

Color-kinematics duality from an algebra of superforms

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Color-kinematics duality states that the kinematic numerators of the cubic tree-level Yang-Mills scattering amplitudes obey the same symmetry properties that the color factors obey due to the Jacobi identity. We present a novel strategy for deriving this duality, based on the differential forms on a superspace. This space of superforms carries a generalization of a Batalin-Vilkovisky (BV) algebra ($BV^\square$ algebra). We show that the homotopy algebra of color-stripped Yang-Mills theory is obtained as a quotient of this space in which a subspace, which is an ideal “up to homotopy,” is modded out. This algebra is a subsector of a $BV_\infty^\square$ algebra. Deriving the latter would provide a first-principle proof of color-kinematics duality from field theory.

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Introduction. The known fundamental forces in nature are governed by Yang-Mills theory in particle physics on the one hand and by general relativity in gravitational physics on the other. Both theories share broad qualitative features such as gauge invariance, but their detailed Lagrangians appear radically different. Yang-Mills theory is governed by a Lagrangian with a cubic vertex, which is linear in derivatives, and a quartic vertex, which is independent of derivatives. This leads to a renormalizable quantum field theory. In contrast, general relativity expanded about Minkowski space yields interaction vertices of arbitrary order, each featuring two derivatives. This yields a non-renormalizable quantum field theory.

Nevertheless, it was established by Bern, Carrasco, and Johansson (BCJ) [1,2] that there is a deep relation between the tree-level scattering amplitudes of a certain gravity theory and that of Yang-Mills theory. To state this relationship, one writes a tree-level n -point scattering amplitude of Yang-Mills theory as the Feynman diagram expansion but using only *cubic* diagrams Γ :

$$\mathcal{A}_n = g^{n-2} \sum_{\Gamma} \frac{c_{\Gamma} n_{\Gamma}}{D_{\Gamma}}. \quad (1)$$

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This can be achieved, for instance, by replacing the field strength $F_{\mu\nu}$ by an independent auxiliary field or by suitably “blowing up” the quartic vertices, so that all Feynman rules become cubic. Moreover, c_{Γ} is the color factor associated with a given diagram Γ , consisting of $n-2$ structure constants f^a_{bc} , n_{Γ} is the kinematic numerator built from the n polarization vectors ϵ_i^μ and momenta k_i^μ , and D_{Γ} encodes the inverse propagators. The BCJ or color-kinematics (CK) duality states that whenever $c_{\Gamma_1} + c_{\Gamma_2} + c_{\Gamma_3} = 0$ holds as a consequence of the Jacobi identity of the color gauge group, then the corresponding numerators obey $n_{\Gamma_1} + n_{\Gamma_2} + n_{\Gamma_3} = 0$. Once the numerators n_{Γ} have this CK duality obeying property, one may replace in (1) the color factors c_{Γ} by a second copy $\tilde{n}_{\Gamma}$ of the numerators. This BCJ double copy yields the scattering amplitudes of $\mathcal{N} = 0$ supergravity (Einstein gravity coupled to a 2-form B field and a scalar dilaton).

The CK duality and the associated double-copy nature of gravity express a deep relationship between Yang-Mills theory and gravity [3–6]. While there are indirect proofs from string theory of CK duality at tree level [7,8], it is desirable to obtain a first-principle derivation directly from field theory, in particular, to obtain an effective algorithm to compute the kinematic numerators in a manifestly local and CK duality compatible fashion. This latter feature, in particular, will be important in order to obtain full control of CK duality and the double copy at loop level.

A natural idea to prove CK duality from first principles is to identify a hidden “kinematic Lie algebra,” in which case $n_{\Gamma_1} + n_{\Gamma_2} + n_{\Gamma_3} = 0$ would follow from its Jacobi identity [9–12]. Results in recent years have shown, however, that a more sophisticated algebraic structure is at the heart of CK duality [13–20]. To this end, one uses the formulation of

field theory in terms of homotopy algebras such as homotopy Lie or L_∞ algebras [21–28] or homotopy associative commutative or C_∞ algebras [29]. A C_∞ algebra is defined by maps m_k , $k = 1, 2, 3, \dots$, with k inputs, satisfying certain identities. For instance, the algebra of differential forms on any space forms a C_∞ algebra, where $m_1 = d$ is the de Rham differential and m_2 is the wedge product, while all m_k for $k \geq 3$ are zero. Such a C_∞ algebra is called strict. Upon taking the tensor product with a color Lie algebra $\mathfrak{g}$ this gives rise, on a 3-manifold, to topological Chern-Simons theory. Yang-Mills theory is similarly encoded in a genuine C_∞ algebra with a nontrivial m_3 . The action reads

$$S_{\text{YM}} = \frac{1}{2} \langle A_a, m_1(A^a) \rangle + \frac{1}{3!} f_{abc} \langle A^a, m_2(A^b, A^c) \rangle + \frac{1}{12} f_{abc} f_{cd}^e \langle A^a, m_3(A^b, A^c, A^d) \rangle, \quad (2)$$

where the color factors f_{abc} are stripped off explicitly. In the case that the C_∞ algebra is the strict algebra of differential forms on a 3-manifold, for which the quartic terms in the second line vanish, this is the Chern-Simons action. When instead the maps (m_1, m_2, m_3) define the color-stripped Yang-Mills C_∞ algebra, the above is the familiar Yang-Mills action.

In this paper, we show that, remarkably, the C_∞ algebra of Yang-Mills theory can be obtained from the algebra of differential forms on the superspace that extends spacetime by one odd coordinate. To this end, we identify a subspace of this space of superforms as an “ideal” that is modded out. More precisely, while this subspace is not an ideal of the strict algebra of differential forms, it is an ideal “up to homotopy,” and the quotient space then inherits the Yang-Mills C_∞ algebra.

It should be emphasized, however, that this algebra is only the first layer of the full homotopy algebra that is needed to explain CK duality [13,17]. This algebra is a generalization of a Batalin-Vilkovisky (BV $_\infty$) algebra [30] called BV $_\infty^\square$. In order to define this algebra, one uses that there is a second differential b , satisfying $b^2 = 0$, in terms of which the propagator is $\frac{b}{\square}$. While the full propagator is nonlocal, in our formulation b is local and can be used to define a local Lie-type bracket b_2 as the failure of b to act as a derivation on m_2 . This, in turn, gives rise to an infinite tower of higher (local) maps. The idea of deriving the maps of the BV $_\infty^\square$ algebra from a strict algebra was pursued previously in [31,32] using algebras of worldline vertex operators. What makes the new approach introduced here particularly promising is that we explicitly identify a strict BV $_\infty^\square$ algebra for which we conjecture that the full BV $_\infty^\square$ algebra is obtained as a suitable quotient. While in contrast to the C_∞ subsector we do not yet know how to construct the full BV $_\infty^\square$ algebra, we will use the opportunity to explain in Sec. IV how it encodes CK duality and comment in the conclusion Sec. V on the ongoing program for deriving this algebra.

Algebra of superforms. We consider the algebra of differential forms (the de Rham complex) on the superspace $\mathbb{R}^{D|1}$ with coordinates (x^μ, c) , where x^μ are the bosonic space-time coordinates and c is an odd coordinate satisfying $c^2 = 0$. This type of construction has been previously studied in [33]. Viewing the basis one-forms as new (odd and even) coordinates via

$$C^\mu \equiv dx^\mu, \quad \mathcal{M} \equiv dc, \quad (3)$$

where $C^\mu C^\nu = -C^\nu C^\mu$, the de Rham complex $\Omega^\bullet(\mathbb{R}^{D|1})$ becomes the space of functions or superfields

$$F(x^\mu, c; C^\mu, \mathcal{M}), \quad (4)$$

whose dependence on c , C^μ , and $\mathcal{M}$ is polynomial. The degrees (ghost number G) of the coordinates are given by $G(x^\mu) = 0$, $G(c) = +1$, $G(C^\mu) = +1$, and $G(\mathcal{M}) = +2$. Note, in particular, that, since $\mathcal{M}$ is bosonic, the functions (4) may depend on arbitrary powers of $\mathcal{M}$. This space is actually a bicomplex, as depicted in Fig. 1: It decomposes according to two degrees, the ghost number G and the superform degree N , which counts the number of C^μ and $\mathcal{M}$.

The wedge product $\wedge$ of differential superforms is encoded in the pointwise product of superfields (4), while the de Rham differential can be written as a vector field

$$d := C^\mu \partial_\mu + \mathcal{M} \frac{\partial}{\partial c}, \quad (5)$$

which satisfies $d^2 = 0$ and acts on the product via the Leibniz rule. The space of superforms $(\Omega^\bullet, d, \wedge)$, where we use the shorthand $\Omega^\bullet \equiv \Omega^\bullet(\mathbb{R}^{D|1})$ in the following, forms a differential graded (dg) commutative associative algebra (dgCom).

Assuming the existence of a metric, the above space carries further structure: a (strict) BV $^\square$ algebra. To define this recall the adjoint of d ,

$$d^\dagger := \frac{\partial^2}{\partial x_\mu \partial C^\mu}, \quad (6)$$

which obeys $[d, d^\dagger] = \square$, where $\square$ is the d’Alembert operator. $d^\dagger$ is a second-order operator and, hence, does not obey the Leibniz rule with respect to $\wedge$; rather, its failure defines a (graded) Lie bracket, the Schouten-Nijenhuis bracket. It was shown by Ben-Shahar and Johansson that this BV $^\square$ algebra $(\Omega^\bullet, d, d^\dagger, \wedge)$ explains CK duality for the toy model of Chern-Simons theory [14].

We now apply an isomorphism to the above algebra that brings it to an equivalent form in which the Yang-Mills kinematic space is more easily identified. This isomorphism is written as e^h , with inverse e^{-h} , where

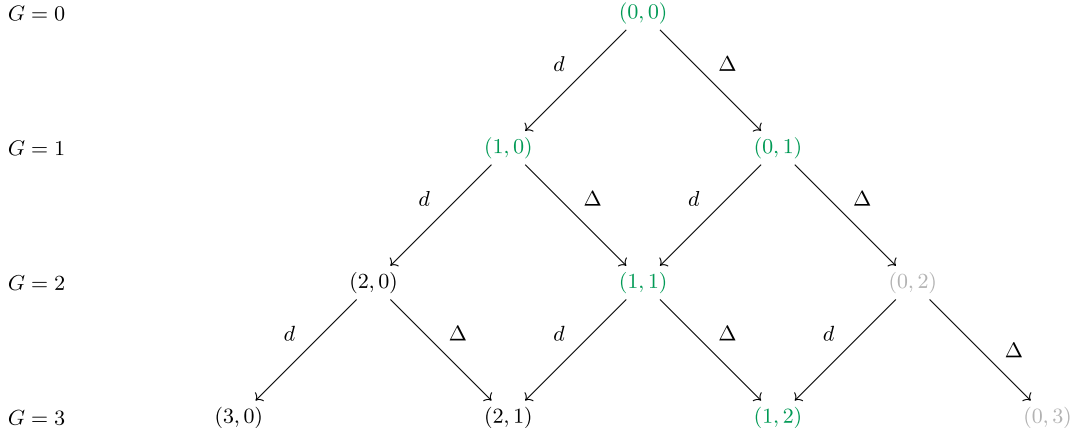


FIG. 1. Bicomplex of superforms. The degrees (p, q) in the above diagram are given by $(N, G - N)$. The spaces that descend to the Yang-Mills kinematic algebra $\mathcal{K}$ are displayed in green. Spaces in gray are zero, due to $c^2 = 0$.

$$h := cd^\dagger, \quad (7)$$

which obeys $h^2 = 0$ and $G(h) = 0, N(h) = -1$. The isomorphism transports the de Rham differential to

$$Q := e^h d e^{-h} = d + \Delta, \quad (8)$$

where Δ is a second differential given by

$$\Delta = c\Box - \mathcal{M}d^\dagger, \quad (9)$$

which obeys $\Delta^2 = 0$ and (anti)commutes with d . The wedge product is transported to

$$\mu := e^h \circ \wedge \circ (e^{-h} \otimes e^{-h}), \quad (10)$$

which explicitly is given by

$$\mu(F_1, F_2) = F_1 \cdot F_2 + c \left[\frac{\partial F_1}{\partial C^\mu} \cdot \partial^\mu F_2 \pm (1 \leftrightarrow 2) \right]. \quad (11)$$

Thus, the transported product is the wedge product shifted by a BV antibracket, which coincides with the above mentioned Schouten-Nijenhuis bracket.

Since above we applied an isomorphism, the resulting algebra $(\Omega^\bullet, Q, \mu)$ is still a strict dgCom algebra. Moreover, it also carries a strict $\text{BV}^\square$ algebra, which arises as follows. We define the vector field

$$b := \frac{\partial}{\partial c}, \quad (12)$$

which is manifestly first order with respect to the wedge product and commutes with d . However, with respect to the transported μ , b turns out to be *second order* and to have the following (anti)commutator with Q :

$$[Q, b] = \Box. \quad (13)$$

These properties can be inferred from the identity

$$\frac{\partial}{\partial c} = e^h \left(\frac{\partial}{\partial c} + d^\dagger \right) e^{-h} \quad (14)$$

and the fact that $d^\dagger$ has the same properties. Thus, the dgCom algebra (Q, μ) is here augmented by a second differential b that is second order with respect to μ and obeys (13); i.e., $(\Omega^\bullet, Q, b, \mu)$ is a (strict) $\text{BV}^\square$ algebra.

Ideal and quotient. We now decompose the de Rham complex of superforms into a subspace that will be related to the kinematic space of Yang-Mills theory and its complement. In terms of the superform degree N , counting the number of C^μ and $\mathcal{M}$, the de Rham complex $\Omega^\bullet$ decomposes as

$$\bigoplus_{n=0}^{\infty} \Omega^n = \Omega^0 \oplus \Omega^1 \oplus \mathcal{I}, \quad \mathcal{I} = \bigoplus_{n=2}^{\infty} \Omega^n, \quad (15)$$

where $\mathcal{I}$ defines the subspace that will become an ideal. More precisely, this space is a subchain complex of $(\Omega^\bullet, Q)$, because Q maps this subspace to itself:

$$\forall \chi \in \mathcal{I}: \quad Q\chi \in \mathcal{I}. \quad (16)$$

This, in turn, allows us to define the Yang-Mills kinematic space as the quotient space

$$\mathcal{K} := \Omega^\bullet / \mathcal{I} \simeq \Omega^0 \oplus \Omega^1. \quad (17)$$

Its elements are the equivalence classes $[F] = [F + \chi]$ for any $\chi \in \mathcal{I}$. One can then define the differential on $\mathcal{K}$ via

$$Q[F] := [QF], \quad (18)$$

where we use the same symbol Q . $(\mathcal{K}, Q)$ is the chain complex of color-stripped Yang-Mills theory as in [15,17].

More explicitly, the single space at $G = 0$ consists of color-stripped gauge parameters $\lambda(x)$, while the $(1, 0)$ and $(0, 1)$ spaces at $G = 1$ contain the fields $C^\mu A_\mu(x)$ and $c\varphi(x)$, respectively, where φ is an auxiliary scalar. The rest of $\mathcal{K}$ consists of the super one-forms in the spaces $(1, 1)$ and $(1, 2)$, where $(1, 1)$ contains the vector and scalar equations as $cC^\mu E_\mu(x) + \mathcal{M}E(x)$, while the Noether identities $c\mathcal{M}N(x)$ belong to the space $(1, 2)$.

The quotient space (17) does not inherit a strict $\text{BV}^\square$ algebra from $(\Omega^\bullet, Q, b, \mu)$, because $\mathcal{I}$ is not an ideal. For instance, the transported wedge product μ of a general element with an element in $\mathcal{I}$ is generally not in $\mathcal{I}$. However, for our purposes it suffices that $\mathcal{I}$ is an ideal up to homotopy so that the quotient space inherits a homotopy version of a $\text{BV}^\square$ algebra: a $\text{BV}_\infty^\square$ algebra. By this, we mean that one can apply a suitable morphism of $\text{BV}_\infty^\square$ algebras to $(\Omega^\bullet, Q, b, \mu)$, after which $\mathcal{I}$ becomes an ideal in the usual sense (all maps with at least one argument in the ideal take values in the ideal). The quotient space (17) then automatically inherits a $\text{BV}_\infty^\square$ algebra.

To this end, we redefine μ by a C_∞ isomorphism:

$$m_2 := \mu + \partial f_2, \quad (19)$$

where f_2 is a graded symmetric bilinear map of degree -1 and we have introduced the shorthand notation

$$\partial f_2 := Q \circ f_2 + f_2 \circ (Q \otimes 1 + 1 \otimes Q). \quad (20)$$

In the realm of C_∞ algebras, the shifted product (19) should be considered equivalent to μ for any f_2 . There are several equivalent choices of f_2 that make $\mathcal{I}$ an ideal with respect to (19). For instance, setting $f_2(F_1, F_2) = c\theta_2(F_1, F_2)$, with θ_2 a graded symmetric bilinear map of degree $G = -2$, we can take, with the χ 's in $\mathcal{I}$ and F in its complement,

$$\begin{aligned} \theta_2(F, \chi) &= (-1)^F \frac{\partial F}{\partial C^\mu} \frac{\partial \chi}{\partial C_\mu}, \\ \theta_2(\chi_1, \chi_2) &= -\frac{1}{2} \mathcal{M} \frac{\partial^2 \chi_1}{\partial C^\mu \partial C^\nu} \cdot \frac{\partial^2 \chi_2}{\partial C_\mu \partial C_\nu}, \end{aligned} \quad (21)$$

while θ_2 is zero otherwise. Since $\mathcal{I}$ is then an ideal with respect to the resulting m_2 , the quotient space $\mathcal{K}$ inherits the C_∞ product

$$m_2([F_1], [F_2]) := [m_2(F_1, F_2)]. \quad (22)$$

The nonvanishing components of m_2 in $\mathcal{K}$ are found to be

$$\begin{aligned} m_2(\lambda_1, \lambda_2) &= \lambda_1 \lambda_2, \\ m_2(\lambda, \mathcal{A}) &= C^\mu(\lambda A_\mu) + c(\lambda \varphi + A^\mu \partial_\mu \lambda), \\ m_2(\lambda, \mathcal{E}) &= cC^\mu(\lambda E_\mu) + \mathcal{M}(\lambda E), \\ m_2(\lambda, N) &= c\mathcal{M}(\lambda N), \\ m_2(\mathcal{A}_1, \mathcal{A}_2) &= 2cC_\mu(\varphi_{[1} A_{2]}^\mu + 2A_{[1}^\rho \partial_\rho A_{2]}^\mu + \partial^\mu A_{[1}^\rho A_{2]\rho}), \\ m_2(\mathcal{A}, \mathcal{E}) &= c\mathcal{M}(\varphi E - A^\mu E_\mu + 2A^\mu \partial_\mu E), \end{aligned} \quad (23)$$

where we denoted by $\mathcal{A} = (A_\mu, \varphi)$ and $\mathcal{E} = (E_\mu, E)$ the doublets of fields and their corresponding equations.

Because of the shift by f_2 in (19), the resulting algebra $(\Omega^\bullet, Q, m_2)$ is no longer a strict dgCom algebra. The associativity of m_2 is violated by f_2 contributions, but it is homotopy associative in the sense of C_∞ algebras. The needed trilinear 3-product is given by

$$\begin{aligned} m_3 &:= \mu \circ (f_2 \otimes 1 - 1 \otimes f_2) + f_2 \circ (\mu \otimes 1 - 1 \otimes \mu) \\ &\quad + \partial f_2 \circ (f_2 \otimes 1 - 1 \otimes f_2). \end{aligned} \quad (24)$$

Remarkably, $\mathcal{I}$ is still an ideal with respect to this m_3 . As a consequence, no further shifts and no higher m_n with $n \geq 4$ are needed. The space $(\Omega^\bullet, m_1 := Q, m_2, m_3)$ then defines a C_∞ algebra, and $\mathcal{I}$ is an ideal of this C_∞ algebra. Consequently, the quotient space (17) inherits a C_∞ algebra with highest nontrivial product being m_3 , whose only nonvanishing component in $\mathcal{K}$ takes three vector fields A_i^μ as inputs and reads

$$m_3^\mu(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = A_1^\rho A_{2\rho} A_3^\mu + A_3^\rho A_{2\rho} A_1^\mu - 2A_1^\rho A_{3\rho} A_2^\mu. \quad (25)$$

This is the kinematic C_∞ algebra of Yang-Mills theory [15, 29, 34] that, thereby, we have derived from a simple algebra of superforms.

Kinematic algebra. As mentioned in the introduction, the C_∞ algebra on $\Omega^\bullet$, and, hence, the C_∞ algebra on the quotient $\mathcal{K}$, is just the first layer of the full $\text{BV}_\infty^\square$ algebra underlying CK duality, to which we turn now. In addition to the graded symmetric and homotopy associative product m_2 , the $\text{BV}_\infty^\square$ algebra features the ‘‘Lie-type’’ bracket:

$$b_2 := [b, m_2], \quad (26)$$

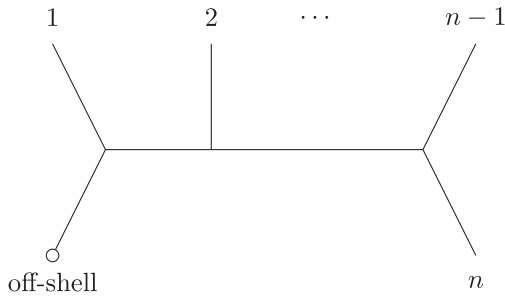
which encodes the failure of b to act as a derivation on m_2 . For a strict $\text{BV}^\square$ algebra, b_2 is the antibracket, which is a graded Lie bracket satisfying the graded Jacobi identity and a Poisson compatibility condition with m_2 . However, since here m_2 is associative only up to homotopy and since b commutes with Q to $\square$ [cf. (13)], the Jacobi identity and Poisson relation get corrected. This defines the $\text{BV}_\infty^\square$ algebra. Specifically, the Jacobiator of b_2 , i.e., its failure

to obey the Jacobi identity, obeys

$$\text{Jac}_{b_2} + \partial b_3 + 6\theta_3 \circ (\partial^\mu \otimes \partial_\mu \otimes 1) \circ \pi = 0, \quad (27)$$

where b_3 is the three-bracket of the kinematic algebra, θ_3 is a new map related to relaxing the Poisson compatibility, and π is a certain Young-tableaux projection [17]. It is straightforward to determine b_3 and θ_3 for the full $\text{BV}_\infty^\square$ algebra on Ω^* , but the challenge is to find a representative in the class of maps related by $\text{BV}_\infty^\square$ isomorphisms for which $\mathcal{I}$ is an ideal. Once such a representative has been identified the quotient space $\mathcal{K} = \Omega^*/\mathcal{I}$ in (17) automatically inherits a $\text{BV}_\infty^\square$ algebra. We will comment in the conclusions on the ongoing research program to determine these maps algorithmically.

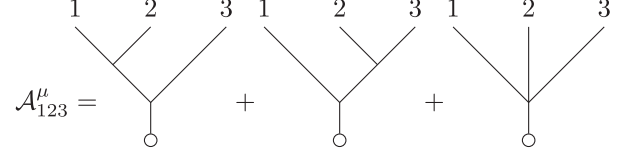
For the remainder of this subsection, we explain how the above relations are related to CK duality, for which the following formulation due to Mafra and Schlotterer is convenient [35–40]. One considers the so-called partial or color-ordered amplitudes, which can be computed from color-ordered Feynman rules (or, equivalently, via homotopy transfer of the kinematic C_∞ algebra [41–45]). One then defines the Berends-Giele currents [46] or multiparticle fields $\mathcal{A}_{12\dots n}^\mu$, with $i = 1, 2, \dots, n$ single-particle labels, as given by the same color-ordered Feynman diagrams but with one leg kept off-shell and dressed with the corresponding propagator. In terms of the local numerators of these diagrams (multiparticle polarizations), CK duality means that the numerators obey the Lie symmetries of nested commutators, where the composition of commutators is dictated by the structure of the cubic diagram. For instance, the numerator $N_{12\dots n}^\mu$ associated with the cubic half-ladder diagram below should have the same symmetry as $[[[T_1, T_2], \dots], T_n]$ in its particle labels:



To understand this in more detail, note from the action (2) that the color-ordered cubic and quartic vertices are described by the two- and three-products m_2 and m_3 of the C_∞ algebra, while the propagator is given by $\frac{b}{\square}$. Upon gauge fixing, the fields obey $b(A) = 0$, so that the numerator contribution $bm_2(A_i, A_j) = b_2(A_i, A_j)$ coincides with the kinematic bracket (26). In particular, the numerator of a cubic half-ladder diagram is given by the successive nesting $b_2(b_2(\dots))$ of two-brackets. Thus, if b_2 were a strict

Lie bracket, it would immediately follow that the numerator has the required Lie symmetries, hence implying CK duality.

However, b_2 does not define a strict Lie algebra, but its failure disappears at the amplitude level, at least at four points. As an example, consider the color-ordered four-point amplitude related to the three-particle field $\mathcal{A}_{123}^\mu$, given by the following diagrams:



which reads analytically

$$\mathcal{A}_{123} = \frac{b}{s_{123}} \left(\frac{m_2(bm_2(A_1, A_2), A_3)}{s_{12}} + \frac{m_2(A_1, bm_2(A_2, A_3))}{s_{23}} + m_3(A_1, A_2, A_3) \right), \quad (28)$$

where we have introduced the Mandelstam variables

$$s_{i_1\dots i_n} = (k_{i_1} + \dots + k_{i_n})^2, \quad (29)$$

and all single particle fields $\mathcal{A}_i^\mu$ are on-shell and transverse, with definite momentum k_i^μ . One next uses that the Young-tableaux symmetries of m_3 are such that one may write

$$m_3(A_1, A_2, A_3) = m_{3h}(A_1, A_2|A_3) - m_{3h}(A_2, A_3|A_1), \quad (30)$$

where m_{3h} is antisymmetric in the first two inputs, in terms of which we have $[b, m_{3h}] = -\theta_{3h}$ [17]. Introducing the three-particle polarizations

$$N_{ijk} := b_2(b_2(A_i, A_j), A_k) - s_{ij}\theta_{3h}(A_i, A_j|A_k), \quad (31)$$

the three-particle field (28) is then given by

$$\mathcal{A}_{123} = \frac{1}{s_{123}} \left(\frac{N_{123}}{s_{12}} - \frac{N_{231}}{s_{23}} \right). \quad (32)$$

CK duality now requires $N_{ijk} = -N_{jik}$, which holds by definition, and $N_{123} + N_{231} + N_{312} = 0$. This sum is given by the Jacobiator of b_2 , plus the terms with θ_3 . In $\text{BV}_\infty^\square$, the two-bracket obeys the deformed homotopy Jacobi identity (27), and, using the fact that $s_{12} + s_{23} + s_{31} = s_{123}$, the sum of the numerators becomes

$$N_{123} + N_{231} + N_{312} = -\partial b_3(A_1, A_2, A_3). \quad (33)$$

The term on the right-hand side is exact and drops out from the amplitude, thereby establishing CK duality at the level of four-point amplitudes. In the framework of [36,37], the exact term $\partial b_3(A_1, A_2, A_3)$ is viewed as a multiparticle

gauge transformation, used to shift the numerators as $N_{123} \rightarrow N_{123} + \frac{1}{3}\partial b_3(A_1, A_2, A_3)$ in order to obey CK duality with one off-shell leg.

Summary and outlook. In this paper, we have introduced a new paradigm for deriving the “kinematic algebra” of Yang-Mills theory, starting from the algebra of differential forms on a superspace. While it has been appreciated in recent years that the algebraic structure underlying color-kinematics duality is a homotopy generalization of a Batalin-Vilkovisky algebra, named $BV_\infty^\square$ by Reiterer [13], so far it has remained elusive to define this algebra for standard Yang-Mills theory explicitly to all orders. Ideally, one would like to obtain this algebra by a derived construction from a strict algebra that carries only linear and bilinear maps.

Here, we have identified an algebra of differential forms on a superspace that carries a strict $BV^\square$ algebra. Remarkably, this space of superforms contains the kinematic space of genuine Yang-Mills theory as a subspace. Moreover, its complement $\mathcal{I}$ within the full space is an ideal up to homotopy. The C_∞ algebra of Yang-Mills theory is then realized as the quotient algebra.

It remains to generalize this derived construction of the C_∞ subsector to the full $BV_\infty^\square$ algebra. One challenge here is to realize the $BV_\infty^\square$ isomorphism on the full space that transforms all higher maps to a form where $\mathcal{I}$ is an ideal. To this end, the recent work [47] in pure mathematics might be important. Establishing an effective procedure for obtaining these transformations would yield an algorithm for

determining the CK duality obeying kinematic numerators in a manifestly local fashion, thereby proving CK duality of Yang-Mills theory at tree level. It would also be interesting to generalize this to more complicated Yang-Mills theories such as supersymmetric theories [48,49].

The algorithmic determination of the higher $BV_\infty^\square$ maps would also be crucial in order to realize gravity as the double copy of Yang-Mills theory. Specifically, one would like to obtain the off-shell and local Lagrangian of gravity, and its diffeomorphism transformations, purely from the Yang-Mills data. (See [50–54] for earlier work.) A promising shortcut might be to double copy the full algebra of superforms as in [55–57] and then to determine $\mathcal{N} = 0$ supergravity, respectively, double field theory [58–61] as in [54], as the appropriate quotient.

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Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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