

Gradient flow coupling of three- and four-dimensional QED

Lars Georg , Robert V. Harlander , and Robert H. Mason 

TTK, RWTH Aachen University, 52056 Aachen, Germany



(Received 30 January 2026; accepted 10 March 2026; published 16 April 2026)

We evaluate the coupling of quantum electrodynamics (QED) in the gradient-flow scheme in three and four space-time dimensions. Our general result applies to any theory with a $U(1)$ gauge field coupled to arbitrary other fields via arbitrary interactions. As an example, we consider QED with n_f flavors in three and four space-time dimensions and evaluate the corresponding β functions. In four dimensions, we find that the perturbative expansion of the β function behaves much better than the corresponding expression in quantum chromodynamics. In three dimensions, we recover both the ultraviolet as well as the infrared fixed points of the QED coupling in the large- n_f limit.

DOI: [10.1103/n586-vj2q](https://doi.org/10.1103/n586-vj2q)

I. INTRODUCTION

As first-principle descriptions of some central aspects of quantum chromodynamics (QCD) such as dynamical chiral symmetry breaking or confinement are still missing, simpler theories with similar properties may be helpful in gaining insight into theoretical concepts that can induce such phenomena. A very popular such theory is quantum electrodynamics (QED) in three space-time dimension, henceforth referred to as QED_3 in this paper. Its coupling has positive mass dimension, which makes the theory superrenormalizable. On the other hand, the theory becomes strongly interacting at low energies. This prohibits a naive perturbative approach to QED_3 . However, the infrared (IR) behavior is softened when considering a large number of fermion flavors n_f [1], in which case the theory develops an IR fixed point [2]. Systematically resumming powers of $1/n_f$ thus allows one to resort to a perturbative approach. A comprehensive review of large- n_f quantum field theory can be found in Ref. [3].

A particularly attractive feature of QED_3 is the dynamical generation of a fermion mass term via spontaneous chiral symmetry breaking [4]. However, QED_3 also receives a lot of attention from condensed matter physics, for example in the context of high-temperature superconductors (see, e.g., Ref. [5]).

The resemblance to QCD, in particular its nonperturbative character, suggests a lattice approach to QED_3 [6]. Traditionally, perturbative and lattice results are difficult to reconcile due to their different ways of regularizing the

divergences of quantum field theory (QFT). However, since the advent of the gradient-flow formalism (GFF), this obstacle can be overcome, because it provides a regularization/renormalization scheme which is accessible both perturbatively and on the lattice [7–11].

In QCD, the GFF is an established tool for practical lattice calculations, allowing for an efficient determination of the overall mass scale in a particular gauge configuration [9]. It extends the four-dimensional gauge fields to an auxiliary fifth dimension, the flow time t . The dependence of the flowed fields on this parameter is determined by the flow equation, which drives the fields from the unflowed gauge configuration toward the classical solution of the field equations.

One of the main features of the GFF is that composite operators of (renormalized) flowed fields do not require operator renormalization. Taken at finite flow time t , matrix elements of such operators can therefore be extrapolated to the continuum limit individually. The connection to the physical limit at $t = 0$ can be done with the help of suitable matching coefficients which can be obtained by a perturbative solution of the flow equations. Examples for applications of this method are found in Refs. [12–18].

The GFF provides a renormalization condition for the gauge coupling which can be implemented both perturbatively and on the lattice. In QCD, this allows one to derive the strong coupling $\alpha_s(M_Z)$, one of the central input parameters for perturbative calculations, from low-energy hadronic observables [19–24]. It was found that, while the conversion between the coupling from the $\overline{\text{MS}}$ to the GF scheme is perturbatively well behaved, the corresponding β function suggests a much smaller radius of convergence compared to its $\overline{\text{MS}}$ equivalent [19,25,26].

In this paper, we study the analogous quantity in QED, both in four and in three space-time dimensions. We show that, due to the linearity of the QED flow equations,

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

this coupling is given by the Laplace transform of the full photon propagator in the unflowed theory. Given the latter, the GF QED coupling can thus be obtained without any approximation. As an example, we evaluate this quantity in four-dimensional QED (QED₄), using the perturbative QED expression for the photon polarization function through four-loop level. We compare the resulting β function to the $\overline{\text{MS}}$ scheme and find that, as opposed to QCD, their perturbative behavior is very similar.

We then proceed to QED₃, where we adopt the large- n_f limit in order to derive the GF QED₃ coupling and the corresponding β function. Using suitable expansions of the result at short and long flow times, we recover the Gaussian fixed point in the ultraviolet (UV), as well as the nontrivial IR fixed point.

II. FLOWED QED

We consider a theory in D Euclidean space-time dimensions with a U(1) gauge factor and n_f massless fermion flavors. We write this theory as

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \sum_{i=1}^{n_f} \bar{\psi}_i \not{D} \psi_i + \frac{1}{2\xi} (\partial_\mu A_\mu)^2 + \Delta\mathcal{L}, \quad (2.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.2)$$

is the U(1) field-strength tensor, ψ_i is the fermion field of flavor i , and ξ is the U(1) gauge parameter. The dependence of all fields on the Euclidean space-time variable x is suppressed here and in the following, unless required for clarity. The term $\Delta\mathcal{L}$ contains arbitrary other sectors of the theory to which the fields A_μ , $\bar{\psi}_i$, and ψ_i may or may not be coupled. For example, some of the ψ_i could be massless quarks, and $\Delta\mathcal{L}$ describes their weak and strong interactions. In the following, the U(1) part of the theory will be most relevant; we will refer to it as QED, even though in the Standard Model (SM) it would correspond to the hypercharge interactions.

The flowed QED gauge field $B_\mu(t)$ is defined via the flow equation

$$\partial_t B_\mu(t) = \square B_\mu(t) + (\kappa - 1) \partial_\mu \partial_\nu B_\nu(t), \quad (2.3)$$

and the boundary condition

$$B_\mu(t=0) = A_\mu, \quad (2.4)$$

where κ is a gauge parameter which drops out of physical observables [27]; throughout this paper, we will set it to $\kappa = 1$.

As opposed to flowed QCD, the flow equation for the QED vector boson is linear and can be solved exactly in

terms of the unflowed field. For our choice $\kappa = 1$, the Fourier transform of the flowed field is given by

$$\tilde{B}_\mu(t, p) = \tilde{A}_\mu(p) e^{-tp^2}, \quad (2.5)$$

In principle one may also consider an electron flow equation for a complete gradient flow treatment of QED, but this is not required for the purpose of this paper. Due to (2.5), one can trivially express the matrix elements of purely photonic flowed QED operators in terms of regular QED fields. This can be seen in particular by focusing on the vacuum expectation value (VEV) of the flowed QED action density, obtained by

$$\begin{aligned} E(t) &= \frac{e_B^2}{4} \int d^D x \langle B_{\mu\nu}(t, x) B_{\mu\nu}(t, x) \rangle \\ &= \frac{e_B^2}{4} \int \frac{d^D p}{(2\pi)^D} e^{-2tp^2} \langle F_{\mu\nu}(p) F_{\mu\nu}(-p) \rangle \\ &= \frac{e^2}{2} (D-1) \int \frac{d^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 + \Pi_R(p)}, \end{aligned} \quad (2.6)$$

with the renormalized electric charge e and the renormalized photon polarization function Π_R . In diagrammatical form, Eq. (2.6) looks as follows¹:

$$E(t) = \text{diagram}, \quad (2.7)$$

where the shaded blob denotes the regular photon polarization function. It is important to note that the relation in Eq. (2.7) holds for the full Lagrangian in Eq. (2.1), including the non-QED interactions of $\Delta\mathcal{L}$. The latter only enter via the polarization function though.

In the language of the perturbative gradient flow in QCD, Eq. (2.7) can be understood as follows: Since the QED flow equation is linear, there are no outgoing photon *flow lines* from any of the *flowed vertices* (cf. Ref. [30] for the notation). Therefore, the flowed photons $B_\mu(t)$ occurring in the flowed action density must be connected to the polarization function at vanishing flow time, for which one can therefore use the expression of the unflowed theory.

Following the definition of the gradient-flow coupling in QCD [31], we may define the QED GF coupling [32,33]

$$a_e^{\text{GF}}(\mu) = \frac{(8\pi t)^{D/2}}{2\pi^2(D-1)} E(t) \Big|_{t=(c_{t\mu}/\mu)^2}, \quad (2.8)$$

¹The Feynman diagram has been produced with FeynGame [28,29].

where $c_{t\mu}$ is a fixed, but arbitrary constant, which is required to fully define the scheme. It may be worth pointing out that, according to Eq. (2.6) and up to a global factor, the gradient-flow coupling in QED is thus just the Laplace transform of $p^{D-2}/(1 + \Pi_R(p))$ with respect to p^2 .

Let us stress that Eq. (2.6)(2.7) are exact and hold for arbitrary space-time dimension D and the general renormalized photon polarization Π_R . In the following, we will consider $E(t)$ for two specific cases: (i) The perturbative approximation in four-dimensional QED through N_3 LO in the QED coupling for general flow time t ; (ii) The large- n_f limit for three-dimensional QED through next-to-leading order (NLO) in $1/n_f$ in the limit of short and long flow times. The following integral will be helpful in this respect:

$$I(t, \beta) = \int \frac{d^D p}{(2\pi)^D} p^{-\beta} e^{-2tp^2} = \frac{(2t)^{\beta/2}}{(8\pi t)^{D/2}} \frac{\Gamma((D-\beta)/2)}{\Gamma(D/2)}, \quad (2.9)$$

where $\Gamma(z)$ is Euler's Γ function, i.e., $z\Gamma(z) = \Gamma(z+1)$ and $\Gamma(1) = 1$. In writing the right-hand side (rhs) of Eq. (2.9), we assumed that the parameters D , β , and t are such that the integral exists, which can always be assumed in dimensional regularization.

III. FOUR-DIMENSIONAL QED

As an example for the application of Eq. (2.6), we consider QED with n_f massless fermions in four space-time dimensions and write the perturbative expansion of the $\overline{\text{MS}}$ -renormalized polarization function as

$$\Pi_R(p) = \sum_{n \geq 1} c_n(l_{\mu p}) a_e^n(\mu) = \sum_{n \geq 1} \sum_{m=0}^n c_{n,m} l_{\mu p}^m a_e^n(\mu), \quad (3.1)$$

where

$$l_{\mu p} = \ln \frac{\mu^2}{p^2} \quad \text{and} \quad a_e(\mu) = \frac{e^2(\mu)}{4\pi^2}, \quad (3.2)$$

with the $\overline{\text{MS}}$ renormalized QED₄ coupling $e(\mu)$ depending on the renormalization scale μ . The coefficients $c_{n,m}$ have been obtained through four loops (i.e., $n = 4$) in Refs. [34–38]. If we insert this expression into Eq. (2.6) and reexpand in a_e , we arrive at integrals of the form

$$I_m(t) = \int \frac{d^D p}{(2\pi)^D} e^{-2tp^2} l_{\mu p}^m = \frac{\partial^m}{\partial \beta^m} \mu^{2\beta} I(t, 2\beta) \Big|_{\beta=0}, \quad (3.3)$$

where one may insert the result of Eq. (2.9) in order to arrive at an analytical expression.

Following Eq. (2.8), we obtain

$$a_e^{\text{GF}}(\mu) = a_e(\mu) \sum_{n \geq 0} \sum_{m=0}^n e_{n,m} a_e^n(\mu) L_{\mu t}^m, \quad (3.4)$$

where

$$L_{\mu t} = \ln 2c_{t\mu}^2 + \gamma_E, \quad (3.5)$$

with $\gamma_E = -\Gamma'(1) = 0.577216\dots$, and the constant $c_{t\mu}$ introduced in Eq. (2.8). The result for the nonlogarithmic coefficients reads

$$\begin{aligned} e_{0,0} &= 1, \quad e_{1,0} = -\frac{2}{9}n_f, \\ e_{2,0} &= \left(\zeta(3) - \frac{43}{48}\right)n_f + \left(\frac{\pi^2}{54} - \frac{5}{81}\right)n_f^2, \\ e_{3,0} &= n_f \left[\frac{67}{144} + \frac{37}{24}\zeta(3) - \frac{5}{2}\zeta(5) \right] \\ &\quad + n_f^2 \left[\frac{3113}{2592} + \frac{5\pi^2}{144} - \frac{7}{6}\zeta(3) \right] \\ &\quad + n_f^3 \left[\frac{100}{729} - \frac{\pi^2}{81} - \frac{2}{27}\zeta(3) \right], \\ e_{4,0} &= n_f \left[-\frac{107}{768} - \frac{13}{32}\zeta(3) - \frac{245}{32}\zeta(5) + \frac{35}{4}\zeta(7) \right] \\ &\quad + n_f^2 \left[\frac{299}{512} + \frac{\pi^2}{96} - \frac{11\pi^4}{8640} - \frac{97}{12}\zeta(3) + 2\zeta^2(3) + \frac{115}{18}\zeta(5) \right] \\ &\quad + n_f^3 \mathbf{si} \left[-\frac{299}{432} + \frac{\pi^4}{360} + \frac{31}{48}\zeta(3) + \frac{2}{3}\zeta^2(3) - \frac{5}{4}\zeta(5) \right] \\ &\quad + n_f^4 \left[-\frac{26611}{31104} - \frac{347\pi^2}{2592} + \frac{47}{216}\zeta(3) + \frac{\pi^2}{9}\zeta(3) + \frac{5}{9}\zeta(5) \right] \\ &\quad + n_f^4 \left[-\frac{875}{6561} - \frac{5\pi^2}{729} + \frac{\pi^4}{540} + \frac{16}{243}\zeta(3) \right], \end{aligned} \quad (3.6)$$

where $\zeta(z)$ is Riemann's ζ function with values $\zeta(3) = 1.20206\dots$, $\zeta(5) = 1.03693\dots$ $\mathbf{si} = 1$ is a marker used to separate singlet and nonsinglet contributions, analogous to Ref. [38]. This conversion between the $\overline{\text{MS}}$ and the GF scheme is illustrated in Fig. 1 for $n_f = 1$ and $n_f = 5$.

Instead of quoting the logarithmic coefficients explicitly, we remark that they can be easily obtained from the renormalization group (RG) invariance of the action density,

$$\mu \frac{d}{d\mu} E(t) = 0, \quad (3.7)$$

leading to the recurrence relations

$$e_{0,k} = 0, \quad k e_{n,k} = \sum_{l=0}^{n-1} (n-l) e_{n-l-1,k-1} \beta_{e,l}, \quad k \geq 1, \quad n \geq 1, \quad (3.8)$$

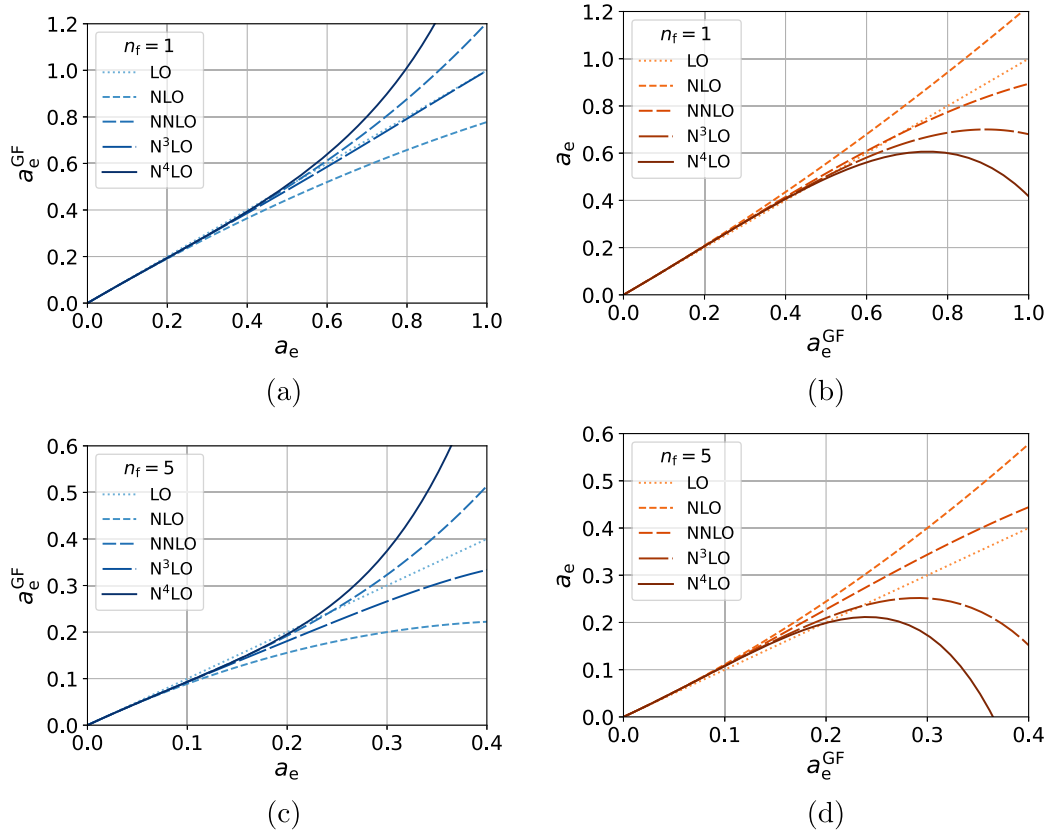


FIG. 1. The conversion of the coupling from the $\overline{\text{MS}}$ scheme to the GF scheme (a), (c) and back (b), (d) for QED₄. Here we used $c_{t\mu} = e^{-\gamma_E/2}/\sqrt{2}$ and $n_f = 1$ (a), (b) or $n_f = 5$ (c), (d).

with the QED₄ β function given by

$$\beta_e = -a_e \sum_{k \geq 0} \beta_{e,k} a_e^k \quad (3.9)$$

where [39–44]

$$\begin{aligned} \beta_{e,0} &= -\frac{1}{3}n_f, & \beta_{e,1} &= -\frac{1}{4}n_f, & \beta_{e,2} &= \frac{11}{144}n_f^2 + \frac{1}{32}n_f, \\ \beta_{e,3} &= \frac{77}{3888}n_f^3 + \left(\frac{13}{36}\zeta(3) - \frac{95}{864}\right)n_f^2 + \frac{23}{128}n_f, \\ \beta_{e,4} &= -\left(\frac{1}{216}\zeta(3) + \frac{107}{31104}\right)n_f^4 \\ &\quad + \left(\frac{5}{12}\zeta(5) - \frac{125}{216}\zeta(3) + \frac{13\pi^4}{8640} + \frac{10879}{41472}\right)n_f^3 \\ &\quad - \left(\frac{85}{32}\zeta(5) - \frac{31}{32}\zeta(3) - \frac{3731}{4608}\right)n_f^2 \\ &\quad - \left(\frac{4157}{6144} + \frac{1}{8}\zeta(3)\right)n_f. \end{aligned} \quad (3.10)$$

While the conversion from the $\overline{\text{MS}}$ to the GF scheme of the coupling according to Eq. (3.4) depends on the choice of

the constant $c_{t\mu}$, the GF β function is independent of it

$$\mu \frac{d}{d\mu} a_e^{\text{GF}} = a_e^{\text{GF}} \beta_e^{\text{GF}}, \quad \beta_e^{\text{GF}} = -a_e^{\text{GF}} \sum_{k \geq 0} \beta_{e,k}^{\text{GF}} (a_e^{\text{GF}})^k. \quad (3.11)$$

The first two coefficients are universal, i.e.,

$$\beta_{e,0}^{\text{GF}} = \beta_{e,0}, \quad \beta_{e,1}^{\text{GF}} = \beta_{e,1}, \quad (3.12)$$

see Eq. (3.10), while the higher orders are easily obtained from the results above

$$\begin{aligned} \beta_{e,2}^{\text{GF}} &= \beta_{e,2} - \beta_{e,1}e_1 + \beta_{e,0}(e_2 - e_1^2), \\ \beta_{e,3}^{\text{GF}} &= \beta_{e,3} - 2\beta_{e,2}e_1 + \beta_{e,1}e_1^2 + 2\beta_{e,0}(e_3 - 3e_2e_1 + 2e_1^3), \\ \beta_{e,4}^{\text{GF}} &= \beta_{e,4} - 3\beta_{e,3}e_1 - \beta_{e,2}(e_2 - 4e_1^2) + \beta_{e,1}(e_3 - 2e_1e_2) \\ &\quad + \beta_{e,0}(3e_4 - 12e_3e_1 - 5e_2^2 + 28e_2e_1^2 - 14e_1^4), \end{aligned} \quad (3.13)$$

with

$$e_n = \sum_{m=0}^n e_{n,m} L_{\mu t}^m \quad (3.14)$$

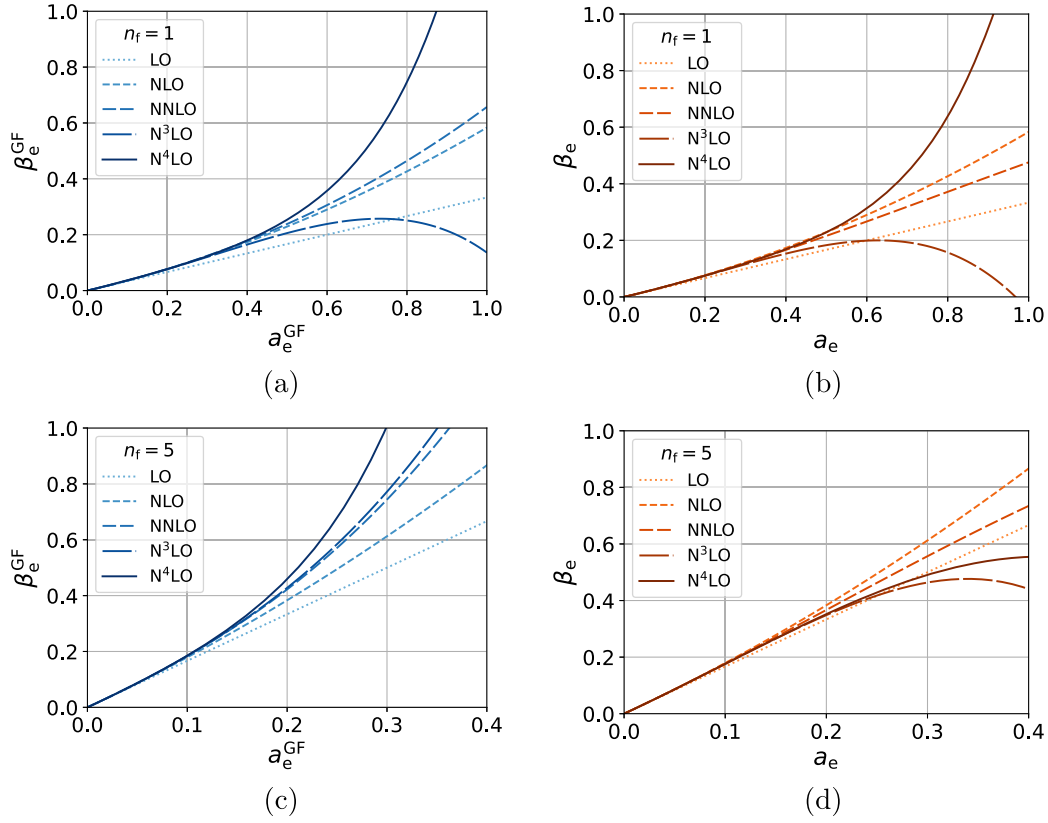


FIG. 2. Comparison of the β function in the GF scheme (a), (c) to the $\overline{\text{MS}}$ scheme (b), (d) for $n_f = 1$ (a), (b) and $n_f = 5$ (c), (d).

and $L_{\mu l}$ from Eq. (3.5). The coefficient $\beta_{e,2}^{\text{GF}}$ agrees with the QCD result in the Abelian limit [19,25,26]; the other coefficients are new.

The behavior of the different orders of the GF β function, together with the $\overline{\text{MS}}$ β function for $n_f = 1$ and $n_f = 5$ for comparison, is shown in Fig. 2. As opposed to the case of the QCD β function, the perturbative behavior in the GF scheme is rather similar, if not better than in the $\overline{\text{MS}}$ scheme [19,25,26].

Let us stress that the results of this section should only be considered as an example for the perturbative application of Eq. (2.6). For example, one could include the effect of other interactions in the photon polarization function. In this case, the c_n in Eq. (3.1) become functions of the coupling constants of these additional interactions. In particular, one may include QCD corrections in the massless-quark limit, which are currently known through order α_s^4 [45]. Neglecting higher orders in a_e , the polarization function would then take the form

$$\begin{aligned} \Pi_R(p) &= a_e(\mu) c_1(a_s, l_{\mu p}) \\ &= a_e(\mu) \left[\Pi^{(1)} + \sum_{n \geq 1} \sum_{m=0}^n d_{n,m} l_{\mu p}^m a_s^n(\mu) \right], \end{aligned} \quad (3.15)$$

and therefore

$$\begin{aligned} a_e^{\text{GF}}(\mu) &= a_e(\mu) (8\pi t)^2 \int \frac{d^4 p}{(2\pi)^4} e^{-2tp^2} [1 - \Pi_R(p)] \Big|_{t=(c_{\eta l}/\mu)^2} \\ &\quad + \mathcal{O}(a_e^3). \end{aligned} \quad (3.16)$$

Here $\Pi^{(1)}$ denotes the one-loop result, which is of course identical to the $\mathcal{O}(a_e)$ term in Eq. (3.1), i.e., $\Pi^{(1)} = c_{1,0} + c_{1,1} l_{\mu p}$. Inserting the coefficients $d_{n,m}$ from the literature [45] and using again Eq. (3.3), one arrives at the corresponding perturbative expansion for a_e^{GF} given in Appendix B.

IV. THREE-DIMENSIONAL QED

Now we turn to QED in three space-time dimensions. In this case, the coupling constant a_e has mass dimension one. The theory is superrenormalizable and strongly coupled in the IR.

A. Naive perturbation theory

Despite the nonperturbative IR behavior of QED₃, let us consider a naive perturbative approach for defining the gradient-flow coupling. We will see that this indeed works up to second order in the coupling, while it fails beyond that.

The renormalized polarization function is known for general number of space-time dimensions D through two loops, with the result [46]

$$\Pi_R(q) = a_e \Pi^{(1)}(q) + a_e^2 \Pi^{(2)}(q) + \dots, \quad (4.1)$$

with

$$\Pi^{(1)}(q) = \frac{1}{1-D} \left(\frac{4\pi}{q^2} \right)^{\frac{4-D}{2}} \frac{D-2}{2} G(1,1) \quad (4.2)$$

and

$$\Pi^{(2)}(q) = \frac{1}{4(1-D)} \left(\frac{4\pi}{q^2} \right)^{4-D} \frac{4D-8}{8-2D} \left\{ G^2(1,1) \left[-4 + \frac{7}{4}D - \frac{1}{4}D^2 \right] \right. \quad (4.3)$$

$$\left. + G(1,1)G\left(1, \frac{4-D}{2}\right) \frac{4}{4-D} \left[6 - 5D + \frac{7}{4}D^2 - \frac{1}{4}D^3 \right] \right\}, \quad (4.4)$$

where

$$G(n_1, n_2) = (4\pi)^{D/2} (p^2)^{n_1+n_2-D/2} \int_k \frac{1}{((p+k)^2)^{n_1} (k^2)^{n_2}} \quad (4.5)$$

$$= \frac{\Gamma(n_1 + n_2 - D/2) \Gamma(D/2 - n_1) \Gamma(D/2 - n_2)}{\Gamma(n_1) \Gamma(n_2) \Gamma(D - n_1 - n_2)}. \quad (4.6)$$

At $D = 3$, this gives

$$\Pi^{(1)}(q) = \frac{n_f}{8q} \quad \text{and} \quad \Pi^{(2)}(q) = \frac{n_f}{16\pi^2 q^2} (\pi^2 - 10). \quad (4.7)$$

Inserting this into Eq. (2.6) with $D = 3$, one may be tempted to define the QED₃ coupling in the GF scheme as

$$a_e^{\text{GF}}(\mu) = a_e(\mu) (1 + a_e(\mu) \sqrt{t} e_1 + a_e^2(\mu) t e_2 + \dots)_{t=(c_{i\mu}/\mu)^2}, \quad (4.8)$$

with

$$e_1 = -n_f \pi^2 \sqrt{\frac{2}{\pi}}, \quad e_2 = n_f \pi^2 (n_f \pi^2 - 4\pi^2 + 40), \quad (4.9)$$

and $c_{i\mu}$ again arbitrary, but fix. However, these first two orders are exceptional in the sense that all higher orders beyond that are IR divergent, as can be seen as follows. For dimensional reasons, it is

$$\Pi^{(n)}(q) \sim 1/q^n, \quad (4.10)$$

and thus Eq. (2.6) results in integrals of the form Eq. (2.9) with $\beta = n \in \mathbb{N}$ which do not exist for $n \geq D = 3$. This is a

consequence of the well-known property of QED₃ that a_e is not a viable expansion parameter.

B. Large- n_f limit

Instead, one may consider QED₃ in the limit of large n_f . Resumming the dominant terms in $1/n_f$ leads to the following photon polarization function:

$$\Pi_R(q) = \frac{a_e}{q} h_f, \quad (4.11)$$

where [47–50]

$$h_f = \frac{\pi^2 n_f}{2} \left(1 + \frac{c^{(1)}}{\pi^2 n_f} + \mathcal{O}(1/n_f^2) \right), \quad \text{with } c^{(1)} = \frac{184}{9} - 2\pi^2. \quad (4.12)$$

Let us define the dimensionless couplings as

$$\hat{a}_e^{\text{GF}}(\mu) \equiv c_{a\mu} \frac{a_e^{\text{GF}}}{\mu}, \quad \hat{a}_e(\mu) \equiv c_{a\mu} \frac{a_e}{\mu}, \quad (4.13)$$

with some constant $c_{a\mu}$. In the following, we choose

$$c_{a\mu} = 2c_{i\mu} h_f \sqrt{\frac{2}{\pi}}. \quad (4.14)$$

Setting $D = 3$ and using Eq. (4.11), the integral of Eq. (2.6) evaluates to

$$\hat{a}_e^{\text{GF}} = \hat{a}_e \left[1 - \hat{a}_e + \frac{\pi}{2} \hat{a}_e^2 - \frac{\pi}{4} \hat{a}_e^3 \mathcal{G}\left(\frac{\pi}{4} \hat{a}_e^2\right) \right] \quad (4.15)$$

with

$$\mathcal{G}(x) \equiv e^{-x} \left[\Gamma(0, -x) + \sqrt{\pi} i \Gamma\left(\frac{1}{2}, -x\right) \right], \quad (4.16)$$

where

$$\Gamma(s, z) = \int_z^\infty dt t^{s-1} e^{-t} \quad (4.17)$$

is the incomplete Γ function. Regulating this integral for $z = -x < 0$ by suitable subtractions, one can write

$$\begin{aligned} \Gamma(0, -x) &= -i\pi + \int_{-x}^1 dt \frac{e^{-t} - 1}{t} - \ln x + \Gamma(0, 1), \\ i\Gamma\left(\frac{1}{2}, -x\right) &= i\sqrt{\pi} + 2\sqrt{x} + \int_0^x dt \frac{e^t - 1}{\sqrt{t}}, \end{aligned} \quad (4.18)$$

which shows that $\mathcal{G}(x)$ in Eq. (4.16) is real-valued. Its large- and small- x limits can be obtained from the usual expansions of the incomplete Γ function.

In this way, we obtain the limits of short and long flow time of Eq. (4.15)²

$$\begin{aligned} \hat{a}_e^{\text{GF}} &= \hat{a}_e \left[1 - \hat{a}_e + \frac{\pi}{2} \hat{a}_e^2 + \frac{\pi}{4} \hat{a}_e^3 \left(\gamma_E + 2 \ln \hat{a}_e + \ln \frac{\pi}{4} \right) + \mathcal{O}(\hat{a}_e^4) \right], \\ \hat{a}_e^{\text{GF}} &= \frac{4}{\pi} \left[1 - \frac{3}{2\hat{a}_e} + \frac{8}{\pi \hat{a}_e^2} - \frac{15}{\pi \hat{a}_e^3} + \mathcal{O}(\hat{a}_e^{-4}) \right]. \end{aligned} \quad (4.19)$$

The behavior of $\hat{a}_e^{\text{GF}}$ in comparison with these limiting cases is shown in Fig. 3. The short flow-time limit agrees with the naive expansion of Eq. (4.8) at leading order in $1/n_f$.

The dimensionless gradient-flow coupling $\hat{a}_e^{\text{GF}}$ then clearly has an IR fixed point at

$$\hat{a}_e^{\text{GF}}(\mu = 0) = \frac{4}{\pi}. \quad (4.20)$$

For the following, it is useful to define

$$\hat{a}_{e,*}^{\text{GF}}(\mu) \equiv \hat{a}_e^{\text{GF}}(\mu) - \frac{4}{\pi} \quad (4.21)$$

²A derivation of these expressions using the strategy of regions is outlined in Appendix A.

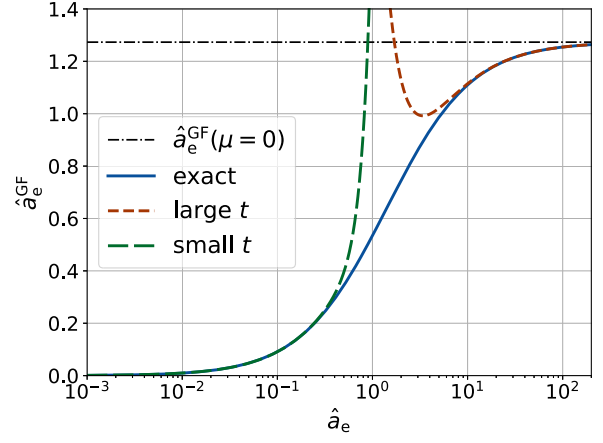


FIG. 3. The conversion between the GF and $\overline{\text{MS}}$ scheme together with the limiting cases for small/large $\hat{a}_e$.

with which the inverse relations of Eq. (4.19) become

$$\begin{aligned} \hat{a}_e &= \hat{a}_e^{\text{GF}} \left[1 + \hat{a}_e^{\text{GF}} + (\hat{a}_e^{\text{GF}})^2 \left(2 - \frac{\pi}{2} \right) \right. \\ &\quad + (\hat{a}_e^{\text{GF}})^3 \left[5 - \frac{5\pi}{2} - \frac{\pi}{4} \left(\gamma_E + 2 \ln \hat{a}_e^{\text{GF}} + \ln \frac{\pi}{4} \right) \right. \\ &\quad \left. \left. + \mathcal{O}((\hat{a}_e^{\text{GF}})^4) \right] \right], \\ \frac{1}{\hat{a}_e} &= -\frac{\pi}{6} \hat{a}_{e,*}^{\text{GF}} \left[1 - \frac{8}{9} \hat{a}_{e,*}^{\text{GF}} + \frac{256 - 45\pi^2}{162} (\hat{a}_{e,*}^{\text{GF}})^2 + \dots \right]. \end{aligned} \quad (4.22)$$

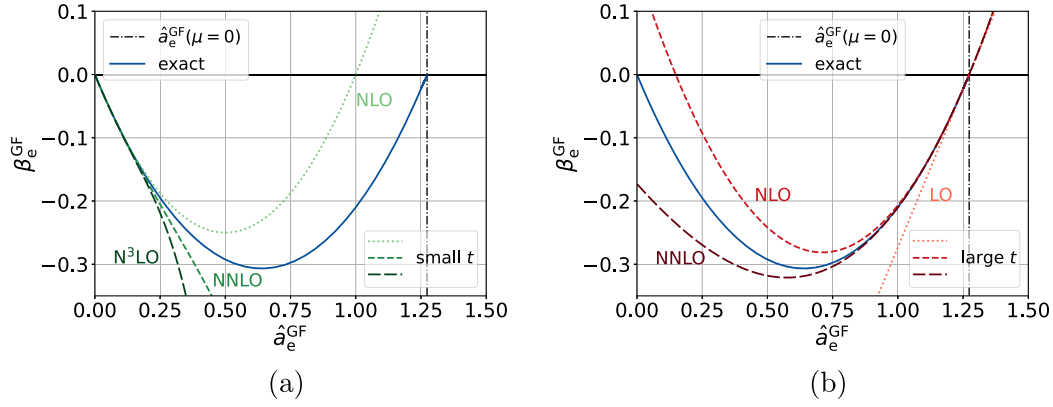
Using Eq. (4.13)(4.15), we can obtain

$$\begin{aligned} \beta_e^{\text{GF}} &= \mu \frac{d}{d\mu} \hat{a}_e^{\text{GF}} = -\hat{a}_e \frac{\partial}{\partial \hat{a}_e} \hat{a}_e^{\text{GF}} \\ &= -\hat{a}_e \left\{ 1 - 2\hat{a}_e + \frac{3\pi}{2} \hat{a}_e^2 + \frac{\pi}{2} \hat{a}_e^3 \left[1 - 2\mathcal{G}\left(\frac{\pi}{4} \hat{a}_e^2\right) \right] \right. \\ &\quad \left. - \frac{\pi^2}{4} \hat{a}_e^4 + \frac{\pi^2}{8} \hat{a}_e^5 \mathcal{G}\left(\frac{\pi}{4} \hat{a}_e^2\right) \right\}. \end{aligned} \quad (4.23)$$

Of course, we are actually interested in β_e^{GF} as a function of $\hat{a}_e^{\text{GF}}$ rather than $\hat{a}_e$. However, obtaining an analytic expression for this would require solving Eq. (4.15) for $\hat{a}_e(\hat{a}_e^{\text{GF}})$, which is beyond analytic reach. Therefore, the β function can only be determined numerically.

To gain insight into the behavior near the fixed points at $\hat{a}_e^{\text{GF}} = 0$ and $\hat{a}_e^{\text{GF}} = \frac{4}{\pi}$, we define an approximate β function valid in the limits of short and long flow time. Expanding Eq. (4.23) in the respective limits and inserting Eq. (4.22), we obtain

$$\mu \frac{d}{d\mu} \hat{a}_e^{\text{GF}} = \beta_e^{\text{GF}}, \quad (4.24)$$


 FIG. 4. The comparison of the β function to the limiting cases: (a) small $\hat{a}_e^{\text{GF}}$; (b) large $\hat{a}_e^{\text{GF}}$.

with

$$\begin{aligned} \beta_e^{\text{GF}} = & -\hat{a}_e^{\text{GF}} + (\hat{a}_e^{\text{GF}})^2 + (\hat{a}_e^{\text{GF}})^3(2 - \pi) \\ & + (\hat{a}_e^{\text{GF}})^4 \left[5 - \frac{3\pi}{2} \left(3 + \frac{\gamma_E}{2} + \ln \hat{a}_e^{\text{GF}} + \frac{1}{2} \ln \frac{\pi}{4} \right) \right] + \dots \end{aligned} \quad (4.25)$$

and

$$\beta_e^{\text{GF}} = \hat{a}_{e,*}^{\text{GF}} \left[1 + \frac{8}{9} \hat{a}_{e,*}^{\text{GF}} - \frac{128 - 45\pi}{81} (\hat{a}_{e,*}^{\text{GF}})^2 + \dots \right]. \quad (4.26)$$

These limits, together with the full β function of Eq. (4.23), are shown in Fig. 4. Note that, for Eq. (4.25), neglecting terms of order $(\hat{a}_e^{\text{GF}})^3$ and higher would imply a non-Gaussian fixed point at $\hat{a}_e^{\text{GF}} = 1$. However, we are not supposed to use this result in the IR limit, which corresponds to $\sqrt{t} \rightarrow \infty$, as is clearly visible in Fig. 4(a).

The critical slope, defined as the gradient of the β -function with respect to the coupling evaluated at the fixed point, is a scheme independent number. Previous studies have found in the large n_f expansion that, to leading order, $\beta'(a_*) = 1$ [2,51]. By inspection, Eq. (4.26) agrees with this result, as does the combination of the complete analytic results given in Eq. (4.15) and Eq. (4.23). $\hat{a}_e^{\text{GF}}$ approaches the fixed point value as $\hat{a}_e \rightarrow \infty$, and the n_f dependence in these equations is absorbed into our definition of $\hat{a}_e$, therefore we do not expect higher-order corrections to the gradient-flow estimate of this value, in agreement with the regular perturbative expansion to $\mathcal{O}(1/n_f)$ in Ref. [51], for example. The evaluation of other critical exponents, such as the fermion-mass anomalous dimension, would require the computation of flowed fermionic operators, which is outside of the scope of this work.

V. CONCLUSIONS

We have evaluated the coupling of four- and three-dimensional QED with n_f massless fermions in the

gradient-flow scheme. Due to its linear form, the flow equation can be solved exactly for QED, and the GF coupling is fully determined as the Laplace transform of a function that only depends on the photon polarization function of the unflowed theory. In the latter, one can include arbitrary additional interactions, so that the U(1) gradient-flow coupling of a more general theory can be obtained from a one-dimensional integral over purely unflowed results within that theory.

As applications of this observation, we have considered four-dimensional QED through four-loop level, as well as three-dimensional QED in the large- n_f limit through next-to-next-to-leading order (NNLO). We have evaluated the corresponding conversions from the regular to the GF couplings, as well as the associated β functions. For QED₃, we recover the well-known UV and IR fixed points.

We hope that our results will be useful in further field theoretical studies, in particular in the context of lattice formulations of QED.

ACKNOWLEDGMENTS

We thank Curtis Peterson for bringing our attention to this problem. We also thank him, Nobuyuki Matsumoto and John Gracey for helpful comments on the manuscript, and Jonas Kohnen and Henry Werthenbach for useful discussions. This research was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Grant No. 396021762-TRR 257.

DATA AVAILABILITY

The data are not publically available. The data are available from the authors upon reasonable request.

APPENDIX A: LONG- AND SHORT-FLOW-TIME LIMITS OF $E(t)$ IN QED₃

In QED₃, the integral in Eq. (2.6) depends on the two dimensionful scales $a_e h_f$ and t . Let us first evaluate it in the

limit $a_e h_f \ll 1/\sqrt{t}$. Following the strategy-of-regions [52], we identify two integration regions³:

(1) $p \sim 1/\sqrt{t}$: here we can expand the integrand in the limit of small $a_e h_f$

$$\begin{aligned} E^{(1)}(t) &= \frac{e^2(D-1)}{2} \int \frac{d^D p}{(2\pi)^D} e^{-2tp^2} \sum_{n=0}^{\infty} \left(-\frac{a_e h_f}{p} \right)^n \\ &= \frac{e^2(D-1)}{2(8\pi t)^{D/2} \Gamma(D/2)} \sum_{n=0}^{\infty} (-a_e h_f \sqrt{2t})^n \Gamma\left(\frac{D-n}{2}\right), \end{aligned} \quad (\text{A1})$$

where we have used Eq. (2.9). Note that, in the limit $D \equiv 3 - 2\epsilon \rightarrow 3$, $E^{(1)}$ has poles for odd n with $n \geq 3$.

(2) $p \ll 1/\sqrt{t}$: here we can expand the exponential⁴

$$\begin{aligned} E^{(2)}(t) &= \frac{e^2(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} dp p^{D-1} \frac{(-2p^2 t)^n}{1 + a_e h_f/p} \\ &= \frac{e^2(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} \sum_{n=0}^{\infty} \frac{1}{n!} (-2t)^n (a_e h_f)^{D+2n} \Gamma(D-1+2n) \Gamma(2(1-n)-D). \end{aligned} \quad (\text{A2})$$

Therefore, this region only contributes whenever region 1 develops poles, in which case they are exactly canceled in the sum of both regions. Inserting this sum into Eq. (2.8) results in the first equation of Eq. (4.19).

In the opposite limit, $a_e h_f \gg 1/\sqrt{t}$, the region $p \gg 1/\sqrt{t}$ vanishes due to the exponential. Therefore, only the region $p \sim 1/\sqrt{t}$ contributes, leading to

$$\begin{aligned} E(t) &= -\frac{e^2(D-1)}{2} \int \frac{d^D p}{(2\pi)^D} e^{-2tp^2} \sum_{n=0}^{\infty} \left(-\frac{p}{a_e h_f} \right)^{n+1} \\ &= -\frac{e^2(D-1)}{2(8\pi t)^{D/2} \Gamma(D/2)} \sum_{n=0}^{\infty} \left(-\frac{1}{a_e h_f \sqrt{2t}} \right)^{n+1} \Gamma\left(\frac{D+n+1}{2}\right). \end{aligned} \quad (\text{A3})$$

Inserting this into Eq. (2.8) results in the second equation of Eq. (4.19).

APPENDIX B: QED₄ GRADIENT-FLOW COUPLING INCLUDING QCD CORRECTIONS

Here we provide the explicit result for the QCD corrections to the QED₄ coupling in the GF scheme. As a consequence of Eq. (3.16), we can write

$$a_e^{\text{GF}}(\mu) = a_e(\mu) \left\{ 1 + a_e(\mu) e_1 + a_e(\mu) \left[\sum_{n \geq 1} e_n^{\text{QCD}} a_s^n(\mu) \right] \right\}, \quad (\text{B1})$$

where the coefficients $e_n^{\text{QCD}} = e_n^{\text{QCD}}(L_{\mu t})$ can be extracted using $d_{n,m}$ from Ref. [45] and, up to $n = 2$, take the form

$$\begin{aligned} e_1^{\text{QCD}} &= -d_{1,0} + d_{1,1}(1 - L_{\mu t}) = \left(\zeta(3) - \frac{43}{48} - \frac{1}{4} L_{\mu t} \right) d_R n_f C_F, \\ e_2^{\text{QCD}} &= -d_{2,0} + d_{2,1}(1 - L_{\mu t}) + d_{2,2} \left(-\frac{\pi^2}{6} + 2L_{\mu t} - L_{\mu t}^2 \right) \\ &= d_R n_f C_F C_A \left[\frac{5}{12} \zeta(5) + \frac{161}{72} \zeta(3) - \frac{11\pi^2}{576} - \frac{30931}{10368} + L_{\mu t} \left(\frac{11}{12} \zeta(3) - \frac{101}{96} \right) - \frac{11}{96} L_{\mu t}^2 \right] \\ &\quad + d_R n_f C_F^2 \left(-\frac{5}{2} \zeta(5) + \frac{37}{24} \zeta(3) + \frac{67}{144} + \frac{1}{32} L_{\mu t} \right) \\ &\quad + d_R n_f^2 C_F T_R \left[-\frac{13}{18} \zeta(3) + \frac{\pi^2}{144} + \frac{2513}{2592} - L_{\mu t} \left(\frac{1}{3} \zeta(3) - \frac{3}{8} \right) + \frac{1}{24} L_{\mu t}^2 \right]. \end{aligned} \quad (\text{B2})$$

³For the application of the strategy-of-regions to GF integrals, see also Ref. [26]; for an alternative way to expand GF integrals, see Ref. [53].

⁴The integral was solved with *Mathematica* [54] and using the identity $\pi/\sin(s\pi) = \Gamma(s)\Gamma(1-s)$ [55].

The coefficient e_1 coincides with the one in QED₄, cf. Eq. (3.14). The relevant SU(3) group factors required for QCD are

$$d_R = 3, \quad C_F = \frac{4}{3}, \quad C_A = 3, \quad T_R = \frac{1}{2}. \quad (\text{B3})$$

The derivation of higher-order terms in Eq. (B1) proceeds analogously.

-
- [1] T. Appelquist and R. D. Pisarski, High-temperature Yang-Mills theories and three-dimensional quantum chromodynamics, *Phys. Rev. D* **23**, 2305 (1981).
 - [2] T. W. Appelquist, M. J. Bowick, D. Karabali, and L. C. R. Wijewardhana, Spontaneous chiral symmetry breaking in three-dimensional QED, *Phys. Rev. D* **33**, 3704 (1986).
 - [3] J. A. Gracey, Large N_f quantum field theory, *Int. J. Mod. Phys. A* **33**, 1830032 (2019).
 - [4] R. D. Pisarski, Chiral symmetry breaking in three-dimensional electrodynamics, *Phys. Rev. D* **29**, 2423 (1984).
 - [5] M. Franz and Z. Tešanović, Algebraic Fermi liquid from phase fluctuations: Topological fermions, vortex berryons, and QED₃ theory of cuprate superconductors, *Phys. Rev. Lett.* **87**, 257003 (2001).
 - [6] E. Dagotto, J. B. Kogut, and A. Kocic, A computer simulation of chiral symmetry breaking in $(2+1)$ -dimensional QED with N flavors, *Phys. Rev. Lett.* **62**, 1083 (1989).
 - [7] R. Narayanan and H. Neuberger, Infinite N phase transitions in continuum Wilson loop operators, *J. High Energy Phys.* **03** (2006) 064.
 - [8] M. Lüscher, Trivializing maps, the Wilson flow and the HMC algorithm, *Commun. Math. Phys.* **293**, 899 (2010).
 - [9] M. Lüscher, Properties and uses of the Wilson flow in lattice QCD, *J. High Energy Phys.* **08** (2010) 071; **03** (2014) 092 (E).
 - [10] M. Lüscher and P. Weisz, Perturbative analysis of the gradient flow in non-Abelian gauge theories, *J. High Energy Phys.* **02** (2011) 051.
 - [11] M. Lüscher, Chiral symmetry and the Yang–Mills gradient flow, *J. High Energy Phys.* **04** (2013) 123.
 - [12] H. Suzuki, Energy–momentum tensor from the Yang–Mills gradient flow, *Prog. Theor. Exp. Phys.* **2013**, 083B03 (2013); **2015**, 079201(E) (2015).
 - [13] H. Makino and H. Suzuki, Lattice energy–momentum tensor from the Yang–Mills gradient flow—inclusion of fermion fields, *Prog. Theor. Exp. Phys.* **2014**, 063B02 (2014); **2015**, 079202(E) (2015).
 - [14] Y. Taniguchi, S. Ejiri, K. Kanaya, M. Kitazawa, H. Suzuki, and T. Umeda (WHOT-QCD Collaboration), $N_f = 2 + 1$ QCD thermodynamics with gradient flow using two-loop matching coefficients, *Phys. Rev. D* **102**, 014510 (2020); **102**, 059903(E) (2020).
 - [15] M. Black, R. Harlander, F. Lange, A. Rago, A. Shindler, and O. Witzel, Using gradient flow to renormalise matrix elements for meson mixing and lifetimes, *Proc. Sci. LATTICE2023* (**2024**) 263 [arXiv:2310.18059].
 - [16] M. Black, R. Harlander, F. Lange, A. Rago, A. Shindler, and O. Witzel, Gradient flow renormalisation for meson mixing and lifetimes, *Proc. Sci. LATTICE2024* (**2025**) 243 [arXiv:2409.18891].
 - [17] M. Black, R. V. Harlander, A. Hasenfratz, A. Rago, and O. Witzel, Renormalized quark masses using gradient flow, arXiv:2506.16327.
 - [18] A. Francis *et al.*, Gradient flow for parton distribution functions: first application to the pion, arXiv:2509.02472.
 - [19] A. Hasenfratz, C. T. Peterson, J. van Sickle, and O. Witzel, Λ parameter of the SU(3) Yang-Mills theory from the continuous β function, *Phys. Rev. D* **108**, 014502 (2023).
 - [20] C. H. Wong, S. Borsanyi, Z. Fodor, K. Holland, and J. Kuti, Toward a novel determination of the strong QCD coupling at the Z-pole, *Proc. Sci. LATTICE2022* (**2023**) 043 [arXiv:2301.06611].
 - [21] G. Schierholz, QCD lambda parameter from gradient flow, *Eur. Phys. J. C* **85**, 1007 (2025).
 - [22] Y. Aoki *et al.* (Flavour Lattice Averaging Group (FLAG) Collaboration), FLAG review 2024, *Phys. Rev. D* **113**, 014508 (2026).
 - [23] R. Larsen, S. Mukherjee, P. Petreczky, H.-T. Shu, and J. H. Weber, Scale setting and strong coupling determination in the gradient flow scheme for $2 + 1$ flavor lattice QCD, arXiv:2502.08061.
 - [24] M. D. Brida, R. Höllwieser, F. Knechtli, T. Korzec, A. Ramos, S. Sint, and R. Sommer, The strength of the interaction between quarks and gluons, arXiv:2501.06633.
 - [25] R. V. Harlander and T. Neumann, The perturbative QCD gradient flow to three loops, *J. High Energy Phys.* **06** (2016) 161.
 - [26] R. Harlander, The gradient flow at higher orders in perturbation theory, *Proc. Sci. LATTICE2021* (**2022**) 489 [arXiv:2111.14376].
 - [27] D. Zwanziger, Covariant quantization of gauge fields without Gribov ambiguity, *Nucl. Phys.* **B192**, 259 (1981).
 - [28] R. V. Harlander, S. Y. Klein, and M. Lipp, FeynGame, *Comput. Phys. Commun.* **256**, 107465 (2020).
 - [29] L. Bündgen, R. V. Harlander, S. Y. Klein, and M. C. Schaaf, FeynGame 3.0, *Comput. Phys. Commun.* **314**, 109662 (2025).
 - [30] J. Artz, R. V. Harlander, F. Lange, T. Neumann, and M. Prausa, Results and techniques for higher order calculations within the gradient-flow formalism, *J. High Energy Phys.* **06** (2019) 121; **10** (2019) 032(E).
 - [31] M. Lüscher, Future applications of the Yang-Mills gradient flow in lattice QCD, *Proc. Sci. LATTICE2013* (**2014**) 016 [arXiv:1308.5598].

- [32] S. Borsanyi *et al.* (BMW Collaboration), Ab initio calculation of the neutron-proton mass difference, *Science* **347**, 1452 (2015).
- [33] L. Bushnaq, I. Campos, M. Catillo, A. Cotellucci, M. Dale, P. Fritzsch, J. Lücke, M. Krstić Marinković, A. Patella, and N. Tantalo (RCstar Collaboration), First results on QCD + QED with C^* boundary conditions, *J. High Energy Phys.* **03** (2023) 012.
- [34] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, New approach to evaluation of multiloop Feynman integrals: The gegenbauer polynomial x space technique, *Nucl. Phys.* **B174**, 345 (1980).
- [35] S. G. Gorishnii, A. L. Kataev, and S. A. Larin, Analytical four loop result for beta function in QED in $\overline{MS}$ and Mom schemes, *Phys. Lett. B* **194**, 429 (1987).
- [36] S. G. Gorishnii, A. L. Kataev, S. A. Larin, and L. R. Surguladze, The analytical four loop corrections to the QED Beta function in the $\overline{MS}$ scheme and to the QED psi function: Total reevaluation, *Phys. Lett. B* **256**, 81 (1991).
- [37] S. G. Gorishnii, A. L. Kataev, and S. A. Larin, The three loop QED contributions to the photon vacuum polarization function in the $\overline{MS}$ scheme and the four loop corrections to the QED beta function in the on-shell scheme, *Phys. Lett. B* **273**, 141 (1991); **275**, 512(E) (1992); **341**, 448(E) (1995).
- [38] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, and C. Sturm, The relation between the QED charge renormalized in $\overline{MS}$ and on-shell schemes at four loops, the QED on-shell β -function at five loops and asymptotic contributions to the muon anomaly at five and six loops, *Nucl. Phys.* **B867**, 182 (2013).
- [39] O. V. Tarasov, A. A. Vladimirov, and A. Y. Zharkov, The Gell-Mann-Low function of QCD in the three loop approximation, *Phys. Lett.* **93B**, 429 (1980).
- [40] S. A. Larin and J. A. M. Vermaseren, The three-loop QCD β function and anomalous dimensions, *Phys. Lett. B* **303**, 334 (1993).
- [41] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, The four-loop β function in quantum chromodynamics, *Phys. Lett. B* **400**, 379 (1997).
- [42] M. Czakon, The four-loop QCD β -function and anomalous dimensions, *Nucl. Phys.* **B710**, 485 (2005).
- [43] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Five-loop running of the QCD coupling constant, *Phys. Rev. Lett.* **118**, 082002 (2017).
- [44] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, The five-loop β function of Yang-Mills theory with fermions, *J. High Energy Phys.* **02** (2017) 090.
- [45] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, and J. Rittinger, Vector correlator in massless QCD at order $\mathcal{O}(\alpha_s^4)$ and the QED β -function at five loop, *J. High Energy Phys.* **07** (2012) 017.
- [46] A. Grozin, Lectures on QED and QCD, in *3rd Dubna International Advanced School of Theoretical Physics* (2005), [arXiv:hep-ph/0508242](https://arxiv.org/abs/hep-ph/0508242).
- [47] V. Gusynin, A. Hams, and M. Reenders, Nonperturbative infrared dynamics of three-dimensional QED with four fermion interaction, *Phys. Rev. D* **63**, 045025 (2001).
- [48] S. Teber, Electromagnetic current correlations in reduced quantum electrodynamics, *Phys. Rev. D* **86**, 025005 (2012).
- [49] A. V. Kotikov and S. Teber, Note on an application of the method of uniqueness to reduced quantum electrodynamics, *Phys. Rev. D* **87**, 087701 (2013).
- [50] S. Metayer and S. Teber, Critical properties of three-dimensional many-flavor QEDs, *Symmetry* **15**, 1806 (2023).
- [51] A. Palanques-Mestre and P. Pascual, The $1/N_F$ expansion of the γ and β functions in Q.E.D., *Commun. Math. Phys.* **95**, 277 (1984).
- [52] M. Beneke and V. A. Smirnov, Asymptotic expansion of Feynman integrals near threshold, *Nucl. Phys.* **B522**, 321 (1998).
- [53] H. Takaura, R. V. Harlander, and F. Lange, A new approach to quark mass determination using the gradient flow, *J. High Energy Phys.* **09** (2025) 025.
- [54] W. R. Inc., *Mathematica*, Version 14.3, Champaign, IL, <https://www.wolfram.com/mathematica>, 2025.
- [55] S. J. Miller, *Chapter 15. The Gamma Function and Related Distributions* (Princeton University Press, Princeton, 2017), pp. 398–426, [10.1515/9781400885381-017](https://doi.org/10.1515/9781400885381-017).