Unification of gravity and standard model: Weyl-Dirac-Born-Infeld action

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We construct a unified (quantum) description, by the gauge principle, of gravity and Standard Model (SM), that generalizes the Dirac-Born-Infeld action to the SM and Weyl geometry, hereafter called Weyl-Dirac-Born-Infeld action (WDBI). The theory is formulated in $d = 4 - 2\epsilon$ dimensions. The WDBI action is a general gauge theory of SM and Weyl group (of dilatations and Poincaré symmetry), in the Weyl gauge covariant (metric) formulation of Weyl geometry. The theory is SM and Weyl gauge invariant in $d = 4 - 2\epsilon$ dimensions and there is no Weyl anomaly. The WDBI action has the unique elegant feature, not present in other gauge theories or even in string theory, that it is mathematically well defined in $d = 4 - 2\epsilon$ dimensions with no need to introduce in the action a UV regulator scale or field. This action actually *predicts* that gravity, through (Weyl covariant) space-time curvature $\hat{R}$, acts as UV regulator of both SM and gravity in $d = 4$. A series expansion of the WDBI action (in dimensionless couplings) recovers in the leading order a Weyl gauge invariant version of SM and the Weyl (gauge theory of) quadratic gravity. The SM and Einstein-Hilbert gravity are recovered in the Stueckelberg broken phase of Weyl gauge symmetry, which restores Riemannian geometry below Planck scale. Subleading orders are suppressed by powers of (dimensionless) gravitational coupling (ξ) of Weyl quadratic gravity.

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I. INTRODUCTION

In this work we search for a unified (quantum) description, by the gauge principle [1], of Standard Model (SM) and gravity. On the SM side this principle was extremely successful. Since gravity "is" geometry, applying this principle to gravity means to consider a gauged space-time symmetry; this dictates the underlying geometry and gravity action as a gauge theory action. Then what space-time symmetry can we consider beyond Poincaré symmetry?

Again, the SM points us in the right direction: SM with a vanishing Higgs mass *parameter* is scale invariant [2], which is a hint that this symmetry may be more fundamental,[1] in which case we should gauge it. This means gauging the Weyl group of dilatations and Poincaré symmetry [3–6]. Actually, there is not much else one can do: this is the *only true gauge theory* of a space-time symmetry beyond Poincaré [5] i.e., with a dynamical/physical gauge boson.[2] In the absence of matter, the gauge theory of the Weyl group "is" Weyl geometry (WG) [12–14] which has this gauge symmetry by construction: WG is defined by classes of equivalence of the metric and of Weyl gauge field (ω_μ) of dilatations, related by Weyl gauge symmetry transformations. The associated gravity action is then constructed as a (vector-tensor) gauge theory of the Weyl group [13], see [5,6] for an update.

No prior knowledge of Weyl geometry is needed here. Given its gauged dilatation invariance beyond Poincaré, Weyl geometry can be regarded as Riemannian geometry "covariantized" with respect to gauged dilatation symmetry (also known as Weyl gauge symmetry) [11,15]. More exactly, there exists a Weyl gauge covariant formulation of Weyl geometry, which is the only physical formulation and which is automatically *metric*, i.e., $\hat{\nabla}_\mu g_{\alpha\beta} = 0$ (but nonaffine) [5,15,16],[3] something overlooked for a century

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[1]Also at high energies or in the early universe, states are effectively massless, endorsing this idea.

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[2]One can also gauge the larger, full conformal group of Weyl plus special conformal symmetry [7] to obtain conformal gravity [8,9]. But this is *not* a true gauge theory since its action cannot have dynamical (physical) gauge bosons of Weyl dilatations and of special conformal symmetry [7], thus this theory does not have the spectrum of a gauge theory! It is for this reason its action is actually a *particular limit* of the action of the gauge theory of (smaller) Weyl group discussed here, when the Weyl gauge boson is "pure gauge" [6,10,11].

[3]The norm of a vector is invariant under Weyl-gauge-covariant parallel transport [5,17], [18] (Appendix B).

of Weyl geometry despite Dirac's suggestion [19]. The associated gauge theory is quadratic in curvatures, known as Weyl gauge theory of quadratic gravity ("Weyl quadratic gravity"), see reviews in [11,20].

This action is spontaneously broken *à la* Stueckelberg [21], in which the Weyl gauge field of dilatations ω_μ acquires mass proportional to Planck mass M_p after eating the would-be-Goldstone ϕ (or "dilaton" ghost) propagated by the higher derivative $\hat{R}^2$ term in the action, and then decouples [22,23]. As a result, Weyl geometry (connection) becomes Riemannian (Levi-Civita), respectively, and at scales below M_p one recovers Einstein-Hilbert action [22] with $\Lambda > 0$. Thus, the phase transition where Weyl gauge symmetry is broken is interpreted as a change of the underlying geometry. No moduli fields are added *ad hoc* for this breaking. All scales have geometric origin [18,23], being related to the vacuum expectation value (vev) of field ϕ from geometric $\hat{R}^2$ term, and since this mode was eaten by ω_μ, its vev does not need to be stabilized.

With these encouraging results, one can also add matter and consider the SM in Weyl geometry, giving the so-called SMW [23]. This is an interesting Weyl gauge invariant theory that describes gravity and SM, and respects current constraints, with Starobinsky-like inflation [24,25], good fits of galaxy rotation curves [26] and black-hole solutions [27]. However, this theory (SMW) does not seem the most general one, since it is ultimately "gluing" together in a sum the (Weyl gauge invariant) actions of SM and of Weyl geometry, i.e., Weyl quadratic gravity. It would be good to *derive* this action from a more fundamental one.

The goal of this paper is to find a more general, unified gauge theory action implementing the Weyl gauge symmetry, beyond the SMW scenario. The theory should make no distinction between SM and Weyl geometry field operators (curvatures, etc), in which these fields *and* their derivatives must transform *covariantly* with respect to both SM and Weyl gauge symmetries [much like SM fields do with respect to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$]. External and internal symmetries must be treated on equal footing in building the action.

Such unified gauge theory can be realized by a version of Dirac-Born-Infeld action [28–32] due to *both* SM and Weyl geometry, called here Weyl-Dirac-Born-Infeld (WDBI). The WDBI action is a space-time integral in $d = 4 - 2\epsilon$ dimensions of $\sqrt{\det A_{\mu\nu}}$ where $A_{\mu\nu}$ is a linear combination of operators of mass dimension 2, that are SM and Weyl gauge invariant; these operators are products of fields of SM and of Weyl geometry (curvatures) and of their covariant derivatives. This action gives a unified framework of internal (SM) and external (Weyl) gauge symmetries, with manifest covariance/invariance with respect to both symmetries. This action is obviously more general than the aforementioned sum of a Weyl gauge invariant version of SM action and of Weyl geometry action (Weyl quadratic gravity).

This WDBI action is a truly special gauge theory: by construction, it is automatically SM and Weyl gauge invariant in $d = 4 - 2\epsilon$ dimensions—a special feature due to Weyl geometry; the WDBI action is mathematically well-defined and *does not require an ultraviolet (UV) regulator* (be it a DR subtraction scale μ, field, etc) and all couplings do remain dimensionless. Introducing a DR scale μ would be a big problem since it would actually break Weyl gauge symmetry. The WDBI action actually *predicts* that the Weyl-gauge-covariant space-time curvature $\hat{R}^\epsilon$, i.e., geometry/gravity acts as UV regulator scale/field in[4] $d = 4 - 2\epsilon$ for SM and gravitational interactions; no DR scale μ or field are added by hand. This is the only gauge theory with this property, showing the importance of this WDBI action.

This mechanism does not work in Riemannian geometry where Weyl gauge covariance does not exist. For example, in ordinary gauge theories a UV regulator scale (DR scale μ, etc) or field is required and added by hand. In conformal gravity a dilaton field is added by hand as regulator, to maintain its symmetry [9]. Not even in string theory can *local* Weyl invariance (on Riemannian world sheet, not in physical space-time as here) be preserved by regularization, being broken by the DR scale μ that is needed/added in $d = 2 + \epsilon$; local Weyl symmetry can then be restored by a condition of vanishing Ricci tensor in target space [33].

The Weyl gauge invariance in $d = 4 - 2\epsilon$ dimensions of the WDBI action of SM and Weyl geometry is important, since it implies that this action is automatically Weyl anomaly-free [15,34–38]. Hence, the WDBI action is a consistent (quantum) gauge theory of gravity and SM. The WDBI action opens a new perspective on physics beyond SM and gravity, based on the gauge principle, that goes beyond the usual *quadratic* actions of gauge theories.

The plan of the paper is this: Section II reviews the formalism of WG as a gauge theory. Section III constructs the WDBI action and shows how SM and Weyl quadratic gravity are obtained in a leading order expansion. Einstein-Hilbert action is recovered in the broken phase, with subleading order corrections suppressed by M_p. Conclusions are in Sec. IV.

II. WEYL GEOMETRY AS A GAUGE THEORY OF WEYL GROUP

Let us first review briefly Weyl geometry as a gauge theory of the Weyl group, in the Weyl gauge covariant (metric, nonaffine) formulation [15], also [5,6,16] for more details. The formalism in this section is actually more general and valid in arbitrary d dimensions, but in the remaining Sec. III, where the SM operators and action are added, we obviously have $d = 4 - 2\epsilon$, $(\epsilon \to 0)$. Weyl

[4]This supports, *a posteriori*, the regularization used in SMW [23] showing it is Weyl-anomaly free [15].

geometry is defined by classes of equivalence of the metric ($g_{\mu\nu}$) and gauge field of dilatations (ω_μ), related by a Weyl gauge transformation, shown below (in the absence of matter)

$$g'_{\mu\nu} = \Sigma^2 g_{\mu\nu}, \qquad \omega'_\mu = \omega_\mu - \partial_\mu \ln \Sigma,$$
$$\sqrt{g'} = \Sigma^d \sqrt{g}, \qquad g'^{\mu\nu} = \Sigma^{-2} g^{\mu\nu} \tag{1}$$

where $\Sigma = \Sigma(x) > 0$. The Weyl charge q of $g_{\mu\nu}$ was set to $q = 2$—such normalization for an Abelian symmetry is a choice. To work with an arbitrary charge for $g_{\mu\nu}$, and also restore the Weyl gauge coupling α, replace $\Sigma^2 \to \Sigma^q$ and $\omega_\mu \to (\alpha q/2)\omega_\mu$ in our results.

Transformation (1) defines Weyl gauge symmetry. The definition of the geometry is completed by the so-called "nonmetricity" condition

$$\tilde{\nabla}_\mu g_{\alpha\beta} + 2\omega_\mu g_{\alpha\beta} = 0, \text{ where } \tilde{\nabla}_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}^\rho_{\lambda\mu} g_{\rho\nu} - \tilde{\Gamma}^\rho_{\lambda\nu} g_{\rho\nu}. \tag{2}$$

Assuming a symmetric connection $\tilde{\Gamma}^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\nu\mu}$, from (2) one finds $\tilde{\Gamma}$, invariant under (1)

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}|_{\partial_\mu \to \partial_\mu + 2\omega_\mu} = \Gamma^\rho_{\mu\nu} + (\delta^\rho_\mu \omega_\nu + \delta^\rho_\nu \omega_\mu - g_{\mu\nu}\omega^\rho), \tag{3}$$

with Γ the familiar Levi-Civita (LC) connection $\Gamma^\rho_{\mu\nu} = (1/2) g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$.

Further, one associates a Riemann tensor of Weyl geometry $\tilde{R}^\rho{}_{\mu\nu\sigma}$ to $\tilde{\Gamma}$, via the commutator of two $\tilde{\nabla}_\mu(\tilde{\Gamma})$ acting on a vector v^ρ: $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] v^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} v^\sigma$. This gives a Riemann tensor of Weyl geometry, defined by $\tilde{\Gamma}$, by a formula similar to that in Riemannian geometry, but with Γ replaced by $\tilde{\Gamma}$.[5] One then computes the Ricci tensor of Weyl geometry $\tilde{R}_{\mu\nu} = \tilde{R}^\sigma_{\mu\sigma\nu}$, etc. This gives the well-known affine, nonmetric ($\tilde{\nabla}_\mu g_{\alpha\beta} \neq 0$) formulation of Weyl geometry. This formulation, used for a century, is not physical since it is *not* Weyl gauge covariant. Indeed, with $\tilde{\Gamma}$ invariant under (1), one shows that $\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu}$ transforms like $g^{\mu\nu}$ i.e., $\tilde{R}' = \Sigma^{-2}\tilde{R}$, but $\tilde{\nabla}_\mu \tilde{R}$ is not Weyl covariant: $\tilde{\nabla}'_\mu \tilde{R}' \neq \Sigma^{-2}\tilde{\nabla}_\mu \tilde{R}$.

However, there does exist a Weyl gauge covariant formulation [15,19] of this geometry, as required for a gauge theory. One defines a gauge covariant derivative $\hat{\nabla}_\mu$ of a tensor field $T^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_p}$ of space-time charge $\tilde{q}_T$ with $T' = \Sigma^{\tilde{q}_T} T$, then [15][6]

$$\hat{\nabla}_\mu T = [\tilde{\nabla}_\mu(\tilde{\Gamma}) + \tilde{q}_T \omega_\mu] T \Rightarrow \hat{\nabla}'_\mu T' = \Sigma^{\tilde{q}_T} \hat{\nabla}_\mu T. \tag{4}$$

where we did not display the indices of the tensor $T^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_p}$.

Since $\hat{\nabla}_\mu$ depends on the charge $\tilde{q}_T$ of field T, no connection $\hat{\Gamma}$ can be associated to $\hat{\nabla}$ for all fields on which it acts; hence, this Weyl covariant formulation is *nonaffine*, but it is *metric* since we now have $\hat{\nabla}_\mu g_{\alpha\beta} = 0$. Thus one can do all calculations directly in this geometry, without going to a (metric) Riemannian geometry as done in the past (for a modern, rigorous interpretation of Weyl geometry as a gauge theory see [5,6,16]).

One then defines the Riemann tensor of Weyl geometry $\hat{R}^\lambda{}_{\mu\nu\sigma}$, using $\hat{\nabla}_\mu$ (instead of $\tilde{\nabla}_\mu$) in the standard definition of this tensor: $[\hat{\nabla}_\mu, \hat{\nabla}_\nu] v^\lambda = \hat{R}^\lambda{}_{\mu\nu\sigma} v^\mu$, where $v^\mu = e^\mu_a v^a$ is a vector with vanishing Weyl charge on the tangent space, $q_{v^a} = 0$.[7] With this, one can compute the Riemann tensor $\hat{R}^\mu{}_{\nu\rho\sigma}$, Ricci tensor $\hat{R}_{\mu\sigma} = \hat{R}^\lambda{}_{\mu\lambda\sigma}$ and Ricci scalar $\hat{R} = \hat{R}_{\mu\sigma} g^{\mu\sigma}$ of Weyl geometry, in terms of their Riemannian geometry counterparts. These relations are presented in the Appendix, Eq. (A1), and will be used later on.[8] One also shows [15] [Eq. (A-25)] that in this Weyl gauge covariant formulation, the Weyl tensor $\hat{C}^\mu_{\nu\rho\sigma}$ associated to $\hat{R}_{\mu\nu\rho\sigma}$ is equal to its Riemannian geometry version ($C^\mu_{\nu\rho\sigma}$), so $\hat{C}^\mu_{\nu\rho\sigma} = C^\mu_{\nu\rho\sigma}$.

In the Weyl gauge covariant (metric) formulation of Weyl geometry, under transformation (1) we have [15]

$$\hat{R}' = \Sigma^{-2}\hat{R}, \quad \hat{R}'_{\mu\nu} = \hat{R}_{\mu\nu}, \quad \hat{R}'^\sigma{}_{\mu\nu\rho} = \hat{R}^\sigma{}_{\mu\nu\rho},$$
$$\hat{\nabla}'_\mu \hat{R}' = \Sigma^{-2}\hat{\nabla}_\mu \hat{R}, \quad \hat{\nabla}'_\alpha \hat{R}'_{\mu\nu} = \hat{\nabla}_\alpha \hat{R}_{\mu\nu}, \quad \hat{\nabla}'_\alpha \hat{R}'^\sigma{}_{\mu\nu\rho} = \hat{\nabla}_\alpha \hat{R}^\sigma_{\mu\nu\rho},$$
$$X' = \Sigma^{-4} X, \quad X = \hat{R}^2_{\mu\nu\rho\sigma}, \quad \hat{R}^2_{\mu\nu}, \quad \hat{C}^2_{\mu\nu\rho\sigma}, \quad \hat{G}, \quad \hat{F}^2_{\mu\nu}, \tag{5}$$

Here the square of a tensor denotes contraction by the metric of indices in the same position. The field strength of ω_μ is $\hat{F}_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$, also invariant under (1). $\hat{G}$ is the Chern-Euler-Gauss-Bonnet term of Weyl geometry (hereafter Euler term), see Appendix, Eq. (A3).

We see now that the curvature tensors/scalar *and* $\hat{\nabla}_\mu$ acting on them do transform covariantly under (1), with the same Weyl charge as the operator itself. This property of Weyl geometry operators is similar to the implementation of (internal) gauge symmetries of the SM with respect to which fields and their derivatives transform covariantly.

This formulation of Weyl geometry may be seen as a covariantized version of Riemannian geometry with respect to the gauged dilatation symmetry [11,15]; since this formulation is metric, one can use it in applications [16], compute quantum corrections [15], etc.

[5]One has $\tilde{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \tilde{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \tilde{\Gamma}^\rho_{\mu\sigma} + \tilde{\Gamma}^\rho_{\mu\lambda}\tilde{\Gamma}^\lambda_{\nu\sigma} - \tilde{\Gamma}^\rho_{\nu\lambda}\tilde{\Gamma}^\lambda_{\mu\sigma}$.

[6]In general $\tilde{q}_T = p - r + q_T$ where q_T is the tangent space charge, see e.g., the review in Sec. 2 of [16].

[7]This definition can be extended if the tangent-space charge of this vector is nonzero [16] (Sec. II).

[8]The relation of Weyl covariant (metric) formulation to the nonmetric one is: $\hat{R}^\rho_{\sigma\mu\nu} = \tilde{R}^\rho_{\sigma\mu\nu} - \delta^\rho_\sigma \hat{F}_{\mu\nu}$.

Let us add that if one is not familiar with Weyl geometry, one may just regard Eq. (A1) as redefinitions of Riemann and Ricci tensors and scalar of Riemannian geometry, such that these redefined expressions and their derivative $\hat{\nabla}_\mu$ transform covariantly, as in Eq. (5).

Using the last equation in (5), the action of Weyl gauge theory of gravity ("Weyl quadratic gravity") associated to Weyl geometry in $d = 4$ dimensions, that is invariant under (1), is then [13] (see also more recent developments in [5,6,11,15,16,22,23,39])[9]

$$S_{\mathbf{w}} = \int d^4x\sqrt{g}\left\{\frac{1}{4!\xi^2}\hat{R}^2 - \frac{1}{\eta^2}\hat{C}^2_{\mu\nu\rho\sigma} - \frac{1}{4\alpha^2}\hat{F}^2_{\mu\nu} + \hat{G}\right\} \quad (6)$$

with perturbative couplings ξ, α, $\eta < 1$; for more on topological terms like $\hat{G}$ see [6,15].

Action (6) undergoes a Stueckelberg breaking of Weyl gauge symmetry, in which ω_μ becomes massive and decouples. One is left at low scales with Riemannian geometry and Einstein-Hilbert action and a positive cosmological constant [22,23] (we return to this action later in the text). Correspondingly, there is a conserved Weyl gauge current, $j_\mu \propto \hat{\nabla}_\mu \hat{R}$ with $\hat{\nabla}^\mu j_\mu = 0$ [16,22], which generalises a similar current in global scale invariant theories [40–44].

At geometric level, one can define a more general Weyl gauge invariant action than (6). Inspired by the Dirac-Born-Infeld (DBI) action [28,29] (for a review [31,32]) one can write a DBI-like action due to Weyl geometry itself, in d dimensions. This action is [30][10]

$$S'_{\mathbf{w}} = \int d^dx\{-\det[a_0\hat{R}g_{\mu\nu} + a_1\hat{R}_{\mu\nu} + a_2\hat{F}_{\mu\nu}]\}^{\frac{1}{2}}. \quad (7)$$

This action is rather special: unlike all other DBI-like actions, it has two new features: first, it has a gauge symmetry (Weyl gauge symmetry) in curved space-time in d dimensions; each term under det is invariant, see (5), while $a_{0,1,2}$ are some *dimensionless* coefficients. Second, another important feature is that no UV regulator scale (or field, tension, etc) is needed here to make this action well defined in $d = 4 - 2\epsilon$ dimensions.

A particular expansion of $S'_{\mathbf{w}}$ in ratios of (dimensionless) couplings a_j/a_0, $j = 1, 2$, recovers in the leading order the Weyl gauge theory of quadratic gravity, Eq. (6), while subleading orders might account for some quantum corrections to (6), see [30]. The immediate natural question is whether one can extend $S'_{\mathbf{w}}$ to include matter (Standard Model)?

III. WDBI ACTION: UNIFICATION OF GRAVITY AND SM

In this section we construct the Weyl-Dirac-Born-Infeld (WDBI) action of SM and Weyl geometry and study its properties, inspired by action (7). The goal is to write a gauge theory action that includes SM interactions alongside the gravitational interactions, on equal footing, while respecting both SM and Weyl gauge symmetries in $d = 4 - 2\epsilon$ dimensions.

To achieve this goal, we must identify all operators constructed from SM and Weyl geometry (curvatures) fields, that have mass dimension 2 (in $d = 4 - 2\epsilon$ dimensions) and are both SM and Weyl gauge invariant (Weyl charge $q = 0$). Why operators of mass dimension 2? Using these operators, the square root of the d-dimensional determinant of their linear combination (denoted $A_{\mu\nu}$) has mass dimension d. Therefore, the associated WDBI action is automatically dimensionless and mathematically well defined in $d = 4 - 2\epsilon$ dimensions, with dimensionless couplings, *without any additional regulator* like a DR scale μ, required in all other gauge theories in $d = 4 - 2\epsilon$. This has important implications discussed later.

The WDBI action sets on equal footing SM operators and Weyl geometry operators ($\hat{R}_{\mu\nu}$, $\hat{R}$, etc), internal and external gauge symmetries, and gives a unified description, by the gauge principle, of gravity and SM. This is a far more general gauge theory action than the (quadratic) gauge theory action of SM in Weyl geometry (SMW) [23], see shortly.

First let us specify the transformation of SM scalars ϕ and fermions ψ under (1)

$$\phi' = \Sigma^{q_\phi}\phi, \qquad \psi' = \Sigma^{q_\psi}\psi, \qquad q_\phi = -\frac{1}{2}(d-2),$$
$$q_\psi = -\frac{1}{2}(d-1), \qquad \Sigma = \Sigma(x) \quad (8)$$

The Weyl charges of ϕ, ψ are found from their (invariant) kinetic terms in curved space-time in d dimensions (see, e.g., the appendix in [23]). This is possible since SM with a vanishing Higgs mass *parameter* is scale invariant and gauging this scale symmetry (to obtain SM with Weyl gauge symmetry) is then immediate [23]. If $d = 4$ we have $q_\phi = -1$ and $q_\psi = -3/2$, i.e., Weyl charges coincide with their inverse mass dimension. The Weyl gauge covariant

[9]Action $S_{\mathbf{w}}$ is easily extended to $d = 4 - 2\epsilon$ dimensions by multiplying its integrand by $\hat{R}^{d/2-2}$ which does maintain the Weyl gauge symmetry of each term in the action.

[10]The structure of action (7) and subsequent (20) in the presence of SM, also has some remote similarity to string action (e.g., D3 brane effective action) [30]. The strings action is related to the integral of the world sheet area $\propto \sqrt{\det \gamma_{\alpha\beta}}$ generated by the propagating string. The counterpart to the induced metric $\gamma_{\alpha\beta}$ is here due not only to geometry (e.g., $\hat{R}g_{\mu\nu}$ term in (7), with $\hat{R}$ a string tension counterpart), but also due to matter/SM that modifies the geometry/metric (backreaction), see (20). This excludes related actions like $\int d^d\sigma \prod_i \det^{n_i}(A^{(i)}_{\mu\nu})$, $\sum_i n_i = \frac{1}{2}$, that could otherwise give, when expanded, a similar leading order action.

derivatives of ϕ, ψ are covariantized versions of their Riemannian version with respect to the gauged dilatation symmetry and transform covariantly with same charge, as shown below.

A. Weyl invariant operators of mass dimension two

Let us write the operators defined by the fields of SM and Weyl conformal geometry, that in $d = 4 - 2\epsilon$ dimensions have a mass dimension 2 and are both SM and Weyl gauge invariant (Weyl charge $q = 0$). In doing so, we include operators suppressed by powers of the Weyl scalar curvature $\hat{R}$. Using (1), (5), and (8), the list of such operators includes:

(i) Weyl geometry operators

$$\hat{R}g_{\mu\nu}, \qquad \hat{R}_{\mu\nu}, \qquad \hat{F}_{\mu\nu}. \tag{9}$$

(ii) SM gauge sector

$$F^{(1)}_{\mu\nu}, \qquad F^{(j)}_{\alpha\beta}F^{(j)}_{\rho\sigma}g^{\alpha\rho}g^{\beta\sigma}\hat{R}^{-1}g_{\mu\nu}. \tag{10}$$

Here $F^{(1)}_{\mu\nu}$ is the field strength of SM hypercharge field B_μ, $F^{(1)}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and $F^{(i)}_{\alpha\beta}F^{(i)\alpha\beta}$, $i = 1$, 2, 3 are SM gauge kinetic terms for U(1), SU(2), SU(3), in this order.

(iii) Higgs sector

$$(\widehat{\nabla}_\alpha H)(\widehat{\nabla}^\alpha H)^\dagger \hat{R}^{1-d/2}g_{\mu\nu}, \qquad H^\dagger H\hat{R}^{2-d/2}g_{\mu\nu}, \qquad (H^\dagger H)^2\hat{R}^{3-d}g_{\mu\nu}, \tag{11}$$

where

$$\widehat{\nabla}_\alpha H = (D_\alpha + q_H\omega_\alpha)H, \qquad q_H = -\frac{1}{2}(d-2). \tag{12}$$

Here $D_\alpha H = (\partial_\alpha - i\mathcal{A}_\alpha)H$ is the SM covariant derivative of the Higgs doublet, $\mathcal{A}_\alpha = (g/2)\vec{\sigma}.\vec{\mathcal{A}}_\alpha + (g'/2)B_\alpha$, with $\vec{\mathcal{A}}_\alpha$ the SU(2) gauge boson, B_α the U(1) of hypercharge, of gauge couplings g and g' respectively. One checks that $\widehat{\nabla}_\alpha H$ transforms covariantly under SM and Weyl gauge symmetry with the same charges as H.

(iv) SM fermionic sector (sum over SM fermions understood)

$$(i\bar{\psi}\gamma^a e^\alpha_a \widehat{\nabla}_\alpha\psi + \text{h.c.})\hat{R}^{1-d/2}g_{\mu\nu}, \tag{13}$$

with

$$\widehat{\nabla}_\alpha\psi = \left[D_\alpha + q_\psi\omega_\alpha + \frac{1}{2}\tilde{s}^{ab}_\alpha\sigma_{ab}\right]\psi, \quad q_\psi = -\frac{1}{2}(d-1). \tag{14}$$

D_α is the usual SM-covariant derivative of fermions, $\sigma_{ab} = (1/4)[\gamma_a, \gamma_b]$, ($a$, b are tangent space indices), and $\tilde{s}^{ab}_\mu$ is the spin connection in Weyl geometry; this has an expression given by the covariantized version (with respect to gauged dilatations) of the Riemannian spin connection $s^{ab}_\alpha = -e^{\lambda b}(\partial_\alpha e^a_\lambda - \Gamma^\nu_{\alpha\lambda}e^a_\nu)$ see, e.g., [5,23]

$$\tilde{s}^{ab}_\alpha = s^{ab}_\alpha|_{\partial_\alpha e^a_\nu \to [\partial_\alpha + \omega_\alpha]e^a_\nu} = s^{ab}_\alpha + (e^a_\alpha e^{\nu b} - e^b_\alpha e^{\nu a})\omega_\nu, \tag{15}$$

where we used that e^a_ν has Weyl charge $q = 1$ (half of that of $g_{\mu\nu}$). Note $\tilde{s}^{ab}_\alpha$ is invariant under (1), therefore $\widehat{\nabla}_\alpha\psi$ transforms covariantly under (1) with the same Weyl charge as ψ. Further, one notices $\gamma^\alpha\tilde{s}^{ab}_\alpha\sigma_{ab} = \gamma^\alpha s^{ab}_\alpha\sigma_{ab} + (d-1)\gamma^\alpha\omega_\alpha$ with $d = 4 - 2\epsilon$; therefore, in (14) the dependence on ω_α of the spin connection is canceled by that from $q_\psi\omega_\alpha$; then $\gamma^\alpha\widehat{\nabla}_\alpha\psi = \gamma^\alpha\nabla_\alpha\psi$ and then the expression in (13), invariant under (1), becomes

$$(i\bar{\psi}\gamma^a e^\alpha_a\nabla_\alpha\psi + \text{h.c.})\hat{R}^{1-d/2}g_{\mu\nu}, \tag{16}$$

with Riemannian operator $\nabla_\alpha = D_\alpha + (1/2)s^{ab}_\alpha\sigma_{ab}$. So even though they are charged under (1), in $d = 4 - 2\epsilon$ dimensions fermions do not couple directly to ω_α at tree-level except through $\hat{R}$, see Eq. (A1) (for $d = 4$ see [23,45]).

(v) Yukawa sector

$$[(\bar{\psi}_L Y_\psi H\psi_R + \bar{\psi}_L Y'_\psi\tilde{H}\psi'_R) + \text{h.c.}]\hat{R}^{2-3d/4}g_{\mu\nu}. \tag{17}$$

with $\tilde{H} = i\sigma_2 H^\dagger$ and Y, Y' Yukawa matrices. It is easily checked that the sum of the Weyl charges of the fields present is zero and this operator has mass dimension 2.

(vi) Gauge kinetic mixing term (ω_μ -hypercharge)

$$\hat{F}_{\alpha\beta}F^{(1)\alpha\beta}\hat{R}^{-1}g_{\mu\nu}. \tag{18}$$

This is invariant under SM group; it is also Weyl gauge invariant, with mass dimension 2.

(vii) Gauge kinetic term of ω_μ

$$\hat{F}_{\alpha\beta}\hat{F}^{\alpha\beta}\hat{R}^{-1}g_{\mu\nu}, \tag{19}$$

which is also Weyl gauge invariant, with mass dimension 2.

Additional operators of Weyl charge $q = 0$ and mass dimension two are possible and will be discussed later. Note also that we considered operators suppressed at most by one power of $\hat{R}$ for $d = 4$; higher suppression powers

can be considered, but they will not introduce new terms in the leading order action (Sec. III F).

B. WDBI action in $d = 4 - 2\epsilon$ dimensions

Using operators (9) to (19), we write a linear combination ($A_{\mu\nu}$) of these and integrate $\sqrt{\det A_{\mu\nu}}$ in $d = 4 - 2\epsilon$ dimensions. This gives a version of Dirac-Born-Infeld action of both SM and Weyl geometry, which we call Weyl-Dirac-Born-Infeld action (WDBI). The action is then

$$S_{\mathbf{d}} = \int d^d x [-\det A_{\mu\nu}]^{\frac{1}{2}}, \tag{20}$$

$$\begin{aligned} A_{\mu\nu} = {} & a_0 \hat{R} g_{\mu\nu} + a_1 \hat{R}_{\mu\nu} + a_2 \hat{F}_{\mu\nu} + a_3 F^{(1)}_{\mu\nu} + a^{(i)}_4 F^{(i)}_{\alpha\beta} F^{(i)\alpha\beta} g_{\mu\nu} \hat{R}^{-1} \\ & + a_5 |\widehat{\nabla}_\alpha H|^2 \hat{R}^{1-d/2} g_{\mu\nu} + a_6 |H|^2 \hat{R}^{2-d/2} g_{\mu\nu} + a_7 |H|^4 \hat{R}^{3-d} g_{\mu\nu} \\ & + a_8 (i \bar{\psi} \gamma^a e^\alpha_a \widehat{\nabla}_\alpha \psi + \text{h.c.}) \hat{R}^{1-d/2} g_{\mu\nu} \\ & + a_9 (\bar{\psi}_L Y_\psi H \psi_R + \bar{\psi}_L Y'_\psi \tilde{H} \psi'_R + \text{h.c.}) \hat{R}^{2-3d/4} g_{\mu\nu}, \\ & + a_{10} \hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta} \hat{R}^{-1} g_{\mu\nu} + a_{11} \hat{F}_{\alpha\beta} F^{(1)\alpha\beta} \hat{R}^{-1} g_{\mu\nu}, \end{aligned} \tag{21}$$

Action $S_{\mathbf{d}}$ has both SM and Weyl gauge invariances in $d = 4 - 2\epsilon$ dimensions, with dimensionless coefficients $a_0, \ldots, a_{11}$. Note that no UV regulator, DR subtraction scale μ or field, etc., is present in this action (a scale, if present, would actually break Weyl symmetry). We return to this issue shortly. Next, define

$$X^\lambda_\nu = \frac{g^{\lambda\rho}}{a_0 \hat{R}} A_{\rho\nu} - \delta^\lambda_\nu, \tag{22}$$

and expand $S_{\mathbf{d}}$[11]

$$S_{\mathbf{d}} = \int d^d x \sqrt{g} (a_0 |\hat{R}|)^{d/2} \left\{ 1 + \frac{1}{2} \text{tr} X + \frac{1}{4} \left(\frac{1}{2} (\text{tr} X)^2 - \text{tr} X^2 \right) + \mathcal{O}\left[\left(\frac{a_j}{a_0} \right)^3 \right] \right\}, \tag{23}$$

with $g = -\det g_{\mu\nu}$. X^λ_ν depends on ratios of coefficients, a_j/a_0 ($j = 1, \ldots, 11$) assumed to be small $|a_j/a_0| \ll 1$, as required for phenomenological reasons, that we verify later.[12] We find

$$\begin{aligned} S_{\mathbf{d}} = \int d^d x \sqrt{g} \Big\{ & \hat{R}^{d/2-2} [c_0 \hat{R}^2 + c_1 (\hat{C}^2_{\mu\nu\rho\sigma} - \hat{G}) + c_2 \hat{F}^2_{\mu\nu} + c_3 \hat{F}^{\mu\nu} F^{(1)}_{\mu\nu} + c^{(j)}_4 F^{(j)}_{\mu\nu} F^{(j)\mu\nu}] \\ & + c_5 |\widehat{\nabla}_\mu H|^2 + c_6 |H|^2 \hat{R} + c_7 |H|^4 \hat{R}^{2-d/2} + c_8 \left(\frac{i}{2} \bar{\psi}_L \gamma^a e^\alpha_a \nabla_\alpha \psi_R + \text{h.c.} \right) \\ & + c_9 (\bar{\psi}_L Y_\psi H \psi_R + \bar{\psi}_L Y'_\psi \tilde{H} \psi'_R + \text{h.c.}) \hat{R}^{1-d/4} + \mathcal{O}\left(\frac{1}{\hat{R}^3} \right) \Big\} + a_0^{d/2} \mathcal{O}\left(\frac{a_i}{a_0} \right)^3. \end{aligned} \tag{24}$$

The dimensionless coefficients c_j, $j = 1, \ldots, 9$ are functions of a_k ($k = 1, \ldots, 11$), found in Appendix, Eqs. (A6) to (A12) and show how terms in action (20) contribute to (24). The terms of coefficients a_{10} and a_{11} are redundant in the leading order, since they do not bring new operators in the action. Similarly for the term of coefficient a_7, but its presence ensures coefficient c_7 (of $|H|^4 \hat{R}^{2-d/2}$) is independent of c_6 (of $|H|^2 \hat{R}$), for phenomenological reasons.

Action (24) is brought to canonical form in Weyl geometry, shown below, with dimensionless physical perturbative couplings of gravity $\xi, \eta, \alpha < 1$, SM couplings $\alpha_j < 1$, ($j = 1, 2, 3$), nonminimal coupling $\xi_H < 1$, correct signs of kinetic terms and no gauge kinetic mixing ω_μ-hypercharge (investigated elsewhere [23]); we assume below $\xi \ll \eta \sim \alpha < 1$ for physical reasons detailed later. Then we obtain

[11] We use $[\det(1 + X)]^{1/2} = 1 + \frac{1}{2}\text{tr}X + \frac{1}{4}[\frac{1}{2}(\text{tr}X)^2 - \text{tr}X^2] + [\frac{1}{48}(\text{tr}X)^3 - \frac{1}{8}\text{tr}X\text{tr}X^2 + \frac{1}{6}\text{tr}X^3] + \mathcal{O}(X^4)$.

[12] In (23) and below, to simplify notation we wrote $\mathcal{O}[(a_j/a_0)^3]$ but we actually mean $\mathcal{O}(a_j a_k a_m / a_0^3)$.

$$S_{\mathbf{d}} = \int d^dx\sqrt{g}\Big\{\hat{R}^{d/2-2}\Big[\frac{1}{4!\xi^2}\hat{R}^2 - \frac{1}{\eta^2}(\hat{C}^2_{\mu\nu\rho\sigma} - \hat{G}) - \frac{1}{4\alpha^2}\hat{F}^2_{\mu\nu} - \frac{1}{4\alpha_j^2}F^{(j)}_{\mu\nu}F^{(j)\mu\nu}\Big]$$
$$+ |\widehat{\nabla}_\mu H|^2 - \frac{\xi_H}{6}|H|^2\hat{R} - \lambda|H|^4\hat{R}^{2-d/2} + \Big(\frac{i}{2}\bar{\psi}_L\gamma^a e^\alpha_a\nabla_\alpha\psi_R + \text{h.c.}\Big)$$
$$+ (\bar{\psi}_L Y_\psi H\psi_R + \bar{\psi}_L Y'_\psi\tilde{H}\psi'_R + \text{h.c.})\hat{R}^{1-d/4} + \mathcal{O}\Big(\frac{1}{\hat{R}^3}\Big)\Big\} + a_0^{d/2}\mathcal{O}\Big(\frac{a_i}{a_0}\Big)^3. \tag{25}$$

This is one of the main results of the paper that we discuss in detail shortly (Sec. III C). First, demanding that coefficients c_j have the values shown in (25), (A13), we find a solution for coefficients a_j in action (20) that brings (24) to canonical form (25). We have that a_0, a_1 are fixed by the two equations below

$$a_0^{d/2} = \frac{1}{\eta^2}\frac{16(d-3)}{d-2}\Big(\frac{a_0}{a_1}\Big)^2, \tag{26}$$

$$\frac{a_0}{a_1} = \frac{-1}{4}(1 \pm \sqrt{1+16\kappa} \approx \mp\sqrt{\kappa},$$
$$\kappa \equiv \frac{(d-2)}{16(d-1)}\Big[\frac{\eta^2}{24\xi^2}\frac{d-1}{d-3} - 1\Big] \gg 1. \tag{27}$$

so $a_0 \sim \xi^{-4/d}$. Assuming for simplicity $a_{10} = 0$ (this is easily relaxed), then we find a_2

$$\frac{a_2}{a_1} = \frac{d-2}{2}(-1 \pm \sqrt{1-z}) \sim \mathcal{O}(1),$$
$$z \equiv \frac{\eta^2}{4\alpha^2}\frac{1}{(d-2)(d-3)}. \tag{28}$$

$z < 1$ for $\eta^2 < 4\alpha^2(d-2)(d-3)$. The physical couplings ξ, η, α in (25) are then fixed by $a_{0,1,2}$ above. The rest of physical couplings are obtained for the following a_j, $j = 4, \ldots, 11$:

$$a_4^{(j)} = \frac{1}{4f}\Big[-\frac{a_0^{2-d/2}}{\alpha_j^2} - a_3^2\delta_{j1}\Big], (j = 1,2,3); \qquad a_5 = a_8 = a_9 = \frac{a_0^{2-d/2}}{f}, \qquad a_6 = -\frac{\xi_H}{6}\frac{a_0^{2-d/2}}{f}$$
$$a_7 = \frac{-1}{f}\Big[\lambda a_0^{2-d/2} + a_6^2\frac{d(d-2)}{8}\Big], \qquad a_{11} = \frac{-1}{f}(2a_2 + a_1(d-2)), \qquad f \equiv \frac{a_0 d}{2} + a_1\frac{d-2}{4}. \tag{29}$$

We see that $a_{1,2} \sim a_0\xi \sim \xi^{1-4/d}$ and $a_j \sim a_0^{1-d/2} \sim \xi^{2-4/d}$, $(j = 4, \ldots, 11;\ d = 4-2\epsilon)$; next, we also impose this last relation to a_3, which is possible since the above a_{11} enforces $c_3 = 0$, leaving a_3 arbitrary. To conclude, $|a_{1,2}/a_0| \sim \xi \ll 1$, $|a_j/a_0| \sim a_0^{-d/2} \sim \xi^2 \ll 1$, $j = 3, \ldots, 11$, and the convergence of expansion (23) is then assured for our solution for a_k, giving action (25) in $d = 4-2\epsilon$ dimensions.

C. Properties of WDBI action

Action (25) is still in the Weyl geometry formulation. To obtain this action in a Riemannian formulation, one simply replaces $\hat{R}$ of Weyl geometry by its Riemannian expression shown in Eq. (A1). All other terms, except $\hat{G}$, are unchanged: indeed, $\hat{F}_{\mu\nu}$ has the same expression in Riemannian and also in flat case, and in the Weyl covariant formulation used here the term $\hat{C}^2_{\mu\nu\rho\sigma}$ is equal to its Riemannian version, so $\hat{C}^2_{\mu\nu\rho\sigma} = C^2_{\mu\nu\rho\sigma}$, Eq. (A2).

Regarding $\hat{G}$ (Euler term), it is a topological term (total derivative) if $d = 4$ (hence it does not affect the equations of motion), but this changes in $d = 4-2\epsilon$ dimensions (for a discussion see [15]); its expression in Riemannian notation is found in (A3) with $\hat{R}_{\mu\nu\rho\sigma}$, $\hat{R}_{\mu\nu}$ and $\hat{R}$ replaced by their Riemannian counterparts, Eq. (A1).

Action (25) has interesting properties:

(i) In the leading order of $S_{\mathbf{d}}$ we obtained Weyl gauge invariant actions of the SM and of Weyl quadratic gravity [Eq. (6)] in $d = 4-2\epsilon$ dimensions; there are also nonminimal couplings of SM to gravity (ξ_H and those induced by $\hat{R}$ which contains ω_μ). If $d = 4$, the geometric part of this action [first three terms in (25)] recovers the Einstein-Hilbert gravity after a Stueckelberg mechanism [22,23], as reviewed in the next section.

(ii) With $d = 4-2\epsilon$, we see that in (25) the exact WDBI action *predicts* that the scalar curvature $\hat{R}^{-\epsilon}$, i.e., geometry acts as the UV regulator "scale"[13] for the leading order action of expanded $S_{\mathbf{d}}$. This is possible

[13]This requires $\hat{R}$ be nonzero, see later.

due to the Weyl gauge covariance of $\hat{R}$. The leading order action is thus mathematically well defined and needs no UV regulator (field or scale); the regularization is "built-in" exact $S_{\mathbf{d}}$. Being Weyl gauge invariant in $d = 4 - 2\epsilon$ dimensions, the leading order action is Weyl anomaly-free,[14] as discussed in [15] with the regularization derived here. Actually, at each order in the expansion, $S_{\mathbf{d}}$ is Weyl gauge invariant and Weyl-anomaly free.

(iii) The (exact) WDBI action, being itself Weyl gauge invariant in $d = 4 - 2\epsilon$ dimensions, is Weyl anomaly-free, too. Thus, the WDBI action is a consistent (quantum) gauge theory.

If one starts with the WDBI action in $d = 4$, its analytical continuation to $d = 4 - 2\epsilon$ does not require a DR scale μ—this is replaced by $\hat{R}$; the action is then mathematically well defined and Weyl gauge invariant, with no added UV regulator scale/field. Quantum calculations can be performed in this Weyl gauge invariant phase, respecting all symmetries of the theory. This shows the power of Weyl geometry as a gauge theory.

This elegant behavior is unique, not seen in theories in Riemannian geometry, where a UV regulator (scale or field) is necessarily added "by hand," to ensure the theory is mathematically well defined in $d = 4 - 2\epsilon$ dimensions. In particular, in conformal gravity a dilaton field is added *ad hoc* as regulator to maintain its symmetry [9] in $d = 4 - 2\epsilon$ dimensions. Finally, unlike here, in string theory local Weyl invariance (on the Riemannian world sheet, not in physical space-time as here) cannot be preserved by the DR scheme which breaks it in $d = 2 + \epsilon$. It is restored by the condition of vanishing Ricci tensor in target space, e.g., [33]. As a side-remark, this condition may not be necessary if world sheet geometry is that of Weyl geometry where this symmetry is natural in d dimensions, see Appendix.

D. WDBI action in $d = 4$ and the broken phase

Let us consider now the case of $d = 4$ dimensions in action (20), (25). We have

$$S_{\mathbf{4}} = \int d^4x\sqrt{g}[-\det A_{\mu\nu}]^{1/2} \tag{30}$$

$$\begin{aligned}\text{with}\quad A_{\mu\nu} = {} & a_0\hat{R}g_{\mu\nu} + a_1\hat{R}_{\mu\nu} + a_2\hat{F}_{\mu\nu} + a_3F^{(1)}_{\mu\nu} + a_4^{(j)}F^{(j)}_{\alpha\beta}F^{(j)\alpha\beta}g_{\mu\nu}\hat{R}^{-1} \\ & + a_5|\widehat{\nabla}_\alpha H|^2\hat{R}^{-1}g_{\mu\nu} + a_6|H|^2g_{\mu\nu} + a_7|H|^4\hat{R}^{-1}g_{\mu\nu} \\ & + a_8(i\bar{\psi}\gamma^a e_a^\alpha\widehat{\nabla}_\alpha\psi + \text{h.c.})\hat{R}^{-1}g_{\mu\nu} \\ & + a_9(\bar{\psi}_L Y_\psi H\psi_R + \bar{\psi}_L Y'_\psi\tilde{H}\psi'_R + \text{h.c.})\hat{R}^{-1}g_{\mu\nu} \\ & + a_{10}\hat{F}_{\alpha\beta}\hat{F}^{\alpha\beta}\hat{R}^{-1}g_{\mu\nu} + a_{11}\hat{F}_{\alpha\beta}F^{(1)\alpha\beta}\hat{R}^{-1}g_{\mu\nu}.\end{aligned} \tag{31}$$

Action (25) becomes

$$\begin{aligned}S_{\mathbf{4}} = \int d^4x\sqrt{g}\Big\{ & \frac{1}{4!\xi^2}\hat{R}^2 - \frac{1}{\eta^2}\hat{C}^2_{\mu\nu\rho\sigma} - \frac{1}{4\alpha^2}\hat{F}^2_{\mu\nu} - \frac{1}{4\alpha_j^2}F^{(j)}_{\mu\nu}F^{(j)\mu\nu} + |\widehat{\nabla}_\mu H|^2 - \frac{\xi_H}{6}|H|^2\hat{R} - \lambda|H|^4 + \left(\frac{i}{2}\bar{\psi}_L\gamma^a e_a^\alpha\nabla_\alpha\psi_R + \text{h.c.}\right) \\ & + (\bar{\psi}_L Y_\psi H\psi_R + \bar{\psi}_L Y'_\psi\tilde{H}\psi'_R + \text{h.c.}) + \mathcal{O}\left(\frac{1}{\hat{R}^3}\right)\Big\} + \mathcal{O}\left(\frac{a_i}{a_0}\right)^3\end{aligned} \tag{32}$$

provided that

$$\begin{aligned}& a_1 = \pm\frac{2\sqrt{2}}{\eta}, \qquad a_0 = \frac{-a_1}{4}(1 \pm \sqrt{1+16\kappa}), \qquad \kappa = \frac{1}{24}\left[\frac{\eta^2}{8\xi^2} - 1\right] \gg 1, \\ & a_2 = (-1 \pm \sqrt{1-z})a_1; \qquad z = \frac{\eta^2}{8\alpha^2}, \qquad a_4^{(j)} = \frac{-1}{4f}\left[\frac{1}{\alpha_j^2} - a_3^2\delta_{j1}\right]; \qquad j = 1,2,3. \\ & a_5 = a_8 = a_9 = \frac{1}{f}, \qquad a_6 = -\frac{\xi_H}{6f}, \qquad a_7 = \left[-\lambda - \frac{\xi_H^2}{36f^2}\right]\frac{1}{f}, \qquad a_{11} = \frac{-2}{f}(a_2 + a_1).\end{aligned} \tag{33}$$

[14] In Riemannian case Weyl anomaly [9,34–38] appears from μ-dependent terms with (local) Weyl symmetry broken by regularization in $d = 4 - 2\epsilon$ and from μ-independent Euler term. This situation changes in Weyl geometry [15] where in $d = 4 - 2\epsilon$, Weyl gauge symmetry is preserved, with Euler term Weyl gauge covariant.

where $f = 2a_0 + a_1/2$. The Weyl gauge covariant derivatives of Higgs and fermions in (32) are immediate from their expressions in Eqs. (12), (14), (16) evaluated for $d = 4$. The topological term $\hat{G}$ was removed from S_4, being a total derivative.

As mentioned, S_4 of (30) has an immediate analytical continuation (regularization), by replacing $d = 4 \to d = 4 - 2\epsilon$, to obtain the exact WDBI action S_d of (20) which is Weyl gauge invariant and Weyl anomaly-free. No regulator is introduced, $\hat{R}$ plays here this role.

The leading order of S_4 contains the SM action with a mild change in the Higgs sector to make it Weyl gauge invariant, with nonminimal gravitational couplings, plus the Weyl quadratic gravity action, first line in (32). We thus recovered in this leading order the action of SM in Weyl geometry (SMW), studied in [23]. The exact WDBI action is however more general and has additional contributions: these appear in its series expansion as subleading orders, which are higher dimensional (nonpolynomial) operators, discussed in Sec. III E.

It is well known that the leading order action shown in (32) has a Stueckelberg breaking mechanism of Weyl gauge symmetry [22,23]. Since this is relevant for the subleading orders of S_4, we briefly review this mechanism by considering only the geometric part of S_4, shown in (32),[15] which is

$$S_{\mathbf{w}} = \int d^4x\sqrt{g}\left\{\frac{1}{4!\xi^2}\hat{R}^2 - \frac{1}{\eta^2}\hat{C}^2_{\mu\nu\rho\sigma} - \frac{1}{4\alpha^2}\hat{F}^2_{\mu\nu}\right\}. \quad (34)$$

First, replace in this action $\hat{R}^2 \to -2\phi^2\hat{R} - \phi^2$, to obtain a new action which gives an equation of motion for ϕ of solution: $\phi^2 = -\hat{R}$ $(\hat{R} < 0)$[16] which replaced back in the action recovers $S_{\mathbf{w}}$; hence the two actions are equivalent. Next, one goes to the Riemannian picture, using (A1) for $d = 4$, to write $\hat{R}$ in terms of Riemannian scalar curvature R. After some arrangements the action in Riemannian geometry notation becomes [23]

$$S_{\mathbf{w}} = \int d^4x\sqrt{g}\left\{\frac{-1}{2\xi^2}\left[\frac{1}{6}\phi^2 R + (\partial_\mu\phi)^2\right] - \frac{\phi^4}{4!\xi^2} + \frac{\alpha^2 q^2}{8\xi^2}\phi^2[\omega_\mu - \partial_\mu \ln\phi]^2 - \frac{1}{4}F^2_{\mu\nu} - \frac{1}{\eta^2}C^2_{\mu\nu\rho\sigma}\right\} \quad (35)$$

where $F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu = \hat{F}_{\mu\nu}$ and we used Eq. (A2). The action remains invariant under (1). By applying transformation (1) with $\Sigma = \phi^2/\langle\phi^2\rangle$ one is fixing ϕ to its vev, assumed to exist. Naively, one sets $\phi \to \langle\phi\rangle$ in $S_{\mathbf{w}}$. In terms of transformed ("primed") fields the above action gives in the broken phase

$$S_{\mathbf{w}} = \int d^4x\sqrt{g'}\left[-\frac{1}{2}M_p^2 R' + \frac{1}{2}m_\omega^2\omega'_\mu\omega'^\mu - \Lambda M_p^2 - \frac{1}{4}\hat{F}'^2_{\mu\nu} - \frac{1}{\eta^2}C^2_{\mu\nu\rho\sigma}\right], \quad (36)$$

where we rescaled $\omega_\mu \to \alpha\omega_\mu$ and introduced the cosmological constant, Planck scale, and the mass of ω_μ

$$\Lambda \equiv \frac{1}{4}\langle\phi\rangle^2, \qquad M_p^2 \equiv \frac{\langle\phi^2\rangle}{6\xi^2}, \qquad m_\omega^2 \equiv 6\alpha^2 M_p^2. \quad (37)$$

All mass scales have geometric origin due to the field ϕ (from the $\hat{R}^2$ geometric term) that generates them [11,18]. As seen from (35), the gauge field ω_μ becomes massive in a Stueckelberg mechanism, by eating the derivative of $\ln\phi$ field which is the would-be-Goldstone of gauged dilatations.[17] This is the Weyl gauge symmetry breaking in the absence of matter. In the presence of the SM, the new would-be-Goldstone is a mixing (radial direction in the field space) of ϕ and the (neutral) Higgs field (h), since now both contribute to the Planck mass and m_ω in (37); in this sense, in action (35) one replaces $(1/\xi^2)\phi^2 \to (1/\xi^2)\phi^2 + \xi_H h^2$. The real (neutral) Higgs field is then the angular direction in the field space of initial ϕ and h. For details see [23] (Sec. 2.5 and Appendix C). This ends our review of the of breaking of Weyl gauge symmetry.

Since Λ and M_p are related, with $\xi^2 \sim \Lambda/M_p^2$, this explains our initial assumption $\xi \ll 1$. One has $m_\omega \sim M_p$ for α not far below 1, so massive ω_μ decouples below M_p and Weyl connection (3) and geometry become Levi-Civita connection and Riemannian geometry, respectively [22,23].[18] Further, for η near 1, we also have that the spin-two state due to the $C^2_{\mu\nu\rho\sigma}$ term in the presence of the Einstein term in (36), has a mass ηM_p [46] and thus it also decouples not far below M_p. Thus, for $\eta \sim \alpha < 1$ not too small, as we assumed, one is left below M_p with the Einstein-Hilbert action, with $\Lambda > 0$ and SM action with a Higgs sector with a coupling to ω_μ. The phenomenology of this action was discussed in [23].

To conclude, the WDBI action in $d = 4$, which is Weyl anomaly free, recovers, in the leading order, a Weyl gauge invariant action of SM and Weyl quadratic gravity. This gauge symmetry is broken in this order, the massive Weyl gauge boson and spin-two state decouple near M_p and one then recovers Einstein-Hilbert gravity, with $\Lambda > 0$, and the SM action.

[15] To see the breaking including the effects from SM action shown in (32), see Sec. 2.5 in [23].

[16] $\hat{R} < 0$ is consistent with $R = -12H_0^2$ $(\Lambda = 3H_0^2)$ obtained for a Friedmann-Robertson-Walker metric.

[17] $\ln\phi$ transforms with a shift under (1).

[18] If one is tuning α to ultraweak values ($\ll 1$), ω_μ can in principle be light (TeV scale or even lower) [23].

E. Subleading orders

What about the subleading orders of the expanded WDBI action? These are Weyl gauge invariant operators $\mathcal{O}(1/\hat{R}^3)$ (part of $\mathcal{O}[(a_i/a_0)^2]$) and $\mathcal{O}[(a_i/a_0)^3]$, see (24), (25), (32).

Concerning $\mathcal{O}(1/\hat{R}^3)$ terms, their origin is in $\mathrm{tr}X^2$ and $(\mathrm{tr}X)^2$; they arise from multiplying two SM-like operators of coefficients $a_j \propto a_0^{1-d/2} = \xi^{2-4/d}$, $(j = 3, 4, \ldots, 11; d = 4 - 2\epsilon)$. They have extra suppression relative to other terms $\mathcal{O}[(a_i/a_0)^2]$ due to mixed contributions SM—gravity, shown in (24), (25), (32). Examples of such operators are

$$\frac{a_4 a_6}{a_0^2}|H|^2 F_{\mu\nu}^{(i)2}\hat{R}^{-1-d/2}, \qquad \frac{a_6 a_7}{a_0^2}|H|^6\hat{R}^{3-3d/2}, \qquad \frac{a_6 a_9}{a_0}|H|^2\bar{\Psi}_L Y_\psi H\psi_R\hat{R}^{2-5d/4}, \tag{38}$$

The coefficients of these operators are of order $\sim\xi^4$. The first operator gives a term in $S_{\mathbf{d}}$

$$S_{\mathbf{d}} \sim \xi^2\int d^dx\sqrt{g}\frac{|H|^2F_{\mu\nu}^{(i)2}}{\hat{R}} \to \frac{1}{M_p^2}\int d^4x\sqrt{g}|H|^2F_{\mu\nu}^{(i)2}. \tag{39}$$

In the last step we used the broken phase in $d = 4$ with M_p of (37). Relative to the rest of $\mathcal{O}(a_j^2/a_0^2)$ operators that we kept in the leading order action, this contribution is strongly suppressed by $\xi^2 \ll 1$, (or by M_p^2 in the broken phase). Similar for the other two operators above. In general, $\mathcal{O}(1/\hat{R}^3)$ operators bring $\mathcal{O}(\xi^2)$ corrections to the physical couplings of the terms shown in the leading order action (recall $\xi^2 \sim \Lambda/M_p^2$).

Concerning $\mathcal{O}[(a_i/a_0)^3]$ operators, they generate corrections such as $\mathcal{O}(a_6^3/a_0^3)$ that contributes to the action a term like

$$S_{\mathbf{d}} \sim \xi^4\int d^dx\sqrt{g}\frac{|H|^6}{\hat{R}^{d-3}} \to \frac{\xi^2}{M_p^2}\int d^4x\sqrt{g}|H|^6, \tag{40}$$

which is more suppressed than (39). Since such operators respect the gauge symmetry, they may be generated as quantum corrections, if one computed these starting from the leading order action as tree-level action. In other words, the WDBI action may include some quantum effects, at least on geometric side [30]. To conclude, the expansion of the WDBI action generates subleading orders which are higher dimensional operators strongly suppressed by powers of gravitational coupling, $\xi^2 \ll 1$ (or by M_p^2 in the broken phase).

F. Other corrections

The list of Weyl gauge invariant operators of mass dimension 2, used to build the WDBI action was minimal, sufficient to recover in the leading order a Weyl gauge invariant SM action and Weyl quadratic gravity action. Additional similar operators could be present in $A_{\mu\nu}$, with new dimensionless coefficients. For example another operator is

$$\hat{R}^{\alpha\beta}\hat{F}_{\alpha\beta}\hat{R}^{-1}g_{\mu\nu} \propto \hat{F}_{\alpha\beta}\hat{F}^{\alpha\beta}\hat{R}^{-1}g_{\mu\nu} \tag{41}$$

since the antisymmetric part of $\hat{R}_{\alpha\beta}$ is $\hat{F}_{\alpha\beta}$. This operator generates a gauge kinetic term for ω_μ in the leading order action, already present in our action; up to a redefinition of Weyl gauge coupling, this operator brings no additional physics. Similarly, the operator obtained from the lhs of (41) with $\hat{F} \to F^{(1)}$, generates a gauge kinetic mixing (hypercharge—ω_μ), already discussed in the leading order and it can also be ignored.

A more general form of $A_{\mu\nu}$ is

$$A'_{\mu\nu} = A_{\mu\nu}[a_k g_{\mu\nu} \to a_k(g_{\mu\nu} + z_k\kappa_{\mu\nu})] \tag{42}$$

where $k = 4, 5, \ldots, 11$, and $z_4, \ldots z_{11}$ are new dimensionless coefficients, with $\kappa_{\mu\nu} \equiv \hat{R}_{\mu\nu}\hat{R}^{-1}$ which transforms under (1) just like the metric. With the new $A'_{\mu\nu}$ one shows that the same action is found in the leading order, up to a redefinition of coefficients c_k, without generating new terms. One can also extend $\kappa_{\mu\nu}$ to include corrections to it like $(1/\hat{R}^2)\hat{R}_{\alpha\beta}\hat{R}^{\alpha\beta}g_{\mu\nu}$, which has the same Weyl charge as the metric, and so on. Such corrections do not bring new terms in the leading order action discussed, but this may change in higher orders of the expanded action.

IV. CONCLUSIONS

In this work we constructed a general gauge theory beyond SM and gravity in $d = 4 - 2\epsilon$ dimensions, based on Weyl gauge group (of dilatations and Poincaré symmetries). The natural framework for such gauge symmetry is Weyl geometry where Weyl gauge symmetry is present by definition. We used the Weyl gauge covariant (metric) formulation of this geometry, which we reviewed. The action we found is a generalized version of the Dirac-Born-Infeld action for SM and Weyl geometry, which we called Weyl-Dirac-Born-Infeld (WDBI) action.

To find this action, one constructs a linear combination ($A_{\mu\nu}$) of all Weyl-gauge-invariant terms (in $d = 4 - 2\epsilon$) that have mass dimension two and are products of SM operators, Weyl geometry operators and their covariant derivatives. The space-time integral in $d = 4 - 2\epsilon$ of $\sqrt{\det A_{\mu\nu}}$ gives the WDBI action. To our knowledge, this is the most general gauge theory of the SM and of gravity based on the Weyl gauge group, in $d = 4 - 2\epsilon$ dimensions.

By construction, the WDBI action is mathematically well defined in $d = 4 - 2\epsilon$ dimensions, with SM and Weyl gauge invariance, and does not require a UV regulator scale (like a DR scale μ) or field added "by hand," as done in

ordinary (quadratic) gauge theories. Actually, a DR scale μ would be a problem since it breaks Weyl gauge symmetry. The WDBI action actually *predicts* that in $d = 4 - 2\epsilon$ the Weyl gauge covariant scalar curvature $\hat{R}^{\epsilon}$, i.e., geometry/gravity acts as a UV regulator for the $d = 4$ theory, as we saw in particular in a leading order of its series expansion. This is a special feature of the WDBI action that maintains Weyl gauge invariance in $d = 4 - 2\epsilon$, and shows that this action is more fundamental than ordinary (quadratic) gauge theories.

This special behavior is not possible in Riemannian geometry where Weyl gauge covariance does not exist; in ordinary gauge theories a regulator (DR scale μ, etc) is added by hand. Further, in conformal gravity action a dilaton is also added by hand as regulator field (to preserve its symmetry in $d = 4 - 2\epsilon$). Not even in string theory can local Weyl invariance (on Riemannian world sheet, not in space-time as here) be respected by regularization (in $d = 2 + \epsilon$), with this symmetry broken by the added DR scale μ; this symmetry is restored by a condition of vanishing Ricci tensor but this may not be necessary if the world sheet geometry is Weyl geometry (then Weyl scalar curvature could act as regulator and preserve the symmetry, as here).

Since the WDBI action has manifest Weyl gauge symmetry in $d = 4 - 2\epsilon$ dimensions, there is no Weyl anomaly, so this action is a consistent (quantum) gauge theory of gravity. In the leading order of a series expansion (in ξ) of the WDBI action, one recovers a Weyl gauge invariant version of SM action plus Weyl (gauge theory of) quadratic gravity; this theory undergoes a Stueckelberg breaking mechanism in which the Weyl gauge boson ω_μ becomes massive and Weyl gauge symmetry is broken. After ω_μ decouples below Planck scale, Riemannian geometry (Levi-Civita connection) is recovered in the broken phase, together with the Einstein-Hilbert gravity, SM action and a positive Λ.

Regarding the subleading orders of the expansion of WDBI action, these are operators suppressed by powers of dimensionless gravitational coupling (ξ), with a structure that has some similarities to quantum corrections to the leading order action. In other words, the WDBI action may encode some quantum corrections. In the broken phase, these operators are higher dimensional operators suppressed by powers of Planck scale, familiar in the SM.

To conclude, the WDBI action is a general gauge theory of SM and gravity, mathematically well defined and Weyl gauge invariant in $d = 4 - 2\epsilon$ dimensions and thus Weyl anomaly-free. This is an interesting unified (quantum) description, by the gauge principle, of SM and gravity, that deserves further study.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX

1. Weyl geometry formulas

We present some formulas in Weyl geometry and the relation to Riemannian geometry, in arbitrary d dimensions; in the text, in the WDBI action, we have $d = 4 - 2\epsilon$ ($\epsilon \to 0$). The relations of curvature tensors/scalar (with a hat) in the Weyl gauge covariant formulation of Weyl geometry, to their Riemannian geometry counterparts, are found by using their definitions in the text, see [15] (Appendix) and [6,16]

$$\begin{aligned}
\hat{R}_{\alpha\mu\nu\sigma} &= R_{\alpha\mu\nu\sigma} + \{g_{\alpha\sigma}\nabla_\nu\omega_\mu - g_{\alpha\nu}\nabla_\sigma\omega_\mu - g_{\mu\sigma}\nabla_\nu\omega_\alpha + g_{\mu\nu}\nabla_\sigma\omega_\alpha\} \\
&\quad + \{\omega^2(g_{\alpha\sigma}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\sigma}) + \omega_\alpha(\omega_\nu g_{\sigma\mu} - \omega_\sigma g_{\mu\nu}) + \omega_\mu(\omega_\sigma g_{\alpha\nu} - \omega_\nu g_{\alpha\sigma})\} \\
\hat{R}_{\mu\sigma} &= R_{\mu\sigma} + \left[\frac{1}{2}(d-2)F_{\mu\sigma} - (d-2)\nabla_{(\mu}\omega_{\sigma)} - g_{\mu\sigma}\nabla_\lambda\omega^\lambda\right] + (d-2)(\omega_\mu\omega_\sigma - g_{\mu\sigma}\omega_\lambda\omega^\lambda) \\
\hat{R} &= g^{\mu\sigma}\hat{R}_{\mu\sigma} = R - 2(d-1)\nabla_\mu\omega^\mu - (d-1)(d-2)\omega_\mu\omega^\mu.
\end{aligned} \tag{A1}$$

with $\hat{R}_{\alpha\mu\nu\sigma} = g_{\alpha\lambda}\hat{R}^{\lambda}_{\mu\nu\sigma}$. Here $R_{\alpha\mu\nu\sigma} = g_{\alpha\lambda}R^{\lambda}_{\mu\nu\sigma}$, $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$, $R = g^{\mu\nu}R_{\mu\nu}$ are the Riemann and Ricci tensor and scalar of Riemannian geometry, respectively, in d dimensions. The right-hand side (rhs) of these equations is in Riemannian notation, with $\nabla_\mu\omega_\nu = \partial_\mu\omega_\nu - \Gamma^{\rho}_{\mu\nu}\omega_\rho$, and Γ the Levi-Civita connection: $\Gamma^{\rho}_{\mu\nu} = (1/2)g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$.

Note $\hat{R}_{\mu\nu} - \hat{R}_{\nu\mu} = (d-2)\hat{F}_{\mu\nu}$, so $\hat{R}_{\mu\nu}$ is not symmetric if $d \neq 2$. The field strength $F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu = \hat{F}_{\mu\nu}$ has the same expression as in Weyl geometry.

One shows that in the Weyl gauge covariant formulation used in this work, the Weyl tensor $\hat{C}^{\mu}_{\nu\rho\sigma}$ associated to the Riemann tensor of Weyl geometry ($\hat{R}^{\mu}_{\nu\rho\sigma}$) is actually equal to its Riemannian counterpart ($C^{\mu}_{\nu\rho\sigma}$) [15] [Eq. (A-25)]

$$\hat{C}^{\mu}_{\nu\rho\sigma} = C^{\mu}_{\nu\rho\sigma}. \tag{A2}$$

In the text we used the following identities of Weyl conformal geometry (in the "hat" notation) that are similar to those of Riemannian geometry, but in a Weyl gauge covariant form [15,16]

$$\hat{G} = \hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\rho\sigma\mu\nu} - 4\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + \hat{R}^2 \tag{A3}$$

and

$$\hat{C}^2_{\mu\nu\rho\sigma} = \hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\rho\sigma\mu\nu} - \frac{4}{d-2}\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + \frac{2}{(d-1)(d-2)}\hat{R}^2, \tag{A4}$$

giving

$$\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} = \frac{d-2}{4(d-3)}(\hat{C}^2_{\mu\nu\rho\sigma} - \hat{G}) + \frac{d}{4(d-1)}\hat{R}^2. \tag{A5}$$

The last equation is used in Eq. (23) to replace the dependence on the Ricci tensor ($\hat{R}_{\mu\nu}$) of Weyl geometry by that on the Weyl tensor of Weyl geometry in the covariant formulation ($\hat{C}_{\mu\nu\rho\sigma}$) since this is identical to the Weyl tensor of Riemannian geometry, $C_{\mu\nu\rho\sigma}$.

2. Coefficients c_j

The coefficients c_j in action (24) have the following expressions in terms of a_j ($d = 4 - 2\epsilon$):

$$c_0 = \left[a_0^2 + \frac{1}{2}a_1 a_0 + a_1^2\frac{d-2}{16(d-1)}\right]a_0^{d/2-2} \tag{A6}$$

$$c_1 = -\frac{a_1^2(d-2)}{16(d-3)}a_0^{d/2-2} \tag{A7}$$

$$c_2 = \frac{a_2}{4}[a_2 + a_1(d-2) + a_{10}f]a_0^{d/2-2} \tag{A8}$$

$$c_3 = \frac{a_3}{4}[2a_2 + a_1(d-2) + a_{11}f]a_0^{d/2-2}, \tag{A9}$$

$$c_4^{(j)} = \left[a_4^{(j)}f + \delta_{1j}\frac{a_3^2}{4}\right]a_0^{d/2-2}, \qquad j = 1,2,3, \tag{A10}$$

$$c_k = [a_k f + \frac{1}{8}d(d-2)\delta_{k7}a_6^2]a_0^{d/2-2}, \qquad k = 5,6,\ldots 9. \tag{A11}$$

$$\text{with the notation: } f = a_0\frac{d}{2} + a_1\frac{(d-2)}{4}. \tag{A12}$$

The physical couplings in (25) are related to c_j as seen by comparing actions (24) and (25)

$$c_0 = \frac{1}{4!\xi^2}, \qquad c_1 = \frac{-1}{\eta^2}, \qquad c_2 = \frac{-1}{4\alpha^2}, \qquad c_3 = 0, \qquad c_4^{(i)} = \frac{-1}{4\alpha_i^2}, \qquad (i = 1,2,3)$$
$$c_5 = c_8 = c_9 = 1, \qquad c_6 = \frac{-\xi_H}{6}, \qquad c_7 = -\lambda, \tag{A13}$$

with α_i ($i = 1, 2, 3$) the gauge couplings of the SM and α the Weyl gauge coupling of dilatations.

From (A13) with (A6) to (A12) one finds the values of initial a_k that lead to physical couplings shown in action (25); these values are presented in Eqs. (26) to (29).

3. Weyl invariance in strings

While this is not important for our study, let us justify the last remark at the end of Sec. III C. Consider the string action below (S_s), with σ^α, $g_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) as world sheet coordinates and metric, respectively. This action has *local* (rather than gauged) Weyl invariance, i.e., the classical action is invariant under metric rescalling $g_{\alpha\beta} \to g'_{\alpha\beta} = \Sigma^2 g_{\alpha\beta}$; the difference from gauged Weyl invariance is that, unlike in (1), there is no gauge field ω_μ in this case ($d = 2$). In a standard notation

$$S_s = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X). \tag{A14}$$

At one-loop, local Weyl symmetry is broken. In a DR scheme in $d = 2 + \epsilon$, a regularized S_s is found by replacing $d^2\sigma \to d^{2+\epsilon}\sigma\mu^\epsilon$ in (A14). The DR scale μ ensures S_s be dimensionless, but initial, classical local Weyl symmetry of S_s is broken, since $\sqrt{g}g^{\alpha\beta}$ has now a nonzero Weyl charge $d - 2 = \epsilon$, see (1). Then the renormalized $G_{\mu\nu}(X)$ receives a correction of the form $\alpha'\mathcal{R}_{\mu\nu}(X)\ln(\Box/\mu^2)$ which is not Weyl invariant, with $\mathcal{R}_{\mu\nu}(X)$ the Ricci tensor in target space. Weyl symmetry is only restored by a (sufficient) condition of vanishing beta function of $G_{\mu\nu}(X)$, defined as a derivative with respect to $\ln\mu$, giving $\alpha'\mathcal{R}_{\mu\nu}(X) = 0$ [33].

However, if the world sheet geometry is actually Weyl geometry rather than Riemannian, a Weyl invariant regularized S_s exists, found by replacing in (A14): $d^2\sigma \to d^{2+\epsilon}\sigma\hat{R}^{\epsilon/2}$

$$S_s = \frac{1}{4\pi\alpha'}\int d^{2+\epsilon}\sigma\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X)\hat{R}^{\epsilon/2} \tag{A15}$$

With $\hat{R}$ as the world sheet scalar curvature of Weyl charge -2, [Eq. (5)], this regularized action, with no regulator scale μ needed/added, is now Weyl invariant in $d = 2 + \epsilon$ and thus, so are the counterterms and the renormalized action. One expects at one-loop a Weyl (gauge) invariant correction of the form $\alpha' \mathcal{R}_{\mu\nu}(X) \ln(\widehat{\Box}/\hat{R})$ to $G_{\mu\nu}(X)$. In any case, there is no need to demand $\alpha' \mathcal{R}_{\mu\nu}(X) = 0$ to maintain local Weyl symmetry in d dimensions, i.e., this is not a necessary condition. Note from (A1) that for $d = 2$: $\hat{R}_{\alpha\beta} - \frac{1}{2}\hat{R} g_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0$, just like in Riemannian case. It may be interesting to study further this observation.

[1] E. Noether, Invariant variation problems, Gott. Nachr. **1918**, 235 (1918) [Trans. Theor. Stat. Phys. **1**, 186 (1971)].
[2] W. A. Bardeen, On naturalness in the standard model, Fermi National Accelerator Laboratory (FNAL), Report No. FERMILAB-CONF-95-391-T.
[3] A. Bregman, Weyl transformations and Poincare gauge invariance, Prog. Theor. Phys. **49**, 667 (1973).
[4] J. M. Charap and W. Tait, A gauge theory of the Weyl group, Proc. R. Soc. A **340**, 249 (1974).
[5] C. Condeescu, D. M. Ghilencea, and A. Micu, Weyl quadratic gravity as a gauge theory and non-metricity vs torsion duality, Eur. Phys. J. C **84**, 292 (2024).
[6] C. Condeescu and A. Micu, The gauge theory of Weyl group and its interpretation as Weyl quadratic gravity, Classical Quantum Gravity **42**, 065011 (2025).
[7] M. Kaku, P. K. Townsend, and P. van Nieuwenhuizen, Gauge theory of the conformal and superconformal group, Phys. Lett. B **69**, 304 (1977).
[8] P. D. Mannheim and D. Kazanas, Exact vacuum solution to conformal Weyl gravity and galactic rotation curves, Astrophys. J. **342**, 635 (1989); D. Kazanas and P. D. Mannheim, General structure of the gravitational equations of motion in conformal Weyl gravity, Astrophys. J. Suppl. Ser. **76**, 431 (1991).
[9] F. Englert, C. Truffin, and R. Gastmans, Conformal invariance in quantum gravity, Nucl. Phys. **B117**, 407 (1976).
[10] D. M. Ghilencea and C. T. Hill, Standard model in conformal geometry: Local vs gauged scale invariance, Ann. Phys. (Amsterdam) **460**, 169562 (2024).
[11] D. M. Ghilencea, Quantum gravity from Weyl conformal geometry, Eur. Phys. J. C **85**, 815 (2025).
[12] Hermann Weyl, Gravitation und elektrizität, Sitz. Kön. Preuss. Akad. Wiss. 465 (1918) [An English version by D. H. Delphenich is currently available at: http://www.neo-classical-physics.info/spacetime-structure.html].
[13] Hermann Weyl Eine neue Erweiterung der Relativitätstheorie (A new extension of the theory of relativity), Ann. Phys. (Leipzig) **59**, 101 (1919) [An English version by D. H. Delphenich is currently available at this link: http://www.neo-classical-physics.info/spacetime-structure.html].
[14] H. Weyl, *Raum, Zeit, Materie, vierte erweiterte Auflage. Julius* (Springer, Berlin, 1921) [Space-time-matter, translated from German by Henry L. Brose, Methuen & Co Ltd, London, 1922, www.gutenberg.org/ebooks/43006 (Project Gutenberg License)].
[15] D. M. Ghilencea, Weyl conformal geometry vs Weyl anomaly, J. High Energy Phys. 10 (2023) 113.
[16] C. Condeescu, D. M. Ghilencea, and A. Micu, Conformal geometry as a gauge theory of gravity: Covariant equations of motion & conservation laws, Ann. Phys. (Amsterdam) **480**, 170125 (2025).
[17] M. P. Hobson and A. N. Lasenby, Weyl gauge theories of gravity do not predict a second clock effect, Phys. Rev. D **102**, 084040 (2020).
[18] D. M. Ghilencea, Non-metric geometry as the origin of mass in gauge theories of scale invariance, Eur. Phys. J. C **83**, 176 (2023).
[19] P. A. M. Dirac, Long range forces and broken symmetries, Proc. R. Soc. A **333**, 403 (1973).
[20] For a historical review, see E. Scholz, The unexpected resurgence of Weyl geometry in the late 20-th century physics, Einstein Stud. **14**, 261 (2018).
[21] E. C. G. Stueckelberg, Interaction forces in electrodynamics and in the field theory of nuclear forces, Helv. Phys. Acta **11**, 299 (1938).
[22] D. M. Ghilencea, Spontaneous breaking of Weyl quadratic gravity to Einstein action and Higgs potential, J. High Energy Phys. 03 (2019) 049; D. M. Ghilencea, Stueckelberg breaking of Weyl conformal geometry and applications to gravity, Phys. Rev. D **101**, 045010 (2020). For a brief review see Section 2.1 in [23].
[23] D. M. Ghilencea, Standard model in Weyl conformal geometry, Eur. Phys. J. C **82**, 23 (2022).
[24] P. G. Ferreira, C. T. Hill, J. Noller, and G. G. Ross, Scale-independent R^2 inflation, Phys. Rev. D **100**, 123516 (2019).
[25] D. M. Ghilencea, Weyl R^2 inflation with an emergent Planck scale, J. High Energy Phys. 10 (2019) 209.
[26] M. Crăciun and T. Harko, Testing Weyl geometric gravity with the SPARC galactic rotation curves database, Phys. Dark Universe **43**, 101423 (2024); P. Burikham, T. Harko, K. Pimsamarn, and S. Shahidi, Dark matter as a Weyl geometric effect, Phys. Rev. D **107**, 064008 (2023).
[27] J. Z. Yang, S. Shahidi, and T. Harko, Black hole solutions in the quadratic Weyl conformal geometric theory of gravity, Eur. Phys. J. C **82**, 1171 (2022).
[28] M. Born and L. Infeld, Foundations of the new field theory, Proc. R. Soc. A **144**, 425 (1934).
[29] P. A. M. Dirac, An extensible model of the electron, Proc. R. Soc. A **268**, 57 (1962).
[30] D. M. Ghilencea, Weyl gauge invariant DBI action in conformal geometry, Phys. Rev. D **111**, 085019 (2025).
[31] D. P. Sorokin, Introductory notes on non-linear electrodynamics and its applications, Fortschr. Phys. **70**, 2200092 (2022).

[32] A. A. Tseytlin, Born-Infeld action, supersymmetry and string theory, *The Many Faces of the Superworld* (World Scientific, Singapore, 2000).
[33] See, for example, D. Tong, String theory, arXiv:0908.0333.
[34] M. J. Duff, Twenty years of the Weyl anomaly, Classical Quantum Gravity **11**, 1387 (1994); Observations on conformal anomalies, Nucl. Phys. **B125**, 334 (1977).
[35] D. M. Capper, M. J. Duff, and L. Halpern, Photon corrections to the graviton propagator, Phys. Rev. D **10**, 461 (1974).
[36] D. M. Capper and M. J. Duff, Trace anomalies in dimensional regularization, Nuovo Cimento A **23**, 173 (1974).
[37] S. Deser, M. J. Duff, and C. J. Isham, Nonlocal conformal anomalies, Nucl. Phys. **B111**, 45 (1976).
[38] S. Deser and A. Schwimmer, Geometric classification of conformal anomalies in arbitrary dimensions, Phys. Lett. B **309**, 279 (1993).
[39] D. M. Ghilencea and H. M. Lee, Weyl gauge symmetry and its spontaneous breaking in the standard model and inflation, Phys. Rev. D **99**, 115007 (2019).
[40] P. G. Ferreira, C. T. Hill, and G. G. Ross, Scale-independent inflation and hierarchy generation, Phys. Lett. B **763**, 174 (2016).
[41] P. G. Ferreira, C. T. Hill, and G. G. Ross, No fifth force in a scale invariant universe, Phys. Rev. D **95**, 064038 (2017).
[42] P. G. Ferreira, C. T. Hill, and G. G. Ross, Inertial spontaneous symmetry breaking and quantum scale invariance, Phys. Rev. D **98**, 116012 (2018).
[43] P. G. Ferreira, C. T. Hill, and G. G. Ross, Weyl current, scale-invariant inflation and Planck scale generation, Phys. Rev. D **95**, 043507 (2017).
[44] J. Garcia-Bellido, J. Rubio, M. Shaposhnikov, and D. Zenhausern, Higgs-dilaton cosmology: From the early to the late universe, Phys. Rev. D **84**, 123504 (2011).
[45] K. Hayashi and T. Kugo, Everything about Weyl's gauge field, Prog. Theor. Phys. **61**, 334 (1979).
[46] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lüst, and A. Riotto, Aspects of quadratic gravity, Fortschr. Phys. **64**, 176 (2016).