively. Utilizing Clebsch–Gordan coefficients, we can express the contraction $J_j^{\mu\nu} \Pi_{1\mu} \Pi_{1\nu}$, which projects the final angular momentum j [97]

$$\begin{aligned} J_0^{\mu\nu} &= \frac{1}{\sqrt{3}} \eta^{\mu\nu}, \quad J_1^{\mu\nu} = -\frac{i}{\sqrt{2}p^2} \epsilon^{\mu\nu\rho\sigma} \epsilon_\rho p_\sigma, \\ J_2^{\mu\nu} &= \epsilon_{\rho\sigma} \left[\frac{1}{2} (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}) - \frac{1}{3} \eta^{\mu\nu} \eta^{\rho\sigma} \right], \end{aligned} \quad (10)$$

where $\eta^{\mu\nu} = -g^{\mu\nu} + p^\mu p^\nu / p^2$, ϵ_ρ is the polarization vector of $T_{4c}^{(1)}$, and $\epsilon_{\rho\sigma}$ is the polarization tensor of $T_{4c}^{(2)}$. ϵ_ρ and $\epsilon_{\rho\sigma}$ satisfy the following summations:

$$\begin{aligned} \sum_\lambda \epsilon_\rho(\lambda) \epsilon_\sigma^*(\lambda) &= \eta_{\rho\sigma}, \\ \sum_\lambda \epsilon_{\mu\nu}(\lambda) \epsilon_{\rho\sigma}^*(\lambda) &= \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) - \frac{1}{3} \eta_{\mu\nu} \eta_{\rho\sigma}. \end{aligned} \quad (11)$$

Next, let us turn our attention to the light hadron pion. It is straightforward to project the correct quantum numbers for the pion using the following replacement:

$$v_i^a(\bar{x}Q) \bar{u}_j^b(xQ) \rightarrow \frac{1}{2} (\gamma_5 \not{Q})_{ij} \delta^{ab},$$

$$\text{which is compatible with } \Phi_\pi(x, \mu_F) \xrightarrow{\mu_F \rightarrow \infty} \frac{f_\pi}{2\sqrt{2}N_c} 6x\bar{x}, \quad (12)$$

where Q is the momentum of π and f_π is the decay constant for the pion.

We utilize the program HEPLIB [98,99] to perform the operations described above. HEPLIB is a C++ library designed for high-energy physics calculations, which interfaces with QGRAF [100] to generate Feynman diagrams and amplitudes. In this context, we generate the partonic process: $c(p)c(p)\bar{c}(p)\bar{c}(p) \rightarrow u(xQ)\bar{d}(\bar{x}Q) + \bar{u}(yQ')d(\bar{y}Q')$, resulting in the generation of 150 diagrams at leading order.

This number is much more than that found for the process $J/\psi \rightarrow p\bar{p}$, which involves only a few Feynman diagrams.

Each diagram consists of four fermion lines and at least three gluon lines. The general structure for each diagram can be expressed as

$$\mathcal{A} = u_{i'}^{a'}(p)u_{j'}^{b'}(p)\bar{v}_{k'}^{c'}(p)\bar{v}_{l'}^{d'}(p)\mathcal{H}_{a'b'c'd'abcd}^{i'j'k'l'ijkl} v_i^a(\bar{x}Q)\bar{u}_j^b(xQ)u_k^c(yQ')\bar{v}_l^d(\bar{y}Q'), \quad (13)$$

using the method outlined in this section, we replace $u_{i'}^{a'}u_{j'}^{b'}\bar{v}_{k'}^{c'}\bar{v}_{l'}^{d'}$ with the expression from (7), and $v_i^a\bar{u}_j^b$ and $u_k^c\bar{v}_l^d$ with (12) and its charge-conjugated version, respectively. After contracting all Dirac- γ matrix indices, the two heavy quark lines and two light quark lines yield Dirac traces. Following the computation of these traces and contractions (with HEPLIB internally invoking the FORM [101] program to perform operations automatically), we obtain the final expressions for the hard kernels $T_{3\otimes 3}^{(J)}$ and $T_{6\otimes 6}^{(J)}$ as follows:

$$\begin{aligned} T_{3\otimes 3}^{(0)} = & (\alpha_s\pi)^3(2048x^5y^3 - 3072x^5y^2 + 1408x^5y - 192x^5 - 512x^4y^4 - 4096x^4y^3 + 5888x^4y^2 \\ & - 2240x^4y + 256x^4 + 2048x^3y^5 - 4096x^3y^4 + 8320x^3y^3 - 6848x^3y^2 + 1684x^3y \\ & - 106x^3 - 3072x^2y^5 + 5888x^2y^4 - 6848x^2y^3 + 4072x^2y^2 - 742x^2y + 17x^2 + 1408xy^5 \\ & - 2240xy^4 + 1684xy^3 - 742xy^2 + 114xy - 2x - 192y^5 + 256y^4 - 106y^3 + 17y^2 - 2y) \\ & \div [9m_c^6x(2x-1)^2\bar{x}y(2y-1)^2\bar{y}(4xy-x-y)(4\bar{x}\bar{y}-\bar{x}-\bar{y})], \end{aligned} \quad (14)$$

$$\begin{aligned} T_{6\otimes 6}^{(0)} = & (\alpha_s\pi)^3(2048x^5y^3 - 3072x^5y^2 + 1408x^5y - 192x^5 - 512x^4y^4 - 4096x^4y^3 + 5888x^4y^2 \\ & - 2240x^4y + 256x^4 + 2048x^3y^5 - 4096x^3y^4 + 8320x^3y^3 - 6848x^3y^2 + 1620x^3y \\ & - 74x^3 - 3072x^2y^5 + 5888x^2y^4 - 6848x^2y^3 + 4200x^2y^2 - 774x^2y + x^2 + 1408xy^5 \\ & - 2240xy^4 + 1620xy^3 - 774xy^2 + 146xy - 2x - 192y^5 + 256y^4 - 74y^3 + y^2 - 2y) \\ & \div [3\sqrt{6}m_c^6x(2x-1)^2\bar{x}y(2y-1)^2\bar{y}(4xy-x-y)(4\bar{x}\bar{y}-\bar{x}-\bar{y})], \end{aligned} \quad (15)$$

$$\begin{aligned} T_{3\otimes 3}^{(2)} = & -(\alpha_s\pi)^3(4m_c^2g^{\mu\nu} + 3K^\mu K^\nu)\epsilon_{\mu\nu}(5120x^5y^3 - 7680x^5y^2 + 3520x^5y - 480x^5 - 512x^4y^4 - 11776x^4y^3 \\ & + 18944x^4y^2 - 9056x^4y + 1264x^4 + 5120x^3y^5 - 11776x^3y^4 + 20800x^3y^3 - 20960x^3y^2 + 8956x^3y \\ & - 1198x^3 - 7680x^2y^5 + 18944x^2y^4 - 20960x^2y^3 + 13168x^2y^2 - 4282x^2y + 485x^2 + 3520xy^5 \\ & - 9056xy^4 + 8956xy^3 - 4282xy^2 + 972xy - 71x - 480y^5 + 1264y^4 - 1198y^3 + 485y^2 - 71y) \\ & \div [36\sqrt{3}m_c^8x(2x-1)^2\bar{x}y(2y-1)^2\bar{y}(4xy-x-y)(4\bar{x}\bar{y}-\bar{x}-\bar{y})], \end{aligned} \quad (16)$$

the other contributions are zero, i.e.,

$$T_{3\otimes 3}^{(1)} = T_{6\otimes 6}^{(1)} = T_{6\otimes 6}^{(2)} = 0. \quad (17)$$

Note that the $i\epsilon$ -prescription is necessary for the denominators to ensure the correct identification of the branching cuts. The explicit forms of the $i\epsilon$ terms are as follows:

$$\begin{aligned} (2x-1)^2 & \rightarrow (2x-1+i\epsilon)(2x-1-i\epsilon) \\ & = -(2x-1+i\epsilon)(2\bar{x}-1+i\epsilon), \end{aligned} \quad (18)$$

$$\begin{aligned} 4xy-x-y & \rightarrow 4xy-x-y+i\epsilon, \\ 4\bar{x}\bar{y}-\bar{x}-\bar{y} & \rightarrow 4\bar{x}\bar{y}-\bar{x}-\bar{y}+i\epsilon. \end{aligned} \quad (19)$$

It is easy to verify that the hard kernels possess the following symmetry:

$$\begin{aligned} T_{3\otimes 3}^{(J)}(x, y) & = T_{3\otimes 3}^{(J)}(1-x, 1-y), \\ T_{6\otimes 6}^{(J)}(x, y) & = T_{6\otimes 6}^{(J)}(1-x, 1-y). \end{aligned} \quad (20)$$

Note that this symmetry also holds in the case with the $i\epsilon$ -prescription.

C. Master formula for amplitude

The leading-twist pion LCDA can be conveniently expanded in the Gegenbauer polynomial basis

$$\begin{aligned} \Phi_\pi(x, \mu_F) & = \frac{f_\pi}{2\sqrt{2N_c}} \sum_{n=0}' a_n(\mu_F) \psi_n(x), \quad \text{with} \\ \psi_n(x) & \equiv 6x\bar{x}C_n^{3/2}(2x-1) \end{aligned} \quad (21)$$

where the pion decay constant $f_\pi = 0.131$ GeV, and $\sum'$ signifies the sum over even integers. All the

nonperturbative dynamics are encoded in the Gegenbauer moments $a_n(\mu_F)$. Those nonperturbative parameters $a_n(\mu_F)$ at the factorization scale μ_F can be evolved from an initial scale μ_0 using the one-loop evolution equation

$$a_n^{1\text{-loop}}(\mu_F) = a_n(\mu_0) \left[\frac{\alpha_s(\mu_F^2)}{\alpha_s(\mu_0^2)} \right]^{\gamma(n)} \quad \text{with} \quad \gamma(n) \equiv \frac{\gamma_0(n)}{2b_0}, \quad (22)$$

where the standard β -function coefficient b_0 is given by $b_0 = (11N_c - 2n_f)/3$, and the one-loop anomalous dimensions $\gamma_0(n)$ are expressed as

$$\gamma_0(n) = 2C_F \left[4 \sum_{i=1}^{n+1} \frac{1}{i} - 3 - \frac{2}{(n+1)(n+2)} \right]. \quad (23)$$

For the kaon, one also needs to sum over the odd integers in addition to the even integers, i.e.,

$$\Phi_K(x, \mu_F) = \frac{f_K}{2\sqrt{2N_c}} \sum_{n=0} a_n(\mu_F) \psi_n(x), \quad (24)$$

where the kaon decay constant is given by $f_K = 0.160$ GeV.

By substituting the expansion in (21) into the factorization formula in (1), we obtain the following integrated master formula for the amplitude of the process $T_{4c}^{(0)} \rightarrow \pi^+ \pi^-$:

$$\begin{aligned} \mathcal{A}(T_{4c}^{(0)} \rightarrow \pi^+ \pi^-) &= \sqrt{2m_H} \frac{(\alpha_s \pi)^3}{m_c^6} f_\pi^2 \left[\frac{\langle \mathcal{O}_{\bar{3}\bar{3}3}^{(0)} \rangle}{4} \sum_{m,n} \mathcal{D}_{mn}^{(0)} a_m(\mu_F) a_n(\mu_F) \right. \\ &\quad \left. + \frac{\langle \mathcal{O}_{\bar{6}\bar{6}\bar{6}}^{(0)} \rangle}{4} \sum_{m,n} \mathcal{F}_{mn}^{(0)} a_m(\mu_F) a_n(\mu_F) \right] \end{aligned} \quad (25)$$

where the integrated coefficients $\mathcal{D}_{mn}^{(0)}$ and $\mathcal{F}_{mn}^{(0)}$ are introduced as

$$\begin{aligned} \frac{(\alpha_s \pi)^3}{m_c^6} \mathcal{D}_{mn}^{(0)} &= \int_0^1 dx \int_0^1 dy \frac{\psi_m(x)}{2\sqrt{2N_c}} T_{\bar{3}\bar{3}3}^{(J)}(x, y) \frac{\psi_n(y)}{2\sqrt{2N_c}}, \\ \frac{(\alpha_s \pi)^3}{m_c^6} \mathcal{F}_{mn}^{(0)} &= \int_0^1 dx \int_0^1 dy \frac{\psi_m(x)}{2\sqrt{2N_c}} T_{\bar{6}\bar{6}\bar{6}}^{(J)}(x, y) \frac{\psi_n(y)}{2\sqrt{2N_c}} \end{aligned} \quad (26)$$

For the process $T_{4c}^{(2)} \rightarrow \pi^+ \pi^-$, we have

$$\begin{aligned} \mathcal{A}(T_{4c}^{(2)} \rightarrow \pi^+(P) + \pi^-(K)) &= \sqrt{2m_H} \frac{(\alpha_s \pi)^3}{m_c^8} f_\pi^2 (4m_c^2 g^{\mu\nu} + 3P^\mu P^\nu) \epsilon_{\mu\nu} \\ &\quad \times \left[\frac{\langle \mathcal{O}_{\bar{3}\bar{3}3}^{(2)} \rangle}{4} \sum_{m,n} \mathcal{D}_{mn}^{(2)} a_m(\mu_F) a_n(\mu_F) \right], \end{aligned} \quad (27)$$

where $\mathcal{D}_{mn}^{(2)}$ is defined as

$$\begin{aligned} \frac{(\alpha_s \pi)^3}{m_c^8} (4m_c^2 g^{\mu\nu} + 3P^\mu P^\nu) \epsilon_{\mu\nu} \mathcal{D}_{mn}^{(2)} &= \int_0^1 dx \int_0^1 dy \frac{\psi_m(x)}{2\sqrt{2N_c}} T_{\bar{3}\bar{3}3}^{(2)}(x, y) \frac{\psi_n(y)}{2\sqrt{2N_c}}, \end{aligned} \quad (28)$$

and the polarization tensor $\epsilon_{\mu\nu}$ satisfies the following summation:

$$\sum_{\text{pol.}} |(4m_c^2 g^{\mu\nu} + 3P^\mu P^\nu) \epsilon_{\mu\nu}|^2 = 96m_c^4. \quad (29)$$

Note that the convolutions in (26) and (28) cannot be integrated straightforwardly using POLYLOGTOOLS [102] package due to the presence of $i\epsilon$ in the denominators. While one might attempt to transform the $i\epsilon$ into x as $x + i\epsilon$ and $\bar{x}$ as $\bar{x} + i\epsilon$, this leads to contradictions with the constraint $x + \bar{x} = 1$. Therefore, we employ the sector decomposition method [103,104], with the assistance of the Cheng-Wu theorem [105,106], to numerically integrate

TABLE I. Numerical values for $\mathcal{D}_{mn}^{(0)}$, $\mathcal{F}_{mn}^{(0)}$, and $\mathcal{D}_{mn}^{(2)}$, where the * case refers that $(m+n)$ is odd.

(m, n)	$\mathcal{D}_{mn}^{(0)}$	$\mathcal{F}_{mn}^{(0)}$	$\mathcal{D}_{mn}^{(2)}$	(m, n)	$\mathcal{D}_{mn}^{(0)}$	$\mathcal{F}_{mn}^{(0)}$	$\mathcal{D}_{mn}^{(2)}$
(0, 0)	0.9718 - 0.01863i	1.705 - 0.6430i	0.2361 - 0.3257i	(3, 3)	16.10 - 56.55i	17.42 - 74.67i	2.094 - 9.631i
(0, 2)	-5.257 + 3.416i	-7.071 + 7.514i	-0.9829 + 1.351i	(3, 5)	-28.31 + 120.2i	-32.41 + 156.7i	-3.922 + 19.56i
(0, 4)	14.19 - 17.05i	17.55 - 26.88i	2.200 - 4.052i	(3, 7)	39.95 - 215.7i	45.85 - 275.9i	5.570 - 34.00i
(0, 6)	-29.96 + 49.75i	-36.66 + 71.74i	-4.470 + 9.750i	(4, 4)	49.99 - 105.2i	63.93 - 144.9i	8.035 - 19.10i
(0, 8)	52.72 - 112.1i	64.26 - 152.5i	7.651 - 19.89i	(4, 6)	-83.63 + 213.0i	-106.9 + 283.0i	-13.12 + 36.14i
(1, 1)	2.518 - 5.502i	2.405 - 9.078i	0.3080 - 1.184i	(4, 8)	133.1 - 387.6i	166.8 - 504.4i	20.19 - 63.10i
(1, 3)	-5.745 + 18.53i	-6.701 + 26.33i	-0.8781 + 3.424i	(5, 5)	53.46 - 242.6i	64.13 - 312.3i	7.292 - 38.39i
(1, 5)	9.728 - 42.11i	12.27 - 57.26i	1.445 - 7.168i	(5, 7)	-79.10 + 419.2i	-94.08 + 531.8i	-11.00 + 64.95i
(1, 7)	-12.39 + 79.14i	-16.83 + 102.9i	-2.088 + 12.74i	(6, 6)	123.7 - 383.6i	158.1 - 497.3i	19.21 - 62.40i
(2, 2)	13.36 - 14.70i	19.33 - 24.90i	2.372 - 3.785i	(6, 8)	-181.7 + 643.0i	-228.4 + 823.8i	-27.77 + 101.8i
(2, 4)	-28.97 + 45.51i	-38.05 + 65.99i	-4.777 + 9.226i	(7, 7)	123.5 - 702.5i	144.3 - 885.1i	17.00 - 107.4i
(2, 6)	54.70 - 108.9i	70.24 - 148.6i	8.450 - 19.63i	(8, 8)	246.5 - 1012.i	310.2 - 1287.i	37.49 - 157.3i
(2, 8)	-93.35 + 219.9i	-115.5 + 290.2i	-13.94 + 37.18i	*	0	0	0

these convolutions. The details can be found in Appendix. This method can be easily extended to other process, such as $T_{4c,b} \rightarrow p\bar{p}$, and to various types of integration of phase-space in high-energy physics.

The convolution results for a few leading values of m and n are tabulated in the Table I. As observed, when $m+n$ is odd, the convolutions yield zero, it can be easily understood since $T_{3\otimes 3}^{(0)}(x, y) = T_{3\otimes 3}^{(0)}(1-x, 1-y)$ and $\psi_m(x) = (-1)^m \psi_m(1-x)$. A similar reasoning applies to the cases of $T_{6\otimes 6}^{(0)}(x, y)$ and $T_{3\otimes 3}^{(2)}(x, y)$.

III. PHENOMENOLOGICAL EXPLORATION

Now, we are ready to present the decay widths for the process $T_{4c,b}^{(J)} \rightarrow \pi^+\pi^-(K^+K^-)$ utilizing the calculated amplitude $\mathcal{A}(T_{4c,b}^{(J)} \rightarrow \pi^+\pi^-(K^+K^-))$, i.e.,

$$\Gamma(T_{4c,b}^{(J)} \rightarrow \pi^+\pi^-(K^+K^-)) = \frac{1}{2m_H} \frac{\Phi_2}{2J+1} |\mathcal{A}(T_{4c,b}^{(J)} \rightarrow \pi^+\pi^-(K^+K^-))|^2 \quad (30)$$

where $\Phi_2 = 1/(8\pi)$ is the massless two-body phase space.

Before making phenomenological predictions, we need to fix the various input parameters. We take the charm quark mass to be $m_c = 1.5$ GeV. The QCD running coupling constant at the renormalization scale μ_R is evaluated with the aid of the package RUNDEC [107]. Furthermore, we also take $\mu_R = \mu_F = \mu$.

Similar to the previous work [77], the LDMEs for T_{4c} are taken or extracted from Refs. [24] and [108], which we will refer to as Model I and Model II, respectively. These values for the LDMEs of T_{4c} are tabulated in Table II.

Similar to Refs. [109,110], the input nonperturbative parameters $a_n(\mu_F)$ for π and K are taken from RQCD [111] and LPC [112] Collaborations, and these values are listed in Table III.

In Fig. 2, we present the decay widths of $T_{4c} \rightarrow \pi^+\pi^-$ and $T_{4c} \rightarrow K^+K^-$ as functions of the renormalization/factorization scale μ . As can be seen, the dependence on the factorization/renormalization scale μ is relatively large at small values of μ . This dependence is expected to diminish with the inclusion of higher-order corrections. Furthermore, the predicted decay widths using the input parameters from the LPC collaboration are significantly larger than those obtained from RQCD for both the pion and the kaon. It is noteworthy that the predictions for $T_{4c}^{(0)}$ using the LDMEs from Model II are negative and unphysical. This is directly attributed to the negative value of the LDME $\langle \mathcal{O}_{3\otimes 3}^{(0)} \rangle \langle \mathcal{O}_{6\otimes 6}^{(0)} \rangle$ in Model II as can be observed in Table II. This finding also indicates that the contribution from the mixing between the $\bar{3} \otimes 3$ and $6 \otimes \bar{6}$ is significant compared to other contributions. To illustrate this concretely, we take $T_{4c}^{(0)} \rightarrow \pi^+\pi^-$ with the LPC input as an example. The corresponding decay width at $\mu = 2m_c$ is given by

$$\Gamma(T_{4c}^{(0)} \rightarrow \pi^+\pi^-) = \left[\frac{(\pi\alpha_s)^3 f_\pi^2}{m_c^6} \right]^2 \left\{ 0.013 |\langle \mathcal{O}_{3\otimes 3}^{(0)} \rangle|^2 + 0.038 \langle \mathcal{O}_{3\otimes 3}^{(0)} \rangle \langle \mathcal{O}_{6\otimes 6}^{(0)} \rangle + 0.028 |\langle \mathcal{O}_{6\otimes 6}^{(0)} \rangle|^2 \right\}_{\mu=2m_c} \quad (31)$$

where it can be observed that the coefficient of $\langle \mathcal{O}_{3\otimes 3}^{(0)} \rangle \langle \mathcal{O}_{6\otimes 6}^{(0)} \rangle$ is larger than the coefficients for $|\langle \mathcal{O}_{3\otimes 3}^{(0)} \rangle|^2$ and $|\langle \mathcal{O}_{6\otimes 6}^{(0)} \rangle|^2$.

TABLE II. Numerical values of the LDMEs for T_{4c} in Model I [24] and Model II [108], in unit of GeV^9 .

LDMEs	$ \langle \mathcal{O}_{3\otimes 3}^{(0)} \rangle ^2$	$\langle \mathcal{O}_{3\otimes 3}^{(0)} \rangle \langle \mathcal{O}_{6\otimes 6}^{(0)} \rangle$	$ \langle \mathcal{O}_{6\otimes 6}^{(0)} \rangle ^2$	$ \langle \mathcal{O}_{3\otimes 3}^{(1)} \rangle ^2$	$ \langle \mathcal{O}_{3\otimes 3}^{(2)} \rangle ^2$
Model I [24]	0.0347	0.0211	0.0128	0.0260	0.0144
Model II [108]	0.0187	-0.0161	0.0139	0.0160	0.0126

TABLE III. The nonperturbative input parameters $a_n(\mu_F)$ for π and K from RQCD [111] and LPC [112] Collaborations.

π	$a_2(2 \text{ GeV})$	$a_4(2 \text{ GeV})$	$a_6(2 \text{ GeV})$	$a_8(2 \text{ GeV})$
RQCD [111]	$0.116^{+0.019}_{-0.020}$	—	—	—
LPC [112]	0.258 ± 0.087	0.122 ± 0.056	0.068 ± 0.038	—
K	$a_1(2 \text{ GeV})$	$a_2(2 \text{ GeV})$	$a_3(2 \text{ GeV})$	$a_4(2 \text{ GeV})$
RQCD [111]	-0.0525^{+31}_{-33}	0.106^{+15}_{-16}	—	—
LPC [112]	$-0.108 \pm 0.014 \pm 0.051$	$0.170 \pm 0.014 \pm 0.044$	$-0.043 \pm 0.006 \pm 0.022$	$0.073 \pm 0.008 \pm 0.021$

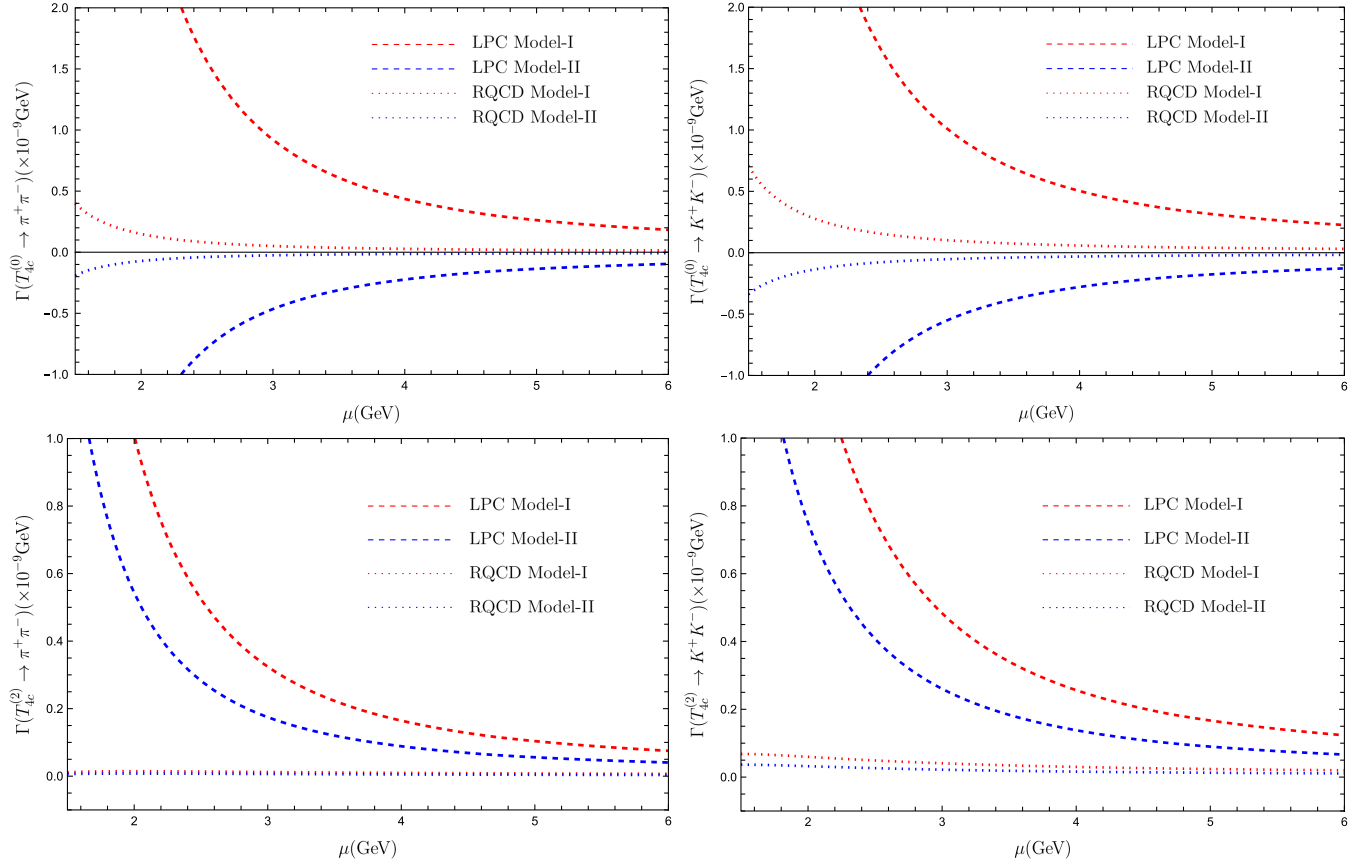


FIG. 2. Decay widths for $T_{4c} \rightarrow \pi^+\pi^-$ and $T_{4c} \rightarrow K^+K^-$ as a function of the renormalization/factorization scale μ . The input LDMEs for T_{4c} are taken from Model I [24] and Model II [108], while the input nonperturbative parameters $a_n(\mu_F)$ for π and K are taken from the RQCD [111] and LPC [112] Collaborations.

TABLE IV. The decay widths and branching ratios for $T_{4c,b} \rightarrow \pi^+\pi^- (K^+K^-)$, using input nonperturbative parameters $a_n(\mu)$ from the RQCD [111] and LPC [112] Collaborations, as well as LDMEs from Model I [24] and Model II [108]. The renormalization/factorization scale is set to $\mu = 2m_c$ for T_{4c} and $\mu = 2m_b$ for T_{4b} .

	LPC + Model I	LPC + Model II	RQCD + Model I	RQCD + Model II
$\Gamma(T_{4c}^{(0)} \rightarrow \pi^+\pi^-)$	9.2×10^{-10} GeV	-4.6×10^{-10} GeV	5.1×10^{-11} GeV	-2.5×10^{-11} GeV
$\text{Br}(T_{4c}^{(0)} \rightarrow \pi^+\pi^-)$	7.7×10^{-9}	-3.9×10^{-9}	4.2×10^{-10}	-2.1×10^{-10}
$\Gamma(T_{4c}^{(0)} \rightarrow K^+K^-)$	1.0×10^{-9} GeV	-5.5×10^{-10} GeV	1.0×10^{-10} GeV	-5.3×10^{-11} GeV
$\text{Br}(T_{4c}^{(0)} \rightarrow K^+K^-)$	8.4×10^{-9}	-4.6×10^{-9}	8.5×10^{-10}	-4.4×10^{-10}
$\Gamma(T_{4c}^{(2)} \rightarrow \pi^+\pi^-)$	3.2×10^{-9} GeV	1.7×10^{-10} GeV	1.2×10^{-11} GeV	6.6×10^{-12} GeV
$\text{Br}(T_{4c}^{(2)} \rightarrow \pi^+\pi^-)$	2.7×10^{-9}	1.5×10^{-9}	1.0×10^{-10}	5.5×10^{-11}
$\Gamma(T_{4c}^{(2)} \rightarrow K^+K^-)$	4.8×10^{-10} GeV	2.6×10^{-10} GeV	4.1×10^{-11} GeV	2.2×10^{-11} GeV
$\text{Br}(T_{4c}^{(2)} \rightarrow K^+K^-)$	4.0×10^{-9}	2.2×10^{-9}	3.4×10^{-10}	1.8×10^{-10}
$\Gamma(T_{4b}^{(0)} \rightarrow \pi^+\pi^-)$	1.7×10^{-14} GeV	-9.2×10^{-15} GeV	1.6×10^{-15} GeV	-9.0×10^{-16} GeV
$\Gamma(T_{4b}^{(0)} \rightarrow K^+K^-)$	2.3×10^{-14} GeV	-1.3×10^{-14} GeV	3.8×10^{-15} GeV	-2.2×10^{-15} GeV
$\Gamma(T_{4b}^{(2)} \rightarrow \pi^+\pi^-)$	7.7×10^{-15} GeV	4.1×10^{-15} GeV	1.0×10^{-15} GeV	5.4×10^{-16} GeV
$\Gamma(T_{4b}^{(2)} \rightarrow K^+K^-)$	1.3×10^{-14} GeV	7.1×10^{-15} GeV	2.8×10^{-15} GeV	1.5×10^{-15} GeV

Furthermore, to predict the branching ratio of the exclusive decays of T_{4c} into pions or kaons, we need to determine the total decay width of T_{4c} . Here, we assume that the decay of T_{4c} is dominated by its decay into double J/ψ , allowing us to approximate the total decay width of T_{4c} as follows:

$$\Gamma_{\text{total}}(T_{4c}) \approx \Gamma(T_{4c} \rightarrow J/\psi J/\psi) \approx 0.12 \text{ GeV}, \quad (32)$$

To make concrete numerical predictions, we present the detailed decay widths in Table IV. The decay widths of $T_{4c}^{(J)} \rightarrow \pi^+ \pi^-$ and $T_{4c}^{(J)} \rightarrow K^+ K^-$ are around 10^{-9} GeV, and the branching ratio is approximately 10^{-8} , which remains unobservable in current experiments.

Due to the lack of LDMEs $\langle \mathcal{O} \rangle_{T_{4b}}$ (including $\langle \mathcal{O}_{3\otimes 3}^{(J)} \rangle$ and $\langle \mathcal{O}_{6\otimes 6}^{(J)} \rangle$) of T_{4b} , we use a very crude estimate for the LDMEs of T_{4b} as done in Refs. [64,77], i.e.,

$$\langle \mathcal{O} \rangle_{T_{4b}} \approx \langle \mathcal{O} \rangle_{T_{4c}} \times \left(\frac{m_b \alpha_s^b}{m_c \alpha_s^c} \right)^{9/2}, \quad (33)$$

where α_s^b and α_s^c represent the strong couplings $\alpha_s(m_c v_c) \sim v_c$ and $\alpha_s(m_b v_b) \sim v_b$, respectively, with v_c and v_b denoting the typical velocities of the charm and bottom quarks. In this work, we take $m_b = 4.8$ GeV, $v_b = \sqrt{0.1}$ and $v_c = \sqrt{0.3}$. The decay widths for $T_{4b} \rightarrow \pi^+ \pi^-$ and $T_{4b} \rightarrow K^+ K^-$ are listed in Table IV. The decay widths of $T_{4b}^{(J)} \rightarrow \pi^+ \pi^-$ and $T_{4b}^{(J)} \rightarrow K^+ K^-$ are around 10^{-14} GeV, which fall far below the current experimental limits.

IV. SUMMARY AND OUTLOOK

In this work, we calculate the decay widths of the fully heavy tetraquark T_{4c} and T_{4b} into light mesons and their antiparticles. The decay width for T_{4c} into pions or kaons is on the order of 10^{-9} GeV, while for T_{4b} , it is approximately 10^{-14} GeV. The branching ratio for T_{4c} decaying into pions or kaons is about 10^{-8} , which is currently unobservable in experiments. The exclusive decays of $T_{4c,b}$ into pions or kaons differ from the process $J/\psi \rightarrow p\bar{p}$ in that one must retain the $i\epsilon$ -prescription in the denominator due to potential divergences in the integration domain. We have developed a systematic method to tackle these integrals, utilizing the Cheng-Wu theorem and the QHULL package. This method can be easily extended to other processes, such as $T_{4c,b} \rightarrow p\bar{p}$ and various phase-space integrations in high-energy physics.

ACKNOWLEDGMENTS

The authors are very grateful to Long-Bing Chen for his assistance with utilizing the POLYLOGTOOLS [102] package. We also thank Yu Jia and Wen-Long Sang for many useful discussions. The work of F. F. is supported by the National

Natural Science Foundation of China under Grant No. 12275353.

DATA AVAILABILITY

The data that support the findings of this article are openly available [113].

APPENDIX: TRIANGULARIZATION OF THE INTEGRATION DOMAIN

Let us consider the following simple integral $\mathcal{I}_0$:

$$\mathcal{I}_0 = \int_{-1}^1 \frac{dx}{x + i\epsilon} = \int_{-\delta}^{\delta} \frac{dx}{x + i\epsilon} = -i\pi. \quad (\text{A1})$$

In the sector decomposition method [103,104], to handle the divergence from the denominator, we add ϵ into the exponents, i.e.,

$$\begin{aligned} \mathcal{I}_0 &= \lim_{\epsilon \rightarrow 0} \int_{-1}^1 dx (x + i\epsilon)^{-1+\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \left[\int_{-1}^{-\delta} dx + \int_{\delta}^1 dx + \int_{-\delta}^{\delta} dx \right] (x + i\epsilon)^{-1+\epsilon}. \end{aligned} \quad (\text{A2})$$

Due to the existence of ϵ , the integration from $-\delta$ to δ becomes zero as $\delta \rightarrow 0$. Thus, we have

$$\begin{aligned} \mathcal{I}_0 &= \lim_{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0}} \left[\int_{-1}^{-\delta} dx + \int_{\delta}^1 dx \right] (x + i\epsilon)^{-1+\epsilon} \\ &\equiv \left[\int_{-1}^0 dx + \int_0^1 dx \right] (x + i\epsilon)^{-1+\epsilon}. \end{aligned} \quad (\text{A3})$$

For each integration domain, x has a definite sign, allowing us to explicitly remove the $i\epsilon$, i.e.,

$$\begin{aligned} \mathcal{I}_0 &= \int_{-1}^0 dx (x + i\epsilon)^{-1+\epsilon} + \int_0^1 dx (x + i\epsilon)^{-1+\epsilon} \\ &= (-1 + i\epsilon)^{-1+\epsilon} \int_{-1}^0 dx (-x)^{-1+\epsilon} + \int_0^1 dx x^{-1+\epsilon}. \end{aligned} \quad (\text{A4})$$

Next, we can integrate over x and obtain

$$\begin{aligned} \mathcal{I}_0 &= (-1 + i\epsilon)^{-1+\epsilon} \frac{1}{\epsilon} + \frac{1}{\epsilon} = \frac{1}{\epsilon} [e^{i\pi(-1+\epsilon)} + 1] \\ &= -i\pi + \mathcal{O}(\epsilon), \end{aligned} \quad (\text{A5})$$

showing that we have explicitly recovered the original result in (A1).

The key idea is to separate the integration domain into different parts so that the corresponding integrand has a definite sign in each part. This approach can be extended to integrals with multiple variables, including those involving δ -functions, with the assistance of the Cheng-Wu theorem [105,106].

Now, consider the following general integral, which commonly arises in phase-space integration in high-energy physics

$$\int_0^\infty dx_1 \dots \int_0^\infty dx_n \delta\left(1 - \sum_{i=1}^n x_i\right) \mathcal{P}_1^{n_1} \dots \mathcal{P}_m^{n_m}, \quad (\text{A6})$$

where $\mathcal{P}_i$ represents a linear combination of $\{x_1, \dots, x_n\}$. Generally, this type of integral falls under the category of complex-type integrals, which traditional sector decomposition methods with contour deformation are unable to handle. With the help of the Cheng-Wu theorem [105,106] and QHULL¹ package, one can convert this specific type of integration from complex type to Euclidean one, significantly improving the efficiency of numerical integration. The algorithm is implemented in the TRIANGULARIZE method in the latest version² of HEPLIB [98].

To illustrate the basic idea, let us consider the following example:

$$\mathcal{I}_1 = \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 (x_1 + x_2 - 2x_3 + i\varepsilon)^{-1+\epsilon} \times (x_1 - x_2 - x_3 + i\varepsilon)^{-1+\epsilon} \delta\left(1 - \sum_{i=1}^3 x_i\right) \quad (\text{A7})$$

The key point is to divide the integration domain, where $x_i \geq 0$, into subdomains such that the denominators $(x_1 + x_2 - 2x_3)$ and $(x_1 - x_2 - x_3)$ both have definite signs, i.e., are always positive or negative in each subdomain. This procedure is analogous to the triangularization of polyhedra in the geometry. This triangulation becomes more apparent when projecting the entire integration domain $x_i \geq 0$ onto a specific plane, for example, $x_1 + x_2 + x_3 = 1$, represented as the green triangle in Fig. 3.

Now the planes $(x_1 + x_2 - 2x_3) = 0$ and $(x_1 - x_2 - x_3) = 0$ intersect the green triangle as two line segments, shown as blue lines in Fig. 3. Those two blue line segments divide the triangle into four parts, resulting in both denominators $(x_1 + x_2 - 2x_3)$ and $(x_1 - x_2 - x_3)$ have definite signs within each division. To complete the triangulation, we add three additional line segments (shown in red in Fig. 3), resulting in the original triangle being divided into seven small triangles. Each small triangle becomes a tetrahedron by connecting to the origin. We can express the three faces of a tetrahedron that pass through the origin in the following general form, and these three faces also correspond to the boundaries of an integral region:

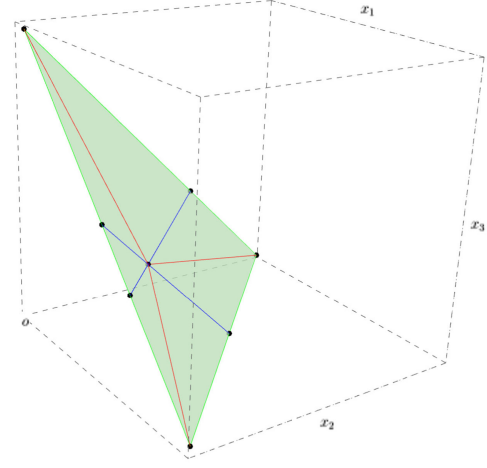


FIG. 3. Visualization for the triangularization.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 \end{cases} \quad (\text{A8})$$

We can then perform the following variable change:

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{cases} \quad (\text{A9})$$

With the help of Cheng-Wu theorem [105,106,114], which states,

$$\mathcal{I} = \int_0^\infty dx_1 \dots \int_0^\infty dx_n \mathcal{F}(x_1, \dots, x_n) \delta\left(1 - \sum_{i=1}^n a_i x_i\right), \quad (\text{A10})$$

where $\mathcal{F}(\lambda x_1, \dots, \lambda x_n) = \lambda^{-n} \mathcal{F}(x_1, \dots, x_n)$, the integral $\mathcal{I}$ remains unchanged for all cases with non-negative coefficients and at least one positive coefficient, i.e., $(\forall i, a_i \geq 0 \text{ and } \exists k, a_k > 0)$. One can transform the integration in each subdomain into the following form:

$$\mathcal{I}_1|_{\text{subdomain}} = \int_0^\infty dy_1 \int_0^\infty dy_2 \int_0^\infty dy_3 (\mathcal{C} + i\varepsilon)^{-1+\epsilon} \times (\mathcal{D} + i\varepsilon)^{-1+\epsilon} \delta\left(1 - \sum_{i=1}^3 y_i\right) \quad (\text{A11})$$

where $\mathcal{C}$ and $\mathcal{D}$ are the two original denominators, expressed as

$$\mathcal{C} = c_1 y_1 + c_2 y_2 + c_3 y_3, \quad \mathcal{D} = d_1 y_1 + d_2 y_2 + d_3 y_3. \quad (\text{A12})$$

¹QHULL: <http://www.qhull.org>.

²HEPLIB: <https://github.com/HepLib>.

Note that, within a specific subdomain, both $\mathcal{C}$ and $\mathcal{D}$ have definite signs, allowing us to remove the $i\epsilon$. For example, in the case where $\mathcal{C} > 0$ and $\mathcal{D} < 0$, we have

$$\mathcal{I}_1|_{\mathcal{C}>0, \mathcal{D}<0} = e^{i\pi(-1+\epsilon)} \int_0^\infty dy_1 \int_0^\infty dy_2 \int_0^\infty dy_3 \mathcal{C}^{-1+\epsilon} \times \mathcal{D}^{-1+\epsilon} \delta\left(1 - \sum_{i=1}^3 y_i\right). \quad (\text{A13})$$

Such an integral corresponds to a Euclidean type and can be easily calculated to high precision using the sector decomposition method. By repeating the procedures described above for all these subdomains, we arrive at the final numerical integration

$$\mathcal{I}_1 = -0.935008093(9) + 0.21230103415(0)i, \quad (\text{A14})$$

which can be compared with the exact analytical result obtained through direct integration using the *Mathematica* program

$$\mathcal{I}_1 = \frac{\text{Li}_2(-\frac{1}{2})}{6} + \frac{\text{Li}_2(4)}{6} - \frac{5\pi^2}{36} + \frac{\ln^2 2}{12} + \frac{1}{12} \ln 3 \ln 4 + \frac{1}{6} i\pi \ln 6 \quad (\text{A15})$$

$$= -0.93500809283492521156 + 0.21230103415326675516i. \quad (\text{A16})$$

The same idea can be easily extended to integrals involving two δ -functions, such as those in Eqs. (14)–(16) in this work.

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